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# Equilibrium fluctuation for an anharmonic chain with boundary conditions in the Euler scaling limit

Stefano OLLA      Lu XU

## Abstract

We study the evolution in equilibrium of the fluctuations for the conserved quantities of a chain of anharmonic oscillators in the hyperbolic space-time scaling limit. Boundary conditions are determined by applying a constant tension at one side, while the position of the other side is kept fixed. The Hamiltonian dynamics is perturbed by random terms conservative of such quantities. We prove that these fluctuations evolve macroscopically following the linearized Euler equations with the corresponding boundary conditions. Furthermore, we prove that such linearized evolution holds in some time scales larger than the hyperbolic one.

## 1. Introduction

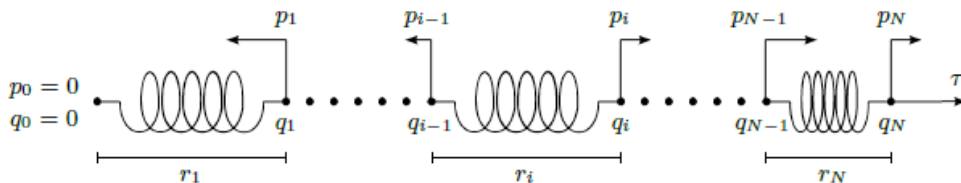
The deduction of Euler equations for a compressible gas from the microscopic dynamics in some space-time scaling limit is one of the main problems in statistical mechanics [17]. With a generic assumption of *local equilibrium*, Euler equations can be formally obtained in the limit, but a mathematical proof starting from deterministic Hamiltonian dynamics is still an open problem. The eventual appearance of shock waves complicates further the task, and in this case, it is expected the convergence to weak entropic solutions of Euler equations.

Some mathematical results have been obtained by perturbing the Hamiltonian dynamics with random terms that conserve only energy, momentum and density, in such a way that the dynamics has enough ergodicity to generate some form of local equilibrium (cf. [19, 7]). These results are obtained by relative entropy techniques and restricted to the smooth regime of the Euler equations. The conservative noise introduced in these works is essentially constituted by random collisions between close particles that exchange their velocities. Under such random perturbations, the only conserved quantities are those that evolve macroscopically with the Euler equations [12]. Actually, random dynamics and local equilibrium are only tools to obtain the separation of scales between microscopic and macroscopic modes necessary in order to close the Euler equations. In the deterministic dynamics of harmonic oscillators with random masses (a non-ergodic dynamics), Anderson localization provides such separation of scales [6].

In the present article we study the evolution of the fluctuations of the conserved quantities in a chain of anharmonic oscillators perturbed by a stochastic noise. The equilibrium states are characterized by certain average values of these conserved quantities. When the system is in equilibrium, these conserved quantities have Gaussian macroscopic fluctuations. Our aim is to prove that these fluctuations, in the macroscopic space-time scaling limit, evolve deterministically following the linearized Euler equations. It turns out that this is more difficult than proving the hydrodynamic limit, as it requires the control of the space-time variance of the currents of the conserved quantities. More precisely it demands to prove that the currents are *equivalent* (in the norm introduced by the space-time variance) to linear functions of the conserved quantities. This step is usually called *Boltzmann-Gibbs principle* (cf. [8, 14]). This is

the main part of the proof, and it forces us to consider *elliptic* type of stochastic perturbations, i.e., noise terms that act also on the positions, not only on the velocities, still maintaining the same three conserved quantities.

We consider a system of  $N$  coupled anharmonic oscillators, similar to the one studied in [7]. For  $i = 0, \dots, N$ , the momentum (or velocity, since we set the masses equal to 1) of the particle  $i$  is denoted by  $p_i \in \mathbb{R}$ , while  $q_i \in \mathbb{R}$  denotes its position. Particle 0 is attached to some fixed point, thus  $p_0 = 0, q_0 = 0$ . Meanwhile, particle  $N$  is pulled (or pushed) by a force  $\tau \in \mathbb{R}$ , which is constant in time.



Each pair of consecutive particles  $(i-1, i)$  is connected by a (nonlinear) spring with potential  $V(q_i - q_{i-1})$ . We need to assume certain assumptions for the potential energy  $V : \mathbb{R} \rightarrow \mathbb{R}$ . The energy of the system is then given by

$$H_N(\mathbf{p}, \mathbf{q}) = \sum_{i=1}^N \left[ \frac{p_i^2}{2} + V(q_i - q_{i-1}) \right].$$

Therefore, the inter-particle distances  $\{r_i = q_i - q_{i-1}; 1 \leq i \leq N\}$  are the essentially relevant variables. Notice that here  $r_i$  can also assume negative values. Let  $e_i = p_i^2/2 + V(r_i)$  be the energy assigned to  $i$ -th particle, then  $H_N = \sum e_i$ . The corresponding Hamiltonian dynamics locally conserves the sums of  $p_i, r_i$  and  $e_i$ . By adding proper stochastic perturbations on the deterministic dynamics, we can make them the only conserved quantities.

Let  $w_i = (p_i, r_i, e_i)$  be the vector of conserved quantities. The hydrodynamic limit is given by the convergence, for any continuous  $G$  on  $[0, 1]$ ,

$$\frac{1}{N} \sum_{i=1}^N w_i(Nt) G\left(\frac{i}{N}\right) \xrightarrow{N \rightarrow \infty} \int_0^1 \mathbf{w}(t, x) G(x) dx,$$

where  $\mathbf{w} = (\mathbf{p}, \mathbf{r}, \mathbf{e})$  solves the *compressible Euler equations*

$$\partial_t \mathbf{w} = \partial_x F(\mathbf{w}), \quad F(\mathbf{w}) = (\boldsymbol{\tau}(\mathbf{r}, \mathbf{u}), \mathbf{p}, \boldsymbol{\tau}(\mathbf{r}, \mathbf{u})\mathbf{p}), \quad \mathbf{u} = \mathbf{e} - \mathbf{p}^2/2, \quad (1.1)$$

with boundary conditions given by

$$\mathbf{p}(0, t) = 0, \quad \boldsymbol{\tau}(\mathbf{r}(1, t), \mathbf{u}(1, t)) = \tau,$$

where  $\boldsymbol{\tau}(r, e)$  is the tension function defined in (2.8) later. In the smooth regime of (1.1), this is proven by relative entropy techniques in [7].

We consider here the system in equilibrium, starting with the Gibbs measure

$$\prod_{i=1}^N \exp\{\lambda \cdot (r_i, e_i) - \mathcal{G}(\lambda)\} dp_i dr_i, \quad (1.2)$$

for given  $\lambda = (\beta\tau, -\beta) \in \mathbb{R} \times \mathbb{R}_-$ , where  $\mathcal{G}$  is the Gibbs potential given by

$$\mathcal{G}(\lambda) = \ln \left( \int_{\mathbb{R}} \exp\{-\beta V(r) + \beta\tau r\} dr \right) + \frac{1}{2} \ln \left( \frac{2\pi}{\beta} \right). \quad (1.3)$$

Denote by  $E_{\lambda,N}$  the expectation with respect to the measure in (1.2). Correspondingly, there are equilibrium values  $0 = E_{\lambda,N}[p_i]$ ,  $\bar{r} = E_{\lambda,N}[r_i]$ ,  $\bar{e} = E_{\lambda,N}[e_i]$  for the conserved quantities. The empirical distribution of the fluctuations of the conserved quantities is defined by

$$\frac{1}{\sqrt{N}} \sum_{i=1}^N \begin{pmatrix} p_i(Nt) \\ r_i(Nt) - \bar{r} \\ e_i(Nt) - \bar{e} \end{pmatrix} \delta \left( x - \frac{i}{N} \right).$$

Formally, it is expected to converge to the solution  $\tilde{\mathbf{w}}(t, x) = (\tilde{\mathbf{p}}, \tilde{\mathbf{r}}, \tilde{\mathbf{e}})(t, x)$  of

$$\partial_t \tilde{\mathbf{w}} = F'(0, \bar{r}, \bar{e}) \partial_x \tilde{\mathbf{w}}, \quad x \in [0, 1], \quad (1.4)$$

where  $F'(\bar{w})$  is the Jacobian matrix of  $F$ , with boundary conditions

$$\tilde{\mathbf{p}}(t, 0) = 0, \quad \left. \frac{\partial \boldsymbol{\tau}}{\partial r} \right|_{(\bar{r}, \bar{e})} \tilde{\mathbf{r}}(t, 1) + \left. \frac{\partial \boldsymbol{\tau}}{\partial e} \right|_{(\bar{r}, \bar{e})} \tilde{\mathbf{e}}(t, 1) = 0, \quad (1.5)$$

and a proper Gaussian stationary initial distribution. Notice that  $\tilde{\mathbf{w}}(t)$  takes values as distributions on  $[0, 1]$ , so (1.4) with the boundary conditions (1.5) should be intended in the weak sense, as rigorously defined in Section 3.

While the non-equilibrium hydrodynamic limit can be proven by adding a simple exchange of  $p_i$  with  $p_{i+1}$  at random independent times (cf. [7]), in order to prove (1.4) we need to add, for each bond  $(i, i+1)$ , a stochastic perturbation on the values of  $p_i$ ,  $p_{i+1}$ ,  $r_i$  and  $r_{i+1}$ , in such way that  $r_i + r_{i+1}$ ,  $p_i + p_{i+1}$ ,  $e_i + e_{i+1}$  are conserved. The corresponding microcanonical surface is a one-dimensional circle, where we add a Wiener process. This stochastic perturbation corresponds to adding a symmetric second order differential operator  $\mathcal{S}_N$  defined by (2.2) that is elliptic on the corresponding microcanonical surfaces. The main part of the article is the proof of a lower bound of order  $N^{-2}$  on the spectral gap of  $\mathcal{S}_N$  that is independent of the values of the conserved quantities. This is an important ingredient for proving the Boltzmann-Gibbs linearization for the dynamics.

The present article contains the first result on equilibrium fluctuations for anharmonic chain of oscillators with multiple conserved quantities. Previous results concerned only linear dynamics or vanishing anharmonicity (e.g. [5] for a system with two conserved quantities). Another novelty of the present article is the presence of nonlinear boundary conditions (tension at the border), as previous results on equilibrium fluctuations concern systems with no boundary conditions, or linear in the conserved quantities.

The hyperbolic scale describes the time for the system to reach its mechanical equilibrium. Beyond that, it takes more time to reach the thermal equilibrium. It is a natural question to investigate the behaviour of the equilibrium fluctuations in larger time scales. In Theorem 3.3 we prove for our anharmonic system that the equilibrium fluctuations of the three conserved quantities continue to evolve deterministically according to the linearized Euler equations up to any time scale  $N^a t$  with  $a \in [1, 6/5)$ . For harmonic chain with two conserved quantities and no boundary conditions an analogous result can be found in [4]. Superdiffusion of energy fluctuations is conjectured in [22] and should appear for some  $a \geq 3/2$ . This has been proven rigorously for harmonic chains with conservative noise (cf. [13] for dynamics with 3 conserved quantities and [3] with two conserved quantities). Results in [13] extends also to the non-stationary superdiffusive evolution of the energy density, while the other two quantities evolve diffusively [16]. See also the review [2] and the other articles in the same volume about the numerical evidence in nonlinear dynamics. The extension of such superdiffusive results to the nonlinear dynamics is one of the most challenging problem. Some results for vanishing anharmonicity can be found in [5].

We believe that such macroscopic behaviour of the equilibrium fluctuations should be valid also for the deterministic (nonlinear) dynamics, but even the case with stochastic perturbations acting only on the velocities remains an open problem.

Another important open problem concerns the evolution of fluctuations out of equilibrium. In the context of the hyperbolic scaling limit, for system with one conserved quantity, like the asymmetric simple exclusion, non-equilibrium fluctuation has been studied in [21].

The present article is organized as follows. In Section 2 we give the precise definition of the microscopic dynamics. In Section 3 we define the solution of (1.4)–(1.5) rigorously and state our main results. In Section 4, 5 and 6 we show the finite-dimensional convergence and the tightness of the fluctuation field, which completes the proof of the main results. The arguments we used are based on an estimate for the spectral gap of  $\mathcal{S}_N$  and the equivalence of ensembles. They are established in Section 7 and 8, respectively.

## 2. The microscopic model

In this section we state the rigorous definition of the microscopic dynamics. Let  $V$  be a convex,  $C^4$ -smooth function on  $\mathbb{R}$  with quadratic growth:

$$\inf_{r \in \mathbb{R}} V''(r) > 0, \quad \sup_{r \in \mathbb{R}} V''(r) < \infty. \quad (2.1)$$

Observe that (2.1) assures that  $V(r)$  acquires its minimum at some unique point  $r_0 \in \mathbb{R}$ . By replacing  $V$  with  $V_* = V(\cdot + r_0) - V(r_0)$ , we can assume without loss of generality that  $V \geq 0$ ,  $V(0) = 0$  and  $V'(0) = 0$ .

For  $N \geq 1$ , let  $\Omega_N = \mathbb{R}^{2N}$  be the configuration space. Its elements are denoted by

$$\eta = (\mathbf{p}, \mathbf{r}); \quad \mathbf{p} = (p_1, \dots, p_N), \quad \mathbf{r} = (r_1, \dots, r_N).$$

Fix  $\tau \in \mathbb{R}$ ,  $p_0 = 0$ , and define first-order differential operators  $\mathcal{X}_i$  acting on smooth functions on  $\Omega_N$  by

$$\begin{aligned} \mathcal{X}_i &= (p_i - p_{i-1}) \frac{\partial}{\partial r_i} + (V'(r_{i+1}) - V'(r_i)) \frac{\partial}{\partial p_i}, \quad \text{for } 1 \leq i \leq N-1, \\ \mathcal{X}_N &= (p_N - p_{N-1}) \frac{\partial}{\partial r_N} + (\tau - V'(r_N)) \frac{\partial}{\partial p_N}. \end{aligned}$$

In addition, define  $\mathcal{Y}_{i,i+1}$  for  $1 \leq i \leq N-1$  as

$$\mathcal{Y}_{i,i+1} = (p_{i+1} - p_i) \left( \frac{\partial}{\partial r_{i+1}} - \frac{\partial}{\partial r_i} \right) - (V'(r_{i+1}) - V'(r_i)) \left( \frac{\partial}{\partial p_{i+1}} - \frac{\partial}{\partial p_i} \right).$$

For any  $\gamma > 0$ , the generator  $\mathcal{L}_N$  is given by

$$\mathcal{L}_N = \mathcal{A}_N + \gamma \mathcal{S}_N, \quad \mathcal{A}_N = \sum_{i=1}^N \mathcal{X}_i, \quad \mathcal{S}_N = \frac{1}{2} \sum_{i=1}^{N-1} \mathcal{Y}_{i,i+1}^2. \quad (2.2)$$

In (2.2), the Liouville operator  $\mathcal{A}_N$  generates the Hamiltonian system introduced in Section 1. Meanwhile,  $\mathcal{Y}_{i,i+1}$  acts on  $(p_i, p_{i+1}, r_i, r_{i+1})$ , maintaining the values of  $p_i + p_{i+1}$ ,  $r_i + r_{i+1}$  and  $e_i + e_{i+1}$ . As  $\mathcal{Y}_{i,i+1}$  generates the tangent space of the one-dimensional microcanonical surface,  $\mathcal{Y}_{i,i+1}^2$  generates a Brownian motion on it. This choice of noise assures that  $p_i$ ,  $r_i$  and  $e_i$  are the only locally conserved quantities.

For each  $N \geq 1$ , denote by  $\{\eta_t = (\mathbf{p}(t), \mathbf{r}(t)) \in \Omega_N; t \geq 0\}$  the Markov process generated by  $N\mathcal{L}_N$ . Observe that  $\eta_t$  can be equivalently expressed by the solution to the following system of

stochastic differential equations:

$$\begin{cases} dp_1(t) = N\nabla_N V'(r_1)dt + dJ_1^p, \\ dp_i(t) = N\nabla_N V'(r_i)dt - \nabla_N^* dJ_i^p, & \text{for } 2 \leq i \leq N-1, \\ dp_N(t) = N[\tau - V'(r_N)]dt - dJ_{N-1}^p, \\ dr_1(t) = Np_1 dt + dJ_1^r, \\ dr_i(t) = N\nabla_N p_{i-1} dt - \nabla_N^* dJ_i^r, & \text{for } 2 \leq i \leq N-1, \\ dr_N(t) = N\nabla_N p_{N-1} dt - dJ_{N-1}^r. \end{cases} \quad (2.3)$$

Here in (2.3),  $J_i^p$ ,  $J_i^r$  stand for the currents of  $p_i$  and  $r_i$  associated to the stochastic dynamics, respectively given by

$$\begin{aligned} dJ_i^p &= \frac{\gamma N}{2} [V''(r_{i+1}) + V''(r_i)] \nabla_N p_i dt + \sqrt{\gamma N} (\nabla_N V'(r_i)) dB_t^i, \\ dJ_i^r &= \gamma N \nabla_N V'(r_i) dt - \sqrt{\gamma N} (\nabla_N p_i) dB_t^i, \end{aligned}$$

where for any sequence  $\{f_i\}$ ,  $\nabla_N f_i = f_{i+1} - f_i$ ,  $\nabla_N^* f_i = f_{i-1} - f_i$ , and  $\{B^i; i \geq 1\}$  is an infinite system of independent, standard Brownian motions. Notice that  $J_p^i$  and  $J_r^i$  share the same Brownian noise  $B_t^i$ .

Denote by  $\pi_{\lambda, N}$  the Gibbs measure in (1.2). The class of bounded, smooth functions on  $\Omega_N$  forms a core of  $\mathcal{A}_N$  and  $\mathcal{S}_N$  in  $L^2(\pi_{\lambda, N})$ , and for such  $f$  and  $g$ ,

$$E_{\lambda, N}[(\mathcal{A}_N f)g] = -E_{\lambda, N}[f(\mathcal{A}_N g)], \quad E_{\lambda, N}[(\mathcal{S}_N f)g] = E_{\lambda, N}[f(\mathcal{S}_N g)].$$

In particular,  $\pi_{\lambda, N}$  is stationary with respect to  $\mathcal{L}_N$ . Moreover,

$$E_{\lambda, N}[f(-\mathcal{L}_N f)] = \gamma E_{\lambda, N}[f(-\mathcal{S}_N f)] = \frac{\gamma}{2} \sum_{i=1}^N E_{\lambda, N}[(\mathcal{Y}_{i, i+1} f)^2].$$

Recall that  $w_i = (p_i, r_i, e_i)$  and let  $\bar{w} = E_{\lambda, N}[w_i]$ , then  $\bar{w} = (0, \bar{r}(\lambda), \bar{e}(\lambda))$ , where

$$(\bar{r}(\lambda), \bar{e}(\lambda)) = \nabla_\lambda \mathcal{G}(\lambda) = \left( \frac{1}{\beta} \frac{\partial}{\partial \tau}, \frac{\tau}{\beta} \frac{\partial}{\partial \tau} - \frac{\partial}{\partial \beta} \right) \mathcal{G}(\lambda). \quad (2.4)$$

where  $\mathcal{G}(\lambda)$  is defined in (1.3). It is also worth noticing that the tension in equilibrium is  $E_{\lambda, N}[V'(r_i)] = \tau$ . Furthermore, the covariance matrix  $\Sigma = \Sigma(\lambda)$  of  $w_i$  under  $\pi_{\lambda, N}$  is given by

$$\Sigma = E_{\lambda, N}[(w_i - \bar{w}) \otimes (w_i - \bar{w})] = \left[ \begin{array}{c|cc} \beta^{-1} & 0 & 0 \\ \hline 0 & \mathcal{G}''(\lambda) & \\ 0 & & \end{array} \right], \quad (2.5)$$

where  $\mathcal{G}''(\lambda)$  stands for the Hessian matrix of  $\mathcal{G}$  calculated in  $\lambda$ .

Define the *thermodynamic entropy*  $\mathcal{S}$  for  $r \in \mathbb{R}$  and  $e > 0$  by

$$\mathcal{S}(r, e) = -\mathcal{G}^*(r, e), \quad \mathcal{G}^*(r, e) = \sup_{\lambda \in \mathbb{R} \times \mathbb{R}_-} \{\lambda \cdot (r, e) - \mathcal{G}(\lambda)\}.$$

Under our assumptions,  $\mathcal{G}$  is strictly convex and so is its Legendre transform  $\mathcal{G}^*$ . Hence,  $\mathcal{S}$  is strictly concave. By the properties of Legendre transform (see, e.g. [1, p.63, Theorem]),

$$\lambda(r, e) \triangleq \nabla_{r, e} \mathcal{G}^*(r, e) = -\nabla_{r, e} \mathcal{S}(r, e) \in \mathbb{R} \times \mathbb{R}_- \quad (2.6)$$

gives the inverse of  $\lambda \mapsto \nabla_\lambda \mathcal{G}(\lambda)$ . In view of (2.4),

$$\mathcal{G}''(\lambda) \mathcal{S}''(\bar{r}(\lambda), \bar{e}(\lambda)) = \mathcal{G}''(\lambda(r, e)) \mathcal{S}''(r, e) = -I_{2 \times 2}. \quad (2.7)$$

For convenience, we denote  $\lambda = (\beta\tau, -\beta)$ , where  $(\beta, \tau)$  are the equilibrium temperature and tension, respectively given by

$$\beta(r, e) = \partial_e \mathcal{S}(r, e), \quad \tau(r, e) = -\frac{\partial_r \mathcal{S}(r, e)}{\partial_e \mathcal{S}(r, e)}. \quad (2.8)$$

By (2.6),  $\beta(r, e)$  is always positive, and

$$\begin{aligned} \frac{\partial \tau}{\partial r} + \tau \frac{\partial \tau}{\partial e} &= \frac{1}{\beta} \left( -\frac{\partial^2 \mathcal{S}}{(\partial r)^2} + \frac{1}{\beta} \frac{\partial \beta}{\partial r} \frac{\partial \mathcal{S}}{\partial r} \right) - \frac{1}{\beta^2} \frac{\partial \mathcal{S}}{\partial r} \left( -\frac{\partial^2 \mathcal{S}}{\partial r \partial e} + \frac{1}{\beta} \frac{\partial \beta}{\partial e} \frac{\partial \mathcal{S}}{\partial r} \right) \\ &= -\frac{1}{\beta^3} \left( \beta^2 \frac{\partial^2 \mathcal{S}}{(\partial r)^2} - \beta \frac{\partial^2 \mathcal{S}}{\partial r \partial e} \frac{\partial \mathcal{S}}{\partial r} - \beta \frac{\partial^2 \mathcal{S}}{\partial r \partial e} \frac{\partial \mathcal{S}}{\partial r} + \frac{\partial^2 \mathcal{S}}{\partial e^2} \left[ \frac{\partial \mathcal{S}}{\partial r} \right]^2 \right) \\ &= \frac{1}{\beta^3} \left( \frac{\partial \mathcal{S}}{\partial e}, -\frac{\partial \mathcal{S}}{\partial r} \right) \cdot (-\mathcal{S})'' \left( \frac{\partial \mathcal{S}}{\partial e}, -\frac{\partial \mathcal{S}}{\partial r} \right). \end{aligned}$$

Since  $\mathcal{S}$  is strictly concave, one can conclude that

$$\frac{\partial \tau}{\partial r} + \tau \frac{\partial \tau}{\partial e} > 0. \quad (2.9)$$

Later in (3.3) one can observe that the expression in (2.9) defines the square the sound speed in the macroscopic hyperbolic system.

Let  $\mathbb{P}_{\lambda, N}$  be the probability measure on the path space  $C([0, \infty), \Omega_N)$  induced by (2.3) and initial condition  $\pi_{\lambda, N}$ . The corresponding expectation is denoted by  $\mathbb{E}_{\lambda, N}$ . We are interested in the evolution of the fluctuations of the balanced quantities of  $\{\eta_t\}$  in macroscopic time. For a smooth function  $H : [0, 1] \rightarrow \mathbb{R}^3$ , define the *empirical distribution of conserved quantities fluctuation field* on  $H$  as

$$Y_N(t, H) = \frac{1}{\sqrt{N}} \sum_{i=1}^N H \left( \frac{i}{N} \right) \cdot (w_i(\eta_t) - \bar{w}), \quad \forall t \geq 0, \quad (2.10)$$

Notice that we consider in (2.10) the *hyperbolic scale*, where the space and time variables are rescaled by the same order of  $N$ .

We close this section with some useful notations. Throughout this article,  $|\cdot|$  and  $\cdot$  always refer to the standard Euclidean norm and inner product in  $\mathbb{R}^d$ . Let  $\mathcal{H}$  be the space of three-dimensional functions  $f = (f_1, f_2, f_3)$  on  $[0, 1]$ , where each  $f_i$  is square integrable. The scalar product and norm on  $\mathcal{H}$  are given by

$$\langle f, g \rangle = \int_0^1 f(x) \cdot g(x) dx, \quad \|f\|^2 = \int_0^1 |f(x)|^2 dx.$$

Then  $\mathcal{H}$  is a Hilbert space, and denote by  $\mathcal{H}'$  its dual space, consisting of all bounded linear functionals on  $\mathcal{H}$ . Note that the definition in (2.10) satisfies that:

$$\mathbb{E}_{\lambda, N} [Y_N^2(t, H)] \leq |\text{Tr } \Sigma(\lambda)| \cdot \frac{1}{N} \sum_{i=1}^N \left| H \left( \frac{i}{N} \right) \right|^2.$$

Thus, one can easily extend the definition of  $Y_N(t, H)$  to all  $H \in \mathcal{H}$ . For all  $N \geq 1$ ,  $t \geq 0$  and  $H \in \mathcal{H}$ ,  $Y_N(t, H) \in L^2(\Omega_N; \pi_{\lambda, N})$ .

### 3. Euler system with boundary conditions

In this section we state the precise definition of the solution to (1.4)–(1.5) with proper random distribution-valued initial condition. The equation (1.4) can be written explicitly as

$$\partial_t \tilde{\mathbf{p}} = \tau_r \partial_x \tilde{\mathbf{c}} + \tau_e \partial_x \tilde{\mathbf{e}}, \quad \partial_t \tilde{\mathbf{c}} = \partial_x \tilde{\mathbf{p}}, \quad \partial_t \tilde{\mathbf{e}} = \tau \partial_x \tilde{\mathbf{p}},$$

where  $(\tau_r, \tau_e)$  are constants given by

$$\tau_r(\lambda) = \frac{\partial}{\partial r} \boldsymbol{\tau}(\bar{r}(\lambda), \bar{e}(\lambda)), \quad \tau_e(\lambda) = \frac{\partial}{\partial e} \boldsymbol{\tau}(\bar{r}(\lambda), \bar{e}(\lambda)). \quad (3.1)$$

Recall that  $(\boldsymbol{\beta}, \boldsymbol{\tau})(\bar{r}(\lambda), \bar{e}(\lambda)) = (\beta, \tau)$  are constants, and by (2.8),  $\partial_r \mathcal{S} = -\boldsymbol{\beta} \boldsymbol{\tau}$ ,  $\partial_e \mathcal{S} = \boldsymbol{\beta}$ . Formally define the linear transformation

$$\tilde{\tau} = \tau_r \tilde{\mathbf{t}} + \tau_e \tilde{\mathbf{e}}, \quad \tilde{S} = -\beta \tau \tilde{\mathbf{t}} + \beta \tilde{\mathbf{e}}.$$

The new coordinates  $\tilde{\tau}, \tilde{S}$  can be viewed as the fluctuation field of tension and thermodynamic entropy, respectively. From (1.4),  $(\tilde{\mathbf{p}}, \tilde{\tau}, \tilde{S})$  evolves with the equation

$$\partial_t \tilde{\mathbf{p}} = \partial_x \tilde{\tau}, \quad \partial_t \tilde{\tau} = c^2 \partial_x \tilde{\mathbf{p}}, \quad \partial_t \tilde{S} = 0, \quad (3.2)$$

where the constant  $c > 0$  is the speed of sound given by

$$c^2 = \tau_r + \tau \tau_e > 0, \quad (3.3)$$

cf. (2.9) and [22, (3.10)]. This transformation also decouples the boundary conditions:

$$\tilde{\mathbf{p}}(t, 0) = 0, \quad \tilde{\tau}(t, 1) = 0. \quad (3.4)$$

Clearly,  $(\tilde{\mathbf{p}}, \tilde{\tau})$  are two coupled sound modes with mixed boundaries, while  $\tilde{S}$  is independent of  $(\tilde{\mathbf{p}}, \tilde{\tau})$  and does not evolve in time. In the case where initial data is smooth and satisfies the boundary conditions, one can easily obtain the smooth solution  $\tilde{\mathbf{w}} = \tilde{\mathbf{w}}(t, x)$  to (1.4)–(1.5) by applying the inverse transformation.

Since  $\tilde{\mathbf{w}}(0)$  is a Gaussian random field, in order to define the inverse transformation rigorously in the non-smooth case, we have to consider the weak solution of (3.2) and (3.4). Define a subspace  $\mathcal{C}(\lambda)$  of  $\mathcal{H}$  by

$$\mathcal{C}(\lambda) = \{g = (g_1, g_2, g_3) \mid g_i \in C^1([0, 1]), g_1(0) = 0, \tau_r g_2(1) + \tau_e g_3(1) = 0\}.$$

Define the first-order differential operator  $L$  on  $\mathcal{C}(\lambda)$  by

$$L = B \left( \frac{d}{dx} \right), \quad \text{where } B = F'(\bar{w}) = \begin{bmatrix} 0 & \tau_r & \tau_e \\ 1 & 0 & 0 \\ \tau & 0 & 0 \end{bmatrix}.$$

Observe that  $B$  has three real eigenvalues  $\{0, \pm c\}$ , thus generates a hyperbolic system. With some abuse of notations, denote the closure of  $L$  on  $\mathcal{H}$  still by  $L$ . For  $i = 1, 2$ , let  $\{\mu_{i,n}; n \geq 0\}$  be two Fourier bases of  $L^2([0, 1])$  given by

$$\mu_{1,n}(x) = \sqrt{2} \sin(\theta_n x), \quad \mu_{2,n}(x) = \sqrt{2} \cos(\theta_n x), \quad \theta_n = \frac{(2n+1)\pi}{2}. \quad (3.5)$$

Notice that  $\mu_{1,n}(0) = \mu_{2,n}(1) = 0$ , in accordance with the boundary conditions in (3.4). For  $k \geq 1$ , define the Sobolev spaces

$$H_k = \left\{ f = (f_1, f_2) \mid \sum_{i=1}^2 \sum_{n=0}^{\infty} \theta_n^{2k} \left( \int_0^1 f_i(x) \mu_{i,n}(x) dx \right)^2 < \infty \right\}.$$

Then  $\text{dom}(L) = \{(g_1, \tau_r g_2 + \tau_e g_3) \in H_1\}$ . To identify the adjoint  $L^*$  of  $L$ , observe that for any  $g \in \mathcal{C}(\lambda)$  and  $h \in \mathcal{H}$ ,

$$\langle Lg, h \rangle = \int_0^1 [g_1'(h_2 + \tau h_3) + (\tau_r g_2' + \tau_e g_3') h_1] dx.$$



Therefore,  $\text{dom}(L^*) = \{(h_1, h_2 + \tau h_3) \in H_1\}$ . In particular,

$$\mathcal{C}_*(\tau) = \{h = (h_1, h_2, h_3) \mid h_i \in C^1([0, 1]), h_1(0) = 0, h_2(1) + \tau h_3(1) = 0\}$$

is a core of  $L^*$  and  $L^*h = -B^T h'$  for  $h \in \mathcal{C}_*(\tau)$ . Notice that  $\mathcal{C}_*(\tau)$  depends only on  $\tau$ , while  $\mathcal{C}(\lambda)$  depends on both  $\beta$  and  $\tau$ .

Now we can define the solution of (1.4)–(1.5) precisely. Let  $\{\tilde{\mathbf{w}}(t) = \tilde{\mathbf{w}}(t, \cdot); t \geq 0\}$  be a stochastic process taking values in  $\mathcal{H}'$ , such that for all  $h \in \mathcal{C}_*(\tau)$ ,

$$\tilde{\mathbf{w}}(t, h) - \tilde{\mathbf{w}}(0, h) = \int_0^t \tilde{\mathbf{w}}(s, L^*h) ds, \quad \forall t > 0, \quad (3.6)$$

and  $\tilde{\mathbf{w}}(0)$  is a Gaussian variable such that for  $h, g \in \mathcal{H}$ ,

$$E[\tilde{\mathbf{w}}(0, h)] = 0, \quad E[\tilde{\mathbf{w}}(0, h)\tilde{\mathbf{w}}(0, g)] = \langle h, \Sigma g \rangle, \quad (3.7)$$

where  $\Sigma$  is the covariance matrix defined in (2.5).

To see the existence and uniqueness of  $\tilde{\mathbf{w}}(t)$ , consider the weak form of equation (3.2): for  $f = (f_1, f_2, f_3)$ ,  $f_i \in C^1([0, 1])$ ,  $f_1(0) = f_2(1) = 0$ ,

$$\tilde{\mathbf{u}}(t, f) - \tilde{\mathbf{u}}(0, f) + \int_0^t \tilde{\mathbf{u}}(s, A^T f') ds = 0, \quad A = \begin{bmatrix} 0 & 1 & 0 \\ c^2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad (3.8)$$

and  $\tilde{\mathbf{u}}(0)$  is a centred Gaussian variable with covariance

$$E[(\tilde{\mathbf{u}}(0, f))^2] = \langle f, Qf \rangle, \quad Q = \text{diag}(\beta^{-1}, \beta^{-1}c^2, \beta^2\partial_\beta^2\mathcal{G}).$$

Suppose that  $\{\boldsymbol{\mu}_{i,n}, \boldsymbol{\nu}_{i,n}; n \geq 0, i = 1, 2\}$  is the three-dimensional Fourier basis given by  $\boldsymbol{\mu}_{1,n} = (\mu_{1,n}, 0, 0)$ ,  $\boldsymbol{\mu}_{2,n} = (0, \mu_{2,n}, 0)$ , and<sup>1</sup>

$$\boldsymbol{\nu}_{1,n}(x) = \sqrt{2}(0, 0, \sin(\kappa_n x)), \quad \boldsymbol{\nu}_{2,n}(x) = \sqrt{2}(0, 0, \cos(\kappa_n x)), \quad \kappa_n = 2n\pi. \quad (3.9)$$

The solution  $\tilde{\mathbf{u}}(t)$  is a stationary Gaussian process, satisfying that

$$\begin{aligned} \tilde{\mathbf{u}}(t, \boldsymbol{\mu}_{1,n}) &= \frac{1}{\sqrt{\beta}}(X_{1,n} \cos(c\theta_n t) + X_{2,n} \sin(c\theta_n t)), \\ \tilde{\mathbf{u}}(t, \boldsymbol{\mu}_{2,n}) &= \frac{c}{\sqrt{\beta}}(X_{1,n} \sin(c\theta_n t) - X_{2,n} \cos(c\theta_n t)), \\ \tilde{\mathbf{u}}(t, \boldsymbol{\nu}_{i,n}) &= \beta \sqrt{\partial_\beta^2 \mathcal{G}(\beta, \tau)} Y_{i,n}, \end{aligned}$$

where  $\{X_{i,n}, Y_{i,n}; n \geq 0, i = 1, 2\}$  is an independent system of standard Gaussian random variables. The sample paths  $\tilde{\mathbf{u}}(\cdot) \in C([0, T]; \mathcal{H}_{-1})$  a.s., where

$$\mathcal{H}_{-k} = \left\{ \tilde{\mathbf{u}} \mid \sum_{i=1}^2 \sum_{n=0}^{\infty} \{\theta_n^{-2k} \tilde{\mathbf{u}}^2(\boldsymbol{\mu}_{i,n}) + \kappa_n^{-2k} \tilde{\mathbf{u}}^2(\boldsymbol{\nu}_{i,n})\} < \infty \right\}.$$

For each  $h \in \mathcal{C}_*(\tau)$ , define  $\tilde{\mathbf{w}}(t, h)$  by

$$\tilde{\mathbf{w}}(t, h) = \tilde{\mathbf{u}}(t, R^{-1}h), \quad R = R(\lambda) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \tau_r & -\beta\tau \\ 0 & \tau_e & \beta \end{bmatrix}. \quad (3.10)$$

<sup>1</sup> $\boldsymbol{\nu}_{i,n}$  are chosen arbitrarily, since this coordinate is a constant Gaussian random field.

Observing that  $A^T R^{-1} = R^{-1} B^T$ , and  $f_1(0) = f_2(1) = 0$  for  $f \in R^{-1}[\mathcal{C}_*(\tau)]$ ,

$$\tilde{\mathbf{w}}(t, h) - \tilde{\mathbf{w}}(0, h) = - \int_0^t \tilde{\mathbf{u}}(s, A^T R^{-1} h') ds = - \int_0^t \tilde{\mathbf{w}}(s, B^T h') ds,$$

and (3.6) is fulfilled. On the other hand, from (2.8) and (3.1),

$$\begin{pmatrix} \tau_r \\ \tau_e \end{pmatrix} = -\frac{1}{\beta} \begin{pmatrix} \partial_r^2 \mathcal{L} \\ \partial_r \partial_e \mathcal{L} \end{pmatrix} \Big|_{(\bar{r}, \bar{e})} - \frac{\tau}{\beta} \begin{pmatrix} \partial_r \partial_e \mathcal{L} \\ \partial_e^2 \mathcal{L} \end{pmatrix} \Big|_{(\bar{r}, \bar{e})}.$$

Combining this with (2.7), one obtains that

$$\mathcal{G}''(\lambda) \begin{pmatrix} \tau_r \\ \tau_e \end{pmatrix} = \begin{pmatrix} \beta^{-1} \\ \beta^{-1} \tau \end{pmatrix}.$$

By this and some direct calculations,

$$R^T \Sigma R = \text{diag}(\beta^{-1}, \beta^{-1} c^2, \beta^2 \partial_\beta^2 \mathcal{G}) = Q,$$

therefore (3.7) also holds. In consequence,  $\{\tilde{\mathbf{w}}(t); t \in [0, T]\}$  uniquely exists in the path space  $C([0, T]; \mathcal{H}_{-k}(\lambda))$  for  $k \geq 1$ , where

$$\mathcal{H}_{-k}(\lambda) = \left\{ \tilde{\mathbf{w}} \mid \|\tilde{\mathbf{w}}\|_{-k}^2 = \sum_{i=1}^2 \sum_{n=0}^{\infty} \{ \theta_n^{-2k} \tilde{\mathbf{w}}^2(R\boldsymbol{\mu}_{i,n}) + \kappa_n^{-2k} \tilde{\mathbf{w}}^2(R\boldsymbol{\nu}_{i,n}) \} < \infty \right\},$$

with three-dimensional Fourier bases  $\boldsymbol{\mu}_{i,n}$  and  $\boldsymbol{\nu}_{i,n}$  given in (3.5) and (3.9), and the Gaussian distribution determined by (3.7) is stationary for  $\tilde{\mathbf{w}}(t)$ .

For  $T > 0$  and  $k > 5/2$ , denote by  $\mathbb{Q}_N$  the distribution of  $\{Y_N(t); t \in [0, T]\}$  on the path space  $C([0, T], \mathcal{H}_{-k}(\lambda))$  induced by  $\mathbb{P}_{\lambda, N}$ . Denote by  $\mathbb{Q}$  the distribution of  $\{\tilde{\mathbf{w}}(t); t \in [0, T]\}$  defined above. Our first result is stated below.

**Theorem 3.1.** *Assume (2.1), then the sequence of probability measures  $\{\mathbb{Q}_N\}$  converges weakly, as  $N \rightarrow \infty$ , to the probability measure  $\mathbb{Q}$ .*

*Remark 3.2.* The condition  $k > 5/2$  is necessary only for the tightness, see Section 6.

Indeed, by the tightness of  $\{\mathbb{Q}_N\}$  proved in Section 6, we can pick an arbitrary limit point of  $\mathbb{Q}_N$ . Denote it by  $\mathbb{Q}$  and let  $\{Y(t)\}$  be a process subject to  $\mathbb{Q}$ . From classical central limit theorem, the distribution of  $Y(0)$  satisfies (3.7). By virtue of the uniqueness of the solution, to prove Theorem 3.1 it suffices to verify (3.6), or equivalently,

$$|Y_N(t, H(t, \cdot)) - Y_N(0, h)| \rightarrow 0 \quad \text{in probability,}$$

where  $H(t, x)$  solves the *backward system*:

$$\partial_t H(t, x) + L^* H(t, x) = 0, \quad H(0, \cdot) = h, \quad (3.11)$$

for smooth initial data  $h \in \mathcal{C}_*(\tau)$ , with the following additional compatibility conditions also assumed at the space-time edges:

$$\begin{aligned} \lim_{x \rightarrow 0^+} \partial_x H_1(0, x) &= 0, & \lim_{x \rightarrow 1^-} \partial_x (H_2(0, x) + \tau H_3(0, x)) &= 0, \\ \lim_{t \rightarrow 0^+} \partial_t^2 H_1(t, 0) &= 0, & \lim_{t \rightarrow 0^+} \partial_t^2 (H_2(t, 1) + \tau H_3(t, 1)) &= 0. \end{aligned} \quad (3.12)$$

Note that (3.12) assures that  $H(t, \cdot) \in \mathcal{C}_*(\tau)$  is differentiable in  $x$  up to the second order, and there exists a finite constant  $C$  such that

$$|H(t, x)| \leq C, \quad |\partial_x H(t, x)| \leq C, \quad |\partial_x^2 H(t, x)| \leq C \quad (3.13)$$

for any  $t \geq 0$  and  $x \in [0, 1]$ . As a further result of Theorem 3.1, we are able to prove that the fluctuation field keeps evolving with the linearized system for time scales beyond hyperbolic, under some additional assumptions.

**Theorem 3.3.** *Assume (2.1). There exists some universal  $\delta > 0$ , such that if*

$$\sup_{r \in \mathbb{R}} V''(r) < (1 + \delta) \inf_{r \in \mathbb{R}} V''(r), \quad (3.14)$$

then for any  $\alpha < 1/5$ ,  $T > 0$  and  $\epsilon > 0$ ,

$$\lim_{N \rightarrow \infty} \mathbb{P}_{\lambda, N} \left\{ \exists t \in [0, T], |Y_N(N^\alpha t, H(N^\alpha t)) - Y_N(0, H(0))| > \epsilon \right\} = 0, \quad (3.15)$$

where  $H(t) = H(t, x)$  solves the backward equation (3.11) and (3.12).

*Remark 3.4.* Theorem 3.3 shows that the fluctuation of thermodynamic entropy  $\tilde{S}$  remains stationary for any time scales  $N^\alpha t$  with  $\alpha < 6/5$ . It is expected that  $\tilde{S}$  would evolve in superdiffusive time scale  $N^\alpha t$  with some  $6/5 \leq \alpha < 2$  following a fractional heat equation.

*Remark 3.5.* Let  $\mathbb{T}_N = \mathbb{Z}/(N\mathbb{Z})$  be the lattice torus with length  $N$ . One can also put the chain on  $\mathbb{T}_N$  by applying the periodic boundary condition  $(p_0, r_0) = (p_N, r_N)$  instead of the ones introduced in Section 1. Then, the equilibrium Gibbs measures become

$$\pi_{\beta, \bar{p}, \tau}^N = \prod_{i \in \mathbb{T}_N} \exp \left\{ \tilde{\lambda} \cdot w_i - \tilde{\mathcal{G}}(\tilde{\lambda}) \right\} dp_i dr_i,$$

for given  $\tilde{\lambda} = (\beta \bar{p}, \beta \tau, -\beta) \in \mathbb{R}^2 \times \mathbb{R}_-$ , where  $\bar{p} \in \mathbb{R}$  denotes the momenta in equilibrium. For  $(p, r) \in \mathbb{R}^2$  and  $e \geq p^2/2 + V(r)$ , we can define the internal energy  $U = e - p^2/2$ , then the thermodynamic entropy and tension function are given by  $\mathcal{S}(r, U)$  and  $\tau(r, U)$ .

Start the dynamics from some equilibrium state  $\pi_{\beta, \bar{p}, \tau}^N$ . Let  $\mathbb{T} = [0, 1)$  stand for the one-dimensional torus. For a bounded smooth function  $H : \mathbb{T} \rightarrow \mathbb{R}$ , the equilibrium fluctuation field is given by

$$Y_N(t, H) = \frac{1}{\sqrt{N}} \sum_{i=1}^N H \left( \frac{i}{N} \right) \cdot \begin{pmatrix} p_i(Nt) - \bar{p} \\ r_i(Nt) - \bar{r} \\ e_i(Nt) - \bar{e} \end{pmatrix}.$$

With similar argument used to prove Theorem 3.1, we can show that  $Y_N(t, H) \rightarrow \tilde{\mathfrak{w}}(t, H)$ . Here  $\tilde{\mathfrak{w}}(t, \cdot)$  solves the following linearized Euler system on torus:

$$\partial_t \tilde{\mathfrak{w}}(t, x) = \begin{bmatrix} -\bar{p}\tau_u & \tau_r & \tau_u \\ 1 & 0 & 0 \\ \tau - \bar{p}^2\tau_u & \bar{p}\tau_r & \bar{p}\tau_u \end{bmatrix} \partial_x \tilde{\mathfrak{w}}(t, x),$$

where the linear coefficients are given by

$$(\tau_r, \tau_u) = (\partial_r, \partial_u) \tau \left( \bar{r}, \bar{e} - \frac{\bar{p}^2}{2} \right).$$

Similar to (3.2), we have  $\tilde{\mathfrak{p}}$  and  $\tilde{\tau} = -\bar{p}\tau_u\tilde{\mathfrak{p}} + \tau_r\tilde{\mathfrak{r}} + \tau_u\tilde{\mathfrak{e}}$  form a system of two coupled wave equations with common sound speed  $c = \tau_r + \tau\tau_u$ , while  $\tilde{S} = \beta(\tilde{\mathfrak{e}} - \bar{p}\tilde{\mathfrak{p}} - \tau\tilde{\mathfrak{r}})$  does not evolve in time.

#### 4. Equilibrium fluctuation

In this section, let  $H(t, x)$  be a bounded and smooth function on  $[0, \infty) \times [0, 1]$ . For any  $T > 0$ , we define two norms  $|H|_T$  and  $\|H\|_T$  of  $H$  as below:

$$|H|_T = \sup_{[0, T] \times [0, 1]} |H(t, x)|,$$

$$\|H\|_T^2 = \sup_{t \in [0, T]} \|H(t)\|^2 = \sup_{t \in [0, T]} \int_0^1 |H(t, x)|^2 dx.$$

For  $Y_N(t, H(t, \cdot))$ , the following decomposition holds  $\mathbb{P}_{\lambda, N}$  almost surely:

$$\begin{aligned} & Y_N(t, H(t)) - Y_N(0, H(0)) - \int_0^t Y_N(s, \partial_s H(s)) ds \\ &= I_{N,1}(t, H) + \gamma I_{N,2}(t, H) + \sqrt{\gamma} M_N(t, H), \quad \forall t > 0, \end{aligned} \quad (4.1)$$

where  $I_{N,1}$  and  $I_{N,2}$  are integrals given by

$$I_{N,1}(t, H) = N \int_0^t \mathcal{A}_N[Y_N(s, H(s))] ds, \quad I_{N,2}(t, H) = N \int_0^t \mathcal{S}_N[Y_N(s, H(s))] ds,$$

and  $M_N$  is a martingale with quadratic variation given by

$$\langle M_N \rangle(t, H) = N \int_0^t \{ \mathcal{S}_N[Y_N^2(s, H(s))] - 2Y_N(s, H(s)) \mathcal{S}_N[Y_N(s, H(s))] \} ds.$$

As the first step to prove Theorem 3.3, the next lemma guarantees that the last two terms in (4.1) vanish uniformly in macroscopic time for equilibrium dynamics.

**Lemma 4.1.** *There exists a constant  $C = C(\lambda, V)$ , such that*

$$\mathbb{E}_{\lambda, N} \left[ \sup_{t \in [0, T]} \gamma |I_{N,2}(t, H)|^2 + \sup_{t \in [0, T]} |M_N(t, H)|^2 \right] \leq \frac{CT}{N} \|\partial_x H\|_T^2.$$

The proof of Lemma 4.1 is standard and we postpone it to the end of this section. To identify the boundary conditions of  $H$ , noting that  $p_0 = 0$ , and

$$\begin{aligned} & N \mathcal{A}_N[Y_N(t, H(t))] \\ &= \sqrt{N} \sum_{i=1}^{N-1} H \left( t, \frac{i}{N} \right) \cdot (J_{\mathcal{A}, i} - J_{\mathcal{A}, i-1}) + \sqrt{N} H(t, 1) \cdot \begin{pmatrix} \tau - V(r_N) \\ p_N - p_{N-1} \\ p_N \tau - p_{N-1} V'(r_N) \end{pmatrix} \\ &= \frac{1}{\sqrt{N}} \sum_{i=1}^{N-1} \nabla_{N,i} H(t) \cdot (-J_{\mathcal{A}, i}(\eta_t)) \\ &\quad - \sqrt{N} \left[ H_1 \left( t, \frac{1}{N} \right) (V'(r_1(t)) - \tau) - (H_2(t, 1) + \tau H_3(t, 1)) p_N(t) \right], \end{aligned}$$

where  $J_{\mathcal{A}, i}$  is the centred instantaneous currents of  $\mathcal{A}_N$ :

$$J_{\mathcal{A}, i} = (V'(r_{i+1}) - \tau, p_i, p_i V'(r_{i+1}))^T,$$

and  $\nabla_{N,i}$  is the discrete derivative operator:

$$\nabla_{N,i} H = N \left[ H \left( \frac{i+1}{N} \right) - H \left( \frac{i}{N} \right) \right].$$

Thus, we can drop the right boundary if  $H(t) \in \mathcal{C}_*(\tau)$  for all  $t$ :

$$I_{N,1}(t, H) = -\frac{1}{\sqrt{N}} \int_0^t \sum_{i=0}^{N-1} \nabla_{N,i} H(s) \cdot J_{\mathcal{A}, i}(\eta_s) ds. \quad (4.2)$$

The next lemma shows that  $I_{N,1}$  can be linearized as  $N \rightarrow \infty$ .

**Lemma 4.2.** Assume (2.1), (3.14), and  $H(t) \in \mathcal{C}_*(\tau)$  for  $t \in [0, T]$ , then

$$\mathbb{E}_{\lambda, N} \left[ \sup_{t \in [0, T]} \left| I_{N,1}(t, H) - \int_0^t Y_N(s, L^* H(s)) ds \right|^2 \right] \leq C \left( \frac{T}{N^{\frac{1}{5}}} + \frac{T^2}{N^{\frac{2}{5}}} \right) \quad (4.3)$$

holds with a constant  $C$ . Furthermore,

$$C \leq C(\lambda, \gamma, V) \|H\|_T^2, \quad \text{where} \quad \|H\|_T^2 = |\partial_x H|_T^2 + |\partial_x^2 H|_T^2 + \|\partial_x H\|_T^2.$$

*Remark 4.3.* The bound (4.3) in Lemma 4.2 is proven under the assumption (3.14). Without assuming (3.14) we have only that, for every fixed  $T > 0$ ,

$$\mathbb{E}_{\lambda, N} \left[ \sup_{t \in [0, T]} \left| I_{N,1}(t, H) - \int_0^t Y_N(s, L^* H(s)) ds \right|^2 \right] \leq o_N(1) \|H\|_T^2. \quad (4.4)$$

This is clear from Remark 5.3 below. The bound (4.4) is enough for proving Theorem 3.1, while (4.3) is necessary in order to prove (3.15).

Lemma 4.2 follows from the Boltzmann-Gibbs principle, proven in Section 5. Here we first give the proof of Theorem 3.3.

*Proof of Theorem 3.3.* Let  $H(t, x)$  be the solution of (3.11). From (4.1) and Lemma 4.1,

$$\mathbb{P}_{\lambda, N} \left\{ \exists t \in [0, T], \left| Y_N(N^\alpha t, H(N^\alpha t)) - Y_N(0, H(0)) - \int_0^{N^\alpha t} Y_N(s, \partial_s H(s)) ds - I_{N,1}(N^\alpha t, H) \right| > \epsilon \right\} \rightarrow 0$$

for any  $\epsilon > 0$ . Lemma 4.2 and (3.13) then yield that for any  $\alpha < 1/5$ ,

$$\mathbb{E}_{\lambda, N} \left[ \sup_{t \in [0, T]} \left| I_{N,1}(N^\alpha t, H) - \int_0^{N^\alpha t} Y_N(s, L^* H(s)) ds \right|^2 \right] \rightarrow 0.$$

Theorem 3.3 then follows from (3.11).  $\square$

For Theorem 3.1, since tightness is shown in Section 6, we only need to take  $\alpha = 0$  in the proof above, and apply Remark 4.3 instead of Lemma 4.2 in the last step.

We now proceed to the proof of Lemma 4.1. Denote by  $\langle \cdot, \cdot \rangle_{\lambda, N}$  the scalar product of two functions  $f, g \in L^2(\pi_{\lambda, N})$ . We make use of a well-known estimate on the space-time variance of a stationary Markov process. For  $f(s, \cdot) \in L^2(\pi_{\lambda, N})$ ,

$$\mathbb{E}_{\lambda, N} \left[ \sup_{t \in [0, T]} \left| \int_0^t f(s, \eta_s) ds \right|^2 \right] \leq 14 \int_0^T \|f(t)\|_{-1, N}^2 dt, \quad (4.5)$$

where  $\|f\|_{-1, N}$  is defined for all  $f$  on  $\Omega_N$  by

$$\|f\|_{-1, N}^2 = \sup_h \left\{ 2\langle f, h \rangle_{\lambda, N} - \gamma N \langle h, -\mathcal{S}_N h \rangle_{\lambda, N} \right\},$$

with the superior taken over all bounded smooth functions  $h$  on  $\Omega_N$ . A proof of (4.5) can be found in [15, Sec. 2.5].

*Proof of Lemma 4.1.* To begin with, note that

$$\begin{aligned} N\mathcal{S}_N[Y_N(t, H(t))] &= \frac{\sqrt{N}}{2} \sum_{i=1}^{N-1} \left[ H\left(t, \frac{i}{N}\right) \cdot \mathcal{Y}_{i,i+1}^2[w_i] + H\left(t, \frac{i+1}{N}\right) \cdot \mathcal{Y}_{i,i+1}^2[w_{i+1}] \right] \\ &= -\frac{1}{\sqrt{N}} \sum_{i=1}^{N-1} \nabla_{N,i} H(t) \cdot J_{\mathcal{S},i}(\eta_t), \end{aligned}$$

where  $J_{\mathcal{S},i}$  is the instantaneous current corresponding to  $\mathcal{S}_N$ :

$$J_{\mathcal{S},i} = \frac{1}{2} \mathcal{Y}_{i,i+1}^2[w_i] = -\frac{1}{2} \mathcal{Y}_{i,i+1}^2[w_{i+1}].$$

By applying (4.5) on  $I_{N,2}(t, H)$ , we get that

$$\begin{aligned} &\mathbb{E}_{\lambda,N} \left[ \sup_{t \in [0,T]} |I_{N,2}(t, H)|^2 \right] \\ &\leq \frac{14}{N} \int_0^T \sup_h \left\{ 2 \sum_{i=1}^{N-1} \langle \nabla_{N,i} H(t) \cdot J_{\mathcal{S},i}, h \rangle_{\lambda,N} - \gamma N \langle h, -\mathcal{S}_N h \rangle_{\lambda,N} \right\} dt. \end{aligned} \tag{4.6}$$

As  $\mathcal{Y}_{i,i+1}$  is anti-symmetric, with  $m_i = \mathcal{Y}_{i,i+1}[w_i] = -\mathcal{Y}_{i,i+1}[w_{i+1}]$ ,

$$\left| \sum_{i=1}^{N-1} \langle \nabla_{N,i} H(t) \cdot J_{\mathcal{S},i}, h \rangle_{\lambda,N} \right|^2 = \frac{1}{4} \left| \sum_{i=1}^{N-1} \langle \nabla_{N,i} H(t) \cdot m_i, \mathcal{Y}_{i,i+1} h \rangle_{\lambda,N} \right|^2.$$

Using Cauchy-Schwarz inequality, the expression above is bounded by

$$\frac{1}{2} \sum_{i=1}^{N-1} |\nabla_{N,i} H(t)|^2 E_{\lambda,N} [ |m_i|^2 ] \langle h, -\mathcal{S}_N h \rangle_{\lambda,N} \leq C_1 N \|\partial_x H(t)\|^2 \langle h, -\mathcal{S}_N h \rangle_{\lambda,N}.$$

Substituting this in (4.6) and optimizing  $h$ , we obtain that

$$\mathbb{E}_{\lambda,N} \left[ \sup_{t \in [0,T]} |I_{N,2}(t, H)|^2 \right] \leq \frac{14C_1}{\gamma N} \int_0^T \|\partial_x H\|^2 dt \leq \frac{C_2 T}{\gamma N} \|\partial_x H\|_T^2.$$

We are left with  $M_N(t, H)$ . Note that for any smooth  $f$  on  $\Omega_N$ ,

$$\mathcal{S}_N[f^2] = \frac{1}{2} \sum_{i=1}^{N-1} \mathcal{Y}_{i,i+1}^2[f^2] = f \sum_{i=1}^{N-1} \mathcal{Y}_{i,i+1}^2 f + \sum_{i=1}^{N-1} (\mathcal{Y}_{i,i+1} f)^2.$$

Since  $\mathcal{Y}_{i,i+1}[Y_N(s, H(s))] = -N^{-3/2} \nabla_{N,i} H(s) \cdot m_i$ , we have

$$\mathcal{S}_N[Y_N^2(s, H(s))] - 2Y_N(s, H(s))\mathcal{S}_N[Y_N(s, H(s))] = \frac{1}{N^3} \sum_{i=1}^{N-1} (\nabla_{N,i} H(s) \cdot m_i)^2.$$

Therefore, the Doob's maximal inequality yields that

$$\begin{aligned} \mathbb{E}_{\lambda,N} \left[ \sup_{t \in [0,T]} |M_N(t, H)|^2 \right] &\leq 4\mathbb{E}_{\lambda,N} [\langle M_N \rangle(T, H)] \\ &\leq \frac{4}{N^2} \int_0^T \sum_{i=1}^{N-1} E_{\lambda,N} [(\nabla_{N,i} H(t) \cdot m_i)^2] dt \\ &\leq \frac{C_3}{N} \int_0^T \|\partial_x H\|^2 dt \leq \frac{C_4 T}{N} \|\partial_x H\|_T^2. \end{aligned}$$

As all constants above depend only on  $\lambda$  and  $V$ , Lemma 4.1 follows.  $\square$

## 5. Boltzmann-Gibbs principle

This section is devoted to the proof of Lemma 4.2. In this section, we denote by  $\{\iota_i; 0 \leq i \leq N\}$  the shift operator semigroup on  $\Omega_N$ , which is given by

$$(\iota_i \eta)_j = \begin{cases} (p_{i+j}, r_{i+j}), & 1 \leq j \leq N-i, \\ (0, 0), & N-i < j \leq N, \end{cases}$$

for all  $\eta \in \Omega_N$  and  $0 \leq i \leq N$ . For function  $F$  on  $\Omega_N$ , define  $\iota_i F = F \circ \iota_i$ . If  $F$  is supported by  $\{\eta_j, 1 \leq j \leq m\}$  for some  $m \leq N$ , then

$$E_{\lambda, N}[\iota_i F] = E_{\lambda, N}[F], \quad \forall 0 \leq i \leq N-m.$$

First notice that  $\nabla_{N,i} H$  in (4.2) can be replaced by  $\partial_x H$ . The difference is

$$\begin{aligned} & \mathbb{E}_{\lambda, N} \left[ \sup_{t \in [0, T]} \left| I_{N,1}(t, H) + \frac{1}{\sqrt{N}} \int_0^t \sum_{i=1}^{N-1} \partial_x H \left( s, \frac{i}{N} \right) \cdot J_{\mathcal{A}, i}(\eta_s) ds \right|^2 \right] \\ & \leq \frac{T}{N} \int_0^T E_{\lambda, N} \left[ \left( \sum_{i=1}^{N-1} \left[ \nabla_{N,i} H(t) - \partial_x H \left( t, \frac{i}{N} \right) \right] \cdot J_{\mathcal{A}, i} \right)^2 \right] dt. \end{aligned}$$

Let  $\otimes$  stand for the tensor product of vectors. As  $E_{\lambda, N}[J_{\mathcal{A}, i} \otimes J_{\mathcal{A}, j}] = \mathbf{0}$  for all  $|i-j| \geq 2$ , the expression in the last line is bounded from above by

$$\frac{3T}{N} \int_0^T \sum_{i=1}^{N-1} E_{\lambda, N} \left[ \left( \left[ \nabla_{N,i} H(t) - \partial_x H \left( t, \frac{i}{N} \right) \right] \cdot J_{\mathcal{A}, i} \right)^2 \right] dt \leq \frac{CT^2 \|\partial_x^2 H\|_T^2}{N^2}.$$

Clearly, it decays faster than what is needed for Lemma 4.2.

Now our aim is to replace the local random field  $J_{\mathcal{A}, i}$  with its linear approximation. The corresponding error can be expressed by

$$\iota_i \Phi = J_{\mathcal{A}, i} - \begin{bmatrix} 0 & \tau_r & \tau_e \\ 1 & 0 & 0 \\ \tau & 0 & 0 \end{bmatrix} (w_i - \bar{w}(\lambda)) = \begin{pmatrix} V'(r_{i+1}) - \tau - \tau_r(r_i - \bar{r}) - \tau_e(e_i - \bar{e}) \\ 0 \\ p_i V'(r_{i+1}) - p_i \tau \end{pmatrix}.$$

Lemma 4.2 follows from the following Boltzmann-Gibbs principle.

**Proposition 5.1.** *Assume (2.1) and (3.14), then*

$$\mathbb{E}_{\lambda, N} \left[ \sup_{t \in [0, T]} \left| \frac{1}{\sqrt{N}} \int_0^t \sum_{i=1}^{N-1} \partial_x H \left( s, \frac{i}{N} \right) \cdot \iota_i \Phi(\eta_s) ds \right|^2 \right] \leq C \left( \frac{T}{N^{\frac{1}{5}}} + \frac{T^2}{N^{\frac{2}{5}}} \right) \quad (5.1)$$

for bounded smooth  $H = H(t, x)$  on  $[0, T] \times [0, 1]$ , where  $C = C(\lambda, \gamma, V) \|H\|_T^2$ .

Boltzmann-Gibbs principle, firstly established for zero range jump process (see [8]), aims at determining the space-time fluctuation of a local function by its linear approximation on the conserved fields. To show this proposition, we need a spectral gap bound of  $\mathcal{S}_N$ , which is the main difficulty here. This is established later in Section 7.

*Remark 5.2.* Notice that the upper bound in Proposition 5.1 is not optimal. Indeed, with the proof below, one can actually obtain an upper bound of

$$C \left( \frac{T}{N^{1-2b}} + \frac{T^2}{N^b} \right), \quad \forall b < \frac{1}{2}.$$

However, this does not improve the time scale in Theorem 3.3.

*Proof.* The first step is to take some  $1 \leq K \ll N$ , and define

$$\Phi_K = \frac{1}{K} \sum_{i=1}^K \iota_i \Phi. \quad (5.2)$$

We want to replace  $\iota_i \Phi$  with  $\iota_{i-1} \Phi_K$  in the left hand side of (5.1). The error is

$$\sum_{i=1}^{N-1} a_i(t) \cdot \iota_i \Phi - \sum_{i=1}^{N-K} a_i(t) \cdot \iota_{i-1} \Phi_K = F_1(t) + F_2(t),$$

where we write  $a_i(t) = \partial_x H(t, i/N)$  for short, and  $F_1, F_2$  are given by

$$\begin{aligned} F_1(t) &= \frac{1}{K} \sum_{i=1}^{K-1} [(K-i)a_i(t) \cdot \iota_i \Phi + ia_{N-K+i}(t) \cdot \iota_{N-K+i} \Phi], \\ F_2(t) &= \frac{1}{K} \sum_{i=1}^{N-1} \sum_{j=i_{K,-}}^{i_{K,+}} (a_i(t) - a_j(t)) \cdot \iota_i \Phi, \end{aligned}$$

where  $i_{K,+} = \max\{i+1-K, 1\}$ ,  $i_{K,-} = \min\{i, N-K\}$ . Since  $E_{\lambda,N}[\iota_i \Phi \otimes \iota_j \Phi] = \mathbf{0}$  for every pair of  $(i, j)$  such that  $|i-j| \geq 2$ ,

$$E_{\lambda,N}[F_1^2(t) + F_2^2(t)] \leq C_1 K (|\partial_x H|_T^2 + |\partial_x^2 H|_T^2),$$

with a constant  $C_1$  depending on  $\lambda$  and  $V$ . Hence,

$$\mathbb{E}_{\lambda,N} \left[ \sup_{t \in [0, T]} \left| \frac{1}{\sqrt{N}} \int_0^t F_1(s, \eta_s) + F_2(s, \eta_s) ds \right|^2 \right] \leq \frac{C_1 T^2 K}{N} (|\partial_x H|_T^2 + |\partial_x^2 H|_T^2). \quad (5.3)$$

The second step is to replace furthermore  $\Phi_K$  with its microcanonical centre. To do so, observe that  $\Phi_K$  is supported by  $\{\eta_j; 1 \leq j \leq K+1\}$ , and define

$$\langle \Phi_K \rangle = E_{\lambda,N} \left[ \Phi \mid \frac{w_1 + w_2 + \dots + w_{K+1}}{K+1} \right],$$

where  $w_i = (p_i, r_i, e_i)$  is the vector of conserved quantities. Due to the equivalence of ensembles proved in Corollary 8.4, the second moment of  $\langle \Phi_K \rangle$  with respect to  $\pi_{\lambda,N}$  is of order  $K^{-2}$ . On the other hand, the second moment of  $\Phi_K$  is  $O(K^{-1})$ :

$$E_{\lambda,N}[|\Phi_K|^2] \leq \frac{1}{K} (E_{\lambda,N}[|\iota_1 \Phi|^2] + 2E_{\lambda,N}[\iota_1 \Phi \cdot \iota_2 \Phi]),$$

Define  $\varphi_K = \Phi_K - \langle \Phi_K \rangle$ . Since  $\varphi_K$  and  $\langle \Phi_K \rangle$  are orthogonal,

$$E_{\lambda,N}[|\varphi_K|^2] = E_{\lambda,N}[|\Phi_K|^2] - E_{\lambda,N}[|\langle \Phi_K \rangle|^2] \leq \frac{C_2}{K}. \quad (5.4)$$

By applying the estimate (4.5), we obtain that

$$\begin{aligned} & \mathbb{E}_{\lambda,N} \left[ \sup_{t \in [0, T]} \left| \frac{1}{\sqrt{N}} \int_0^t \sum_{i=1}^{N-K} a_i(s) \cdot \iota_{i-1} \varphi_K(\eta_s) ds \right|^2 \right] \\ & \leq \frac{14}{N} \int_0^T \sup_h \left\{ 2 \sum_{i=1}^{N-K} \langle a_i(t) \cdot \iota_{i-1} \varphi_K, h \rangle_{\lambda,N} - \gamma N \langle h, -\mathcal{S}_N h \rangle_{\lambda,N} \right\} dt, \end{aligned} \quad (5.5)$$



where the supremum is taken over all bounded smooth functions on  $\Omega_N$ . As  $\varphi_K$  is supported by  $\{\eta_i; 1 \leq i \leq K+1\}$ , by the spectral gap in Proposition 7.1,

$$-\mathcal{S}_{K+1}G_{a,K} = a \cdot \varphi_K, \quad a \in \mathbb{R}^3$$

can be solved by some function  $G_{a,K}$  satisfying that

$$\langle G_{a,K}, -\mathcal{S}_{K+1}G_{a,K} \rangle_{\lambda,N} \leq C(K+1)^2 E_{\lambda,N}[(a \cdot \varphi_K)^2] \leq C_3 K |a|^2, \quad (5.6)$$

where the last step follows from (5.4). For  $1 \leq i \leq N-K$  and  $a \in \mathbb{R}^3$ ,

$$\langle a \cdot \iota_{i-1} \varphi_K, h \rangle_{\lambda,N} = \frac{1}{2} \sum_{j=1}^K \langle \mathcal{Y}_{i+j-1, i+j} [\iota_{i-1} G_{a,K}], \mathcal{Y}_{i+j-1, i+j} h \rangle_{\lambda,N}.$$

Hence, Cauchy-Schwarz inequality yields that

$$\begin{aligned} & \left| \sum_{i=1}^{N-K} \langle a_i(t) \cdot \iota_{i-1} \varphi_K, h \rangle_{\lambda,N} \right|^2 \\ & \leq \left( \frac{1}{2} \sum_{i=1}^{N-K} \sum_{j=1}^K E_{\lambda,N}[(\mathcal{Y}_{i+j-1, i+j} h)^2] \right) \left( \frac{1}{2} \sum_{i=1}^{N-K} \sum_{j=1}^K E_{\lambda,N}[(\mathcal{Y}_{j, j+1} G_{a_i(t), K})^2] \right) \\ & \leq K \langle h, -\mathcal{S}_N h \rangle_{\lambda,N} \sum_{i=1}^{N-K} \langle G_{a_i(t), K}, -\mathcal{S}_{K+1} G_{a_i(t), K} \rangle_{\lambda,N}. \end{aligned}$$

Using the estimate obtained in (5.6), the expression above is bounded by

$$C_3 K^2 \langle h, -\mathcal{S}_N h \rangle_{\lambda,N} \sum_{i=1}^{N-K} |a_i(t)|^2 \leq C_4 K^2 N \|\partial_x H(t)\|^2 \langle h, -\mathcal{S}_N h \rangle_{\lambda,N}.$$

Substituting this estimate into (5.5) and optimizing in  $h$ ,

$$\mathbb{E}_{\lambda,N} \left[ \sup_{t \in [0, T]} \left| \frac{1}{\sqrt{N}} \int_0^t \sum_{i=1}^{N-K} a_i(s) \cdot \iota_{i-1} \varphi_K(\eta_s) ds \right|^2 \right] \leq \frac{C_5 T K^2}{\gamma N} \|\partial_x H\|_T^2. \quad (5.7)$$

Finally,  $\langle \Phi_K \rangle$  is supported by  $\{\eta_i; 1 \leq i \leq K+1\}$ , so that  $E_{\lambda,N}[\iota_i \langle \Phi_K \rangle \otimes \iota_j \langle \Phi_K \rangle] = 0$  for  $|i-j| \geq K+2$ , and therefore,

$$\begin{aligned} & \mathbb{E}_{\lambda,N} \left[ \sup_{t \in [0, T]} \left| \frac{1}{\sqrt{N}} \int_0^t \sum_{i=1}^{N-K} a_i(s) \cdot \iota_{i-1} \langle \Phi_K \rangle(\eta_s) ds \right|^2 \right] \\ & \leq \frac{T}{N} \int_0^T \sum_{|i-j| \leq K+1} E_{\lambda,N} [(a_i(t) \cdot \iota_{i-1} \langle \Phi_K \rangle)(a_j(t) \cdot \iota_{j-1} \langle \Phi_K \rangle)] dt \\ & \leq \frac{T^2 \|\partial_x H\|_T^2}{N} \sum_{i=1}^{N-K} \sum_{j=-K-1}^{K+1} E_{\lambda,N} [|\langle \Phi_K \rangle| |\iota_j \langle \Phi_K \rangle|] \leq \frac{C_6 T^2}{K} \|\partial_x H\|_T^2, \end{aligned} \quad (5.8)$$

where the last line is due to that  $E_{\lambda,N}[\langle \Phi_K \rangle^2] = O(K^{-2})$ .

In conclusion, by summing up (5.3), (5.7), (5.8), and taking  $K = N^{2/5}$ , we get the estimate in Proposition 5.1, with a constant  $C$  satisfying that

$$C \leq C(\lambda, \gamma, V) (\|\partial_x H\|_T^2 + |\partial_x H|_T^2 + |\partial_x^2 H|_T^2).$$

This completes the proof of the proposition.  $\square$

*Remark 5.3.* If only (2.1) is assumed, we can apply Remark 7.6 instead of Proposition 7.1 in the proof of (5.7). By doing this, we can prove Proposition 5.1 for any fixed  $T > 0$ , with a weaker upper bound  $o_N(1) \|H\|_T^2$ .

## 6. Tightness

In Section 4 we have proved the convergence of the finite-dimensional distribution of  $\{\mathbb{Q}_N\}$ . In order to complete the proof of Theorem 3.1, we need its tightness in  $C([0, T], \mathcal{H}_{-k}(\lambda))$ . The proof is standard, and we summarize it here.

It suffices to show the two statements below:

$$\lim_{M \rightarrow \infty} \limsup_{N \rightarrow \infty} \mathbb{P}_{\lambda, N} \left\{ \sup_{t \in [0, T]} \|Y_N(t)\|_{-k} \geq M \right\} = 0, \quad (6.1)$$

$$\lim_{\delta \downarrow 0} \limsup_{N \rightarrow \infty} \mathbb{P}_{\lambda, N} \{w_{-k}(Y_N, \delta) \geq \epsilon\} = 0, \quad \forall \epsilon > 0, \quad (6.2)$$

where  $w_{-k}(Y_N, \delta)$  is the modulus of continuity in  $C([0, T], \mathcal{H}_{-k}(\lambda))$ . Recall that

$$\|Y_N\|_{-k}^2 = \sum_{i=1}^2 \sum_{n=0}^{\infty} \{ \theta_n^{-2k} Y_N^2(R\boldsymbol{\mu}_{i,n}) + \kappa_n^{-2k} Y_N^2(R\boldsymbol{\nu}_{i,n}) \},$$

where  $R$  is the rotation matrix in (3.10), and  $\boldsymbol{\mu}_{i,n}$ ,  $\boldsymbol{\nu}_{i,n}$  are the three-dimensional Fourier bases defined in (3.5) and (3.9).

Take  $f = \boldsymbol{\mu}_{i,n}$  or  $\boldsymbol{\nu}_{i,n}$  for some  $(i, n)$ . Applying (4.1) with  $H(t) \equiv Rf$ ,

$$Y_N(t, Rf) = Y_0(0, Rf) + \int_0^t Y_N(s, L^*[Rf]) ds + \epsilon_N(t, f),$$

and by Lemma 4.1 and Remark 4.3,  $\epsilon_N$  satisfies that

$$\mathbb{E}_{\lambda, N} \left[ \sup_{t \in [0, T]} \epsilon_N^2(t, f) \right] = o_N(1) (|f'|_{\infty}^2 + |f''|_{\infty}^2 + \|f'\|^2).$$

On the other hand, it is easy to see that

$$\mathbb{E}_{\lambda, N} \left[ \sup_{t \in [0, T]} \left| \int_0^t Y(s, L^*[Rf]) ds \right|^2 \right] \leq CT^2 \|f'\|^2.$$

Observe that  $|f''_{i,n}(x)| \leq \sqrt{2} \max\{\theta_n^2, \kappa_n^2\}$ . Then, for  $k > 5/2$ , (6.1) and (6.2) can be proved by standard arguments (cf. [14, 11.3]).

## 7. Spectral gap

In this section, we state and prove the spectral gap estimate for the dynamics. The main result, Proposition 7.1, plays a central role in the proof of Proposition 5.1.

Since we want to consider dynamics without boundary conditions in this section, the notations would be slightly different. Recall (2.1) and denote

$$\delta_- = \inf_{r \in \mathbb{R}} V''(r), \quad \delta_+ = \sup_{r \in \mathbb{R}} V''(r).$$

For  $\beta > 0$ ,  $(\bar{p}, \tau) \in \mathbb{R}^2$ , let  $\pi_{\beta, \bar{p}, \tau}^K$  be the product measure on  $\Omega_K$  given by

$$\pi_{\beta, \bar{p}, \tau}^K(d\mathbf{p} \, d\mathbf{r}) = \prod_{i=1}^K \frac{1}{Z_{\beta, \tau}} \exp \left\{ -\frac{\beta(p_i - \bar{p})^2}{2} - \beta V(r_i) + \beta \tau r_i \right\} dp_i \, dr_i,$$

where  $Z_{\beta,\tau}$  is the normalization constant. Note that the additional coefficient  $\bar{p}$  refers to a nonzero average speed. For  $K \geq 2$  and  $w = (p, r, e)$  such that  $e > p^2/2 + V(r)$ , the *microcanonical manifold*  $\Omega_{w,K}$  is defined as

$$\Omega_{w,K} = \left\{ (p_k, r_k), 1 \leq k \leq K \mid \frac{1}{K} \sum_{k=1}^K w_k = w \right\}.$$

In view of (2.1),  $\Omega_{w,K}$  is a compact and connected manifold. The *microcanonical expectation* on  $\Omega_{w,K}$  is defined as the conditional expectation

$$E_{w,K} = E_{\pi_{\beta,\bar{p},\tau}^K}[\cdot | \Omega_{w,K}]$$

Notice that the definition of  $E_{w,K}$  is independent of the choice of  $\beta$ ,  $\bar{p}$  or  $\tau$ . For two functions  $f_1, f_2$  such that  $E_{w,K}[f_i^2] < \infty$ , we write  $\langle f_1, f_2 \rangle_{w,K} = E_{w,K}[f_1 f_2]$ . For each pair  $(i, j)$  such that  $1 \leq i < j \leq K$ , let  $\mathcal{F}_{i,j}$  be the  $\sigma$ -algebra over  $\Omega_{w,K}$  given by

$$\mathcal{F}_{i,j} = \sigma(\{(p_k, r_k); 1 \leq k \leq K, k \neq i, j\}).$$

**Proposition 7.1.** *Suppose that the potential  $V$  satisfies (2.1). There exists a universal constant  $\delta > 0$ , such that if  $V$  fulfils furthermore (3.14), then*

$$E_{w,K} [(f - E_{w,K}[f])^2] \leq C_K \sum_{k=1}^{K-1} E_{w,K} [(\mathcal{Y}_{k,k+1} f)^2] \quad (7.1)$$

for all  $(w, K)$  and bounded smooth function  $f$ , and  $C_K \leq CK^2$ .

The proof of Proposition 7.1 is divided into Lemma 7.2, 7.3 and 7.4 below.

**Lemma 7.2.** *Assume (2.1), then there exists constant  $C$ , such that*

$$E_{w,2} [(f - E_{w,2}[f])^2] \leq CE_{w,2} [(\mathcal{Y}_{1,2} f)^2]$$

for all  $w$  and bounded smooth function  $f$  on  $(p_1, r_1, p_2, r_2)$ .

**Lemma 7.3.** *Assume (2.1), then there exists constant  $C$ , such that*

$$\sum_{1 \leq i < j \leq K} E_{w,K} [(f - E_{w,K}[f | \mathcal{F}_{i,j}])^2] \leq CK^3 \sum_{k=1}^{K-1} E_{w,K} [(f - E_{w,K}[f | \mathcal{F}_{k,k+1}])^2]$$

for all  $K \geq 3$ ,  $w$  and bounded smooth function  $f$ .

**Lemma 7.4.** *Assume (2.1) and (3.14), then*

$$E_{w,K} [(f - E_{w,K}[f])^2] \leq C'_K \sum_{1 \leq i < j \leq K} E_{w,K} [(f - E_{w,K}[f | \mathcal{F}_{i,j}])^2] \quad (7.2)$$

for all  $K \geq 3$ ,  $w$  and bounded smooth function  $f$ , and  $C'_K \leq C'K^{-1}$ .

Indeed, for each  $k = 1, \dots, K-1$ , by applying Lemma 7.2 to the variables  $p_k, r_k, p_{k+1}$  and  $r_{k+1}$ , one obtains that

$$E_{w,K} [(f - E_{w,K}[f | \mathcal{F}_{k,k+1}])^2 | \mathcal{F}_{k,k+1}] \leq CE_{w,K} [(\mathcal{Y}_{k,k+1} f)^2 | \mathcal{F}_{k,k+1}].$$

Then, Proposition 7.1 turns to be the direct consequence of this, Lemma 7.3 and Lemma 7.4. We now prove these lemmas in turn.

*Proof of Lemma 7.2.* For  $(p_1, r_1, p_2, r_2) \in \mathbb{R}^4$ , define

$$p = p(p_1, p_2) = \frac{p_1 + p_2}{2}, \quad r = r(r_1, r_2) = \frac{r_1 + r_2}{2},$$

and the *internal energy*  $E = E(p_1, r_1, p_2, r_2) \geq 0$  given by

$$E = \frac{e_1 + e_2}{2} - \frac{p^2}{2} - V(r) = \frac{(p_1 - p_2)^2}{8} + \frac{V(r_1) + V(r_2)}{2} - V\left(\frac{r_1 + r_2}{2}\right).$$

Furthermore, let  $\theta \in [0, 2\pi)$  satisfy that  $\sqrt{E} \cos \theta = \sqrt{2}(p_1 - p_2)/4$  and

$$\sqrt{E} \sin \theta = \operatorname{sgn}(r_1 - r_2) \sqrt{\frac{V(r_1) + V(r_2)}{2} - V\left(\frac{r_1 + r_2}{2}\right)}.$$

Observe that if  $V = r^2/2$ ,  $(\sqrt{E}, \theta)$  is the usual polar coordinates on the two-dimensional plane centred at  $(p, r)$ . The Jacobian determinant of the bijection  $(p_1, r_1, p_2, r_2) \rightarrow (p, r, E, \theta)$  is

$$\mathfrak{J}(p, r, E, \theta) = \sqrt{2} \cdot \frac{\sqrt{V(r_1) + V(r_2) - 2V(r)}}{|V'(r_1) - V'(r_2)|}.$$

Recall that  $0 < \delta_- \leq V''(r) \leq \delta_+ < \infty$ , we have

$$0 < \frac{\sqrt{\delta_-}}{\sqrt{2}\delta_+} \leq \mathfrak{J}(p, r, E, \theta) \leq \frac{\sqrt{\delta_+}}{\sqrt{2}\delta_-}. \quad (7.3)$$

For a bounded smooth function  $f = f(p_1, r_1, p_2, r_2)$ , define  $f_*(p, r, E, \theta) = f(p_1, r_1, p_2, r_2)$ , and let  $\langle f_* \rangle = \int_0^{2\pi} f_*(p, r, E, \theta) d\theta$ . Since  $E = e - p^2/2 - V(r)$  remains a constant in the surface  $\Omega_{w,2}$  for fixed  $w = (p, r, e)$ , simple calculation yields that

$$E_{w,2} [(f - \langle f_* \rangle)^2] = \frac{\int_0^{2\pi} [f_*(p, r, E, \theta) - \langle f_* \rangle]^2 \mathfrak{J}(p, r, E, \theta) d\theta}{\int_0^{2\pi} \mathfrak{J}(p, r, E, \theta) d\theta}.$$

On the other hand, since  $\mathcal{Y}_{1,2} f = \mathfrak{J}^{-1} \partial_\theta f_*$ , we have

$$E_{w,2} [(\mathcal{Y}_{1,2} f)^2] = \frac{\int_0^{2\pi} [\partial_\theta f_*(p, r, E, \theta)]^2 \mathfrak{J}^{-1}(p, r, E, \theta) d\theta}{\int_0^{2\pi} \mathfrak{J}(p, r, E, \theta) d\theta}.$$

By virtue of the Poincaré inequality on one-dimensional torus:

$$\int_0^{2\pi} (f_* - \langle f_* \rangle)^2 d\theta \leq C \int_0^{2\pi} (\partial_\theta f_*)^2 d\theta,$$

and the uniform bound of  $\mathfrak{J}$  in (7.3), we obtain that

$$E_{w,2} [(f - E_{w,2}[f])^2] \leq E_{w,2} [(f - \langle f_* \rangle)^2] \leq \frac{C\delta_+}{2\delta_-^2} E_{w,2} [(\mathcal{Y}_{1,2} f)^2]$$

holds with some universal constant  $C < \infty$ . □

*Proof of Lemma 7.3.* This lemma is proved along the idea in [18, Lemma 12.4]. Below are some notations only used in this proof. All of the subscripts  $i, j, k$  are taken from  $\{1, \dots, K\}$ . We write  $x_k = (p_k, r_k)$  and  $\mathbf{x} = (x_1, \dots, x_K)$ . Recall the bijection defined in the proof of the Lemma 7.2. For simplicity we write

$$(p_{i,j}, r_{i,j}, E(i,j), \theta_{i,j}) = (p, r, E, \theta)(x_i, x_j), \quad \forall i < j.$$

For  $\theta \in [0, 2\pi]$ , denote the Jacobian determinant by

$$\mathfrak{J}_{x_i, x_j}(\theta) = \mathfrak{J}(p_{i,j}, r_{i,j}, E(i, j), \theta).$$

For  $i < j$ ,  $\theta \in [0, 2\pi]$  and  $\mathbf{x} = (x_1, \dots, x_K)$ , define a vector  $\rho_{i,j}^\theta \mathbf{x}$  by

$$(\rho_{i,j}^\theta \mathbf{x})_k = \begin{cases} g_1(p_{i,j}, r_{i,j}, E(i, j), \theta), & k = i; \\ g_2(p_{i,j}, r_{i,j}, E(i, j), \theta), & k = j; \\ x_k, & k \neq i, j, \end{cases}$$

where  $(g_1, g_2)$  denotes the inverse map of  $(x_1, x_2) \rightarrow (p, r, E, \theta)$ . Observe that  $\rho_{i,j}^\theta \mathbf{x} = \mathbf{x}$  when  $\theta = \theta_{i,j}$ , and for every smooth function  $f$ ,

$$E_{w,K}[f|\mathcal{F}_{i,j}] = \frac{1}{J_{x_i+x_j}} \int_0^{2\pi} f(\rho_{i,j}^\theta \mathbf{x}) \mathfrak{J}_{x_i, x_j}(\theta) d\theta,$$

where  $J_{x_i+x_j} = \int_0^{2\pi} \mathfrak{J}_{x_i, x_j}(\theta) d\theta$ . On the other hand, let  $\tau_{i,j} \mathbf{x}$  be the vector given by

$$(\tau_{i,j} \mathbf{x})_i = x_j, \quad (\tau_{i,j} \mathbf{x})_j = x_i, \quad (\tau_{i,j} \mathbf{x})_k = x_k, \quad \forall k \neq i, j.$$

Moreover for  $1 \leq i < j \leq K$ , we inductively define that

$$\sigma_{i,i} = \tilde{\sigma}_{i,i} = id, \quad \sigma_{i,j} = \tau_{j-1,j} \circ \sigma_{i,j-1}, \quad \tilde{\sigma}_{i,j} = \tilde{\sigma}_{i,j-1} \circ \tau_{j-1,j}.$$

Observe that for any  $i < j$  and  $\theta \in [0, 2\pi]$ ,  $\rho_{i,j}^\theta \equiv \tilde{\sigma}_{i,j-1} \circ \rho_{j-1,j}^\theta \circ \sigma_{i,j-1}$ .

For a smooth function  $f$ , by Cauchy-Schwarz inequality,

$$(f - E_{w,K}[f|\mathcal{F}_{i,j}])^2 \leq \frac{1}{J_{x_i+x_j}} \int_0^{2\pi} [f(\rho_{i,j}^\theta \mathbf{x}) - f(\mathbf{x})]^2 \mathfrak{J}_{x_i, x_j}(\theta) d\theta.$$

The right-hand side is bounded from above by  $3(f_1 + f_2 + f_3)$ , where

$$\begin{aligned} f_1 &= \frac{1}{J_{x_i+x_j}} \int_0^{2\pi} [f(\sigma_{i,j-1} \mathbf{x}) - f(\mathbf{x})]^2 \mathfrak{J}_{x_i, x_j}(\theta) d\theta, \\ f_2 &= \frac{1}{J_{x_i+x_j}} \int_0^{2\pi} [f(\rho_{j-1,j}^\theta \circ \sigma_{i,j-1} \mathbf{x}) - f(\sigma_{i,j-1} \mathbf{x})]^2 \mathfrak{J}_{x_i, x_j}(\theta) d\theta, \\ f_3 &= \frac{1}{J_{x_i+x_j}} \int_0^{2\pi} [f(\tilde{\sigma}_{i,j-1} \circ \rho_{j-1,j}^\theta \circ \sigma_{i,j-1} \mathbf{x}) - f(\rho_{j-1,j}^\theta \circ \sigma_{i,j-1} \mathbf{x})]^2 \mathfrak{J}_{x_i, x_j}(\theta) d\theta. \end{aligned}$$

For  $f_1$ , noticing that  $f_1 = (f(\sigma_{i,j-1} \mathbf{x}) - f(\mathbf{x}))^2$ , hence

$$\begin{aligned} E_{w,K}[f_1] &\leq K \sum_{k=i}^{j-2} E_{w,K} [(f \circ \sigma_{i,k+1} - f \circ \sigma_{i,k})^2] \\ &= K \sum_{k=i}^{j-2} E_{w,K} [(f \circ \tau_{k,k+1} - f)^2]. \end{aligned}$$

Notice that  $E_{w,K}[f \circ \tau_{k,k+1} | \mathcal{F}_{k,k+1}] = E_{w,K}[f | \mathcal{F}_{k,k+1}]$ , so that

$$E_{w,K} [(f \circ \tau_{k,k+1} - E_{w,K}[f | \mathcal{F}_{k,k+1}])^2] = E_{w,K} [(f - E_{w,K}[f | \mathcal{F}_{k,k+1}])^2].$$

This together with the convex inequality  $(a+b)^2 \leq 2(a^2 + b^2)$  yields that

$$E_{w,K}[f_1] \leq 4K \sum_{k=i}^{j-2} E_{w,K} [(f - E_{w,K}[f | \mathcal{F}_{k,k+1}])^2].$$

For  $f_2$ , by applying the change of variable  $\mathbf{y} = \sigma_{i,j-1}\mathbf{x}$ , we obtain that

$$E_{w,K}[f_2] = E_{w,K} \left[ \frac{1}{J_{y_{j-1}+y_j}} \int_0^{2\pi} [f(\rho_{j-1,j}^\theta \mathbf{y}) - f(\mathbf{y})]^2 \mathfrak{J}_{y_i, y_j}(\theta) d\theta \right].$$

Therefore, we can calculate this term as

$$\begin{aligned} E_{w,K}[f_2] &= 2E_{w,K}[f^2] - 2E_{w,K}[fE_{w,K}[f|\mathcal{F}_{j-1,j}]] \\ &= 2E_{w,K}[(f - E_{w,K}[f|\mathcal{F}_{j-1,j}])^2]. \end{aligned}$$

For  $f_3$ , the same change of variable yields that

$$E_{w,K}[f_3] = E_{w,K} [E_{w,K}[(f \circ \tilde{\sigma}_{i,j-1} - f)^2 | \mathcal{F}_{j-1,j}]] = E_{w,K} [(f \circ \tilde{\sigma}_{i,j-1} - f)^2].$$

Since  $\tilde{\sigma}_{k,j-1} = \tau_{k,k+1} \circ \tilde{\sigma}_{k+1,j-1}$ , by repeating the calculation in  $f_1$ ,

$$E_{w,K}[f_3] \leq 4K \sum_{k=i}^{j-2} E_{w,K} [(f - E_{w,K}[f|\mathcal{F}_{k,k+1}])^2].$$

Hence, with some universal constant  $C < \infty$  we have

$$E_{w,K} [(f - E_{w,K}[f|\mathcal{F}_{i,j}])^2] \leq CK \sum_{k=i}^{j-1} E_{w,K} [(f - E_{w,K}[f|\mathcal{F}_{k,k+1}])^2].$$

Lemma 7.3 follows by summing up this estimate with  $i$  and  $j$ .  $\square$

To show Lemma 7.4, we need the following pre-estimate.

**Lemma 7.5.** *Assume (2.1), then (7.2) holds with constants  $C'_K$  satisfying*

$$C'_K \leq \frac{C'}{K} \left( \frac{\delta_+}{\delta_-} \right)^{3(K-1)}.$$

*Remark 7.6.* In view of Lemma 7.5, the spectral gap in (7.1) also holds without the assumption (3.14). In this case, the constant  $C_K$  satisfies that

$$C_K \leq CK^2 \left( \frac{\delta_+}{\delta_-} \right)^{3(K-1)}.$$

This estimate turns out to be sufficient for Theorem 3.1.

We first prove Lemma 7.4 from Lemma 7.5. The proof of Lemma 7.5 is put in the end of this section. Consider the bounded operator

$$L_K f = \frac{1}{K} \sum_{1 \leq i < j \leq K} (E_{w,K}[f|\mathcal{F}_{i,j}] - f), \quad \forall f \text{ s.t. } E_{w,K}[f^2] < \infty.$$

Let  $\lambda_{w,K}$  be the spectral gap of  $L_K$  with respect to  $E_{w,K}$ :

$$\lambda_{w,K} \triangleq \inf \{ \langle f, -L_K f \rangle_{w,K} \mid E_{w,K}[f] = 0, E_{w,K}[f^2] = 1 \},$$

and let  $\lambda_K = \inf \{ \lambda_{w,K}; w \in \mathbb{R}^2 \times \mathbb{R}_+ \}$ . Then (7.2) is equivalent to

$$\inf \{ \lambda_K; K \geq 3 \} > 0. \tag{7.4}$$

We prove (7.4) through an induction argument, firstly established for  $K = 3, 4$  in [9].

**Lemma 7.7.** *If  $k\lambda_k \geq 1$  holds for some  $k \geq 3$ , then for all  $K \geq k$ ,*

$$\lambda_K \geq (k\lambda_k - 1) \left( \frac{1}{k-2} - \frac{2}{K(k-2)} \right) + \frac{1}{K}.$$

In view of (3.14) and Lemma 7.5, for some fixed  $k$  which is large enough,

$$k\lambda_k > \frac{k}{C'} \left( \frac{\delta_-}{\delta_+} \right)^{3k-3} \geq \frac{k}{C'} \frac{1}{(1+\delta)^{3k-3}} \geq 1,$$

provided that  $\delta > 0$  is small enough. Then, with Lemma 7.7 we can show that the sequence  $\{\lambda_K; K \geq 3\}$  is uniformly bounded from below.

*Proof of Lemma 7.7.* We make use of the equivalent characterization of  $\lambda_{w,K}$  that

$$\lambda_{w,K} = \inf \left\{ \frac{\langle L_K f, L_K f \rangle_{w,K}}{\langle f, -L_K f \rangle_{w,K}} \mid \langle f, -L_K f \rangle_{w,K} \neq 0 \right\}.$$

In this proof we denote by  $B$  the set of all pairs  $b = (i, j)$  such that  $1 \leq i < j \leq K$ , and write  $D_b f = E_{w,K}[f | \mathcal{F}_b] - f$  for all  $b \in B$ , then

$$\langle L_K f, L_K f \rangle_{w,K} = \frac{1}{K^2} \sum_{b, b' \in B} \langle D_b f, D_{b'} f \rangle_{w,K},$$

$$\langle f, -L_K f \rangle_{w,K} = \frac{1}{K} \sum_{b \in B} \langle D_b f, D_b f \rangle_{w,K}.$$

We write  $b \sim b'$  if two pairs  $b$  and  $b'$  have at least one common point. We also consider all the  $k$ -particle subsets  $T_k \subseteq \{1, \dots, K\}$ . Notice that if  $b \sim b'$  but  $b \neq b'$ , there are  $\binom{K-3}{k-3}$  different  $T_k$ 's containing both  $b$  and  $b'$ . Hence,

$$\binom{K-3}{k-3} \sum_{\substack{b, b' \in B \\ b \neq b', b \sim b'}} \langle D_b f, D_{b'} f \rangle_{w,K} = \sum_{T_k} \sum_{\substack{b, b' \subseteq T_k \\ b \neq b', b \sim b'}} \langle D_b f, D_{b'} f \rangle_{w,K}.$$

If  $b \not\sim b'$ , there are  $\binom{K-4}{k-4}$  different  $T_k$ 's contain both  $b$  and  $b'$ , while for the case  $b = b'$  it is  $\binom{K-2}{k-2}$ . Therefore, the right-hand side of the equation above equals to

$$\sum_{T_k} \sum_{b, b' \subseteq T_k} \langle D_b f, D_{b'} f \rangle_{w,K} - \binom{K-4}{k-4} \sum_{b \not\sim b'} \langle D_b f, D_{b'} f \rangle_{w,K} - \binom{K-2}{k-2} \sum_{b \in B} \langle D_b f, D_b f \rangle_{w,K}.$$

The definition of  $\lambda_k$  yields that

$$\frac{1}{k} \sum_{b, b' \subseteq T_k} \langle D_b f, D_{b'} f \rangle_{w,K} \geq \lambda_k \sum_{b \subseteq T_k} \langle D_b f, D_b f \rangle_{w,K}.$$

And for  $b \not\sim b'$ ,  $\langle D_b f, D_{b'} f \rangle_{w,K} = E_{w,K}[(D_{b'} D_b f)^2] \geq 0$ . Therefore,

$$\sum_{\substack{b, b' \in B \\ b \neq b', b \sim b'}} \langle D_b f, D_{b'} f \rangle_{w,K} \geq \frac{(k\lambda_k - 1)(K-2)}{k-2} \sum_{b \in B} \langle D_b f, D_b f \rangle_{w,K}.$$

By the condition  $k\lambda_k > 1$ , the right-hand side is positive. In conclusion,

$$\begin{aligned} \langle L_K f, L_K f \rangle_{w,K} &\geq \frac{1}{K^2} \sum_{b \in B} \langle D_b f, D_b f \rangle_{w,K} + \frac{1}{K^2} \sum_{b \neq b', b \sim b'} \langle D_b f, D_{b'} f \rangle_{w,K} \\ &\geq \frac{1}{K^2} \left[ \frac{(k\lambda_k - 1)(K - 2)}{k - 2} + 1 \right] \sum_{b \in B} \langle D_b f, D_b f \rangle_{w,K} \\ &= \left[ (k\lambda_k - 1) \left( \frac{1}{k - 2} - \frac{2}{K(k - 2)} \right) + \frac{1}{K} \right] \langle f, -L_K f \rangle_{w,K}. \end{aligned}$$

Notice that this estimate is independent of the choice of  $w$ .  $\square$

Finally, to complete the proof of Proposition 7.1, we are left to show Lemma 7.5. To do this, we make use of the spectral gap bound of Kac walk. For  $a \in \mathbb{R}^2$  and  $R \geq |a|^2$ , consider the  $(2K - 3)$ -dimensional sphere

$$S_K(a, R) = \left\{ x_1, \dots, x_K \in \mathbb{R}^2 \mid \frac{1}{K} \sum_{k=1}^K x_k = a, \frac{1}{K} \sum_{k=1}^K |x_k|^2 = R \right\}.$$

Denote by  $\mu_K(a, R)$  the uniform measure on  $S_K(a, R)$ . With a little abuse of notations, let  $\mathcal{F}_{i,j} = \sigma\{x_k; k \neq i, j\}$  for  $1 \leq i < j \leq K$ .

**Lemma 7.8.** *There exists a constant  $C$  such that*

$$E_{\mu_K(a,R)} [(f - E_{\mu_K(a,R)}[f])^2] \leq \frac{C}{K} \sum_{1 \leq i < j \leq K} E_{\mu_K(a,R)} [(f - E_{\mu_K(a,R)}[f|\mathcal{F}_{i,j}])^2]$$

for all  $(a, R, K)$  and bounded smooth function  $f$ .

Lemma 7.8 is a special case of the spectral gap proved in [11], see also [10]. We here prove Lemma 7.5 by applying a perturbation on Lemma 7.8.

*Proof.* To begin with, from (2.1) we know that for  $r \neq r'$  and  $K \geq 1$ ,

$$\frac{\sqrt{2(K+1)}}{\sqrt{K}} c_- \leq \frac{|V'(r) - V'(r')|}{\sqrt{V(r) + KV(r') - (K+1)V\left(\frac{r+Kr'}{K+1}\right)}} \leq \frac{\sqrt{2(K+1)}}{\sqrt{K}} c_+, \quad (7.5)$$

where  $c_- = \delta_-/\sqrt{\delta_+}$  and  $c_+ = \delta_+/\sqrt{\delta_-}$ . For each  $K \geq 2$ , we construct a bijection  $\tau_K : \Omega_K \rightarrow \Omega_K$ , satisfying the following two conditions.

- (i) For  $w = (p, r, e)$ ,  $\tau_K(\Omega_{w,K}) = S_K(a, R)$ , where  $a = (p, r)$ ,  $R = 2e - 2V(r) + r^2$ ;
- (ii) The Jacobian matrix  $\tau'_K$  of  $\tau_K$  satisfies that  $c_-^{K-1} \leq |\det(\tau'_K)| \leq c_+^{K-1}$ .

Indeed, given a bounded, measurable, positive function  $g$  on  $\Omega_{w,K}$ , by (i) we know that  $\tau_K^{-1}g := g \circ \tau_K^{-1}$  defines a function on  $S_K(a, R)$ , and (ii) yields that

$$c_0^{-(K-1)} E_{\mu_K(a,R)} [\tau_K^{-1}g] \leq E_{w,K}[g] \leq c_0^{K-1} E_{\mu_K(a,R)} [\tau_K^{-1}g],$$

where  $c_0 = c_+/c_-$ . For bounded and smooth function  $f$ , write  $g = (f - E_{\mu_K(a,R)}[\tau_K^{-1}f])^2$ . As  $E_{w,K}[(f - a)^2]$  reaches its minimum in  $a \in \mathbb{R}$  at  $a = E_{w,K}[f]$ , we have

$$E_{w,K} [(f - E_{w,K}[f])^2] \leq E_{w,K}[g] \leq c_0^{K-1} E_{\mu_K(a,R)} [\tau_K^{-1}g]. \quad (7.6)$$



Similarly, by taking  $h_{i,j} = (f - E_{w,K}[f|\mathcal{F}_{i,j}])^2$ ,

$$\begin{aligned} & E_{\mu_K(a,R)} [(\tau_K^{-1}f - E_{\mu_K(a,R)}[\tau_K^{-1}f|\mathcal{F}_{i,j}])^2] \\ & \leq E_{\mu_K(a,R)} [\tau_K^{-1}h_{i,j}] \leq c_0^{K-1} E_{w,K}[h_{i,j}]. \end{aligned} \quad (7.7)$$

Meanwhile, substitute  $\tau_K^{-1}f$  for  $f$  in Lemma 7.8 and we obtain

$$E_{\mu_K(a,R)}[\tau_K^{-1}g] \leq \frac{C}{K} \sum_{i < j} E_{\mu_K(a,R)} [(\tau_K^{-1}f - E_{\mu_K(a,R)}[\tau_K^{-1}f|\mathcal{F}_{i,j}])^2]. \quad (7.8)$$

Recall that  $c_0 = (\delta_+/\delta_-)^{3/2}$ , Lemma 7.5 then follows from (7.6)–(7.8).

Now we construct the map  $\tau_K$ . Observe that for the case  $K = 2$ , we can define

$$\tau_2(p_1, p_2, r_1, r_2) = (p_1, p_2, r - r', r + r'),$$

where  $r = (r_1 + r_2)/2$  and  $r'$  is uniquely determined by the relation

$$\frac{(r - r')^2}{2} + \frac{(r + r')^2}{2} - r^2 = V(r_1) + V(r_2) - 2V(r).$$

To extend the above formula to general  $K$ , define

$$\alpha_k = \frac{1}{k} \sum_{i=1}^k r_i, \quad \forall 1 \leq k \leq K.$$

Consider two maps  $\zeta, \zeta_* : \mathbb{R}^K \rightarrow \mathbb{R}^K$ . The first map  $\zeta$  is given by

$$\zeta(r_1, \dots, r_K) = (r'_1, \dots, r'_K),$$

such that  $r'_K = \alpha_K$ , and for  $1 \leq k \leq K - 1$ ,

$$(r'_k)^2 = \frac{2k}{k+1} (V(r_{k+1}) + kV(\alpha_k) - (k+1)V(\alpha_{k+1})),$$

where the sign of  $r'_k$  is chosen in accordance with  $r_k - \alpha_K$ . Meanwhile,  $\zeta_*$  is given by

$$\zeta_*(r'_1, \dots, r'_K) = (r''_1, \dots, r''_K),$$

such that

$$r''_k = \begin{cases} r'_K - \sum_{i=1}^{K-1} \frac{r'_i}{i}, & \text{for } k = 1, \\ r'_K + r'_{k-1} - \sum_{i=k}^{K-1} \frac{r'_i}{i}, & \text{for } 2 \leq k \leq K - 1, \\ r'_K + r'_{K-1}, & \text{for } k = K. \end{cases}$$

Denote by  $J$  and  $J_*$  the Jacobian matrices of  $\zeta$  and  $\zeta_*$ , respectively. To compute  $J$ , noticing that  $\partial_{r_i} r'_k = \partial_{r_k} r'_k$  for all  $i \leq k$ , and  $\partial_{r_i} r'_k = 0$  for all  $i > k + 1$ , we have

$$J = \begin{bmatrix} \frac{\partial r'_1}{\partial r_1} & \frac{\partial r'_1}{\partial r_2} & 0 & \cdots & 0 \\ \frac{\partial r'_2}{\partial r_1} & \frac{\partial r'_2}{\partial r_2} & \frac{\partial r'_2}{\partial r_3} & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ \frac{\partial r'_{K-1}}{\partial r_{K-1}} & \frac{\partial r'_{K-1}}{\partial r_{K-1}} & \frac{\partial r'_{K-1}}{\partial r_{K-1}} & \cdots & \frac{\partial r'_{K-1}}{\partial r_K} \\ \frac{\partial r'_K}{\partial r_K} & \frac{\partial r'_K}{\partial r_K} & \frac{\partial r'_K}{\partial r_K} & \cdots & \frac{\partial r'_K}{\partial r_K} \end{bmatrix}.$$

Hence, its determinant reads

$$|\det(J)| = \left| \frac{\partial r'_K}{\partial r_K} \right| \cdot \prod_{k=1}^{K-1} \left| \frac{\partial r'_k}{\partial r_k} - \frac{\partial r'_k}{\partial r_{k+1}} \right|.$$

Since  $\partial_{r_K} r'_K = 1/K$  and for  $k = 1, \dots, K-1$  we have

$$\frac{\partial r'_k}{\partial r_i} = \begin{cases} \frac{k}{(k+1)r'_k} [V'(\alpha_k) - V'(\alpha_{k+1})], & \text{if } 1 \leq i \leq k, \\ \frac{k}{(k+1)r'_k} [V'(r_{k+1}) - V'(\alpha_{k+1})], & \text{if } i = k+1. \end{cases}$$

In consequence,  $|\det(J)|$  equals to

$$\frac{1}{K} \prod_{k=1}^{K-1} \frac{\sqrt{k}}{\sqrt{2(k+1)}} \frac{|V'(r_{k+1}) - V'(\alpha_k)|}{\sqrt{V(r_{k+1}) + kV(\alpha_k) - (k+1)V(\alpha_{k+1})}}.$$

Applying the estimate in (7.5) to obtain that

$$\frac{c_-^{K-1}}{K} \leq |\det(J)| \leq \frac{c_+^{K-1}}{K}.$$

Meanwhile it is easy to calculate that  $|\det(J_*)| = K$ . Therefore, define

$$\tau_K : (p_1, \dots, p_K, r_1, \dots, r_K) \mapsto (p_1, \dots, p_K, r'_1, \dots, r'_K),$$

then  $|\det(\tau'_K)|$  satisfies (ii). On the other hand, suppose that  $\{x_k = (p_k, r_k)\}_{1 \leq k \leq K}$  belongs to the microcanonical manifold  $\Omega_{w,K}$ , then  $r'_K = r$  and

$$\frac{1}{K} \sum_{k=1}^{K-1} \frac{k+1}{k} \frac{(r'_k)^2}{2} = \frac{1}{K} \sum_{k=1}^K V(r_k) - V(r'_K) = e - \frac{1}{K} \sum_{k=1}^K \frac{p_k^2}{2} - V(r).$$

Then, it follows from the definition of  $r''_k$  that

$$\frac{1}{K} \sum_{k=1}^K (r''_k)^2 = (r'_K)^2 + \frac{1}{K} \sum_{k=1}^{K-1} \frac{k+1}{k} (r'_k)^2 = 2e - 2V(r) + r^2 - \frac{1}{K} \sum_{k=1}^K p_k^2.$$

Hence,  $\tau_K(x_1, \dots, x_K) \in S_K(a, R)$  with  $R = 2e - V(r) + r^2$ , and (i) is also verified. The proof of Lemma 7.5 is then completed.  $\square$

## 8. Equivalence of ensembles

In this section we prove the equivalence of ensembles for the dynamics with multi-dimensional conserved quantities. By applying Proposition 8.3 to the model in this paper, we obtain Corollary 8.4, which is necessary in the proof of Lemma 4.2.

The notations in this section are different from the former part. Let  $\pi$  be a Borel measure on  $\Omega = \mathbb{R}^m$  with smooth density function with respect to the Lebesgue measure, and  $\mathbf{f} = (f_1, \dots, f_d)$  be a  $d$ -dimensional function on  $\Omega$  with compact level sets. Suppose that there is some domain  $D \subseteq \mathbb{R}^d$ , such that

$$Z(\lambda) \triangleq \log \left[ \int_{\Omega} \exp \{ \lambda \cdot \mathbf{f}(\omega) \} \pi(d\omega) \right] < \infty, \quad \forall \lambda \in D.$$

To avoid the problem of regularity, we assume that  $Z$  is four times continuously differentiable on  $D$ , and its Hessian matrix  $\Sigma_{\lambda} = Z''(\lambda)$  is always positive-definite. To simplify the notations, we denote  $\mathbf{u}_{\lambda} = \nabla_{\lambda} Z(\lambda)$ .

For  $\lambda \in D$  we can define the *tilted probability measure* by

$$\pi_\lambda(d\omega) \triangleq \exp\{\lambda \cdot \mathbf{f}(\omega) - Z(\lambda)\} \pi(d\omega).$$

Observe that  $E_{\pi_\lambda}[\mathbf{f}] = \mathbf{u}_\lambda$ , and  $E_{\pi_\lambda}[(\mathbf{f} - \mathbf{u}_\lambda) \otimes (\mathbf{f} - \mathbf{u}_\lambda)] = \Sigma_\lambda$ . Let  $\Phi_\lambda$  be the centred characteristic function of  $\mathbf{f}$  with respect to  $\pi_\lambda$ , given by

$$\Phi_\lambda(\mathbf{h}) = \int_{\Omega} \exp\{i\mathbf{h} \cdot (\mathbf{f}(\omega) - \mathbf{u}_\lambda)\} \pi_\lambda(d\omega), \quad \forall \mathbf{h} \in \mathbb{R}^d.$$

We also assume that there exists some  $\epsilon_0 > 0$ , such that

$$\sup_{\mathbf{h} \in \mathbb{R}^d} |\mathbf{h}|^{\epsilon_0} |\Phi_\lambda(\mathbf{h})| < \infty.$$

The main methods we use here is a multi-dimensional local central limit theorem with an edge expansion and a large deviation property for  $\mathbf{f}$ . We state them in Lemma 8.1 and Lemma 8.2 respectively. Let  $\phi_\lambda = \phi_\lambda(\mathbf{x})$  be the Gaussian density function on  $\mathbb{R}^d$ , whose mean is  $\mathbf{0}$  and variance matrix is  $\Sigma_\lambda$ :

$$\phi_\lambda(\mathbf{x}) = \frac{1}{(2\pi)^{d/2}} \frac{1}{\sqrt{\det \Sigma_\lambda}} \exp\left\{-\frac{\mathbf{x}' \Sigma_\lambda^{-1} \mathbf{x}}{2}\right\}, \quad \forall \mathbf{x} \in \mathbb{R}^d.$$

For  $k \in \mathbb{N}_+$ , define the  $d$ -variable polynomials  $P_{\lambda,k}$  by

$$P_{\lambda,k}(\mathbf{h}) = \sum_{|\alpha|=k} \frac{\partial_\alpha Z(\lambda)}{\alpha!} \mathbf{h}^\alpha,$$

where  $\alpha = (\alpha_1, \dots, \alpha_d)$  is multiple index,  $\alpha_j \geq 0$ , and

$$|\alpha| = \sum_{j=1}^d \alpha_j, \quad \alpha! = \prod_{j=1}^d \alpha_j!, \quad \partial_\alpha = \prod_{j=1}^d \frac{\partial^{\alpha_j}}{\partial \lambda_j^{\alpha_j}}, \quad \mathbf{h}^\alpha = \prod_{j=1}^d h_j^{\alpha_j}.$$

Also define the polynomials  $Q_{\lambda,3}$  and  $Q_{\lambda,4}$  by

$$Q_{\lambda,3} = \frac{1}{(2\pi)^d \phi_\lambda} \int_{\mathbb{R}^d} \exp\left\{-i\mathbf{x} \cdot \mathbf{h} - \frac{\mathbf{h}' \Sigma_\lambda \mathbf{h}}{2}\right\} P_{\lambda,3}(i\mathbf{h}) d\mathbf{h};$$

$$Q_{\lambda,4} = \frac{1}{(2\pi)^d \phi_\lambda} \int_{\mathbb{R}^d} \exp\left\{-i\mathbf{x} \cdot \mathbf{h} - \frac{\mathbf{h}' \Sigma_\lambda \mathbf{h}}{2}\right\} \left(P_{\lambda,4} + \frac{P_{\lambda,3}^2}{2}\right)(i\mathbf{h}) d\mathbf{h}.$$

Let  $\Omega_n$  be the  $n$ -product space of  $\Omega$ . Define

$$\mathbf{f}_{(n)}(\vec{\omega}) = \frac{1}{n} \sum_{j=1}^n \mathbf{f}(\omega_j), \quad \forall \vec{\omega} = (\omega_1, \dots, \omega_n) \in \Omega_n.$$

Equip  $\Omega_n$  with the product measure  $\pi_{\lambda,n} = \otimes_j \pi_\lambda(d\omega_j)$ . We have the following local central limit theorem. The proof is standard [20, Theorem VII.15].

**Lemma 8.1.** *Let  $f_{\lambda,n}$  be the density function of the random vector*

$$\sqrt{n}(\mathbf{f}_{(n)} - \mathbf{u}_\lambda) = \frac{1}{\sqrt{n}} \sum_{j=1}^n (\mathbf{f}(\omega_j) - \mathbf{u}_\lambda)$$

*with respect to the product measure  $\pi_{\lambda,n}$  for  $n$  large enough. Then,*

$$\left| f_{\lambda,n}(\mathbf{x}) - \phi_\lambda(\mathbf{x}) \left(1 + \frac{Q_{\lambda,3}(\mathbf{x})}{\sqrt{n}} + \frac{Q_{\lambda,4}(\mathbf{x})}{n}\right) \right| \leq \frac{K_{\lambda,n}}{n}, \quad \forall \mathbf{x} \in \mathbb{R}^d, \quad (8.1)$$

*where  $\lim_{n \rightarrow \infty} K_{\lambda,n} = 0$ , uniformly in any compact subset of  $D$ .*

As  $Z$  is strictly convex, consider its *Fenchel-Legendre transform*:

$$Z^*(\mathbf{u}) = \sup_{\lambda \in D} \{\lambda \cdot \mathbf{u} - Z(\lambda)\}.$$

Let  $D^* = \{\mathbf{u} \in \mathbb{R}^d : Z^*(\mathbf{u}) < \infty\}$ . The superior is reached at a unique  $\lambda(\mathbf{u}) \in D$ , given by the convex conjugate

$$\lambda(\mathbf{u}) = \nabla_{\mathbf{u}} Z^*(\mathbf{u}), \quad \mathbf{u}_\lambda = \nabla_\lambda Z(\lambda).$$

Notice that  $\mathbf{u} \mapsto \lambda(\mathbf{u})$  and  $\lambda \mapsto \mathbf{u}_\lambda$  are a pair of inverse maps between  $D$  and  $D^*$ . For  $\lambda \in D$  and  $\mathbf{u} \in D^*$ , define the rate function  $I_\lambda(\mathbf{u})$  by

$$I_\lambda(\mathbf{u}) = Z^*(\mathbf{u}) - Z^*(\mathbf{u}_\lambda) - \nabla_{\mathbf{u}} Z^*(\mathbf{u}_\lambda) \cdot (\mathbf{u} - \mathbf{u}_\lambda). \quad (8.2)$$

Denote by  $M_\lambda$  the largest eigenvalue of  $\Sigma_\lambda$ . By the arguments above it is not hard to conclude that for any constant  $M > M_\lambda$ , we have

$$I_\lambda(\mathbf{u}) \geq (2M)^{-1} |\mathbf{u} - \mathbf{u}_\lambda|^2 \quad (8.3)$$

holds if  $|\mathbf{u} - \mathbf{u}_\lambda|$  is small enough. By virtue of (8.3), we can also obtain the following large deviation property for  $\mathbf{f}_{(n)}$ .

**Lemma 8.2.** *For any  $M > M_\lambda$ , there exists some  $\delta_M$  such that*

$$\pi_{\lambda,n} \{|\mathbf{f}_{(n)} - \mathbf{u}_\lambda| \geq \delta\} \leq 2^d \exp\left(-\frac{nM\delta^2}{d}\right),$$

holds for all  $n \geq 1$  when  $|\delta| < \delta_M$ .

*Proof.* Let  $\Gamma \subseteq \mathbb{R}^d$  be the collection of vectors whose coordinates are all  $\pm 1$ . Notice that the following inequality holds for all  $\mathbf{x} \in \mathbb{R}^d$ :

$$e^{|\mathbf{x}|} \leq \prod_{j=1}^d e^{|x_j|} \leq \prod_{j=1}^d (e^{-x_j} + e^{x_j}) = \sum_{\gamma \in \Gamma} e^{\gamma \cdot \mathbf{x}}.$$

By exponential Chebyshev's inequality and the above estimate, for  $\theta > 0$ ,

$$\begin{aligned} \pi_{\lambda,n} \{|\mathbf{f}_{(n)} - \mathbf{u}_\lambda| \geq \delta\} &\leq \sum_{\gamma \in \Gamma} e^{-n\theta\delta} \int_{|\mathbf{f}_{(n)} - \mathbf{u}_\lambda| \geq \delta} \exp\{n\theta\gamma \cdot (\mathbf{f}_{(n)} - \mathbf{u}_\lambda)\} \pi_{\lambda,n}(d\bar{\omega}) \\ &\leq \sum_{\gamma \in \Gamma} \exp\{-n\theta u' + nZ(\lambda + \theta\gamma) - nZ(\lambda)\}, \end{aligned}$$

where  $u' = \gamma \cdot \mathbf{u}_\lambda + \delta$ . To optimize this estimate, define

$$I_{\lambda,\gamma}(u') = \sup_{\theta > 0} \{\theta u' - Z(\lambda + \theta\gamma) + Z(\lambda)\} = \sup_{\theta \in \mathbb{R}} \{\theta u' - Z(\lambda + \theta\gamma) + Z(\lambda)\}.$$

The last equality is due to the fact that  $u' - \partial_\theta Z(\lambda + \theta\gamma)|_{\theta=0} = \delta > 0$ . Notice that  $I_{\lambda,\gamma}$  is the rate function defined in (8.2) corresponding to the measure  $\pi_\lambda$  and the function  $\gamma \cdot \mathbf{f}$ . By the arguments which have been used to derive (8.3), one obtains that  $I_{\lambda,\gamma}(u') \geq M_\lambda |\gamma|^{-2} \delta^2$ . The estimate in Lemma 8.2 then follows directly.  $\square$

Now fix some  $k \in \mathbb{N}_+$ . For an integrable function  $G$  on  $\Omega_k$ , any  $n \geq k$  and  $\mathbf{u} \in D^*$ , define the microcanonical expectation  $\langle G | \mathbf{u} \rangle_n$  by

$$\langle G | \mathbf{u} \rangle_n = E_{\pi_{\lambda,n}} [G \mid \mathbf{f}_{(n)} = \mathbf{u}].$$

It is easy to see that the definition of  $\langle G|\mathbf{u}\rangle_n$  does not depend on  $\lambda$ . Notice that though the conditional expectation can usually be defined only in an almost sure sense, under the regularity of  $\mathbf{f}$ , the microcanonical surface

$$\Omega_{\mathbf{u},n} = \{\vec{\omega} \in \Omega_n; \mathbf{f}_{(n)}(\vec{\omega}) = \mathbf{u}\},$$

is smooth enough to define the regular conditional expectation for everywhere in  $D^*$ . Recall that  $\mathbf{u}_\lambda = E_{\pi_\lambda}[\mathbf{f}]$ . The following estimate (cf. [14, p.353, Corollary A2.1.4]) holds.

**Proposition 8.3.** *Suppose that for some compact subset  $D_0$  of  $D$ ,*

$$C_j \triangleq \sup_{\lambda \in D_0} E_{\pi_\lambda} [|\mathbf{f} - \mathbf{u}_\lambda|^j] < \infty, \quad j = 1, 2, 3, 4,$$

and  $G : \Omega_k \rightarrow \mathbb{R}$  satisfies that  $E_{\pi_{\lambda,k}}[G^2] < \infty$  for all  $\lambda \in D_0$ . Then,

$$\limsup_{n \rightarrow \infty} n |\langle G|\mathbf{u}_\lambda \rangle_n - E_{\pi_{\lambda,k}}[G]| \leq Ck \sqrt{E_{\pi_{\lambda,k}}[(G - E_{\pi_{\lambda,k}}[G])^2]}, \quad (8.4)$$

with a uniform constant  $C$  for every  $\lambda \in D_0$ .

*Proof.* The proof is exactly parallel to [14, Corollary A2.1.4]. We sketch it for completeness. Without loss of generality we can assume that  $E_{\pi_{\lambda,k}}[G] = 0$  for some fixed  $\lambda \in D_0$ . Denote by  $F_{\lambda,n}$  the density function of  $\mathbf{f}_{(n)}$  under  $\pi_{\lambda,n}$ :

$$\int_{\Omega_n} g(\mathbf{f}_{(n)}) d\pi_{\lambda,n} = \int_{\mathbb{R}^d} g(\mathbf{u}) F_{\lambda,n}(\mathbf{u}) d\mathbf{u}$$

for all integrable function  $g$  on  $\mathbb{R}^d$ . We can write  $\langle G|\mathbf{u}\rangle_n$  as

$$\int_{\mathbb{R}^k} G(\omega_1, \dots, \omega_k) \left( \frac{F_{\lambda,n-k}(\mathbf{u}_{k,n})}{F_{\lambda,n}(\mathbf{u})} - 1 \right) \pi_{\lambda,k}(d\omega_1 \dots d\omega_k),$$

where  $\mathbf{u}_{k,n} = (n-k)^{-1}(n\mathbf{u} - k\mathbf{f}_{(k)})$ . Schwarz inequality then yields that

$$\langle G|\mathbf{u}\rangle_n^2 \leq E_{\pi_{\lambda,k}}[G^2] E_{\pi_{\lambda,k}} \left[ \left| \frac{F_{\lambda,n-k}(\mathbf{u}_{k,n})}{F_{\lambda,n}(\mathbf{u})} - 1 \right|^2 \right].$$

Take  $\mathbf{u} = \mathbf{u}_\lambda$  in the above expression. By Lemma 8.1,

$$\left| \frac{F_{\lambda,n-k}(\mathbf{u}_{k,n})}{F_{\lambda,n}(\mathbf{u}_\lambda)} - 1 \right| \leq \frac{Ck}{n} (1 + |\mathbf{f}_{(k)} - \mathbf{u}_\lambda| + k|\mathbf{f}_{(k)} - \mathbf{u}_\lambda|^2).$$

where  $C$  is a constant depending on  $\{C_j; j = 1, 2, 3, 4\}$ , the polynomials  $Q_{\lambda,3}$ ,  $Q_{\lambda,4}$  and the sequence  $K_{\lambda,n}$  appeared in (8.1). Hence, (8.4) holds for the fixed  $\lambda$  we chosen. Since the polynomials  $Q_{\lambda,3}$  and  $Q_{\lambda,4}$  are continuously dependent on  $\lambda$ , and  $K_{\lambda,n}$  vanishes uniformly in  $D_0$ , we can extend the result to every  $\lambda \in D_0$ .  $\square$

Now we apply Proposition 8.3 to the model established in Section 1. Let  $\Omega = \mathbb{R}^2$ ,  $\pi$  be the Lebesgue measure, and  $\mathbf{f}$  be the three-dimensional function on  $\Omega$  given by

$$\mathbf{f}(\omega) = (p, r, -p^2/2 - V(r)), \quad \text{for } \omega = (p, r) \in \Omega,$$

where  $V$  is a  $C^4$ -smooth function with quadratic growth (2.1). It is not hard to obtain that  $D = \mathbb{R}^2 \times \mathbb{R}_+$  and  $D^* = \mathbb{R}^2 \times \mathbb{R}_-$ . For  $\lambda \in D$ ,

$$Z(\lambda) = \ln \left( \int_{\mathbb{R}} e^{-\lambda_3 V(r) + \lambda_2 r} dr \right) + \frac{\lambda_1^2}{2\lambda_3} + \frac{1}{2} \ln \left( \frac{2\pi}{\lambda_3} \right), \quad \lambda = (\lambda_1, \lambda_2, \lambda_3).$$

So  $Z$  is four times differentiable and all of its partial derivatives are uniformly bounded in  $[-K, K]^2 \times [\epsilon, \infty)$  for  $K, \epsilon > 0$ . Furthermore, the assumptions in Proposition 8.3 holds in the same set. Recall the continuous map  $\mathbf{u} \rightarrow \lambda(\mathbf{u})$  from  $D^*$  and  $D$ , which gives the inverse of  $\lambda \rightarrow \mathbf{u}_\lambda$ . With Proposition 8.3, we have the following estimate.

**Corollary 8.4.** *Suppose that  $F$  is a function on  $\Omega_k$ , such that  $E_{\pi_{\lambda,k}}[F]$  is twice continuously differentiable in  $\lambda$ , and  $E_{\pi_{\lambda',k}}[F^4] < \infty$  for some fixed  $\lambda' \in D$ . Define*

$$G = F - E_{\pi_{\lambda',k}}[F] - \nabla_{\mathbf{u}} E_{\pi_{\lambda(\mathbf{u}),k}}|_{\mathbf{u}=\mathbf{u}'} \cdot (\mathbf{f}(\omega_j) - \mathbf{u}'),$$

where  $\mathbf{u}' = \mathbf{u}_{\lambda'} \in D^*$ . Then for  $n$  large enough, we have

$$E_{\lambda',n}[\langle G|\mathbf{u}\rangle_n^2] \leq Cn^{-2},$$

where  $C$  is a finite constant depending only on  $F$  and  $\lambda'$ .

*Proof.* Fix some  $\delta \in (0, \delta_M)$ , where  $\delta_M$  is the constant appeared in Lemma 8.2. By Schwarz inequality and Lemma 8.2,

$$E_{\lambda',n}[\langle G|\mathbf{u}\rangle_n^2 \mathbf{1}_{\{|\mathbf{u}-\mathbf{u}'|>\delta\}}] \leq 2^{\frac{d}{2}} \exp\left\{-\frac{nM\delta^2}{2d}\right\} \sqrt{E_{\lambda',n}[\langle G|\mathbf{u}\rangle_n^4]},$$

so, it suffices only consider the compact set  $\{|\mathbf{u} - \mathbf{u}'| \leq \delta\}$ . Observe that

$$\langle G|\mathbf{u}\rangle_n = \langle F|\mathbf{u}\rangle_n - E_{\pi_{\lambda',k}}[F] - \nabla_{\mathbf{u}} E_{\pi_{\lambda(\mathbf{u}),k}}|_{\mathbf{u}=\mathbf{u}'} \cdot (\mathbf{u} - \mathbf{u}'), \quad \forall \mathbf{u} \in D^*.$$

Recall that  $\lambda(\mathbf{u})$  continuous in  $D^*$ , so  $\{\lambda(\mathbf{u}); |\mathbf{u} - \mathbf{u}'| \leq \delta\}$  is a compact subset of  $D$ . Apply Proposition 8.3 with  $\lambda = \lambda(\mathbf{u})$  to obtain that

$$|\langle F|\mathbf{u}\rangle_n - E_{\pi_{\lambda(\mathbf{u}),k}}[F]| \leq Cn^{-1}, \quad \forall \mathbf{u} \in \{|\mathbf{u} - \mathbf{u}'| \leq \delta\},$$

where a constant  $C = C(F, \delta)$ , so its square integral is bounded by  $C'n^{-2}$ . We are left with the second moment in  $\{|\mathbf{u} - \mathbf{u}'| \leq \delta\}$  of

$$E_{\pi_{\lambda(\mathbf{u}),k}}[F] - E_{\pi_{\lambda',k}}[F] - \nabla_{\mathbf{u}} E_{\pi_{\lambda(\mathbf{u}),k}}|_{\mathbf{u}=\mathbf{u}'} \cdot (\mathbf{u} - \mathbf{u}').$$

Since  $E_{\pi_{\lambda,n}}[F]$  is smooth in  $\lambda$  and  $\lambda(\mathbf{u})$  is smooth in  $\mathbf{u}$ , we know that this function is bounded by  $C|\mathbf{u} - \mathbf{u}_\lambda|$  with a constant  $C = C(F, \lambda')$ . The desired estimate then follows from the fact that  $E_{\lambda',n}[|\mathbf{f}(\mathbf{u}) - \mathbf{u}'|^4] \leq C'n^{-2}$ .  $\square$

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