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# Equilibrium fluctuation for an anharmonic chain with boundary conditions in the Euler scaling limit

Stefano OLLA      Lu XU

## Abstract

We study the evolution in equilibrium of the fluctuations for the conserved quantities of a chain of anharmonic oscillators in the hyperbolic space-time scaling. Boundary conditions are determined by applying a constant tension at one side, while the position of the other side is kept fixed. The Hamiltonian dynamics is perturbed by random terms conservative of such quantities. We prove that these fluctuations evolve macroscopically following the linearized Euler equations with the corresponding boundary conditions, even in some time scales larger than the hyperbolic one.

## 1 Introduction

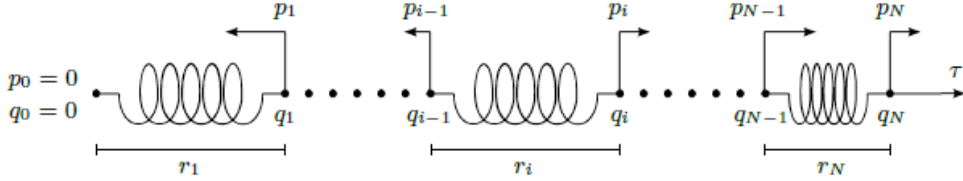
The deduction of Euler equations for a compressible gas from the microscopic dynamics under a space-time scaling limit is one of the main problems in statistical mechanics [11]. With a generic assumption of *local equilibrium*, Euler equations can be formally obtained in the limit, but a mathematical proof starting from deterministic Hamiltonian dynamics is still an open problem. The eventual appearance of shock waves complicates further the task, and in this case, it is expected the convergence to weak entropic solutions of Euler equations.

Some mathematical results have been obtained by perturbing the Hamiltonian dynamics by random terms that conserve energy and momentum, in such way that the dynamics has enough ergodicity to generate some form of local equilibrium (cf. [14, 2]). These results are obtained by relative entropy techniques and restricted to the smooth regime of the Euler equations. The noise introduced in these works are essentially random collisions between close particles and acts only on the velocities. Under such random perturbations, the only conserved quantities are those that evolve macroscopically with the Euler equations. Actually, random dynamics and local equilibrium are only tools to obtain the separation of scales between microscopic and macroscopic modes necessary in order to close the Euler equations. In the deterministic dynamics of harmonic oscillators with random masses (not ergodic), Anderson localization provides such separation of scales [1].

In this article we study the evolution of the fluctuations of the conserved quantities. When the system is in equilibrium at certain averaged values of the conserved quantities, these have Gaussian macroscopic fluctuations. The aim is to prove that these fluctuations, in the macroscopic space-time scaling limit, evolve deterministically following the linearized Euler equations. It turns out that this is more difficult than proving the hydrodynamic limit, as it requires the control of the space-time variance of the currents of the conserved quantities. More precisely it demands to prove that the currents are *equivalent* (in the norm introduced by the space-time variance) to linear functions of the conserved quantities. This step is usually called *Boltzmann-Gibbs principle* (cf. [3, 9]). This is the main part of the proof, and it forces us to consider

*elliptic* type of stochastic perturbations, i.e., noise terms that act also on the positions, not only on the velocities, still maintaining the same conserved quantities.

The system we consider is  $N+1$  coupled anharmonic oscillators, similar to the one considered in [2]. For  $i = 0, \dots, N$ , the momenta (or velocity, since we set the masses equal to 1) of the particle  $i$  is denoted by  $p_i \in \mathbb{R}$ , while  $q_i \in \mathbb{R}$  denotes its position. Particle 0 is attached to some fixed point, thus  $p_0 = 0, q_0 = 0$ . Meanwhile, particle  $N$  is pulled (or pushed) by a force  $\tau \in \mathbb{R}$ , which is constant in time.



Each pair of consecutive particles  $(i-1, i)$  is connected by a (nonlinear) spring with potential  $V(q_i - q_{i-1})$ . We need to assume certain assumptions for the potential energy  $V : \mathbb{R} \rightarrow \mathbb{R}$ . The energy of the system is then given by

$$H_N(\mathbf{p}, \mathbf{q}) = \sum_{i=1}^N \left[ \frac{p_i^2}{2} + V(q_i - q_{i-1}) \right].$$

Therefore, the inter-particle distances  $\{r_i = q_i - q_{i-1}; 1 \leq i \leq N\}$  are the essentially relevant variables. Notice that here  $r_i$  can also assume negative values. Let  $e_i = p_i^2/2 + V(r_i)$  be the energy assigned to  $i$ -th particle, then  $H_N = \sum e_i$ . The corresponding Hamiltonian dynamics locally conserved the sums of  $p_i, r_i$  and  $e_i$ . By adding proper stochastic perturbations on the deterministic dynamics, we can make them the only conserved quantities.

Let  $w_i = (p_i, r_i, e_i)$  be the vector of conserved quantities. The hydrodynamic limit is given by the convergence, for any continuous  $G$  on  $[0, 1]$ ,

$$\frac{1}{N} \sum_{i=1}^N w_i(Nt) G\left(\frac{i}{N}\right) \xrightarrow{N \rightarrow \infty} \int_0^1 \mathbf{w}(t, x) G(x) dx,$$

where  $\mathbf{w} = (\mathbf{p}, \mathbf{r}, \mathbf{e})$  solves the *compressible Euler equations*

$$\partial_t \mathbf{w} = \partial_x F(\mathbf{w}), \quad F(\mathbf{w}) = (\boldsymbol{\tau}(\mathbf{r}, \mathbf{u}), \mathbf{p}, \boldsymbol{\tau}(\mathbf{r}, \mathbf{u})\mathbf{p}), \quad \mathbf{u} = \mathbf{e} - \mathbf{p}^2/2, \quad (1.1)$$

with boundary conditions given by

$$\mathbf{p}(0, t) = 0, \quad \boldsymbol{\tau}(\mathbf{r}(1, t), \mathbf{u}(1, t)) = \tau,$$

where  $\boldsymbol{\tau}(r, e)$  is the tension function defined in (2.4) later. In the smooth regime of (1.1), this is proven by relative entropy techniques in [2].

We consider here the system in equilibrium, starting with the Gibbs measure

$$\prod_{i=1}^N \exp \{ \lambda \cdot (r_i, e_i) - \mathcal{G}(\lambda) \} dp_i dr_i, \quad (1.2)$$

for given  $\lambda = (\beta\tau, -\beta) \in \mathbb{R} \times \mathbb{R}_-$ , where  $\mathcal{G}$  is the Gibbs potential given by

$$\mathcal{G}(\lambda) = \ln \left( \int_{\mathbb{R}} \exp \{ -\beta V(r) + \beta\tau r \} dr \right) + \frac{1}{2} \ln \left( \frac{2\pi}{\beta} \right). \quad (1.3)$$

Denote by  $E_{\lambda,N}$  the expectation with respect to the measure in (1.2). Correspondingly, there are equilibrium values  $\bar{w} = E_{\lambda,N}[w_i]$  for the conserved quantities. The empirical distribution of the fluctuations of the conserved quantities is defined by

$$\frac{1}{\sqrt{N}} \sum_{i=1}^N (w_i(Nt) - \bar{w}) \delta \left( x - \frac{i}{N} \right).$$

Formally, it is expected to converge to the solution  $\tilde{\mathbf{w}}$  of the linearized system

$$\partial_t \tilde{\mathbf{w}} = F'(\bar{w}) \partial_x \tilde{\mathbf{w}}, \quad (1.4)$$

where  $F'(\bar{w})$  is the Jacobian matrix of  $F$  computed at  $\bar{w}$ , with boundary conditions

$$\tilde{\mathbf{p}}(t, 0) = 0, \quad \tau_r \tilde{\mathbf{r}}(t, 1) + \tau_e \tilde{\mathbf{e}}(t, 1) = 0, \quad (1.5)$$

and a proper Gaussian stationary initial distribution. Notice that the limit fluctuation fields  $\tilde{\mathbf{w}}$  are only random distributions on  $[0, 1]$ , so the equations (1.4) with the boundary conditions (1.5) should be intended in the weak sense, as defined in Section 3.

While the non-equilibrium hydrodynamic limit can be proven by adding a simple exchange of  $p_i$  with  $p_{i+1}$  at random independent times (cf. [2]), in order to prove (1.4) we need to add, for each bond  $(i, i+1)$ , a stochastic perturbation that exchanges  $(p_i, p_{i+1}, r_i, r_{i+1})$  in such way that  $r_i + r_{i+1}$ ,  $p_i + p_{i+1}$ ,  $e_i + e_{i+1}$  are conserved. The corresponding microcanonical surface is a one-dimensional circle, where we add a Wiener process. This stochastic perturbation corresponds to adding a symmetric second order differential operator  $\mathcal{S}_N$  defined by (2.2) that is elliptic on the corresponding microcanonical surfaces. The main part of the article is the proof of a lower bound of order  $N^{-2}$  on the spectral gap of  $\mathcal{S}_N$  that is independent of the values of the conserved quantities. This is an important ingredient for proving the Boltzmann-Gibbs linearization for the dynamics.

The hyperbolic scale describes the time for the system to reach its mechanical equilibrium. Beyond that, it takes more time to reach the thermal equilibrium. Generally speaking, superdiffusion of energy is conjectured for one-dimensional Hamiltonian system conserving momentum, see [16], and proven rigorously for harmonic chains with conservative noise in [8]. In Theorem 3.2 we prove for our anharmonic system that the equilibrium fluctuations on the three conserved quantities continue to evolve deterministically according to the linearized Euler equations up to a time scale  $N^a t$  with  $a \in [1, 6/5)$ . According to [16], superdiffusive effects should appear for some  $a \geq 3/2$ .

We believe that such macroscopic behavior of the equilibrium fluctuations should be valid also for the deterministic (non-linear) dynamics, but even the case with a stochastic perturbation acting only on the velocities remains an open problem. In the diffusive scaling with stochastic perturbations acting only on velocities but conserving neither momentum nor energy, equilibrium fluctuations for the density has been proven in [13], while if noise conserve only energy the convergence of the fluctuations have been proven in [12].

Another important open problem concerns the evolution of fluctuations out of equilibrium. For system with one conserved quantity, like the asymmetric simple exclusion, in the context of the hyperbolic scaling this has been proven in [15].

## 2 The microscopic model

In this section we define the microscopic dynamics rigorously. Let  $V$  be a twice differentiable function on  $\mathbb{R}$  with quadratic growth:

$$\inf_{r \in \mathbb{R}} V''(r) > 0, \quad \sup_{r \in \mathbb{R}} V''(r) < \infty. \quad (2.1)$$

Observe that (2.1) assures that  $V(r)$  acquires its minimum at some unique point  $r_0 \in \mathbb{R}$ . By replacing  $V$  with  $V_* = V(\cdot + r_0) - V(r_0)$ , we can assume without loss of generality that  $V \geq 0$ ,  $V(0) = 0$  and  $V'(0) = 0$ .

For  $N \geq 1$ , let  $\Omega_N = \mathbb{R}^{2N}$  be the configuration space. Its elements are denoted by

$$\eta = (\mathbf{p}, \mathbf{r}); \quad \mathbf{p} = (p_1, \dots, p_N), \quad \mathbf{r} = (r_1, \dots, r_N).$$

Fix  $\tau \in \mathbb{R}$ ,  $p_0 = 0$ , and define first-order differential operators  $\mathcal{X}_i$  on  $\Omega_N$  as

$$\begin{aligned} \mathcal{X}_i &= (p_i - p_{i-1}) \frac{\partial}{\partial r_i} + (V'(r_{i+1}) - V'(r_i)) \frac{\partial}{\partial p_i}, \quad \text{for } 1 \leq i \leq N-1, \\ \mathcal{X}_N &= (p_N - p_{N-1}) \frac{\partial}{\partial r_N} + (\tau - V'(r_N)) \frac{\partial}{\partial p_N}. \end{aligned}$$

In addition, define  $\mathcal{Y}_{i,i+1}$  for  $1 \leq i \leq N-1$  as

$$\mathcal{Y}_{i,i+1} = (p_{i+1} - p_i) \left( \frac{\partial}{\partial r_{i+1}} - \frac{\partial}{\partial r_i} \right) - (V'(r_{i+1}) - V'(r_i)) \left( \frac{\partial}{\partial p_{i+1}} - \frac{\partial}{\partial p_i} \right).$$

For any  $\gamma > 0$ , the generator  $\mathcal{L}_N$  is given by

$$\mathcal{L}_N = \mathcal{A}_N + \gamma \mathcal{S}_N, \quad \mathcal{A}_N = \sum_{i=1}^N \mathcal{X}_i, \quad \mathcal{S}_N = \frac{1}{2} \sum_{i=1}^{N-1} \mathcal{Y}_{i,i+1}^2. \quad (2.2)$$

The Liouville operator  $\mathcal{A}_N$  generates the Hamiltonian system introduced in Section 1, while each  $\mathcal{Y}_{i,i+1}$  generates a continuous stochastic perturbation on  $(p_i, p_{i+1}, r_i, r_{i+1})$ , preserving the amounts of  $p_i + p_{i+1}$ ,  $r_i + r_{i+1}$  and  $e_i + e_{i+1}$ . This choice of noises assures that  $p_i$ ,  $r_i$  and  $e_i$  are the only locally conserved quantities.

Denote by  $\pi_{\lambda,N}$  the Gibbs measure in (1.2). The class of bounded, smooth functions on  $\Omega_N$  forms a core of  $\mathcal{A}_N$  and  $\mathcal{S}_N$  in  $L^2(\pi_{\lambda,N})$ , and for such  $f$  and  $g$ ,

$$E_{\lambda,N}[(\mathcal{A}_N f)g] = -E_{\lambda,N}[f(\mathcal{A}_N g)], \quad E_{\lambda,N}[(\mathcal{S}_N f)g] = E_{\lambda,N}[f(\mathcal{S}_N g)].$$

In particular,  $\pi_{\lambda,N}$  is stationary with respect to  $\mathcal{L}_N$ . Moreover,

$$E_{\lambda,N}[f(-\mathcal{L}_N f)] = \gamma E_{\lambda,N}[f(-\mathcal{S}_N f)] = \frac{\gamma}{2} \sum_{i=1}^N E_{\lambda,N}[(\mathcal{Y}_{i,i+1} f)^2].$$

Recall that  $\bar{w} = E_{\lambda,N}[w_i]$ . We have  $\bar{w} = (0, \bar{r}(\lambda), \bar{e}(\lambda))$ , where

$$(\bar{r}(\lambda), \bar{e}(\lambda)) = \left( \frac{1}{\beta} \frac{\partial}{\partial \tau}, \frac{\tau}{\beta} \frac{\partial}{\partial \tau} - \frac{\partial}{\partial \beta} \right) \mathcal{G}.$$

where  $\mathcal{G}(\lambda)$  is defined in (1.3). It is also worth noticing that the tension in equilibrium is  $E_{\lambda,N}[V'(r_i)] = \tau$ . Furthermore, the covariance matrix  $\Sigma = \Sigma(\lambda)$  of  $w_i$  under  $\pi_{\lambda,N}$  is given by

$$\Sigma = E_{\lambda,N}[(w_i - \bar{w}) \otimes (w_i - \bar{w})] = \left[ \begin{array}{c|cc} \beta^{-1} & 0 & 0 \\ \hline 0 & \mathcal{G}''(\lambda) & \\ 0 & & \end{array} \right], \quad (2.3)$$

where  $\mathcal{G}''(\lambda)$  is the Hessian matrix of  $\mathcal{G}$  with respect to  $\lambda$ .

The *thermodynamic entropy*  $\mathcal{S}$  is defined as the Legendre transform of  $\mathcal{G}$ :

$$\mathcal{S}(r, e) = - \sup_{\lambda \in \mathbb{R} \times \mathbb{R}_-} \{ \lambda \cdot (r, e) - \mathcal{G}(\lambda) \}, \quad \forall r \in \mathbb{R}, \quad e \geq 0.$$

Then  $\beta$  and  $\tau$  can be expressed by functions of  $\bar{r}$  and  $\bar{e}$  as

$$\beta(r, e) = \partial_e \mathcal{S}, \quad \tau(r, e) = -\frac{\partial_r \mathcal{S}}{\partial_e \mathcal{S}}. \quad (2.4)$$

For convenience, take  $\lambda = (\beta\tau, -\beta)$  to be the convex conjugate variables of  $(r, e)$ :

$$\lambda(r, e) = -\nabla \mathcal{S}(r, e), \quad (\bar{r}(\lambda), \bar{e}(\lambda)) = \nabla \mathcal{G}(\lambda). \quad (2.5)$$

Since  $\lambda(\bar{r}(\lambda), \bar{e}(\lambda)) \equiv \lambda$ , we have the relation

$$\mathcal{G}''(\lambda) \mathcal{S}''(\bar{r}(\lambda), \bar{e}(\lambda)) = \mathcal{G}''(\lambda(r, e)) \mathcal{S}''(r, e) = I_{2 \times 2}. \quad (2.6)$$

Furthermore, the definition of  $\tau$  shows that

$$\begin{aligned} \frac{\partial \tau}{\partial r} + \tau \frac{\partial \tau}{\partial e} &= \frac{1}{\beta} \left( -\frac{\partial^2 \mathcal{S}}{(\partial r)^2} + \frac{1}{\beta} \frac{\partial \beta}{\partial r} \frac{\partial \mathcal{S}}{\partial r} \right) - \frac{1}{\beta^2} \frac{\partial \mathcal{S}}{\partial r} \left( -\frac{\partial^2 \mathcal{S}}{\partial r \partial e} + \frac{1}{\beta} \frac{\partial \beta}{\partial e} \frac{\partial \mathcal{S}}{\partial r} \right) \\ &= -\frac{1}{\beta^3} \left( \beta^2 \frac{\partial^2 \mathcal{S}}{(\partial r)^2} - \beta \frac{\partial^2 \mathcal{S}}{\partial r \partial e} \frac{\partial \mathcal{S}}{\partial r} - \beta \frac{\partial^2 \mathcal{S}}{\partial r \partial e} \frac{\partial \mathcal{S}}{\partial r} + \frac{\partial^2 \mathcal{S}}{\partial e^2} \left[ \frac{\partial \mathcal{S}}{\partial r} \right]^2 \right) \\ &= \frac{1}{\beta^3} \left( \frac{\partial \mathcal{S}}{\partial e}, -\frac{\partial \mathcal{S}}{\partial r} \right) \cdot (-\mathcal{S})'' \left( \frac{\partial \mathcal{S}}{\partial e}, -\frac{\partial \mathcal{S}}{\partial r} \right). \end{aligned}$$

By (2.1),  $\beta > 0$  and  $\mathcal{S}$  is strictly concave. Hence, we conclude that

$$\frac{\partial \tau}{\partial r} + \tau \frac{\partial \tau}{\partial e} > 0. \quad (2.7)$$

For each  $N \geq 1$ , denote by  $\{\eta_t \in \Omega_N; t \geq 0\}$  the Markov process generated by  $N\mathcal{L}_N$ . Observe that  $\eta_t = (\mathbf{p}(t), \mathbf{r}(t))$  can be equivalently expressed by the solution to the following system of stochastic differential equations:

$$\begin{cases} dp_1(t) = N \nabla V'(r_1) dt + dJ_1^p, \\ dp_i(t) = N \nabla V'(r_i) dt - \nabla^* dJ_i^p, \quad \text{for } 2 \leq i \leq N-1, \\ dp_N(t) = N [\tau - V'(r_N)] dt - dJ_{N-1}^p, \\ dr_1(t) = N p_1 dt + dJ_1^r, \\ dr_i(t) = N \nabla p_{i-1} dt - \nabla^* dJ_i^r, \quad \text{for } 2 \leq i \leq N-1, \\ dr_N(t) = N \nabla p_{N-1} dt - dJ_{N-1}^r, \end{cases} \quad (2.8)$$

where for any sequence  $\{f_i\}$ ,  $\nabla f_i = f_{i+1} - f_i$ ,  $\nabla^* f_i = f_{i-1} - f_i$ ,

$$\begin{aligned} dJ_i^p &= \frac{\gamma N}{2} [V''(r_{i+1}) + V''(r_i)] \nabla p_i dt + \sqrt{\gamma N} (\nabla V'(r_i)) dB_t^i, \\ dJ_i^r &= \gamma N \nabla V'(r_i) dt - \sqrt{\gamma N} (\nabla p_i) dB_t^i, \end{aligned}$$

and  $\{B^i; i \geq 1\}$  is an infinite system of independent, standard Brownian motions. Let  $\mathbb{P}_{\lambda, N}$  be the probability measure on the path space  $C([0, \infty), \Omega_N)$  induced by (2.8) and initial condition  $\pi_{\lambda, N}$ . The corresponding expectation is denoted by  $\mathbb{E}_{\lambda, N}$ .

We are interested in the evolution of the fluctuations of the balanced quantities of  $\{\eta_t\}$  in macroscopic time. For a smooth function  $H : [0, 1] \rightarrow \mathbb{R}^3$ , define the *empirical distribution of conserved quantities fluctuation field* on  $H$  as

$$Y_N(t, H) = \frac{1}{\sqrt{N}} \sum_{i=1}^N H \left( \frac{i}{N} \right) \cdot (w_i(\eta_t) - \bar{w}), \quad \forall t \geq 0. \quad (2.9)$$

Notice that we consider in (2.9) the *hyperbolic scaling*, where the space and time variables are rescaled by the same order of  $N$ .

Let  $\mathcal{H}$  be the space of all functions  $f = (f_1, f_2, f_3)$  on  $[0, 1]$ , where each  $f_i$  is square integrable. Define the scalar product and norm on  $\mathcal{H}$  by

$$\langle f, g \rangle = \int_0^1 f(x) \cdot g(x) dx, \quad \|f\|^2 = \int_0^1 |f(x)|^2 dx.$$

Then,  $\mathcal{H}$  becomes a Hilbert space. Note that the definition in (2.9) satisfies that:

$$\mathbb{E}_{\lambda, N} [Y_N^2(t, H)] \leq |\text{Tr } \Sigma(\lambda)| \cdot \frac{1}{N} \sum_{i=1}^N \left| H \left( \frac{i}{N} \right) \right|^2.$$

Thus, one can easily extend the definition of  $Y_N(t, H)$  to all  $H \in \mathcal{H}$ . For all  $N \geq 1$ ,  $t \geq 0$  and  $H \in \mathcal{H}$ ,  $Y_N(t, H) \in L^2(\Omega_N; \pi_{\lambda, N})$ .

### 3 Euler system with boundary conditions

In this section we state the precise definition of the solution to (1.4), (1.5) with proper random distribution-valued initial condition. The equation (1.4) can be written explicitly as

$$\partial_t \tilde{\mathbf{p}} = \tau_r \partial_x \tilde{\mathbf{e}} + \tau_e \partial_x \tilde{\mathbf{e}}, \quad \partial_t \tilde{\mathbf{e}} = \partial_x \tilde{\mathbf{p}}, \quad \partial_t \tilde{\mathbf{e}} = \tau \partial_x \tilde{\mathbf{p}},$$

where  $(\tau_r, \tau_e)$  are constants given by

$$\tau_r(\lambda) = \frac{\partial}{\partial r} \boldsymbol{\tau}(\bar{r}(\lambda), \bar{e}(\lambda)), \quad \tau_e(\lambda) = \frac{\partial}{\partial e} \boldsymbol{\tau}(\bar{r}(\lambda), \bar{e}(\lambda)). \quad (3.1)$$

Observe that by formally applying the linear transformation

$$\tilde{\tau} = \tau_r \tilde{\mathbf{e}} + \tau_e \tilde{\mathbf{e}}, \quad \tilde{S} = \beta^{-1}(\tilde{\mathbf{e}} - \tau \tilde{\mathbf{e}}),$$

we have the new coordinates  $(\tilde{\mathbf{p}}, \tilde{\tau}, \tilde{S})$  satisfy that

$$\partial_t \tilde{\mathbf{p}} = \partial_x \tilde{\tau}, \quad \partial_t \tilde{\tau} = c^2 \partial_x \tilde{\mathbf{p}}, \quad \partial_t \tilde{S} = 0, \quad (3.2)$$

where the constant  $c > 0$  is the speed of sound given by

$$c^2 = \tau_r + \tau \tau_e > 0,$$

cf. (2.7) and [16, (3.10)]. This transformation also decouples the boundary conditions:

$$\tilde{\mathbf{p}}(t, 0) = 0, \quad \tilde{\tau}(t, 1) = 0. \quad (3.3)$$

It turns to be clear that  $(\tilde{\mathbf{p}}, \tilde{\tau})$  are two coupled sound modes with mixed boundaries, while  $\tilde{S}$  is independent of  $(\tilde{\mathbf{p}}, \tilde{\tau})$  and does not evolve in time. Suppose that the initial data is smooth and satisfies the boundary conditions, one can easily obtain the smooth solution  $\tilde{\mathbf{w}} = \tilde{\mathbf{w}}(t, x)$  to (1.4), (1.5) by applying the inverse transformation.

Since  $\tilde{\mathbf{w}}(0)$  is a random Gaussian distribution on  $[0, 1]$ , to present the idea above rigorously we have to consider the weak solution. Define a subspace  $\mathcal{C}(\lambda)$  of  $\mathcal{H}$  by

$$\mathcal{C}(\lambda) = \{g = (g_1, g_2, g_3) \mid g_i \in C^1([0, 1]), g_1(0) = 0, \tau_r g_2(1) + \tau_e g_3(1) = 0\}.$$

Define the first-order differential operator  $L$  on  $\mathcal{C}(\lambda)$  by

$$L = B \left( \frac{d}{dx} \right), \quad \text{where } B = F'(\bar{w}) = \begin{bmatrix} 0 & \tau_r & \tau_e \\ 1 & 0 & 0 \\ \tau & 0 & 0 \end{bmatrix}.$$

Observe that  $B$  has three real eigenvalues  $\{0, \pm c\}$ , thus generates a hyperbolic system. With some abuse of notations, denote the closure of  $L$  on  $\mathcal{H}$  still by  $L$ . For  $i = 1, 2$ , let  $\{\mu_{i,n}; n \geq 0\}$  be two Fourier bases of  $L^2([0, 1])$  given by

$$\mu_{1,n}(x) = \sqrt{2} \sin(\theta_n x), \quad \mu_{2,n}(x) = \sqrt{2} \cos(\theta_n x), \quad \theta_n = \frac{(2n+1)\pi}{2}. \quad (3.4)$$

Notice that  $\mu_{1,n}(0) = \mu_{2,n}(1) = 0$ . For  $k \geq 1$ , define the Sobolev spaces

$$H_k = \left\{ f = (f_1, f_2) \mid \sum_{i=1}^2 \sum_{n=0}^{\infty} \theta_n^{2k} \left( \int_0^1 f_i(x) \mu_{i,n}(x) dx \right)^2 < \infty \right\}.$$

Then  $\text{dom}(L) = \{(g_1, \tau_r g_2 + \tau_e g_3) \in H_1\}$ . To identify the adjoint  $L^*$  of  $L$ , notice that for any  $g \in \mathcal{C}(\lambda)$  and  $h \in \mathcal{H}$ ,

$$\langle Lg, h \rangle = \int_0^1 g_1'(h_2 + \tau h_3) + (\tau_r g_2' + \tau_e g_3') h_1 dx.$$

Therefore,  $\text{dom}(L^*) = \{(h_1, h_2 + \tau h_3) \in H_1\}$ . In particular,

$$\mathcal{C}_*(\tau) = \{h = (h_1, h_2, h_3) \mid h_i \in C^1([0, 1]), h_1(0) = 0, h_2(1) + \tau h_3(1) = 0\}$$

is a core of  $L^*$  and  $L^*h = -B^T h'$  for  $h \in \mathcal{C}_*(\tau)$ . Notice that  $\mathcal{C}_*(\tau)$  depends only on  $\tau$ , while  $\mathcal{C}(\lambda)$  depends on both  $\beta$  and  $\tau$ .

Now we can state the definition of (1.4) and (1.5) precisely. Let  $\{\tilde{\mathfrak{w}}(t) = \tilde{\mathfrak{w}}(t, \cdot); t \geq 0\}$  be a stochastic process with values on  $\mathcal{H}'$ , such that for every  $h \in \mathcal{C}_*(\tau)$ ,

$$\tilde{\mathfrak{w}}(t, h) - \tilde{\mathfrak{w}}(0, h) = \int_0^t \tilde{\mathfrak{w}}(s, L^*h) ds, \quad \forall t > 0, \quad (3.5)$$

and  $\tilde{\mathfrak{w}}(0)$  is a Gaussian variable such that for all  $h, g \in \mathcal{H}$ ,

$$E[\tilde{\mathfrak{w}}(0, h)] = 0, \quad E[\tilde{\mathfrak{w}}(0, h) \tilde{\mathfrak{w}}(0, g)] = \langle h, \Sigma g \rangle, \quad (3.6)$$

where  $\Sigma$  is the covariance matrix defined in (2.3).

To see the existence and uniqueness of  $\tilde{\mathfrak{w}}(t)$ , let  $\tilde{\mathfrak{u}} = \tilde{\mathfrak{u}}(t)$  be the solution to the weak form of (3.2): for  $f = (f_1, f_2, f_3)$ ,  $f_i \in C^1([0, 1])$ ,  $f_1(0) = f_2(1) = 0$ ,

$$\tilde{\mathfrak{u}}(t, f) - \tilde{\mathfrak{u}}(0, f) + \int_0^t \tilde{\mathfrak{u}}(s, A^T f') ds = 0, \quad A = \begin{bmatrix} 0 & 1 & 0 \\ c^2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad (3.7)$$

and  $\tilde{\mathfrak{u}}(0)$  is a centered Gaussian variable with covariance

$$E[(\tilde{\mathfrak{u}}(0, f))^2] = \langle f, Qf \rangle, \quad Q = \text{diag}(\beta^{-1}, \beta^{-1}c^2, \beta^{-2}\partial_\beta^2 \mathcal{G}).$$

The solution  $\tilde{\mathfrak{u}}(t)$  uniquely exists in the space  $\mathcal{H}_{-1}$  defined as

$$\mathcal{H}_{-k} = \left\{ \tilde{\mathfrak{u}} \mid \sum_{i=1}^2 \sum_{n=0}^{\infty} \{\theta_n^{-2k} \tilde{\mathfrak{u}}^2(\mu_{i,n}) + \kappa_n^{-2k} \tilde{\mathfrak{u}}^2(\nu_{i,n})\} < \infty \right\},$$

where  $\mu_{1,n} = (\mu_{1,n}, 0, 0)$ ,  $\mu_{2,n} = (0, \mu_{2,n}, 0)$ , and

$$\nu_{1,n}(x) = \sqrt{2}(0, 0, \sin(\kappa_n x)), \quad \nu_{2,n}(x) = \sqrt{2}(0, 0, \cos(\kappa_n x)), \quad \kappa_n = 2n\pi. \quad (3.8)$$



Moreover, the distribution of  $\tilde{\mathbf{u}}(t)$  is stationary. For  $h \in \mathcal{C}_*(\tau)$ , define

$$\tilde{\mathbf{w}}(t, h) = \tilde{\mathbf{u}}(t, R^{-1}h), \quad R = R(\lambda) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \tau_r & -\beta^{-1}\tau \\ 0 & \tau_e & \beta^{-1} \end{bmatrix}. \quad (3.9)$$

Observing that  $A^T R^{-1} = R^{-1}B^T$ , and  $f_1(0) = f_2(1) = 0$  for  $f \in R^{-1}[\mathcal{C}_*(\tau)]$ ,

$$\tilde{\mathbf{w}}(t, h) - \tilde{\mathbf{w}}(0, h) = - \int_0^t \tilde{\mathbf{u}}(s, A^T R^{-1}h') ds = - \int_0^t \tilde{\mathbf{w}}(s, B^T h') ds,$$

and (3.5) is fulfilled. On the other hand, from (2.5) and (3.1),

$$\begin{pmatrix} \tau_r \\ \tau_e \end{pmatrix} = \frac{1}{\beta} \begin{pmatrix} \partial_r^2 \mathcal{L} \\ \partial_r \partial_e \mathcal{L} \end{pmatrix} \Big|_{(\bar{r}, \bar{e})} + \frac{\tau}{\beta} \begin{pmatrix} \partial_r \partial_e \mathcal{L} \\ \partial_e^2 \mathcal{L} \end{pmatrix} \Big|_{(\bar{r}, \bar{e})}.$$

Combining this with (2.6), one obtains that

$$\mathcal{G}''(\lambda) \begin{pmatrix} \tau_r \\ \tau_e \end{pmatrix} = \begin{pmatrix} \beta^{-1} \\ \beta^{-1}\tau \end{pmatrix}.$$

By this and some direct calculations,

$$R^T \Sigma R = \text{diag}(\beta^{-1}, \beta^{-1}c^2, \beta^{-2}\partial_\beta^2 \mathcal{G}) = Q,$$

therefore (3.6) also holds. In consequence, for  $k \geq 1$ , the solution  $\tilde{\mathbf{w}}(t)$  of (3.5) exists uniquely in the space  $\mathcal{H}_{-k}(\lambda)$  defined as

$$\mathcal{H}_{-k}(\lambda) = \left\{ \tilde{\mathbf{w}} \mid \|\tilde{\mathbf{w}}\|_{-k}^2 = \sum_{i=1}^2 \sum_{n=0}^{\infty} \{ \theta_n^{-2k} \tilde{\mathbf{w}}^2(R\boldsymbol{\mu}_{i,n}) + \kappa^{-2k} \tilde{\mathbf{w}}^2(R\boldsymbol{\nu}_{i,n}) \} < \infty \right\},$$

where  $\boldsymbol{\mu}_{i,n}$  and  $\boldsymbol{\nu}_{i,n}$  are three-dimensional Fourier bases given in (3.4) and (3.8). Moreover, almost all sample paths of  $\{\tilde{\mathbf{w}}(t); t \geq 0\}$  are continuous, and the Gaussian distribution determined by (3.6) is stationary for  $\tilde{\mathbf{w}}(t)$ .

For  $T > 0$  and  $k > 5/2$ , denote by  $\mathbb{Q}_N$  the distribution of  $\{Y_N(t); t \in [0, T]\}$  on the path space  $C([0, T], \mathcal{H}_{-k}(\lambda))$  induced by  $\mathbb{P}_{\lambda, N}$ . Denote by  $\mathbb{Q}$  the distribution of  $\{\tilde{\mathbf{w}}(t); t \in [0, T]\}$  defined above. Our first result is stated as below.

**Theorem 3.1.** *Assume (2.1), then the sequence of probability measures  $\{\mathbb{Q}_N\}$  converges weakly, as  $N \rightarrow \infty$ , to the probability measure  $\mathbb{Q}$ .*

Indeed, by the tightness of  $\{\mathbb{Q}_N\}$  in Section 7, we can pick an arbitrary limit point of  $\mathbb{Q}_N$ . Denote it by  $\mathbb{Q}$  and let  $\{Y(t)\}$  be a process subject to  $\mathbb{Q}$ . From classical central limit theorem, the distribution of  $Y(0)$  satisfies (3.6). By virtue of the uniqueness of the solution, to prove Theorem 3.1 it suffices to verify (3.5), or equivalently, for any  $h \in C_*(\tau)$ ,

$$|Y_N(t, H(t, \cdot)) - Y_N(0, h)| \rightarrow 0$$

in probability, where  $H(t, x)$  solves the *backward Euler system*:

$$\partial_t H(t, x) + L^* H(t, x) = 0, \quad H(0, x) = h(x). \quad (3.10)$$

Notice that  $H(t, \cdot) \in \mathcal{C}_*(\tau)$  for all  $t \geq 0$ . With some additional assumptions, we can prove a stronger result saying that the macroscopic fluctuation evolves with the linearized system for time scales beyond hyperbolic.

**Theorem 3.2.** *Assume (2.1). There exists some universal  $\delta > 0$ , such that if*

$$\sup_{r \in \mathbb{R}} V''(r) < (1 + \delta) \inf_{r \in \mathbb{R}} V''(r), \quad (3.11)$$

then for any  $\alpha < 1/5$ ,  $h \in \mathcal{C}_*(\tau)$ ,  $T > 0$  and  $\epsilon > 0$ ,

$$\lim_{N \rightarrow \infty} \mathbb{P}_{\lambda, N} \left\{ \exists t \in [0, T], |Y_N(N^\alpha t, H(N^\alpha t)) - Y_N(0, h)| > \epsilon \right\} = 0, \quad (3.12)$$

where  $H(t) = H(t, x)$  solves the backward equation (3.10).

*Remark 3.3.* Theorem 3.2 shows that the fluctuation of thermodynamic entropy  $\tilde{S}$  keeps stationary for any time scales  $N^\alpha t$  with  $\alpha < 6/5$ . It is expected that  $\tilde{S}$  would evolve under some superdiffusive scaling  $\alpha < 2$  following a fractional heat equation.

*Remark 3.4.* Instead of the boundary conditions introduced in Section 1, one can also apply the periodic boundary condition  $(p_0, q_0) = (p_N, q_N)$ . Under this setting, the dynamics is in equilibrium under the Gibbs measures is given by

$$\pi_{\beta, \bar{p}, \tau}^N = \prod_{i=1}^N \exp \left\{ \tilde{\lambda} \cdot w_i - \tilde{\mathcal{G}}(\tilde{\lambda}) \right\} dp_i dr_i,$$

for given  $\tilde{\lambda} = (\beta \bar{p}, \beta \tau, -\beta) \in \mathbb{R}^2 \times \mathbb{R}_-$ , where  $\bar{p} \in \mathbb{R}$  denotes the momenta in equilibrium. For  $(p, r) \in \mathbb{R}^2$  and  $e \geq p^2/2 + V(r)$ , we can define the internal energy  $U = e - p^2/2$ , then the thermodynamic entropy and tension function are still given by  $\mathcal{S}(r, U)$  and  $\tau(r, U)$ .

Start the dynamics from some equilibrium state  $\pi_{\beta, \bar{p}, \tau}^N$ . For a bounded smooth function  $H : \mathbb{T} \rightarrow \mathbb{R}^3$ , the equilibrium fluctuation field is given by

$$Y_N(t, H) = \frac{1}{\sqrt{N}} \sum_{i \in \mathbb{T}_N} H \left( \frac{i}{N} \right) \cdot \begin{pmatrix} p_i(Nt) - \bar{p} \\ r_i(Nt) - \bar{r} \\ e_i(Nt) - \bar{e} \end{pmatrix}.$$

By similar argument used to prove Theorem 3.1, one can show that  $Y_N(t, H) \rightarrow \tilde{\mathfrak{w}}(t, H)$ . Here  $\tilde{\mathfrak{w}}(t, \cdot)$  solves the following linearized Euler system on torus:

$$\partial_t \tilde{\mathfrak{w}}(t, x) = \begin{bmatrix} -\bar{p}\tau_u & \tau_r & \tau_u \\ 1 & 0 & 0 \\ \tau - \bar{p}^2\tau_u & \bar{p}\tau_r & \bar{p}\tau_u \end{bmatrix} \partial_x \tilde{\mathfrak{w}}(t, x),$$

where the linear coefficients are given by

$$(\tau_r, \tau_u) = (\partial_r, \partial_u) \tau \left( \bar{r}, \bar{e} - \frac{\bar{p}^2}{2} \right).$$

Similar to (3.2), we have  $\tilde{\mathfrak{p}}$  and  $\tilde{\tau} = -\bar{p}\tau_u\tilde{\mathfrak{p}} + \tau_r\tilde{\mathfrak{r}} + \tau_u\tilde{\mathfrak{e}}$  forms a system of two coupled wave equations with common sound speed  $c = \tau_r + \tau\tau_u$ , while  $\tilde{S} = \beta^{-1}(\tilde{\mathfrak{e}} - \bar{p}\tilde{\mathfrak{p}} - \tau\tilde{\tau})$  does not evolve in time.

## 4 Equilibrium fluctuation

In this section, let  $H(t, x)$  be a bounded and smooth function on  $[0, \infty) \times [0, 1]$ . For any  $T > 0$ , we define two norms  $|H|_T$  and  $\|H\|_T$  of  $H$  as below:

$$|H|_T = \sup_{[0, T] \times [0, 1]} |H(t, x)|,$$

$$\|H\|_T^2 = \sup_{t \in [0, T]} \|H(t)\|^2 = \sup_{t \in [0, T]} \int_0^1 H^2(t, x) dx.$$

For  $Y_N(t, H(t, \cdot))$ , the following decomposition holds  $\mathbb{P}_{\lambda, N}$  almost surely:

$$\begin{aligned} Y_N(t, H(t)) - Y_N(0, H(0)) - \int_0^t Y_N(s, \partial_s H(s)) ds \\ = I_{N,1}(t, H) + \gamma I_{N,2}(t, H) + \sqrt{\gamma} M_N(t, H), \quad \forall t > 0, \end{aligned} \quad (4.1)$$

where  $I_{N,1}$  and  $I_{N,2}$  are integrals given by

$$I_{N,1}(t, H) = N \int_0^t \mathcal{A}_N[Y_N(s, H(s))] ds, \quad I_{N,2}(t, H) = N \int_0^t \mathcal{S}_N[Y_N(s, H(s))] ds,$$

and  $M_N$  is a martingale with quadratic variation given by

$$\langle M_N \rangle(t, H) = N \int_0^t \{ \mathcal{S}_N[Y_N^2(s, H(s))] - 2Y_N(s, H(s)) \mathcal{S}_N[Y_N(s, H(s))] \} ds.$$

As the first step to prove Theorem 3.2, the next lemma guarantees that the last two terms in (4.1) vanish uniformly in macroscopic time for equilibrium dynamics.

**Lemma 4.1.** *There exists a constant  $C = C(\lambda, V)$ , such that*

$$\mathbb{E}_{\lambda, N} \left[ \sup_{t \in [0, T]} \gamma |I_{N,2}(t, H)|^2 + \sup_{t \in [0, T]} |M_N(t, H)|^2 \right] \leq \frac{CT}{N} \|\partial_x H\|_T^2.$$

The proof of Lemma 4.1 is standard and we postpone it to the end of this section. To identify the boundary conditions needed for  $H$ , noting that  $p_0 = 0$ , and

$$\begin{aligned} & N \mathcal{A}_N[Y_N(t, H(t))] \\ &= \sqrt{N} \sum_{i=1}^{N-1} H \left( t, \frac{i}{N} \right) \cdot (J_{\mathcal{A}, i} - J_{\mathcal{A}, i-1}) + \sqrt{N} H(t, 1) \cdot \begin{pmatrix} \tau - V(r_N) \\ p_N - p_{N-1} \\ p_N \tau - p_{N-1} V'(r_N) \end{pmatrix} \\ &= \frac{1}{\sqrt{N}} \sum_{i=1}^{N-1} \nabla_{N,i} H(t) \cdot (-J_{\mathcal{A}, i}(\eta_t)) \\ &\quad - \sqrt{N} \left[ H_1 \left( t, \frac{1}{N} \right) (V'(r_1(t)) - \tau) - (H_2(t, 1) + \tau H_3(t, 1)) p_N(t) \right], \end{aligned}$$

where  $J_{\mathcal{A}, i}$  is the centered instantaneous currents of  $\mathcal{A}_N$ :

$$J_{\mathcal{A}, i} = (V'(r_{i+1}) - \tau, p_i, p_i V'(r_{i+1}))^T,$$

and  $\nabla_{N,i}$  is the discrete derivative operator:

$$\nabla_{N,i} H = N \left[ H \left( \frac{i+1}{N} \right) - H \left( \frac{i}{N} \right) \right].$$

Thus, we can drop the right boundary if  $H(t) \in \mathcal{C}_*(\tau)$  for all  $t$ :

$$I_{N,1}(t, H) = -\frac{1}{\sqrt{N}} \int_0^t \sum_{i=0}^{N-1} \nabla_{N,i} H(s) \cdot J_{\mathcal{A}, i}(\eta_s) ds. \quad (4.2)$$

The next lemma shows that  $I_{N,1}$  can be linearized as  $N \rightarrow \infty$ .

**Lemma 4.2.** *Assume (2.1), (3.11), and  $H(t) \in \mathcal{C}_*(\tau)$  for  $t \in [0, T]$ , then*

$$\mathbb{E}_{\lambda, N} \left[ \sup_{t \in [0, T]} \left| I_{N,1}(t, H) - \int_0^t Y_N(s, L^* H(s)) ds \right|^2 \right] \leq C \left( \frac{T}{N^{\frac{1}{5}}} + \frac{T^2}{N^{\frac{2}{5}}} \right) \quad (4.3)$$

holds with some constant  $C$ . Furthermore,

$$C \leq C(\lambda, \gamma, V) \|H\|_T^2, \quad \text{where} \quad \|H\|_T = |\partial_x H|_T + |\partial_x^2 H|_T + \|\partial_x H\|_T.$$

*Remark 4.3.* The bound (4.3) in Lemma 4.2 is proven under the assumption (3.11). Without assuming (3.11) we have only that, for every fixed  $T > 0$ ,

$$\mathbb{E}_{\lambda, N} \left[ \sup_{t \in [0, T]} \left| I_{N,1}(t, H) - \int_0^t Y_N(s, L^* H(s)) ds \right|^2 \right] \leq o_N(1) \|H\|_T^2. \quad (4.4)$$

This is clear from Remark 5.2 below. The bound (4.4) is enough for proving Theorem 3.1, while (4.3) is necessary in order to prove (3.12).

Lemma 4.2 follows from the Boltzmann-Gibbs principle, proven in Section 5. Here we first give the proof of Theorem 3.2.

*Proof of Theorem 3.2.* Let  $H(t, x)$  be the solution of (3.10). From (4.1) and Lemma 4.1,

$$\mathbb{P}_{\lambda, N} \left\{ \exists t \in [0, T], \left| Y_N(N^\alpha t, H(N^\alpha t)) - Y_N(0, H_0) - \int_0^{N^\alpha t} Y_N(s, \partial_s H(s)) ds - I_{N,1}(N^\alpha t, H) \right| > \epsilon \right\} \rightarrow 0$$

for any  $\epsilon > 0$ . Recall that  $H(t, \cdot) \in \mathcal{C}_*(\tau)$  and an elementary estimate:

$$|\partial_x^\ell H(t, x)| \leq \sup \left\{ |\partial_y^\ell H(s, y)|; 0 \leq s < c^{-1}, 0 \leq y \leq 1 \right\}$$

for any  $t \geq 0$  and  $x \in [0, 1]$ . Therefore, by Lemma 4.2, for any  $\alpha < 1/5$ ,

$$\mathbb{E}_{\lambda, N} \left[ \sup_{t \in [0, T]} \left| I_{N,1}(N^\alpha t, H) - \int_0^{N^\alpha t} Y_N(s, L^* H(s)) ds \right|^2 \right] \rightarrow 0.$$

Theorem 3.2 then follows from (3.10).  $\square$

For Theorem 3.1, since tightness is shown in Section 7, we only need to take  $\alpha = 0$  in the proof above, and apply Remark 4.3 instead of Lemma 4.2 in the last step.

We now proceed to the proof of Lemma 4.1. Denote by  $\langle \cdot, \cdot \rangle_{\lambda, N}$  the scalar product of two functions  $f, g \in L^2(\pi_{\lambda, N})$ . We make use of a well-known estimate on the space-time variance of a stationary Markov process. For  $f(s, \cdot) \in L^2(\pi_{\lambda, N})$ ,

$$\mathbb{E}_{\lambda, N} \left[ \sup_{t \in [0, T]} \left| \int_0^t f(s, \eta_s) ds \right|^2 \right] \leq 14 \int_0^T \|f(t)\|_{-1, N}^2 dt, \quad (4.5)$$

where  $\|f\|_{-1, N}$  is defined for all  $f$  on  $\Omega_N$  by

$$\|f\|_{-1, N}^2 = \sup_h \left\{ 2\langle f, h \rangle_{\lambda, N} - \gamma N \langle h, -\mathcal{S}_N h \rangle_{\lambda, N} \right\},$$

with the superior taken over all bounded smooth functions  $h$  on  $\Omega_N$ . A proof of (4.5) can be found in [10, Sec. 2.5].

*Proof of Lemma 4.1.* To begin with, note that

$$\begin{aligned} N\mathcal{S}_N[Y_N(t, H(t))] &= \frac{\sqrt{N}}{2} \sum_{i=1}^{N-1} \left[ H\left(t, \frac{i}{N}\right) \cdot \mathcal{Y}_{i,i+1}^2[w_i] + H\left(t, \frac{i+1}{N}\right) \cdot \mathcal{Y}_{i,i+1}^2[w_{i+1}] \right] \\ &= -\frac{1}{\sqrt{N}} \sum_{i=1}^{N-1} \nabla_{N,i} H(t) \cdot J_{\mathcal{S},i}(\eta_t), \end{aligned}$$

where  $J_{\mathcal{S},i}$  is the instantaneous current corresponding to  $\mathcal{S}_N$ :

$$J_{\mathcal{S},i} = \frac{1}{2} \mathcal{Y}_{i,i+1}^2[w_i] = -\frac{1}{2} \mathcal{Y}_{i,i+1}^2[w_{i+1}].$$

By applying (4.5) on  $I_{N,2}(t, H)$ , one obtains that

$$\begin{aligned} &\mathbb{E}_{\lambda,N} \left[ \sup_{t \in [0, T]} |I_{N,2}(t, H)|^2 \right] \\ &\leq \frac{14}{N} \int_0^T \sup_h \left\{ 2 \sum_{i=1}^{N-1} \langle \nabla_{N,i} H(t) \cdot J_{\mathcal{S},i}, h \rangle_{\lambda,N} - \gamma N \langle h, -\mathcal{S}_N h \rangle_{\lambda,N} \right\} dt. \end{aligned}$$

By Cauchy-Schwarz inequality, with  $m_i = \mathcal{Y}_{i,i+1}[w_i]$  for  $i = 1$  to  $N-1$ ,

$$\begin{aligned} \left| \sum \langle \nabla_{N,i} H(t) \cdot J_{\mathcal{S},i}, h \rangle_{\lambda,N} \right|^2 &= \frac{1}{4} \left| \sum \langle \nabla_{N,i} H(t) \cdot m_i, \mathcal{Y}_{i,i+1} h \rangle_{\lambda,N} \right|^2 \\ &\leq \frac{1}{4} \sum_{i=1}^{N-1} |\nabla_{N,i} H(t)|^2 E_{\lambda,N} [m_i^2] \sum_{i=1}^{N-1} E_{\lambda,N} [(\mathcal{Y}_{i,i+1} h)^2] \\ &\leq C_1 N \|\partial_x H(t)\|^2 \langle h, -\mathcal{S}_N h \rangle_{\lambda,N}. \end{aligned}$$

Substituting this and optimizing  $h$ , we obtain that

$$\mathbb{E}_{\lambda,N} \left[ \sup_{t \in [0, T]} |I_{N,2}(t, H)|^2 \right] \leq \frac{14C_1}{\gamma N} \int_0^T \|\partial_x H\|^2 dt \leq \frac{C_2 T}{\gamma N} \|\partial_x H\|_T^2.$$

On the other hand, recall that  $m_i = \mathcal{Y}_{i,i+1}[w_i]$  and

$$\mathcal{S}_N [Y_N^2(s, H(s))] - 2Y_N(s, H(s)) \mathcal{S}_N [Y_N(s, H(s))] = \frac{1}{N^3} \sum_{i=1}^{N-1} [\nabla_{N,i} H(s) \cdot m_i(\eta_s)]^2.$$

Therefore, by Doob's maximal inequality,

$$\begin{aligned} \mathbb{E}_{\lambda,N} \left[ \sup_{t \in [0, T]} |M_N(t, H)|^2 \right] &\leq 4\mathbb{E}_{\lambda,N} [\langle M_N \rangle(T, H)] \\ &\leq \frac{4}{N^2} \int_0^T \sum_{i=1}^{N-1} E_{\lambda,N} [(\nabla_{N,i} H(t) \cdot m_i)^2] dt \\ &\leq \frac{C_3}{N} \int_0^T \|\partial_x H\|^2 dt \leq \frac{C_4 T}{N} \|\partial_x H\|_T^2. \end{aligned}$$

Since the constants depend only on  $\lambda$  and  $V$ , Lemma 4.1 follows.  $\square$

## 5 Boltzmann-Gibbs principle

This section devotes to the proof of Lemma 4.2. In this section, we denote by  $\{\iota_i; 0 \leq i \leq N\}$  the shift operator semigroup on  $\Omega_N$ , which is given by

$$(\iota_i \eta)_j = \begin{cases} (p_{i+j}, r_{i+j}), & 1 \leq j \leq N-i, \\ (0, 0), & N-i < j \leq N, \end{cases}$$

for all  $\eta \in \Omega_N$  and  $0 \leq i \leq N$ . For function  $F$  on  $\Omega_N$ , define  $\iota_i F = F \circ \iota_i$ . If  $F$  is supported by  $\{\eta_j, 1 \leq j \leq m\}$  for some  $m \leq N$ , then

$$E_{\lambda, N}[\iota_i F] = E_{\lambda, N}[F], \quad \forall 0 \leq i \leq N-m.$$

First notice that  $\nabla_{N, i} H$  in (4.2) can be replaced by  $\partial_x H$ . The difference is

$$\begin{aligned} & \mathbb{E}_{\alpha, \lambda, N} \left[ \sup_{t \in [0, T]} \left| I_{N, 1}(t, H) - \frac{1}{\sqrt{N}} \int_0^t \sum_{i=1}^{N-1} \partial_x H \left( s, \frac{i}{N} \right) \cdot J_{\mathcal{A}, i}(\eta_s) ds \right|^2 \right] \\ & \leq \frac{T}{N} \int_0^T E_{\lambda, N} \left[ \left( \sum \left[ \nabla_{N, i} H(t) - \partial_x H \left( t, \frac{i}{N} \right) \right] \cdot J_{\mathcal{A}, i} \right)^2 \right] dt. \end{aligned}$$

Since  $E_{\lambda, N}[J_{\mathcal{A}, i} \otimes J_{\mathcal{A}, j}] = \mathbf{0}$  for  $i \neq j$ , it is bounded from above by

$$\frac{T}{N} E_{\lambda, N} [ |J_{\mathcal{A}, i}|^2 ] \int_0^T \sum_{i=1}^{N-1} \left| \nabla_{N, i} H(t) - H' \left( t, \frac{i}{N} \right) \right|^2 dt \leq \frac{CT^2 |\partial_x^2 H|_T^2}{N^2}.$$

Clearly the order is better than what is needed for Lemma 4.2.

Now we want to replace the local random field  $J_{\mathcal{A}, i}$  with its linear approximation. The corresponding error is

$$\Phi = J_{\mathcal{A}, 0} - B(\lambda)(w_0 - \bar{w}(\lambda)) = \begin{pmatrix} V'(r_1) - \tau_r r_0 - \tau_e e_0 \\ 0 \\ p_0 V'(r_1) - p_0 \tau \end{pmatrix}.$$

Lemma 4.2 follows from the following Boltzmann-Gibbs principle.

**Proposition 5.1.** *Assume (2.1) and (3.11), then*

$$\mathbb{E}_{\lambda, N} \left[ \sup_{t \in [0, T]} \left| \frac{1}{\sqrt{N}} \int_0^t \sum_{i=1}^{N-1} \partial_x H \left( s, \frac{i}{N} \right) \cdot \iota_i \Phi(\eta_s) ds \right|^2 \right] \leq C \left( \frac{T}{N^{\frac{1}{5}}} + \frac{T^2}{N^{\frac{2}{5}}} \right)$$

for all bounded smooth  $H = H(t, x)$  on  $[0, T] \times [0, 1]$ .

Boltzmann-Gibbs principle, firstly established for zero range jump process (see [3]), aims at determining the space-time fluctuation of a local function by its linear approximation on the conserved fields. To show this proposition, we need a spectral gap bound of  $\mathfrak{S}_N$ , which is the main difficult point, as well as the equivalence of canonical and microcanonical ensembles. They are established in Section 6 and Section 8, respectively.

*Proof.* The first step is to take some  $1 \leq K \ll N$ , and define

$$\Phi_K = \frac{1}{K} \sum_{i=1}^K \iota_i \Phi.$$

Observe that  $\Phi_K$  is supported by  $\{\eta_j; 1 \leq j \leq K+1\}$ , and

$$\sum_{i=1}^{N-1} a_i(t) \cdot \iota_i \Phi = F_1(t) + F_2(t) + \sum_{i=1}^{N-K} a_i(t) \cdot \iota_{i-1} \Phi_K,$$

where we write  $a_i(t) = \partial_x H(t, i/N)$  for short, and  $F_1, F_2$  are given by

$$\begin{aligned} F_1(t) &= \frac{1}{K} \left( \sum_{i=1}^{K-1} + \sum_{i=N-K+1}^{N-1} \right) (K-i) a_i(t) \cdot \iota_i \Phi, \\ F_2(t) &= \frac{1}{K} \left( \sum_{i=1}^K \sum_{j=1}^k + \sum_{i=K+1}^{N-K-1} \sum_{j=i+1-K}^i + \sum_{i=N-K}^{N-1} \sum_{j=i+1-K}^{N-K} \right) (a_i(t) - a_j(t)) \cdot \iota_i \Phi. \end{aligned}$$

Since  $E_{\lambda, N}[\iota_i \Phi \otimes \iota_j \Phi] = \mathbf{0}$  for  $i \neq j$ , we have for each  $t \in [0, T]$ ,

$$E_{\lambda, N}[F_1^2(t) + F_2^2(t)] \leq \left( C_1 + \frac{C_2}{N} \right) K (|\partial_x H|_T^2 + |\partial_x^2 H|_T^2),$$

with constants  $C_1$  and  $C_2$  depending on  $\lambda$  and  $V$ . Hence,

$$\mathbb{E}_{\lambda, N} \left[ \sup_{t \in [0, T]} \left| \frac{1}{\sqrt{N}} \int_0^t F_1(s, \eta_s) + F_2(s, \eta_s) ds \right|^2 \right] \leq \frac{C_3 T^2 K}{N} (|\partial_x H|_T^2 + |\partial_x^2 H|_T^2). \quad (5.1)$$

For the second step, define  $\varphi_K = \Phi_K - \langle \Phi_K \rangle$ , where

$$\langle \Phi_K \rangle = E_{\lambda, N} \left[ \Phi \mid \frac{w_1 + w_1 + \dots + w_{K+1}}{K+1} \right].$$

Due to the equivalence of ensembles in Proposition 8.3, the square variance of  $\langle \Phi_K \rangle$  with respect to  $\pi_{\lambda, N}$  is of order  $K^{-2}$ . Therefore,

$$E_{\lambda, N}[|\varphi_K|^2] = E_{\lambda, N}[|\Phi_K|^2] - E_{\lambda, N}[|\langle \Phi_K \rangle|^2] \leq C_4 K^{-1}.$$

By applying the estimate (4.5), we obtain that

$$\begin{aligned} & \mathbb{E}_{\lambda, N} \left[ \sup_{t \in [0, T]} \left| \frac{1}{\sqrt{N}} \int_0^t \sum_{i=1}^{N-K} a_i(s) \cdot \iota_{i-1} \varphi_K(\eta_s) ds \right|^2 \right] \\ & \leq \frac{14}{N} \int_0^T \sup_h \left\{ 2 \sum_{i=1}^{N-K} \langle a_i(t) \cdot \iota_{i-1} \varphi_K, h \rangle_{\lambda, N} - \gamma N \langle h, -\mathcal{S}_N h \rangle_{\lambda, N} \right\} dt, \end{aligned} \quad (5.2)$$

where the superior is taken over all bounded smooth functions on  $\Omega_N$ . As  $\varphi_K$  is supported by  $\{\eta_i; 1 \leq i \leq K+1\}$ , by the spectral gap in Proposition 6.1,

$$-\mathcal{S}_K G_{\alpha, K} = a \cdot \varphi_K, \quad a \in \mathbb{R}^3$$

can be solved by some function  $G_{a, K}$  satisfying that

$$\langle G_{a, K}, -\mathcal{S}_{K+1} G_{a, K} \rangle_{\lambda, N} \leq C(K+1)^2 E_{\lambda, N}[(a \cdot \varphi_K)^2] \leq C_5 K |a|^2.$$

With these notations, for  $i = 1, \dots, N-K$  and  $a \in \mathbb{R}^3$ ,

$$\langle a \cdot \iota_{i-1} \varphi_K, h \rangle_{\lambda, N} = \frac{1}{2} \sum_{j=1}^K \langle \mathcal{Y}_{i+j-1, i+j} [\iota_{i-1} G_{a, K}], \mathcal{Y}_{i+j-1, i+j} h \rangle_{\lambda, N}.$$

Hence, by Cauchy-Schwarz inequality,

$$\begin{aligned}
& \left| \sum_{i=1}^{N-K} \langle a_i(t) \cdot \iota_{i-1} \varphi_K, h \rangle_{\lambda, N} \right|^2 \\
& \leq \frac{1}{4} \left( \sum_{i=1}^{N-K} \sum_{j=1}^K E_{\lambda, N} [(\mathcal{Y}_{i+j-1, i+j} h)^2] \right) \left( \sum_{i=1}^{N-K} \sum_{j=1}^K E_{\lambda, N} [(\mathcal{Y}_{j, j+1} G_{a_i(t), K})^2] \right) \\
& \leq K \langle h, -\mathcal{S}_N h \rangle_{\lambda, N} \sum_{i=1}^{N-K} \langle G_{a_i(t), K}, -\mathcal{S}_{K+1} G_{a_i(t), K} \rangle_{\lambda, N} \\
& \leq C_5 K^2 \langle h, -\mathcal{S}_N h \rangle_{\lambda, N} \sum_{i=1}^{N-K} |a_i(t)|^2 \leq C_6 K^2 N \|\partial_x H(t)\|^2 \langle h, -\mathcal{S}_N h \rangle_{\lambda, N}.
\end{aligned}$$

Substituting this into (5.2) and optimizing in  $h$ ,

$$\mathbb{E}_{\lambda, N} \left[ \sup_{t \in [0, T]} \left| \frac{1}{\sqrt{N}} \int_0^t \sum_{i=1}^{N-K} a_i(s) \cdot \iota_{i-1} \varphi_K(\eta_s) ds \right|^2 \right] \leq \frac{C_7 T K^2}{\gamma N} \|\partial_x H\|_T^2. \quad (5.3)$$

Finally, since  $E_{\lambda, N}[\iota_i \langle \Phi_K \rangle \otimes \iota_j \langle \Phi_K \rangle] = 0$  for  $|i - j| > K$ , by utilizing the equivalence of ensembles in Proposition 8.3 again, we obtain that

$$\mathbb{E}_{\lambda, N} \left[ \sup_{t \in [0, T]} \left| \frac{1}{\sqrt{N}} \int_0^t \sum_{i=1}^{N-K} a_i(s) \cdot \iota_{i-1} \langle \Phi_K \rangle(\eta_s) ds \right|^2 \right] \leq \frac{C_8 T^2}{K} \|\partial_x H\|_T^2. \quad (5.4)$$

In conclusion, by summing up (5.1), (5.3), (5.4), and taking  $K = N^{2/5}$ , we get the estimate in Proposition 5.1, with the constant satisfying that

$$C \leq C(\lambda, \gamma, V) (\|\partial_x H\|_T^2 + |\partial_x H|_T^2 + |\partial_x^2 H|_T^2).$$

This completes the proof of the proposition.  $\square$

*Remark 5.2.* If only (2.1) is assumed, we can apply Remark 6.6 instead of Proposition 6.1 in the proof of (5.3). By doing this, we can prove Proposition 5.1 for any fixed  $T > 0$ , with a weaker upper bound  $o_N(1) \|H\|_T^2$ .

## 6 Spectral gap

In this section, we state and prove the spectral gap estimate for the dynamics without boundary condition. The main result, Proposition 6.1, plays a central role in the proof of Proposition 5.1.

We use the following notations in this section. Recall (2.1) and denote

$$\delta_- = \inf_{r \in \mathbb{R}} V''(r), \quad \delta_+ = \sup_{r \in \mathbb{R}} V''(r).$$

For  $K \geq 2$  and  $w \in \mathbb{R}^3$ , the *microcanonical manifold*  $\Omega_{w, K}$  is defined as

$$\Omega_{w, K} = \left\{ (p_k, r_k), 1 \leq k \leq K \mid \frac{1}{K} \sum_{k=1}^K w_k = w \right\}.$$

In view of (2.1),  $\Omega_{w, K}$  is a compact and connected manifold if and only if  $w = (p, r, e)$ ,  $e \geq p^2/2 + V(r)$ . The Gibbs measure  $\pi_{\lambda, K}$  induces the uniform measure on  $\Omega_{w, K}$ , which is called the *microcanonical measure*, and is denoted by

$$\pi_{w, K} = \pi_{\lambda, K} [ \cdot \mid \Omega_{w, K} ].$$



Notice that the definition of  $\pi_{w,K}$  does not depend on the choice of  $\lambda$ . We denote by  $E_{w,K}$  the expectation with respect to  $\pi_{w,K}$ . For each pair  $(i, j)$  such that  $1 \leq i < j \leq K$ , let  $\mathcal{F}_{i,j}$  be the  $\sigma$ -algebra over  $\Omega_{w,K}$  given by

$$\mathcal{F}_{i,j} = \sigma(\{(p_k, r_k); 1 \leq k \leq K, k \neq i, j\}).$$

**Proposition 6.1.** *Suppose that the potential  $V$  satisfies (2.1). There exists a universal constant  $\delta > 0$ , such that if  $V$  fulfills furthermore (3.11), then*

$$E_{w,K} [(f - E_{w,K}[f])^2] \leq C_K \sum_{k=1}^{K-1} E_{w,K} [(\mathcal{X}_{k,k+1} f)^2] \quad (6.1)$$

for all  $(w, K)$  and bounded smooth function  $f$ , and  $C_K \leq CK^2$ .

The proof of Proposition 6.1 is divided into Lemma 6.2, 6.3 and 6.4 below.

**Lemma 6.2.** *Assume (2.1), then there exists constant  $C$ , such that*

$$E_{w,2} [(f - E_{w,2}[f])^2] \leq CE_{w,2} [(\mathcal{X}_{1,2} f)^2]$$

for all  $w$  and bounded smooth function  $f$  on  $(p_1, r_1, p_2, r_2)$ .

**Lemma 6.3.** *Assume (2.1), then there exists constant  $C$ , such that*

$$\sum_{1 \leq i < j \leq K} E_{w,K} [(f - E_{w,K}[f|\mathcal{F}_{i,j}])^2] \leq CK^3 \sum_{k=1}^{K-1} E_{w,K} [(f - E_{w,K}[f|\mathcal{F}_{k,k+1}])^2]$$

for all  $K \geq 3$ ,  $w$  and bounded smooth function  $f$ .

**Lemma 6.4.** *Assume (2.1) and (3.11), then*

$$E_{w,K} [(f - E_{w,K}[f])^2] \leq C'_K \sum_{1 \leq i < j \leq K} E_{w,K} [(f - E_{w,K}[f|\mathcal{F}_{i,j}])^2] \quad (6.2)$$

for all  $K \geq 3$ ,  $w$  and bounded smooth function  $f$ , and  $C'_K \leq C'K^{-1}$ .

Indeed, for each  $k = 1, \dots, K-1$ , Lemma 6.2 yields that

$$E_{w,K} [(f - E_{w,K}[f|\mathcal{F}_{k,k+1}])^2|\mathcal{F}_{i,i+1}] \leq CE_{w,K} [(\mathcal{X}_{i,i+1} f)^2|\mathcal{F}_{i,i+1}].$$

Then, Proposition 6.1 turns to be the direct consequence of this, Lemma 6.3 and Lemma 6.4. We now prove these lemmas in turn.

*Proof of Lemma 6.2.* For  $(p_1, r_1, p_2, r_2) \in \mathbb{R}^4$ , define

$$p = p(p_1, p_2) = \frac{p_1 + p_2}{2}, \quad r = r(r_1, r_2) = \frac{r_1 + r_2}{2},$$

and the *internal energy*  $E = E(p_1, r_1, p_2, r_2) \geq 0$  given by

$$E = \frac{e_1 + e_2}{2} - \frac{p^2}{2} - V(r) = \frac{(p_1 - p_2)^2}{8} + \frac{V(r_1) + V(r_2)}{2} - V\left(\frac{r_1 + r_2}{2}\right).$$

Furthermore, let  $\theta \in [0, 2\pi)$  satisfy that  $\sqrt{E} \cos \theta = \sqrt{2}(p_1 - p_2)/4$  and

$$\sqrt{E} \sin \theta = \operatorname{sgn}(r_1 - r_2) \sqrt{\frac{V(r_1) + V(r_2)}{2} - V\left(\frac{r_1 + r_2}{2}\right)}.$$

The Jacobian determinant of the bijection  $(p_1, r_1, p_2, r_2) \rightarrow (p, r, E, \theta)$  is

$$\mathfrak{J}(p, r, E, \theta) = \sqrt{2} \cdot \frac{\sqrt{V(r_1) + V(r_2) - 2V(r)}}{|V'(r_1) - V'(r_2)|}.$$

Recall that  $0 < \delta_- \leq V''(r) \leq \delta_+ < \infty$ , we have

$$0 < \frac{\sqrt{\delta_-}}{\sqrt{2}\delta_+} \leq \mathfrak{J}(p, r, E, \theta) \leq \frac{\sqrt{\delta_+}}{\sqrt{2}\delta_-}. \quad (6.3)$$

For a bounded smooth function  $f = f(p_1, r_1, p_2, r_2)$ , define  $f_*(p, r, E, \theta) = f(p_1, r_1, p_2, r_2)$ , and let  $\langle f_* \rangle = \int_0^{2\pi} f_*(p, r, E, \theta) d\theta$ . By simple calculations,

$$E_{w,2} [(f - \langle f_* \rangle)^2] = \frac{\int_0^{2\pi} [f_*(p, r, E, \theta) - \langle f_* \rangle]^2 \mathfrak{J}(p, r, E, \theta) d\theta}{\int_0^{2\pi} \mathfrak{J}(p, r, E, \theta) d\theta}.$$

On the other hand, since  $\mathcal{X}_{1,2}f = \mathfrak{J}^{-1} \partial_\theta f_*$ , we have

$$E_{w,2} [(\mathcal{X}_{1,2}f)^2] = \frac{\int_0^{2\pi} [\partial_\theta f_*(p, r, E, \theta)]^2 \mathfrak{J}^{-1}(p, r, E, \theta) d\theta}{\int_0^{2\pi} \mathfrak{J}(p, r, E, \theta) d\theta}.$$

Applying Poincaré inequality and (6.3), we obtain that

$$E_{w,2} [(f - E_{w,2}[f])^2] \leq E_{w,2} [(f - \langle f_* \rangle)^2] \leq \frac{C\delta_+}{2\delta_-^2} E_{w,2} [(X_{1,2}f)^2]$$

holds with some universal constant  $C < \infty$ .  $\square$

*Proof of Lemma 6.3.* We prove this lemma along the standard idea. Here are some notations used only in this proof. For  $1 \leq k \leq K$ , write  $x_k = (p_k, r_k)$  and  $\mathbf{x} = (x_1, \dots, x_K)$ . Recall the bijection defined in the proof of the Lemma 6.2. For simplicity we write

$$(p_{i,j}, r_{i,j}, E(i,j), \theta_{i,j}) = (p, r, E, \theta)(x_i, x_j), \quad \forall i < j.$$

For  $\theta \in [0, 2\pi]$ , denote the Jacobian determinant by

$$\mathfrak{J}_{\mathbf{x},i,j}(\theta) = \mathfrak{J}(p_{i,j}, r_{i,j}, E(i,j), \theta).$$

For  $i < j$ ,  $\theta \in [0, 2\pi]$  and  $\mathbf{x} = (x_1, \dots, x_K)$ , define a vector  $\rho_{i,j}^\theta \mathbf{x}$  by

$$(\rho_{i,j}^\theta \mathbf{x})_k = \begin{cases} g_1(p_{i,j}, r_{i,j}, E(i,j), \theta), & k = i; \\ g_2(p_{i,j}, r_{i,j}, E(i,j), \theta), & k = j; \\ x_k, & k \neq i, j, \end{cases}$$

where  $(g_1, g_2)$  denotes the inverse map of  $(x_1, x_2) \rightarrow (p, r, E, \theta)$ . Observe that  $\rho_{i,j}^\theta \mathbf{x} = \mathbf{x}$  when  $\theta = \theta_{i,j}$ , and for every smooth function  $f$ ,

$$E_{w,K}[f|\mathcal{F}_{i,j}] = \frac{1}{J_{x_i+x_j}} \int_0^{2\pi} f(\rho_{i,j}^\theta \mathbf{x}) \mathfrak{J}_{\mathbf{x},i,j}(\theta) d\theta,$$

where  $J_{x_i+x_j} = \int_0^{2\pi} \mathfrak{J}_{\mathbf{x},i,j}(\theta) d\theta$ . On the other hand, let  $\tau_{i,j} \mathbf{x}$  be the vector given by

$$(\tau_{i,j} \mathbf{x})_i = x_j, \quad (\tau_{i,j} \mathbf{x})_j = x_i, \quad (\tau_{i,j} \mathbf{x})_k = x_k, \quad \forall k \neq i, j.$$

Moreover for  $1 \leq i < j \leq K$ , we inductively define that

$$\sigma_{i,i} = \tilde{\sigma}_{i,i} = id, \quad \sigma_{i,j} = \tau_{j-1,j} \circ \sigma_{i,j-1}, \quad \tilde{\sigma}_{i,j} = \tilde{\sigma}_{i,j-1} \circ \tau_{j-1,j}.$$

Observe that for any  $i < j$  and  $\theta \in [0, 2\pi)$ ,  $\rho_{i,j}^\theta \equiv \tilde{\sigma}_{i,j-1} \circ \rho_{j-1,j}^\theta \circ \sigma_{i,j-1}$ .

For a smooth function  $f$ , by Cauchy-Schwarz inequality,

$$(f - E_{w,K}[f|\mathcal{F}_{i,j}])^2 \leq \frac{1}{J_{x_i+x_j}} \int_0^{2\pi} [f(\rho_{i,j}^\theta \mathbf{x}) - f(\mathbf{x})]^2 \mathfrak{J}_{\mathbf{x},i,j}(\theta) d\theta.$$

The right-hand side is bounded from above by  $3(f_1 + f_2 + f_3)$ , where

$$\begin{aligned} f_1 &= \frac{1}{J_{x_i+x_j}} \int_0^{2\pi} [f(\sigma_{i,j-1} \mathbf{x}) - f(\mathbf{x})]^2 \mathfrak{J}_{\mathbf{x},i,j}(\theta) d\theta, \\ f_2 &= \frac{1}{J_{x_i+x_j}} \int_0^{2\pi} [f(\rho_{j-1,j}^\theta \circ \sigma_{i,j-1} \mathbf{x}) - f(\sigma_{i,j-1} \mathbf{x})]^2 \mathfrak{J}_{\mathbf{x},i,j}(\theta) d\theta, \\ f_3 &= \frac{1}{J_{x_i+x_j}} \int_0^{2\pi} [f(\tilde{\sigma}_{i,j-1} \circ \rho_{j-1,j}^\theta \circ \sigma_{i,j-1} \mathbf{x}) - f(\rho_{j-1,j}^\theta \circ \sigma_{i,j-1} \mathbf{x})]^2 \mathfrak{J}_{\mathbf{x},i,j}(\theta) d\theta. \end{aligned}$$

For  $f_1$ , noticing that  $f_1 = (f(\sigma_{i,j-1} \mathbf{x}) - f(\mathbf{x}))^2$ , hence

$$\begin{aligned} E_{w,K}[f_1] &\leq K \sum_{k=i}^{j-2} E_{w,K} [(f \circ \sigma_{i,k+1} - f \circ \sigma_{i,k})^2] \\ &= K \sum_{k=i}^{j-2} E_{w,K} [(f \circ \tau_{k,k+1} - f)^2]. \end{aligned}$$

Since  $E_{w,K}[f \circ \tau_{k,k+1} | \mathcal{F}_{k,k+1}] = E_{w,K}[f | \mathcal{F}_{k,k+1}]$ , by the estimate above and Cauchy-Schwarz inequality, we obtain that

$$E_{w,K}[f_1] \leq 4K \sum_{k=i}^{j-2} E_{w,K} [(f - E_{w,K}[f | \mathcal{F}_{k,k+1}])^2].$$

For  $f_2$ , by applying the change of variable  $\mathbf{y} = \sigma_{i,j-1} \mathbf{x}$ , we obtain that

$$E_{w,K}[f_2] = E_{w,K} \left[ \frac{1}{J_{y_{j-1}+y_j}} \int_0^{2\pi} [f(\rho_{j-1,j}^\theta \mathbf{y}) - f(\mathbf{y})]^2 \mathfrak{J}_{\mathbf{y},i,j}(\theta) d\theta \right].$$

Therefore, we can calculate this term as

$$\begin{aligned} E_{w,K}[f_2] &= 2E_{w,K}[f^2] - 2E_{w,K}[f E_{w,K}[f | \mathcal{F}_{j-1,j}]] \\ &= 2E_{w,K} [(f - E_{w,K}[f | \mathcal{F}_{j-1,j}])^2]. \end{aligned}$$

For  $f_3$ , the same change of variable yields that

$$E_{w,K}[f_3] = E_{w,K} [E_{w,K}[(f \circ \tilde{\sigma}_{i,j-1} - f)^2 | \mathcal{F}_{j-1,j}]] = E_{w,K} [(f \circ \tilde{\sigma}_{i,j-1} - f)^2].$$

Since  $\tilde{\sigma}_{k,j-1} = \tau_{k,k+1} \circ \tilde{\sigma}_{k+1,j-1}$ , by repeating the calculation in  $f_1$ ,

$$E_{w,K}[f_3] \leq 4K \sum_{k=i}^{j-2} E_{w,K} [(f - E_{w,K}[f | \mathcal{F}_{k,k+1}])^2].$$

Hence, with some universal constant  $C < \infty$  we have

$$E_{w,K} [(f - E_{w,K}[f | \mathcal{F}_{i,j}])^2] \leq CK \sum_{k=i}^{j-1} E_{w,K} [(f \circ \tau_{k,k+1} - f)^2].$$

Lemma 6.3 follows by summing up this estimate with  $i$  and  $j$ .  $\square$

To show Lemma 6.4, we need the following pre-estimate.

**Lemma 6.5.** *Assume (2.1), then (6.2) holds with constants  $C'_K$  satisfying*

$$C'_K \leq \frac{C'}{K} \left( \frac{\delta_+}{\delta_-} \right)^{3(K-1)}.$$

*Remark 6.6.* In view of Lemma 6.5, the spectral gap in (6.1) also holds without the assumption (3.11). In this case, the constants  $C_K$  satisfies that

$$C_K \leq CK^2 \left( \frac{\delta_+}{\delta_-} \right)^{3(K-1)}.$$

We first prove Lemma 6.4 from Lemma 6.5. The proof of Lemma 6.5 is put in the end of this section. Consider the bounded operator

$$\mathcal{L}_K f = \frac{1}{K} \sum_{1 \leq i < j \leq K} (E_{w,K}[f|\mathcal{F}_{i,j}] - f), \quad f \in L^2(\Omega_{w,K}, \pi_{w,K}).$$

Let  $\lambda_{w,K}$  be the spectral gap of  $\mathcal{L}_K$  on  $L^2(\Omega_{w,K}, \pi_{w,K})$ :

$$\lambda_{w,K} \triangleq \inf \{ \langle f, -\mathcal{L}_K f \rangle_{\pi_{w,K}} \mid E_{w,K}[f] = 0, E_{w,K}[f^2] = 1 \},$$

and let  $\lambda_K = \inf \{ \lambda_{w,K}; w \in \mathbb{R}^2 \times \mathbb{R}_+ \}$ . Then (6.2) is equivalent to

$$\inf \{ \lambda_K; K \geq 3 \} > 0.$$

We prove it through an induction argument, firstly established for  $k = 3$  in [5].

**Lemma 6.7.** *If  $k\lambda_k \geq 1$  holds for some  $k \geq 3$ , then for all  $K \geq k$ ,*

$$\lambda_K \geq (k\lambda_k - 1) \left( \frac{1}{k-2} - \frac{2}{K(k-2)} \right) + \frac{1}{K}.$$

By Lemma 6.5, for some fixed  $k$  which is large enough,

$$k\lambda_k > \frac{k}{C'} \left( \frac{\delta_-}{\delta_+} \right)^{3k-3} \geq \frac{k}{C'} \frac{1}{(1+\delta)^{3k-3}} \geq 1,$$

provide that  $\delta_+ \leq (1+\delta)\delta_-$  with some  $\delta$  small enough. Then with Lemma 6.7 we can show that the sequence  $\{ \lambda_K; K \geq 3 \}$  is uniformly bounded from below.

*Proof.* We make use of the equivalent characterization of  $\lambda_{w,K}$  that

$$\lambda_{w,K} = \inf \left\{ \frac{\langle \mathcal{L}_K f, \mathcal{L}_K f \rangle_{\pi_{w,K}}}{\langle f, -\mathcal{L}_K f \rangle_{\pi_{w,K}}} \mid \langle f, -\mathcal{L}_K f \rangle_{\pi_{w,K}} \neq 0 \right\}.$$

In this proof we denote by  $B$  the set of all pairs  $b = (i, j)$  such that  $1 \leq i < j \leq K$ , and write  $D_b f = E_{w,K}[f|\mathcal{F}_{i,j}] - f$  for all  $b \in B$ , then

$$\langle \mathcal{L}_K f, \mathcal{L}_K f \rangle_{\pi_{w,K}} = \frac{1}{K^2} \sum_{b, b' \in B} \langle D_b f, D_{b'} f \rangle_{\pi_{w,K}},$$

$$\langle f, -\mathcal{L}_K f \rangle_{\pi_{w,K}} = \frac{1}{K} \sum_{b \in B} \langle D_b f, D_b f \rangle_{\pi_{w,K}}.$$

We write  $b \sim b'$  if two pairs  $b$  and  $b'$  have at least one common point. We also consider all the  $k$ -particle subsets  $T_k \subseteq \{1, \dots, K\}$ . Notice that if  $b \sim b'$  but  $b \neq b'$ , there are  $\binom{K-3}{k-3}$  different  $T_k$ 's containing both  $b$  and  $b'$ . Hence,

$$\binom{n-3}{k-3} \sum_{\substack{b, b' \in B \\ b \neq b', b \sim b'}} \langle D_b f, D_{b'} f \rangle_{\pi_{w, K}} = \sum_{T_k} \sum_{\substack{b, b' \subseteq T_k \\ b \neq b', b \sim b'}} \langle D_b f, D_{b'} f \rangle_{\pi_{w, K}}.$$

If  $b \not\sim b'$ , there are  $\binom{K-4}{k-4}$  different  $T_k$ 's contain both  $b$  and  $b'$ , while for the case  $b = b'$  it is  $\binom{K-2}{k-2}$ . Therefore, the right-hand side of the equation above equals to

$$\sum_{T_k} \sum_{b, b' \subseteq T_k} \langle D_b f, D_{b'} f \rangle_{\pi_{w, K}} - \binom{K-4}{k-4} \sum_{b \not\sim b'} \langle D_b f, D_{b'} f \rangle_{\pi_{w, K}} - \binom{K-2}{k-2} \sum_{b \in B} \langle D_b f, D_b f \rangle_{\pi_{w, K}}.$$

The definition of  $\lambda_k$  yields that

$$\frac{1}{k} \sum_{b, b' \subseteq T_k} \langle D_b f, D_{b'} f \rangle_{\pi_{w, K}} \geq \lambda_k \sum_{b \subseteq T_k} \langle D_b f, D_b f \rangle_{\pi_{w, K}}.$$

And for  $b \not\sim b'$ ,  $\langle D_b f, D_{b'} f \rangle_{\pi_{w, K}} = E_{w, K} [(D_{b'} D_b f)^2] \geq 0$ . Therefore,

$$\sum_{\substack{b, b' \in B \\ b \neq b', b \sim b'}} \langle D_b f, D_{b'} f \rangle_{\pi_{w, K}} \geq \frac{(k\lambda_k - 1)(K-2)}{k-2} \sum_{b \in B} \langle D_b f, D_b f \rangle_{\pi_{w, K}}.$$

By the condition  $k\lambda_k > 1$ , the right-hand side is positive. In conclusion,

$$\begin{aligned} \langle \mathcal{L}_K f, \mathcal{L}_K f \rangle_{\pi_{w, K}} &\geq \frac{1}{K^2} \sum_{b \in B} \langle D_b f, D_b f \rangle_{\pi_{w, K}} + \frac{1}{K^2} \sum_{b \neq b', b \sim b'} \langle D_b f, D_{b'} f \rangle_{\pi_{w, K}} \\ &\geq \frac{1}{K^2} \left[ \frac{(k\lambda_k - 1)(K-2)}{k-2} + 1 \right] \sum_{b \in B} \langle D_b f, D_b f \rangle_{\pi_{w, K}} \\ &= \left[ (k\lambda_k - 1) \left( \frac{1}{k-2} - \frac{2}{K(k-2)} \right) + \frac{1}{K} \right] \langle f, -\mathcal{L}_K f \rangle_{\pi_{w, K}}. \end{aligned}$$

Notice that this estimate is independent of the choice of  $w$ .  $\square$

Finally, to complete the proof of Proposition 6.1, we are left to show Lemma 6.5. To do this, we make use of the spectral gap bound of Kac walk. For  $a \in \mathbb{R}^2$  and  $R > |a|^2$ , consider the  $(2K-3)$ -dimensional sphere

$$S_K(a, R) = \left\{ x_1, \dots, x_K \in \mathbb{R}^2 \mid \frac{1}{K} \sum_{k=1}^K x_k = a, \frac{1}{K} \sum_{k=1}^K |x_k|^2 = R \right\}.$$

Denote by  $\mu_K(a, R)$  the uniform measure on  $S_K(a, R)$ . With a little abuse of notations, let  $\mathcal{F}_{i, j} = \sigma\{x_k; k \neq i, j\}$  for  $1 \leq i < j \leq K$ .

**Lemma 6.8.** *There exists a constant  $C$  such that*

$$E_{\mu_K(a, R)} [(f - E_{\mu_K(a, R)}[f])^2] \leq \frac{C}{K} \sum_{1 \leq i < j \leq n} E_{\mu_K(a, R)} [(f - E_{\mu_K(a, R)}[f | \mathcal{F}_{i, j}])^2]$$

for all  $(a, R, K)$  and bounded smooth function  $f$ .

Lemma 6.8 can be proved by the arguments in [6] and [7]. We here prove Lemma 6.4 by applying a perturbation on the spectral gap in Lemma 6.8.

*Proof.* To begin with, from (2.1) we know that for  $r \neq r'$  and  $K \geq 1$ ,

$$\frac{\sqrt{2(K+1)}}{\sqrt{K}} c_- \leq \frac{|V'(r) - V'(r')|}{\sqrt{V(r) + KV(r') - (K+1)V(\frac{r+Kr'}{K+1})}} \leq \frac{\sqrt{2(K+1)}}{\sqrt{K}} c_+, \quad (6.4)$$

where  $c_- = \delta_-/\sqrt{\delta_+}$  and  $c_+ = \delta_+/\sqrt{\delta_-}$ . For each  $K \geq 3$ , we construct a bijection  $\tau_K : \Omega_K \rightarrow \Omega_K$ , satisfying the following two conditions.

- (i) For  $w = (p, r, e)$ ,  $a = (p, r)$  and  $R = 2e - 2V(r) + r^2$ ,  $\tau_K(S_K(a, R)) = \Omega_{w,K}$ ;
- (ii) The Jacobian matrix  $\tau'_K$  of  $\tau_K$  satisfies that  $c_-^{K-1} \leq |\det(\tau'_K)| \leq c_+^{K-1}$ .

Indeed, given a bounded, measurable, positive function  $g$  on  $\Omega_{w,K}$ , by (i) we know that  $\tau_K g := g \circ \tau_K$  defines a function on  $S_K(a, R)$ , and (ii) yields that

$$c_0^{-(K-1)} E_{\mu_K(a,R)}[\tau_K g] \leq E_{w,K}[g] \leq c_0^{K-1} E_{\mu_K(a,R)}[\tau_K g],$$

where  $c_0 = c_+/c_-$ . For bounded and smooth function  $f$ , we can apply the estimate above to  $g = (f - E_{\mu_K(a,R)}[\tau_K f])^2$  to obtain

$$E_{w,K} [(f - E_{w,K}[f])^2] \leq E_{w,K}[g] \leq c_0^{K-1} E_{\mu_K(a,R)}[\tau_K g].$$

On the other hand, take  $h_{i,j} = (f - E_{w,K}[f|\mathcal{F}_{i,j}])^2$  and similarly,

$$E_{\mu_K(a,R)} [(\tau_K f - E_{\mu_K(a,R)}[\tau_K f|\mathcal{F}_{i,j}])^2] \leq E_{\mu_K(a,R)}[\tau_K h_{i,j}] \leq c_0^{K-1} E_{w,K}[h_{i,j}].$$

Substituting  $\tau_K f$  for  $f$  in Lemma 6.8, we get

$$\begin{aligned} E_{w,K} [(f - E_{w,K}[f])^2] &\leq c_0^{K-1} E_{\mu_K(a,R)} [(\tau_K f - E_{\mu_K(a,R)}[\tau_K f])^2] \\ &\leq \frac{C c_0^{K-1}}{K} \sum_{i < j} E_{\mu_K(a,R)} [(\tau_K f - E_{\mu_K(a,R)}[\tau_K f|\mathcal{F}_{i,j}])^2] \\ &\leq \frac{C c_0^{2(K-1)}}{K} \sum_{i < j} E_{w,K} [(f - E_{w,K}[f|\mathcal{F}_{i,j}])^2]. \end{aligned}$$

Since  $c_0 = (\delta_+/\delta_-)^{3/2}$ , Lemma 6.5 then follows.

Now fix  $K \geq 3$  and we construct the map  $\tau_K$ . Write  $x_k = (p_k, r_k)$  and define

$$\alpha_k = \frac{1}{k} \sum_{i=1}^k r_i, \quad \forall 1 \leq k \leq K.$$

Consider two maps  $\zeta, \zeta_* : \mathbb{R}^K \rightarrow \mathbb{R}^K$ . The first map  $\zeta$  is given by

$$\zeta(r_1, \dots, r_K) = (r'_1, \dots, r'_K),$$

such that  $r'_K = \alpha_K$ , and for  $1 \leq k \leq K-1$ ,

$$(r'_k)^2 = \frac{2k}{k+1} (V(r_{k+1}) + kV(\alpha_k) - (k+1)V(\alpha_{k+1})),$$

where the sign of  $r'_k$  is chosen in accordance with  $r_k - \alpha_K$ . Meanwhile,  $\zeta_*$  is given by

$$\zeta_*(r'_1, \dots, r'_K) = (r''_1, \dots, r''_K),$$

such that

$$r''_k = \begin{cases} r'_K - \sum_{i=1}^{K-1} \frac{r'_i}{i}, & \text{for } k = 1, \\ r'_K + r'_{k-1} - \sum_{i=k}^{K-1} \frac{r'_i}{i}, & \text{for } 2 \leq k \leq K-1, \\ r'_K + r'_{K-1}, & \text{for } k = K. \end{cases}$$

Denote by  $J$  and  $J_*$  the Jacobian matrixes of  $\zeta$  and  $\zeta_*$ , respectively. To compute  $J$ , noticing that  $\partial_{r_i} r'_k = \partial_{r_k} r'_k$  for all  $i \leq k$ , and  $\partial_{r_i} r'_k = 0$  for all  $i > k$ , we have

$$J = \begin{bmatrix} \frac{\partial r'_1}{\partial r_1} & \frac{\partial r'_1}{\partial r_2} & 0 & \cdots & 0 \\ \frac{\partial r'_2}{\partial r_2} & \frac{\partial r'_2}{\partial r_2} & \frac{\partial r'_2}{\partial r_3} & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ \frac{\partial r'_{K-1}}{\partial r_{K-1}} & \frac{\partial r'_{K-1}}{\partial r_{K-1}} & \frac{\partial r'_{K-1}}{\partial r_K} & \cdots & \frac{\partial r'_{K-1}}{\partial r_K} \\ \frac{\partial r'_K}{\partial r_K} & \frac{\partial r'_K}{\partial r_K} & \frac{\partial r'_K}{\partial r_K} & \cdots & \frac{\partial r'_K}{\partial r_K} \end{bmatrix}.$$

Hence, its determinant reads

$$|\det(J)| = \left| \frac{\partial r'_K}{\partial r_K} \right| \cdot \prod_{k=1}^{K-1} \left| \frac{\partial r'_k}{\partial r_k} - \frac{\partial r'_k}{\partial r_{k+1}} \right|.$$

Since  $\partial_{r_K} r'_K = 1/K$  and for  $k = 1, \dots, K-1$  we have

$$\frac{\partial r'_k}{\partial r_i} = \begin{cases} \frac{k}{(k+1)r'_k} [V'(\alpha_k) - V'(\alpha_{k+1})], & \text{if } 1 \leq i \leq k, \\ \frac{k}{(k+1)r'_k} [V'(r_{k+1}) - V'(\alpha_{k+1})], & \text{if } i = k+1. \end{cases}$$

In consequence,  $|\det(J)|$  equals to

$$\frac{1}{K} \prod_{k=1}^{K-1} \frac{\sqrt{k}}{\sqrt{2(k+1)}} \frac{|V'(r_{k+1}) - V'(\alpha_k)|}{\sqrt{V(r_{k+1}) + kV(\alpha_k) - (k+1)V(\alpha_{k+1})}}.$$

Applying the estimate in (6.4) to obtain that

$$\frac{c_-^{K-1}}{K} \leq |\det(J)| \leq \frac{c_+^{K-1}}{K}.$$

Meanwhile it is easy to calculate that  $|\det(J_*)| = K$ . Therefore, define

$$\tau_K : (p_1, \dots, p_K, r_1, \dots, r_K) \mapsto (p_1, \dots, p_K, (\zeta_* \circ \zeta)^{-1}(r_1, \dots, r_K)),$$

then  $|\det(\tau'_K)|$  satisfies (ii). On the other hand, by the definition of  $\zeta$ ,

$$\sum_{k=1}^K r_k = Kr'_K, \quad \sum_{k=1}^K V(r_k) = KV(r'_K) + \sum_{k=1}^{K-1} \frac{k+1}{k} \frac{(r'_k)^2}{2}.$$

Similarly, by the definition of  $\zeta_*$ ,

$$\sum_{k=1}^K r''_k = Kr'_K, \quad \sum_{k=1}^K \frac{(r''_k)^2}{2} = \frac{K(r'_K)^2}{2} + \sum_{k=1}^{K-1} \frac{k+1}{k} \frac{(r'_k)^2}{2},$$

hence (i) is also fulfilled by  $\tau_K$ . This completes the proof.  $\square$

## 7 Tightness

In Section 4 we have proved the convergence of the finite-dimensional distribution of  $\{\mathbb{Q}_N\}$ . In order to complete the proof of Theorem 3.1, we need its tightness in  $C([0, T], \mathcal{H}_{-k}(\lambda))$ . The proof is standard, and we summarize it here.

It suffices to show the two statements below:

$$\lim_{M \rightarrow \infty} \limsup_{N \rightarrow \infty} \mathbb{P}_{\lambda, N} \left\{ \sup_{t \in [0, T]} \|Y_N(t)\|_{-k} \geq M \right\} = 0, \quad (7.1)$$

$$\lim_{\delta \downarrow 0} \limsup_{N \rightarrow \infty} \mathbb{P}_{\lambda, N} \{w_{-k}(Y_N, \delta) \geq \epsilon\} = 0, \quad \forall \epsilon > 0, \quad (7.2)$$

where  $w_{-k}(Y_N, \delta)$  is the modulus of continuity in  $C([0, T], \mathcal{H}_{-k}(\lambda))$ . Recall that

$$\|Y_N\|_{-k}^2 = \sum_{i=1}^2 \sum_{n=0}^{\infty} \{\theta_n^{-2k} Y_N^2(R\boldsymbol{\mu}_{i,n}) + \kappa_n^{-2k} Y_N^2(R\boldsymbol{\nu}_{i,n})\},$$

where  $R$  is the rotation matrix in (3.9), and  $\boldsymbol{\mu}_{i,n}, \boldsymbol{\nu}_{i,n}$  are the three-dimensional Fourier bases defined in (3.4) and (3.8).

Take  $f = \boldsymbol{\mu}_{i,n}$  or  $\boldsymbol{\nu}_{i,n}$  for some  $(i, n)$ . Applying (4.1) with  $H(t) \equiv Rf$ ,

$$Y_N(t, Rf) = Y_0(0, Rf) + \int_0^t Y_N(s, L^*[Rf]) ds + \epsilon_N(t, f),$$

and by Lemma 4.1 and Remark 4.3,  $\epsilon_N$  satisfies that

$$\mathbb{E}_{\lambda, N} \left[ \sup_{t \in [0, T]} \epsilon_N^2(t, f) \right] = o_N(1) (|f'|_{\infty}^2 + |f''|_{\infty}^2 + \|f'\|^2).$$

On the other hand, it is easy to see that

$$\mathbb{E}_{\lambda, N} \left[ \sup_{t \in [0, T]} \left| \int_0^t Y(s, L^*[Rf]) ds \right|^2 \right] \leq CT^2 \|f'\|^2.$$

Observe that  $|f''_{i,n}(x)| \leq \sqrt{2}(\theta_n \wedge \kappa_n)^2$ . Then, for  $k > 5/2$ , (7.1) and (7.2) can be proved by standard arguments (cf. [9, 11.3]).

## 8 Equivalence of ensembles

In this section we deal with the equivalence of ensembles in a multi-dimensional framework, which has been used in the proof of Lemma 4.2. In the main result, Proposition 8.3, we prove a Lebowitz–Percus–Verlet type formula (cf. [4]).

The notations in this section are independent of the other part. Let  $\Omega$  be a general topological space equipped with a positive measure  $\nu$ . Let  $\mathbf{f} = (f_1, \dots, f_d)$  be a random vector on  $\Omega$  with compact level sets. For a  $d$ -dimensional coefficient  $\lambda$ , define

$$Z(\lambda) = \log \left[ \int_{\Omega} \exp \{ \lambda \cdot \mathbf{f}(\omega) \} \nu(d\omega) \right].$$

Assume a nonempty domain  $D \subseteq \mathbb{R}^d$  such that  $Z(\lambda) < \infty$  for  $\lambda \in D$ , and

- (a)  $Z \in C_b^4(D)$ ,  $Z''(\lambda)$  is positive-definite for  $\lambda \in D$ .



For simplicity we denote  $\mathbf{u}_\lambda = Z''(\lambda)$  and  $\Sigma_\lambda = Z''(\lambda)$ . For  $\lambda \in D$ , let  $\nu_\lambda$  be the *tilted probability measure* on  $\Omega$ , given by the formula

$$\nu_\lambda(d\omega) \triangleq \exp\{\lambda \cdot \mathbf{f}(\omega) - Z(\lambda)\} \nu(d\omega).$$

Observe that under  $\nu_\lambda$ , the average of  $\mathbf{f}$  is  $\mathbf{u}_\lambda$  and the covariance matrix is  $\Sigma_\lambda$ . Let  $\Phi_\lambda$  denote the characteristic function of  $\mathbf{f} - \mathbf{u}_\lambda$ :

$$\Phi_\lambda(\mathbf{h}) = \int_{\Omega} \exp\{i\mathbf{h} \cdot (\mathbf{f}(\omega) - \mathbf{u}_\lambda)\} \nu_\lambda(d\omega), \quad \forall \mathbf{h} \in \mathbb{R}^d.$$

We also assume that

(b) there exists some  $\epsilon_0 > 0$  such that  $|\Phi_\lambda(\mathbf{h})| \leq |\mathbf{h}|^{-\epsilon_0}$  if  $|\mathbf{h}|$  is large enough.

*Remark 8.1.* In the model introduced in Section 1,  $\Omega = \mathbb{R}^2$  and  $\nu$  is the Lebesgue measure on  $\Omega$ . By taking  $d = 3$  and  $\lambda = (0, \beta\tau, -\beta)$ ,  $Z(\lambda)$  is the Gibbs potential in (1.3), and  $\nu_\lambda$  is the corresponding marginal distribution.

Notice that  $Z$  is convex, we consider its *Fenchel-Legendre transform*

$$Z^*(\mathbf{u}) = \sup_{\lambda \in D} \{\lambda \cdot \mathbf{u} - Z(\lambda)\}.$$

Let  $D^* = \{\mathbf{u} \in \mathbb{R}^d : Z^*(\mathbf{u}) < \infty\}$ . The superior is reached at a unique  $\lambda(\mathbf{u}) \in D$ , satisfying that  $\mathbf{u} = \nabla Z|_{\lambda=\lambda(\mathbf{u})}$  and  $\lambda(\mathbf{u}) = \nabla Z^*(\mathbf{u})$ . Notice that the map  $\mathbf{u} \mapsto \lambda(\mathbf{u})$  is a one-to-one correspondence from  $D^*$  to  $D$ , and its inverse is given by  $\lambda \mapsto \mathbf{u}_\lambda$ . The Hessian matrices of  $Z$  and  $Z^*$  then satisfy the relation

$$(Z^*)''|_{\mathbf{u}=\mathbf{u}_\lambda} = [Z''(\lambda)]^{-1} = \Sigma_\lambda^{-1}.$$

We define the rate function  $I_\lambda(\mathbf{u})$  by

$$I_\lambda(\mathbf{u}) = Z^*(\mathbf{u}) - Z^*(\mathbf{u}_\lambda) - \nabla Z^*(\mathbf{u}_\lambda) \cdot (\mathbf{u} - \mathbf{u}_\lambda). \quad (8.1)$$

Denote by  $M_\lambda$  the largest eigenvalue of  $\Sigma_\lambda$ . By the arguments above it is not hard to conclude that for any constant  $M > M_\lambda$ , we have

$$I_\lambda(\mathbf{u}) \geq (2M)^{-1} |\mathbf{u} - \mathbf{u}_\lambda|^2 \quad (8.2)$$

holds if  $|\mathbf{u} - \mathbf{u}_\lambda|$  is small enough.

For  $n \geq 1$ , equip the product space  $\Omega^n$  with measure  $\nu_\lambda^n = \otimes_j d\nu_\lambda(\omega_j)$ , and define

$$\mathbf{f}_{(n)}(\omega_1, \dots, \omega_n) = \frac{1}{n} \sum_{j=1}^n \mathbf{f}(\omega_j), \quad \forall (\omega_1, \dots, \omega_n) \in \Omega^n.$$

In view of 8.2 we can obtain the following large deviation property.

**Lemma 8.2.** *For any  $M > M_\lambda$ , there exists some  $\delta_M$  such that*

$$\nu_\lambda^n \{|\mathbf{f}_{(n)} - \mathbf{u}_\lambda| \geq \delta\} \leq 2^d \exp\left(-\frac{nM\delta^2}{d}\right),$$

*holds for all  $n \geq 1$  when  $|\delta| < \delta_M$ .*

*Proof.* Let  $\Gamma \subseteq \mathbb{R}^d$  be the collection of vectors whose coordinates are all  $\pm 1$ . Notice that the following inequality holds for all  $\mathbf{x} \in \mathbb{R}^d$ :

$$e^{|\mathbf{x}|} \leq \prod_{j=1}^d e^{|x_j|} \leq \prod_{j=1}^d (e^{-x_j} + e^{x_j}) = \sum_{\gamma \in \Gamma} e^{\gamma \cdot \mathbf{x}}.$$

By exponential Chebyshev's inequality and the above estimate, for  $\theta > 0$ ,

$$\begin{aligned} \nu_\lambda^n \{ |\mathbf{f}_{(n)} - \mathbf{u}_\lambda| \geq \delta \} &\leq \sum_{\gamma \in \Gamma} e^{-n\theta\delta} \int_{|\mathbf{f}_{(n)} - \mathbf{u}_\lambda| \geq \delta} \exp \{ n\theta\gamma \cdot (\mathbf{f}_{(n)} - \mathbf{u}_\lambda) \} d\nu_\lambda^n \\ &\leq \sum_{\gamma \in \Gamma} \exp \{ -n\theta u' + nZ(\lambda + \theta\gamma) - nZ(\lambda) \}, \end{aligned}$$

where  $u' = \gamma \cdot \mathbf{u}_\lambda + \delta$ . To optimize this estimate, define

$$I_{\lambda,\gamma}(u') = \sup_{\theta > 0} \{ \theta u' - Z(\lambda + \theta\gamma) + Z(\lambda) \} = \sup_{\theta \in \mathbb{R}} \{ \theta u - Z(\lambda + \theta\gamma) + Z(\lambda) \}.$$

The last equality is due to the fact that  $u' - \partial_\theta Z(\lambda + \theta\gamma)|_{\theta=0} = \delta > 0$ . Notice that  $I_{\lambda,\gamma}$  is the rate function defined in (8.1) corresponding to the measure  $\nu_\lambda$  and the function  $\gamma \cdot \mathbf{f}$ . By the arguments which has been used to derive (8.2), one obtains that  $I_{\lambda,\gamma}(u') \geq M_\lambda |\gamma|^{-2} \delta^2$ . The estimate in Lemma 8.2 then follows directly.  $\square$

For a nice measurable function  $G$  on  $\Omega^n$ , we consider the canonical expectation  $\varphi_{n,G}$  and the microcanonical expectation  $\psi_{n,G}$ , respectively given by

$$\varphi_{n,G}(\lambda) \triangleq E_{\nu_\lambda^n}[G], \quad \psi_{n,G}(\mathbf{u}) \triangleq E_{\nu_\lambda^n}[G | \mathbf{f}_{(n)} = \mathbf{u}].$$

By equivalence of ensembles we mean that  $|\varphi_{n,G}(\lambda) - \psi_{n,G}(\mathbf{u}_\lambda)|$  vanishes when volume grows to infinity. To state the result, let  $\|G\|_{\lambda,n}$  be the  $L^2$  norm of  $G$  with respect to  $\nu_\lambda^n$ , and assume that there is a constant  $C_\lambda < \infty$  such that

**(c.1)**  $\|G\|_{\lambda,n} \leq C_\lambda$ ,  $|\psi_{n,G}(\mathbf{u}_\lambda)| \leq C_\lambda \|G\|_{\lambda,n}$ ;

**(c.2)**  $\psi_{n,G}$  is three times differentiable on  $D$ , and for all  $\gamma \in \mathbb{R}^d$ ,

$$|\nabla \psi_{n,G}|_{\mathbf{u}=\mathbf{u}_\lambda}| \leq C_\lambda \sqrt{n} \|G\|_{2,\lambda,n}, \quad |[\psi_{n,G}''|_{\mathbf{u}=\mathbf{u}_\lambda}] \gamma| \leq C_\lambda n \|G\|_{\lambda,n} \cdot |\gamma|;$$

**(c.3)** with a constant  $b_\lambda > \sqrt{8dM_\lambda}$  and a sequence  $c_{\lambda,n} \rightarrow 0$  as  $n \rightarrow \infty$ ,

$$\sup_{|\mathbf{u}-\mathbf{u}_\lambda| < \delta_n} \frac{1}{3!} \sum_{|\alpha|=3} \partial_\alpha \psi_{n,G}(\mathbf{u}) \gamma^\alpha \leq B_{\lambda,n} \|G\|_{\lambda,n} |\gamma|^3, \quad \forall \gamma \in \mathbb{R}^d,$$

where  $\delta_n$  and  $B_{\lambda,n}$  are constants given by

$$\delta_n = b_\lambda \sqrt{\frac{\log n}{n}}, \quad B_{\lambda,n} = \frac{c_{\lambda,n}}{\log \log n} \sqrt{\frac{n}{\log n}}.$$

The equivalence of ensembles is stated below.

**Proposition 8.3.** *Assume  $\mathbf{a}$ ,  $\mathbf{b}$  and **c.1-c.3** above. Then*

$$\left| \varphi_{n,G}(\lambda) - \psi_{n,G}(\mathbf{u}_\lambda) - \frac{1}{2n} \text{Tr} \left[ \nabla (\Sigma_\lambda^{-1} \nabla \varphi_{n,G}(\lambda)) \right] \right| \leq \frac{K_n}{n} \|G\|_{\lambda,n},$$

where the sequence  $\{K_n\}$  satisfies that  $\lim_{n \rightarrow \infty} K_n = 0$ .

*Proof.* The proof goes exactly the same as the one-dimensional case showed in [4], based on a local central limit theorem with an expansion of error, as well as an estimate on the large deviation probability of  $\mathbf{f}_{(n)}$  in Lemma 8.2.  $\square$

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