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BOUNDARY FEEDBACK STABILIZATION OF THE BOUSSINESQ SYSTEM WITH MIXED BOUNDARY CONDITIONS

Mythily Ramaswamy*, Jean-Pierre Raymond†, Arnab Roy*

Abstract

We study the feedback stabilization of the Boussinesq system in a two dimensional domain, with mixed boundary conditions. After ascertaining the precise loss of regularity of the solution in such models, we prove first Green's formulas for functions belonging to weighted Sobolev spaces and then correctly define the underlying control system. This provides a rigorous mathematical framework for models studied in the engineering literature. We prove the stabilizability by extending to the linearized Boussinesq system a local Carleman estimate already established for the Oseen system. Then we determine a feedback control law able to stabilize the linearized system around the stationary solution, with any prescribed exponential decay rate, and able to stabilize locally the nonlinear system.

Key words. Boussinesq system, mixed boundary conditions, boundary Feedback Stabilization, finite dimensional controllers, fractional Sobolev spaces.

AMS subject classifications. 93B52, 93C20, 93D15, 76D05, 76D55, 26B20.

1 Introduction

In the recent years, there has been a considerable interest in feedback stabilization of fluid flows around stationary solutions. However most of the works focus on either Dirichlet or Neumann boundary conditions on the entire boundary. Mixed boundary conditions do arise naturally in applications, for example, in energy efficient building structures.

In this paper, we would like to establish rigorous stabilization results for the Boussinesq system, around a stationary solution, by feedback controls of finite dimension, for geometrical domains and boundary conditions including models studied numerically in [1]. Stabilization problems for the Navier–Stokes system with mixed boundary conditions are already studied in [2]. Those results could be extended without major difficulties to the Boussinesq system. But

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in [2], only the case of a right angle junction between Dirichlet and Neumann boundary conditions is considered. Therefore the results obtained in [2] cannot be applied to the geometrical configurations studied in [1].

The goal of this paper is to fill this gap. An extension of the results of [2] is not obvious. Indeed the analysis of such control systems relies on fine regularity results and on Green's formulas to characterize adjoint operators. These Green's formulas, which are known for velocity fields and temperatures with $H^{3/2+\varepsilon}$ spatial regularity with $\varepsilon > 0$, may be wrong for $H^{3/2-\varepsilon}$ spatial regularity when $\varepsilon > 0$ (see [3]). In addition to that, the $H^{3/2-\varepsilon}$ spatial regularity with $\varepsilon > 0$ that we have for the models considered here, is not sufficient to deal with the nonlinear term of the control Navier–Stokes equations. This is why a careful analysis of these difficulties is carried out in the present paper. The main idea to overcome the drawbacks linked to the loss of regularity is to work with weighted Sobolev spaces. The analysis of the stationary case can be handled with results proved in [4] for the Laplace and Stokes equations separately, but the approach proposed in the present paper is totally new both for the analysis and for the control of such instationary systems, even for the case of the Navier–Stokes system alone.

The precise assumptions on Ω are stated in Section 2.1. Its boundary $\Gamma = \partial\Omega$ is split as follows $\Gamma = \overline{\Gamma_w} \cup \overline{\Gamma_c} \cup \overline{\Gamma_n}$, where Γ_w , Γ_c and Γ_n are relatively open subsets in Γ , two by two disjoint. An example of domain satisfying the assumptions stated in Section 2.1 is given in Fig. 1. The control Boussinesq system satisfied by the temperature τ , the fluid velocity \mathbf{u} and

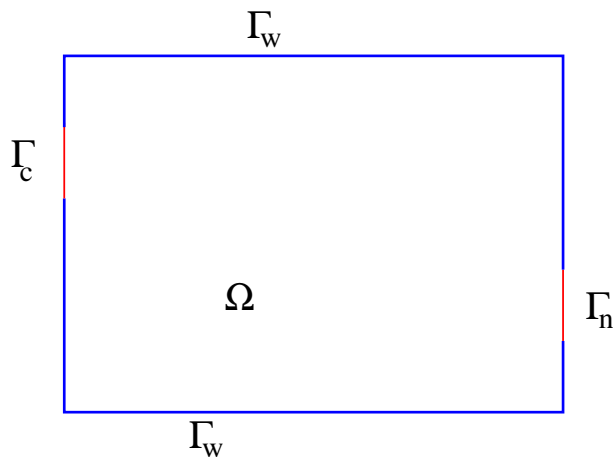


Figure 1: Rectangular domain with inflow and outflow boundary condition.

pressure q is

$$\begin{aligned}
\frac{\partial \tau}{\partial t} - \kappa \Delta \tau + \mathbf{u} \cdot \nabla \tau &= f_s \text{ in } Q, \quad \tau(0) = \tau_0 \text{ in } \Omega, \\
\tau &= g_{\tau,s} + \theta_c \text{ on } \Sigma_c, \quad \frac{\partial \tau}{\partial n} = 0 \text{ on } \Sigma \setminus \Sigma_c, \\
\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} - \operatorname{div} \sigma(\mathbf{u}, q) &= \beta \tau, \quad \operatorname{div} \mathbf{u} = 0 \text{ in } Q, \\
\mathbf{u} &= 0 \text{ on } \Sigma_w, \quad \mathbf{u} = \mathbf{g}_{\mathbf{u},s} + \mathbf{v}_c \text{ on } \Sigma_c, \quad \sigma(\mathbf{u}, q) \mathbf{n} = 0 \text{ on } \Sigma_n, \\
\Pi \mathbf{u}(0) &= \Pi \mathbf{u}_0,
\end{aligned} \tag{1.1}$$

where κ is the thermal diffusivity, ν is the fluid viscosity, $\sigma(\mathbf{u}, q) = \nu(\nabla \mathbf{u} + (\nabla \mathbf{u})^T) - qI$ is the

Cauchy stress tensor, f_s , $g_{\tau,s}$ and $\mathbf{g}_{\mathbf{u},s}$ are stationary data, $Q = \Omega \times (0, \infty)$, $\Sigma = \Gamma \times (0, \infty)$, $\Sigma_n = \Gamma_n \times (0, \infty)$, $\Sigma_w = \Gamma_w \times (0, \infty)$, $\Sigma_c = \Gamma_c \times (0, \infty)$, θ_c and \mathbf{v}_c are control functions, $\beta \in L^\infty(\Omega; \mathbb{R}^2)$, and Π is the Leray projector introduced in Section 2.2.

Let $(\tau_s, \mathbf{u}_s, q_s)$ be a solution to the stationary Boussinesq system

$$\begin{aligned} -\kappa \Delta \tau_s + \mathbf{u}_s \cdot \nabla \tau_s &= f_s \text{ in } \Omega, \\ \tau_s &= g_{\tau,s} \text{ on } \Gamma_c, \quad \frac{\partial \tau_s}{\partial n} = 0 \text{ on } \Gamma \setminus \Gamma_c, \\ (\mathbf{u}_s \cdot \nabla) \mathbf{u}_s - \operatorname{div}(\sigma(\mathbf{u}_s, q_s)) &= \beta \tau_s, \quad \operatorname{div} \mathbf{u}_s = 0 \text{ in } \Omega, \\ \mathbf{u}_s &= \mathbf{g}_{\mathbf{u},s} \text{ on } \Gamma_c, \quad \mathbf{u}_s = 0 \text{ on } \Gamma_w, \quad \sigma(\mathbf{u}_s, q_s) \mathbf{n} = 0 \text{ on } \Gamma_n. \end{aligned} \tag{1.2}$$

We assume that (τ, \mathbf{u}, q) is an unstable solution for the instationary Boussinesq system (1.1). Our goal is to stabilize system (1.1) in a neighborhood of the stationary solution $(\tau_s, \mathbf{u}_s, q_s)$ at a given decay rate ω , by finite dimensional boundary controls in the form

$$\theta_c(x, t) = \sum_{i=1}^N f_i(t) g_{\theta,i}(x) \quad \text{and} \quad \mathbf{v}_c(x, t) = \sum_{i=1}^N f_i(t) \mathbf{g}_{\mathbf{v},i}(x). \tag{1.3}$$

The functions $\mathbf{g}_i = (g_{\theta,i}, \mathbf{g}_{\mathbf{v},i})^T$ are localized in Γ_c , and they play the role of actuators. They have to be chosen in order to satisfy some stabilizability properties. The function $\mathbf{f} = (f_i)_{1 \leq i \leq N} \in L^2(0, \infty; \mathbb{R}^N)$ is the control variable. In order to determine the control \mathbf{f} in feedback form, we write below the system satisfied by $(\theta, \mathbf{v}, p) = (\tau - \tau_s, \mathbf{u} - \mathbf{u}_s, q - q_s)$:

$$\begin{aligned} \frac{\partial \theta}{\partial t} - \kappa \Delta \theta + \mathbf{u}_s \cdot \nabla \theta + \mathbf{v} \cdot \nabla \tau_s + \mathbf{v} \cdot \nabla \theta &= 0 \text{ in } Q, \\ \theta &= \theta_c \text{ on } \Sigma_c, \quad \frac{\partial \theta}{\partial n} = 0 \text{ on } \Sigma \setminus \Sigma_c, \quad \theta(0) = \theta_0 \text{ in } \Omega, \\ \frac{\partial \mathbf{v}}{\partial t} - \operatorname{div} \sigma(\mathbf{v}, p) + (\mathbf{u}_s \cdot \nabla) \mathbf{v} + (\mathbf{v} \cdot \nabla) \mathbf{u}_s + (\mathbf{v} \cdot \nabla) \mathbf{v} &= \beta \theta \text{ in } Q, \\ \operatorname{div} \mathbf{v} &= 0 \text{ in } Q, \quad \mathbf{v} = \mathbf{v}_c \text{ on } \Sigma_c, \quad \mathbf{v} = 0 \text{ on } \Sigma_w, \\ \sigma(\mathbf{v}, p) \mathbf{n} &= 0 \text{ on } \Sigma_n, \quad \Pi \mathbf{v}(0) = \Pi \mathbf{v}_0 \text{ in } \Omega, \end{aligned} \tag{1.4}$$

where $\theta_0 = \tau_0 - \tau_s$ and $\mathbf{v}_0 = \mathbf{u}_0 - \mathbf{u}_s$. The linearized system around the steady state solution $(\tau_s, \mathbf{u}_s, q_s)$ is given by

$$\begin{aligned} \frac{\partial \theta}{\partial t} - \kappa \Delta \theta + \mathbf{u}_s \cdot \nabla \theta + \mathbf{v} \cdot \nabla \tau_s &= 0 \text{ in } Q, \\ \theta &= \theta_c \text{ on } \Sigma_c, \quad \frac{\partial \theta}{\partial n} = 0 \text{ on } \Sigma \setminus \Sigma_c, \quad \theta(0) = \theta_0 \text{ in } \Omega, \\ \frac{\partial \mathbf{v}}{\partial t} - \operatorname{div} \sigma(\mathbf{v}, p) + (\mathbf{u}_s \cdot \nabla) \mathbf{v} + (\mathbf{v} \cdot \nabla) \mathbf{u}_s &= \beta \theta, \quad \operatorname{div} \mathbf{v} = 0 \text{ in } Q, \\ \mathbf{v} &= \mathbf{v}_c \text{ on } \Sigma_c, \quad \mathbf{v} = 0 \text{ on } \Sigma_w, \quad \sigma(\mathbf{v}, p) \mathbf{n} = 0 \text{ on } \Sigma_n, \\ \Pi \mathbf{v}(0) &= \Pi \mathbf{v}_0 \text{ in } \Omega. \end{aligned} \tag{1.5}$$

The main issues that we have to deal with are

– obtaining regularity results for the linearized system, for the closed-loop homogeneous and nonhomogeneous linearized systems,

- writing the linearized system as a control system,
- studying the stabilizability of the control system, and determining stabilizing feedback control laws,
- studying the stability of the closed-loop nonlinear system.

Let us address successively these issues. In Section 3, by using the variational formulation of the stationary Boussinesq system, we define the Boussinesq operator $(\mathcal{A}, D(\mathcal{A}))$ in $Z = L^2(\Omega) \times \mathbf{V}_{n, \Gamma_d}^0(\Omega)$, where $\mathbf{V}_{n, \Gamma_d}^0(\Omega) = \{\mathbf{v} \in L^2(\Omega; \mathbb{R}^2) \mid \operatorname{div} \mathbf{v} = 0 \text{ in } \Omega, \mathbf{v} \cdot \mathbf{n} = 0 \text{ on } \Gamma_d\}$ and $\Gamma_d = \Gamma_w \cup \Gamma_c$ (see Section 2.1). The control operator \mathcal{B} is defined in Section 3.5, and in Section 4, we show that the linearized system (1.5) satisfied by $\mathbf{z} = (\theta, \Pi \mathbf{v})^T$ can be written in the form

$$\mathbf{z}' = \mathcal{A}\mathbf{z} + \mathcal{B}\mathbf{f}, \quad \mathbf{z}(0) = \mathbf{z}_0,$$

and that $(I - \Pi)\mathbf{v}$ satisfies an algebraic equation. This representation is essential to apply the stabilizability Hautus test. We also need the precise characterization of the adjoint operators \mathcal{A}^* , \mathcal{B}^* . This is obtained by using Green's formulas and regularity results for functions belonging to $D(\mathcal{A})$ and to $D(\mathcal{A}^*)$. In particular, we prove in Section 3 that

$$D(\mathcal{A}) \subset H_\delta^2(\Omega) \times H_\delta^2(\Omega; \mathbb{R}^2) \quad \text{and} \quad D(\mathcal{A}^*) \subset H_\delta^2(\Omega) \times H_\delta^2(\Omega; \mathbb{R}^2),$$

where $H_\delta^2(\Omega) \times H_\delta^2(\Omega; \mathbb{R}^2)$ is a weighted Sobolev space introduced in Section 2.1. We emphasize that these regularity results are obtained by using the mixed (variational) formulation of the stationary Boussinesq system, the Green's formulas are obtained by using the PDE formulation of this system and the variational formulation is used in writing the PDE in operator form. That is why we show the equivalence between these three formulations. Due to the loss of regularity induced by the Dirichlet–Neumann, Neumann–Neumann and Dirichlet–Dirichlet junctions, the analysis is more tricky.

Stabilizability results may be deduced from null controllability results. Local boundary controllability to trajectories of the Boussinesq system is established in [5], [6], [7]. See also [8], for additional results of local exact controllability to the trajectories of the Boussinesq systems with interior controls. But all these controllability results are for Dirichlet conditions for both velocity and temperature and when the boundary of the domain is regular. As far as we know, there are no similar controllability or stabilizability results in the case of mixed boundary conditions and the analysis of our problem is more delicate. Here, adapting to the Boussinesq system the approach used in [9] for the Oseen system, we prove in Appendix a local Carleman estimate for the adjoint stationary Boussinesq system. Using this result, we prove that the control system corresponding to (1.5) is stabilizable. Then the feedback control law is defined as in [10] or [2]. The regularity of solutions to the nonhomogeneous linearized closed loop system is also established in Section 5.

In particular, we show in Section 5.3 that if the initial condition (θ_0, \mathbf{v}_0) belongs to $H^\varepsilon(\Omega) \times \mathbf{V}_{n, \Gamma_d}^\varepsilon(\Omega)$, where $\mathbf{V}_{n, \Gamma_d}^\varepsilon(\Omega) = \mathbf{V}_{n, \Gamma_d}^0(\Omega) \cap H^\varepsilon(\Omega; \mathbb{R}^2)$, and if the right hand side of the nonhomogeneous linearized closed-loop system belongs to $L^2(0, \infty; H_{\Gamma_c}^{-1+\varepsilon}(\Omega)) \times L^2(0, \infty; \mathbf{H}_{\Gamma_d}^{-1+\varepsilon}(\Omega))$, for $\varepsilon \in (0, 1/2)$, then the solution to the closed-loop linearized system belongs to the Hilbert space

$$\begin{aligned} X = & \left(L^2(0, \infty; D((\lambda_0 I - \mathcal{A})^{\frac{1}{2} + \frac{\varepsilon}{2}})) \cap H^{\frac{1}{2} + \frac{\varepsilon}{2}}(0, \infty; Z) \right) \\ & + H^1(0, \infty; H^{\frac{3}{2} - \varepsilon}(\Omega) \times \mathbf{H}^{\frac{3}{2} - \varepsilon}(\Omega)). \end{aligned}$$

In Section 6, we carefully estimate the nonlinear terms $\mathbf{v} \cdot \nabla \theta$ and $(\mathbf{v} \cdot \nabla) \mathbf{v}$ in order to show, via a fixed point method, that the closed-loop nonlinear system (6.1) admits a unique solution in some ball of X . More precisely we prove the following theorem in Section 6.2.

Theorem 1.1. *Let ε belong to $(0, 1/2)$. For a given $\omega > 0$, there exist a family of actuators $\mathbf{g}_i = (g_{\theta,i}, \mathbf{g}_{\mathbf{v},i}) \in H_{00}^{3/2}(\Gamma_c) \times \mathbf{H}_{00}^{3/2}(\Gamma_c)$ explicitly given in (5.5), a constant $\mu_0 > 0$ and a constant $C_0 > 0$ such that if $\mu \in (0, \mu_0)$ and if $\|(\theta_0, \mathbf{v}_0)\|_{H^\varepsilon(\Omega) \times \mathbf{V}_{n,\Gamma_d}^\varepsilon(\Omega)} \leq C_0 \mu$, then the system (6.1) admits a unique solution in the ball $B_\mu = \{(\theta, \mathbf{v}) \in X : \|e^{\omega t}(\theta, \mathbf{v})\|_X \leq \mu\}$, and it satisfies*

$$\|(\theta(t), \mathbf{v}(t))\|_{H^\varepsilon(\Omega) \times \mathbf{H}^\varepsilon(\Omega)} \leq C e^{-\omega t},$$

where C depends on $\|\theta_0\|_{H^\varepsilon(\Omega)}$ and $\|\mathbf{v}_0\|_{\mathbf{H}^\varepsilon(\Omega)}$.

For the definition of the spaces $H_{00}^{3/2}(\Gamma_c)$ and $\mathbf{H}_{00}^{3/2}(\Gamma_c)$ see Section 2.1.

Therefore, our paper provides a rigorous framework for models studied in [1]. Let us mention some additional references linked to our paper. The idea of using finite dimensional controllers for the stabilization of linear parabolic systems goes back to [11]. The stabilization of the Navier–Stokes system by means of finite dimensional feedback controllers is obtained in [12], in the case of an internal control. The case of a boundary control is studied in [13], [10], [14]. See also additional results in [15], under the more restrictive assumption that the unstable spectrum of the linearized operator is semi-simple.

2 Preliminaries

2.1 Functional framework and assumptions

We set $\mathbf{L}^2(\Omega) = H^0(\Omega; \mathbb{R}^2)$, $\mathbf{H}^s(\Omega) = H^s(\Omega; \mathbb{R}^2)$, $\forall s > 0$ and the same notation conventions will be used for trace spaces. We denote by $\mathbf{H}_{00}^{1/2}(\Gamma_c)$ (respectively $\mathbf{H}_{00}^{3/2}(\Gamma_c)$) the subset of functions belonging to $\mathbf{H}^{1/2}(\Gamma_c)$ (respectively $\mathbf{H}^{3/2}(\Gamma_c)$) whose extension by 0 to the whole boundary Γ belongs to $\mathbf{H}^{1/2}(\Gamma)$ (respectively $\mathbf{H}^{3/2}(\Gamma)$). A similar definition is valid for $H_{00}^{1/2}(\Gamma_d)$. Now we introduce the spaces

$$\mathbf{V}_{\Gamma_d}^1(\Omega) = \{ \mathbf{v} \in \mathbf{H}^1(\Omega) \mid \operatorname{div} \mathbf{v} = 0 \text{ in } \Omega, \mathbf{v} = \mathbf{0} \text{ on } \Gamma_d \},$$

$$\mathbf{V}_{n,\Gamma_d}^0(\Omega) \text{ is the closure of } \mathbf{V}_{\Gamma_d}^1(\Omega) \text{ in } \mathbf{L}^2(\Omega),$$

$$H_{\Gamma_n}^1(\Omega) = \{ p \in H^1(\Omega) \mid p = 0 \text{ on } \Gamma_n \}, \quad Z = L^2(\Omega) \times \mathbf{V}_{n,\Gamma_d}^0(\Omega).$$

From the above definition of $\mathbf{V}_{n,\Gamma_d}^0(\Omega)$, it follows that, for all $\mathbf{v} \in \mathbf{V}_{n,\Gamma_d}^0(\Omega)$, $\mathbf{v} \cdot \mathbf{n}|_{\Gamma_d} = 0$ as element in $(H_{00}^{1/2}(\Gamma_d))'$.

For $\varepsilon \in [0, 1/2)$, we define the spaces $H_{\Gamma_c}^{-1+\varepsilon}(\Omega)$ as the dual space of $H_{\Gamma_c}^{1-\varepsilon}(\Omega)$ with respect to $L^2(\Omega)$ and $\mathbf{H}_{\Gamma_d}^{-1+\varepsilon}(\Omega)$ as the dual space of $\mathbf{H}_{\Gamma_d}^{1-\varepsilon}(\Omega)$ with respect to $\mathbf{L}^2(\Omega)$.

Throughout the paper, the following assumptions are assumed to be satisfied.

(H₁) For the two dimensional domain Ω , the boundary Γ is split in the form $\Gamma = \bigcup_{i=1}^{N_s} \Gamma_i$, where each Γ_i is a submanifold of class C^2 and the boundary condition on Γ_i for the fluid velocity (or temperature) is only of one type, either Dirichlet or Neumann.

(H₂) The set of junction points between these submanifolds is $J = \{J_k \mid j = 1, \dots, N_J\}$. For each vertex $J_k \in J$, there exists $r_k > 0$ such that $\{x \in \mathbb{R}^2 \mid \text{dist}(x, J_k) \leq r_k\} \cap \Gamma$ is the union of two segments.

(H₃) If $\bar{\Gamma}_i \cap \bar{\Gamma}_j = \{J_k\}$, for some $1 \leq i, j \leq N_s$ and some $1 \leq k \leq N_J$, the angle at J_k interior to Ω is at most π if it corresponds to a Dirichlet–Neumann junction, and it is strictly less than 2π otherwise.

(H₄) The control zone Γ_c corresponds to one of the submanifolds of the family $(\Gamma_i)_{1 \leq i \leq N_s}$. We assume that Γ_c is of class C^3 .

(H₅) The solution $(\tau_s, \mathbf{u}_s, q_s)$ of the stationary problem (1.2) belongs to $H^{1+\varepsilon_0}(\Omega) \times \mathbf{H}^{1+\varepsilon_0}(\Omega) \times H^{\varepsilon_0}(\Omega)$ and $(\tau_s, \mathbf{u}_s)|_{\Omega_{c,\varepsilon}} \in H^{2+\varepsilon_0}(\Omega_{c,\varepsilon}) \times \mathbf{H}^{2+\varepsilon_0}(\Omega_{c,\varepsilon})$ for all $\varepsilon > 0$ and some $\varepsilon_0 > 0$, where $\Omega_{c,\varepsilon} = \{x \in \Omega \mid \text{dist}(x, \Gamma \setminus \Gamma_c) > \varepsilon\}$.

In our work, we need weighted Sobolev spaces for further analysis. For all $-1 < \delta < 1$, $s \in \mathbb{N}$, we introduce $H_\delta^s(\Omega; \mathbb{R}^2)$, the closure of $C^\infty(\bar{\Omega})$ in the norms

$$\|v\|_{H_\delta^s(\Omega; \mathbb{R}^2)}^2 = \sum_{|\alpha| \leq s} \int_{\Omega} r^{2\delta} |\partial_\alpha v|^2,$$

where r stands for the distance to J , $\alpha = (\alpha_1, \alpha_2) \in \mathbb{N}^2$ denotes a two-index, $|\alpha| = \alpha_1 + \alpha_2$ and ∂_α denotes the corresponding partial differential operator. We denote the weighted Sobolev spaces as

$$\mathbf{L}_\delta^2(\Omega) = H_\delta^0(\Omega; \mathbb{R}^2) \quad \text{and} \quad \mathbf{H}_\delta^s(\Omega) = H_\delta^s(\Omega; \mathbb{R}^2), \quad \forall s > 0.$$

Note that, following [16], we can also define the space $H_\delta^s(\Omega)$ when $s > 0$ is not an integer. We also recall that [16, Theorem 3]

$$H_\delta^s(\Omega) \hookrightarrow H^{s-\delta}(\Omega) \quad \text{for all } s \geq \delta \geq 0,$$

and the trace operator γ_0 is continuous from $H_\delta^2(\Omega)$ into $H^{3/2-\delta}(\Gamma)$ for all $0 < \delta < 3/2$. We also introduce the following time-dependent weighted Sobolev space

$$\mathbf{H}_\delta^{2,1}(\Omega) = L^2(0, T; \mathbf{H}_\delta^2(\Omega)) \cap H^1(0, T; \mathbf{L}^2(\Omega)).$$

2.2 Some useful results

The orthogonal projection in $\mathbf{L}^2(\Omega)$ onto $\mathbf{V}_{n,\Gamma_d}^0(\Omega)$ is denoted by Π . The following result can be proved as in [2, Lemma 2.2]:

Lemma 2.1. *We have the following orthogonal decomposition*

$$\mathbf{L}^2(\Omega) = \mathbf{V}_{n,\Gamma_d}^0(\Omega) \oplus \nabla H_{\Gamma_n}^1(\Omega).$$

The orthogonal projection Π from $\mathbf{L}^2(\Omega)$ to $\mathbf{V}_{n,\Gamma_d}^0(\Omega)$ is defined by

$$\begin{aligned} \Pi \mathbf{u} &= \mathbf{u} - \nabla q_1 - \nabla q_2, \quad \text{where} \\ q_1 &\in H_0^1(\Omega), \quad \Delta q_1 = \operatorname{div} \mathbf{u} \text{ in } \Omega, \\ q_2 &\in H_{\Gamma_n}^1(\Omega), \quad \Delta q_2 = 0 \text{ in } \Omega, \quad \frac{\partial q_2}{\partial \mathbf{n}} = (\mathbf{u} - \nabla q_1) \cdot \mathbf{n} \text{ on } \Gamma_d. \end{aligned}$$

We have introduced weighted Sobolev spaces in order to study the regularity of solutions to the following elliptic equations

$$\lambda_0 \theta - \kappa \Delta \theta = f \text{ in } \Omega, \quad \theta = 0 \text{ on } \Gamma_c, \quad \frac{\partial \theta}{\partial \mathbf{n}} = 0 \text{ on } \Gamma \setminus \Gamma_c, \quad (2.1)$$

and

$$\begin{aligned} \lambda_0 \mathbf{v} - \operatorname{div} \sigma(\mathbf{v}, p) &= \mathbf{h} \text{ in } \Omega, \\ \operatorname{div} \mathbf{v} &= 0 \text{ in } \Omega, \quad \mathbf{v} = \mathbf{0} \text{ on } \Gamma_d, \quad \sigma(\mathbf{v}, p) \mathbf{n} = \mathbf{0} \text{ on } \Gamma_n, \end{aligned} \quad (2.2)$$

where $\lambda_0 \geq 0$, $f \in L^2(\Omega)$, and $\mathbf{h} \in \mathbf{L}^2(\Omega)$. It is well known that equation (2.1) admits a unique solution θ in $H^1(\Omega)$, and that the system (2.2) admits a unique solution $(\mathbf{v}, p) \in \mathbf{H}^1(\Omega) \times L^2(\Omega)$. Under the assumptions made on Γ in Section 2.1, we can prove the following proposition.

Proposition 2.2. *The solution $\theta \in H^1(\Omega)$ to equation (2.1) belongs to $H_\delta^2(\Omega)$ and satisfies*

$$\|\theta\|_{H_\delta^2(\Omega)} \leq C_\delta \|f\|_{L^2(\Omega)},$$

for all $\delta \in (1/2, 1)$. The solution $(\mathbf{v}, p) \in \mathbf{H}^1(\Omega) \times L^2(\Omega)$ to system (2.2) belongs to $\mathbf{H}_\delta^2(\Omega) \times H_\delta^1(\Omega)$ for all $\delta \in (1/2, 1)$, and it satisfies

$$\|\mathbf{v}\|_{\mathbf{H}_\delta^2(\Omega)} + \|p\|_{H_\delta^1(\Omega)} \leq C_\delta \|\mathbf{h}\|_{\mathbf{L}^2(\Omega)}.$$

Proof. Since the angle of junctions between Dirichlet–Neumann (respectively Neumann–Neumann and Dirichlet–Dirichlet) is less than or equal to π (respectively less than 2π), the regularity for θ follows from [4, Theorem 6.4.6], and the regularity for (\mathbf{v}, p) follows from [4, Theorem 9.4.5]. \square

The existence result for stationary Navier–Stokes equation for polyhedral domain is given in [4, Theorem 11.2.1]. We can adapt this idea to establish the existence and regularity result for nonlinear stationary Boussinesq system (1.2).

Theorem 2.3. *Let $f_s \in H_{\Gamma_c}^{-1}$, $(g_{\tau,s}, \mathbf{g}_{\mathbf{u},s}) \in H_{00}^{1/2}(\Gamma_c) \times \mathbf{H}_{00}^{1/2}(\Gamma_c)$. There exist constants $\varepsilon_1 > 0$ and $C_1 > 0$ such that if*

$$\|(g_{\tau,s}, \mathbf{g}_{\mathbf{u},s})\|_{H^{1/2}(\Gamma_c) \times \mathbf{H}^{1/2}(\Gamma_c)} + \|f_s\|_{H_{\Gamma_c}^{-1}(\Omega)} \leq C_1 \varepsilon_1, \quad (2.3)$$

then the problem (1.2) admits a solution in the ball

$$B_{\varepsilon_1} = \{(\tau, \mathbf{u}, q) \in H^1(\Omega) \times \mathbf{H}^1(\Omega) \times L^2(\Omega) : \|(\tau, \mathbf{u}, q)\|_{H^1(\Omega) \times \mathbf{H}^1(\Omega) \times L^2(\Omega)} \leq \varepsilon_1\}.$$

If moreover $f_s \in L^2(\Omega)$, $(g_{\tau,s}, \mathbf{g}_{\mathbf{u},s}) \in H_{00}^{3/2}(\Gamma_c) \times \mathbf{H}_{00}^{3/2}(\Gamma_c)$, then $(\tau, \mathbf{u}, q) \in H^{1+\varepsilon_0}(\Omega) \times \mathbf{H}^{1+\varepsilon_0}(\Omega) \times H^{\varepsilon_0}(\Omega)$, for some $\varepsilon_0 > 0$.

3 Stationary linearized system

To define the different operators involved in the Boussinesq system, we first consider the following stationary equation

$$\begin{aligned}
& \lambda_0 \theta - \kappa \Delta \theta + \mathbf{u}_s \cdot \nabla \theta + \mathbf{v} \cdot \nabla \tau_s = f \text{ in } \Omega, \\
& \theta = \theta_c \text{ on } \Gamma_c, \quad \frac{\partial \theta}{\partial n} = 0 \text{ on } \Gamma \setminus \Gamma_c, \\
& \lambda_0 \mathbf{v} - \operatorname{div} \sigma(\mathbf{v}, p) + (\mathbf{u}_s \cdot \nabla) \mathbf{v} + (\mathbf{v} \cdot \nabla) \mathbf{u}_s - \beta \theta = \mathbf{h} \text{ in } \Omega, \\
& \operatorname{div} \mathbf{v} = 0 \text{ in } \Omega, \\
& \mathbf{v} = \mathbf{v}_c \text{ on } \Gamma_c, \quad \mathbf{v} = \mathbf{0} \text{ on } \Gamma_w, \quad \sigma(\mathbf{v}, p) \mathbf{n} = 0 \text{ on } \Gamma_n,
\end{aligned} \tag{3.1}$$

where $\lambda_0 > 0$ is chosen later on, $\mathbf{v}_c \in \mathbf{H}_{00}^{3/2}(\Gamma_c)$, $\theta_c \in H_{00}^{3/2}(\Gamma_c)$, and $(f, \mathbf{h}) \in L^2(\Omega) \times \mathbf{L}^2(\Omega)$.

Definition 3.1. *Let δ belong to $(1/2, 1)$. A triplet $(\theta, \mathbf{v}, p) \in H_\delta^2(\Omega) \times \mathbf{H}_\delta^2(\Omega) \times H_\delta^1(\Omega)$ is a solution to (3.1) iff the equations (3.1)₁, (3.1)₃, (3.1)₄ are satisfied in the sense of distributions, and (3.1)₂ and (3.1)₅ are satisfied in the sense of traces.*

To prove the existence and uniqueness of solution to system (3.1), via the mixed (variational) formulation, we introduce the continuous bilinear form on $H^1(\Omega) \times \mathbf{H}^1(\Omega)$ defined by

$$\begin{aligned}
a((\theta, \mathbf{v}), (\xi, \phi)) &= \int_{\Omega} (\lambda_0 \theta \xi + \kappa \nabla \theta \cdot \nabla \xi + (\mathbf{u}_s \cdot \nabla \theta) \xi + (\mathbf{v} \cdot \nabla \tau_s) \xi) dx \\
&+ \int_{\Omega} (\lambda_0 \mathbf{v} \cdot \phi + 2\nu D\mathbf{v} : D\phi + [(\mathbf{u}_s \cdot \nabla) \mathbf{v} + (\mathbf{v} \cdot \nabla) \mathbf{u}_s] \cdot \phi - \theta \beta \cdot \phi) dx,
\end{aligned}$$

where $D\mathbf{v} = \frac{1}{2}(\nabla \mathbf{v} + (\nabla \mathbf{v})^T)$, and the continuous linear form on $\mathbf{H}^1(\Omega) \times L^2(\Omega)$ defined by

$$b(\phi, p) = \int_{\Omega} p \operatorname{div} \phi dx.$$

The mixed formulation for system (3.1) is

$$\begin{aligned}
& \text{Find } (\theta, \mathbf{v}, p) \in H^1(\Omega) \times \mathbf{H}^1(\Omega) \times L^2(\Omega) \text{ such that} \\
& \theta = \theta_c \text{ on } \Gamma_c \text{ and } \mathbf{v} = \mathbf{v}_c \text{ on } \Gamma_c, \mathbf{v} = \mathbf{0} \text{ on } \Gamma_w, \\
& a((\theta, \mathbf{v}), (\xi, \phi)) - b(\phi, p) = \int_{\Omega} (f \xi + \mathbf{h} \cdot \phi) \\
& \text{for all } (\xi, \phi) \in H_{\Gamma_c}^1(\Omega) \times \mathbf{H}_{\Gamma_d}^1(\Omega), \\
& b(\mathbf{v}, \psi) = 0 \text{ for all } \psi \in L^2(\Omega).
\end{aligned} \tag{3.2}$$

The variational formulation for system (3.1) is

$$\begin{aligned}
& \text{Find } (\theta, \mathbf{v}) \in H^1(\Omega) \times \mathbf{V}^1(\Omega) \text{ such that} \\
& \theta = \theta_c \text{ on } \Gamma_c \text{ and } \mathbf{v} = \mathbf{v}_c \text{ on } \Gamma_c, \mathbf{v} = \mathbf{0} \text{ on } \Gamma_w, \\
& a((\theta, \mathbf{v}), (\xi, \phi)) = \int_{\Omega} (f \xi + \mathbf{h} \cdot \phi) \quad \forall (\xi, \phi) \in H_{\Gamma_c}^1(\Omega) \times \mathbf{V}_{\Gamma_d}^1(\Omega).
\end{aligned} \tag{3.3}$$

The mixed formulation (3.2) will be used to analyze the regularity of solutions to equation (3.1), while the variational formulation (3.3) will be used to write equation (3.1) in an operator form. Later we will prove the equivalence of these formulations and of Definition 3.1.

3.1 Homogeneous boundary conditions

Proposition 3.2. *There exists $\lambda_0 > 0$ depending only on $\kappa, \nu, \mathbf{u}_s, \tau_s$ and β such that, for all $(\theta, \mathbf{v}) \in H_{\Gamma_c}^1(\Omega) \times \mathbf{V}_{\Gamma_d}^1(\Omega)$, we have*

$$a((\theta, \mathbf{v}), (\theta, \mathbf{v})) \geq \frac{1}{2} \min(\kappa, 2\nu) \|(\theta, \mathbf{v})\|_{H_{\Gamma_c}^1(\Omega) \times \mathbf{V}_{\Gamma_d}^1(\Omega)}^2. \quad (3.4)$$

Proof. The proposition is a direct consequence of classical majorizations combined with Korn's and Young's inequalities. \square

Consider the linearized stationary Boussinesq system with homogeneous boundary conditions

$$\begin{aligned} \lambda_0 \theta - \kappa \Delta \theta + \mathbf{u}_s \cdot \nabla \theta + \mathbf{v} \cdot \nabla \tau_s &= f \text{ in } \Omega, \\ \theta &= 0 \text{ on } \Gamma_c, \quad \frac{\partial \theta}{\partial n} = 0 \text{ on } \Gamma \setminus \Gamma_c, \\ \lambda_0 \mathbf{v} - \operatorname{div} \sigma(\mathbf{v}, p) + (\mathbf{u}_s \cdot \nabla) \mathbf{v} + (\mathbf{v} \cdot \nabla) \mathbf{u}_s - \beta \theta &= \mathbf{h} \text{ in } \Omega, \\ \operatorname{div} \mathbf{v} &= 0 \text{ in } \Omega, \quad \mathbf{v} = 0 \text{ on } \Gamma_d, \quad \sigma(\mathbf{v}, p) \mathbf{n} = 0 \text{ on } \Gamma_n. \end{aligned} \quad (3.5)$$

Proposition 3.3. *Let λ_0 be large so that (3.4) holds. Let (f, \mathbf{h}) belong to $L^2(\Omega) \times \mathbf{L}^2(\Omega)$ and assume that $\theta_c = 0, \mathbf{v}_c = \mathbf{0}$. The variational formulation (3.3) admits a unique solution (θ, \mathbf{v}) .*

The mixed formulation (3.2) admits a unique solution (θ, \mathbf{v}, p) . This solution belongs to $H_\delta^2(\Omega) \times \mathbf{H}_\delta^2(\Omega) \times H_\delta^1(\Omega)$ for all $\delta \in (1/2, 1)$, and we have

$$\|\theta\|_{H_\delta^2(\Omega)} + \|\mathbf{v}\|_{\mathbf{H}_\delta^2(\Omega)} + \|p\|_{H_\delta^1(\Omega)} \leq C_\delta (\|f\|_{L^2(\Omega)} + \|\mathbf{h}\|_{\mathbf{L}^2(\Omega)}). \quad (3.6)$$

Proof. The existence and uniqueness of (θ, \mathbf{v}, p) belonging to $H_{\Gamma_c}^1(\Omega) \times \mathbf{V}_{\Gamma_d}^1(\Omega) \times L^2(\Omega)$ follows from the Lax–Milgram Lemma and from the surjectivity of the divergence map as in [4, Theorem 9.1.5]. To prove the regularity, we can write (3.5) in the form

$$\begin{aligned} \lambda_0 \mathbf{v} - \operatorname{div} \sigma(\mathbf{v}, p) &= \mathbf{h} + \beta \theta - (\mathbf{u}_s \cdot \nabla) \mathbf{v} - (\mathbf{v} \cdot \nabla) \mathbf{u}_s \text{ in } \Omega, \\ \operatorname{div} \mathbf{v} &= 0 \text{ in } \Omega, \quad \mathbf{v} = 0 \text{ on } \Gamma_d, \quad \sigma(\mathbf{v}, p) \mathbf{n} = 0 \text{ on } \Gamma_n, \\ \lambda_0 \theta - \kappa \Delta \theta &= f - \mathbf{u}_s \cdot \nabla \theta - \mathbf{v} \cdot \nabla \tau_s \text{ in } \Omega, \quad \theta = 0 \text{ on } \Gamma_c, \\ \frac{\partial \theta}{\partial n} &= 0 \text{ on } \Gamma \setminus \Gamma_c. \end{aligned}$$

Since $\mathbf{u}_s \in \mathbf{H}^{1+\varepsilon_0}(\Omega)$, we have $(\mathbf{h} + \beta \theta - (\mathbf{u}_s \cdot \nabla) \mathbf{v} - (\mathbf{v} \cdot \nabla) \mathbf{u}_s) \in \mathbf{L}^2(\Omega)$. From Proposition 2.2, it follows that $(\mathbf{v}, p) \in \mathbf{H}_\delta^2(\Omega) \times H_\delta^1(\Omega)$. As $(\mathbf{u}_s, \tau_s) \in \mathbf{H}^{1+\varepsilon_0}(\Omega) \times H^{1+\varepsilon_0}(\Omega)$, we have $(f - \mathbf{u}_s \cdot \nabla \theta - \mathbf{v} \cdot \nabla \tau_s) \in L^2(\Omega)$. From Proposition 2.2, it follows that $\theta \in H_\delta^2(\Omega)$. \square

3.2 Green's formulas

Now we state Green's formulas for the temperature and velocity separately.

Theorem 3.4. Let δ belong to $(1/2, 1)$, and ε be positive. Let us set $\Gamma_{c,\varepsilon} = \{x \in \Gamma \mid \text{dist}(x, \Gamma_c) < \varepsilon\}$, and let us define

$$H_{\Gamma_{c,\varepsilon}}^1(\Omega) = \{\theta \in H_{\Gamma_c}^1 \mid \theta|_{\Gamma_{c,\varepsilon}} = 0\}.$$

For all $\theta \in H_\delta^2(\Omega)$ and for all $\xi \in H_{\Gamma_{c,\varepsilon}}^1(\Omega)$, we have

$$\langle \xi, \Delta \theta \rangle_{L_{-\delta}^2(\Omega), L_\delta^2(\Omega)} = - \int_\Omega \nabla \theta \cdot \nabla \xi + \int_{\Gamma \setminus \Gamma_{c,\varepsilon}} \xi \frac{\partial \theta}{\partial n}. \quad (3.7)$$

If in addition $\frac{\partial \theta}{\partial n} \in L^2(\Gamma \setminus \Gamma_c)$, then

$$\langle \xi, \Delta \theta \rangle_{L_{-\delta}^2(\Omega), L_\delta^2(\Omega)} = - \int_\Omega \nabla \theta \cdot \nabla \xi + \int_{\Gamma \setminus \Gamma_c} \xi \frac{\partial \theta}{\partial n}, \quad \text{for all } \xi \in H_{\Gamma_c}^1(\Omega). \quad (3.8)$$

Proof. Let us first notice that with the help of Hardy's inequality ([4, page 223])

$$\int_\Omega r^{-2\delta} |u|^2 dx \leq C \int_\Omega r^{2(1-\delta)} |\nabla u|^2 dx,$$

we have $H_{\Gamma_c}^1(\Omega) \subset L_{-\delta}^2(\Omega)$, for $\delta \in (1/2, 1)$. The Green's formula (3.7) can be first proved for $\theta \in C^\infty(\overline{\Omega})$ and $\xi \in C^\infty(\overline{\Omega}) \cap H_{\Gamma_{c,\varepsilon}}^1(\Omega)$. Next the theorem follows from the density of $C^\infty(\overline{\Omega})$ in $H_\delta^2(\Omega)$ and that of $C^\infty(\overline{\Omega}) \cap H_{\Gamma_{c,\varepsilon}}^1(\Omega)$ in $H_{\Gamma_{c,\varepsilon}}^1(\Omega)$. The second Green's formula (3.8) can be proved by using the density of $H_{\Gamma_{c,\varepsilon}}^1(\Omega)$ into $H_{\Gamma_c}^1(\Omega)$. \square

Theorem 3.5. Let δ belong to $(1/2, 1)$, and ε be positive. Let us set $\Gamma_{d,\varepsilon} = \{x \in \Gamma \mid \text{dist}(x, \Gamma_d) < \varepsilon\}$, and let us define

$$\mathbf{H}_{\Gamma_{d,\varepsilon}}^1(\Omega) = \{\mathbf{v} \in \mathbf{H}_{\Gamma_d}^1 \mid \mathbf{v}|_{\Gamma_{d,\varepsilon}} = 0\}.$$

For all $(\mathbf{v}, p) \in \mathbf{H}_\delta^2(\Omega) \times H_\delta^1(\Omega)$ and for all $\phi \in \mathbf{H}_{\Gamma_{d,\varepsilon}}^1(\Omega)$, we have

$$\langle \phi, \text{div } \sigma(\mathbf{v}, p) \rangle_{\mathbf{L}_{-\delta}^2(\Omega), \mathbf{L}_\delta^2(\Omega)} = -2\nu \int_\Omega D\mathbf{v} : D\phi + \int_\Omega p \text{div } \phi + \int_{\Gamma \setminus \Gamma_{d,\varepsilon}} \sigma(\mathbf{v}, p) \mathbf{n} \cdot \phi. \quad (3.9)$$

If in addition $\sigma(\mathbf{v}, p) \mathbf{n} \in \mathbf{L}^2(\Gamma_n)$, then we have

$$\langle \phi, \text{div } \sigma(\mathbf{v}, p) \rangle_{\mathbf{L}_{-\delta}^2(\Omega), \mathbf{L}_\delta^2(\Omega)} = -2\nu \int_\Omega D\mathbf{v} : D\phi + \int_\Omega p \text{div } \phi + \int_{\Gamma_n} \sigma(\mathbf{v}, p) \mathbf{n} \cdot \phi, \quad (3.10)$$

for all $\phi \in \mathbf{H}_{\Gamma_d}^1(\Omega)$.

Proof. The Green's formula (3.10) can be first proved for $(\mathbf{v}, p) \in \mathbf{C}^\infty(\overline{\Omega}) \times C^\infty(\overline{\Omega})$ and $\phi \in \mathbf{C}^\infty(\overline{\Omega}) \cap \mathbf{H}_{\Gamma_{d,\varepsilon}}^1(\Omega)$. Next (3.10) follows from the density of $\mathbf{C}^\infty(\overline{\Omega})$ in $\mathbf{H}_\delta^2(\Omega)$, of $\mathbf{C}^\infty(\overline{\Omega}) \cap \mathbf{H}_{\Gamma_{d,\varepsilon}}^1(\Omega)$ in $\mathbf{H}_{\Gamma_{d,\varepsilon}}^1(\Omega)$, and of $C^\infty(\overline{\Omega})$ in $H_\delta^1(\Omega)$. The second Green's formula (3.9) follows from the density of $\mathbf{H}_{\Gamma_{d,\varepsilon}}^1(\Omega)$ into $\mathbf{H}_{\Gamma_d}^1(\Omega)$. \square

These Green's formulas are used to establish the equivalence between Definition 3.1, mixed formulation (3.2), and variational formulation (3.3).

Theorem 3.6. *Let λ_0 be large so that (3.4) holds. Let (f, \mathbf{h}) belong to $L^2(\Omega) \times \mathbf{L}^2(\Omega)$, and assume that $\theta_c = 0$, $\mathbf{v}_c = \mathbf{0}$.*

(1) *A triplet $(\theta, \mathbf{v}, p) \in H_\delta^2(\Omega) \times \mathbf{H}_\delta^2(\Omega) \times H_\delta^1(\Omega)$ is a solution to (3.5) in the sense of Definition 3.1 if and only if (θ, \mathbf{v}, p) is a solution to the mixed formulation (3.2).*

(2) *If (θ, \mathbf{v}, p) is a solution to the mixed formulation (3.2), then (θ, \mathbf{v}) is a solution to the variational formulation (3.3).*

(3) *If (θ, \mathbf{v}) is a solution to the variational formulation (3.3), then there exists $p \in L^2(\Omega)$ such that (θ, \mathbf{v}, p) is a solution to the mixed formulation (3.2).*

Proof. (1) Let $(\theta, \mathbf{v}, p) \in H_\delta^2(\Omega) \times \mathbf{H}_\delta^2(\Omega) \times H_\delta^1(\Omega)$ be solution of equation (3.5). Multiplying (3.5)₁ by $\xi \in H_{\Gamma_c}^1(\Omega)$ and (3.5)₃ by $\phi \in \mathbf{H}_{\Gamma_d}^1(\Omega)$ and using the Green's formulas (3.8)–(3.10), we obtain the mixed variational formulation.

Conversely, assume that $(\theta, \mathbf{v}, p) \in H_\delta^2(\Omega) \times \mathbf{H}_\delta^2(\Omega) \times H_\delta^1(\Omega)$ satisfies the mixed variational formulation (3.2) with $\theta_c = 0$ and $\mathbf{v}_c = \mathbf{0}$. By using the Green's formulas (3.8)–(3.10) successively with test functions $(\xi, \mathbf{0})$ and $(0, \phi)$ for $\xi \in C_c^\infty(\Omega)$ and $\phi \in C_c^\infty(\Omega; \mathbb{R}^2)$ in (3.2), we can prove that (θ, \mathbf{v}, p) satisfies (3.5)₁ and (3.5)₃ in the sense of distributions.

Now, by using equation (3.5)₁, (3.7) and the mixed formulation (3.2) with $(\xi, \mathbf{0})$, $\xi \in H_{\Gamma_{c,\varepsilon}}^1(\Omega)$, we obtain

$$\int_{\Gamma \setminus \Gamma_{c,\varepsilon}} \frac{\partial \theta}{\partial n} \xi = 0, \quad \forall \xi \in H_{\Gamma_{c,\varepsilon}}^1(\Omega). \quad (3.11)$$

Thus $\frac{\partial \theta}{\partial n} = 0$ on $\Gamma \setminus \Gamma_{c,\varepsilon}$. Since this is true for all $\varepsilon > 0$, we conclude that $\frac{\partial \theta}{\partial n} = 0$ on $\Gamma \setminus \Gamma_c$. With similar arguments, we can also recover the boundary condition $\sigma(\mathbf{v}, p)\mathbf{n} = 0$ on Γ_n .

(2) It is very easy to see that if we choose $(\xi, \phi) \in H_{\Gamma_c}^1(\Omega) \times \mathbf{V}_{\Gamma_d}^1(\Omega)$ in (3.2), then we obtain the variational formulation (3.3).

(3) This follows from the uniqueness of solution (θ, \mathbf{v}) to the variational formulation (3.3) and the uniqueness of (θ, \mathbf{v}, p) to the mixed formulation (3.2) proved in Proposition 3.3. \square

We define the unbounded operator $(\mathcal{A}, D(\mathcal{A}))$ in Z by

$$\begin{aligned} D(\mathcal{A}) &= \{(\theta, \mathbf{v}) \in H_{\Gamma_c}^1(\Omega) \times \mathbf{V}_{\Gamma_d}^1(\Omega) \mid (\xi, \phi) \mapsto a((\theta, \mathbf{v}), (\xi, \phi)) \text{ is} \\ &\quad \text{continuous in } Z\}, \quad \text{and} \\ \langle (\lambda_0 I - \mathcal{A})(\theta, \mathbf{v}), (\xi, \phi) \rangle_Z &= a((\theta, \mathbf{v}), (\xi, \phi)) \\ \text{for all } (\theta, \mathbf{v}) \in D(\mathcal{A}), (\xi, \phi) &\in H_{\Gamma_c}^1(\Omega) \times \mathbf{V}_{\Gamma_d}^1(\Omega). \end{aligned}$$

Theorem 3.7. *The operator $(\mathcal{A}, D(\mathcal{A}))$ is the infinitesimal generator of a strongly continuous analytic semigroup on Z , and its resolvent is compact.*

Proof. Using the coercivity of the bilinear form $a((\theta, \mathbf{v}), (\theta, \mathbf{v}))$ in $H_{\Gamma_c}^1(\Omega) \times \mathbf{V}_{\Gamma_d}^1(\Omega)$, we conclude from [17, Theorem 2.12, page 115] that $(\mathcal{A}, D(\mathcal{A}))$ generates an analytic semigroup on Z . From

Proposition 3.3, it follows that $(\lambda_0 I - \mathcal{A})^{-1}$ is bounded from Z into $H_{\Gamma_c}^1(\Omega) \times \mathbf{V}_{\Gamma_d}^1(\Omega)$. Since, the imbedding from $H_{\Gamma_c}^1(\Omega) \times \mathbf{V}_{\Gamma_d}^1(\Omega)$ into Z is compact, the resolvent of \mathcal{A} is compact in Z . \square

3.3 Adjoint problem

Let us introduce the adjoint equation

$$\begin{aligned} \lambda_0 \xi - \kappa \Delta \xi - \mathbf{u}_s \cdot \nabla \xi - \boldsymbol{\beta} \cdot \boldsymbol{\phi} &= f \text{ in } \Omega, \\ \xi &= 0 \text{ on } \Gamma_c, \quad \kappa \frac{\partial \xi}{\partial n} + (\mathbf{u}_s \cdot \mathbf{n}) \xi = 0 \text{ on } \Gamma \setminus \Gamma_c, \\ \lambda_0 \boldsymbol{\phi} - \operatorname{div} \sigma(\boldsymbol{\phi}, \psi) - (\mathbf{u}_s \cdot \nabla) \boldsymbol{\phi} + (\nabla \mathbf{u}_s)^T \boldsymbol{\phi} + (\nabla \tau_s)^T \xi &= \mathbf{h} \text{ in } \Omega, \\ \operatorname{div} \boldsymbol{\phi} &= 0 \text{ in } \Omega, \quad \boldsymbol{\phi} = 0 \text{ on } \Gamma_d, \quad \sigma(\boldsymbol{\phi}, \psi) \mathbf{n} + (\mathbf{u}_s \cdot \mathbf{n}) \boldsymbol{\phi} = 0 \text{ on } \Gamma_n, \end{aligned} \quad (3.12)$$

where $(f, \mathbf{h}) \in L^2(\Omega) \times \mathbf{L}^2(\Omega)$. We define the following bilinear form $a^\#$ and the linear form ℓ , on $H_{\Gamma_c}^1(\Omega) \times \mathbf{V}_{\Gamma_d}^1(\Omega)$, by

$$\begin{aligned} a^\#((\xi, \boldsymbol{\phi}), (\theta, \mathbf{v})) &= \int_{\Omega} (\lambda_0 \xi \theta + \kappa \nabla \xi \cdot \nabla \theta - (\mathbf{u}_s \cdot \nabla \xi) \theta - (\boldsymbol{\beta} \cdot \boldsymbol{\phi}) \theta) dx \\ &+ \int_{\Omega} \left(\lambda_0 \boldsymbol{\phi} \cdot \mathbf{v} + 2\nu D\boldsymbol{\phi} : D\mathbf{v} - ((\mathbf{u}_s \cdot \nabla) \boldsymbol{\phi}) \cdot \mathbf{v} + (\nabla \mathbf{u}_s)^T \boldsymbol{\phi} \cdot \mathbf{v} + (\nabla \tau_s)^T \xi \cdot \mathbf{v} \right) dx \\ &+ \int_{\Gamma \setminus \Gamma_c} (\mathbf{u}_s \cdot \mathbf{n}) \xi \theta dx + \int_{\Gamma_n} \mathbf{v} \cdot (\mathbf{u}_s \cdot \mathbf{n}) \boldsymbol{\phi} dx, \\ \ell(\theta, \mathbf{v}) &= \int_{\Omega} (f\theta + \mathbf{h} \cdot \mathbf{v}). \end{aligned}$$

The mixed (variational) formulation for equation (3.12) is defined by

$$\begin{aligned} \text{Find } (\xi, \boldsymbol{\phi}, \psi) &\in H_{\Gamma_c}^1(\Omega) \times \mathbf{H}_{\Gamma_d}^1(\Omega) \times L^2(\Omega) \text{ such that} \\ a^\#((\xi, \boldsymbol{\phi}), (\theta, \mathbf{v})) - b(\mathbf{v}, \psi) &= \ell(\theta, \mathbf{v}) \quad \forall (\theta, \mathbf{v}) \in H_{\Gamma_c}^1(\Omega) \times \mathbf{H}_{\Gamma_d}^1(\Omega), \\ b(\boldsymbol{\phi}, p) &= 0 \text{ for all } p \in L^2(\Omega). \end{aligned} \quad (3.13)$$

We can prove that the mixed formulation (3.13) admits a unique solution $(\xi, \boldsymbol{\phi}, \psi) \in H_{\delta}^2(\Omega) \times \mathbf{H}_{\delta}^2(\Omega) \times H_{\delta}^1(\Omega)$ for all $\delta \in (\frac{1}{2}, 1)$, as in Proposition 3.3. Associated with $a^\#$, we introduce the operator $(\mathcal{A}^\#, D(\mathcal{A}^\#))$ defined by

$$D(\mathcal{A}^\#) = \{(\xi, \boldsymbol{\phi}) \in H_{\Gamma_c}^1(\Omega) \times \mathbf{V}_{\Gamma_d}^1(\Omega) \mid (\theta, \mathbf{v}) \mapsto a^\#((\xi, \boldsymbol{\phi}), (\theta, \mathbf{v})) \text{ is continuous in } Z\},$$

and $\langle (\lambda_0 I - \mathcal{A}^\#)(\xi, \boldsymbol{\phi}), (\theta, \mathbf{v}) \rangle_Z = a^\#((\xi, \boldsymbol{\phi}), (\theta, \mathbf{v}))$ for all $(\xi, \boldsymbol{\phi}) \in D(\mathcal{A}^\#)$ and $(\theta, \mathbf{v}) \in H_{\Gamma_c}^1(\Omega) \times \mathbf{V}_{\Gamma_d}^1(\Omega)$. Let $(\mathcal{A}^*, D(\mathcal{A}^*))$ be the adjoint of $(\mathcal{A}, D(\mathcal{A}))$. Since $a^\#((\xi, \boldsymbol{\phi}), (\theta, \mathbf{v})) = a((\theta, \mathbf{v}), (\xi, \boldsymbol{\phi}))$, for all $(\xi, \boldsymbol{\phi})$ and all (θ, \mathbf{v}) belonging to $H_{\Gamma_c}^1(\Omega) \times \mathbf{V}_{\Gamma_d}^1(\Omega)$, we have $D(\mathcal{A}^*) = D(\mathcal{A}^\#)$ and $\mathcal{A}^* = \mathcal{A}^\#$.

Let λ_0 be fixed so that (3.4) holds. Let us analyze the fractional powers of the domain of the operator \mathcal{A} and its adjoint \mathcal{A}^* used later on.

Theorem 3.8. *We have*

$$\begin{aligned}
D((\lambda_0 I - \mathcal{A})^{\frac{1}{2}}) &= H_{\Gamma_c}^1(\Omega) \times \mathbf{V}_{\Gamma_d}^1(\Omega), & D((\lambda_0 I - \mathcal{A}^*)^{\frac{1}{2}}) &= H_{\Gamma_c}^1(\Omega) \times \mathbf{V}_{\Gamma_d}^1(\Omega), \\
(D((\lambda_0 I - \mathcal{A}^*)^{\frac{1}{2}}))' &= H_{\Gamma_c}^{-1}(\Omega) \times \mathbf{V}_{\Gamma_d}^{-1}(\Omega).
\end{aligned} \tag{3.14}$$

Furthermore, for $\varepsilon \in (0, \frac{1}{2})$,

$$\begin{aligned}
(D((\lambda_0 I - \mathcal{A}^*)^{1/2-\varepsilon/2}))' &= [L^2(\Omega), H_{\Gamma_c}^{-1}(\Omega)]_{1-\varepsilon} \times [\mathbf{V}_{n, \Gamma_d}^0(\Omega), \mathbf{V}_{\Gamma_d}^{-1}(\Omega)]_{1-\varepsilon} \\
&= H_{\Gamma_c}^{-1+\varepsilon}(\Omega) \times \left(\mathbf{H}_{\Gamma_d}^{-1+\varepsilon}(\Omega) \cap \mathbf{V}_{\Gamma_d}^{-1}(\Omega) \right),
\end{aligned} \tag{3.15}$$

$$D((\lambda_0 I - \mathcal{A})^{1/2+\varepsilon/2}) \subset \left(H_{\Gamma_c}^1(\Omega) \cap H^{1+\eta(\varepsilon)}(\Omega) \right) \times \left(\mathbf{V}_{\Gamma_d}^1(\Omega) \cap \mathbf{H}^{1+\eta(\varepsilon)}(\Omega) \right), \tag{3.16}$$

where $\eta(\varepsilon) = (\frac{\varepsilon}{2} - \varepsilon^2)$.

Proof. Step 1. Proof of (3.14). The first and second identities in (3.14) follow by adapting the proof from [2, Theorem 2.13] to our case. For that it is sufficient to establish

$$[H_{\Gamma_c}^1(\Omega) \times \mathbf{V}_{\Gamma_d}^1(\Omega), H_{\Gamma_c}^{-1}(\Omega) \times \mathbf{V}_{\Gamma_d}^{-1}(\Omega)]_{1/2} = Z, \tag{3.17}$$

and to show that $(\lambda_0 I - \mathcal{A})$ is a closed maximal monotone operator in $H_{\Gamma_c}^{-1}(\Omega) \times \mathbf{V}_{\Gamma_d}^{-1}(\Omega)$. The last identity of (3.14) follows by duality arguments.

Step 2. Proof of (3.15). Now using (3.14), for $\varepsilon \in (0, 1/2)$, we have

$$D((\lambda_0 I - \mathcal{A}^*)^{1/2-\varepsilon/2}) = [H_{\Gamma_c}^1(\Omega) \times \mathbf{V}_{\Gamma_d}^1(\Omega), L^2(\Omega) \times \mathbf{V}_{n, \Gamma_d}^0(\Omega)]_{\varepsilon}.$$

By duality, we get the first identity in (3.15). To get the second identity in (3.15), as in [2, Lemma 2.12], we first note that

$$\mathbf{V}_{n, \Gamma_d}^0(\Omega) = \mathbf{L}^2(\Omega) \cap \mathbf{V}_{\Gamma_d}^{-1}(\Omega).$$

Then, for $\varepsilon \in (0, 1/2)$, we can write

$$\begin{aligned}
[\mathbf{V}_{n, \Gamma_d}^0(\Omega), \mathbf{V}_{\Gamma_d}^{-1}(\Omega)]_{1-\varepsilon} &= [\mathbf{L}^2(\Omega) \cap \mathbf{V}_{\Gamma_d}^{-1}(\Omega), \mathbf{H}_{\Gamma_d}^{-1}(\Omega) \cap \mathbf{V}_{\Gamma_d}^{-1}(\Omega)]_{1-\varepsilon} \\
&= [\mathbf{L}^2(\Omega), \mathbf{H}_{\Gamma_d}^{-1}(\Omega)]_{1-\varepsilon} \cap \mathbf{V}_{\Gamma_d}^{-1}(\Omega) = \mathbf{H}_{\Gamma_d}^{-1+\varepsilon}(\Omega) \cap \mathbf{V}_{\Gamma_d}^{-1}(\Omega).
\end{aligned} \tag{3.18}$$

We also have

$$[L^2(\Omega), H_{\Gamma_c}^{-1}(\Omega)]_{1-\varepsilon} = H_{\Gamma_c}^{-1+\varepsilon}(\Omega).$$

Combining both the results yields the second identity in (3.15).

Step 3. Proof of (3.16). Let us denote by (θ, \mathbf{v}) the solution to the equation

$$(\lambda_0 I - \mathcal{A})(\theta, \mathbf{v}) = (f, \mathbf{h}). \tag{3.19}$$

Let $\varepsilon \in (0, 1/2)$ be fixed. We know that $\lambda_0 I - \mathcal{A}$ is an isomorphism from $(D((\lambda_0 I - \mathcal{A}^*)^{1/2-\varepsilon/2}))'$ into $D((\lambda_0 I - \mathcal{A})^{1/2+\varepsilon/2})$. Thus we have the following estimate

$$\|(f, \mathbf{h})\|_{(D((\lambda_0 I - \mathcal{A}^*)^{1/2-\varepsilon/2}))'} \leq C \|(\theta, \mathbf{v})\|_{D((\lambda_0 I - \mathcal{A})^{1/2+\varepsilon/2})}. \tag{3.20}$$

Notice further that the solution mapping from (f, \mathbf{h}) to (θ, \mathbf{v}) is continuous from $H_{\Gamma_c}^{-1}(\Omega) \times \mathbf{V}_{\Gamma_d}^{-1}(\Omega)$ into $H_{\Gamma_c}^1(\Omega) \times \mathbf{V}_{\Gamma_d}^1(\Omega)$ and also from Z into $(H^{3/2-\varepsilon}(\Omega) \cap H_{\Gamma_c}^1(\Omega)) \times (\mathbf{H}^{3/2-\varepsilon}(\Omega) \cap \mathbf{V}_{\Gamma_d}^1(\Omega))$. Thus by interpolation, it is also continuous from $(D((\lambda_0 I - \mathcal{A}^*))^{1/2-\varepsilon/2})' = [L^2(\Omega) \times \mathbf{V}_{n, \Gamma_d}^0(\Omega), H_{\Gamma_c}^{-1}(\Omega) \times \mathbf{V}_{\Gamma_d}^{-1}(\Omega)]_{1-\varepsilon}$ into

$$\begin{aligned} [U] & [(H^{3/2-\varepsilon}(\Omega) \cap H_{\Gamma_c}^1(\Omega)) \times (\mathbf{H}^{3/2-\varepsilon}(\Omega) \cap \mathbf{V}_{\Gamma_d}^1(\Omega)), H_{\Gamma_c}^1 \times \mathbf{V}_{\Gamma_d}^1]_{1-\varepsilon} \\ & = (H^{1+\eta(\varepsilon)}(\Omega) \cap H_{\Gamma_c}^1(\Omega)) \times (\mathbf{H}^{1+\eta(\varepsilon)}(\Omega) \cap \mathbf{V}_{\Gamma_d}^1(\Omega)), \end{aligned}$$

where $\eta(\varepsilon) = \frac{\varepsilon}{2} - \varepsilon^2$. Thus we have

$$\|(\theta, \mathbf{v})\|_{H^{1+\eta(\varepsilon)} \times \mathbf{H}^{1+\eta(\varepsilon)}} \leq C \|(f, \mathbf{h})\|_{(D((\lambda_0 I - \mathcal{A}^*))^{1/2-\varepsilon/2})'}.$$

This continuity result, with estimate (3.20), gives the last inclusion in (3.16). \square

3.4 Nonhomogeneous boundary conditions

We now want to prove the existence of a solution (θ, \mathbf{v}, p) of the mixed formulation for the system (3.1) with nonhomogeneous boundary condition.

Theorem 3.9. *Assume that (f, \mathbf{h}) belongs to $L^2(\Omega) \times \mathbf{L}^2(\Omega)$ and (θ_c, \mathbf{v}_c) belongs to $H_{00}^{\frac{1}{2}}(\Gamma_c) \times \mathbf{H}_{00}^{\frac{1}{2}}(\Gamma_c)$. Then, the mixed formulation (3.2) admits a unique solution (θ, \mathbf{v}, p) .*

Moreover, if (θ_c, \mathbf{v}_c) belongs to $H_{00}^{\frac{3}{2}}(\Gamma_c) \times \mathbf{H}_{00}^{\frac{3}{2}}(\Gamma_c)$, then this solution belongs to $H_{\delta}^2(\Omega) \times \mathbf{H}_{\delta}^2(\Omega) \times H_{\delta}^1(\Omega)$ for all $\delta \in (\frac{1}{2}, 1)$, and it satisfies

$$\begin{aligned} [U] & \|\theta\|_{H_{\delta}^2(\Omega)} + \|\mathbf{v}\|_{\mathbf{H}_{\delta}^2(\Omega)} + \|p\|_{H_{\delta}^1(\Omega)} \\ & \leq C_{\delta} (\|f\|_{L^2(\Omega)} + \|\mathbf{h}\|_{\mathbf{L}^2(\Omega)} + \|\theta_c\|_{H^{\frac{3}{2}}(\Gamma_c)} + \|\mathbf{v}_c\|_{\mathbf{H}^{\frac{3}{2}}(\Gamma_c)}). \end{aligned} \quad (3.21)$$

Proof. The uniqueness follows from the uniqueness result stated in Proposition 3.3. To prove the existence of solution, we first consider the equation

$$\begin{aligned} \lambda_0 \hat{\theta} - \kappa \Delta \hat{\theta} &= f \text{ in } \Omega, \quad \hat{\theta} = \theta_c \text{ on } \Gamma_c, \quad \frac{\partial \hat{\theta}}{\partial \mathbf{n}} = 0 \text{ on } \Gamma \setminus \Gamma_c, \\ \lambda_0 \hat{\mathbf{v}} - \operatorname{div} \sigma(\hat{\mathbf{v}}, \hat{p}) - \beta \hat{\theta} &= \mathbf{h} \text{ in } \Omega, \\ \operatorname{div} \hat{\mathbf{v}} &= 0 \text{ in } \Omega, \quad \hat{\mathbf{v}} = 0 \text{ on } \Gamma_c, \quad \hat{\mathbf{v}} = 0 \text{ on } \Gamma_w, \quad \sigma(\hat{\mathbf{v}}, \hat{p}) \mathbf{n} = 0 \text{ on } \Gamma_n. \end{aligned} \quad (3.22)$$

The equation for $\hat{\theta}$ is decoupled from the equation for $(\hat{\mathbf{v}}, \hat{p})$, thus we can apply results already existing in the literature for each equation separately. Since $\theta_c \in H_{00}^{\frac{3}{2}}(\Gamma_c)$, from [4, Theorem 6.4.6], it follows that system (3.22)₁–(3.22)₂ admits a unique solution $\hat{\theta} \in H^1(\Omega)$, and that this solution belongs to $H_{\delta}^2(\Omega)$ for all $\delta \in (\frac{1}{2}, 1)$. Since $\mathbf{v}_c \in \mathbf{H}_{00}^{\frac{3}{2}}(\Gamma_c)$, from [4, Theorem 9.1.5, Theorem 9.4.5], it follows that system (3.22)₃–(3.22)₄ admits a unique solution $(\hat{\mathbf{v}}, \hat{p}) \in \mathbf{V}^1(\Omega) \times L^2(\Omega)$, and that this solution belongs to $\mathbf{H}_{\delta}^2(\Omega) \times H_{\delta}^1(\Omega)$ for all $\delta \in (1/2, 1)$.

We look for a solution (θ, \mathbf{v}, p) to the mixed formulation (3.2) of the form $(\theta, \mathbf{v}, p) = (\widehat{\theta}, \widehat{\mathbf{v}}, \widehat{p}) + (\widetilde{\theta}, \widetilde{\mathbf{v}}, \widetilde{p})$. Thus, $(\widetilde{\theta}, \widetilde{\mathbf{v}}, \widetilde{p})$ must satisfy

$$\begin{aligned}
\lambda_0 \widetilde{\theta} - \kappa \Delta \widetilde{\theta} + \mathbf{u}_s \cdot \nabla \widetilde{\theta} + \widetilde{\mathbf{v}} \cdot \nabla \tau_s &= -\mathbf{u}_s \cdot \nabla \widehat{\theta} - \widehat{\mathbf{v}} \cdot \nabla \tau_s \text{ in } \Omega, \\
\widetilde{\theta} &= 0 \text{ on } \Gamma_c, \quad \frac{\partial \widetilde{\theta}}{\partial n} = 0 \text{ on } \Gamma \setminus \Gamma_c, \\
\lambda_0 \widetilde{\mathbf{v}} - \operatorname{div} \sigma(\widetilde{\mathbf{v}}, \widetilde{p}) + (\mathbf{u}_s \cdot \nabla) \widetilde{\mathbf{v}} + (\widetilde{\mathbf{v}} \cdot \nabla) \mathbf{u}_s - \beta \widetilde{\theta}_1 &= \\
-(\mathbf{u}_s \cdot \nabla) \widehat{\mathbf{v}} + (\widehat{\mathbf{v}} \cdot \nabla) \mathbf{u}_s &\text{ in } \Omega, \\
\operatorname{div} \widetilde{\mathbf{v}} &= 0 \text{ in } \Omega, \quad \widetilde{\mathbf{v}} = 0 \text{ on } \Gamma_c, \quad \widetilde{\mathbf{v}} = 0 \text{ on } \Gamma_w, \quad \sigma(\widetilde{\mathbf{v}}, \widetilde{p}) n = 0 \text{ on } \Gamma_n.
\end{aligned} \tag{3.23}$$

Since the right hand side in (3.23)₁ (respectively (3.23)₄) belongs to $L^2(\Omega)$ (respectively $\mathbf{L}^2(\Omega)$), from Proposition 3.3, it follows that this system has a unique solution belonging to $H_\delta^2(\Omega) \times \mathbf{H}_\delta^2(\Omega) \times H_\delta^1(\Omega)$ for all $\delta \in (\frac{1}{2}, 1)$. \square

Remark 3.10. *We can prove the equivalence of the three formulations for the solution as before, even when (f, \mathbf{h}) belong to $L^2(\Omega) \times \mathbf{L}^2(\Omega)$ and (θ_c, \mathbf{v}_c) belongs to $H_{00}^{\frac{1}{2}}(\Gamma_c) \times \mathbf{H}_{00}^{\frac{1}{2}}(\Gamma_c)$ by using Theorem 3.9.*

3.5 Rewriting equation (3.1) in the operator form

We define the Dirichlet operators $G \in \mathcal{L}(H_{00}^{1/2}(\Gamma_c) \times \mathbf{H}_{00}^{1/2}(\Gamma_c), H^1(\Omega) \times \mathbf{H}^1(\Omega))$ and $G_p \in \mathcal{L}(H_{00}^{1/2}(\Gamma_c) \times \mathbf{H}_{00}^{1/2}(\Gamma_c), L^2(\Omega))$ by

$$G(\theta_c, \mathbf{v}_c) = (G_\theta(\theta_c, \mathbf{v}_c), G_\mathbf{v}(\theta_c, \mathbf{v}_c))^T = (\theta, \mathbf{v})^T, \quad G_p(\theta_c, \mathbf{v}_c) = p, \tag{3.24}$$

where (θ, \mathbf{v}, p) satisfies equation (3.1) with $(f, \mathbf{h}) = (0, \mathbf{0})$.

Due to Theorem 3.9, the operators G and G_p are continuous linear operators from $(H_{00}^{3/2}(\Gamma_c) \times \mathbf{H}_{00}^{3/2}(\Gamma_c))$ into $(H_\delta^2(\Omega) \times \mathbf{H}_\delta^2(\Omega))$ and from $(H_{00}^{3/2}(\Gamma_c) \times \mathbf{H}_{00}^{3/2}(\Gamma_c))$ into $H_\delta^1(\Omega)$ respectively.

We can extend the operator \mathcal{A} to an unbounded operator $\widetilde{\mathcal{A}}$ with domain $D(\widetilde{\mathcal{A}}) = L^2(\Omega) \times \mathbf{V}_n^0(\Omega)$ in $(D(\mathcal{A}^*))'$, in order that $(\widetilde{\mathcal{A}}, D(\widetilde{\mathcal{A}}))$ is the infinitesimal generator of an analytic semi-group on $(D(\mathcal{A}^*))'$, with compact resolvent. Later on, we shall denote this extension by \mathcal{A} .

Theorem 3.11. *Let $(\theta_c, \mathbf{v}_c) \in H_{00}^{3/2}(\Gamma_c) \times \mathbf{H}_{00}^{3/2}(\Gamma_c)$. Let us set*

$$B(\theta_c, \mathbf{v}_c) = (\lambda_0 I - \mathcal{A}) \widetilde{\Pi} G(\theta_c, \mathbf{v}_c), \tag{3.25}$$

where G is defined in 3.24 and $\widetilde{\Pi} = \begin{pmatrix} \operatorname{Id} & 0 \\ 0 & \Pi \end{pmatrix}$. The operator B belongs to $\mathcal{L}(H_{00}^{3/2}(\Gamma_c) \times \mathbf{H}_{00}^{3/2}(\Gamma_c), (D(\mathcal{A}^*))')$.

If a triplet $(\theta, \mathbf{v}, p) \in H_\delta^2(\Omega) \times \mathbf{H}_\delta^2(\Omega) \times H_\delta^1(\Omega)$ is a solution to (3.1), then (θ, \mathbf{v}) satisfies

$$\begin{aligned}
(\lambda_0 I - \mathcal{A})(\theta, \Pi \mathbf{v})^T - B(\theta_c, \mathbf{v}_c)^T &= (f, \Pi \mathbf{h})^T \text{ in } (D(\mathcal{A}^*))', \\
(I - \Pi) \mathbf{v} &= \sum_{i=1}^N f_i (I - \Pi) G_{\mathbf{v}}(\theta_c, \mathbf{v}_c).
\end{aligned} \tag{3.26}$$

Conversely, if $(\theta, \mathbf{v}) \in H_\delta^2(\Omega) \times \mathbf{H}_\delta^2(\Omega)$ satisfies (3.26), then there exists a unique $p \in H_\delta^1(\Omega)$ such that $(\theta, \mathbf{v}, p) \in H_\delta^2(\Omega) \times \mathbf{H}_\delta^2(\Omega) \times H_\delta^1(\Omega)$ is a solution to (3.1).

Proof. Since G is continuous from $(H_{00}^{3/2}(\Gamma_c) \times \mathbf{H}_{00}^{3/2}(\Gamma_c))$ into $(H_\delta^2(\Omega) \times \mathbf{H}_\delta^2(\Omega))$, it is obvious that $B \in \mathcal{L}(H_{00}^{3/2}(\Gamma_c) \times \mathbf{H}_{00}^{3/2}(\Gamma_c), (D(\mathcal{A}^*))')$.

If (θ, \mathbf{v}, p) satisfies equation (3.1), then we can write

$$(\theta, \mathbf{v}, p) = (\tilde{\theta}, \tilde{\mathbf{v}}, \tilde{p}) + (\hat{\theta}, \hat{\mathbf{v}}, \hat{p}),$$

where $(\tilde{\theta}, \tilde{\mathbf{v}}, \tilde{p})$ is the solution to (3.5) and $(\hat{\theta}, \hat{\mathbf{v}}, \hat{p})$ is the solution to (3.1) with $(f, \mathbf{h}) = (0, \mathbf{0})$. Then we obtain

$$(\lambda_0 I - \mathcal{A})(\tilde{\theta}, \Pi \tilde{\mathbf{v}})^T = (f, \Pi \mathbf{h})^T, \quad (3.27)$$

and

$$(\hat{\theta}, \hat{\mathbf{v}})^T = G(\theta_c, \mathbf{v}_c)^T \text{ implies } (\lambda_0 I - \mathcal{A})(\hat{\theta}, \Pi \hat{\mathbf{v}})^T = B(\theta_c, \mathbf{v}_c)^T \quad (3.28)$$

From (3.27) and (3.28), we obtain (3.26).

The proof of the converse statement is based on the decomposition $(\theta, \mathbf{v}) = (\tilde{\theta}, \tilde{\mathbf{v}}) + (\hat{\theta}, \hat{\mathbf{v}})$ and on Theorem 3.6-(3). It is left to the reader. \square

When (θ_c, \mathbf{v}_c) is in the form (1.3), it is convenient to introduce the operator $\mathcal{B} \in \mathcal{L}(\mathbb{R}^N, (D(\mathcal{A}^*))')$ defined by

$$\mathcal{B}\mathbf{f} = \sum_{i=1}^N f_i B(g_{\theta,i}, \mathbf{g}_{\mathbf{v},i}). \quad (3.29)$$

Thus we can rewrite (3.26)₁ as

$$(\lambda_0 I - \mathcal{A})(\theta, \Pi \mathbf{v})^T - \mathcal{B}\mathbf{f} = (f, \Pi \mathbf{h}) \text{ in } (D(\mathcal{A}^*))'. \quad (3.30)$$

In order to compute the adjoint of the control operator, we now prove a Green's formula between the solution $(G(\theta_c, \mathbf{v}_c), G_p(\theta_c, \mathbf{v}_c))$ to the nonhomogeneous boundary value problem (3.1) with (θ_c, \mathbf{v}_c) as boundary condition and solution (ξ, ϕ, ψ) to equation (3.12). Before proving that we need one technical lemma about the existence of normal derivative in suitable space.

Lemma 3.12. *Let Ω be a bounded open subset of \mathbb{R}^2 satisfying assumptions (H_1) – (H_4) . Let us define*

$$E_1(\Omega) = \left\{ \xi \in H^{2-\delta}(\Omega) \mid \Delta \xi \in L^2(\Omega) \right\},$$

$$E_2(\Omega) = \left\{ [\mathbf{v}] \in H^{1-\delta}(\Omega; \mathbb{R}^{2 \times 2}) \mid \operatorname{div}[\mathbf{v}] \in \mathbf{L}^2(\Omega) \right\},$$

where $[\mathbf{v}]$ is a (2×2) matrix and the divergence is taken line by line. Then

- (1) The linear mapping $\gamma_1 : \xi \mapsto \frac{\partial \xi}{\partial \mathbf{n}}$ is continuous from $E_1(\Omega)$ into $H^{1/2-\delta}(\Gamma)$.
- (2) The linear mapping $\gamma_{\mathbf{n}} : [\mathbf{v}] \mapsto [\mathbf{v}]\mathbf{n}$ is continuous from $E_2(\Omega)$ into $\mathbf{H}^{1/2-\delta}(\Gamma)$.

Proof. Let us set $E(\Omega) = \{ \xi \in H^1(\Omega) \mid \Delta \xi \in L^2(\Omega) \}$. We know that $\gamma_1 \in \mathcal{L}(E(\Omega), H^{-1/2}(\Gamma))$ [18, Chapter VII, Lemma 1, page 381] and $\gamma_{\mathbf{n}} \in \mathcal{L}(H^2(\Omega), H^{1/2}(\Gamma))$ [19, Theorem III.2.23, page 158]. Thus by interpolation, it follows that γ_1 is also bounded from $E_1(\Omega)$ into $H^{1/2-\delta}(\Gamma)$, since $[E(\Omega), H^2(\Omega)]_{1-\delta} = E_1(\Omega)$ and $[H^{-1/2}(\Gamma), H^{1/2}(\Gamma)]_{1-\delta} = H^{1/2-\delta}(\Gamma)$.

The proof for $\gamma_{\mathbf{n}}$ is similar. \square

Theorem 3.13. *Let Ω be a bounded open subset of \mathbb{R}^2 satisfying assumptions $(H_1) - (H_4)$. Let $1/2 < \delta < 1$. Assume that (θ_c, \mathbf{v}_c) belongs to $(H_{00}^{3/2}(\Gamma_c) \times \mathbf{H}_{00}^{3/2}(\Gamma_c))$. Let $(\widehat{g}, \widehat{\mathbf{g}})$ belong to $L^2(\Gamma \setminus \Gamma_c) \times \mathbf{L}^2(\Gamma_n)$.*

For all $\theta \in H_\delta^2(\Omega)$ satisfying $\theta = \theta_c$ on Γ_c , $\frac{\partial \theta}{\partial n} = 0$ on $\Gamma \setminus \Gamma_c$, and for all $\xi \in H_\delta^2(\Omega) \cap H_{\Gamma_c}^1(\Omega)$ satisfying $\frac{\partial \xi}{\partial n} = \widehat{g}$ on $\Gamma \setminus \Gamma_c$, we have

$$\begin{aligned} & \langle \xi, \Delta \theta \rangle_{L_{-\delta}^2(\Omega), L_\delta^2(\Omega)} - \langle \theta, \Delta \xi \rangle_{L_{-\delta}^2(\Omega), L_\delta^2(\Omega)} \\ &= - \left\langle \frac{\partial \xi}{\partial n}, \theta_c \right\rangle_{H^{1/2-\delta}(\Gamma_c), H^{\delta-1/2}(\Gamma_c)} - \int_{\Gamma \setminus \Gamma_c} \widehat{g} \theta. \end{aligned} \quad (3.31)$$

For all $(\mathbf{v}, p) \in \mathbf{H}_\delta^2(\Omega) \times H_\delta^1(\Omega)$ satisfying $\operatorname{div} \mathbf{v} = 0$ in Ω , $\mathbf{v} = \mathbf{v}_c$ on Γ_c , $\mathbf{v} = \mathbf{0}$ on Γ_w , $\sigma(\mathbf{v}, p)\mathbf{n} = \mathbf{0}$ on Γ_n and for all $(\phi, \psi) \in (\mathbf{H}_\delta^2(\Omega) \cap \mathbf{V}_{\Gamma_d}^1(\Omega)) \times H_\delta^1(\Omega)$ satisfying $\sigma(\phi, \psi)\mathbf{n} = \widehat{\mathbf{g}}$ on Γ_n , we have

$$\begin{aligned} & \langle \phi, \operatorname{div} \sigma(\mathbf{v}, p) \rangle_{\mathbf{L}_{-\delta}^2(\Omega), \mathbf{L}_\delta^2(\Omega)} - \langle \mathbf{v} \operatorname{div} \sigma(\phi, \psi) \rangle_{\mathbf{L}_{-\delta}^2(\Omega), \mathbf{L}_\delta^2(\Omega)} \\ &= - \langle \sigma(\phi, \psi)\mathbf{n}, \mathbf{v}_c \rangle_{\mathbf{H}^{1/2-\delta}(\Gamma_c), \mathbf{H}^{\delta-1/2}(\Gamma_c)} - \int_{\Gamma_n} \widehat{\mathbf{g}} \cdot \mathbf{v}. \end{aligned} \quad (3.32)$$

Proof. Step 1. Proof of (3.31). Since θ belongs to $H_\delta^2(\Omega)$ and $\frac{\partial \theta}{\partial n} = 0$ on $\Gamma \setminus \Gamma_c$ and $\xi \in H_\delta^2(\Omega) \cap H_{\Gamma_c}^1(\Omega)$, from Theorem 3.4, it follows that

$$\langle \xi, \Delta \theta \rangle_{L_{-\delta}^2(\Omega), L_\delta^2(\Omega)} = - \int_{\Omega} \nabla \theta \cdot \nabla \xi. \quad (3.33)$$

For $(\theta^k)_k \subset C^\infty(\overline{\Omega})$ and $(\xi^k)_k \subset C^\infty(\overline{\Omega})$, we have

$$\int_{\Omega} \theta^k \Delta \xi^k = - \int_{\Omega} \nabla \theta^k \cdot \nabla \xi^k + \int_{\Gamma} \theta^k \frac{\partial \xi^k}{\partial n}. \quad (3.34)$$

By density of $C^\infty(\overline{\Omega})$ in $H_\delta^2(\Omega)$, passing to the limit when k tends to infinity and using Lemma 3.12, we obtain

$$\langle \theta, \Delta \xi \rangle_{L_{-\delta}^2(\Omega), L_\delta^2(\Omega)} = - \int_{\Omega} \nabla \theta \cdot \nabla \xi + \left\langle \frac{\partial \xi}{\partial n}, \theta \right\rangle_{H^{1/2-\delta}(\Gamma), H^{\delta-1/2}(\Gamma)}. \quad (3.35)$$

Since

$$\begin{aligned} \left\langle \frac{\partial \xi}{\partial n}, \theta \right\rangle_{H^{1/2-\delta}(\Gamma), H^{\delta-1/2}(\Gamma)} &= \left\langle \frac{\partial \xi}{\partial n}, \theta \right\rangle_{H^{1/2-\delta}(\Gamma_c), H^{\delta-1/2}(\Gamma_c)} \\ &+ \left\langle \frac{\partial \xi}{\partial n}, \theta \right\rangle_{H^{1/2-\delta}(\Gamma \setminus \Gamma_c), H^{\delta-1/2}(\Gamma \setminus \Gamma_c)} \\ &= \left\langle \frac{\partial \xi}{\partial n}, \theta \right\rangle_{H^{1/2-\delta}(\Gamma), H^{\delta-1/2}(\Gamma)} + \int_{\Gamma \setminus \Gamma_c} \widehat{g} \theta, \end{aligned}$$

the proof of (3.31) is complete.

Step 2. Proof of (3.32). As $(\mathbf{v}, p) \in \mathbf{H}_\delta^2(\Omega) \times H_\delta^1(\Omega)$ satisfies $\sigma(\mathbf{v}, p)\mathbf{n} = \mathbf{0}$ on Γ_n and $\phi \in \mathbf{H}_\delta^2(\Omega) \cap \mathbf{V}_{\Gamma_d}^1(\Omega)$, by using Theorem 3.5, we obtain

$$\langle \phi, \operatorname{div} \sigma(\mathbf{v}, p) \rangle_{\mathbf{L}_{-\delta}^2(\Omega), \mathbf{L}_{\delta}^2(\Omega)} = -2\nu \int_{\Omega} D\mathbf{v} : D\phi \, dx. \quad (3.36)$$

For a sequence $(\mathbf{v}^k, p^k)_k \subset \mathbf{C}^\infty(\bar{\Omega}) \times C^\infty(\bar{\Omega})$ and a sequence $(\phi^k, \psi^k) \in \mathbf{C}^\infty(\bar{\Omega}) \times C^\infty(\bar{\Omega})$. We have

$$\int_{\Omega} \mathbf{v}^k \cdot \operatorname{div} \sigma(\phi^k, \psi^k) = -2\nu \int_{\Omega} D\mathbf{v}^k : D\phi^k + \int_{\Omega} \psi^k \operatorname{div} \mathbf{v}^k + \int_{\Gamma} \sigma(\phi^k, \psi^k) \mathbf{n} \cdot \mathbf{v}^k. \quad (3.37)$$

By density of $\mathbf{C}^\infty(\bar{\Omega}) \times C^\infty(\bar{\Omega})$ in $\mathbf{H}_{\delta}^2(\Omega) \times H_{\delta}^1(\Omega)$, passing to the limit as $k \rightarrow \infty$, using Lemma 3.12, we obtain

$$\begin{aligned} & \langle \mathbf{v} \operatorname{div} \sigma(\phi, \psi) \rangle_{\mathbf{L}_{-\delta}^2(\Omega), \mathbf{L}_{\delta}^2(\Omega)} \\ &= -2\nu \int_{\Omega} D\mathbf{v} : D\phi + \int_{\Omega} \psi \operatorname{div} \mathbf{v} + \langle \sigma(\phi, \psi) \mathbf{n}, \mathbf{v} \rangle_{\mathbf{H}^{1/2-\delta}(\Gamma), \mathbf{H}^{\delta-1/2}(\Gamma)}. \end{aligned}$$

We can proceed as in Step 1 to complete the proof of (3.32). \square

The Green's formula (3.31)–(3.32) in Theorem 3.13 is needed to characterize the adjoint of the operator \mathcal{B} .

Proposition 3.14. *The adjoint of the control operator $\mathcal{B} \in \mathcal{L}(\mathbb{R}^N, (D(\mathcal{A}^*))')$ is the operator $\mathcal{B}^* \in \mathcal{L}(D(\mathcal{A}^*), \mathbb{R}^N)$ defined by*

$$\langle \mathbf{f}, \mathcal{B}^*(\xi, \phi) \rangle_{\mathbb{R}^N} = \sum_{i=1}^N \int_{\Gamma_c} f_i \left(-\kappa g_{\theta, i} \frac{\partial \xi}{\partial n} - \mathbf{g}_{\mathbf{v}, i} \cdot \sigma(\phi, \psi) \mathbf{n} \right), \quad (3.38)$$

for all $(\xi, \phi) \in D(\mathcal{A}^*)$. Furthermore, the adjoint of $B \in \mathcal{L}(H_{00}^{3/2}(\Gamma_c) \times \mathbf{H}_{00}^{3/2}(\Gamma_c), (D(\mathcal{A}^*))')$, belongs to $\mathcal{L}(D(\mathcal{A}^*), (H_{00}^{3/2}(\Gamma_c) \times \mathbf{H}_{00}^{3/2}(\Gamma_c))')$ and is defined by

$$\langle (\theta_c, \mathbf{v}_c), B^*(\xi, \phi) \rangle = \int_{\Gamma_c} -\kappa \frac{\partial \xi}{\partial n} \theta_c - \sigma(\phi, \psi) \mathbf{n} \cdot \mathbf{v}_c, \quad (3.39)$$

for all $(\xi, \phi) \in D(\mathcal{A}^*)$ and all $(\theta_c, \mathbf{v}_c) \in (H_{00}^{3/2}(\Gamma_c) \times \mathbf{H}_{00}^{3/2}(\Gamma_c))$.

Proof. We know that $(\lambda_0 I - \mathcal{A}^*)$ is an isomorphism from $D(\mathcal{A}^*)$ to $L^2(\Omega) \times \mathbf{V}_{n, \Gamma_d}^0(\Omega)$. Thus for all $(\xi, \phi) \in D(\mathcal{A}^*)$, there exists $(f, \mathbf{h}) \in L^2(\Omega) \times \mathbf{V}_{n, \Gamma_d}^0(\Omega)$ such that $(\lambda_0 I - \mathcal{A}^*)(\xi, \phi) = (f, \mathbf{h})$.

For all $(\xi, \phi) \in D(\mathcal{A}^*)$:

$$\begin{aligned} \langle \mathcal{B}\mathbf{f}, (\xi, \phi) \rangle_{(D(\mathcal{A}^*))', D(\mathcal{A}^*)} &= \sum_{i=1}^N f_i \langle B(g_{\theta, i}, \mathbf{g}_{\mathbf{v}, i}), (\xi, \phi) \rangle_{(D(\mathcal{A}^*))', D(\mathcal{A}^*)} \\ &= \sum_{i=1}^N f_i \left\langle (\lambda_0 I - \mathcal{A}) \tilde{\Pi} G(g_{\theta, i}, \mathbf{g}_{\mathbf{v}, i}), (\xi, \phi) \right\rangle_{(D(\mathcal{A}^*))', D(\mathcal{A}^*)} \\ &= \left\langle G(\theta_c, \mathbf{v}_c), \tilde{\Pi}(\lambda_0 I - \mathcal{A}^*)(\xi, \phi) \right\rangle_{L^2(\Omega) \times L^2(\Omega)} \\ &= \langle (\theta, \mathbf{v}), (f, \mathbf{h}) \rangle_{L^2(\Omega) \times L^2(\Omega)} = \int_{\Omega} \theta f + \int_{\Omega} \mathbf{v} \cdot \mathbf{h}. \end{aligned}$$

Now with the help of equation (3.12) and Green's formula (3.31), we obtain

$$\begin{aligned}
\int_{\Omega} \theta f &= \int_{\Omega} \theta (\lambda_0 \xi - \kappa \Delta \xi - \mathbf{u}_s \cdot \nabla \xi - \boldsymbol{\beta} \cdot \boldsymbol{\phi}) \\
&= \int_{\Omega} \lambda_0 \theta \xi - \kappa \int_{\Omega} \xi \Delta \theta - \kappa \int_{\Gamma_c} \theta_c \frac{\partial \xi}{\partial n} + \int_{\Gamma \setminus \Gamma_c} \theta (\mathbf{u}_s \cdot \mathbf{n}) \xi + \int_{\Omega} (\mathbf{u}_s \cdot \nabla \theta) \xi \\
&\quad - \int_{\Gamma \setminus \Gamma_c} \theta (\mathbf{u}_s \cdot \mathbf{n}) \xi - \int_{\Omega} \boldsymbol{\beta} \theta \cdot \boldsymbol{\phi} \\
&= -\kappa \int_{\Gamma_c} \theta_c \left(\frac{\partial \xi}{\partial n} \right) - \int_{\Omega} \boldsymbol{\beta} \theta \cdot \boldsymbol{\phi} - \int_{\Omega} (\mathbf{v} \cdot \nabla \tau_s) \xi.
\end{aligned}$$

Similarly, by using equation (3.12) and Green's formula (3.32), we have

$$\int_{\Omega} \mathbf{v} \cdot \mathbf{h} = - \int_{\Gamma_c} \mathbf{v}_c \cdot \sigma(\boldsymbol{\phi}, \boldsymbol{\psi}) \mathbf{n} + \int_{\Omega} \boldsymbol{\beta} \theta \cdot \boldsymbol{\phi} + \int_{\Omega} (\nabla \tau_s)^T \xi \cdot \mathbf{v}.$$

Hence, we obtain

$$\begin{aligned}
\langle \mathcal{B}\mathbf{f}, (\xi, \boldsymbol{\phi}) \rangle_{(D(\mathcal{A}^*))', D(\mathcal{A}^*)} &= -\kappa \int_{\Gamma_c} \theta_c \left(\frac{\partial \xi}{\partial n} \right) - \int_{\Omega} \boldsymbol{\beta} \theta \cdot \boldsymbol{\phi} - \int_{\Omega} (\mathbf{v} \cdot \nabla \tau_s) \xi \\
&\quad - \int_{\Gamma_c} \mathbf{v}_c \cdot \sigma(\boldsymbol{\phi}, \boldsymbol{\psi}) \mathbf{n} + \int_{\Omega} \boldsymbol{\beta} \theta \cdot \boldsymbol{\phi} + \int_{\Omega} (\nabla \tau_s)^T \xi \cdot \mathbf{v} \\
&= \sum_{i=1}^N \int_{\Gamma_c} f_i \left(-\kappa g_{\theta, i} \frac{\partial \xi}{\partial n} - \mathbf{g}_{\mathbf{v}, i} \cdot \sigma(\boldsymbol{\phi}, \boldsymbol{\psi}) \mathbf{n} \right). \quad \square
\end{aligned}$$

4 Instationary linearized Boussinesq system

Definition 4.1. A triplet $(\theta, \mathbf{v}, p) \in H_{\delta}^{2,1}(\Omega) \times \mathbf{H}_{\delta}^{2,1}(\Omega) \times L^2(0, T; H_{\delta}^1(\Omega))$ is a solution to (1.5) if the equations (1.5)₁, (1.5)₄ are satisfied in the sense of distributions and (1.5)₂, (1.5)₅ are satisfied in the sense of traces.

Theorem 4.2. Assume that $(\theta_c, \mathbf{v}_c) \in [L^2(0, T; H_{00}^{3/2}(\Gamma_c)) \cap H^1(0, T; L^2(\Gamma_c))] \times [L^2(0, T; \mathbf{H}_{00}^{3/2}(\Gamma_c)) \cap H^1(0, T; \mathbf{L}^2(\Gamma_c))]$ and $(\theta_0, \mathbf{v}_0) \in L^2(\Omega) \times \mathbf{V}_{n, \Gamma_d}^0(\Omega)$. A triplet $(\theta, \mathbf{v}, p) \in H_{\delta}^{2,1}(\Omega) \times \mathbf{H}_{\delta}^{2,1}(\Omega) \times L^2(0, T; H_{\delta}^1(\Omega))$ is a solution to (1.5) iff (θ, \mathbf{v}) satisfies

$$\begin{aligned}
(\theta', \Pi \mathbf{v}')^T &= \mathcal{A}(\theta, \Pi \mathbf{v})^T + \mathcal{B}\mathbf{f} \text{ in } (0, T), \quad (\theta(0), \Pi \mathbf{v}(0)) = (\theta_0, \mathbf{v}_0), \\
(I - \Pi) \mathbf{v} &= \sum_{i=1}^N f_i (I - \Pi) G_{\mathbf{v}}(g_{\theta, i}, \mathbf{g}_{\mathbf{v}, i}),
\end{aligned} \tag{4.1}$$

where $G_{\mathbf{v}}$ is to the second component of the operator $G = (G_{\theta}, G_{\mathbf{v}})^T$.

Proof. The result follows from Theorem 3.11 with $(f, \mathbf{h}) = (\theta', \Pi \mathbf{v}')$. □

Remark 4.3. We can write evolution equation (4.1)₁ in the following form

$$\mathbf{z}' = \mathcal{A}\mathbf{z} + \mathcal{B}\mathbf{f}, \quad \mathbf{z}(0) = (\theta_0, \mathbf{v}_0)^T = \mathbf{z}_0, \tag{4.2}$$

where $\mathcal{B} \in \mathcal{L}(\mathbb{R}^N, (D(\mathcal{A})^*)')$. Actually, as in [2, Theorem 3.2], we can show that

$$(\lambda_0 I - \mathcal{A})^{-\gamma} \mathcal{B} \in \mathcal{L}(\mathbb{R}^N, Z) \quad \text{for any } \frac{3}{4} < \gamma < 1. \quad (4.3)$$

Using this result, we can easily verify that if $\mathbf{z}_0 \in Z$, equation (4.2) admits a unique weak solution in $L^2(0, T; Z)$.

5 Stabilization of the linearized system

Since $(\mathcal{A}, D(\mathcal{A}))$ is the infinitesimal generator of an analytic semigroup on Z with compact resolvent, the spectrum Σ of \mathcal{A} is contained in a sector of the form $S_{a, \vartheta} = \{\lambda \in \mathbb{C} \mid |\arg(\lambda - a)| > \vartheta > \pi/2\}$ and it consists entirely of isolated eigenvalues of finite multiplicity [20, Chap. 3, Theorem 6.29, p. 187]. Thus, for a given $\omega > 0$, the eigenvalues of \mathcal{A} , repeated according to their geometric multiplicity, can be ordered in the following manner

$$\cdots \leq \operatorname{Re} \lambda_{N_u+1} < -\omega < \operatorname{Re} \lambda_{N_u} \leq \operatorname{Re} \lambda_{N_u-1} \leq \cdots \leq \operatorname{Re} \lambda_1. \quad (5.1)$$

If the above inequalities satisfied by $-\omega$ are not strict, without loss of generality we can increase ω to have strict inequalities. We introduce the set of indices $I_u = \{1, \dots, N_u\}$. We have the following decomposition of Z :

$$Z = Z_u \oplus Z_s, \quad Z = Z_u^* \oplus Z_s^*, \quad (5.2)$$

where Z_u is the space generated by $\bigcup_{j \in I_u} G_{\mathbb{R}}(\lambda_j)$ and Z_u^* the space generated by $\bigcup_{j \in I_u} G_{\mathbb{R}}^*(\lambda_j)$. Here we denote by $G_{\mathbb{R}}(\lambda_j)$, the real generalized eigenspace for \mathcal{A} , that is to say the space generated by $\operatorname{Re}(G_{\mathbb{C}}(\lambda_j)) \cup \operatorname{Im}(G_{\mathbb{C}}(\lambda_j))$.

5.1 Stabilizability of the linearized system

In order to stabilize the system

$$\mathbf{z}' = (\mathcal{A} + \omega I)\mathbf{z} + \mathcal{B}\mathbf{f}, \quad (5.3)$$

we want to first fix the family of functions $\{\mathbf{g}_i\}$. For that we consider the adjoint eigenvalue problem for $\lambda_i \in I_u$:

$$\begin{aligned} \lambda_i \xi - \kappa \Delta \xi - \mathbf{u}_s \cdot \nabla \xi - \boldsymbol{\beta} \cdot \boldsymbol{\phi} &= 0 \quad \text{in } \Omega, \\ \xi &= 0 \quad \text{on } \Gamma_c, \quad \kappa \frac{\partial \xi}{\partial n} + (\mathbf{u}_s \cdot \mathbf{n})\xi = 0 \quad \text{on } \Gamma \setminus \Gamma_c, \\ \lambda_i \boldsymbol{\phi} - \operatorname{div} \sigma(\boldsymbol{\phi}, \psi) - (\mathbf{u}_s \cdot \nabla)\boldsymbol{\phi} + (\nabla \mathbf{u}_s)^T \boldsymbol{\phi} + (\nabla \tau_s)^T \xi &= 0 \quad \text{in } \Omega, \\ \operatorname{div} \boldsymbol{\phi} &= 0 \quad \text{in } \Omega, \quad \boldsymbol{\phi} = 0 \quad \text{on } \Gamma_d, \quad \sigma(\boldsymbol{\phi}, \psi)\mathbf{n} + (\mathbf{u}_s \cdot \mathbf{n})\boldsymbol{\phi} = 0 \quad \text{on } \Gamma_n. \end{aligned} \quad (5.4)$$

If $(\xi, \boldsymbol{\phi}, \psi)$ is a non trivial solution to (5.4), we shall say that ψ is the pressure associated with $\boldsymbol{\phi}$. We choose a family of eigenfunctions $(\xi_i, \boldsymbol{\phi}_i)_{i \in I_u}$ of the adjoint problem associated with

the eigenvalue λ_i , for $i \in I_u$, such that $Z_u^* = \text{span}\{\text{Re}(\xi_i, \phi_i), \text{Im}(\xi_i, \phi_i) \mid i \in I_u\}$. Following [2], we set

$$W = \text{span} \left[\left\{ \left(\text{Re} \left(\frac{\partial \xi_i}{\partial n} \right), \text{Re}(\sigma(\phi_i, \psi_i) \mathbf{n}) \right) \mid i \in I_u \right\} \cup \left\{ \left(\text{Im} \left(\frac{\partial \xi_i}{\partial n} \right), \text{Im}(\sigma(\phi_i, \psi_i) \mathbf{n}) \right) \mid i \in I_u \right\} \right],$$

where ψ_i is the pressure associated with ϕ_i . Next, we choose in W , the functions $\tilde{\mathbf{g}}_i = (\tilde{g}_{\theta,i}, \tilde{\mathbf{g}}_{\mathbf{v},i})$, $1 \leq i \leq N$ for some N , such that

$$\text{span}\{\tilde{\mathbf{g}}_1, \tilde{\mathbf{g}}_2, \dots, \tilde{\mathbf{g}}_N\} = W, \quad \mathbf{g}_i = m \tilde{\mathbf{g}}_i, \quad (5.5)$$

where m is a smooth function with compact support in Γ_c such that $m \equiv 1$ in $\tilde{\Gamma}_c$, a nonempty open subset of Γ_c . Since the eigenfunctions of \mathcal{A}^* are smooth, at least $H^3 \times \mathbf{H}^3$ away from the junction points of Γ_c , we have $\mathbf{g}_i \in H_{00}^{3/2}(\Gamma_c) \times \mathbf{H}_{00}^{3/2}(\Gamma_c)$.

In the following theorem, we prove the stabilizability of the system using the above family $(g_{\theta,i}, \mathbf{g}_{\mathbf{v},i})_{1 \leq i \leq N} \subset H_{00}^{3/2}(\Gamma_c) \times \mathbf{H}_{00}^{3/2}(\Gamma_c)$.

Theorem 5.1. *If the family $(g_{\theta,i}, \mathbf{g}_{\mathbf{v},i})_{1 \leq i \leq N} \subset H_{00}^{3/2}(\Gamma_c) \times \mathbf{H}_{00}^{3/2}(\Gamma_c)$ is chosen as in (5.5), then $(\mathcal{A} + \omega I, \mathcal{B})$ is stabilizable in Z .*

Proof. Since condition (4.3) is satisfied, it is known that the pair $(\mathcal{A} + \omega I, \mathcal{B})$ in system (5.3) is stabilizable in Z if the following Hautus condition is satisfied [17, Part III, Chapter 2, Proposition 3.4]:

$$\text{Ker}(\lambda_j I - \mathcal{A}^*) \cap \text{Ker } \mathcal{B}^* = \{0\} \quad \text{for all } j = 1, 2, \dots, N_u. \quad (5.6)$$

If $(\xi, \phi)^T \in \text{Ker}(\lambda_j I - \mathcal{A}^*) \cap \text{Ker } \mathcal{B}^*$, with $j \in I_u$, then $(\xi, \phi)^T$ is a solution to (5.4) and $\mathcal{B}^*(\xi, \phi)^T = 0$. By using Proposition 3.14, we have

$$\begin{aligned} \mathcal{B}^* \begin{pmatrix} \xi \\ \phi \end{pmatrix} &= \left(- \int_{\Gamma_c} \kappa m \tilde{g}_{\theta,j} \text{Re} \left(\frac{\partial \xi}{\partial n} \right) - \int_{\Gamma_c} m \tilde{\mathbf{g}}_{\mathbf{v},j} \cdot \text{Re}(\sigma(\phi, \psi) \mathbf{n}) \right. \\ &\quad \left. - i \int_{\Gamma_c} \kappa m \tilde{g}_{\theta,j} \text{Im} \left(\frac{\partial \xi}{\partial n} \right) - \int_{\Gamma_c} m \tilde{\mathbf{g}}_{\mathbf{v},j} \cdot \text{Im}(\sigma(\phi, \psi) \mathbf{n}) \right)_{j=1}^N. \end{aligned}$$

Since $\mathcal{B}^*(\xi, \phi)^T = 0$ and $(\tilde{\mathbf{g}}_i)_{1 \leq i \leq N}$ form a basis for W , we deduce that

$$\int_{\Gamma_c} \kappa m \left| \frac{\partial \xi}{\partial n} \right|^2 + \int_{\Gamma_c} m |\sigma(\phi, \psi) \mathbf{n}|^2 = 0.$$

Since $m \equiv 1$ in $\tilde{\Gamma}_c$, we obtain

$$\frac{\partial \xi}{\partial n} = 0 \text{ on } \tilde{\Gamma}_c, \quad \sigma(\phi, \psi) \mathbf{n} = 0 \text{ on } \tilde{\Gamma}_c.$$

With Theorem Appendix A.2, we can conclude that $(\xi, \phi) = (0, \mathbf{0})$. Hence $(\mathcal{A} + \omega I, \mathcal{B})$ is stabilizable in Z . \square

5.2 Stabilization by finite dimensional feedback controls

In this Section, we look for the feedback stabilization of the pair $(\mathcal{A} + \omega I, \mathcal{B})$ in Z . Let π_u be the projection onto Z_u along Z_s , $\pi_s = I - \pi_u$, and let us set

$$\begin{aligned} \mathcal{A}_u &= \pi_u(\mathcal{A} + \omega I), & \mathcal{A}_s &= \pi_s(\mathcal{A} + \omega I), & B_u &= \pi_u B, & B_s &= \pi_s B, \\ \mathcal{B}_u \mathbf{f} &= \sum_{i=1}^N f_i B_u \mathbf{g}_i \quad \forall \mathbf{f} = (f_1, \dots, f_N) \in \mathbb{R}^N. \end{aligned}$$

We consider the Riccati equation

$$\mathcal{P}_u = \mathcal{P}_u^* > 0, \quad \mathcal{P}_u \mathcal{A}_u + \mathcal{A}_u^* \mathcal{P}_u - \mathcal{P}_u \mathcal{B}_u \mathcal{B}_u^* \mathcal{P}_u = 0.$$

From [10, 2], it follows that $(\mathcal{A}_u - \mathcal{B}_u \mathcal{B}_u^* \mathcal{P}_u)$ is exponentially stable on Z_u .

Proposition 5.2. *The operator $\mathcal{K} = -\mathcal{B}^* \mathcal{P}$ from Z to \mathbb{R}^n , with $\mathcal{P} = \pi_u^* \mathcal{P}_u \pi_u \in \mathcal{L}(Z)$, provides a stabilizing feedback for $(\mathcal{A} + \omega I, \mathcal{B})$. Moreover, the operator $\mathcal{A} + \omega I - \mathcal{B} \mathcal{B}^* \mathcal{P}$ with domain*

$$D(\mathcal{A} + \omega I - \mathcal{B} \mathcal{B}^* \mathcal{P}) = \{z \in Z \mid (\mathcal{A} + \omega I - \mathcal{B} \mathcal{B}^* \mathcal{P})z \in Z\}$$

is the infinitesimal generator of an analytic semigroup exponentially stable on Z .

Proof. We refer to [21]. □

5.3 Regularity of solutions to the closed loop system

Theorem 5.3. *Let $\varepsilon \in (0, \frac{1}{2})$. Assume that $(\theta_0, \mathbf{v}_0) \in H^\varepsilon(\Omega) \times \mathbf{V}_{n, \Gamma_d}^\varepsilon(\Omega)$, $(f_1, \mathbf{f}_2) \in L^2(0, \infty; H_{\Gamma_c}^{-1+\varepsilon}(\Omega)) \times L^2(0, \infty; \mathbf{H}_{\Gamma_d}^{-1+\varepsilon}(\Omega))$ and $\mathbf{g}_i = (g_{\theta, i}, \mathbf{g}_{\mathbf{v}, i}) \in H_{00}^{3/2}(\Gamma_c) \times \mathbf{H}_{00}^{3/2}(\Gamma_c)$ is chosen as in (5.5), then the solution (θ, \mathbf{v}) to following system:*

$$\begin{aligned} \frac{\partial \theta}{\partial t} - \kappa \Delta \theta + \mathbf{u}_s \cdot \nabla \theta + \mathbf{v} \cdot \nabla \tau_s - \omega \theta &= f_1 \text{ in } Q_\infty = \Omega \times (0, \infty), \\ \frac{\partial \theta}{\partial n} &= 0 \text{ on } (\Gamma \setminus \Gamma_c) \times (0, \infty), \\ \frac{\partial \mathbf{v}}{\partial t} - \operatorname{div} \sigma(\mathbf{v}, p) + (\mathbf{u}_s \cdot \nabla) \mathbf{v} + (\mathbf{v} \cdot \nabla) \mathbf{u}_s - \omega \mathbf{v} - \beta \theta &= \mathbf{f}_2 \text{ in } Q_\infty, \\ \operatorname{div} \mathbf{v} = 0 \text{ in } Q_\infty, \quad \mathbf{v} = 0 \text{ on } \Gamma_w \times (0, \infty), \quad \sigma(\mathbf{v}, p) \mathbf{n} = 0 \text{ on } \Gamma_n \times (0, \infty), \\ (\theta, \mathbf{v})^T &= - \sum_{i=1}^N [\mathcal{B}_u^* \mathcal{P}_u \pi_u (\theta, \mathbf{v})^T]_i \mathbf{g}_i \text{ on } \Gamma_c \times (0, \infty), \\ \theta(0) &= \theta_0 \quad \text{and} \quad \Pi \mathbf{v}(0) = \mathbf{v}_0 \text{ in } \Omega, \end{aligned} \tag{5.7}$$

satisfies

$$\begin{aligned} &\|(\theta, \mathbf{v})\|_{L^2(0, \infty; D((\lambda_0 I - \mathcal{A})^{\frac{1}{2} + \frac{\varepsilon}{2}}) \cap H^{\frac{1}{2} + \frac{\varepsilon}{2}}(0, \infty; Z) + H^1(0, \infty; H^{\frac{3}{2} - \varepsilon}(\Omega) \times \mathbf{H}^{\frac{3}{2} - \varepsilon}(\Omega)))} \\ &\leq C \left(\|(f_1, \mathbf{f}_2)\|_{L^2(0, \infty; H_{\Gamma_c}^{-1+\varepsilon}(\Omega)) \times L^2(0, \infty; \mathbf{H}_{\Gamma_d}^{-1+\varepsilon}(\Omega))} + \|(\theta_0, \mathbf{v}_0)\|_{H^\varepsilon(\Omega) \times \mathbf{V}_{n, \Gamma_d}^\varepsilon(\Omega)} \right). \end{aligned} \tag{5.8}$$

Proof. Step 1. In the first step, our main aim is to prove that $(\theta, \mathbf{v}) \in L^2(0, \infty; L^2(\Omega) \times \mathbf{L}^2(\Omega))$. We can write $(\theta, \mathbf{v}) = (\theta_u, \mathbf{v}_u) + (\theta_s, \mathbf{v}_s) + (0, (I - \Pi)\mathbf{v})$ with $(\theta_u, \mathbf{v}_u)^T = \pi_u(\theta, \mathbf{v})^T$, $(\theta_s, \mathbf{v}_s)^T = \pi_s(\theta, \mathbf{v})^T$ satisfy the following equations

$$\begin{aligned} (\theta'_u, \mathbf{v}'_u)^T &= (\mathcal{A}_u - \mathcal{B}_u \mathcal{B}_u^* \mathcal{P}_u)(\theta_u, \mathbf{v}_u)^T + \pi_u(f_1, \mathbf{f}_2)^T, \\ (\theta'_s, \mathbf{v}'_s)^T &= \mathcal{A}_s(\theta_s, \mathbf{v}_s)^T - \sum_{i=1}^N [\mathcal{B}_u^* \mathcal{P}_u \pi_u(\theta, \mathbf{v})^T]_i B_s \mathbf{g}_i + \pi_s(f_1, \mathbf{f}_2)^T, \\ (\theta_u(0), \mathbf{v}_u(0))^T &= \pi_u(\theta_0, \mathbf{v}_0)^T, \quad (\theta_s(0), \mathbf{v}_s(0))^T = \pi_s(\theta_0, \mathbf{v}_0)^T. \end{aligned} \quad (5.9)$$

$(\mathcal{A}_u - \mathcal{B}_u \mathcal{B}_u^* \mathcal{P}_u)$ is the infinitesimal generator of a stable, analytic semigroup on Z_u . Observe that the operator π_u belongs to $\mathcal{L}((D(\mathcal{A}^*))', Z_u)$ (see [10, Remark 3.9]). As Z_u is finite dimensional, we have $Z_u \subset D(\mathcal{A})$. Here $\pi_u(f_1, \mathbf{f}_2)^T \in L^2(0, \infty; D(\mathcal{A}))$ and $\pi_u(\theta_0, \mathbf{v}_0) \in D(\mathcal{A})$. Thus, we have $(\theta_u, \mathbf{v}_u)^T \in H^1(0, \infty; D(\mathcal{A}))$. The semigroup generated by \mathcal{A}_s is analytic on Z_s . Also $-\sum_{i=1}^N [\mathcal{B}_u^* \mathcal{P}_u \pi_u(\theta, \mathbf{v})^T]_i B_s \mathbf{g}_i + \pi_s(f_1, \mathbf{f}_2)^T \in L^2(0, \infty; (D(\mathcal{A}^*))')$ and $\pi_s(\theta_0, \mathbf{v}_0)^T \in L^2(\Omega) \times \mathbf{V}_n^0(\Omega)$. Thus by [17, theorem 2.2, page 208], we can conclude that $(\theta_s, \mathbf{v}_s) \in L^2(0, \infty; L^2(\Omega) \times \mathbf{L}^2(\Omega))$.

Step 2. The solution of equation (5.7) can be written as $(\theta, \mathbf{v}, p) = (\tilde{\theta}, \tilde{\mathbf{v}}, \tilde{p}) + (\hat{\theta}, \hat{\mathbf{v}}, \hat{p})$, where $(\tilde{\theta}, \tilde{\mathbf{v}}, \tilde{p})$ satisfies

$$\begin{aligned} \lambda_0 \tilde{\theta}(t) - \kappa \Delta \tilde{\theta}(t) + \mathbf{u}_s \cdot \nabla \tilde{\theta}(t) + \tilde{\mathbf{v}}(t) \cdot \nabla \tau_s &= 0 \text{ in } \Omega, \\ \frac{\partial \tilde{\theta}(t)}{\partial n} &= 0 \text{ on } \Gamma \setminus \Gamma_c, \\ \lambda_0 \tilde{\mathbf{v}}(t) - \operatorname{div} \sigma(\tilde{\mathbf{v}}(t), q) + (\mathbf{u}_s \cdot \nabla) \tilde{\mathbf{v}}(t) + (\tilde{\mathbf{v}}(t) \cdot \nabla) \mathbf{u}_s - \beta \tilde{\theta}(t) &= 0 \text{ in } \Omega, \\ \operatorname{div} \tilde{\mathbf{v}}(t) = 0 \text{ in } \Omega, \quad \tilde{\mathbf{v}}(t) = 0 \text{ on } \Gamma_w, \quad \sigma(\tilde{\mathbf{v}}(t), \tilde{p}(t)) \mathbf{n} = 0 \text{ on } \Gamma_n, \\ (\tilde{\theta}(t), \tilde{\mathbf{v}}(t))^T &= - \sum_{i=1}^N [\mathcal{B}_u^* \mathcal{P}_u \pi_u(\theta(t), \mathbf{v}(t))^T]_i \mathbf{g}_i \text{ on } \Gamma_c, \end{aligned} \quad (5.10)$$

and $(\hat{\theta}, \hat{\mathbf{v}}, \hat{p})$ satisfies

$$\begin{aligned} \frac{\partial \hat{\theta}}{\partial t} + \lambda_0 \hat{\theta} - \kappa \Delta \hat{\theta} + \mathbf{u}_s \cdot \nabla \hat{\theta} + \hat{\mathbf{v}} \cdot \nabla \tau_s \\ = - \frac{\partial \tilde{\theta}}{\partial t} + \lambda_0 \tilde{\theta} + \lambda_0 \hat{\theta} + \omega \theta + f_1 \text{ in } Q, \\ \hat{\theta}(0) = \theta_0 - \tilde{\theta}(0) \text{ in } \Omega, \quad \hat{\theta} = 0 \text{ on } \Gamma_c, \quad \frac{\partial \hat{\theta}}{\partial n} = 0 \text{ on } \Gamma \setminus \Gamma_c, \\ \frac{\partial \hat{\mathbf{v}}}{\partial t} + \lambda_0 \hat{\mathbf{v}} - \operatorname{div} \sigma(\hat{\mathbf{v}}, \hat{p}) + (\mathbf{u}_s \cdot \nabla) \hat{\mathbf{v}} + (\hat{\mathbf{v}} \cdot \nabla) \mathbf{u}_s - \beta \hat{\theta} \\ = - \frac{\partial \tilde{\mathbf{v}}}{\partial t} + \lambda_0 \tilde{\mathbf{v}} + \lambda_0 \hat{\mathbf{v}} + \omega \mathbf{v} + \mathbf{f}_2 \text{ in } Q, \\ \operatorname{div} \hat{\mathbf{v}} = 0 \text{ in } Q, \\ \hat{\mathbf{v}} = 0 \text{ on } \Gamma_w, \quad \hat{\mathbf{v}} = 0 \text{ on } \Gamma_c, \quad \sigma(\hat{\mathbf{v}}, \hat{p}) \mathbf{n} = 0 \text{ on } \Gamma_n, \\ \hat{\mathbf{v}}(0) = \mathbf{v}_0 - \Pi \tilde{\mathbf{v}}(0) \text{ in } \Omega. \end{aligned} \quad (5.11)$$

By applying Theorem 3.9, we have $(\tilde{\theta}, \tilde{\mathbf{v}})^T \in H^1(0, \infty; H^{\frac{3}{2}-\varepsilon}(\Omega) \times \mathbf{H}^{\frac{3}{2}-\varepsilon}(\Omega))$. It is easy to see

that $(\widehat{\theta}, \widehat{\mathbf{v}})^T \in L^2(0, \infty; L^2(\Omega) \times \mathbf{L}^2(\Omega))$.

$$\left(\begin{array}{c} -\frac{\partial \widehat{\theta}}{\partial t} + \lambda_0 \widehat{\theta} + \lambda_0 \widehat{\theta} + \omega \theta + f_1 \\ \Pi \left(-\frac{\partial \widehat{\mathbf{v}}}{\partial t} + \lambda_0 \widehat{\mathbf{v}} + \lambda_0 \widehat{\mathbf{v}} + \omega \mathbf{v} + \mathbf{f}_2 \right) \end{array} \right) \in L^2(0, \infty; (D(\lambda_0 I - \mathcal{A}^*)^{1/2 - \varepsilon/2})').$$

We also have

$$(\theta_0 - \widetilde{\theta}(0), \mathbf{v}_0 - \Pi \widetilde{\mathbf{v}}(0))^T \in H^\varepsilon(\Omega) \times \mathbf{V}_{n, \Gamma_d}^\varepsilon(\Omega) = D((\lambda_0 I - \mathcal{A})^{\varepsilon/2}).$$

By [17, Theorem 2.2, page 208], we can conclude that:

$$(\widehat{\theta}, \widehat{\mathbf{v}})^T \in L^2(0, \infty; D((\lambda_0 I - \mathcal{A})^{\frac{1}{2} + \frac{\varepsilon}{2}})) \cap H^{\frac{1}{2} + \frac{\varepsilon}{2}}(0, \infty; Z),$$

with the estimate

$$\begin{aligned} & \|(\widehat{\theta}, \widehat{\mathbf{v}})\|_{L^2(0, \infty; D((\lambda_0 I - \mathcal{A})^{\frac{1}{2} + \frac{\varepsilon}{2}})) \cap H^{\frac{1}{2} + \frac{\varepsilon}{2}}(0, \infty; Z)} \\ & \leq C \left(\|(f_1, \mathbf{f}_2)\|_{L^2(0, \infty; H_{\Gamma_c}^{-1 + \varepsilon}(\Omega)) \times L^2(0, \infty; \mathbf{H}_{\Gamma_d}^{-1 + \varepsilon}(\Omega))} + \|(\theta_0, \mathbf{v}_0)\|_{H^\varepsilon(\Omega) \times \mathbf{V}_{n, \Gamma_d}^\varepsilon(\Omega)} \right). \quad \square \end{aligned}$$

6 Stabilization of full system

We apply the feedback control law as in Proposition 5.2 to system (1.4) and we obtain the following closed loop system:

$$\begin{aligned} \frac{\partial \theta}{\partial t} - \kappa \Delta \theta + \mathbf{u}_s \cdot \nabla \theta + \mathbf{v} \cdot \nabla \tau_s + \mathbf{v} \cdot \nabla \theta &= 0 \text{ in } Q, \\ \frac{\partial \theta}{\partial n} &= 0 \text{ on } \Sigma \setminus \Sigma_c, \\ \frac{\partial \mathbf{v}}{\partial t} - \operatorname{div} \sigma(\mathbf{v}, p) + (\mathbf{u}_s \cdot \nabla) \mathbf{v} + (\mathbf{v} \cdot \nabla) \mathbf{u}_s + (\mathbf{v} \cdot \nabla) \mathbf{v} - \beta \theta &= 0 \text{ in } Q, \\ \operatorname{div} \mathbf{v} &= 0 \text{ in } Q, \quad \mathbf{v} = 0 \text{ on } \Sigma_w, \quad \sigma(\mathbf{v}, p) \mathbf{n} = 0 \text{ on } \Sigma_n, \\ (\theta, \mathbf{v})^T &= - \sum_{i=1}^N [\mathcal{B}_u^* \mathcal{P}_u \pi_u(\theta, \mathbf{v})^T]_i \mathbf{g}_i \text{ on } \Sigma_c, \\ \theta(0) &= \theta_0 \quad \text{and} \quad \Pi \mathbf{v}(0) = \mathbf{v}_0 \text{ in } \Omega. \end{aligned} \tag{6.1}$$

Motivated by the regularity of the solution in Theorem 5.3 of the closed loop linearized system (5.7), let us introduce the following spaces

$$\begin{aligned} X_1 &= L^2(0, \infty; D((\lambda_0 I - \mathcal{A})^{\frac{1}{2} + \frac{\varepsilon}{2}})) \cap H^{\frac{1}{2} + \frac{\varepsilon}{2}}(0, \infty; Z), \\ X_2 &= H^1(0, \infty; H^{\frac{3}{2} - \varepsilon}(\Omega) \times \mathbf{H}^{\frac{3}{2} - \varepsilon}(\Omega)), \end{aligned}$$

and $X = X_1 + X_2$ equipped with the norm

$$\|a\|_X = \inf_{a = a_1 + a_2, a_1 \in X_1, a_2 \in X_2} \left(\|a_1\|_{X_1}^2 + \|a_2\|_{X_2}^2 \right)^{1/2}. \tag{6.2}$$

6.1 Estimates of nonlinear terms

We first state the following lemma regarding the estimates of nonlinear terms.

Lemma 6.1. *Let $\varepsilon \in (0, 1/2)$ and $(\xi, \phi) \in X$. Then $(\phi \cdot \nabla)\phi \in L^2(0, \infty; \mathbf{H}_{\Gamma_d}^{-1+\varepsilon}(\Omega))$ and $(\phi \cdot \nabla)\xi \in L^2(0, \infty; H_{\Gamma_c}^{-1+\varepsilon}(\Omega))$. Also, we have the following estimate:*

$$\|(\phi \cdot \nabla)\phi\|_{L^2(0, \infty; \mathbf{H}_{\Gamma_d}^{-1+\varepsilon}(\Omega))} \leq C \|(\xi, \phi)\|_X^2, \quad (6.3)$$

$$\|(\phi \cdot \nabla)\xi\|_{L^2(0, \infty; H_{\Gamma_c}^{-1+\varepsilon}(\Omega))} \leq C \|(\xi, \phi)\|_X^2. \quad (6.4)$$

Moreover, if $(\xi^1, \phi^1), (\xi^2, \phi^2) \in X$, then we have the following estimate:

$$\begin{aligned} & [l] \|(\phi^1 \cdot \nabla)\xi^1 - (\phi^2 \cdot \nabla)\xi^2\|_{L^2(0, \infty; H_{\Gamma_c}^{-1+\varepsilon}(\Omega))} \\ & + \|(\phi^1 \cdot \nabla)\phi^1 - (\phi^2 \cdot \nabla)\phi^2\|_{L^2(0, \infty; \mathbf{H}_{\Gamma_d}^{-1+\varepsilon}(\Omega))} \\ & \leq C (\|(\xi^1, \phi^1)\|_X + \|(\xi^2, \phi^2)\|_X) \|(\xi^1, \phi^1) - (\xi^2, \phi^2)\|_X. \end{aligned} \quad (6.5)$$

Proof. Step 1. Proof of (6.3). We can write $(\xi, \phi) = (\xi_1, \phi_1) + (\xi_2, \phi_2)$, for any $(\xi, \phi) \in X$ with $(\xi_1, \phi_1) \in X_1$, $(\xi_2, \phi_2) \in X_2$. Recall from Theorem 3.8 that

$$D((\lambda_0 I - \mathcal{A})^{\frac{1}{2} + \frac{\varepsilon}{2}}) \subset \left(H_{\Gamma_c}^1(\Omega) \cap H^{1 + \frac{\varepsilon}{2} - \varepsilon^2}(\Omega) \right) \times \left(\mathbf{V}_{\Gamma_d}^1(\Omega) \cap \mathbf{H}^{1 + \frac{\varepsilon}{2} - \varepsilon^2}(\Omega) \right).$$

Thus, we get:

$$\xi_1 \in L^2(0, \infty; H_{\Gamma_c}^1(\Omega) \cap H^{1 + \frac{\varepsilon}{2} - \varepsilon^2}(\Omega)) \cap H^{\frac{1}{2} + \frac{\varepsilon}{2}}(0, \infty; L^2(\Omega)); \quad (6.6)$$

$$\phi_1 \in L^2(0, \infty; \mathbf{V}_{\Gamma_d}^1(\Omega) \cap \mathbf{H}^{1 + \frac{\varepsilon}{2} - \varepsilon^2}(\Omega)) \cap H^{\frac{1}{2} + \frac{\varepsilon}{2}}(0, \infty; \mathbf{V}_{n, \Gamma_d}^0(\Omega)). \quad (6.7)$$

Further,

$$\xi_2 \in H^1(0, \infty; H^{\frac{3}{2} - \varepsilon}(\Omega)) \quad \text{and} \quad \phi_2 \in H^1(0, \infty; \mathbf{H}^{\frac{3}{2} - \varepsilon}(\Omega)).$$

Now, consider

$$(\phi \cdot \nabla)\phi = (\phi_1 \cdot \nabla)\phi_1 + (\phi_1 \cdot \nabla)\phi_2 + (\phi_2 \cdot \nabla)\phi_1 + (\phi_2 \cdot \nabla)\phi_2. \quad (6.8)$$

We are going to analyse each term separately.

If Ω satisfies cone property in \mathbb{R}^n with $n > 2s > 0$, from [22, Theorem 7.57, Page 217] it follows that

$$\mathbf{H}^s(\Omega) \hookrightarrow \mathbf{L}^r(\Omega) \quad \text{for} \quad 2 \leq r \leq \frac{2n}{n - 2s}. \quad (6.9)$$

Thus, we get $\mathbf{H}_{\Gamma_d}^{1-\varepsilon}(\Omega) \hookrightarrow \mathbf{L}^{\frac{2}{\varepsilon}}(\Omega)$. By duality, we have

$$L^2(0, \infty; \mathbf{L}^{\frac{2}{2-\varepsilon}}(\Omega)) \hookrightarrow L^2(0, \infty; \mathbf{H}_{\Gamma_d}^{-1+\varepsilon}(\Omega)). \quad (6.10)$$

Thus, it is enough to prove that $(\phi \cdot \nabla)\phi \in L^2(0, \infty; \mathbf{L}^{\frac{2}{2-\varepsilon}}(\Omega))$.

Estimate of the first term of (6.8). As $(\xi_1, \phi_1) \in X_1$, by interpolating between the spaces $L^2(0, \infty; D((\lambda_0 I - \mathcal{A})^{\frac{1}{2} + \frac{\varepsilon}{2}}))$ and $H^{\frac{1}{2} + \frac{\varepsilon}{2}}(0, \infty; Z)$, we obtain:

$$\begin{aligned} (\xi_1, \phi_1) \in L^\infty(0, \infty; D((\lambda_0 I - \mathcal{A})^{\frac{\varepsilon}{2}})) &= L^\infty(0, \infty; H^\varepsilon(\Omega) \times \mathbf{V}_{\Gamma_d}^\varepsilon(\Omega)) \\ &\hookrightarrow L^\infty(0, \infty; L^{\frac{2}{1-\varepsilon}}(\Omega) \times \mathbf{L}^{\frac{2}{1-\varepsilon}}(\Omega)), \end{aligned}$$

where the last inclusion will follow from (6.9). Also from (6.7), we have:

$$\nabla \phi_1 \in L^2(0, \infty; \mathbf{H}^{\frac{\varepsilon}{2} - \varepsilon^2}(\Omega)) \hookrightarrow L^2(0, \infty; \mathbf{L}^{\frac{4}{2-\varepsilon+2\varepsilon^2}}(\Omega)). \quad (6.11)$$

As $\left(\frac{1-\varepsilon}{2} + \frac{2-\varepsilon+2\varepsilon^2}{4}\right) = \frac{4-3\varepsilon+2\varepsilon^2}{4}$, by applying Hölder's inequality, we obtain

$$(\phi_1 \cdot \nabla) \phi_1 \in L^2(0, \infty; \mathbf{L}^{\frac{4}{4-3\varepsilon+2\varepsilon^2}}(\Omega)). \quad (6.12)$$

It is immediate from $\frac{4-3\varepsilon+2\varepsilon^2}{4} < \frac{2-\varepsilon}{2}$ that:

$$(\phi_1 \cdot \nabla) \phi_1 \in L^2(0, \infty; \mathbf{L}^{\frac{2}{2-\varepsilon}}(\Omega)).$$

Moreover, from (6.10), (6.11), (6.12):

$$\begin{aligned} \|(\phi_1 \cdot \nabla) \phi_1\|_{L^2(0, \infty; \mathbf{H}^{-1+\varepsilon}(\Omega))} &\leq C \|(\phi_1 \cdot \nabla) \phi_1\|_{L^2(0, \infty; \mathbf{L}^{\frac{2}{2-\varepsilon}}(\Omega))} \\ &\leq C \|(\phi_1 \cdot \nabla) \phi_1\|_{L^2(0, \infty; \mathbf{L}^{\frac{4}{4-3\varepsilon+2\varepsilon^2}}(\Omega))} \\ &\leq C \|\phi_1\|_{L^\infty(0, \infty; \mathbf{L}^{\frac{2}{1-\varepsilon}}(\Omega))} \|\nabla \phi_1\|_{L^2(0, \infty; \mathbf{L}^{\frac{4}{2-\varepsilon+2\varepsilon^2}}(\Omega))} \\ &\leq C \|\phi_1\|_{L^\infty(0, \infty; \mathbf{V}_{\Gamma_d}^\varepsilon(\Omega))} \|\phi_1\|_{L^2(0, \infty; \mathbf{H}^{1+\frac{\varepsilon}{2}-\varepsilon^2}(\Omega))} \\ &\leq C \|(\xi, \phi)\|_X^2. \end{aligned} \quad (6.13)$$

Estimate of the second term of (6.8). We have already established that $\phi_1 \in L^\infty(0, \infty; \mathbf{L}^{\frac{2}{1-\varepsilon}}(\Omega))$. Also we know that $\phi_2 \in H^1(0, \infty; \mathbf{H}^{\frac{3}{2}-\varepsilon}(\Omega))$. Again, (6.9) implies

$$\nabla \phi_2 \in H^1(0, \infty; \mathbf{H}^{\frac{1}{2}-\varepsilon}(\Omega)) \hookrightarrow H^1(0, \infty; \mathbf{L}^{\frac{4}{2\varepsilon+1}}(\Omega)).$$

As $\left(\frac{1-\varepsilon}{2} + \frac{1+2\varepsilon}{4}\right) = \frac{3}{4} < \frac{2-\varepsilon}{2}$, we can conclude as before $(\phi_1 \cdot \nabla) \phi_2 \in L^2(0, \infty; \mathbf{L}^{\frac{2}{2-\varepsilon}}(\Omega))$ with the estimate

$$\|(\phi_1 \cdot \nabla) \phi_2\|_{L^2(0, \infty; \mathbf{H}^{-1+\varepsilon}(\Omega))} \leq C \|(\xi, \phi)\|_X^2. \quad (6.14)$$

Estimate of the third term of (6.8). We have

$$\begin{aligned} \phi_2 \in H^1(0, \infty; \mathbf{H}^{\frac{3}{2}-\varepsilon}(\Omega)) &\hookrightarrow L^\infty((0, \infty) \times \Omega; \mathbb{R}^2), \\ \nabla \phi_1 \in L^2(0, \infty; \mathbf{L}^{\frac{4}{2-\varepsilon+2\varepsilon^2}}(\Omega)) &\hookrightarrow L^2(0, \infty; \mathbf{L}^{\frac{2}{2-\varepsilon}}(\Omega)). \end{aligned}$$

Thus, $(\phi_2 \cdot \nabla) \phi_1 \in L^2(0, \infty; \mathbf{L}^{\frac{2}{2-\varepsilon}}(\Omega))$ with the estimate

$$\|(\phi_2 \cdot \nabla) \phi_1\|_{L^2(0, \infty; \mathbf{H}^{-1+\varepsilon}(\Omega))} \leq C \|(\xi, \phi)\|_X^2. \quad (6.15)$$

Estimate of the fourth term of (6.8). We already know that

$$\begin{aligned}\phi_2 &\in H^1(0, \infty; \mathbf{H}^{\frac{3}{2}-\varepsilon}(\Omega)) \hookrightarrow L^\infty((0, \infty) \times \Omega; \mathbb{R}^2), \\ \nabla \phi_2 &\in H^1(0, \infty; \mathbf{H}^{\frac{1}{2}-\varepsilon}(\Omega)) \hookrightarrow L^\infty(0, \infty; \mathbf{L}^{\frac{4}{2\varepsilon+1}}(\Omega)) \hookrightarrow L^\infty(0, \infty; \mathbf{L}^{\frac{2}{2-\varepsilon}}(\Omega)).\end{aligned}$$

Hence, $(\phi_2 \cdot \nabla)\phi_2 \in L^2(0, \infty; \mathbf{L}^{\frac{2}{2-\varepsilon}}(\Omega))$ with the estimate

$$\|(\phi_2 \cdot \nabla)\phi_2\|_{L^2(0, \infty; \mathbf{H}^{-1+\varepsilon}(\Omega))} \leq C\|(\xi, \phi)\|_X^2. \quad (6.16)$$

Step 2. The proof of (6.4) is similar to the above one. For the Lipschitz estimates, if $(\xi^1, \phi^1), (\xi^2, \phi^2) \in X$, we have

$$\begin{aligned}&\|(\phi^1 \cdot \nabla)\phi^1 - (\phi^2 \cdot \nabla)\phi^2\|_{L^2(0, \infty; \mathbf{H}_{\Gamma_d}^{-1+\varepsilon}(\Omega))} \\ &= \|(\phi^1 \cdot \nabla)(\phi^1 - \phi^2) + ((\phi^1 - \phi^2) \cdot \nabla)\phi^2\|_{L^2(0, \infty; \mathbf{H}_{\Gamma_d}^{-1+\varepsilon}(\Omega))}.\end{aligned}$$

We can write $(\xi^i, \phi^i) = (\xi_1^i, \phi_1^i) + (\xi_2^i, \phi_2^i)$, for any $(\xi^i, \phi^i) \in X$ with $(\xi_1^i, \phi_1^i) \in X_1$, $(\xi_2^i, \phi_2^i) \in X_2$, for $i = 1, 2$. Using this splitting, we can decompose the terms $(\phi^1 \cdot \nabla)(\phi^1 - \phi^2)$, $((\phi^1 - \phi^2) \cdot \nabla)\phi^2$ as in (6.8) and we can estimate these terms following the proof of (6.13)–(6.16) to obtain (6.5). \square

6.2 Proof of the main result

Proof of Theorem 1.1. We want to stabilize (θ, \mathbf{v}) with the prescribed exponential decay rate $e^{-\omega t}$. So, we will make the following change of variables:

$$\hat{\theta} = e^{\omega t}\theta, \quad \hat{\mathbf{v}} = e^{\omega t}\mathbf{v} \quad \text{and} \quad \hat{p} = e^{\omega t}p. \quad (6.17)$$

Then the system satisfied by $(\hat{\theta}, \hat{\mathbf{v}}, \hat{p})$ is

$$\begin{aligned}\frac{\partial \hat{\theta}}{\partial t} - \omega \hat{\theta} - \kappa \Delta \hat{\theta} + \mathbf{u}_s \cdot \nabla \hat{\theta} + \hat{\mathbf{v}} \cdot \nabla \tau_s &= -e^{-\omega t} (\hat{\mathbf{v}} \cdot \nabla \hat{\theta}) \quad \text{in } Q_\infty, \\ \frac{\partial \hat{\theta}}{\partial n} &= 0 \quad \text{on } (\Gamma \setminus \Gamma_c) \times (0, \infty), \\ \frac{\partial \hat{\mathbf{v}}}{\partial t} - \omega \hat{\mathbf{v}} - \operatorname{div} \sigma(\hat{\mathbf{v}}, \hat{p}) + (\mathbf{u}_s \cdot \nabla) \hat{\mathbf{v}} + (\hat{\mathbf{v}} \cdot \nabla) \mathbf{u}_s - \beta \hat{\theta} \\ &= -e^{-\omega t} (\hat{\mathbf{v}} \cdot \nabla) \hat{\mathbf{v}} \quad \text{in } Q_\infty, \\ \operatorname{div} \hat{\mathbf{v}} &= 0 \quad \text{in } Q_\infty, \quad \hat{\mathbf{v}} = 0 \quad \text{on } \Gamma_w \times (0, \infty), \quad \sigma(\hat{\mathbf{v}}, \hat{p}) \mathbf{n} = 0 \quad \text{on } \Gamma_n \times (0, \infty), \\ (\hat{\theta}, \hat{\mathbf{v}})^T &= - \sum_{i=1}^N e^{-\omega t} [\mathcal{B}_u^* \mathcal{P}_u \pi_u(\theta, \mathbf{v})^T]_i \mathbf{g}_i \quad \text{on } \Gamma_c \times (0, \infty), \\ \hat{\theta}(0) &= \theta_0 \quad \text{and} \quad \Pi \hat{\mathbf{v}}(0) = \mathbf{v}_0 \quad \text{in } \Omega.\end{aligned} \quad (6.18)$$

Thus it is enough to prove that equation (6.18) admits a unique solution in the ball $\hat{B}_\mu = \{(\hat{\theta}, \hat{\mathbf{v}}) \in X : \|(\hat{\theta}, \hat{\mathbf{v}})\|_X \leq \mu\}$. We can define the mapping M on X , given by:

$$M : (\xi, \phi) \mapsto (\theta_{\xi, \phi}, \mathbf{v}_{\xi, \phi}),$$

where $(\theta_{\xi,\phi}, \mathbf{v}_{\xi,\phi})$ will satisfy the following system:

$$\begin{aligned}
\frac{\partial \theta_{\xi,\phi}}{\partial t} - \omega \theta_{\xi,\phi} - \kappa \Delta \theta_{\xi,\phi} + \mathbf{u}_s \cdot \nabla \theta_{\xi,\phi} + \mathbf{v}_{\xi,\phi} \cdot \nabla \tau_s \\
&= -e^{-\omega t} (\phi \cdot \nabla \xi) \quad \text{in } Q_\infty, \\
\frac{\partial \theta_{\xi,\phi}}{\partial n} &= 0 \quad \text{on } (\Gamma \setminus \Gamma_c) \times (0, \infty), \\
\frac{\partial \mathbf{v}_{\xi,\phi}}{\partial t} - \omega \mathbf{v}_{\xi,\phi} - \operatorname{div} \sigma(\mathbf{v}_{\xi,\phi}, p_{\xi,\phi}) + (\mathbf{u}_s \cdot \nabla) \mathbf{v}_{\xi,\phi} + (\mathbf{v}_{\xi,\phi} \cdot \nabla) \mathbf{u}_s - \beta \theta_{\xi,\phi} \\
&= -e^{-\omega t} (\phi \cdot \nabla) \phi \quad \text{in } Q_\infty, \\
\operatorname{div} \mathbf{v}_{\xi,\phi} &= 0 \quad \text{in } Q_\infty, \\
\mathbf{v}_{\xi,\phi} &= 0 \quad \text{on } \Gamma_w \times (0, \infty), \quad \sigma(\mathbf{v}_{\xi,\phi}, p_{\xi,\phi}) \mathbf{n} = 0 \quad \text{on } \Gamma_n \times (0, \infty), \\
(\theta_{\xi,\phi}, \mathbf{v}_{\xi,\phi})^T &= - \sum_{i=1}^N e^{-\omega t} [\mathcal{B}_u^* \mathcal{P}_u \pi_u (\theta_{\xi,\phi}, \mathbf{v}_{\xi,\phi})^T]_i \mathbf{g}_i \quad \text{on } \Gamma_c \times (0, \infty), \\
\theta_{\xi,\phi}(0) &= \theta_0 \quad \text{and} \quad \Pi \mathbf{v}_{\xi,\phi}(0) = \mathbf{v}_0 \quad \text{in } \Omega.
\end{aligned}$$

Our first aim is to show that M is a mapping from \widehat{B}_μ to itself. Let $\|(\xi, \phi)\|_X \leq \mu$. We set

$$\mu_0 = \frac{1}{4C^2}, \quad \text{and} \quad C_0 = \frac{3}{4C}. \quad (6.19)$$

If $(\xi, \phi) \in \widehat{B}_\mu$ and $0 < \mu < \mu_0$, with Theorem 5.3 and Lemma 6.1, we obtain:

$$\begin{aligned}
\|M(\xi, \phi)\|_X &= \|(\theta_{\xi,\phi}, \mathbf{v}_{\xi,\phi})\|_X \\
&\leq C \left(\|(\phi \cdot \nabla \xi), (\phi \cdot \nabla) \phi\|_{L^2(0,\infty; H_{\Gamma_c}^{-1+\varepsilon}(\Omega)) \times L^2(0,\infty; \mathbf{H}_{\Gamma_d}^{-1+\varepsilon}(\Omega))} + \|(\theta_0, \mathbf{v}_0)\|_{H^\varepsilon(\Omega) \times \mathbf{V}_{n,\Gamma_d}^\varepsilon(\Omega)} \right) \\
&\leq C \left(\|(\xi, \phi)\|_X^2 + \|(\theta_0, \mathbf{v}_0)\|_{H^\varepsilon(\Omega) \times \mathbf{V}_{n,\Gamma_d}^\varepsilon(\Omega)} \right) \\
&\leq C \left(C\mu^2 + \frac{3\mu}{4C} \right) \leq \mu.
\end{aligned}$$

If $(\xi^1, \phi^1), (\xi^2, \phi^2) \in \widehat{B}_\mu$, with Theorem 5.3 and (6.5), we have:

$$\begin{aligned}
&\|M(\xi^1, \phi^1) - M(\xi^2, \phi^2)\|_X \\
&\leq C \|((\phi^1 \cdot \nabla \xi^1), (\phi^1 \cdot \nabla) \phi^1) - ((\phi^2 \cdot \nabla) \xi^2, (\phi^2 \cdot \nabla) \phi^2)\|_{L^2(0,\infty; H_{\Gamma_c}^{-1+\varepsilon}(\Omega)) \times \mathbf{H}_{\Gamma_d}^{-1+\varepsilon}(\Omega)} \\
&\leq C^2 (\|(\xi^1, \phi^1)\|_X + \|(\xi^2, \phi^2)\|_X) \|(\xi^1, \phi^1) - (\xi^2, \phi^2)\|_X \\
&\leq 2\mu C^2 \|(\xi^1, \phi^1) - (\xi^2, \phi^2)\|_X.
\end{aligned}$$

Thus, if $0 < \mu < \mu_0$ and $\|(\theta_0, \mathbf{v}_0)\|_{H^\varepsilon(\Omega) \times \mathbf{V}_{n,\Gamma_d}^\varepsilon(\Omega)} \leq C_0 \mu$, M is a strict contraction in \widehat{B}_μ and the system (6.18) admits a unique solution in \widehat{B}_μ . \square

A Unique Continuation for the Linearized Boussinesq System

We establish a unique continuation result for the linearized Boussinesq system, adapting the local Carleman inequality proved in [9] for the Oseen equation. Such unique continuation result for Oseen equation has been obtained earlier in [23]. However, the authors in [9] prove different Carleman estimates than [23].

Theorem Appendix A.1. Let \mathcal{O} be a domain in \mathbb{R}^2 , $\lambda \in \mathbb{C}$, $\tilde{\omega}$ be a non empty subset of \mathcal{O} such that $\text{dist}(\partial\tilde{\omega}, \partial\mathcal{O}) > 0$. Assume that $(\tau_s, \mathbf{u}_s) \in H^{2+\varepsilon_0}(\mathcal{O}) \times \mathbf{H}^{2+\varepsilon_0}(\mathcal{O})$. Let $(\xi, \phi, \psi) \in H^2(\mathcal{O}) \times \mathbf{H}^2(\mathcal{O}) \times H^1(\mathcal{O})$ be a solution to the following problem:

$$\begin{aligned} -\beta \cdot \phi - \Delta\xi - \mathbf{u}_s \cdot \nabla\xi &= \lambda\xi \text{ in } \mathcal{O}, \\ -\Delta\phi - (\mathbf{u}_s \cdot \nabla)\phi + (\nabla\mathbf{u}_s)^T\phi + \nabla\psi + (\nabla\tau_s)^T\xi &= \lambda\phi \text{ in } \mathcal{O}, \\ \text{div } \phi &= 0 \text{ in } \mathcal{O}. \end{aligned} \tag{A.1}$$

If $\xi = 0$, $\phi = \mathbf{0}$ in $\tilde{\omega}$, then $\phi = 0 = \xi$ in \mathcal{O} and $\psi \equiv \text{constant}$ in \mathcal{O} .

Proof. This proof consists of mainly three steps. In the first step we construct a cut-off function χ and we write equations satisfied by $\chi\phi, \chi\xi, \chi\psi$. In the second step we establish some Carleman estimates and in the final step we deduce the unique continuation for the Linearized Boussinesq System.

Step 1. Equations for $(\chi\phi, \chi\psi), \chi\xi$. Since $\phi = 0 = \xi$ in $\tilde{\omega}$, $\psi = \text{constant}$ in $\tilde{\omega}$. Without loss of generality, we may assume that $\psi = 0$ in $\tilde{\omega}$. Let us set

$$\mathcal{O}_{-\varepsilon} = \{x \in \mathcal{O} \mid \text{dist}(x, \partial\mathcal{O}) > \varepsilon\}.$$

We choose $\varepsilon > 0$ sufficiently small to have $\text{dist}(\partial\tilde{\omega}, \partial\mathcal{O}_{-\varepsilon}) > 0$.

Let χ be a smooth, non-negative, cut-off function, with values in $[0, 1]$, such that

$$\chi(x) = 1 \text{ if } x \in \mathcal{O}_{-\varepsilon/2} \text{ and } \chi(x) = 0 \text{ if } x \in \mathcal{O} \setminus \mathcal{O}_{-\varepsilon/4}.$$

The equations satisfied by $\chi\phi$ and $\chi\xi$ are

$$\begin{aligned} -\Delta(\chi\phi) - (\mathbf{u}_s \cdot \nabla)(\chi\phi) + (\nabla\mathbf{u}_s)^T(\chi\phi) + \nabla(\chi\psi) - \lambda(\chi\phi) &= \mathbf{f}, \\ -\Delta(\chi\xi) - \mathbf{u}_s \cdot \nabla(\chi\xi) - \lambda(\chi\xi) &= \beta \cdot (\chi\phi) + f_2, \end{aligned} \tag{A.2}$$

where $\mathbf{f} = \mathbf{f}_1 + (\nabla\tau_s)^T(\chi\xi)$, $f_2 = -\xi\Delta\chi - 2\nabla\chi \cdot \nabla\xi - \xi(\mathbf{u}_s \cdot \nabla\chi)$ with

$$\mathbf{f}_1 = -\phi\Delta\chi - 2(\nabla\phi)(\nabla\chi) + \phi(\mathbf{u}_s \cdot \nabla\chi) + \psi\nabla\chi.$$

Also observe that $\text{supp}(\mathbf{f}_1) \subset \mathcal{O}_{-\varepsilon/4} \setminus \mathcal{O}_{-\varepsilon/2}$ and $\text{supp}(f_2) \subset \mathcal{O}_{-\varepsilon/4} \setminus \mathcal{O}_{-\varepsilon/2}$.

Step 2. Carleman estimates. We introduce the function $\varphi_\tau(x) = e^{\tau d(x)}$, where $\tau > 0$ is a free parameter and

- $d(x) \in C^2(\overline{\mathcal{O}})$ function, with no critical points outside $\tilde{\omega}$, i.e. $\inf_{x \in \overline{\mathcal{O}} \setminus \tilde{\omega}} |\nabla d(x)| = k > 0$,
- $d(x) \geq 1$ on $\overline{\mathcal{O}}$, $\max_{x \in \overline{\mathcal{O}} \setminus \mathcal{O}_{-\varepsilon/2}} d(x) = d^* < d_1 = \min_{x \in \overline{\mathcal{O}}_{-\varepsilon}} d(x)$.

Carleman estimate for $(\chi\phi, \chi\psi)$. Observe that $(\chi\phi, \chi\psi) \in \mathbf{H}_0^2(\mathcal{O}) \times H_0^1(\mathcal{O})$ with $(\chi\phi)|_{\partial\mathcal{O}} = \mathbf{0} = \nabla(\chi\phi)|_{\partial\mathcal{O}}$ and $(\chi\psi)|_{\partial\mathcal{O}} = 0 = \nabla(\chi\psi)|_{\partial\mathcal{O}}$. By using local Carleman inequality in [9, (A.9)] for functions $(\chi\phi, \chi\psi) \in \mathbf{H}_0^2(\mathcal{O}) \times H_0^1(\mathcal{O})$ satisfying (A.2)₁, we obtain:

There exist $C > 0$ and $\widehat{\tau} > 1$ such that for all $\tau > \widehat{\tau}$, there exists $\widehat{s}(\tau)$ such that for all $s > \widehat{s}(\tau)$

$$\begin{aligned}
& [ll] \int_{\mathcal{O}} e^{2s\varphi_\tau(x)} \left(|\nabla(\chi\phi)|^2 + s^2\tau^2 e^{2\tau d(x)} |\chi\phi|^2 + s\tau^2 e^{\tau d(x)} |\chi\psi - \operatorname{div}(\chi\phi)|^2 \right) dx \\
& \leq C \int_{\mathcal{O}} e^{2s\varphi_\tau(x)} \left(s e^{\tau d(x)} |\operatorname{div}(\chi\phi)|^2 + |-\lambda(\chi\phi) - \Delta(\chi\phi) - (\mathbf{u}_s \cdot \nabla)(\chi\phi) \right. \\
& \qquad \qquad \qquad \left. + (\nabla \mathbf{u}_s)^T(\chi\phi) + \nabla(\chi\psi)|^2 \right) dx \\
& \leq C \int_{\mathcal{O}} e^{2s\varphi_\tau(x)} \left(s e^{\tau d(x)} |\operatorname{div}(\chi\phi)|^2 + |\mathbf{f}_1|^2 + |\chi\xi|^2 \right) dx, \tag{A.3}
\end{aligned}$$

where $C > 0$ does not depend on ϕ, ψ . Here, we have used the fact that $\tau_s \in H^{2+\varepsilon_0}(\mathcal{O})$ in order to have $\nabla \tau_s \in \mathbf{L}^\infty(\mathcal{O})$.

Carleman estimate for $\chi\xi$. We use the Carleman estimate [24, Theorem 1.2] to equation (A.2)₂. Observe that $\chi\xi \in H_0^1(\mathcal{O})$ and $\chi\xi = 0 = \nabla(\chi\xi)$ in $\widetilde{\omega}$.

There exist $C > 0$ and $\widehat{\tau} > 1$ such that for all $\tau > \widehat{\tau}$, there exists $\widehat{s}(\tau)$ such that for all $s > \widehat{s}(\tau)$,

$$\begin{aligned}
& [ll] \int_{\mathcal{O}} e^{2s\varphi_\tau(x)} \left[|\nabla(\chi\xi)|^2 + s^2\tau^2 e^{2\tau d(x)} |\chi\xi|^2 \right] dx \\
& \leq C \int_{\mathcal{O}} e^{2s\varphi_\tau(x)} s^{-1}\tau^{-2} e^{-\tau d(x)} (|f_2|^2 + |\chi\phi|^2) dx, \tag{A.4}
\end{aligned}$$

where $C > 0$ does not depend on ξ .

Combined Carleman estimates. Now, by using the bound for $\chi\phi$ from (A.3), the estimate (A.4) gives

$$\begin{aligned}
& [ll] \int_{\mathcal{O}} e^{2s\varphi_\tau(x)} \left[|\nabla(\chi\xi)|^2 + s^2\tau^2 e^{2\tau d(x)} |\chi\xi|^2 \right] dx \\
& \leq C \int_{\mathcal{O}} e^{2s\varphi_\tau(x)} s^{-1}\tau^{-2} e^{-\tau d(x)} |f_2|^2 dx \\
& + C \int_{\mathcal{O}} e^{2s\varphi_\tau(x)} s^{-1}\tau^{-2} \left(s e^{\tau d(x)} |\operatorname{div}(\chi\phi)|^2 + |\mathbf{f}_1|^2 + |\chi\xi|^2 \right) dx. \tag{A.5}
\end{aligned}$$

That implies

$$\begin{aligned}
& [ll] \int_{\mathcal{O}} e^{2s\varphi_\tau(x)} \left[|\nabla(\chi\xi)|^2 + (s^2\tau^2 e^{2\tau d(x)} - C s^{-1}\tau^{-2}) |\chi\xi|^2 \right] dx \leq \\
& C \int_{\mathcal{O}} e^{2s\varphi_\tau(x)} s^{-1}\tau^{-2} e^{-\tau d(x)} |f_2|^2 dx + C \int_{\mathcal{O}} e^{2s\varphi_\tau(x)} s^{-1}\tau^{-2} \left(s e^{\tau d(x)} |\operatorname{div}(\chi\phi)|^2 + |\mathbf{f}_1|^2 \right) dx. \tag{A.6}
\end{aligned}$$

By choosing s large enough, there exists $C_0 > 0$ such that

$$(s^2\tau^2 e^{2\tau d(x)} - C s^{-1}\tau^{-2}) |\chi\xi|^2 \geq C_0 s^2 \tau^2 e^{2\tau d(x)}. \tag{A.7}$$

Thus, with the help of (A.7), estimate (A.6) becomes

$$\begin{aligned}
& [ll] \int_{\mathcal{O}} e^{2s\varphi_\tau(x)} \left[|\nabla(\chi\xi)|^2 + s^2\tau^2 e^{2\tau d(x)} |\chi\xi|^2 \right] dx \leq \\
& C \int_{\mathcal{O}} e^{2s\varphi_\tau(x)} s^{-1}\tau^{-2} e^{-\tau d(x)} |f_2|^2 dx + C \int_{\mathcal{O}} e^{2s\varphi_\tau(x)} s^{-1}\tau^{-2} \left(s e^{\tau d(x)} |\operatorname{div}(\chi\phi)|^2 + |\mathbf{f}_1|^2 \right) dx.
\end{aligned} \tag{A.8}$$

Now, by using the bound for $\chi\xi$ from (A.8), the estimate (A.3) gives

$$\begin{aligned}
& [ll] \int_{\mathcal{O}} e^{2s\varphi_\tau(x)} \left(|\nabla(\chi\phi)|^2 + s^2\tau^2 e^{2\tau d(x)} |\chi\phi|^2 + s\tau^2 e^{\tau d(x)} |\chi\psi - \operatorname{div}(\chi\phi)|^2 \right) dx \leq \\
& C \int_{\mathcal{O}} e^{2s\varphi_\tau(x)} s^{-1}\tau^{-2} e^{-\tau d(x)} |f_2|^2 dx + C \int_{\mathcal{O}} e^{2s\varphi_\tau(x)} s^{-1}\tau^{-2} \left(s e^{\tau d(x)} |\operatorname{div}(\chi\phi)|^2 + |\mathbf{f}_1|^2 \right) dx \\
& \quad + C \int_{\mathcal{O}} e^{2s\varphi_\tau(x)} \left(s e^{\tau d(x)} |\operatorname{div}(\chi\phi)|^2 + |\mathbf{f}_1|^2 \right). \tag{A.9}
\end{aligned}$$

Hence, we have

$$\begin{aligned}
& [ll] \int_{\mathcal{O}} e^{2s\varphi_\tau(x)} \left(|\nabla(\chi\phi)|^2 + s^2\tau^2 e^{2\tau d(x)} |\chi\phi|^2 + s\tau^2 e^{\tau d(x)} |\chi\psi - \operatorname{div}(\chi\phi)|^2 \right) dx \\
& \leq C \int_{\mathcal{O}} e^{2s\varphi_\tau(x)} (|\mathbf{f}_1|^2 + |f_2|^2) dx + C \int_{\mathcal{O}} e^{2s\varphi_\tau(x)} s e^{\tau d(x)} |\operatorname{div}(\chi\phi)|^2 dx \\
& = C \int_{\mathcal{O}} e^{2s\varphi_\tau(x)} (|\mathbf{f}_1|^2 + |f_2|^2) dx + C \int_{\mathcal{O}} e^{2s\varphi_\tau(x)} s e^{\tau d(x)} |\nabla\chi \cdot \phi|^2 dx. \tag{A.10}
\end{aligned}$$

In the last equality, we use the fact that $\operatorname{div}(\chi\phi) = \nabla\chi \cdot \phi$ as $\operatorname{div}\phi = 0$.

Step 3. Unique Continuation. Observe that

$$\mathbf{f}_1 = f_2 = \nabla\chi = 0 \quad \text{in } \mathcal{O}_{-\varepsilon/2}. \tag{A.11}$$

In virtue of estimate (A.9) and observation (A.11), we obtain

$$\begin{aligned}
& [ll] \int_{\mathcal{O}_{-\varepsilon}} e^{2s\varphi_\tau(x)} \left(|\nabla(\chi\phi)|^2 + s^2\tau^2 e^{2\tau d(x)} |\chi\phi|^2 + s\tau^2 e^{\tau d(x)} |\chi\psi - \operatorname{div}(\chi\phi)|^2 \right) dx \\
& \leq C \int_{\mathcal{O} \setminus \mathcal{O}_{-\varepsilon/2}} e^{2s\varphi_\tau(x)} (|\mathbf{f}_1|^2 + |f_2|^2) dx + C \int_{\mathcal{O} \setminus \mathcal{O}_{-\varepsilon/2}} e^{2s\varphi_\tau(x)} s e^{\tau d(x)} |\nabla\chi \cdot \phi|^2 dx. \tag{A.12}
\end{aligned}$$

Recall that

$$d(x) \geq 1, x \in \mathcal{O}, \quad \max_{x \in \mathcal{O} \setminus \mathcal{O}_{-\varepsilon/2}} d(x) = d^* < d_1 = \min_{x \in \overline{\mathcal{O}_{-\varepsilon}}} d(x). \tag{A.13}$$

Hence, by using (A.13) and the fact that $\chi \equiv 1$ in $\mathcal{O}_{-\varepsilon}$, estimate (A.12) gives

$$\begin{aligned} & \int_{\mathcal{O}_{-\varepsilon}} (|\phi(x)|^2 + |\psi(x)|^2) \\ & \leq C \exp[2s(e^{\tau d^*} - e^{\tau d_1})] s^{-1} \tau^{-2} e^{-\tau d_1} \int_{\mathcal{O} \setminus \mathcal{O}_{-\varepsilon/2}} (|\mathbf{f}_1|^2 + |f_2|^2) dx \\ & \quad + C \exp[2s(e^{\tau d^*} - e^{\tau d_1})] \tau^{-2} (e^{\tau d^* - \tau d_1}) \int_{\mathcal{O} \setminus \mathcal{O}_{-\varepsilon/2}} |\nabla \chi \cdot \phi|^2 dx. \end{aligned}$$

Since $d^* < d_1$, by taking s large enough, we obtain $|\phi(x)|^2 + |\psi(x)|^2 = 0$ in $\mathcal{O}_{-\varepsilon}$. Hence,

$$\phi(x) = 0 = \psi(x), \quad \text{in } \mathcal{O}_{-\varepsilon}. \quad (\text{A.14})$$

Since, $\varepsilon > 0$ can be chosen arbitrarily small, we obtain

$$\phi(x) = 0 = \psi(x), \quad \text{in } \mathcal{O}. \quad (\text{A.15})$$

Then by using equation (A.1), we can also conclude that $\xi = 0$ in \mathcal{O} . \square

Theorem Appendix A.2. *Suppose (τ_s, \mathbf{u}_s) satisfies assumption (H_5) in Section 2.1. For any $\lambda_j \in \Sigma$ with $\text{Re } \lambda_j > -\omega$, if*

$$\begin{aligned} & \lambda_j \xi - \kappa \Delta \xi - \mathbf{u}_s \cdot \nabla \xi - \beta \cdot \phi = 0 \text{ in } \Omega, \\ & \xi = 0 \text{ on } \Gamma_c, \quad \kappa \frac{\partial \xi}{\partial n} + (\mathbf{u}_s \cdot \mathbf{n}) \xi = 0 \text{ on } \Gamma \setminus \Gamma_c, \\ & \lambda_j \phi - \text{div } \sigma(\phi, \psi) - (\mathbf{u}_s \cdot \nabla) \phi + (\nabla \mathbf{u}_s)^T \phi + (\nabla \tau_s)^T \xi = 0 \text{ in } \Omega, \\ & \text{div } \phi = 0 \text{ in } \Omega, \quad \phi = 0 \text{ on } \Gamma_d, \quad \sigma(\phi, \psi) \mathbf{n} + (\mathbf{u}_s \cdot \mathbf{n}) \phi = 0 \text{ on } \Gamma_n \\ & \text{and } \frac{\partial \xi}{\partial n} = 0 \text{ on } \tilde{\Gamma}_c, \quad \sigma(\phi, \psi) \mathbf{n} = \mathbf{0} \text{ on } \tilde{\Gamma}_c, \end{aligned} \quad (\text{A.16})$$

then $(\xi, \phi) = (0, 0)$ in Ω .

Proof. Let Ω_e be an extension of Ω such that $\hat{\Omega} = \Omega_e \setminus \bar{\Omega}$ satisfies $\partial \hat{\Omega} \cap \Gamma = \tilde{\Gamma}_c$. We extend (ξ, ϕ, ψ) by zero across the boundary $\tilde{\Gamma}_c$ onto $\hat{\Omega}$. For that, we set

$$\hat{\phi} = \begin{cases} \phi & \text{in } \Omega, \\ \mathbf{0} & \text{in } \hat{\Omega}, \end{cases} \quad \hat{\psi} = \begin{cases} \psi & \text{in } \Omega, \\ 0 & \text{in } \hat{\Omega}, \end{cases} \quad \hat{\xi} = \begin{cases} \xi & \text{in } \Omega, \\ 0 & \text{in } \hat{\Omega}. \end{cases}$$

Recall that $(\tau_s, \mathbf{u}_s)|_{\Omega_{c,\varepsilon}} \in H^{2+\varepsilon_0}(\Omega_{c,\varepsilon}) \times \mathbf{H}^{2+\varepsilon_0}(\Omega_{c,\varepsilon})$ for all $\varepsilon > 0$ and for some $\varepsilon_0 > 0$, where $\Omega_{c,\varepsilon} = \{x \in \Omega \mid \text{dist}(x, \Gamma \setminus \Gamma_c) > \varepsilon\}$. Now, we fix $\varepsilon > 0$ small enough so that $\tilde{\Gamma}_c \subset \Gamma_{c,\varepsilon} = \{x \in \Gamma_c \mid \text{dist}(x, \text{Bd}(\Gamma_c)) > \varepsilon\}$, where $\text{Bd}(\Gamma_c) = \bar{\Gamma}_c \setminus \Gamma_c$ is the relative boundary of Γ_c . We set $\mathcal{O} = \{x \in \Omega_e \mid \text{dist}(x, \partial \Omega_e) > \varepsilon\}$. We extend (τ_s, \mathbf{u}_s) to \mathcal{O} in such a way that this extension $(\hat{\tau}_s, \hat{\mathbf{u}}_s)$ belongs to $H^{2+\varepsilon_0}(\mathcal{O}) \times \mathbf{H}^{2+\varepsilon_0}(\mathcal{O})$, see [18, Remark 5, Chapter IV, Page 117], [25, Theorem 5.4].

It is easy to see that the triplet $(\hat{\xi}, \hat{\phi}, \hat{\psi}) \in H^2(\mathcal{O}) \times \mathbf{H}^2(\mathcal{O}) \times H^1(\mathcal{O})$ satisfies

$$\begin{aligned} & -\hat{\beta} \cdot \hat{\phi} - \kappa \Delta \hat{\xi} - \hat{\mathbf{u}}_s \cdot \nabla \hat{\xi} = \lambda_j \hat{\xi} \text{ in } \mathcal{O} \\ & -\nu \Delta \hat{\phi} - (\hat{\mathbf{u}}_s \cdot \nabla) \hat{\phi} + (\nabla \hat{\mathbf{u}}_s)^T \hat{\phi} + \nabla \hat{\psi} + (\nabla \hat{\tau}_s)^T \hat{\xi} = \lambda_j \hat{\phi} \text{ in } \mathcal{O} \\ & \text{div } \hat{\phi} = 0 \text{ in } \mathcal{O} \\ & \hat{\phi} = 0 = \hat{\xi} \text{ in } \tilde{\omega}, \end{aligned} \quad (\text{A.17})$$

where $\tilde{\omega}$ is a subset of \mathcal{O} . Now we are in the same situation as in equation (A.1). We can apply Theorem Appendix A.1 to conclude that $\hat{\phi} = 0 = \hat{\xi}$ in \mathcal{O} . \square

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