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LOCAL STABILIZATION OF COMPRESSIBLE
NAVIER-STOKES EQUATIONS IN ONE DIMENSION
AROUND NON-ZERO VELOCITY

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Abstract. In this paper we study the local stabilization of one dimensional compressible Navier-Stokes equations around a constant steady solution (ρ_s, u_s) , where $\rho_s > 0, u_s \neq 0$. *In the case of periodic boundary conditions*, we determine a distributed control acting only in the velocity equation, able to stabilize the system, locally around (ρ_s, u_s) , with an arbitrary exponential decay rate. *In the case of Dirichlet boundary conditions*, we determine boundary controls for the velocity and for the density at the inflow boundary, able to stabilize the system, locally around (ρ_s, u_s) , with an arbitrary exponential decay rate.

Key words. Compressible Navier-Stokes equations, local stabilization, feedback control, localized interior control.

1. INTRODUCTION

Stabilization of fluid flows around unstable stationary solutions is an important issue in many engineering applications (see e.g. [5]). The case of the incompressible Navier-Stokes equations has been widely studied both in the mathematical and engineering literatures [2, 3, 4, 15, 16, 17, 20]. Similar

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issues for the compressible Navier-Stokes equations are much more recent. There are a few papers studying the controllability of such systems in the one dimensional case [1, 12, 11]. The null controllability of linearized systems ([11]) or nonlinear systems ([1, 12]) implies their stabilization. But they do not give an explicit way for computing stabilizing controls. One of the goals of this paper is to fill this gap. We would like to determine stabilizing controls for the one dimensional Navier-Stokes equations, around a constant steady state $\rho_s > 0, u_s \neq 0$. We first study the local stabilization of the one dimensional compressible Navier-Stokes system, with periodic boundary conditions, by a distributed control. Next we shall see that the stabilization of the one dimensional compressible Navier-Stokes system by Dirichlet boundary controls may be deduced from this first result.

We consider the following compressible isentropic Navier-Stokes equations in the interval $(0, L)$ with periodic boundary conditions

$$\begin{aligned} \rho_t + (\rho u)_y &= 0 \quad \text{in } (0, L) \times (0, \infty), \\ \rho(u_t + uu_y) + (p(\rho))_y - \nu u_{yy} &= \rho f \chi_{(\ell_1, \ell_2)} \quad \text{in } (0, L) \times (0, \infty), \\ \rho(0, \cdot) = \rho(L, \cdot), \quad u(0, \cdot) = u(L, \cdot), \quad u_y(0, \cdot) = u_y(L, \cdot) &\quad \text{in } (0, \infty), \\ \rho(\cdot, 0) = \rho_0(\cdot), \quad u(\cdot, 0) = u_0(\cdot) &\quad \text{in } (0, L). \end{aligned} \tag{1.1}$$

Here $\rho(y, t)$ is the density, $u(y, t)$ is the fluid velocity, $\nu > 0$ is the fluid viscosity, and the pressure p is assumed to satisfy the constitutive law

$$p(\rho) = a \rho^\gamma,$$

for some constants $a > 0$ and $\gamma \geq 1$. Here, f is an interior control with support in (ℓ_1, ℓ_2) , a nonempty interval of $(0, L)$. We set

$$\Omega_y = (0, L), \quad \text{and} \quad Q_y^\infty := \Omega_y \times (0, \infty). \tag{1.2}$$

Let us first notice that any pair of constants (ρ_s, u_s) , with $\rho_s > 0$, is a steady state solution of (1.1) for $f = 0$. By integrating the first equation in (1.1) and using the periodic boundary conditions, we also observe that

$$\int_0^L \rho(y, t) dy = \int_0^L \rho_0(y) dy, \quad \forall t > 0.$$

Thus there is no effect of the control f on the mean value of the density. So we start with an initial density ρ_0 satisfying

$$\frac{1}{L} \int_0^L \rho_0(y) dy = \rho_s \quad \text{and} \quad \min_{y \in \Omega_y} \rho_0(y) > 0. \tag{1.3}$$

To study the local stabilization of (1.1) around the pair of constants (ρ_s, u_s) , where $\rho_s > 0$ and $u_s \neq 0$, we define

$$\sigma = \rho - \rho_s, \quad v = u - u_s.$$

The system satisfied by (σ, v) is

$$\begin{aligned} \sigma_t + \rho_s v_y + u_s \sigma_y + \sigma_y v + \sigma v_y &= 0 \quad \text{in } Q_y^\infty, \\ v_t + v v_y + u_s v_y + a\gamma(\sigma + \rho_s)^{\gamma-2} \sigma_y - \nu \frac{v_{yy}}{\sigma + \rho_s} &= f\chi_{(\ell_1, \ell_2)} \quad \text{in } Q_y^\infty, \\ \sigma(0, \cdot) = \sigma(L, \cdot), \quad v(0, \cdot) = v(L, \cdot), \quad v_y(0, \cdot) = v_y(L, \cdot) &\quad \text{in } (0, \infty), \\ \sigma(\cdot, 0) = \sigma_0(\cdot) = \rho_0(\cdot) - \rho_s, \quad v(\cdot, 0) = v_0(\cdot) = u_0(\cdot) - u_s &\quad \text{in } \Omega_y, \\ \frac{1}{L} \int_0^L \sigma_0(y) dy &= 0. \end{aligned} \tag{1.4}$$

Now note that σ satisfies

$$\int_0^L \sigma(y, t) dy = 0, \quad \forall t > 0.$$

To achieve the stabilization of (1.4) with exponential decay $e^{-\omega t}$, for any $\omega > 0$, it is convenient to introduce the new unknowns

$$\hat{\sigma} = e^{\omega t} \sigma, \quad \hat{v} = e^{\omega t} v, \quad \hat{f} = e^{\omega t} f.$$

We notice that $(\hat{\sigma}, \hat{v}, \hat{f})$ satisfies the system

$$\begin{aligned} \hat{\sigma}_t + u_s \hat{\sigma}_y + \rho_s \hat{v}_y - \omega \hat{\sigma} + e^{-\omega t} \{ \hat{\sigma}_y \hat{v} + \hat{\sigma} \hat{v}_y \} &= 0 \quad \text{in } Q_y^\infty, \\ \hat{v}_t + u_s \hat{v}_y + e^{-\omega t} \hat{v} \hat{v}_y + a\gamma(e^{-\omega t} \hat{\sigma} + \rho_s)^{\gamma-2} \hat{\sigma}_y & \\ - \nu \frac{\hat{v}_{yy}}{e^{-\omega t} \hat{\sigma} + \rho_s} - \omega \hat{v} &= \hat{f} \chi_{(\ell_1, \ell_2)} \quad \text{in } Q_y^\infty, \\ \hat{\sigma}(0, \cdot) = \hat{\sigma}(L, \cdot), \quad \hat{v}(0, \cdot) = \hat{v}(L, \cdot), \quad \hat{v}_y(0, \cdot) = \hat{v}_y(L, \cdot) &\quad \text{in } (0, \infty), \\ \hat{\sigma}(\cdot, 0) = \sigma_0(\cdot), \quad \hat{v}(\cdot, 0) = v_0(\cdot) \quad \text{in } \Omega_y, \quad \frac{1}{L} \int_0^L \sigma_0(y) dy &= 0. \end{aligned} \tag{1.5}$$

Thus we have

$$\frac{1}{L} \int_0^L \hat{\sigma}(y, t) dy = 0, \quad \forall t > 0.$$

To study the associated stabilization problem, we need to introduce the one dimensional Sobolev spaces with periodic boundary conditions. For $s \in \mathbb{N} \cup \{0\}$, we denote by $H_{per}^s(0, L)$, the space of L -periodic functions belonging

to $H_{loc}^s(\mathbb{R})$, and by $\dot{H}_{per}^s(0, L)$ the subspace of the functions belonging to $H_{per}^s(0, L)$, with mean value zero.

Our first main result is regarding stabilization of system (1.5).

Theorem 1.1. *Let ω be any positive number. There exist positive constants $\hat{\mu}_0$ and $\hat{\kappa}$, depending on ω , ρ_s , u_s , ℓ_1 , ℓ_2 and L , such that for all $0 < \hat{\mu} \leq \hat{\mu}_0$ and all initial condition $(\sigma_0, v_0) \in \dot{H}_{per}^1(\Omega_y) \times H_{per}^1(\Omega_y)$ satisfying*

$$\|(\sigma_0, v_0)\|_{\dot{H}_{per}^1(\Omega_y) \times H_{per}^1(\Omega_y)} \leq \hat{\kappa} \hat{\mu},$$

there exists a control $\hat{f} \in L^2(0, \infty; L^2(\Omega_y))$ for which system (1.5) admits a unique solution $(\hat{\sigma}, \hat{v})$ satisfying

$$\|(\hat{\sigma}, \hat{v})\|_{L^\infty(0, \infty; \dot{H}_{per}^1(\Omega_y)) \cap L^2(0, \infty; \dot{H}_{per}^1(\Omega_y)) \times H^1(0, \infty; L^2(\Omega_y)) \cap L^2(0, \infty; H_{per}^2(\Omega_y))} \leq \hat{\mu}.$$

Moreover, $(\hat{\sigma}, \hat{v}) \in C_b([0, \infty); \dot{H}_{per}^1(\Omega_y) \times H_{per}^1(\Omega_y))$, $|\hat{\sigma}(y, t)| \leq \frac{\rho_s}{2}$ for all $(y, t) \in Q_y^\infty$.

The above theorem leads us to the following stabilization result for system (1.1).

Theorem 1.2. (Case of periodic boundary conditions.) *Let ω be any positive number. There exist positive constants $\hat{\mu}_0$ and $\hat{\kappa}$, depending on ω , ρ_s , u_s , ℓ_1 , ℓ_2 and L , such that, for all $0 < \hat{\mu} \leq \hat{\mu}_0$ and all initial condition $(\rho_0, u_0) \in H_{per}^1(\Omega_y) \times H_{per}^1(\Omega_y)$, where ρ_0 satisfies (1.3) and (ρ_0, u_0) obeys*

$$\|(\rho_0 - \rho_s, u_0 - u_s)\|_{\dot{H}_{per}^1(\Omega_y) \times H_{per}^1(\Omega_y)} \leq \hat{\kappa} \hat{\mu},$$

there exists a control $f \in L^2(0, \infty; L^2(\Omega_y))$ for which system (1.1) admits a unique solution (ρ, u) satisfying

$$\|(\rho(\cdot, t) - \rho_s, u(\cdot, t) - u_s)\|_{\dot{H}_{per}^1(\Omega_y) \times H_{per}^1(\Omega_y)} \leq C \hat{\mu} e^{-\omega t},$$

for some positive constant C depending on ω , ρ_s , u_s , ℓ_1 , ℓ_2 and L but independent of $\hat{\mu}$. Moreover, we have $\rho(y, t) \geq \frac{\rho_s}{2}$ for all $(y, t) \in Q_y^\infty$.

The proofs of the above theorems appear in Section 4 and the details about the controls \hat{f} and f obtained via a nonlinear control law are given in Section 4.4.

It is well known that the main difficulty in studying the one dimensional compressible Navier-Stokes equations (1.1) comes from the nonlinear term $\rho_y u$. There are two classical ways to deal with that term. One way consists in using the Schauder fixed point theorem, to prove the existence of a solution to system (1.1). This method is well adapted for finite time interval $[0, T]$

and when there is no feedback control (see e.g. [12, 19]). In our case, since we look for a solution to (1.1) or (1.5) over the time interval $(0, \infty)$, the Schauder fixed point method cannot be used.

The second method consists in using a change of variables and in writing the nonlinear system in Lagrangian variables. We follow that way. Since we deal with the equations satisfied by $\hat{\sigma} = e^{\omega t}(\rho - \rho_s)$ and $\hat{v} = e^{\omega t}(u - u_s)$, we do not use the classical Lagrangian change of variables, but a modified one, adapted to system (1.5). In our situation the change of variables is defined through the solution to a transport equation and not to an ordinary differential equation. We shall refer to the transformed system (2.9), as the Lagrangian system and the transformed variable $(\tilde{\sigma}, \tilde{v})$, as the Lagrangian variables. Similarly, system (1.5) will be referred to as the Eulerian system. Finally our method consists in finding a feedback control operator able to stabilize first the linearized Lagrangian system, and next the nonlinear one. This is done by using a fixed point method. Then, coming back to the Eulerian system, we prove the stabilization of system (1.5) and hence of system (1.1).

The transformed nonlinear system presents two new difficulties. One is that the control zone is also evolving with time. Thus the control operator becomes time dependent. But there is no general stabilization method for finding a feedback control operator for nonautonomous systems. We manage this situation by choosing a fixed control domain, which lies inside each transformed control zone for every $t > 0$ (Lemma 2.7).

The second difficulty is that the nonlinear term F_1 appearing in the right hand side of the density equation of the Lagrangian system, is no longer with mean value zero. Hence the associated density is also not with mean value zero, but the control has no effect on the mean value of density. To handle this difficulty, we split F_1 and $\tilde{\sigma}$ in a unique manner as for all $x \in (0, L)$, $\forall t > 0$,

$$F_1(x, t) = F_{1,m}(x, t) + F_{1,\Omega}(t), \quad \tilde{\sigma}(x, t) = \tilde{\sigma}_m(x, t) + \tilde{\sigma}_\Omega(t).$$

Here $F_{1,m}$ and $\tilde{\sigma}_m$ are with mean value zero, and $F_{1,\Omega}$ and $\tilde{\sigma}_\Omega$ (which are the mean values of F_1 and $\tilde{\sigma}$ respectively) are functions only depending on time. We estimate the two components differently. In our fixed point argument, we will see that it is convenient to deal with a solution $\tilde{\sigma}_\Omega$ and a right hand side $F_{1,\Omega}$ in weighted Lebesgue spaces. Proceeding in this way, we prove that $\tilde{\sigma}_\Omega$ is bounded in the corresponding weighted Lebesgue space. However, we can deduce afterwards that $\tilde{\sigma}_\Omega$ is indeed bounded (see Theorem 4.4).

Next we consider the one dimensional compressible Navier-Stokes equations around (ρ_s, u_s) , $\rho_s > 0, u_s > 0$ with boundary controls

$$\begin{aligned} \rho_t + (\rho u)_y &= 0 \quad \text{in } (0, L) \times (0, \infty), \\ \rho(u_t + uu_y) + (p(\rho))_y - \nu u_{yy} &= 0 \quad \text{in } (0, L) \times (0, \infty), \\ \rho(0, \cdot) &= q_1(\cdot), \quad u(0, \cdot) = q_2(\cdot), \quad u(L, \cdot) = q_3(\cdot) \quad \text{in } (0, \infty), \\ \rho(\cdot, 0) &= \rho_0(\cdot), \quad u(\cdot, 0) = u_0(\cdot) \quad \text{in } (0, L), \\ p(\rho) &= a \rho^\gamma, \end{aligned} \tag{1.6}$$

for some constants $a > 0$ and $\gamma \geq 1$. We prove a local stabilization result for system (1.6) with initial conditions (ρ_0, u_0) close to (ρ_s, u_s) . Since $u_s > 0$, we prove that $u(y, t) \geq \frac{u_s}{2} > 0$ for all $(y, t) \in Q_y^\infty$. Thus the boundary condition for density has to be prescribed only at $y = 0$. For $u_s < 0$ the boundary condition for density should be prescribed at $y = L$ and the result of local stabilization can be easily adapted from the case $u_s > 0$. Our main theorem for this case, reads as follows.

Theorem 1.3. (Case of Dirichlet boundary controls.) *Let ω be any positive number. There exists a positive constant μ_d , depending on $\omega, \rho_s, u_s > 0$, and L , such that for any initial condition $(\rho_0, u_0) \in H^1(0, L) \times H^1(0, L)$ satisfying*

$$\min_{[0, L]} \rho_0 > 0 \quad \text{and} \quad \|(\rho_0 - \rho_s, u_0 - u_s)\|_{H^1(0, L) \times H^1(0, L)} \leq \mu_d, \tag{1.7}$$

there exist controls $q_1 \in L^2(0, \infty) \cap C_b([0, \infty))$ and $q_2, q_3 \in H^{\frac{3}{4}}(0, \infty)$, such that system (1.6) admits a unique solution (ρ, u) satisfying

$$\|(\rho(\cdot, t) - \rho_s, u(\cdot, t) - u_s)\|_{H^1(0, L) \times H^1(0, L)} \leq C e^{-\omega t}, \tag{1.8}$$

for some positive constant C depending on ω, ρ_s, u_s , and L but independent of μ_d . Furthermore, $\rho(y, t) \geq \frac{\rho_s}{2}$ and $u(y, t) \geq \frac{u_s}{2}$ for all $(y, t) \in (0, L) \times (0, \infty)$.

We prove the above stabilization result by extending system (1.6) to $(-L, L)$ with periodic boundary conditions and then using Theorem 1.2 with a control localized in $(-L, 0)$. Finally the traces of the velocity and the density at boundary give the boundary controls for system (1.6).

Chowdhury et. al ([9]) prove the exponential stabilization of compressible Navier-Stokes system in $(0, \pi)$ with homogeneous Dirichlet boundary conditions around $(\rho_s, 0)$, by using a localized control for velocity, for initial

conditions in $H^1(0, \pi) \times H^1(0, \pi)$. Our approach to prove the local stabilization of system (1.1) around (ρ_s, u_s) , $\rho_s > 0, u_s \neq 0$, with periodic boundary conditions, is also using Lagrangian coordinate transformation, similar to [9]. However, there are crucial differences in the behaviour of the two systems and hence in the techniques to handle them. While the transformation is given by an ODE in [9], it is given by a pde of transport type and hence the estimates require more intricate analysis. The linearized system around $(\bar{\rho}, 0)$ in ([9]) is not null controllable by localized control because of the accumulation point ω_0 in the spectrum of the linearized operator (see [8]). That is why in [9], the decay rate has to be chosen strictly less than ω_0 . But in our case, we are able to show that the system is locally stabilizable with exponential decay $e^{-\omega t}$, for any $\omega > 0$. When $u_s \neq 0$, even though the unstable subspace is of infinite dimension for ω arbitrarily large, the unstable eigenvalues are isolated and there is a uniform lower bound for the differences between any two eigenvalues. We are able to manage the infinite dimensional unstable spaces by using the null controllability of the linearized system associated with (1.1) by a localized control (see [11]). Furthermore, in [9], because the unstable subspace of the linearized system is of finite dimension, the feedback control operator turns out to be a Hilbert-Schmidt operator. In contrast, in our case, the infinite dimensional unstable subspace necessitates a totally different argument to get the structure of the feedback operator.

To complete the references, we mention that Ervedoza et. al ([12]) prove the local exact controllability of compressible Navier-Stokes system to constant states (ρ_s, u_s) with $\rho_s > 0, u_s \neq 0$ using boundary controls for density and velocity, when the initial conditions for density and velocity both belong in $H^3(0, L)$. Our stabilization result Theorem 1.3 is also with similar boundary controls but in less regular space. However, our approach is entirely different from that of [12]. In [10], the authors consider the linearized compressible Navier-Stokes system around (ρ_s, u_s) with $\rho_s > 0, u_s \neq 0$ with periodic boundary conditions. By the moment method they prove the null controllability of this system in $\dot{H}_{per}^{s+1} \times H_{per}^s$, for $s > 6.5$, using a localized L^2 -interior control only for the velocity equation. In [11], the null controllability of that system is obtained in $\dot{H}_{per}^1 \times L^2$ by proving an observability inequality. In [14], the authors study the stabilizability of the same linearized system with exponential decay $e^{-\omega t}$, for any $\omega > 0$, using L^2 -control acting only in velocity equation. It is proved that $\dot{H}_{per}^1 \times L^2$ is the largest space in which that system is stabilizable with any arbitrary exponential decay rate.

The plan of the paper is as follows. In Section 2, we introduce the Lagrangian change of variables and study its properties in Section 2.1. We explain how we can choose a fixed control zone in Section 2.2. In Section 3, we study the feedback stabilization of the linearized Lagrangian system. The stabilization of the nonlinear system is treated in Section 4. We state and prove the stabilization results for the Lagrangian system in Section 4.1. The Lagrangian system and its equivalence with the initial one are studied in Section 4.2. Section 4.3 is devoted to the proofs of Theorems 1.1 and 1.2. In Section 4.4 we determine the nonlinear control law for the Eulerian system. The case of Dirichlet boundary controls is studied in Section 5. For the sake of completeness, some classical proofs and estimations are added in an appendix (Section 6).

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2. REWRITING SYSTEM (1.5)

The goal of this section is to explain how we can transform system (1.5) through a change of variables. A similar approach is used in [9] when $u_s = 0$. In the case $u_s \neq 0$, the method is more complicated.

For any $s \in \mathbb{N} \cup \{0\}$, we equip the spaces

$$H_{per}^s(0, L) = \left\{ \varphi \mid \varphi = \sum_{k \in \mathbb{Z}} c_k e^{ik \frac{2\pi x}{L}}, \quad \sum_{k \in \mathbb{Z}} |k|^{2s} |c_k|^2 < \infty \right\},$$

$$\dot{H}_{per}^s(0, L) = \left\{ \varphi \in H_{per}^s(0, L) \mid \int_0^L \varphi(x) dx = 0 \right\},$$

with the norms,

$$\|\varphi\|_{H_{per}^s(0, L)} = \left(\sum_{k \in \mathbb{Z}} (1 + |k|^{2s}) |c_k|^2 \right)^{\frac{1}{2}}, \quad \|\varphi\|_{\dot{H}_{per}^s(0, L)} = \left(\sum_{k \in \mathbb{Z} \setminus \{0\}} |k|^{2s} |c_k|^2 \right)^{\frac{1}{2}}.$$

We mention that the Sobolev space $H_{per}^s(0, L)$ for $s = 0$ corresponds to $L^2(0, L)$.

Let us also recall that for any bounded open interval $(L_1, L_2) \subset \mathbb{R}$, a Sobolev constant s_0 of the embedding $H^1(L_1, L_2) \hookrightarrow L^\infty(L_1, L_2)$ can be

chosen as (see for example, Theorem 8.8 in [7])

$$s_0 = 4\sqrt{2} \left(1 + \frac{1}{L_2 - L_1} \right). \quad (2.1)$$

In particular, we shall use the notation s_0 for the interval $(0, L)$ and in this case

$$s_0 = 4\sqrt{2} \left(1 + \frac{1}{L} \right). \quad (2.2)$$

2.1. Lagrangian variables. To define properly the change of variables, in addition to $\Omega_y = (0, L)$, we introduce the notation

$$\Omega_x := (0, L), \quad \text{and} \quad Q_x^\infty := \Omega_x \times (0, \infty), \quad (2.3)$$

to consider functions depending on the x variable. Since we deal with periodic boundary conditions, it is convenient to identify Ω_y as well as Ω_x with the one dimensional torus $\mathbb{R}/(L\mathbb{Z})$.

For any smooth function \widehat{v} , L -periodic in the space variable and bounded in $L^2(0, \infty; H_{per}^2(\Omega_y))$, we consider the L -periodic mapping $Y_{\widehat{v}}(\cdot, t)$ from Ω_x to Ω_y satisfying the following equation

$$\begin{aligned} \frac{\partial Y_{\widehat{v}}(x, t)}{\partial t} + u_s \frac{\partial Y_{\widehat{v}}(x, t)}{\partial x} &= u_s + e^{-\omega t} \widehat{v}(Y_{\widehat{v}}(x, t), t), \quad \forall (x, t) \in Q_x^\infty, \\ Y_{\widehat{v}}(x, 0) &= I(x), \quad \forall x \in \Omega_x, \\ Y_{\widehat{v}}(x, \cdot) &= Y_{\widehat{v}}(x + L, \cdot), \quad \forall x \in \Omega_x, \end{aligned} \quad (2.4)$$

where $I(x)$ is the identity mapping in $\mathbb{R}/(L\mathbb{Z})$. By the method of characteristics, this is equivalent to the following ordinary differential equation for all $(x, t) \in Q_x^\infty$,

$$\begin{aligned} \frac{d}{d\tau} \left[Y_{\widehat{v}}(x + u_s(\tau - t), \tau) \right] &= u_s + e^{-\omega\tau} \widehat{v}(Y_{\widehat{v}}(x + u_s(\tau - t), \tau), \tau), \quad \tau > 0, \\ Y_{\widehat{v}}(x + u_s(\tau - t), \tau)|_{\tau=0} &= I(x - u_s t), \quad \forall (x, t) \in Q_x^\infty, \end{aligned}$$

and hence to the following integral formulation for all $\tau > 0$,

$$Y_{\widehat{v}}(x + u_s(\tau - t), \tau) = I(x + u_s(\tau - t)) + \int_0^\tau e^{-\omega r} \widehat{v}(Y_{\widehat{v}}(x + u_s(r - t), r), r) dr.$$

Thus to prove the existence of a solution to (2.4), it is enough to prove the existence of a solution to the above integral formulation. In order to do that

we introduce the spaces

$$\begin{aligned}\widehat{V} &= C_b([0, \infty); H_{per}^1(\Omega_y)) \cap L^2(0, \infty; H_{per}^2(\Omega_y)), \\ \widehat{V}_\omega &= \left\{ \widehat{v} \in \widehat{V} \mid \|\widehat{v}\|_{L^2(0, \infty; H_{per}^2(\Omega_y))} \leq \min\left\{ \frac{\sqrt{\omega}}{2\sqrt{2s_0}}, \frac{\sqrt{\omega}}{\sqrt{2Ls_0}} \right\} \right\},\end{aligned}\tag{2.5}$$

where s_0 is defined in (2.2). The following proposition gives the existence and uniqueness of a solution to (2.4) under some conditions of \widehat{v} .

Proposition 2.1. *If $\widehat{v} \in \widehat{V}_\omega$, then there exists a unique function $Y_{\widehat{v}} \in C_b([0, \infty); L^2(\Omega_x))$ satisfying*

$$Y_{\widehat{v}}(x, t) = I(x) + \int_0^t e^{-\omega\tau} \widehat{v}(Y_{\widehat{v}}(x + u_s(\tau - t), \tau), \tau) d\tau.\tag{2.6}$$

For every $t > 0$, the periodic mapping $x \rightarrow Y_{\widehat{v}}(x, t)$ is bijective from the 1d-torus Ω_x to the 1d-torus Ω_y . Moreover, $Y_{\widehat{v}}$ belongs to $C_b([0, \infty); H_{per}^2(\Omega_x)) \cap C_b^1([0, \infty); H_{per}^1(\Omega_x))$ and it satisfies equation (2.4).

The proof of the above proposition follows from the Picard's iteration method and careful estimations of the integrals. For the sake of completeness, the proof is given in Section 6.1.

Lemma 2.2. *Let $\widehat{v} \in \widehat{V}_\omega$ and let $Y_{\widehat{v}}$ be the solution of equation (2.4). Then*

$$\left\| \frac{\partial Y_{\widehat{v}}}{\partial x} - 1 \right\|_{L^\infty(Q_x^\infty)} \leq \frac{1}{2}.$$

The proof is given in Section 6.1.

Corollary 2.3. *Let $\widehat{v} \in \widehat{V}_\omega$ and let $Y_{\widehat{v}}$ be the solution of equation (2.4). Then, for every $t > 0$, the periodic map $x \rightarrow Y_{\widehat{v}}(x, t)$ is C^1 diffeomorphism from the 1d-torus Ω_x to the 1d-torus Ω_y . Denoting by $X_{\widehat{v}}(\cdot, t)$ the L -periodic inverse of $Y_{\widehat{v}}(\cdot, t)$, we have*

$$X_{\widehat{v}}(Y_{\widehat{v}}(x, t), t) = x, \quad \forall (x, t) \in Q_x^\infty, \quad Y_{\widehat{v}}(X_{\widehat{v}}(y, t), t) = y, \quad \forall (y, t) \in Q_y^\infty.$$

□

Let us introduce the constants

$$b := a\gamma\rho_s^{\gamma-2}, \quad \nu_0 := \frac{\nu}{\rho_s}.\tag{2.7}$$

We set for all $(x, t) \in Q_x^\infty$,

$$\tilde{\sigma}(x, t) = \widehat{\sigma}(Y_{\widehat{v}}(x, t), t), \quad \tilde{v}(x, t) = \widehat{v}(Y_{\widehat{v}}(x, t), t), \quad \tilde{f}(x, t) = \widehat{f}(Y_{\widehat{v}}(x, t), t).\tag{2.8}$$

Let us consider the system

$$\begin{aligned}
& \tilde{\sigma}_t + u_s \tilde{\sigma}_x + \rho_s \tilde{v}_x - \omega \tilde{\sigma} = F_1(\tilde{\sigma}, \tilde{v}, t) \quad \text{in } Q_x^\infty, \\
& \tilde{v}_t + u_s \tilde{v}_x + b \tilde{\sigma}_x - \nu_0 \tilde{v}_{xx} - \omega \tilde{v} = \\
& \quad \tilde{f} \chi_{(\tilde{\ell}_{1,\tilde{v}}(t), \tilde{\ell}_{2,\tilde{v}}(t))} + F_2(\tilde{\sigma}, \tilde{v}, t) \quad \text{in } Q_x^\infty, \\
& \tilde{\sigma}(0, \cdot) = \tilde{\sigma}(L, \cdot), \quad \tilde{v}(0, \cdot) = \tilde{v}(L, \cdot), \quad \tilde{v}_x(0, \cdot) = \tilde{v}_x(L, \cdot) \quad \text{in } (0, \infty), \\
& \tilde{\sigma}(\cdot, 0) = \sigma_0(\cdot), \quad \tilde{v}(\cdot, 0) = v_0(\cdot) \quad \text{in } \Omega_x, \quad \int_0^L \sigma_0(x) dx = 0, \\
& Y(x, t) = I(x) + \int_0^t e^{-\omega\tau} \tilde{v}(x + u_s(\tau - t), \tau) d\tau, \quad \forall (x, t) \in Q_x^\infty, \\
& X(Y(x, t), t) = x, \quad \forall (x, t) \in Q_x^\infty, \quad Y(X(y, t), t) = y, \quad \forall (y, t) \in Q_y^\infty, \\
& \tilde{\ell}_{1,\tilde{v}}(t) = X(\ell_1, t), \quad \tilde{\ell}_{2,\tilde{v}}(t) = X(\ell_2, t), \quad \forall t > 0,
\end{aligned} \tag{2.9}$$

where

$$\begin{aligned}
F_1(\tilde{\sigma}, \tilde{v}, t) &= \rho_s \tilde{v}_x \left(1 - \left(\frac{\partial Y}{\partial x} \right)^{-1} \right) - e^{-\omega t} \tilde{\sigma} \tilde{v}_x \left(\frac{\partial Y}{\partial x} \right)^{-1}, \\
F_2(\tilde{\sigma}, \tilde{v}, t) &= \tilde{\sigma}_x \left[b - a\gamma (e^{-\omega t} \tilde{\sigma} + \rho_s)^{\gamma-2} \left(\frac{\partial Y}{\partial x} \right)^{-1} \right] \\
&\quad - \frac{\nu \tilde{v}_x}{(e^{-\omega t} \tilde{\sigma} + \rho_s)} \frac{\partial^2 Y}{\partial x^2} \left(\frac{\partial Y}{\partial x} \right)^{-3} - \nu \tilde{v}_{xx} \left[\frac{1}{\rho_s} - \frac{1}{(e^{-\omega t} \tilde{\sigma} + \rho_s)} \left(\frac{\partial Y}{\partial x} \right)^{-2} \right].
\end{aligned} \tag{2.10}$$

Then we have the following theorem.

Theorem 2.4. *Let*

$$\hat{\sigma} \in L^\infty(0, \infty; \dot{H}_{per}^1(\Omega_y)) \cap L^2(0, \infty; \dot{H}_{per}^1(\Omega_y)),$$

$$\hat{v} \in L^2(0, \infty; H_{per}^2(\Omega_y)) \cap H^1(0, \infty; L^2(\Omega_y)),$$

be the solution of (1.5) with control $\hat{f} \in L^2(0, \infty; L^2(\Omega_y))$. If in addition $\hat{v} \in \hat{V}$, then $(\tilde{\sigma}, \tilde{v}, \tilde{f})$ defined by (2.8), together with $(Y, X) = (Y_{\hat{v}}, X_{\hat{v}})$, satisfies system (2.9). Further, $\tilde{\sigma}$ belongs to $L^\infty(0, \infty; H_{per}^1(\Omega_x)) \cap L^2(0, \infty; H_{per}^1(\Omega_x))$, \tilde{v} belongs to $L^2(0, \infty; H_{per}^2(\Omega_x)) \cap H^1(0, \infty; L^2(\Omega_x))$, $\tilde{f} \in L^2(0, \infty; L^2(\Omega_x))$, and there exists a constant $M_{1,\omega}$, depending on ω , such that

$$\|(\tilde{\sigma}, \tilde{v})\|_{\tilde{D}} \leq M_{1,\omega} \|(\hat{\sigma}, \hat{v})\|_{\hat{D}},$$

where $\|(\tilde{\sigma}, \tilde{v})\|_{\tilde{D}}$ denotes the norm of $(\tilde{\sigma}, \tilde{v})$ in

$$L^\infty(0, \infty; H_{per}^1(\Omega_x)) \cap L^2(0, \infty; H_{per}^1(\Omega_x)) \\ \times (L^2(0, \infty; H_{per}^2(\Omega_x)) \cap H^1(0, \infty; L^2(\Omega_x)))$$

and $\|(\hat{\sigma}, \hat{v})\|_{\hat{D}}$ denotes the norm of $(\hat{\sigma}, \hat{v})$ in

$$(L^\infty(0, \infty; \dot{H}_{per}^1(\Omega_y)) \cap L^2(0, \infty; \dot{H}_{per}^1(\Omega_y))) \\ \times (L^2(0, \infty; H_{per}^2(\Omega_y)) \cap H^1(0, \infty; L^2(\Omega_y))).$$

Proof. For $(\hat{\sigma}, \hat{v}, \hat{f})$ with the L -periodic transformation $Y_{\hat{v}}$ defined in (2.6), by the chain rule differentiation formula, $(\tilde{\sigma}, \tilde{v}, \tilde{f})$ satisfies (2.9) in Q_x^∞ , with L -periodic boundary conditions. From Lemma 2.2, it follows that $\frac{\partial Y_{\hat{v}}}{\partial x} \geq \frac{1}{2}$ for $\hat{v} \in \hat{V}_\omega$. The rest of the proof follows from this and the change of variables formula. \square

The converse of the above theorem will be handled in Section 4.2. We shall need the following spaces

$$\tilde{V} = C_b([0, \infty); H_{per}^1(\Omega_x)) \cap L^2(0, \infty; H_{per}^2(\Omega_x)), \\ \tilde{V}_\omega = \left\{ \tilde{v} \in \tilde{V} \mid \|\tilde{v}\|_{L^2(0, \infty; H_{per}^2(\Omega_x))} \leq \frac{\sqrt{\omega}}{2s_0} \right\}, \quad (2.11)$$

and, for $\tilde{v} \in \tilde{V}$, the transformation

$$Y_{\tilde{v}}(x, t) = I(x) + \int_0^t e^{-\omega\tau} \tilde{v}(x + u_s(\tau - t), \tau) d\tau, \quad \forall (x, t) \in Q_x^\infty. \quad (2.12)$$

We have the following lemma.

Lemma 2.5. *Let $\tilde{v} \in \tilde{V}$ and let $Y_{\tilde{v}}$ be defined by (2.12), then*

$$\left\| \frac{\partial Y_{\tilde{v}}}{\partial x} - 1 \right\|_{L^\infty(0, \infty; L^2(\Omega_x))} \leq \frac{1}{\sqrt{2\omega}} \|\tilde{v}\|_{L^2(0, \infty; H_{per}^2(\Omega_x))}, \quad (2.13)$$

$$\left\| \frac{\partial^2 Y_{\tilde{v}}}{\partial x^2} \right\|_{L^\infty(0, \infty; L^2(\Omega_x))} \leq \frac{1}{\sqrt{2\omega}} \|\tilde{v}\|_{L^2(0, \infty; H_{per}^2(\Omega_x))}, \quad (2.14)$$

$$\left\| \frac{\partial Y_{\tilde{v}}}{\partial x} - 1 \right\|_{L^\infty(0, \infty; H_{per}^1(\Omega_x))} \leq \frac{1}{\sqrt{\omega}} \|\tilde{v}\|_{L^2(0, \infty; H_{per}^2(\Omega_x))}. \quad (2.15)$$

Moreover, if $\tilde{v} \in \tilde{V}_\omega$, we have

$$\left\| \frac{\partial Y_{\tilde{v}}}{\partial x} - 1 \right\|_{L^\infty(Q_x^\infty)} \leq \frac{1}{2}. \quad (2.16)$$

Proof. By differentiating (2.12), we get

$$\frac{\partial Y_{\tilde{v}}}{\partial x}(x, t) - 1 = \int_0^t e^{-\omega s} \tilde{v}_x(x + u_s(s - t), s) ds, \quad \forall (x, t) \in Q_x^\infty. \quad (2.17)$$

Thus

$$\left\| \frac{\partial Y_{\tilde{v}}}{\partial x}(\cdot, t) - 1 \right\|_{L^2(\Omega_x)}^2 \leq \frac{1 - e^{-2\omega t}}{2\omega} \|\tilde{v}_x\|_{L^2(0, \infty; L^2(\Omega_x))}^2,$$

and (2.13) is proved. Estimate (2.14) follows from

$$\frac{\partial^2 Y_{\tilde{v}}}{\partial x^2}(x, t) = \int_0^t e^{-\omega s} \tilde{v}_{xx}(x + u_s(s - t), s) ds.$$

The estimate (2.15) is a direct consequence of (2.13) and (2.14). If $\tilde{v} \in \tilde{V}_\omega$, we have

$$\left\| \frac{\partial Y_{\tilde{v}}}{\partial x} - 1 \right\|_{L^\infty(Q_x^\infty)} \leq s_0 \left\| \frac{\partial Y_{\tilde{v}}}{\partial x} - 1 \right\|_{L^\infty(0, \infty; H_{per}^1(\Omega_x))} \leq \frac{1}{2}.$$

□

Corollary 2.6. *Let $\tilde{v} \in \tilde{V}_\omega$ and let $Y_{\tilde{v}}$ be defined by (2.12). Then, for each $t > 0$, the periodic mapping $x \rightarrow Y_{\tilde{v}}(x, t)$ is C^1 -diffeomorphism from the 1d-torus Ω_x to the 1d-torus Ω_y . Denoting by $X_{\tilde{v}}(\cdot, t)$ the L -periodic inverse of $Y_{\tilde{v}}(\cdot, t)$, we have*

$$X_{\tilde{v}}(Y_{\tilde{v}}(x, t), t) = x, \quad \forall (x, t) \in Q_x^\infty, \quad Y_{\tilde{v}}(X_{\tilde{v}}(y, t), t) = y, \quad \forall (y, t) \in Q_y^\infty.$$

□

2.2. From a moving to a fixed control zone. As mentioned in the introduction, in the transformed system (2.9), the control zone depends on the time variable t . To handle this situation, we choose an open interval $\mathcal{O} \subset \Omega_x$ such that \mathcal{O} lies inside the control zone $(\tilde{\ell}_{1, \tilde{v}}(t), \tilde{\ell}_{2, \tilde{v}}(t))$ for all $t > 0$. This is detailed in the following Lemma.

Lemma 2.7. *Let \tilde{v} belong to $C_b([0, \infty); H_{per}^1(\Omega_x)) \cap L^2(0, \infty; H_{per}^2(\Omega_x))$ and let us also assume that*

$$\|\tilde{v}\|_{L^2(0, \infty; H_{per}^2(\Omega_x))} \leq \min \left\{ \frac{\sqrt{2\omega}(\ell_2 - \ell_1)}{8s_0}, \frac{\sqrt{\omega}}{2s_0} \right\}. \quad (2.18)$$

Then we have

$$|\tilde{\ell}_{j, \tilde{v}}(t) - \ell_j| \leq \frac{|\ell_2 - \ell_1|}{8}, \quad \forall t > 0, \quad j = 1, 2. \quad (2.19)$$

Furthermore, if we choose the open set $\mathcal{O} \subset \Omega_x$ defined by

$$\mathcal{O} := \left(\frac{(7\ell_1 + \ell_2)}{8}, \frac{(7\ell_2 + \ell_1)}{8} \right), \quad (2.20)$$

then we have

$$\mathcal{O} \subset (\tilde{\ell}_{1,\tilde{v}}(t), \tilde{\ell}_{2,\tilde{v}}(t)), \quad \forall t > 0. \quad (2.21)$$

Proof. For every $t > 0$ the moving domain for the control is $(\tilde{\ell}_{1,\tilde{v}}(t), \tilde{\ell}_{2,\tilde{v}}(t))$, where

$$\ell_j = \tilde{\ell}_{j,\tilde{v}}(t) + \int_0^t e^{-\omega\tau} \tilde{v}(\tilde{\ell}_{j,\tilde{v}}(t) + u_s(\tau - t), \tau) d\tau, \quad j = 1, 2, \quad \forall t > 0.$$

For $\|\tilde{v}\|_{L^2(0,\infty;H_{per}^2(\Omega_x))} \leq \frac{\sqrt{2\omega}|\ell_2 - \ell_1|}{8s_0}$, we get

$$|\tilde{\ell}_{j,\tilde{v}}(t) - \ell_j| \leq \frac{1}{\sqrt{2\omega}} \|\tilde{v}\|_{L^2(0,\infty;L^\infty(\Omega_x))} \leq \frac{|\ell_2 - \ell_1|}{8}.$$

For $\delta = \frac{|\ell_2 - \ell_1|}{8}$, we also have $\ell_1 + \delta < \ell_2 - \delta$. Therefore, the lemma follows by choosing

$$\mathcal{O} := (\ell_1 + \delta, \ell_2 - \delta).$$

□

Let us notice that, with (2.21), we have

$$\chi_{(\tilde{\ell}_{1,\tilde{v}}(t), \tilde{\ell}_{2,\tilde{v}}(t))} \chi_{\mathcal{O}} = \chi_{\mathcal{O}}, \quad \forall t > 0.$$

Thus to study the stabilizability of system (2.9), it is enough to study the stabilizability of the system

$$\begin{aligned}
& \tilde{\sigma}_t + u_s \tilde{\sigma}_x + \rho_s \tilde{v}_x - \omega \tilde{\sigma} = F_1(\tilde{\sigma}, \tilde{v}, t) \quad \text{in } Q_x^\infty, \\
& \tilde{v}_t + u_s \tilde{v}_x + b \tilde{\sigma}_x - \nu_0 \tilde{v}_{xx} - \omega \tilde{v} \\
& \quad = \chi_{\mathcal{O}} \tilde{f} + F_2(\tilde{\sigma}, \tilde{v}, t) \quad \text{in } Q_x^\infty, \\
& \tilde{\sigma}(0, \cdot) = \tilde{\sigma}(L, \cdot), \quad \tilde{v}(0, \cdot) = \tilde{v}(L, \cdot), \quad \tilde{v}_x(0, \cdot) = \tilde{v}_x(L, \cdot) \quad \text{in } (0, \infty), \\
& \tilde{\sigma}(\cdot, 0) = \sigma_0(\cdot), \quad \tilde{v}(\cdot, 0) = v_0(\cdot) \quad \text{in } \Omega_x, \quad \int_0^L \sigma_0(x) dx = 0, \\
& Y(x, t) = I(x) + \int_0^t e^{-\omega \tau} \tilde{v}(x + u_s(\tau - t), \tau) d\tau, \quad \forall (x, t) \in Q_x^\infty, \\
& X(Y(x, t), t) = x, \quad \forall (x, t) \in Q_x^\infty, \quad Y(X(y, t), t) = y, \quad \forall (y, t) \in Q_y^\infty, \\
& \tilde{\ell}_{1, \tilde{v}}(t) = X(\ell_1, t), \quad \tilde{\ell}_{2, \tilde{v}}(t) = X(\ell_2, t), \quad \forall t > 0,
\end{aligned} \tag{2.22}$$

where F_1 and F_2 are defined in (2.10).

3. STABILIZATION OF THE LINEARIZED LAGRANGIAN SYSTEM

In this section, we will use the notation Ω and Q^∞ instead of Ω_x and Q_x^∞ , since we are going to study the Lagrangian system (2.22) where the unknowns are functions of (x, t) only. Associated to the transformed system (2.22), with the control zone \mathcal{O} , let us consider the following linearized system

$$\begin{aligned}
& \tilde{\sigma}_t + u_s \tilde{\sigma}_x + \rho_s \tilde{v}_x = 0 \quad \text{in } Q^\infty, \\
& \tilde{v}_t - \nu_0 \tilde{v}_{xx} + u_s \tilde{v}_x + b \tilde{\sigma}_x = \tilde{f} \chi_{\mathcal{O}} \quad \text{in } Q^\infty, \\
& \tilde{\sigma}(0, \cdot) = \tilde{\sigma}(L, \cdot), \quad \tilde{v}(0, \cdot) = \tilde{v}(L, \cdot), \quad \tilde{v}_x(0, \cdot) = \tilde{v}_x(L, \cdot) \quad \text{in } (0, \infty), \\
& \tilde{\sigma}(\cdot, 0) = \tilde{\sigma}_0, \quad \tilde{v}(\cdot, 0) = \tilde{v}_0 \quad \text{in } \Omega, \quad \int_\Omega \tilde{\sigma}_0(x) dx = 0,
\end{aligned} \tag{3.1}$$

where the control \tilde{f} belongs to $L^2(0, \infty; L^2(\Omega))$.

Let us introduce the complex Hilbert space

$$Z = \dot{H}_{per}^1(\Omega) \times L^2(\Omega)$$

endowed with the inner product

$$\left\langle \begin{pmatrix} \rho \\ u \end{pmatrix}, \begin{pmatrix} \sigma \\ v \end{pmatrix} \right\rangle_{\mathbf{z}} := b \int_0^L \rho_x(x) \overline{\sigma_x(x)} dx + \rho_s \int_0^L u(x) \overline{v(x)} dx.$$

We define the unbounded operator $(A, \mathcal{D}(A))$ in Z by

$$\mathcal{D}(A) = \dot{H}_{per}^2(\Omega) \times H_{per}^2(\Omega)$$

and

$$A = \begin{bmatrix} -u_s \frac{d}{dx} & -\rho_s \frac{d}{dx} \\ -b \frac{d}{dx} & \nu_0 \frac{d^2}{dx^2} - u_s \frac{d}{dx} \end{bmatrix}. \quad (3.2)$$

Setting $z(t) = (\tilde{\sigma}(\cdot, t), \tilde{v}(\cdot, t))^T$ and $B\tilde{f} = (0, \tilde{f}\chi_{\mathcal{O}})^T$, system (3.1) can be written as

$$z'(t) = Az(t) + B\tilde{f}(t), \quad z(0) = z_0 \in Z. \quad (3.3)$$

Let us mention that $(A, \mathcal{D}(A))$ generates a C_0 semigroup in Z , denoted by $\{e^{tA}\}_{t \geq 0}$, and the control operator B belongs to $\mathcal{L}(L^2(\Omega), Z)$. We recall Lemma 2.2 from [14] regarding the spectrum of A for $L = 2\pi$ (see also [11]).

Lemma 3.1. *The spectrum of A consists of 0 and two sequence of complex eigenvalues $\{-\lambda_k^h, -\lambda_k^p\}_{k \in \mathbb{Z}^*}$ with $-\lambda_k^h = -\overline{\lambda_{-k}^h}$, $-\lambda_k^p = -\overline{\lambda_{-k}^p}$ for all $k \in \mathbb{Z}^*$.*

Moreover, for $k = 0$, we denote $-\lambda_0^h = 0$. For $k \in \mathbb{Z}^*$ with $k^2 < \frac{4b\rho_s}{\nu_0^2}$,

$$\lambda_k^h = \frac{[k^2\nu_0 - ik(\sqrt{4b\rho_s - k^2\nu_0^2} + 2u_s)]}{2}, \quad \lambda_k^p = \frac{[k^2\nu_0 + ik(\sqrt{4b\rho_s - k^2\nu_0^2} - 2u_s)]}{2},$$

and, for $k \in \mathbb{Z}^*$ with $k^2 \geq \frac{4b\rho_s}{\nu_0^2}$,

$$\lambda_k^h = \frac{[(k^2\nu_0 - |k|\sqrt{k^2\nu_0^2 - 4b\rho_s}) - 2iku_s]}{2}, \quad \lambda_k^p = \frac{[(k^2\nu_0 + |k|\sqrt{k^2\nu_0^2 - 4b\rho_s}) - 2iku_s]}{2}.$$

Let us denote $\omega_0 = \frac{b\rho_s}{\nu_0}$. We have the following asymptotic behaviors

$$\begin{aligned} \operatorname{Re}\lambda_k^h &\rightarrow \omega_0, & \frac{\operatorname{Re}\lambda_k^p}{k^2} &\rightarrow \nu_0 \quad \text{as } |k| \rightarrow \infty, \\ \left| \frac{\operatorname{Im}\lambda_k^h}{k} \right| &\rightarrow u_s & \left| \frac{\operatorname{Im}\lambda_k^p}{k} \right| &\rightarrow u_s \quad \text{as } |k| \rightarrow \infty. \end{aligned}$$

In view of the above lemma, for $\omega > \omega_0$, $(A + \omega I)$ has an infinite number of eigenvalues with positive real part (see [14] and [11]). In spite of having an infinite dimensional unstable space, system (3.1) is stabilizable in Z with exponential decay $e^{-\omega t}$, for any $\omega > 0$, by a L^2 -control acting everywhere in Ω (see [14]). We know that system (3.1) is null controllable in Z at T , by L^2 -localized control, for any $T > \frac{L}{|u_s|}$ (see Theorem 1.2 in [11]). Now from this null controllability result, we obtain the complete stabilization of system (3.1) (see Theorem 3.3 in [21]). One can also get a feedback stabilization result as in the following theorem.

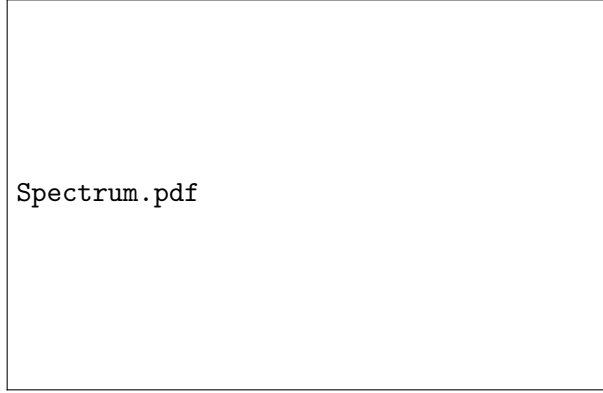


FIGURE 1. Spectrum of A with $u_s = 1 = \rho_s = a = \nu$, $\gamma = 25$, $\omega_0 = 25$ and $\omega = 100$

Theorem 3.2. *Let ω be any positive number. There exists $K_m \in \mathcal{L}(Z, L^2(\Omega))$ such that the semigroup $(e^{t(A+\omega I+BK_m)})_{t \geq 0}$ is exponentially stable. The solution $(\tilde{\sigma}, \tilde{v})$ of*

$$\frac{d}{dt} \begin{pmatrix} \tilde{\sigma} \\ \tilde{v} \end{pmatrix} = (A + \omega I + BK_m) \begin{pmatrix} \tilde{\sigma} \\ \tilde{v} \end{pmatrix}, \quad \begin{pmatrix} \tilde{\sigma} \\ \tilde{v} \end{pmatrix}(\cdot, 0) = \begin{pmatrix} \tilde{\sigma}_0 \\ \tilde{v}_0 \end{pmatrix} \in Z, \quad (3.4)$$

belongs to $C_b([0, \infty); Z) \cap L^2(0, \infty; Z)$ and satisfies

$$\|(\tilde{\sigma}, \tilde{v})(\cdot, t)\|_Z < M e^{-\delta t} \|(\tilde{\sigma}_0, \tilde{v}_0)\|_Z \quad \text{for all } t > 0, \quad (3.5)$$

for some $\delta > 0$ and $M > 0$. Furthermore, K_m can be chosen in the form $K_m = -B^*P$, where P is the solution of the following algebraic Riccati equation

$$\begin{aligned} P &\in \mathcal{L}(Z, Z'), \quad P = P^* > 0, \\ P(A + \omega I) + (A^* + \omega I)P - PBB^*P + I &= 0. \end{aligned} \quad (3.6)$$

The above theorem follows from Theorem 3.3 in [21].

It is convenient to define the feedback operator $K \in \mathcal{L}(H_{per}^1(\Omega) \times L^2(\Omega), L^2(\Omega))$ by

$$K(\tilde{\sigma}, \tilde{v}) = K_m(\tilde{\sigma}_m, \tilde{v}), \quad \forall (\tilde{\sigma}, \tilde{v}) \in H_{per}^1(\Omega) \times L^2(\Omega), \quad (3.7)$$

where $\sigma_m(x, t) = \sigma(x, t) - \frac{1}{L} \int_{\Omega} \sigma(y, t) dy$ and $K_m \in \mathcal{L}(\dot{H}_{per}^1(\Omega) \times L^2(\Omega), L^2(\Omega))$. Now we analyze further the structure of this feedback operator to see if it can be expressed by a kernel in a suitable Sobolev space. We have the following proposition in this direction.

Proposition 3.3. *Let the operator $K \in \mathcal{L}(H_{per}^1(\Omega) \times L^2(\Omega), L^2(\Omega))$ be defined in (3.7). Then there exist two kernel operators $k_\sigma \in L^2(\Omega; H_{per}^{-1}(\Omega))$ and $k_v \in L^2(\Omega \times \Omega)$ such that for all $(\tilde{\sigma}, \tilde{v}) \in H_{per}^1(\Omega) \times L^2(\Omega)$,*

$$K(\tilde{\sigma}, \tilde{v})(x) = \langle k_\sigma(x, \cdot), \tilde{\sigma}(\cdot) \rangle_{H_{per}^{-1}, H_{per}^1} + \int_{\Omega} k_v(x, \xi) \tilde{v}(\xi) d\xi.$$

Proof. The operator $K \in \mathcal{L}(H_{per}^1(\Omega) \times L^2(\Omega), L^2(\Omega))$ can be decomposed in the form

$$K(\tilde{\sigma}(\cdot, t), \tilde{v}(\cdot, t)) = K_\sigma \tilde{\sigma}(\cdot, t) + K_v \tilde{v}(\cdot, t),$$

with

$$\|K_\sigma \tilde{\sigma}\|_{L^2(\Omega)} \leq C \|\tilde{\sigma}\|_{H_{per}^1(\Omega)} \quad \text{for all } \tilde{\sigma} \in H_{per}^1(\Omega), \quad (3.8)$$

and

$$\|K_v \tilde{v}\|_{L^2(\Omega)} \leq C \|\tilde{v}\|_{L^2(\Omega)} \quad \text{for all } \tilde{v} \in L^2(\Omega). \quad (3.9)$$

Let us denote by $\mathcal{D}_{per}(\Omega)$ the set of functions which are the restrictions to Ω of L -periodic C^∞ functions. The space $\mathcal{D}_{per}(\Omega \times \Omega)$ is defined in an analogous manner. The dual of $\mathcal{D}_{per}(\Omega)$ is denoted by $\mathcal{D}'_{per}(\Omega)$ and the dual of $\mathcal{D}_{per}(\Omega \times \Omega)$ is denoted by $\mathcal{D}'_{per}(\Omega \times \Omega)$. From Schwartz's kernel Theorem (see [18, Theorem II], [13]) adapted to $\mathcal{D}_{per}(\Omega)$, it follows that there exist $k_\sigma \in \mathcal{D}'_{per}(\Omega \times \Omega)$ and $k_v \in \mathcal{D}'_{per}(\Omega \times \Omega)$ such that for all $x \in \Omega$,

$$(K_\sigma \tilde{\sigma})(x) = \langle k_\sigma(x, \cdot), \tilde{\sigma}(\cdot) \rangle_{\mathcal{D}'_{per}(\Omega), \mathcal{D}_{per}(\Omega)} \quad \text{for all } \tilde{\sigma} \in \mathcal{D}_{per}(\Omega), \quad (3.10)$$

and

$$(K_v \tilde{v})(x) = \langle k_v(x, \cdot), \tilde{v}(\cdot) \rangle_{\mathcal{D}'_{per}(\Omega), \mathcal{D}_{per}(\Omega)} \quad \text{for all } \tilde{v} \in \mathcal{D}_{per}(\Omega). \quad (3.11)$$

Due to (3.8) it follows that k_σ is a distribution of order 1, and due to (3.9) that k_v is a distribution of order 0.

Since $\mathcal{D}_{per}(\Omega)$ is dense in $H_{per}^1(\Omega)$ as well as in $L^2(\Omega)$, using the calculations so far and the definitions of k_σ and k_v , we have a unique extension $k_\sigma(x, \cdot) \in H_{per}^{-1}(\Omega)$ and $k_v(x, \cdot) \in L^2(\Omega)$ such that

$$(K_\sigma \tilde{\sigma})(x) = \langle k_\sigma(x, \cdot), \tilde{\sigma}(\cdot) \rangle_{H_{per}^{-1}(\Omega), H_{per}^1(\Omega)} \quad \text{for all } \tilde{\sigma} \in H_{per}^1(\Omega),$$

and

$$(K_v \tilde{v})(x) = \langle k_v(x, \cdot), \tilde{v}(\cdot) \rangle_{L^2(\Omega), L^2(\Omega)} \quad \text{for all } \tilde{v} \in L^2(\Omega).$$

Moreover, due to (3.8) and (3.9), k_σ belongs to $L^2(\Omega; H_{per}^{-1}(\Omega))$ and k_v belongs to $L^2(\Omega \times \Omega)$. \square

Let us set

$$V = \dot{H}_{per}^1(\Omega) \times H_{per}^1(\Omega).$$

To use a fixed point argument in later analysis, we need to consider initial condition $(\tilde{\sigma}_0, \tilde{v}_0)$ in V . We have the following regularity theorem for the solution $(\tilde{\sigma}, \tilde{v})$ of (3.4).

Theorem 3.4. *For $(\tilde{\sigma}_0, \tilde{v}_0) \in V$, the solution $(\tilde{\sigma}, \tilde{v})$ of (3.4) satisfies*

$$\begin{aligned} & \|\tilde{\sigma}\|_{L^\infty(0, \infty; \dot{H}_{per}^1(\Omega))} + \|\tilde{\sigma}\|_{L^2(0, \infty; \dot{H}_{per}^1(\Omega))} + \|\tilde{v}\|_{L^2(0, \infty; H_{per}^2(\Omega))} \\ & \quad + \|\tilde{v}\|_{H^1(0, \infty; L^2(\Omega))} \leq C \|(\tilde{\sigma}_0, \tilde{v}_0)\|_V. \end{aligned}$$

Proof. The estimate of $(\tilde{\sigma}, \tilde{v})$ in $L^2(0, \infty; Z) \cap L^\infty(0, \infty; Z)$ follows from the exponential stability of the semigroup $(e^{t(A+\omega I+BK_m)})_{t \geq 0}$, see (3.5). Next, the estimate of \tilde{v} in $L^2(0, \infty; H_{per}^2(\Omega)) \cap H^1(0, \infty; L^2(\Omega))$ follows from regularity results for parabolic equations, with a right hand side in $L^2(0, \infty; L^2(\Omega))$. \square

To handle the nonlinear terms in (2.22), we need to consider the linearized system (3.4) with forcing terms, i.e.,

$$\begin{aligned} & \tilde{\sigma}_t + u_s \tilde{\sigma}_x + \rho_s \tilde{v}_x - \omega \tilde{\sigma} = f_1 \quad \text{in } Q^\infty, \\ & \tilde{v}_t + u_s \tilde{v}_x + b \tilde{\sigma}_x - \nu_0 \tilde{v}_{xx} - \omega \tilde{v} \\ & \quad = \chi_{\mathcal{O}} K_m \left(\tilde{\sigma}(\cdot, t) - \frac{1}{L} \int_{\Omega} \tilde{\sigma}(\xi, t) d\xi, \tilde{v}(\cdot, t) \right) + f_2 \quad \text{in } Q^\infty, \\ & \tilde{\sigma}(0, \cdot) = \tilde{\sigma}(L, \cdot), \quad \tilde{v}(0, \cdot) = \tilde{v}(L, \cdot), \quad \tilde{v}_x(0, \cdot) = \tilde{v}_x(L, \cdot) \quad \text{in } (0, \infty), \\ & \tilde{\sigma}(\cdot, 0) = \sigma_0(\cdot), \quad \tilde{v}(\cdot, 0) = v_0(\cdot) \quad \text{in } \Omega. \end{aligned} \tag{3.12}$$

As explained in the Introduction, to study system (3.12), we decompose f_1 and $\tilde{\sigma}$ uniquely as follows

$$f_1(x, t) = f_{1,m}(x, t) + f_{1,\Omega}(t), \quad \tilde{\sigma}(x, t) = \tilde{\sigma}_m(x, t) + \tilde{\sigma}_\Omega(t), \quad \forall (x, t) \in Q_x^\infty, \tag{3.13}$$

where

$$\begin{aligned} f_{1,\Omega}(t) &:= \frac{1}{L} \int_{\Omega} f_1(x, t) dx, \quad \int_{\Omega} f_{1,m}(x, t) dx = 0, \quad \forall t > 0, \\ \tilde{\sigma}_\Omega(t) &:= \frac{1}{L} \int_{\Omega} \tilde{\sigma}(x, t) dx, \quad \frac{1}{L} \int_{\Omega} \tilde{\sigma}_m(x, t) dx = 0, \quad \forall t > 0. \end{aligned} \tag{3.14}$$

Setting $z_m(t) = (\tilde{\sigma}_m(\cdot, t), \tilde{v}(\cdot, t))^T$, we easily check that $(\tilde{\sigma}, \tilde{v})$ is a solution to (3.12) if and only if $(z_m, \tilde{\sigma}_\Omega)$ is a solution to

$$\begin{aligned} z'_m(t) &= (A + \omega I + BK_m)z_m(t) + (f_{1,m}(\cdot, t), f_2(\cdot, t))^T \quad \text{for all } t > 0, \\ z_m(0) &= z_0, \\ \tilde{\sigma}'_\Omega(t) &= \omega \tilde{\sigma}_\Omega(t) + f_{1,\Omega}(t), \quad \forall t > 0, \quad \tilde{\sigma}_\Omega(0) = 0. \end{aligned} \tag{3.15}$$

This leads to

$$\tilde{\sigma}_\Omega(t) = e^{\omega t} \int_0^t e^{-\omega s} f_{1,\Omega}(s) ds, \quad \forall t > 0. \tag{3.16}$$

Thus, we introduce the weighted Lebesgue spaces

$$\begin{aligned} L^\infty(0, \infty; e^{-\omega(\cdot)}) &= \{h \mid e^{-\omega(\cdot)}h \in L^\infty(0, \infty)\}, \\ L^1(0, \infty; e^{-\omega(\cdot)}) &= \{h \mid e^{-\omega(\cdot)}h \in L^1(0, \infty)\}. \end{aligned} \tag{3.17}$$

We have the following results for the linearized closed loop system (3.12).

Theorem 3.5. *For $(\tilde{\sigma}_0, \tilde{v}_0) \in V$ and $f_{1,m} \in L^2(0, \infty; \dot{H}_{per}^1(\Omega))$, $f_{1,\Omega} \in L^1(0, \infty; e^{-\omega(\cdot)})$ and $f_2 \in L^2(0, \infty; L^2(\Omega))$, the solution $z(t) = (\tilde{\sigma}(\cdot, t), \tilde{v}(\cdot, t))^T$ of system (3.12) satisfies*

$$\begin{aligned} &\|\tilde{\sigma}_m\|_{L^2(0, \infty; \dot{H}_{per}^1(\Omega))} + \|\tilde{\sigma}_m\|_{L^\infty(0, \infty; \dot{H}_{per}^1(\Omega))} + \|\tilde{\sigma}_\Omega\|_{L^\infty(0, \infty; e^{-\omega(\cdot)})} \\ &+ \|\tilde{v}\|_{L^2(0, \infty; H_{per}^2(\Omega))} + \|\tilde{v}\|_{H^1(0, \infty; L^2(\Omega))} \\ &\leq C_1 \left(\|f_{1,m}\|_{L^2(0, \infty; \dot{H}_{per}^1(\Omega))} + \|f_{1,\Omega}\|_{L^1(0, \infty; e^{-\omega(\cdot)})} \right. \\ &\quad \left. + \|f_2\|_{L^2(0, \infty; L^2(\Omega))} + \|(\tilde{\sigma}_0, \tilde{v}_0)\|_V \right). \end{aligned} \tag{3.18}$$

Proof. The estimate of $(\tilde{\sigma}_m, \tilde{v})$ in $L^2(0, \infty; Z) \cap L^\infty(0, \infty; Z)$ follows from the exponential stability of the semigroup $(e^{t(A+\omega I+BK_m)})_{t \geq 0}$, see (3.5), the Duhamel formula and the Young inequality for convolution products. The estimate of $\tilde{\sigma}_\Omega$ in $L^\infty(0, \infty; e^{-\omega(\cdot)})$ is obvious. Next, as in Theorem 3.4, the estimate of \tilde{v} in $L^2(0, \infty; H_{per}^2(\Omega)) \cap H^1(0, \infty; L^2(\Omega))$ follows from regularity results for parabolic equations, with a right hand side in $L^2(0, \infty; L^2(\Omega))$. \square

4. STABILIZATION OF THE NONLINEAR SYSTEM

With the feedback operator $K \in \mathcal{L}(H_{per}^1(\Omega_x) \times L^2(\Omega_x), L^2(\Omega_x))$ defined in (3.7), the closed loop nonlinear system corresponding to (2.9) is

$$\begin{aligned}
\tilde{\sigma}_t + u_s \tilde{\sigma}_x + \rho_s \tilde{v}_x - \omega \tilde{\sigma} &= F_1(\tilde{\sigma}, \tilde{v}, t) \quad \text{in } Q_x^\infty, \\
\tilde{v}_t + u_s \tilde{v}_x + b \tilde{\sigma}_x - \nu_0 \tilde{v}_{xx} - \omega \tilde{v} & \\
&= \chi_O K(\tilde{\sigma}(\cdot, t), \tilde{v}(\cdot, t)) + F_2(\tilde{\sigma}, \tilde{v}, t) \quad \text{in } Q_x^\infty, \\
\tilde{\sigma}(0, \cdot) = \tilde{\sigma}(L, \cdot), \quad \tilde{v}(0, \cdot) = \tilde{v}(L, \cdot), \quad \tilde{v}_x(0, \cdot) = \tilde{v}_x(L, \cdot) &\quad \text{in } (0, \infty), \\
\tilde{\sigma}(\cdot, 0) = \sigma_0(\cdot), \quad \tilde{v}(\cdot, 0) = v_0(\cdot) \quad \text{in } \Omega_x, \quad \int_{\Omega_x} \sigma_0(x) dx = 0, & \\
Y(x, t) = I(x) + \int_0^t e^{-\omega\tau} \tilde{v}(x + u_s(\tau - t), \tau) d\tau, \quad \forall (x, t) \in Q_x^\infty, & \\
X(Y(x, t), t) = x, \quad \forall (x, t) \in Q_x^\infty, \quad Y(X(y, t), t) = y, \quad \forall (y, t) \in Q_y^\infty, & \\
\tilde{\ell}_{1, \bar{x}}(t) = X(\ell_1, t), \quad \tilde{\ell}_{2, \bar{x}}(t) = X(\ell_2, t), \quad \forall t > 0, &
\end{aligned} \tag{4.1}$$

where F_1 and F_2 are defined in (2.10).

In this section, first we show that (4.1) admits a unique solution in a suitable ball defined by some estimates. Next we show that it is possible to come back from system (4.1) to the original system (1.5), since $(\tilde{\sigma}, \tilde{v}, X, Y)$ satisfies the required estimates. Using these results, we finally prove Theorems 1.1 and 1.2.

4.1. Stabilization of the Lagrangian system (4.1). Recall the unique decomposition introduced in (3.13)-(3.14). We first state the following lemma which will be useful in deriving several estimates.

Lemma 4.1. *Let $\tilde{\sigma}(x, t) = \tilde{\sigma}_m(x, t) + \tilde{\sigma}_\Omega(x, t)$ with $\tilde{\sigma}_m \in L^\infty(0, \infty; \dot{H}_{per}^1(\Omega_x))$ and $\tilde{\sigma}_\Omega \in L^\infty(0, \infty; e^{-\omega(\cdot)})$. If*

$$\|\tilde{\sigma}_m\|_{L^\infty(0, \infty; \dot{H}_{per}^1(\Omega_x))} + \|\tilde{\sigma}_\Omega\|_{L^\infty(0, \infty; e^{-\omega(\cdot)})} \leq \frac{\rho_s}{4s_0}, \tag{4.2}$$

then

$$|e^{-\omega t} \tilde{\sigma}(x, t)| \leq \frac{\rho_s}{2}, \quad \forall (x, t) \in Q_x^\infty.$$

Proof. Using (4.2) and $s_0 > 1$, for all $(x, t) \in Q_x^\infty$, we have

$$\begin{aligned}
|e^{-\omega t} \tilde{\sigma}(x, t)| &\leq \|\tilde{\sigma}_m\|_{L^\infty(Q_x^\infty)} + \|\tilde{\sigma}_\Omega\|_{L^\infty(0, \infty; e^{-\omega(\cdot)})} \\
&\leq s_0 \|\tilde{\sigma}_m\|_{L^\infty(0, \infty; \dot{H}_{per}^1(\Omega_x))} + \|\tilde{\sigma}_\Omega\|_{L^\infty(0, \infty; e^{-\omega(\cdot)})} \leq \frac{\rho_s}{2}.
\end{aligned}$$

□

Let us consider the space

$$D = \left\{ (\zeta, \vartheta) \mid \vartheta \in H^1(0, \infty; L^2(\Omega_x)) \cap L^2(0, \infty; H_{per}^2(\Omega_x)), \right. \\ \zeta = \zeta_m + \zeta_\Omega, \quad \zeta_m \in L^2(0, \infty; \dot{H}_{per}^1(\Omega_x)) \cap L^\infty(0, \infty; \dot{H}_{per}^1(\Omega_x)), \\ \left. \zeta_\Omega \in L^\infty(0, \infty; e^{-\omega(\cdot)}) \right\}, \quad (4.3)$$

equipped with the norm

$$\|(\zeta, \vartheta)\|_D = \|\zeta_m\|_{L^2(0, \infty; \dot{H}_{per}^1(\Omega_x))} + \|\zeta_m\|_{L^\infty(0, \infty; \dot{H}_{per}^1(\Omega_x))} + \\ \|\zeta_\Omega\|_{L^\infty(0, \infty; e^{-\omega(\cdot)})} + \|\vartheta\|_{L^2(0, \infty; H_{per}^2(\Omega_x))} + \|\vartheta\|_{H^1(0, \infty; L^2(\Omega_x))}.$$

For any $\mu > 0$, we define

$$D_\mu = \{(\tilde{\sigma}, \tilde{\nu}) \in D \mid \|(\tilde{\sigma}, \tilde{\nu})\|_D \leq \mu\}. \quad (4.4)$$

We set

$$F_1(\zeta, \vartheta, t) = \rho_s \vartheta_x \left(1 - \left(\frac{\partial Y(\zeta, \vartheta)}{\partial x}\right)^{-1}\right) - e^{-\omega t} \zeta \vartheta_x \left(\frac{\partial Y(\zeta, \vartheta)}{\partial x}\right)^{-1}, \\ F_2(\zeta, \vartheta, t) = \zeta_x \left(b - a\gamma \left(e^{-\omega t} \zeta + \rho_s\right)^{\gamma-2} \left(\frac{\partial Y(\zeta, \vartheta)}{\partial x}\right)^{-1}\right) \\ - \frac{\nu \vartheta_x}{e^{-\omega t} \zeta + \rho_s} \frac{\partial^2 Y(\zeta, \vartheta)}{\partial x^2} \left(\frac{\partial Y(\zeta, \vartheta)}{\partial x}\right)^{-3} \\ - \nu \vartheta_{xx} \left(\frac{1}{\rho_s} - \frac{1}{e^{-\omega t} \zeta + \rho_s} \left(\frac{\partial Y(\zeta, \vartheta)}{\partial x}\right)^{-2}\right). \quad (4.5)$$

As mentioned in the Introduction, we use the decomposition

$$F_1(\zeta, \vartheta, t) = F_{1,m}(\zeta, \vartheta, t) + F_{1,\Omega}(\zeta, \vartheta, t), \quad \forall t > 0,$$

where $F_{1,\Omega}(\zeta, \vartheta, t) = \frac{1}{L} \int_{\Omega} F_1(\zeta, \vartheta, t) dx$.

The next lemma gives some useful estimations of (4.5).

Lemma 4.2. *There exists a positive constant C_2 depending on ω , ρ_s , u_s , s_0 , L , ν such that for all (ζ, ϑ) , (ζ^1, ϑ^1) , (ζ^2, ϑ^2) belonging to D_μ , with $\mu = \min \left\{ \frac{\rho_s}{4s_0}, \frac{\sqrt{\omega}}{2s_0} \right\}$, we have the following estimates*

$$\|F_{1,m}(\zeta, \vartheta, \cdot)\|_{L^2(0, \infty; \dot{H}_{per}^1(\Omega_x))} \leq C_2 \|(\zeta, \vartheta)\|_D^2, \quad (4.6)$$

$$\|F_{1,\Omega}(\zeta, \vartheta, \cdot)\|_{L^1(0, \infty; e^{-\omega(\cdot)})} \leq C_2 \|(\zeta, \vartheta)\|_D^2, \quad (4.7)$$

$$\|F_2(\zeta, \vartheta, \cdot)\|_{L^2(0, \infty; L^2(\Omega_x))} \leq C_2 \|(\zeta, \vartheta)\|_D^2, \quad (4.8)$$

$$\begin{aligned} & \|F_{1,m}(\zeta^1, \vartheta^1, \cdot) - F_{1,m}(\zeta^2, \vartheta^2, \cdot)\|_{L^2(0,\infty; \dot{H}_{per}^1(\Omega_x))} \\ & \leq C_2 \left(\|(\zeta^1, \vartheta^1)\|_D + \|(\zeta^2, \vartheta^2)\|_D \right) \|(\zeta^1, \vartheta^1) - (\zeta^2, \vartheta^2)\|_D, \end{aligned} \quad (4.9)$$

$$\begin{aligned} & \|F_{1,\Omega}(\zeta^1, \vartheta^1, \cdot) - F_{1,\Omega}(\zeta^2, \vartheta^2, \cdot)\|_{L^1(0,\infty; e^{-\omega(\cdot)})} \\ & \leq C_2 \left(\|(\zeta^1, \vartheta^1)\|_D + \|(\zeta^2, \vartheta^2)\|_D \right) \|(\zeta^1, \vartheta^1) - (\zeta^2, \vartheta^2)\|_D, \end{aligned} \quad (4.10)$$

$$\begin{aligned} & \|F_2(\zeta^1, \vartheta^1, \cdot) - F_2(\zeta^2, \vartheta^2, \cdot)\|_{L^2(0,\infty; L^2(\Omega_x))} \\ & \leq C_2 \left(\|(\zeta^1, \vartheta^1)\|_D + \|(\zeta^2, \vartheta^2)\|_D \right) \|(\zeta^1, \vartheta^1) - (\zeta^2, \vartheta^2)\|_D. \end{aligned} \quad (4.11)$$

The proof of all these estimates (4.6)-(4.11) is given in the appendix (Section 6.2).

We have the following theorem.

Theorem 4.3. *Let $K \in \mathcal{L}(H_{per}^1(\Omega_x) \times L^2(\Omega_x), L^2(\Omega_x))$ be defined in (3.7) and \mathcal{O} defined by (2.20). There exist constants $\mu_0 > 0$ and $\kappa > 0$, depending on $s_0, \omega, L, \ell_1, \ell_2, u_s, \rho_s$, such that, for $0 < \tilde{\mu} \leq \mu_0$, and any initial conditions (σ_0, v_0) satisfying*

$$\|(\sigma_0, v_0)\|_{\dot{H}_{per}^1(\Omega_x) \times H_{per}^1(\Omega_x)} \leq \kappa \tilde{\mu}, \quad (4.12)$$

the closed loop system (4.1) admits a unique solution $(\tilde{\sigma}, \tilde{v}, X, Y)$ such that $(\tilde{\sigma}, \tilde{v})$ belongs to $D_{\tilde{\mu}}$, $X \in C_b([0, \infty); H_{per}^2(\Omega_y)) \cap C_b^1([0, \infty); H_{per}^1(\Omega_y))$, and $Y \in C_b([0, \infty); H_{per}^2(\Omega_x)) \cap C_b^1([0, \infty); H_{per}^1(\Omega_x))$. Moreover, $\tilde{\sigma} \in C([0, \infty); H_{per}^1(\Omega_x))$, \tilde{v} belongs to \tilde{V}_ω and satisfies (2.18).

Proof. The proof is based on the Banach fixed point Theorem. Let us choose

$$\mu_0 := \min \left\{ \frac{\rho_s}{4s_0}, \frac{\sqrt{\omega}}{2s_0}, \frac{\sqrt{2\omega}|\ell_2 - \ell_1|}{8s_0}, \frac{1}{4C_1C_2} \right\}, \quad \kappa = \frac{1}{2C_1}, \quad (4.13)$$

where C_1 and C_2 are the constants appearing in Theorem 3.5 and Lemma 4.2 respectively. Let $\tilde{\mu}$ belong to $(0, \mu_0]$. For any $(\zeta, \vartheta) \in D_{\tilde{\mu}}$, we denote by

$(\tilde{\sigma}^{(\zeta, \vartheta)}, \tilde{v}^{(\zeta, \vartheta)})$ the solution of the following linear system

$$\begin{aligned}
\tilde{\sigma}_t^{(\zeta, \vartheta)} + u_s \tilde{\sigma}_x^{(\zeta, \vartheta)} + \rho_s \tilde{v}_x^{(\zeta, \vartheta)} - \omega \tilde{\sigma}^{(\zeta, \vartheta)} &= F_1(\zeta, \vartheta, t) \quad \text{in } Q_x^\infty, \\
\tilde{v}_t^{(\zeta, \vartheta)} + u_s \tilde{v}_x^{(\zeta, \vartheta)} + b \tilde{\sigma}_x^{(\zeta, \vartheta)} - \nu_0 \tilde{u}_{xx}^{(\zeta, \vartheta)} - \omega \tilde{v}^{(\zeta, \vartheta)} &= \\
\chi_{\mathcal{O}}(x) K(\tilde{\sigma}^{(\zeta, \vartheta)}(\cdot, t), \tilde{v}^{(\zeta, \vartheta)}(\cdot, t)) + F_2(\zeta, \vartheta, t) &\quad \text{in } Q_x^\infty, \\
\tilde{\sigma}^{(\zeta, \vartheta)}(0, \cdot) = \tilde{\sigma}^{(\zeta, \vartheta)}(L, \cdot), \quad \tilde{v}^{(\zeta, \vartheta)}(0, \cdot) = \tilde{v}^{(\zeta, \vartheta)}(L, \cdot), \\
\tilde{v}_x^{(\zeta, \vartheta)}(0, \cdot) = \tilde{v}_x^{(\zeta, \vartheta)}(L, \cdot) &\quad \text{in } (0, \infty), \\
\tilde{\sigma}^{(\zeta, \vartheta)}(\cdot, 0) = \sigma_0(\cdot), \quad \tilde{v}^{(\zeta, \vartheta)}(\cdot, 0) = v_0(\cdot) &\quad \text{in } \Omega_x, \quad \int_{\Omega_x} \sigma_0(x) dx = 0, \\
Y^{(\zeta, \vartheta)}(x, t) = I(x) + \int_0^t e^{-\omega \tau} \vartheta(x + u_s(\tau - t), \tau) d\tau, &\quad \forall (x, t) \in Q_x^\infty,
\end{aligned} \tag{4.14}$$

where F_1 and F_2 are defined in (4.5).

Let us prove that the mapping

$$(\zeta, \vartheta) \mapsto (\tilde{\sigma}^{(\zeta, \vartheta)}, \tilde{v}^{(\zeta, \vartheta)}) \tag{4.15}$$

is a contraction in $D_{\tilde{\mu}}$. Since $\tilde{\mu}$ is less than or equal to $\frac{\rho_s}{4s_0}$ and $\frac{\sqrt{\omega}}{2s_0}$, from (3.5), (4.6), (4.7), (4.8) and (4.12), it follows that

$$\begin{aligned}
&\|(\tilde{\sigma}^{(\zeta, \vartheta)}, \tilde{v}^{(\zeta, \vartheta)})\|_D \\
&\leq C_1 \left(\|(\sigma_0, v_0)\|_{\dot{H}_{per}^1(\Omega_x) \times H_{per}^1(\Omega_x)} + \|F_{1,m}(\zeta, \vartheta, \cdot)\|_{L^2(0, \infty; \dot{H}_{per}^1(\Omega_x))} \right. \\
&\quad \left. + \|F_{1,\Omega}(\zeta, \vartheta, \cdot)\|_{L^\infty(0, \infty; e^{-\omega(\cdot)})} + \|F_2(\zeta, \vartheta, \cdot)\|_{L^2(0, \infty; L^2(\Omega_x))} \right) \tag{4.16} \\
&\leq C_1 \left(\|(\sigma_0, v_0)\|_{\dot{H}_{per}^1(\Omega_x) \times H_{per}^1(\Omega_x)} + C_2 \|(\zeta, \vartheta)\|_D^2 \right) \\
&\leq \tilde{\mu} (C_1 \kappa + C_1 C_2 \tilde{\mu}) \leq \tilde{\mu}.
\end{aligned}$$

Hence, if $(\zeta, \vartheta) \in D_{\tilde{\mu}}$, then $(\tilde{\sigma}^{(\zeta, \vartheta)}, \tilde{v}^{(\zeta, \vartheta)})$ belongs to $D_{\tilde{\mu}}$. We set

$$\tilde{\Sigma} = \tilde{\sigma}^{(\zeta^1, \vartheta^1)} - \tilde{\sigma}^{(\zeta^2, \vartheta^2)}, \quad \tilde{V} = \tilde{v}^{(\zeta^1, \vartheta^1)} - \tilde{v}^{(\zeta^2, \vartheta^2)}.$$

The couple $(\tilde{\Sigma}, \tilde{V})$ satisfies

$$\begin{aligned} \tilde{\Sigma}_t + u_s \tilde{\Sigma}_x + \rho_s \tilde{V}_x - \omega \tilde{\Sigma} &= F_1(\zeta^1, \vartheta^1, t) - F_1(\zeta^2, \vartheta^2, t) \quad \text{in } Q_x^\infty, \\ \tilde{V}_t + u_s \tilde{V}_x + b \tilde{\Sigma}_x - \nu_0 \tilde{V}_{xx} - \omega \tilde{V} &= \\ \chi_{\mathcal{O}} K(\tilde{\Sigma}(\cdot, t), \tilde{V}(\cdot, t)) + F_2(\zeta^1, \vartheta^1, t) - F_2(\zeta^2, \vartheta^2, t) &\quad \text{in } Q_x^\infty, \\ \tilde{\Sigma}(0, \cdot) = \tilde{\Sigma}(L, \cdot), \quad \tilde{V}(0, \cdot) = \tilde{V}(L, \cdot), \quad \tilde{V}_x(0, \cdot) = \tilde{V}_x(L, \cdot) &\quad \text{in } (0, \infty), \\ \tilde{\Sigma}(\cdot, 0) = 0, \quad \tilde{V}(\cdot, 0) = 0 &\quad \text{in } \Omega_x. \end{aligned} \tag{4.17}$$

If $(\zeta^1, \vartheta^1) \in D_{\tilde{\mu}}$ and $(\zeta^2, \vartheta^2) \in D_{\tilde{\mu}}$, with Theorem 3.5 and estimates (4.9), (4.10), (4.11) and (4.13), we have

$$\begin{aligned} &\left\| \left(\tilde{\sigma}(\zeta^1, \vartheta^1), \tilde{v}(\zeta^1, \vartheta^1) \right) - \left(\tilde{\sigma}(\zeta^2, \vartheta^2), \tilde{v}(\zeta^2, \vartheta^2) \right) \right\|_D \\ &\leq C_1 \left(\|F_{1,m}(\zeta^1, \vartheta^1, \cdot) - F_{1,m}(\zeta^2, \vartheta^2, \cdot)\|_{L^2(0,\infty; \dot{H}_{per}^1(\Omega_x))} \right. \\ &\quad \left. + \|F_{1,\Omega}(\zeta^1, \vartheta^1, \cdot) - F_{1,\Omega}(\zeta^2, \vartheta^2, \cdot)\|_{L^1(0,\infty; e^{-\omega(\cdot)})} \right) \\ &\quad + \|F_2(\zeta^1, \vartheta^1, \cdot) - F_2(\zeta^2, \vartheta^2, \cdot)\|_{L^2(0,\infty; L^2(\Omega_x))} \\ &\leq 2C_1 C_2 \tilde{\mu} \|(\zeta^1, \vartheta^1) - (\zeta^2, \vartheta^2)\|_D \leq \frac{1}{2} \|(\zeta^1, \vartheta^1) - (\zeta^2, \vartheta^2)\|_D. \end{aligned} \tag{4.18}$$

Hence, the mapping defined in (4.15) is a contraction. Further, \tilde{v} obtained from this fixed point argument belongs to \tilde{V}_ω and satisfies (2.18) because of our choice of μ_0 . The proof is complete. \square

4.2. Transformation to original system. Now we want to prove the converse of Theorem 2.4.

Theorem 4.4. *Let $\tilde{\mu}$ belong to $(0, \mu_0]$, where μ_0 is defined by (4.13). Let $(\tilde{\sigma}, \tilde{v}, X, Y)$ be a solution to system (4.1) such that $(\tilde{\sigma}, \tilde{v})$ belongs to $D_{\tilde{\mu}}$. Let us set $\hat{f}(\cdot, t) = \chi_{\mathcal{O}} K((\tilde{\sigma}(\cdot, t), \tilde{v}(\cdot, t)))$,*

$$\hat{\sigma}(y, t) = \tilde{\sigma}(X(y, t), t), \quad \hat{v}(y, t) = \tilde{v}(X(y, t), t), \quad \hat{f}(y, t) = \tilde{f}(X(y, t), t).$$

If $\tilde{v} \in \tilde{V}_\omega$, then $\hat{\sigma}$ belongs to $L^2(0, \infty; \dot{H}_{per}^1(\Omega_y)) \cap L^\infty(0, \infty; \dot{H}_{per}^1(\Omega_y))$, \hat{v} belongs to $L^2(0, \infty; H_{per}^2(\Omega_y)) \cap H^1(0, \infty; L^2(\Omega_y))$, \hat{f} belongs to $L^2(0, \infty; L^2(\Omega_y))$ and there exists a constant $M_{2,\omega}$, depending on ω , such that

$$\|(\hat{\sigma}, \hat{v})\|_{\hat{D}} \leq M_{2,\omega} \|(\tilde{\sigma}, \tilde{v})\|_D.$$

Moreover $(\hat{\sigma}, \hat{v}, \hat{f})$ satisfies system (1.5), and, for $t > 0$, $Y(\cdot, t) = Y_{\hat{v}}(\cdot, t)$ is the solution of (2.4) and $X(\cdot, t) = X_{\hat{v}}(\cdot, t)$ is the inverse of $Y(\cdot, t)$. In addition, $\tilde{\sigma}_\Omega$ belongs to $L^2(0, \infty) \cap L^\infty(0, \infty)$.

Proof. For $\tilde{v} \in \tilde{V}_\omega$, using Lemma 2.5 and the change of variables formula, we get that $(\hat{\sigma}, \hat{v}, \hat{f})$ satisfies (1.5) in Q_y^∞ . Notice that all the solutions to the density equation (1.5)₁ are with mean zero. Consequently, $\hat{\sigma}$ also satisfies this condition. Since

$$\frac{\partial \hat{\sigma}}{\partial y}(y, t) = \frac{\partial \tilde{\sigma}}{\partial x}(X(y, t), t) \frac{\partial X}{\partial y}(y, t),$$

we have

$$\|\hat{\sigma}(\cdot, t)\|_{\dot{H}_{per}^1(\Omega_y)} \leq \sqrt{2} \|\tilde{\sigma}_m(\cdot, t)\|_{\dot{H}_{per}^1(\Omega_x)}, \quad \forall t > 0,$$

if $\tilde{v} \in \tilde{V}_\omega$. Indeed, in that case we have $\left| \frac{\partial X}{\partial y}(y, t) \right| \leq 2$. This inequality provides an estimate of $\hat{\sigma}$ in $L^\infty(0, \infty; \dot{H}_{per}^1(\Omega_y)) \cap L^2(0, \infty; \dot{H}_{per}^1(\Omega_y))$, but also in $L^\infty(0, \infty; H_{per}^1(\Omega_y)) \cap L^2(0, \infty; H_{per}^1(\Omega_y))$ because $\hat{\sigma}$ is with mean value zero.

Let us now recall the identity

$$\tilde{\sigma}_\Omega(t) = \frac{1}{L} \int_0^L \tilde{\sigma}(x, t) dx = \frac{1}{L} \int_0^L \hat{\sigma}(y, t) \frac{\partial X(y, t)}{\partial y} dy, \quad \forall t > 0. \quad (4.19)$$

From (4.19) and the estimate of $\hat{\sigma}$ in $L^\infty(0, \infty; H_{per}^1(\Omega_y)) \cap L^2(0, \infty; H_{per}^1(\Omega_y))$, it follows that $\tilde{\sigma}_\Omega$ also belongs to $L^\infty(0, \infty) \cap L^2(0, \infty)$, because $\left| \frac{\partial X}{\partial y}(y, t) \right| \leq 2$. \square

Remark 4.5. *The Lebesgue spaces $L^1(0, \infty; e^{-\omega(\cdot)})$ and $L^\infty(0, \infty; e^{-\omega(\cdot)})$ are well adapted to study the ordinary differential equation satisfied by $\tilde{\sigma}_\Omega$. We can prove the convergence of the fixed point method by using these spaces. But it is not possible to do it directly in $L^\infty(0, \infty) \cap L^2(0, \infty)$. We have finally deduced from (4.19) that $\tilde{\sigma}_\Omega$ is actually bounded. However, it is not possible to use an identity similar to (4.19) for the different iterates of the fixed point method.*

The next remark will be useful to prove the stabilization of system (1.5) by using the stabilization of system (2.8).

Remark 4.6. *In Section 4.1, we first obtain the unique solution $(\tilde{\sigma}, \tilde{v}, X, Y)$ of system (2.9) satisfying $\tilde{v} \in \tilde{V}_\omega$. Then applying Theorem 4.4, we get a solution $(\hat{\sigma}, \hat{v}, X, Y)$ for system (1.5) where Y is the solution of (2.4) and X its inverse. But the solution $(\hat{\sigma}, \hat{v}, X, Y)$, obtained in this way, is not necessarily unique. To prove the uniqueness, we first need to guarantee that the solutions to system (1.5) provide solutions to system (2.9) by a change of variables. That is obtained by imposing one more condition on the norm*

of $\|\tilde{v}\|_{L^2(0,\infty;H_{per}^2(\Omega_x))}$, in addition to the fact that $\tilde{v} \in \tilde{V}_\omega$. The details are given in the following lemma. The uniqueness of solution for system (1.5) will be obtained as a consequence of the uniqueness of solution for system (2.9) (see the proof of Theorem 1.1).

Lemma 4.7. *Let $\tilde{v} \in \tilde{V}_\omega$ and let us set $\hat{v}(y, t) = \tilde{v}(X(y, t), t)$ for all $(y, t) \in Q_y^\infty$. There exists a positive C_ω such that if*

$$\|\tilde{v}\|_{L^2(0,\infty;H_{per}^2(\Omega_x))} \leq C_\omega,$$

then \hat{v} belongs to \hat{V}_ω .

Proof. For each $t > 0$, we have $\hat{v}(y, t) = \tilde{v}(X(y, t), t)$ and $Y(X(y, t), t) = y$ for $y \in \Omega_y$. Now differentiating these two terms with respect to y , we get

$$\frac{\partial \hat{v}}{\partial y} = \frac{\partial \tilde{v}}{\partial x} \frac{\partial X}{\partial y}, \quad \frac{\partial Y}{\partial x} \frac{\partial X}{\partial y} = 1. \quad (4.20)$$

Using change of variables and the fact that $\frac{\partial X}{\partial y} \leq 2$, we have

$$\|\hat{v}\|_{L^2(Q_y^\infty)} \leq 2\|\tilde{v}\|_{L^2(Q_x^\infty)}. \quad (4.21)$$

Using $\frac{\partial Y}{\partial x} \geq \frac{1}{2}$, from Lemma 2.5, we get

$$\left\| \frac{\partial \hat{v}}{\partial y} \right\|_{L^2(Q_y^\infty)} \leq \left\| \frac{\partial \tilde{v}}{\partial x} \right\|_{L^2(Q_x^\infty)} \left\| \frac{\partial X}{\partial y} \right\|_{L^\infty(Q_y^\infty)} \leq 2 \left\| \frac{\partial \tilde{v}}{\partial x} \right\|_{L^2(Q_x^\infty)}. \quad (4.22)$$

By differentiating (4.20) with respect to y , we get

$$\frac{\partial^2 \hat{v}}{\partial y^2} = \frac{\partial^2 \tilde{v}}{\partial x^2} \left(\frac{\partial X}{\partial Y} \right)^2 + \frac{\partial \tilde{v}}{\partial x} \frac{\partial^2 X}{\partial y^2}, \quad \frac{\partial^2 Y}{\partial x^2} \left(\frac{\partial X}{\partial y} \right)^2 + \frac{\partial Y}{\partial x} \frac{\partial^2 X}{\partial y^2} = 0. \quad (4.23)$$

With Lemma 2.5, we have

$$\begin{aligned} \left\| \frac{\partial^2 X}{\partial y^2} \right\|_{L^\infty(0,\infty;L^2(\Omega_y))} &\leq 8 \left\| \frac{\partial^2 Y}{\partial x^2} \right\|_{L^\infty(0,\infty;L^2(\Omega_x))} \\ &\leq \frac{8}{\sqrt{2\omega}} \|\tilde{v}\|_{L^2(0,\infty;H_{per}^2(\Omega_x))} \leq \frac{2\sqrt{2}}{s_0}. \end{aligned}$$

Thus we get

$$\left\| \frac{\partial^2 \hat{v}}{\partial y^2} \right\|_{L^2(Q_y^\infty)} \leq 4 \left\| \frac{\partial^2 \tilde{v}}{\partial x^2} \right\|_{L^2(Q_x^\infty)} + 2\sqrt{2} \left\| \frac{\partial \tilde{v}}{\partial x} \right\|_{L^2(0,\infty;H_{per}^1(\Omega_x))}. \quad (4.24)$$

Finally by using (4.21), (4.22), (4.24), we obtain

$$\begin{aligned} \|\widehat{v}\|_{L^2(0,\infty;H_{per}^2(\Omega_y))} &= \|\widehat{v}\|_{L^2(Q_x^\infty)} + \left\| \frac{\partial \widehat{v}}{\partial x} \right\|_{L^2(Q_x^\infty)} + \left\| \frac{\partial^2 \widehat{v}}{\partial x^2} \right\|_{L^2(Q_x^\infty)} \\ &\leq 2\sqrt{2}(1 + \sqrt{2})\|\widetilde{v}\|_{L^2(0,\infty;H_{per}^2(\Omega_x))}. \end{aligned}$$

Let us choose

$$C_\omega := \min \left\{ \frac{\sqrt{\omega}}{8(\sqrt{2} + 1)s_0}, \frac{\sqrt{\omega}}{4\sqrt{L}(\sqrt{2} + 1)s_0} \right\}. \quad (4.25)$$

For C_ω defined above, \widehat{v} belongs to \widehat{V}_ω . \square

4.3. Stabilization of the original system. By making a change of variables, we can transform system (4.1) to find a control law for system (1.5). For (σ_0, v_0) satisfying (4.12), let $(\widetilde{\sigma}, \widetilde{v}, X, Y)$ be the unique solution to system (4.1) satisfying $(\widetilde{\sigma}, \widetilde{v}) \in D_{\widetilde{\mu}}$. Associated to this solution, we consider the change of variables

$$\begin{pmatrix} x \\ t \end{pmatrix} \mapsto \begin{pmatrix} Y(x, t) \\ t \end{pmatrix}, \quad \forall (x, t) \in Q_x^\infty. \quad (4.26)$$

Then, for each $t > 0$, the feedback control $K(\widetilde{\sigma}(\cdot, t), \widetilde{v}(\cdot, t))$, is transformed in the form

$$K(\widetilde{\sigma}(t), \widetilde{v}(t)) \circ X(\cdot, t), \quad (4.27)$$

where $X(\cdot, t)$, the inverse of $Y(\cdot, t)$, is also one of the components of the solution to system (4.1). As in Theorem 4.4, we can set

$$\widehat{\sigma}(y, t) = \widetilde{\sigma}(X(y, t), t), \quad \widehat{v}(y, t) = \widetilde{v}(X(y, t), t), \quad \forall (y, t) \in Q_y^\infty, \quad (4.28)$$

and for each $t \geq 0$ we have $X(\cdot, t) = X_{\widehat{v}}(\cdot, t)$, where $X_{\widehat{v}}(\cdot, t)$ is the inverse of $Y_{\widehat{v}}(\cdot, t)$ and $Y_{\widehat{v}}(\cdot, t)$ is the solution to transport equation (2.4). Therefore the feedback law K , transformed with the change of variables (4.26), depends not only on $\widehat{v}(\cdot, t)$ but also on $X_{\widehat{v}}(\cdot, t)$. This is why we set

$$\widehat{K}(\widehat{\sigma}(t), \widehat{v}(t), X_{\widehat{v}}(t))(y) = K(\widetilde{\sigma}(\cdot, t), \widetilde{v}(\cdot, t)) \circ X(y, t), \quad \forall (y, t) \in Q_y^\infty. \quad (4.29)$$

The feedback operator K is linear but, due to the change of variables, \widehat{K} is a nonlinear operator.

With the change of variables (4.26), system (4.1) is transformed into

$$\begin{aligned}
& \widehat{\sigma}_t + u_s \widehat{\sigma}_y + \rho_s \widehat{v}_y - \omega \widehat{\sigma} + e^{-\omega t} \{ \widehat{\sigma}_y \widehat{v} + \widehat{\sigma} \widehat{v}_y \} = 0 \quad \text{in } Q_y^\infty, \\
& \widehat{v}_t + u_s \widehat{v}_y + e^{-\omega t} \widehat{v} \widehat{v}_y + a \gamma (e^{-\omega t} \widehat{\sigma} + \rho_s)^{\gamma-2} \widehat{\sigma}_y - \nu \frac{\widehat{v}_{yy}}{e^{-\omega t} \widehat{\sigma} + \rho_s} - \omega \widehat{v} \\
& \quad = \chi_{(l_1, l_2)} \widehat{K}(\widehat{\sigma}(t), \widehat{v}(t), X_{\widehat{v}}(t)) \quad \text{in } Q_y^\infty, \\
& \widehat{\sigma}(0, \cdot) = \widehat{\sigma}(L, \cdot), \quad \widehat{v}(0, \cdot) = \widehat{v}(L, \cdot), \quad \widehat{v}_y(0, \cdot) = \widehat{v}_y(L, \cdot) \quad \text{in } (0, \infty), \\
& \widehat{\sigma}(\cdot, 0) = \sigma_0(\cdot), \quad \widehat{v}(\cdot, 0) = v_0(\cdot) \quad \text{in } \Omega_y, \quad \frac{1}{L} \int_0^L \sigma_0(y) dy = 0, \\
& \frac{\partial Y_{\widehat{v}}(x, t)}{\partial t} + u_s \frac{\partial Y_{\widehat{v}}(x, t)}{\partial x} = u_s + e^{-\omega t} \widehat{v}(Y_{\widehat{v}}(x, t), t), \quad \forall (x, t) \in Q_x^\infty, \\
& Y_{\widehat{v}}(x, 0) = I(x), \quad \forall x \in \Omega_x, \quad Y_{\widehat{v}}(x, \cdot) = Y_{\widehat{v}}(x + L, \cdot), \quad \forall x \in \Omega_x, \\
& X_{\widehat{v}}(Y_{\widehat{v}}(x, t), t) = x, \quad \forall (x, t) \in Q_x^\infty.
\end{aligned} \tag{4.30}$$

Theorem 4.8. *Let ω be any positive number. There exist positive constants $\widehat{\mu}_0$ and $\widehat{\kappa}$, depending on ω , ρ_s , u_s , l_1 , l_2 and L , such that for all $0 < \widehat{\mu} \leq \widehat{\mu}_0$ and all initial condition $(\sigma_0, v_0) \in \dot{H}_{per}^1(\Omega_y) \times H_{per}^1(\Omega_y)$ satisfying*

$$\|(\sigma_0, v_0)\|_{\dot{H}_{per}^1(\Omega_y) \times H_{per}^1(\Omega_y)} \leq \widehat{\kappa} \widehat{\mu}, \tag{4.31}$$

the nonlinear closed loop system (4.30) admits a unique solution $(\widehat{\sigma}, \widehat{v}, Y_{\widehat{v}}, X_{\widehat{v}})$ satisfying

$$\|(\widehat{\sigma}, \widehat{v})\|_{\widehat{D}} \leq \widehat{\mu}. \tag{4.32}$$

Moreover, $(\widehat{\sigma}, \widehat{v}) \in C_b([0, \infty); \dot{H}_{per}^1(\Omega_y) \times H_{per}^1(\Omega_y))$, $|\widehat{\sigma}(y, t)| \leq \frac{\rho_s}{2}$ for all $(y, t) \in Q_y^\infty$.

Proof. In view of Remark 4.6 and Lemma 4.7, μ_0 in Theorem 4.3 can be reduced further, if necessary, so that $\mu_0 \leq C_\omega$, where C_ω is defined in Lemma 4.7. Then by Theorems 4.3 and 4.4, we deduce that there exist constants $\widehat{\mu}_0 > 0$ and $\widehat{\kappa}$ such that, for $0 < \widehat{\mu} \leq \widehat{\mu}_0$ and any initial condition $(\sigma_0, v_0) \in \dot{H}_{per}^1(\Omega_y) \times H_{per}^1(\Omega_y)$ satisfying

$$\|(\sigma_0, v_0)\|_{\dot{H}_{per}^1(\Omega_y) \times H_{per}^1(\Omega_y)} \leq \widehat{\kappa} \widehat{\mu}, \tag{4.33}$$

the closed loop system (4.30) admits a solution $(\widehat{\sigma}, \widehat{v}, X, Y)$ such that

$$\|(\widehat{\sigma}, \widehat{v})\|_{L^\infty(0, \infty; \dot{H}_{per}^1(\Omega_y)) \cap L^2(0, \infty; \dot{H}_{per}^1(\Omega_y)) \times H^1(0, \infty; L^2(\Omega_y)) \cap L^2(0, \infty; H_{per}^2(\Omega_y))} \leq \widehat{\mu}.$$

Moreover, $X \in C_b([0, \infty); H_{per}^2(\Omega_y)) \cap C_b^1([0, \infty); H_{per}^1(\Omega_y))$, $Y \in C_b([0, \infty); H_{per}^2(\Omega_x)) \cap C_b^1([0, \infty); H_{per}^1(\Omega_x))$, $\widehat{\sigma} \in C([0, \infty); \dot{H}_{per}^1(\Omega_y))$, and $\widehat{v} \in C_b([0, \infty); H_{per}^1(\Omega_y))$.

If the initial condition (σ_0, v_0) satisfies (4.33) and if $(\widehat{\sigma}, \widehat{v}, X_{\widehat{v}}, Y_{\widehat{v}})$ is a solution to (4.30), we can define $(\widetilde{\sigma}, \widetilde{v})$ by change of variables, and $(\widetilde{\sigma}, \widetilde{v}, X, Y)$, with $(X, Y) = (X_{\widehat{v}}, Y_{\widehat{v}})$ is a solution to system (4.1). From the decomposition $\widetilde{\sigma} = \widetilde{\sigma}_m + \widetilde{\sigma}_\Omega$ and the definition of $\widetilde{\sigma}_\Omega$, we deduce that

$$\begin{aligned} \|\widetilde{\sigma}_\Omega\|_{L^\infty(0, \infty; e^{-\omega(\cdot)})} &\leq \frac{1}{L^{1/2}} \|\widetilde{\sigma}\|_{L^\infty(0, \infty; H_{per}^1(\Omega_y)) \cap L^2(0, \infty; H_{per}^1(\Omega_y))}, \\ \|\widetilde{\sigma}_\Omega\|_{L^\infty(0, \infty) \cap L^2(0, \infty)} &\leq \|\widetilde{\sigma}\|_{L^\infty(0, \infty; H_{per}^1(\Omega_y)) \cap L^2(0, \infty; H_{per}^1(\Omega_y))}, \end{aligned}$$

and

$$\begin{aligned} &\|\widetilde{\sigma}_m\|_{L^\infty(0, \infty; H_{per}^1(\Omega_y)) \cap L^2(0, \infty; H_{per}^1(\Omega_y))} + \|\widetilde{\sigma}_\Omega\|_{L^\infty(0, \infty; e^{-\omega(\cdot)})} \\ &\leq \|\widetilde{\sigma}\|_{L^\infty(\mathbb{R}^+; H_{per}^1(\Omega_y)) \cap L^2(\mathbb{R}^+; H_{per}^1(\Omega_y))} + \|\widetilde{\sigma}_\Omega\|_{L^\infty(\mathbb{R}^+) \cap L^2(\mathbb{R}^+) \cap L^\infty(\mathbb{R}^+; e^{-\omega(\cdot)})} \\ &\leq (2 + L^{-1/2}) \|\widetilde{\sigma}\|_{L^\infty(0, \infty; H_{per}^1(\Omega_y)) \cap L^2(0, \infty; H_{per}^1(\Omega_y))}. \end{aligned}$$

Therefore, due to Theorem 2.4 and to the definition of the norms $\|\cdot\|_D$ and $\|\cdot\|_{\widehat{D}}$, we have

$$\|(\widetilde{\sigma}, \widetilde{v})\|_D \leq (2 + L^{-1/2}) \|(\widetilde{\sigma}, \widetilde{v})\|_{\widehat{D}} \leq (2 + L^{-1/2}) M_{1, \omega} \|(\widehat{\sigma}, \widehat{v})\|_{\widehat{D}}.$$

Thus, if we choose $(2 + L^{-1/2}) M_{1, \omega} \widehat{\mu}_0 \leq \mu_0$, we shall have $\|(\widetilde{\sigma}, \widetilde{v})\|_D \leq \mu_0$, and $(\widetilde{\sigma}, \widetilde{v}, X, Y)$ will be the unique solution of system (4.1) in D_{μ_0} . Thus, under the additional condition $(2 + L^{-1/2}) M_{1, \omega} \widehat{\mu}_0 \leq \mu_0$, (4.30) admits a unique solution $(\widehat{\sigma}, \widehat{v}, X_{\widehat{v}}, Y_{\widehat{v}})$ such that $\|(\widehat{\sigma}, \widehat{v})\|_{\widehat{D}} \leq \widehat{\mu}$. The proof is complete. \square

From Theorem 4.8, the stabilization result of Theorem 1.1 is obtained with the control \widehat{f} defined by

$$\widehat{f}(y, t) = \chi_{(\ell_1, \ell_2)}(y) \widehat{K}(\widehat{\sigma}(\cdot, t), \widehat{v}(\cdot, t), X(\cdot, t))(y), \quad \forall (y, t) \in Q_y^\infty. \quad (4.34)$$

Proof of Theorem 1.2. The closed loop nonlinear system corresponding

to system (1.1) reads as follows

$$\begin{aligned}
\rho_t + (\rho u)_y &= 0 \quad \text{in } Q_y^\infty, \\
\rho(u_t + uu_y) + (p(\rho))_y - \nu u_{yy} \\
&= \rho \chi_{(\ell_1, \ell_2)} e^{-\omega t} \widehat{K}(e^{\omega t}(\rho(t) - \rho_s), e^{\omega t}(u(t) - u_s), X(t)) \quad \text{in } Q_y^\infty, \\
\rho(0, \cdot) &= \rho(L, \cdot), \quad u(0, \cdot) = u(L, \cdot), \quad u_y(0, \cdot) = u_y(L, \cdot) \quad \text{in } (0, \infty), \\
\rho(\cdot, 0) &= \rho_0(\cdot), \quad u(\cdot, 0) = u_0(\cdot) \quad \text{in } \Omega_y, \\
\frac{\partial Y(x, t)}{\partial t} + u_s \frac{\partial Y(x, t)}{\partial x} &= u(Y(x, t), t), \quad \forall (x, t) \in Q_x^\infty, \\
Y(x, 0) &= I(x), \quad \forall x \in \Omega_x, \quad Y(x, \cdot) = Y(x + L, \cdot), \quad \forall x \in \Omega_x, \\
X(Y(x, t), t) &= x, \quad \forall (x, t) \in Q_x^\infty.
\end{aligned} \tag{4.35}$$

From the proof of Theorem 1.1, it follows that system (4.35) admits at least a solution (ρ, u, X, Y) defined by

$$\rho = e^{-\omega t} \widehat{\sigma} + \rho_s, \quad u = e^{-\omega t} \widehat{v} + u_s, \quad X = X_{\widehat{v}}, \quad Y = Y_{\widehat{v}},$$

where $(\widehat{\sigma}, \widehat{v}, X_{\widehat{v}}, Y_{\widehat{v}})$ is the solution of system (4.30). The solution (ρ, u, X, Y) is unique in the set of functions satisfying

$$\|(e^{\omega t}(\rho - \rho_s), (e^{\omega t}(u - u_s)))\|_{\widehat{D}} \leq \widehat{\mu}.$$

Thus the stabilization result of Theorem 1.2 is obtained with the control f defined by

$$f(y, t) = \chi_{(\ell_1, \ell_2)}(y) e^{-\omega t} \widehat{K}(e^{\omega t}(\rho(t) - \rho_s), e^{\omega t}(u(t) - u_s), X(t))(y), \tag{4.36}$$

for all $(y, t) \in Q_y^\infty$.

4.4. Control law for the original system. In this section, we explain why the control defined in (4.27) or in (4.29) is a nonlinear control law for system (4.30). The state variable of system (4.30) is $(\widehat{\sigma}, \widehat{v}, Y_{\widehat{v}}, X_{\widehat{v}})$. It is clear that $(\widehat{\sigma}, \widehat{v}, Y_{\widehat{v}})$ are state variables, since the triplet satisfies an evolution equation. The last equation in (4.30) is a nonlinear equation characterizing $X_{\widehat{v}}$ in terms of $Y_{\widehat{v}}$. Since we deal with one dimensional problems, $X_{\widehat{v}}$ can be easily deduced from $Y_{\widehat{v}}$. This is why we can consider $X_{\widehat{v}}$ as an additional component of the state variable for system (4.30).

Due to Proposition 3.3, the feedback K is expressed by two functions $k_\sigma \in L^2(\Omega_x; H_{per}^{-1}(\Omega_x))$ and $k_v \in L^2(\Omega_x \times \Omega_x)$. For all $t > 0$, we compose

$k_\sigma(\cdot, \cdot)$ and $k_v(\cdot, \cdot)$ with $X_{\widehat{v}}(\cdot, t)$ and we set

$$\widehat{k}_\sigma(y, \zeta, t) = |\partial_y X_{\widehat{v}}(\zeta, t)| k_\sigma(X_{\widehat{v}}(y, t), X_{\widehat{v}}(\zeta, t))$$

and

$$\widehat{k}_v(y, \zeta, t) = |\partial_y X_{\widehat{v}}(\zeta, t)| k_v(X_{\widehat{v}}(y, t), X_{\widehat{v}}(\zeta, t)).$$

Since $X_{\widehat{v}}$ belongs to $C_b([0, \infty); H_{per}^2(\Omega_y)) \cap C_b^1([0, \infty); H_{per}^1(\Omega_y))$, it follows that

$$\widehat{k}_\sigma \in C_b([0, \infty); L^2(\Omega_y; H_{per}^{-1}(\Omega_y))) \quad \text{and} \quad \widehat{k}_v \in C_b([0, \infty); L^2(\Omega_y \times \Omega_y)).$$

Therefore, the control law \widehat{K} for the original system is defined by

$$\begin{aligned} & \widehat{K}(\widehat{\sigma}(\cdot, t), \widehat{v}(\cdot, t), X_{\widehat{v}}(t))(y) \\ &= \left\langle \widehat{k}_\sigma(y, \cdot, t), \widehat{\sigma}(\cdot, t) \right\rangle_{H_{per}^{-1}(\Omega_y), H_{per}^1(\Omega_y)} + \left(\widehat{k}_v(y, \cdot, t), \widehat{v}(\cdot, t) \right)_{L^2(\Omega_y)}. \end{aligned} \quad (4.37)$$

Remark 4.9. *The corresponding control law for system (4.35) is*

$$\begin{aligned} & e^{-\omega t} \widehat{K} \left(e^{\omega t} (\rho(t) - \rho_s), e^{\omega t} (u(t) - u_s), X(t) \right) (y) \\ &= \left\langle \widehat{k}_\sigma(y, \cdot, t), (\rho(\cdot, t) - \rho_s) \right\rangle_{H_{per}^{-1}(\Omega_y), H_{per}^1(\Omega_y)} \\ &+ \left(\widehat{k}_v(y, \cdot, t), (u(\cdot, t) - u_s) \right)_{L^2(\Omega_y)}. \end{aligned}$$

From Theorem 4.8, it follows that the first two components of the state of system (4.35), namely (ρ, u) , are stabilized towards the steady state (ρ_s, u_s) , exponentially in the $H_{per}^1(\Omega_y) \times H_{per}^1(\Omega_y)$ norm, even if (Y, X) is not stabilized.

5. DIRICHLET BOUNDARY CONTROL

In this section, we study the local stabilization of the one dimensional compressible Navier-Stokes system around (ρ_s, u_s) , $\rho_s > 0, u_s > 0$ by boundary controls. We prove Theorem 1.3 by using Theorem 1.2. To do that, we need to extend the domain $(0, L)$ to $(-L, L)$ and to consider system (1.6) in $(-L, L)$ with periodic boundary conditions. The next theorem gives the stabilization result for domain $(-L, L) \times (0, \infty)$, analogous to Theorem 1.2 and we skip its proof as it is similar to that of Theorem 1.2.

Theorem 5.1. *Let L be any positive number and (ℓ_1, ℓ_2) is an open subset of $(-L, L)$. Let us consider system (1.1) in $(-L, L) \times (0, \infty)$ with periodic boundary conditions. For any positive number ω , there exist positive constants $\widehat{\mu}_0$ and $\widehat{\kappa}$, depending on $\omega, \rho_s, u_s, \ell_1, \ell_2$ and L , such that, for*

$0 < \widehat{\mu} \leq \widehat{\mu}_0$ and any initial condition $(\rho_0, u_0) \in H_{per}^1(-L, L) \times H_{per}^1(-L, L)$, where ρ_0 satisfies

$$\frac{1}{2L} \int_{-L}^L \rho_0(y) dy = \rho_s \quad \text{and} \quad \min_{y \in [-L, L]} \rho_0(y) > 0$$

and (ρ_0, u_0) obeys

$$\|(\rho_0 - \rho_s, u_0 - u_s)\|_{\dot{H}_{per}^1(-L, L) \times H_{per}^1(-L, L)} \leq \widehat{\kappa} \widehat{\mu},$$

there exists a control $f \in L^2(0, \infty; L^2(-L, L))$ for which system (1.1) admits a unique solution (ρ, u) satisfying

$$\|(\rho(\cdot, t) - \rho_s, u(\cdot, t) - u_s)\|_{\dot{H}_{per}^1(-L, L) \times H_{per}^1(-L, L)} \leq C \widehat{\mu} e^{-\omega t},$$

for some positive constant C depending on $\omega, \rho_s, u_s, \ell_1, \ell_2$ and L but independent of $\widehat{\mu}$. Moreover, we have $\rho(y, t) \geq \frac{\rho_s}{2}$ for all $(y, t) \in (-L, L) \times (0, \infty)$.

Furthermore, if $u_s > 0$, we can choose $\widehat{\mu}_0$ in such a way that $u(y, t) \geq \frac{u_s}{2}$ for all $(y, t) \in (-L, L) \times (0, \infty)$.

To prove the stabilization result for system (1.6) using the above theorem, we extend the initial condition (ρ_0, u_0) as follows.

Proposition 5.2. *Let $\widehat{\mu}_0$ and $\widehat{\kappa}$ be the positive constants defined in Theorem 5.1 corresponding to the domain $(-L, L)$. There exists a positive constant μ_d depending on ω, ρ_s, u_s, L and $\widehat{\mu}_0$, such that any $(\rho_0, u_0) \in H^1(0, L) \times H^1(0, L)$ satisfying (1.7), i.e.*

$$\min_{[0, L]} \rho_0 > 0 \quad \text{and} \quad \|(\rho_0 - \rho_s, u_0 - u_s)\|_{H^1(0, L) \times H^1(0, L)} \leq \mu_d,$$

admits an extension (ρ_0^e, u_0^e) in $H_{per}^1(-L, L) \times H_{per}^1(-L, L)$ obeying

$$\frac{1}{2L} \int_{-L}^L \rho_0^e(x) dx = \rho_s, \quad \min_{[-L, L]} \rho_0^e > 0, \quad (5.1)$$

and

$$\|(\rho_0^e - \rho_s, u_0^e - u_s)\|_{\dot{H}_{per}^1(-L, L) \times H_{per}^1(-L, L)} \leq \widehat{\kappa} \widehat{\mu}_0. \quad (5.2)$$

Proof. We want to determine μ_d and the extension $(\rho_0^e, u_0^e) \in H_{per}^1(-L, L) \times H_{per}^1(-L, L)$ of (ρ_0, u_0) depending on $\widehat{\mu}_0$ in such a way that (1.7) implies (5.1) and (5.2). Recall that $\|\rho_0 - \rho_s\|_{H^1(0, L)} \leq \mu_d$ gives

$$\|\rho_0 - \rho_s\|_{L^\infty(0, L)} \leq s_0 \mu_d, \quad (5.3)$$

where s_0 is the Sobolev constant depending on the domain. To get (5.1), we need the compatibility condition

$$\frac{1}{2L} \int_0^L \rho_0(x) dx < \rho_s. \quad (5.4)$$

In view of (5.3), we have

$$\frac{1}{2L} \int_0^L \rho_0(x) dx \leq \frac{1}{2}(\rho_s + s_0\mu_d).$$

We notice that (5.4) is satisfied if We choose μ_d such that

$$0 < \mu_d < \frac{\rho_s}{s_0}. \quad (5.5)$$

From now on, we assume that (5.5) is true. In order to determine the extension, we consider the minimization problem

$$\min \left\{ \int_{-L}^0 |\varrho'(x)|^2 dx \mid \varrho \in H^1(-L, 0) \text{ and satisfies (5.7)} \right\} \quad (5.6)$$

with

$$\begin{aligned} \varrho(0) &= \rho_0(0), \quad \varrho(-L) = \rho_0(L), \\ \int_{-L}^0 \varrho(x) dx &= (\rho_s + \check{\rho})L, \quad \text{where } \check{\rho} = \rho_s - \frac{1}{L} \int_0^L \rho_0(x) dx. \end{aligned} \quad (5.7)$$

By solving Euler-Lagrange equation associated with (5.6), we obtain the solution

$$\varrho_{min}(x) = ax^2 + bx + \sigma_0, \quad \text{on } (-L, 0)$$

with

$$a = 3 \frac{(\rho_0(L) - \rho_s) + (\rho_0(0) - \rho_s) - 2\check{\rho}}{L^2}, \quad b = 2 \left(\frac{aL}{3} - \frac{\rho_s + \check{\rho} - \rho_0(0)}{L} \right). \quad (5.8)$$

Since $|\rho_0(L) - \rho_s| \leq s_0\mu_d$, $|\rho_0(0) - \rho_s| \leq s_0\mu_d$ and $|\check{\rho}| \leq s_0\mu_d$, we have

$$|a| \leq \frac{12s_0\mu_d}{L^2}, \quad |b| \leq \frac{12s_0\mu_d}{L}. \quad (5.9)$$

Thus we get a positive constant M_1 depending only on L , s_0 such that

$$\int_{-L}^0 |\varrho'_{min}(x)|^2 dx = \int_{-L}^0 |2ax + b|^2 dx \leq 2 \left(\frac{4a^2L^3}{3} + b^2L \right) \leq M_1\mu_d^2. \quad (5.10)$$

Now define

$$\rho_0^e = \begin{cases} \rho_0, & \text{on } (0, L), \\ \varrho_{min}, & \text{on } (-L, 0]. \end{cases} \quad (5.11)$$

Then ρ_0^e belongs to $\dot{H}_{per}^1(-L, L)$ and by (5.10), we have

$$\|\rho_0^e - \rho_s\|_{\dot{H}_{per}^1(-L, L)}^2 \leq (M_1 + 1)\mu_d^2. \quad (5.12)$$

In place of (5.5), we assume that μ_d satisfies the more restrictive condition

$$0 < \mu_d \leq \frac{\rho_s}{s_0\sqrt{M_1 + 1}}. \quad (5.13)$$

In that case, ρ_0^e obeys (5.1). We extend u_0 as follows

$$u_0^e(x) = \begin{cases} u_0(x), & \text{if } x \in (0, L), \\ \frac{x}{L}(u_0(0) - u_0(L)) + u_0(0), & \text{if } x \in (-L, 0]. \end{cases} \quad (5.14)$$

Then u_0^e belongs to $H_{per}^1(-L, L)$ and

$$\|u_0^e - u_s\|_{H_{per}^1(-L, L)}^2 = \|u_0^e - u_s\|_{H_{per}^1(-L, 0)}^2 + \|u_0 - u_s\|_{H_{per}^1(0, L)}^2 \leq M_2\mu_d^2, \quad (5.15)$$

for some positive constant M_2 depending only on L, s_0 . Finally we assume that

$$0 < \mu_d = \min \left\{ \frac{\rho_s}{s_0\sqrt{M_1 + 1}}, \frac{\widehat{\kappa}\widehat{\mu}_0}{\sqrt{M_1 + M_2 + 2}} \right\}. \quad (5.16)$$

In that case, (ρ_0^e, u_0^e) satisfies estimate (5.2), the proposition is proved. \square

Proof of Theorem 1.3. Let us consider (ρ_0, u_0) satisfying (1.7) where $\mu_d > 0$ obeys (5.16). Then (ρ_0, u_0) admits an extension (ρ_0^e, u_0^e) to $(-L, L)$, satisfying (5.1) and (5.2) (see Proposition 5.2). We consider system (1.1) in $(-L, L) \times (0, \infty)$ with the initial condition (ρ_0^e, u_0^e) and an interior control localized in $\left(\frac{-L}{2}, \frac{-L}{4}\right)$. Now, from Theorem 5.1 applied to the interval $(-L, L)$, it follows that there exists a control $f \in L^2(0, \infty; L^2(-L, L))$, with support in $\left(\frac{-L}{2}, \frac{-L}{4}\right)$, such that system (1.1) in the domain $(-L, L) \times (0, \infty)$ admits a unique solution (ρ^e, u^e) satisfying

$$\|(\rho^e(\cdot, t) - \rho_s, u^e(\cdot, t) - u_s)\|_{\dot{H}_{per}^1(-L, L) \times H_{per}^1(-L, L)} \leq Ce^{-\omega t},$$

for some positive constant C independent of μ_d . Furthermore,

$$\rho^e \in L^2(0, \infty; H_{per}^1(-L, L)) \cap C_b([0, \infty); H_{per}^1(-L, L))$$

and

$$u^e \in H^1(0, \infty; L^2(-L, L)) \cap L^2(0, \infty; H_{per}^2(-L, L)) \cap C_b([0, \infty); H_{per}^1(-L, L)).$$

We also have

$$\rho^e(y, t) \geq \frac{\rho_s}{2} \quad \text{and} \quad u^e(y, t) \geq \frac{u_s}{2}, \quad \forall (y, t) \in (-L, L) \times (0, \infty).$$

By setting

$$(\rho, u) = (\rho^e, u^e)|_{(0, L) \times (0, \infty)}, \quad (5.17)$$

(ρ, u) is the solution of (1.6) with the controls

$$q_1(t) = \rho^e(0, t), \quad q_2(t) = u^e(0, t), \quad q_3(t) = u^e(L, t). \quad (5.18)$$

Moreover (ρ, u) satisfies (1.8), $\rho(y, t) \geq \frac{\rho_s}{2}$ and $u(y, t) \geq \frac{u_s}{2}$ for all $(y, t) \in Q_y^\infty$. With (5.18) and the regularity of the trace of (ρ^e, u^e) at $x = 0, L$, we have $q_1 \in L^2(0, \infty) \cap C_b([0, \infty))$ and $q_j \in H^{\frac{3}{4}}(0, \infty)$, for $j = 2, 3$. Hence, Theorem 1.3 is proved. \square

6. APPENDIX

6.1. Proof of Proposition 2.1 and Lemma 2.2. Proof of Proposition 2.1.

Proof. In order to use the method of successive approximations for the existence of a L -periodic solution $Y_{\widehat{v}}$ to equation (2.6), we define the sequence of functions $\{Y_{\widehat{v}}^{(n)}\}_{n \in \mathbb{N}}$ by

$$Y_{\widehat{v}}^{(1)}(x, t) = I(x), \quad \forall (x, t) \in Q_x^\infty, \quad (6.1)$$

and for every $n \in \mathbb{N}$,

$$Y_{\widehat{v}}^{(n+1)}(x, t) = I(x) + \int_0^t e^{-\omega\tau} \widehat{v}(Y_{\widehat{v}}^{(n)}(x + u_s(\tau - t), \tau) + I(x + u_s(\tau - t)), \tau) d\tau, \quad (6.2)$$

for all $(x, t) \in Q_x^\infty$. By induction, it follows that $Y_{\widehat{v}}^{(n+1)}$ is L -periodic and $Y_{\widehat{v}}^{(n+1)} \in C_b([0, \infty); L^2(\Omega_x))$ for every $n \in \mathbb{N}$. Now, for every $t > 0$, we

obtain

$$\begin{aligned}
& \|Y_{\widehat{v}}^{(n+1)}(\cdot, t) - Y_{\widehat{v}}^{(n)}(\cdot, t)\|_{L^2(\Omega_x)}^2 \\
& \leq \frac{1}{2\omega} \int_0^\infty \left\| \frac{\partial \widehat{v}}{\partial y}(\cdot, \tau) \right\|_{L^\infty(\Omega_y)}^2 \times \\
& \quad \int_0^L |Y_{\widehat{v}}^{(n)}(x + u_s(\tau - t), \tau) - Y_{\widehat{v}}^{(n-1)}(x + u_s(\tau - t), \tau)|^2 dx d\tau.
\end{aligned} \tag{6.3}$$

By using $\xi = x + u_s(\tau - t)$, for fixed $\tau > 0$, $t > 0$, $n \in \mathbb{N}$ and the periodicity of $\{Y_{\widehat{v}}^{(n)}\}_{n \in \mathbb{N}}$, we have that

$$\begin{aligned}
& \int_0^L |Y_{\widehat{v}}^{(n)}(x + u_s(\tau - t), \tau) - Y_{\widehat{v}}^{(n-1)}(x + u_s(\tau - t), \tau)|^2 dx \\
& = \int_{u_s(\tau-t)}^{L+u_s(\tau-t)} |Y_{\widehat{v}}^{(n)}(\xi, \tau) - Y_{\widehat{v}}^{(n-1)}(\xi, \tau)|^2 d\xi \\
& = \int_0^L |Y_{\widehat{v}}^{(n)}(\xi, \tau) - Y_{\widehat{v}}^{(n-1)}(\xi, \tau)|^2 d\xi.
\end{aligned} \tag{6.4}$$

From (6.3) and (6.4), we obtain

$$\begin{aligned}
& \|Y_{\widehat{v}}^{(n+1)}(\cdot, t) - Y_{\widehat{v}}^{(n)}(\cdot, t)\|_{L^2(\Omega_x)}^2 \\
& \leq \frac{1}{2\omega} \int_0^\infty \left\| \frac{\partial \widehat{v}}{\partial y}(\cdot, \tau) \right\|_{L^\infty(\Omega_y)}^2 \int_0^L |Y_{\widehat{v}}^{(n)}(\xi, \tau) - Y_{\widehat{v}}^{(n-1)}(\xi, \tau)|^2 d\xi d\tau \\
& \leq \frac{s_0^2}{2\omega} \|\widehat{v}\|_{L^2(0, \infty; H_{per}^2(\Omega_y))}^2 \|Y_{\widehat{v}}^{(n)} - Y_{\widehat{v}}^{(n-1)}\|_{C_b(0, \infty; L^2(\Omega_x))}^2.
\end{aligned}$$

Similarly, for every $t > 0$, we also have

$$\|Y_{\widehat{v}}^{(2)}(\cdot, t) - Y_{\widehat{v}}^{(1)}(\cdot, t)\|_{L^2(\Omega_x)}^2 \leq \frac{Ls_0^2}{2\omega} \|\widehat{v}\|_{L^2(0, \infty; H_{per}^2(\Omega_y))}^2.$$

As $\widehat{v} \in \widehat{V}_\omega$, this yields

$$\|Y_{\widehat{v}}^{(2)} - Y_{\widehat{v}}^{(1)}\|_{C_b([0, \infty); L^2(\Omega_x))} \leq \frac{1}{2},$$

$$\|Y_{\widehat{v}}^{(n+1)} - Y_{\widehat{v}}^{(n)}\|_{C_b([0, \infty); L^2(\Omega_x))} \leq \frac{1}{2^n}, \quad \forall n \in \mathbb{N}.$$

From the above estimates, it follows that $\{Y_{\widehat{v}}^{(n)}\}_{n \in \mathbb{N}}$ converges to a L -periodic function $Y_{\widehat{v}}$ in the space $C_b([0, \infty); L^2(\Omega_x))$ and $Y_{\widehat{v}}$ is the unique solution to

equation (2.6).

In order to conclude the regularity of this solution, we check that

$$\begin{aligned} & \frac{d}{d\tau} \left[\frac{\partial Y_{\widehat{v}}}{\partial x}(x + u_s(\tau - t), \tau) \right] \\ &= e^{-\omega\tau} \frac{\partial \widehat{v}}{\partial y} \left(Y_{\widehat{v}}(x + u_s(\tau - t), \tau), \tau \right) \left[\frac{\partial Y_{\widehat{v}}}{\partial x}(x + u_s(\tau - t), \tau) \right], \quad \tau > 0, \\ & \frac{\partial Y_{\widehat{v}}}{\partial x}(x + u_s(\tau - t), \tau)|_{\tau=0} = 1. \end{aligned} \tag{6.5}$$

The unique solution of equation (6.5) is

$$\frac{\partial Y_{\widehat{v}}}{\partial x}(x, t) = \exp \left(\int_0^t e^{-\omega\tau} \frac{\partial \widehat{v}}{\partial y} (Y_{\widehat{v}}(x + u_s(\tau - t), \tau), \tau) d\tau \right), \quad \forall (x, t) \in Q_x^\infty. \tag{6.6}$$

From this expression, we estimate that

$$\left\| \frac{\partial Y_{\widehat{v}}}{\partial x} \right\|_{L^\infty(Q_x^\infty)} \leq \frac{3}{2}. \tag{6.7}$$

From (6.6) and (6.7), it follows that

$$\left\| \frac{\partial Y_{\widehat{v}}}{\partial x}(\cdot, t) \right\|_{L^2(\Omega_x)}^2 \leq \frac{9L}{4}, \quad \forall t > 0. \tag{6.8}$$

In a similar manner, we obtain

$$\left\| \frac{\partial^2 Y_{\widehat{v}}}{\partial x^2}(\cdot, t) \right\|_{L^2(\Omega_x)}^2 \leq \frac{27}{16\omega} \|\widehat{v}\|_{L^2(0, \infty; H_{per}^2(\Omega_y))}^2, \quad \forall t > 0. \tag{6.9}$$

Using the fact that $\widehat{v} \in \widehat{V}$, we obtain

$$\left\| \frac{\partial^2 Y_{\widehat{v}}}{\partial t \partial x}(\cdot, t) \right\|_{L^2(\Omega_x)} \leq \frac{3}{2} \|\widehat{v}\|_{C_b([0, \infty); H_{per}^1(\Omega_y))} + \frac{3\sqrt{3}u_s}{4\sqrt{\omega}} \|\widehat{v}\|_{L^2(0, \infty; H_{per}^2(\Omega_y))}, \tag{6.10}$$

for all $t > 0$. Finally, (6.8), (6.9) and (6.10) yield that

$$Y_{\widehat{v}} \in C_b([0, \infty); H_{per}^2(\Omega_x)) \cap C_b^1([0, \infty); H_{per}^1(\Omega_x))$$

and hence $Y_{\widehat{v}}$ is the solution of (2.4).

For every $t > 0$, the injectivity of the mapping $x \mapsto Y_{\widehat{v}}(x, t)$ follows from the uniqueness of the solution to equation (2.6). The existence of a unique solution to the backward equation corresponding to (2.4) implies the surjectivity of the mapping $x \mapsto Y_{\widehat{v}}(x, t)$ for every $t > 0$. \square

Proof of Lemma 2.2.

Proof. From the estimates in the above proof, we obtain

$$\left| \frac{\partial Y_{\widehat{v}}}{\partial x}(x, t) - 1 \right| = \left| \exp \left(\int_0^t e^{-\omega\tau} \frac{\partial \widehat{v}}{\partial y}(Y_{\widehat{v}}(x + u_s(\tau - t), \tau), \tau) d\tau \right) - 1 \right| \leq \frac{1}{2},$$

for all $(x, t) \in Q_x^\infty$. Thus

$$\left\| \frac{\partial Y_{\widehat{v}}}{\partial x} - 1 \right\|_{L^\infty(Q_x^\infty)} \leq \frac{1}{2}.$$

□

6.2. Estimates of the nonlinear terms. In this section, we assume that (ζ, ϑ) , (ζ^1, ϑ^1) and (ζ^2, ϑ^2) belong to D_μ with $\mu = \min \left\{ \frac{\rho_s}{4s_0}, \frac{\sqrt{\omega}}{2s_0} \right\}$. Here, to avoid heavy notation, we denote the change of variables by Y instead of $Y(\zeta, \vartheta)$.

Proof of (4.6) and (4.9). To estimate $F_{1,m}$ in $L^2(0, \infty; \dot{H}_{per}^1(\Omega_x))$, it is enough to estimate $\frac{\partial F_1}{\partial x}$ in $L^2(0, \infty; L^2(\Omega_x))$. We have

$$\begin{aligned} \frac{\partial F_1}{\partial x} &= \rho_s \vartheta_{xx} \left(1 - \frac{1}{\frac{\partial Y}{\partial x}} \right) + \rho_s \vartheta_x \frac{\frac{\partial^2 Y}{\partial x^2}}{\left(\frac{\partial Y}{\partial x} \right)^2} - e^{-\omega t} \frac{(\zeta_m)_x \vartheta_x}{\frac{\partial Y}{\partial x}} \\ &\quad - e^{-\omega t} \frac{\zeta_m \vartheta_{xx}}{\frac{\partial Y}{\partial x}} - e^{-\omega t} \frac{\zeta_\Omega \vartheta_{xx}}{\frac{\partial Y}{\partial x}} + e^{-\omega t} (\zeta_m + \zeta_\Omega) \vartheta_x \frac{\frac{\partial^2 Y}{\partial x^2}}{\left(\frac{\partial Y}{\partial x} \right)^2}. \end{aligned} \quad (6.11)$$

The different terms in (6.11) may be estimated as follows

$$\begin{aligned} &\left\| \rho_s \vartheta_{xx} \left(1 - \left(\frac{\partial Y}{\partial x} \right)^{-1} \right) \right\|_{L^2(0, \infty; L^2(\Omega_x))} \\ &\leq 2\rho_s s_0 \left\| \frac{\partial Y}{\partial x} - 1 \right\|_{L^\infty(0, \infty; H_{per}^1(\Omega_x))} \|\vartheta\|_{L^2(0, \infty; H_{per}^2(\Omega_x))} \\ &\leq \frac{2\rho_s s_0}{\sqrt{\omega}} \|\vartheta\|_{L^2(0, \infty; H_{per}^2(\Omega_x))}^2, \end{aligned} \quad (6.12)$$

$$\begin{aligned}
& \left\| \rho_s \vartheta_x \frac{\frac{\partial^2 Y}{\partial x^2}}{\left(\frac{\partial Y}{\partial x}\right)^2} \right\|_{L^2(0, \infty; L^2(\Omega_x))} \\
& \leq 4\rho_s s_0 \|\vartheta_x\|_{L^2(0, \infty; H_{per}^1(\Omega_x))} \left\| \frac{\partial^2 Y}{\partial x^2} \right\|_{L^\infty(0, \infty; L^2(\Omega_x))} \\
& \leq \frac{2\sqrt{2}\rho_s s_0}{\sqrt{\omega}} \|\vartheta\|_{L^2(0, \infty; H_{per}^2(\Omega_x))}^2,
\end{aligned} \tag{6.13}$$

$$\begin{aligned}
& \left\| e^{-\omega t} \frac{(\zeta_m)_x \vartheta_x}{\frac{\partial Y}{\partial x}} \right\|_{L^2(0, \infty; L^2(\Omega_x))} = \left\| e^{-\omega t} \frac{(\zeta_m)_x \vartheta_x}{\frac{\partial Y}{\partial x}} \right\|_{L^2(0, \infty; L^2(\Omega_x))} \\
& \leq 2s_0 \|\zeta_m\|_{L^\infty(0, \infty; \dot{H}_{per}^1(\Omega_x))} \|\vartheta\|_{L^2(0, \infty; H_{per}^2(\Omega_x))},
\end{aligned} \tag{6.14}$$

$$\begin{aligned}
& \left\| e^{-\omega t} \frac{\zeta_m \vartheta_{xx}}{\frac{\partial Y}{\partial x}} \right\|_{L^2(0, \infty; L^2(\Omega_x))} \\
& \leq 2s_0 \|\zeta_m\|_{L^\infty(0, \infty; \dot{H}_{per}^1(\Omega_x))} \|\vartheta\|_{L^2(0, \infty; H_{per}^2(\Omega_x))},
\end{aligned} \tag{6.15}$$

$$\left\| e^{-\omega t} \frac{\zeta_\Omega \vartheta_{xx}}{\frac{\partial Y}{\partial x}} \right\|_{L^2(0, \infty; L^2(\Omega_x))} \leq 2\|\zeta_\Omega\|_{L^\infty(0, \infty; e^{-\omega(\cdot)})} \|\vartheta\|_{L^2(0, \infty; H_{per}^2(\Omega_x))}, \tag{6.16}$$

and

$$\begin{aligned}
& \left\| e^{-\omega t} (\zeta_m + \zeta_\Omega) \vartheta_x \frac{\frac{\partial^2 Y}{\partial x^2}}{\left(\frac{\partial Y}{\partial x}\right)^2} \right\|_{L^2(0, \infty; L^2(\Omega_x))} \\
& \leq 4 \left(s_0 \|\zeta_m\|_{L^\infty(0, \infty; \dot{H}_{per}^1(\Omega_x))} + \|\zeta_\Omega\|_{L^\infty(0, \infty; e^{-\omega(\cdot)})} \right) \left\| \vartheta_x \frac{\partial^2 Y}{\partial x^2} \right\|_{L^2(0, \infty; L^2(\Omega_x))} \\
& \leq \frac{\sqrt{2}s_0\rho_s}{\sqrt{\omega}} \|\vartheta\|_{L^2(0, \infty; H_{per}^2(\Omega_x))}^2.
\end{aligned} \tag{6.17}$$

Using (6.12), (6.13), (6.14), (6.15), (6.16) and (6.17), we can choose a constant $C_2 = C_2(\omega, s_0, L, \rho_s, u_s)$ such that (4.6) holds.

To prove the Lipschitz estimate in (4.9), we have to estimate $\frac{\partial F_1}{\partial x}(\zeta^1, \vartheta^1, t) - \frac{\partial F_1}{\partial x}(\zeta^2, \vartheta^2, t)$ where $Y^j(x, t) = x + \int_0^t e^{-\omega s} \vartheta^j(x, s) ds$, $j = 1, 2$. For that, we have to consider different terms. We are going to estimate

$$e^{-\omega t} \frac{\zeta_m^1 \vartheta_{xx}^1}{\frac{\partial Y^1}{\partial x}} - e^{-\omega t} \frac{\zeta_m^2 \vartheta_{xx}^2}{\frac{\partial Y^2}{\partial x}} \quad \text{and} \quad \rho_s \vartheta_x^1 \frac{\frac{\partial^2 Y^1}{\partial x^2}}{\left(\frac{\partial Y^1}{\partial x}\right)^2} - \rho_s \vartheta_x^2 \frac{\frac{\partial^2 Y^2}{\partial x^2}}{\left(\frac{\partial Y^2}{\partial x}\right)^2}. \tag{6.18}$$

The other terms could be estimated similarly. We write

$$\begin{aligned} & e^{-\omega t} \frac{\zeta_m^1 \vartheta_{xx}^1}{\frac{\partial Y^1}{\partial x}} - e^{-\omega t} \frac{\zeta_m^2 \vartheta_{xx}^2}{\frac{\partial Y^2}{\partial x}} \\ &= e^{-\omega t} \frac{\vartheta_{xx}^1}{\frac{\partial Y^1}{\partial x}} (\zeta_m^1 - \zeta_m^2) + e^{-\omega t} \frac{\zeta_m^2}{\frac{\partial Y^1}{\partial x}} (\vartheta_{xx}^1 - \vartheta_{xx}^2) + e^{-\omega t} \zeta_m^2 \vartheta_{xx}^2 \left(\frac{1}{\frac{\partial Y^1}{\partial x}} - \frac{1}{\frac{\partial Y^2}{\partial x}} \right). \end{aligned}$$

For the first term we have

$$\begin{aligned} & \left\| e^{-\omega t} \frac{\vartheta_{xx}^1}{\frac{\partial Y^1}{\partial x}} (\zeta_m^1 - \zeta_m^2) \right\|_{L^2(0, \infty; L^2(\Omega_x))} \\ & \leq 2s_0 \|\zeta_m^1 - \zeta_m^2\|_{L^\infty(0, \infty; \dot{H}_{per}^1(\Omega_x))} \|\vartheta^1\|_{L^2(0, \infty; H_{per}^2(\Omega_x))}. \end{aligned}$$

The second term can be estimated as

$$\begin{aligned} & \left\| e^{-\omega t} \frac{\zeta_m^2}{\frac{\partial Y^1}{\partial x}} (\vartheta_{xx}^1 - \vartheta_{xx}^2) \right\|_{L^2(0, \infty; L^2(\Omega_x))} \\ & \leq 2s_0 \|\zeta_m^2\|_{L^\infty(0, \infty; \dot{H}_{per}^1(\Omega_x))} \|\vartheta^1 - \vartheta^2\|_{L^2(0, \infty; H_{per}^2(\Omega_x))}. \end{aligned}$$

For the third term, we have

$$\begin{aligned} & \left\| e^{-\omega t} \zeta_m^2 \vartheta_{xx}^2 \left(\frac{1}{\frac{\partial Y^1}{\partial x}} - \frac{1}{\frac{\partial Y^2}{\partial x}} \right) \right\|_{L^2(0, \infty; L^2(\Omega_x))} \\ & \leq 4 \left\| e^{-\omega t} \zeta_m^2 \vartheta_{xx}^2 \left(\frac{\partial Y^2}{\partial x} - \frac{\partial Y^1}{\partial x} \right) \right\|_{L^2(0, \infty; L^2(\Omega_x))} \\ & \leq \frac{\rho_s s_0}{\sqrt{\omega}} \|\vartheta^2\|_{L^2(0, \infty; H_{per}^2(\Omega_x))} \|\vartheta^1 - \vartheta^2\|_{L^2(0, \infty; H_{per}^2(\Omega_x))}. \end{aligned}$$

Hence we have

$$\begin{aligned} & \left\| e^{-\omega t} \frac{\zeta_m^1 \vartheta_{xx}^1}{\frac{\partial Y^1}{\partial x}} - e^{-\omega t} \frac{\zeta_m^2 \vartheta_{xx}^2}{\frac{\partial Y^2}{\partial x}} \right\|_{L^2(0, \infty; L^2(\Omega_x))} \\ & \leq \max \left\{ 2s_0, \frac{\rho_s s_0}{\sqrt{\omega}} \right\} \left(\|(\zeta^1, \vartheta^1)\|_D + \|(\zeta^2, \vartheta^2)\|_D \right) \times \\ & \quad \left(\|\zeta_m^1 - \zeta_m^2\|_{L^\infty(0, \infty; \dot{H}_{per}^1(\Omega_x))} + \|\vartheta^1 - \vartheta^2\|_{L^2(0, \infty; H_{per}^2(\Omega_x))} \right). \end{aligned}$$

This completes the estimate of the first expression in (6.18). For the second one, we write

$$\rho_s \vartheta_x^1 \frac{\frac{\partial^2 Y^1}{\partial x^2}}{\left(\frac{\partial Y^1}{\partial x}\right)^2} - \rho_s \vartheta_x^2 \frac{\frac{\partial^2 Y^2}{\partial x^2}}{\left(\frac{\partial Y^2}{\partial x}\right)^2}$$

$$\begin{aligned}
&= \rho_s \frac{\frac{\partial^2 Y^1}{\partial x^2}}{\left(\frac{\partial Y^1}{\partial x}\right)^2} (\vartheta_x^1 - \vartheta_x^2) + \rho_s \frac{\vartheta_x^2}{\left(\frac{\partial Y^1}{\partial x}\right)^2} \left(\frac{\partial^2 Y^1}{\partial x^2} - \frac{\partial^2 Y^2}{\partial x^2} \right) \\
&\quad + \rho_s \vartheta_x^2 \frac{\partial^2 Y^2}{\partial x^2} \left(\frac{1}{\left(\frac{\partial Y^1}{\partial x}\right)^2} - \frac{1}{\left(\frac{\partial Y^2}{\partial x}\right)^2} \right).
\end{aligned}$$

We can estimate the first term as follows

$$\begin{aligned}
&\left\| \rho_s \frac{\frac{\partial^2 Y^1}{\partial x^2}}{\left(\frac{\partial Y^1}{\partial x}\right)^2} (\vartheta_x^1 - \vartheta_x^2) \right\|_{L^2(0, \infty; L^2(\Omega_x))} \\
&\leq \frac{4\rho_s s_0}{\sqrt{2\omega}} \|\vartheta^1\|_{L^2(0, \infty; H_{per}^2(\Omega_x))} \|\vartheta^1 - \vartheta^2\|_{L^2(0, \infty; H_{per}^2(\Omega_x))}.
\end{aligned}$$

The second term can be estimated as

$$\begin{aligned}
&\left\| \rho_s \frac{\vartheta_x^2}{\left(\frac{\partial Y^1}{\partial x}\right)^2} \left(\frac{\partial^2 Y^1}{\partial x^2} - \frac{\partial^2 Y^2}{\partial x^2} \right) \right\|_{L^2(0, \infty; L^2(\Omega_x))} \\
&\leq \frac{4\rho_s s_0}{\sqrt{2\omega}} \|\vartheta^2\|_{L^2(0, \infty; H_{per}^2(\Omega_x))} \|\vartheta^1 - \vartheta^2\|_{L^2(0, \infty; H_{per}^2(\Omega_x))}.
\end{aligned}$$

For the last term, the estimate is

$$\begin{aligned}
&\left\| \rho_s \vartheta_x^2 \frac{\partial^2 Y^2}{\partial x^2} \left(\frac{1}{\left(\frac{\partial Y^1}{\partial x}\right)^2} - \frac{1}{\left(\frac{\partial Y^2}{\partial x}\right)^2} \right) \right\|_{L^2(0, \infty; L^2(\Omega_x))} \\
&\leq 16\rho_s \left\| \vartheta_x^2 \frac{\partial^2 Y^2}{\partial x^2} \left(\left(\frac{\partial Y^1}{\partial x}\right)^2 - \left(\frac{\partial Y^2}{\partial x}\right)^2 \right) \right\|_{L^2(0, \infty; L^2(\Omega_x))} \\
&\leq 36\rho_s \left\| \vartheta_x^2 \frac{\partial^2 Y^2}{\partial x^2} \left(\frac{\partial Y^1}{\partial x} - \frac{\partial Y^2}{\partial x} \right) \right\|_{L^2(0, \infty; L^2(\Omega_x))} \\
&\leq \frac{36\rho_s s_0^2}{\omega\sqrt{2}} \|\vartheta^2\|_{L^2(0, \infty; H_{per}^2(\Omega_x))}^2 \|\vartheta^1 - \vartheta^2\|_{L^2(0, \infty; H_{per}^2(\Omega_x))} \\
&\leq \frac{18\rho_s s_0}{\sqrt{2\omega}} \|\vartheta^2\|_{L^2(0, \infty; H_{per}^2(\Omega_x))} \|\vartheta^1 - \vartheta^2\|_{L^2(0, \infty; H_{per}^2(\Omega_x))}.
\end{aligned}$$

In the last inequality, we have used that $\|\vartheta^2\|_{L^2(0, \infty; H_{per}^2(\Omega_x))} \leq \frac{\sqrt{\omega}}{2s_0}$. Hence we have

$$\begin{aligned}
&\left\| \rho_s \vartheta_x^1 \frac{\frac{\partial^2 Y^1}{\partial x^2}}{\left(\frac{\partial Y^1}{\partial x}\right)^2} - \rho_s \vartheta_x^2 \frac{\frac{\partial^2 Y^2}{\partial x^2}}{\left(\frac{\partial Y^2}{\partial x}\right)^2} \right\|_{L^2(0, \infty; L^2(\Omega_x))} \\
&\leq \frac{22\rho_s s_0}{\sqrt{2\omega}} \left(\|(\zeta^1, \vartheta^1)\|_D + \|(\zeta^2, \vartheta^2)\|_D \right) \|\vartheta^1 - \vartheta^2\|_{L^2(0, \infty; H_{per}^2(\Omega_x))}.
\end{aligned}$$

The other terms can be estimated similarly. Therefore, there exists a positive constant $C_2 = C_2(\omega, s_0, L, \rho_s, u_s)$ such that

$$\begin{aligned} & \|F_{1,m}(\zeta^1, \vartheta^1, \cdot) - F_{1,m}(\zeta^2, \vartheta^2, \cdot)\|_{L^2(0,\infty; \dot{H}_{per}^1(\Omega_x))} \\ & \leq C_2 \left(\|(\zeta^1, \vartheta^1)\|_D + \|(\zeta^2, \vartheta^2)\|_D \right) \|(\zeta^1, \vartheta^1) - (\zeta^2, \vartheta^2)\|_D. \end{aligned}$$

□

Proof of (4.7) and (4.10). From Section 4.1, we have

$$F_{1,\Omega}(\zeta, \vartheta, t) = \int_{\Omega_x} \rho_s \vartheta_x \left(1 - \left(\frac{\partial Y}{\partial x} \right)^{-1} \right) dx - \int_{\Omega_x} e^{-\omega t} (\tilde{\zeta}_m + \tilde{\zeta}_\Omega) \vartheta_x \left(\frac{\partial Y}{\partial x} \right)^{-1} dx.$$

Using Lemma 4.1, to estimate the first term we write

$$\begin{aligned} & \left\| \int_{\Omega_x} \rho_s \vartheta_x \left(1 - \left(\frac{\partial Y}{\partial x} \right)^{-1} \right) dx \right\|_{L^1(0,\infty; e^{-\omega(\cdot)})} \\ & \leq 2\rho_s \int_0^\infty e^{-\omega t} \left| \int_{\Omega_x} \vartheta_x \left(\frac{\partial Y}{\partial x} - 1 \right) dx \right| dt \\ & \leq 2\rho_s \int_0^\infty e^{-\omega t} \|\vartheta_x(\cdot, t)\|_{L^2(\Omega_x)} \left\| \frac{\partial Y}{\partial x}(\cdot, t) - 1 \right\|_{L^2(\Omega_x)} dt \\ & \leq \frac{2\rho_s}{\sqrt{2\omega}} \|\vartheta\|_{L^2(0,\infty; H_{per}^2(\Omega_x))} \int_0^\infty e^{-\omega t} \|\vartheta_x(\cdot, t)\|_{L^2(\Omega_x)} dt \\ & \leq \frac{\rho_s}{\omega} \|\vartheta\|_{L^2(0,\infty; H_{per}^2(\Omega_x))}^2. \end{aligned}$$

The second term can be estimated as follows

$$\begin{aligned}
& \left\| \int_{\Omega_x} e^{-\omega t} (\tilde{\zeta}_m + \tilde{\zeta}_\Omega) \vartheta_x \left(\frac{\partial Y}{\partial x} \right)^{-1} dx \right\|_{L^1(0, \infty; e^{-\omega(\cdot)})} \\
&= \int_0^\infty e^{-\omega t} \left| \int_{\Omega_x} e^{-\omega t} (\tilde{\zeta}_m + \tilde{\zeta}_\Omega) \vartheta_x \left(\frac{\partial Y}{\partial x} \right)^{-1} dx \right| dt \\
&\leq 2s_0 \|\zeta_m\|_{L^\infty(0, \infty; \dot{H}_{per}^1(\Omega_x))} \int_0^\infty e^{-\omega t} \int_{\Omega_x} |\vartheta_x| dx dt \\
&\quad + 2\|\zeta_\Omega\|_{L^\infty(0, \infty; e^{-\omega(\cdot)})} \int_0^\infty e^{-\omega t} \int_{\Omega_x} |\vartheta_x| dx dt \\
&\leq 2\sqrt{L}s_0 \|\zeta_m\|_{L^\infty(0, \infty; \dot{H}_{per}^1(\Omega_x))} \int_0^\infty e^{-\omega t} \|\vartheta_x(\cdot, t)\|_{L^2(\Omega_x)} dt \\
&\quad + 2\sqrt{L}\|\zeta_\Omega\|_{L^\infty(0, \infty; e^{-\omega(\cdot)})} \int_0^\infty e^{-\omega t} \|\vartheta_x(\cdot, t)\|_{L^2(\Omega_x)} dt \\
&\leq \frac{2\sqrt{L}s_0}{\sqrt{2\omega}} \|\zeta_m\|_{L^\infty(0, \infty; \dot{H}_{per}^1(\Omega_x))} \|\vartheta_x\|_{L^2(0, \infty; L^2(\Omega_x))} \\
&\quad + \frac{2\sqrt{L}}{\sqrt{2\omega}} \|\zeta_\Omega\|_{L^\infty(0, \infty; e^{-\omega(\cdot)})} \|\vartheta_x\|_{L^2(0, \infty; L^2(\Omega_x))} \\
&\leq (1 + s_0) \frac{\sqrt{L}}{\sqrt{2\omega}} \left(\|\zeta_m\|_{L^\infty(0, \infty; \dot{H}_{per}^1(\Omega_x))}^2 + \|\zeta_\Omega\|_{L^\infty(0, \infty; e^{-\omega(\cdot)})}^2 \right. \\
&\quad \left. + \|\vartheta\|_{L^2(0, \infty; H_{per}^2(\Omega_x))}^2 \right).
\end{aligned}$$

Therefore, there exists a positive constant $C_2 = C_2(\rho_s, \omega, s_0, L)$ such that

$$\|F_{1, \Omega}(\zeta, \vartheta, \cdot)\|_{L^1(0, \infty; e^{-\omega(\cdot)})} \leq C_2 \|(\zeta, \vartheta)\|_D^2.$$

The Lipschitz estimate (4.10) can be proved in a similar way. \square

Proof of (4.8) and (4.11). Let us recall that

$$\begin{aligned}
F_2(\zeta, \vartheta, t) &= (\tilde{\sigma}_m)_x \left(b - a\gamma(\rho_s + e^{-\omega t}\zeta)^{\gamma-2} \frac{1}{\frac{\partial Y}{\partial x}} \right) + \frac{\nu}{\rho_s + e^{-\omega t}\zeta} \frac{-\frac{\partial^2 Y}{\partial x^2}}{\left(\frac{\partial Y}{\partial x}\right)^3} \vartheta_x \\
&\quad - \nu \vartheta_{xx} \left(\frac{1}{\rho_s} - \frac{1}{\rho_s + e^{-\omega t}\zeta} \frac{1}{\left(\frac{\partial Y}{\partial x}\right)^2} \right).
\end{aligned}$$

To estimate $F_2(\zeta, \vartheta, t)$ we need a lower bound for $\rho_s + e^{-\omega t}\zeta$. Due to Lemma 4.1, we notice that

$$|\rho_s + e^{-\omega t}\zeta(x, t)| \geq \rho_s - |e^{-\omega t}\zeta(x, t)| \geq \frac{\rho_s}{2}, \quad \forall (x, t) \in Q_x^\infty. \quad (6.19)$$

By the fundamental theorem of calculus, we have

$$(\rho_s + e^{-\omega t} \zeta(x, t))^{\gamma-2} - \rho_s^{\gamma-2} = (\gamma-2)e^{-\omega t} \zeta(x, t) \int_0^1 (\rho_s + \tau e^{-\omega t} \zeta(x, t))^{\gamma-3} d\tau,$$

for all $(x, t) \in Q_x^\infty$. Therefore, it yields

$$|(\rho_s + e^{-\omega t} \zeta(x, t))^{\gamma-2} - \rho_s^{\gamma-2}| \leq C_3 e^{-\omega t} |\zeta(x, t)|, \quad \forall (x, t) \in Q_x^\infty, \quad (6.20)$$

for some positive constant C_3 . Using Lemma 4.1, (6.20), (2.15), the first term can be estimated as follows

$$\begin{aligned} & \left\| a\gamma(\zeta_m)_x \left(\rho_s^{\gamma-2} - (\rho_s + e^{-\omega t} \zeta)^{\gamma-2} \left(\frac{\partial Y}{\partial x} \right)^{-1} \right) \right\|_{L^2(0, \infty; L^2(\Omega_x))} \\ & \leq C_3 \left(\|(\zeta_m)_x \left(\frac{\partial Y}{\partial x} \right)^{-1}\|_{L^2(0, \infty; L^2(\Omega_x))} + \|(\zeta_m)_x e^{-\omega t} (\zeta_m + \zeta_\Omega)\|_{L^2(0, \infty; L^2(\Omega_x))} \right) \\ & \leq C_3 \|(\zeta_m)_x\|_{L^2(0, \infty; L^2(\Omega_x))} \left(\left\| \frac{\partial Y}{\partial x} - 1 \right\|_{L^\infty(Q_x^\infty)} + \|\zeta_\Omega\|_{L^\infty(0, \infty; e^{-\omega(\cdot)})} \right. \\ & \quad \left. + \|(\zeta_m)\|_{L^\infty(0, \infty; \dot{H}_{per}^1(\Omega_x))} \right) \\ & \leq C_3 \left(\|\vartheta\|_{L^2(H_{per}^2(\Omega_x))}^2 + \|\zeta_m\|_{(L^2 \cap L^\infty)(\dot{H}_{per}^1(\Omega_x))}^2 + \|\zeta_\Omega\|_{L^\infty(0, \infty; e^{-\omega(\cdot)})}^2 \right). \end{aligned}$$

For the second term we have the following estimate

$$\begin{aligned} & \left\| \frac{\nu}{\rho_s + e^{-\omega t} \zeta} \frac{-\frac{\partial^2 Y}{\partial x^2}}{\left(\frac{\partial Y}{\partial x} \right)^3} \vartheta_x \right\|_{L^2(0, \infty; L^2(\Omega))} \leq \frac{16\nu}{\rho_s} \left\| \frac{\partial^2 Y}{\partial x^2} \vartheta_x \right\|_{L^2(0, \infty; L^2(\Omega_x))} \\ & \leq \frac{16\nu s_0}{\rho_s \sqrt{2\omega}} \|\vartheta\|_{L^2(0, \infty; H_{per}^2(\Omega_x))}^2. \end{aligned}$$

To estimate the third term first note that

$$\begin{aligned} & \frac{1}{\rho_s} - \frac{1}{\rho_s + e^{-\omega t} \zeta} \frac{1}{\left(\frac{\partial Y}{\partial x} \right)^2} \\ & = \frac{1}{\rho_s} - \frac{1}{\rho_s + e^{-\omega t} \zeta} + \frac{1}{\rho_s + e^{-\omega t} \zeta} - \frac{1}{\rho_s + e^{-\omega t} \zeta} \frac{1}{\frac{\partial Y}{\partial x}} + \frac{1}{\rho_s + e^{-\omega t} \zeta} \frac{1}{\frac{\partial Y}{\partial x}} - \frac{1}{\rho_s + e^{-\omega t} \zeta} \frac{1}{\left(\frac{\partial Y}{\partial x} \right)^2} \\ & = \frac{e^{-\omega t} \zeta}{\rho_s(\rho_s + e^{-\omega t} \zeta)} + \frac{1}{\rho_s + e^{-\omega t} \zeta} \left(\frac{\partial Y}{\partial x} - 1 \right) \frac{1}{\frac{\partial Y}{\partial x}} + \frac{1}{\rho_s + e^{-\omega t} \zeta} \left(\frac{\partial Y}{\partial x} - 1 \right) \frac{1}{\left(\frac{\partial Y}{\partial x} \right)^2}. \end{aligned}$$

Using Lemma 4.1 and (2.15) we have the following estimate for third term

$$\begin{aligned}
& \left\| \nu \vartheta_{xx} \left(\frac{1}{\rho_s} - \frac{1}{\rho_s + e^{-\omega t} \zeta} \frac{1}{\left(\frac{\partial Y}{\partial x}\right)^2} \right) \right\|_{L^2(0, \infty; L^2(\Omega_x))} \\
& \leq \frac{2\nu}{\rho_s^2} \|e^{-\omega t} (\tilde{\sigma}_m + \tilde{\sigma}_\Omega) \vartheta_{xx}\|_{L^2(0, \infty; L^2(\Omega_x))} + \frac{4\nu}{\rho_s} \left\| \vartheta_{xx} \left(\frac{\partial Y}{\partial x} - 1 \right) \right\|_{L^2(0, \infty; L^2(\Omega_x))} \\
& \quad + \frac{8\nu}{\rho_s} \left\| \vartheta_{xx} \left(\frac{\partial Y}{\partial x} - 1 \right) \right\|_{L^2(0, \infty; L^2(\Omega_x))} \\
& \leq C_3 \left(\|\tilde{\sigma}_m\|_{L^\infty(0, \infty; \dot{H}_{per}^1(\Omega_x))} + \|\tilde{\sigma}_\Omega\|_{L^\infty(0, \infty; e^{-\omega(\cdot)})} \right) \|\vartheta\|_{L^2(0, \infty; H_{per}^2(\Omega_x))} \\
& \quad + \frac{12\nu s_0}{\rho_s \sqrt{\omega}} \|\vartheta\|_{L^2(0, \infty; H_{per}^2(\Omega_x))}^2 \\
& \leq C_3 \left(\|\tilde{\sigma}_m\|_{L^\infty(0, \infty; \dot{H}_{per}^1(\Omega_x))}^2 + \|\tilde{\sigma}_\Omega\|_{L^\infty(0, \infty; e^{-\omega(\cdot)})}^2 + \|\vartheta\|_{L^2(0, \infty; H_{per}^2(\Omega_x))}^2 \right).
\end{aligned}$$

Therefore, there exists a positive constant $C_2 = C_2(s_0, \rho_s, u_s, L, \omega, \nu)$ such that

$$\|F_2(\zeta, \vartheta, \cdot)\|_{L^2(0, \infty; L^2(\Omega_x))} \leq C_2 \|(\zeta, \vartheta)\|_D^2.$$

The Lipschitz estimate (4.11) can be proved in a similar way. \square

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STABILIZATION OF COMPRESSIBLE NAVIER-STOKES EQUATIONS