Asymptotic behavior of a solutions of ordinary differential equations defined by a trigonometric polynomial field

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November 15, 2021

Abstract

We prove in this paper that any solution x, of an ordinary differential equations defined by a trigonometric polynomial field, admits a *rotation vector* $\rho \in \mathbb{R}^n$. More precisely, the function $t \mapsto x(t) - \rho t$ is uniformly bounded on time and it is a *weak almost periodic* function of *slope* ρ.

Keywords: Periodic system, differential equation, rotation vector, rotation number, almost-periodic function, trigonometric polynomial.

1 Introduction

In this paper we study the asymptotic behavior of a solutions of the ordinary differential equations defined by a trigonometric polynomial field. The idea comes from the scalar case; H. Poincaré defined the *rotation number* for circle homeomorphisms [7]. The simple example is a scalar differential equation

$$\dot{x} = f(x), \quad x(0) \in \mathbb{R}, \quad t \in \mathbb{R},$$

where $f : \mathbb{R} \to \mathbb{R}$ is lipschitz and 1-periodic. There exists a rotation number $\lambda \in \mathbb{R}$ such that the function $t \mapsto x(t) - \lambda t$ is periodic. For two dimension [5] showed that the solution x, of the scalar differential equation

$$\dot{x} = f(t, x), \quad x(0) \in \mathbb{R}, \quad t \in \mathbb{R},$$

where $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is lipschitz and 1-periodic relative to each variable, admits a rotation number. We know that any non-autonomous system can be written as an autonomous system. Our result is a generalization of this asymptotic behavior to any dimension. In this case λ is a vector and it is called a *rotation vector* or *rotation set* as defined in [6]. Under some assumptions of stability [[8], [3]] prove the existence of the rotation vector. We mention relative results about the rotation vector in [2]. Some biological works use the ordinary differential equations defined by a trigonometric polynomial field as in [[1], [4]].

2 Definition and Main result

We study in this article the following system

$$\dot{x} = f(x), \quad t \in \mathbb{R}, \quad x \in \mathbb{R}^n, \quad x(0) = x_0,$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ is a trigonometric polynomial in the following sense

Definition 1. [Trigonometric polynomial function] A function $g : \mathbb{R}^n \to \mathbb{R}$ is called *a trigonometric polynomial* if there exists a finite sequence $(c_p)_{p \in 2\pi\mathbb{Z}^n} \subset \mathbb{C}$ such that

$$\forall x \in \mathbb{R}^n : \quad g(x) = \sum_{p \in 2\pi\mathbb{Z}^n} c_p \exp(i\langle x, p\rangle),$$

where $\langle ., . \rangle$ is the usual scalar product on $\mathbb{R}^n$. A function $g : \mathbb{R}^n \to \mathbb{R}^n$ is *a trigonometric polynomial* if each component is a trigonometric polynomial function.

To formulate the Mains results let us introduce the following definitions. We use the usual following norm $\|y\| := \max_{1 \le j \le n} \|y_j\|$ for every $y := (y_j)_{j=1}^n \in \mathbb{C}^n$.

Definition 2. [Rotation vector] Let $\lambda \in \mathbb{R}^n$ and $\phi : \mathbb{R} \to \mathbb{R}^n$ be a function. We say that ϕ admits λ as the *rotation vector* if

$$\sup_{t \in \mathbb{R}} \|\phi(t) - \lambda t\| < \infty.$$

Definition 3. [Periodic modulo $\mathbb{Z}^n$ function] A function $g : \mathbb{R}^n \to \mathbb{R}^n$ is called *periodic modulo* $\mathbb{Z}^n$, if

$$g(z_1 + k_1, \ldots, z_n + k_n) = g(z_1 \ldots, z_n), \quad \forall (k_j)_{j=1}^n \in \mathbb{Z}^n, \ \forall (z_j)_{j=1}^n \in \mathbb{R}^n.$$

Definition 4. [Weakly almost-periodic function] Let be $r \in \mathbb{R}^n$. A function $h : \mathbb{R} \to \mathbb{R}^n$ is *weakly almost periodic of slope* r if it is C^∞ and if there exists a uniformly bounded sequence for the sup-norm of C^∞ functions $(g_k : \mathbb{R}^n \to \mathbb{R}^n)_{k\in\mathbb{N}}$ that are periodic modulo $\mathbb{Z}^n$ and there exists a sequence $(r_k)_{k\in\mathbb{N}} \subset \mathbb{Q}^n$ such that

$$\lim_{k\to\infty} r_k = r \quad \text{and} \quad \forall t > 0 : \lim_{k\to\infty} \sup_{s\in[-t,t]} \|g_k(r_k s) - h(s)\| = 0.$$

Main Result. *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a trigonometric polynomial function. For every $x_0 \in \mathbb{R}^n$ the unique solution $x : \mathbb{R} \to \mathbb{R}^n$ of*

$$\dot{x} = f(x), \quad t \in \mathbb{R}, \quad x(0) = x_0,$$

admits a rotation vector $\rho \in \mathbb{R}^n$. In addition, the function

$$t \mapsto x(t) - \rho t,$$

is weakly almost periodic of slope ρ.

3 Space of C^∞ periodic modulo $\mathbb{Z}^n$ functions

We define in this Section the space and the norm used to prove the Main result. In order to use the Fourier development, let us introduce the following notations

Notation 5. For every continuous function $g : \mathbb{R}^n \to \mathbb{R}^n$ and every $p \in 2\pi\mathbb{Z}^n$ we denote $a_p[g] \in \mathbb{C}^n$ the following limit if it exists

$$a_p[g] := \lim_{t\to+\infty} \frac{1}{(2t)^n} \int_{-t}^{t} \dots \int_{-t}^{t} g(z) \exp(-i\langle z, p\rangle) dz_1 \dots dz_n.$$

The following constant ω will be used only as change of variable in order to find a contraction in Lemmas 13 and 14. For every $\omega \in \mathbb{N}^*$, we denote $E_\omega(\mathbb{R}^n)$ the set of C^∞ function $g : \mathbb{R}^n \to \mathbb{R}^n$ such that $z \mapsto g(\omega z)$ is a periodic modulo $\mathbb{Z}^n$ function. We remark that for very $\omega \in \mathbb{N}^*$ and $g \in E_\omega(\mathbb{R}^n)$ we have

$$a_p[g_\omega] = \int_0^1 \dots \int_0^1 g_\omega(z) \exp(-i\langle z, p\rangle) dz_1 \dots dz_n, \quad g_\omega(z) = g(\omega z), \qquad (1)$$

which is the Fourier coefficient of the function g_ω. Since $g \in E_\omega(\mathbb{R}^n)$ is C^∞, by Dirichlet Theorem, we get

$$g(z) = \sum_{p \in 2\pi\mathbb{Z}^n} a_p[g_\omega] \exp(i\frac{1}{\omega}\langle z, p\rangle), \tag{2}$$
$$\forall q \geq 0 : \quad \sum_{p \in 2\pi\mathbb{Z}^n/\mathbb{0}} \|a_p[g_\omega]\| \|\frac{p}{\omega}\|^q < +\infty.$$

We are now in position to define the following seminorm in $E_\omega(\mathbb{R}^n)$: Let $\omega \in \mathbb{N}^*$ and $g \in E_\omega(\mathbb{R}^n)$, we denote for every $\omega \in \mathbb{N}^*$ and $q \geq 0$

$$\|g\|_{\omega,q} := 2 \sum_{p \in 2\pi\mathbb{Z}^n/\mathbb{0}} \|a_p[g_\omega]\| \|\frac{p}{\omega}\|^q,$$

where we recall that $\|y\| := \max_{1\leq j\leq n}\|y_j\|$ for every $y := (y_j)_{j=1}^n \in \mathbb{C}^n$. We prove in the following Lemma that a periodic modulo $\mathbb{Z}^n$ function g is C^∞ if it is *uniformly bounded for the seminorm*, i.e

$$\forall q \geq 0 : \quad \|g\|_{\omega,q} < +\infty,$$

In other words, the set $E_\omega(\mathbb{R}^n)$ coincide with the set of the periodic modulo $\mathbb{Z}^n$ functions uniformly bounded for the seminorm.

Lemma 6. *Let $\omega \in \mathbb{N}^*$. Let $(c_p)_p$ be a complex-valued family such that*

$$\forall q \geq 0 : \quad \sum_{p \in 2\pi\mathbb{Z}^n/\mathbb{0}} \|c_p\| \|\frac{p}{\omega}\|^q < +\infty.$$

Then the following series is normally convergent

$$g(z) := \sum_{p \in 2\pi\mathbb{Z}^n} c_p \exp(i\frac{1}{\omega}\langle z, p\rangle), \quad c_p \in \mathbb{C}^n,$$

and $c_p = a_p[g_\omega]$ for every $p \in 2\pi\mathbb{Z}^n$. Further, $g \in E_\omega(\mathbb{R}^n)$.

Proof. Appendix. A □

We denote $\mathbb{0} := (0, \ldots, 0) \in \mathbb{R}^n$. In the following Lemma we prove that the seminorm $\|.\|_{\omega,0}$ is a norm in the space $\{g \in E_\omega(\mathbb{R}^n) : \ g(\mathbb{0}) = \mathbb{0}\}$ and we compare it to the uniform norm topology.

Lemma 7. *Let be $\omega \in \mathbb{N}^*$ and $g \in E_\omega(\mathbb{R}^n)$ such that $g(\mathbb{0}) = \mathbb{0}$ then*

$$\|a_{\mathbb{0}}[g_\omega]\| \leq \frac{1}{2}\|g\|_{\omega,0} \quad \textit{and} \quad \|g\|_\infty \leq \|g\|_{\omega,0}.$$

Proof. Appendix. B □

Finally, the following Lemma gives an upper-bound of the quantity $\|g\|_{\omega,q}$ when g is a trigonometric polynomial.

Lemma 8. *Let $g : \mathbb{R}^n \to \mathbb{R}^n$ be a trigonometric polynomial function. Then there exists $\beta := \beta(g) > 0$ such that for every $\omega \in \mathbb{N}^*$ we have*

$$\|d^k g_\omega\|_{\omega,q} < n^k \beta (\frac{\beta}{\omega})^{q+k}, \quad \forall q, k \geq 0,$$

where $g_\omega(z) := g(\frac{z}{\omega})$.

Proof. Appendix. C □

4 Main proposition

Notation 9. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a trigonometric polynomial. We denote the finite subset $\Lambda_f \subset 2\pi\mathbb{Z}^n$ as

$$\Lambda_f := \{p \in 2\pi\mathbb{Z}^n : \quad \|a_p[f]\| \neq 0\},$$

and we denote

$$|\Lambda_f| := \max\{\|p\|, \quad p \in \Lambda_f\}.$$

Let be $y \in \mathbb{R}^n/\{\mathbb{0}\}$. Define,

$$\Lambda(f, y) := \{p \in 2\pi\mathbb{Z}^n : \ \|p\| \leq 2\pi + |\Lambda_f|, \ \langle y, p\rangle \neq 0\},$$

Remark that $\Lambda(f, y) \neq \emptyset$. We denote

$$\tau(f, y) := \max\Big\{\frac{1}{|\langle y, p\rangle|} : \quad p \in \Lambda(f, y)\}.$$

Let be $y \in \mathbb{Q}^n/\{\mathbb{0}\}$, we denote

$$\tau(y) = \max\Big\{\frac{1}{|\langle y, p\rangle|} : \quad p \in 2\pi\mathbb{Z}^n, \ \langle y, p\rangle \neq 0\Big\}.$$

We denote β the constant $\beta(f)$ of the function f defined in Lemma 8.

In order to prove the Main result, the aim is to find the periodic modulo $\mathbb{Z}^n$ functions g_k of the Definition 4. Using the Fourier development and

Equation 2, remark that for every C^∞ periodic modulo $\mathbb{Z}^n$ function g we have

$$f(z+g(z)) = \sum_{p\in 2\pi\mathbb{Z}^n} a_p[H[g]]\exp(i\langle z,p\rangle),\ H[g](z) := f(z+g(z)),$$

under some convergence assumption of the series, by integration we get

$$\forall v\in\mathbb{R}^n:\quad \int_0^t f(vs+g(vs))ds = t\sum_{p\in 2\pi\mathbb{Z}^n,\ \langle v,p\rangle=0} a_p[H[g]] + \sum_{p\in 2\pi\mathbb{Z}^n,\ \langle v,p\rangle\neq 0} \frac{a_p[H[g]]]}{\langle v,p\rangle}\Big(\exp(i\langle v,p\rangle t)-1\Big).$$

We need remove the linear part, hence the last term of the right member of the last equality play the role of the periodic modulo $\mathbb{Z}^n$ functions g_k of the Definition 4. In the following Proposition we prove that it is bounded by $2\beta\tau(f,v)$.

Proposition 10. *[Main proposition] Let $f:\mathbb{R}^n\to\mathbb{R}^n$ be a trigonometric polynomial function. Then for every $r\in\mathbb{Q}^n/\{\mathbb{0}\}$ and every $\epsilon>0$ there exists a C^∞ periodic modulo $\mathbb{Z}^n$ function $\phi_{r,\epsilon}:\mathbb{R}^n\to\mathbb{R}^n$ such that $\|\phi_{r,\epsilon}\|_\infty < 2\beta\tau(f,r)$ and such that*

$$\sup_{z\in\mathbb{R}^n}\|\phi_{r,\epsilon}(z) - \sum_{p\in 2\pi\mathbb{Z}^n,\ \langle r,p\rangle\neq 0}\frac{a_p[H[\phi_{r,\epsilon}]]}{\langle r,p\rangle}(\exp(i\langle z,p\rangle)-1)\| < \epsilon,$$

where

$$H[\phi_{r,\epsilon}](z) := f(z+\phi_{r,\epsilon}(z)),\ \forall z\in\mathbb{R}^n.$$

As is state in the above Section, the following constant ω is used as change of variable in order to find a contraction.

Definition 11. For every $r\in\mathbb{R}^n$ and every $\omega\in\mathbb{N}^*$, define the set $K_{r,\omega}$ as $g\in K_{r,\omega}$ if

- there exists a complex-valued family $(c_p)_{p\in 2\pi\mathbb{Z}^n}$ such that

$$g(z) = \sum_{p\in 2\pi\mathbb{Z}^n} c_p(\exp(i\frac{1}{\omega}\langle z,p\rangle)-1),\ \forall z\in\mathbb{R}^n,$$

- $\|g\|_{\omega,0}\leq 2\beta\tau(f,r)$,

- $\|g\|_{\omega,q} < \infty$ for every $q > 1$.

Lemma 12. *The set $K_{r,\omega}$ is a nonempty subset of $E_\omega(\mathbb{R}^n)$.*

Proof. The set $K_{r,\omega} \neq \emptyset$ because it contain the function $z \mapsto g(z) = \mathbb{0}$. Let be $r \in \mathbb{R}^n$ and $\omega \in \mathbb{N}^*$. Let be $g \in K_{r,\omega}$. Since $\|g\|_{\omega,0} < \infty$ then $z \mapsto g(\omega z)$ is a continuous periodic modulo $\mathbb{Z}^n$. By definition of $K_{r,\omega}$ and by Lemma 6 the function g is C^∞. □

For every $r \in \mathbb{R}^n$, for every $\omega \in \mathbb{N}^*$, and every $g \in K_{r,\omega}$ define the function $\Psi[r,\omega,g] : \mathbb{R}^n \to \overline{\mathbb{R}^n}$ as

$$\Psi[r,\omega,g](z) := \sum_{p\in 2\pi\mathbb{Z}^n,\ \langle r,p\rangle\neq 0} \frac{a_p[H[\omega,g]]}{\langle r,p\rangle}(\exp(i\frac{1}{\omega}\langle z,p\rangle) - 1),$$

where

$$H[\omega,g](z) := f\Big(\frac{1}{\omega}z + g(z)\Big).$$

In the following Lemma we prove that $K_{r,\omega}$ is invariant under the operator $\Psi[r,\omega,.]$.

Lemma 13. *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a trigonometric polynomial function. For every $r \in \mathbb{Q}^n/\{\mathbb{0}\}$, there exists $\omega_r > 0$ such that for every $\omega > \omega_r$ we have*

$$g \in K_{r,\omega} \implies \Psi[r,\omega,g] \in K_{r,\omega}.$$

Proof. Prove that

$$\forall g \in K_{r,\omega} : \quad \|\Psi[r,\omega,g]\|_{\omega,q} < \infty, \quad \forall q \geq 0.$$

Let be $g \in K_{r,\omega}$ and denote

$$f_\omega(z) := f(\frac{z}{\omega}), \text{ and } \tilde{H}[\omega,g](z) := f\Big(\frac{1}{\omega}z + g(z)\Big) - f(\frac{z}{\omega}).$$

By definition, we get

$$a_p[H[\omega,g]] = a_p[f_\omega] + a_p[\tilde{H}[\omega,g]].$$

By definition of $\Psi[r,\omega,.]$ for every $g \in K_{r,\omega}$ we have

$$\begin{aligned}\Psi[r,\omega,g](z) = &\sum_{p\in 2\pi\mathbb{Z}^n,\ \langle r,p\rangle\neq 0} \frac{a_p[f_\omega]}{\langle r,p\rangle}(\exp(i\frac{1}{\omega}\langle z,p\rangle) - 1)\\ &+ \sum_{p\in 2\pi\mathbb{Z}^n,\ \langle r,p\rangle\neq 0} \frac{a_p[\tilde{H}[\omega,g]]}{\langle r,p\rangle}(\exp(i\frac{1}{\omega}\langle z,p\rangle) - 1)\end{aligned}$$

Recall that $|\Lambda_f|$ is defined on the Notations 9. For every $p \in 2\pi\mathbb{Z}^n$ such that $\langle r, p\rangle \neq 0$ and such that $\|p\| > |\Lambda_f|$ we have $a_p[f_\omega] = 0$. By definition of $\tau(f, r)$ and $\tau(r)$ in Notation 9, we get

$$\begin{aligned}\|\Psi[r,\omega,g]\|_{\omega,q} \leq 2\tau(f,r) &\sum_{p\in 2\pi\mathbb{Z}^n,\ \langle r,p\rangle\neq 0} \|a_p[f_\omega]\|\|\frac{p}{\omega}\|^q \\ &+ 2\tau(r) \sum_{p\in 2\pi\mathbb{Z}^n,\ \langle r,p\rangle\neq 0} \|a_p[\tilde{H}[\omega,g]]\|\|\frac{p}{\omega}\|^q.\end{aligned}$$

By definition of the norm, we have,

$$\|\Psi[r,\omega,g]\|_{\omega,q} \leq \tau(f,r)\|f_\omega\|_{\omega,q} + \tau(r)\|\tilde{H}[\omega,g]\|_{\omega,q}. \tag{3}$$

By Lemma 8, we have

$$\|f_\omega\|_{\omega,q} \leq \frac{\beta^{q+1}}{\omega^q}. \tag{4}$$

Now, we estimate the quantity $\|\tilde{H}[\omega,g]\|_{\omega,q}$. By definition, we have

$$\tilde{H}[\omega,g](z) = f\Big(\frac{1}{\omega}z + g(z)\Big) - f(\frac{1}{\omega}z) = \sum_{k=1}^{\infty} \frac{d^k f(\frac{z}{\omega})}{k!}(g(z))^{(k)},$$

where

$$d^k f(\frac{z}{\omega})(g(z))^{(k)} := \sum_{i_1,\ldots,i_k=1}^{n} \frac{\partial f}{\partial z_{i_1}\ldots\partial z_{i_k}}(\frac{z}{\omega})g_{i_1}(z)\ldots g_{i_k}(z).$$

For every $k \geq 1$ we have

$$\|a_{\mathbb{0}}[d^k f_\omega]\| = 0,$$

and since $g \in K_{r,\omega}$ then $g(\mathbb{0}) = \mathbb{0}$ by Lemma 7 we get

$$\|a_{\mathbb{0}}[g]\| \leq \|g\|_{\omega,0} \leq \alpha_0 := 2\beta\tau(f,r).$$

By Lemma 15 [See Appendix D], we have for all $q \geq 0$

$$\begin{aligned}\|\tilde{H}[\omega,g]\|_{\omega,q} &\leq \sum_{k=1}^{\infty} \frac{1}{k!}\|\Big[(d^k f_\omega)(g)^{(k)}\Big]\|_{\omega,q} \\ &\leq \sum_{k=1}^{\infty} \frac{(k+1)^q\omega^{kq}}{k!2^k}\|d^k f_\omega\|_{\omega,q}(\|g\|_{\omega,q} + \alpha_0)^k.\end{aligned}$$

By Lemma 8, we get

$$\forall q \geq 0: \quad \|\tilde{H}[\omega, g]\|_{\omega,q} \leq \sum_{k=1}^{\infty} \frac{\exp(qk)\omega^{kq}}{k!2^k} n^k \frac{\beta^{k+q+1}}{\omega^{k+q}} [\alpha_0 + \|g\|_q]^k.$$

By hypothesis $g \in K_{r,\omega}$, then

$$\forall q \geq 0, \exists \alpha_q > 0: \quad \|g\|_{\omega,q} \leq \alpha_q \text{ where } \alpha_0 := 2\beta\tau(f, r).$$

we deduce that for all $q \geq 0$,

$$\begin{aligned}
\|\tilde{H}[\omega, g]\|_{\omega,q} &\leq \sum_{k=1}^{\infty} \frac{\exp(qk)\omega^{kq}}{k!2^k} n^k \frac{\beta^{k+q+1}}{\omega^{k+q}} (\alpha_0 + \alpha_q)^k \\
&= \frac{\beta^{q+1}}{\omega^q} \sum_{k=1}^{\infty} \frac{1}{k!} \Big(n \exp(q)\omega^q \frac{\beta}{2\omega} (\alpha_0 + \alpha_q) \Big)^k \\
&\leq \frac{\beta^{q+1}}{\omega^q} \Big[\exp\Big(n \exp(q)\omega^q \frac{\beta}{2\omega} (\alpha_0 + \alpha_q) \Big) - 1 \Big] < \infty.
\end{aligned}$$

By Equations (4) and Equation (3), we obtain

$$\forall g \in K_{r,\omega}: \quad \|\Psi[r, \omega, g]\|_{\omega,q} < \infty, \quad \forall q \geq 0.$$

Choose $\omega > \omega_r > 0$, where $\omega_r \in \mathbb{N}^*$ satisfies

$$\beta \Big[\exp\Big(n \frac{\beta}{\omega_r} \alpha_0 \Big) - 1 \Big] < \frac{\beta\tau(f, r)}{\tau(r)}.$$

We obtain

$$\forall \omega > \omega_r: \quad \|\tilde{H}[\omega, g]\|_{\omega,0} < \frac{\beta\tau(f, r)}{\tau(r)}. \tag{5}$$

Replace both Equations (4) and (5) on Equation (3), one obtain

$$\forall \omega > \omega_r: \quad \|\Psi[r, \omega, g]\|_{\omega,0} < \beta\tau(f, r) + \beta\tau(f, r) = 2\beta\tau(f, r) = \alpha_0.$$

□

Lemma 14. *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a trigonometric polynomial function. For every $r \in \mathbb{Q}^n/\{\mathbb{0}\}$ there exists $\omega := \omega(r) > 0$ such that for every $\epsilon > 0$, there exists $\phi_{r,\omega,\epsilon} \in K_{r,\omega}$ satisfying*

$$\|\phi_{r,\omega,\epsilon} - \Psi[r, \omega, \phi_{r,\omega,\epsilon}]\|_\infty < \epsilon.$$

Proof. Let be $\omega > \beta$ and $h, g \in K_{r,\omega}$, we have

$$\Psi[r,\omega,h](z) - \Psi[r,\omega,g](z)$$
$$= \sum_{p\in 2\pi\mathbb{Z}^n,\ \langle r,p\rangle\neq 0} \frac{a_p[\phi[\omega,h,g](h-g)]}{\langle r,p\rangle}(\exp(i\frac{1}{\omega}\langle z,p\rangle)-1),$$

where

$$\phi[\omega,h,g](z) := \int_0^1 df\Big(\frac{1}{\omega}z + sh(z) + (1-s)g(z)\Big)ds.$$

The function

$$z \mapsto \phi[\omega,h,g](\omega z),$$

is a C^∞ periodic modulo $\mathbb{Z}^n$ function. As in Proof of Lemma 13, for every $u \geq 0$,

$$\|\Psi[r,\omega,h] - \Psi[r,\omega,g]\|_{\omega,0} \leq \tau(r)\|\phi[h,g](h-g)\|_{\omega,0}.$$

Since $h, g \in K_{r,\omega}$, then

$$\|h-g\|_{\omega,0} < \|h\|_{\omega,0} + \|g\|_{\omega,0} < \infty.$$

Prove that there exists $\omega_r > 0$ such that for every $\omega > \omega_r$ there exists $\epsilon_\omega \in (0,1)$ such that

$$\|\Psi[r,\omega,h] - \Psi[r,\omega,g]\|_{\omega,0} \leq \epsilon_\omega\|h-g\|_{\omega,0}.$$

By Lemma 15 [See Appendix D], we have

$$\|\phi[h,g](h-g)\|_{\omega,0} \leq \|\phi[\omega,h,g]\|_{\omega,0}\|h-g\|_{\omega,0}.$$

Then

$$\|\Psi[r,\omega,h] - \Psi[r,\omega,g]\|_{\omega,0} \leq \tau(r)\|\phi[\omega,h,g]\|_{\omega,0}\|h-g\|_{\omega,0}. \tag{6}$$

Prove that there exists $\omega_r > 0$ such that for every $\omega > \omega_r$ we have

$$\forall g, h \in K_{r,\omega}: \quad 2\tau(r)\|\phi[\omega,h,g]\|_{\omega,0} < 1.$$

As in Proof of Lemma 13, we have

$$\phi[\omega,h,g](z) = \int_0^1 \sum_{k=0}^{\infty} \frac{d^{k+1}f_\omega(z)}{k!}\Big(sh(z)+(1-s)g(z)\Big)^{(k)}ds,$$

By Lemma 15 [See Appendix D] and 7,

$$\|\phi[\omega,h,g]\|_{\omega,0} \leq \int_0^1 \sum_{k=0}^{\infty} \frac{1}{k!2^k} \|d^{k+1} f_\omega\|_{\omega,0} \|V_s\|_{\omega,0}^{2k}.$$

where $f_\omega(z) := f(\frac{1}{\omega}z)$ and where $V_s : \mathbb{R}^n \to \mathbb{R}^n$ is the C^∞ periodic modulo $\mathbb{Z}^n$ defined by

$$V_s(z) := sh(z) + (1-s)g(z), \quad \forall z \in \mathbb{Z}^n, \quad s \in [0,1].$$

By hypothesis $g, h \in K_{r,\omega}$, by consequence

$$\sup_{s\in[0,1]} \|V_s\|_{\omega,0} \leq \|g\|_{\omega,0} + \|h\|_{\omega,0} \leq 4\beta\tau(f,r).$$

By Lemma 15, we have

$$\begin{aligned}\|\phi[\omega,h,g]\|_{\omega,0} &\leq \sum_{k=0}^{\infty} \frac{1}{k!2^k} \frac{\beta^{k+2}}{\omega^{k+1}} (4\beta\tau(f,r))^k = \frac{\beta^2}{\omega} \sum_{k=0}^{\infty} \frac{1}{k!} \Big[\frac{1}{2}\Big(\frac{\beta}{\omega} 4\beta\tau(f,r)\Big)^2\Big]^k \\ &= \frac{\beta^2}{\omega} \exp\Big(\frac{1}{2}\Big(\frac{\beta}{\omega} 4\beta\tau(f,r)\Big)^2\Big).\end{aligned}$$

Choose $\omega > \omega_r > 0$ large such that

$$2\tau(r)\frac{\beta^2}{\omega_r} \exp\Big(\frac{1}{2}\Big(\frac{\beta}{\omega} 4\beta\tau(f,r)\Big)^2\Big) < 1.$$

By Equation (6), we have proved that for every $\omega > \omega_r$ we have

$$\|[\Psi[r,\omega,h] - \Psi[r,\omega,g]\|_{\omega,0} \leq \frac{1}{2}\|h-g\|_{\omega,0}. \tag{7}$$

Now, choose $\omega_r >> 1$ fixed and very large. By Equation (7), for every $\epsilon > 0$ there exists $k_\epsilon \geq 0$ such that

$$\|\Psi[r,\omega_r,\phi_{r,\omega,\epsilon}] - \phi_{r,\omega,\epsilon}\|_{\omega,0} < \epsilon,$$

where

$$\phi_{r,\omega,\epsilon} := \Psi^{k_\epsilon}[r,\omega_r,g].$$

By Lemma 7 we obtain

$$\|\Psi[r,\omega_r,\phi_{r,\omega,\epsilon}] - \phi_{r,\omega,\epsilon}\|_{\infty} < \epsilon,$$

By Lemma 13 we have $\phi_{r,\omega,\epsilon} \in K_{r,\omega}$. □

Proof of Proposition 10. Let be $r \in \mathbb{Q}^n/\{\mathbb{0}\}$. By Lemma 14, there exists $\omega := \omega_r > 0$ such that for every $\epsilon > 0$ there exists $\phi_{r,\omega,\epsilon} \in K_{r,\omega}$ such that

$$\|\Psi[r, \omega, \phi_{r,\omega,\epsilon}] - \phi_{r,\omega,\epsilon}\|_\infty < \epsilon. \tag{8}$$

Define the functions,

$$\tilde{\phi}_{r,\epsilon}(z) := \ \phi_{r,\omega,\epsilon}(\omega z), \text{ and } H[g](z) := f(z + g(z)).$$

We recall that,

$$H[\omega, \phi_{r,\omega,\epsilon}](z) = f\Big(\frac{1}{\omega}z + \phi_{r,\omega,\epsilon}r(z)\Big).$$

By Equation 1,

$$a_p[H[\omega, \phi_{r,\omega,\epsilon}]] = a_p[H[\tilde{\phi}_{r,\epsilon}]],$$

Using the definition of $\Psi[r, \omega, \tilde{\phi}_{r,\epsilon}]$ and replace on Equation (8), the function $\tilde{\phi}_{r,\epsilon}$ satisfies

$$\sup_{z\in\mathbb{R}^n} \|\tilde{\phi}_{r,\epsilon}(z) - \sum_{p\in i2\pi\mathbb{Z}^n,\ \langle r,p\rangle\neq 0} \frac{a_p[H[\tilde{\phi}_{r,\epsilon}]]}{\langle r, p\rangle}(\exp(\langle z, p\rangle) - 1)\| < \epsilon.$$

By Lemma 14, $\phi_{r,\omega,\epsilon} \in K_{r,\omega}$ then

$$\|\tilde{\phi}_{r,\epsilon}]\|_\infty \leq \|\phi_{r,\omega,\epsilon}\|_\infty < 2\beta\tau(f, r).$$

By Lemma 12 the set $K_{r,\omega}$ is a subset of $E_\omega(\mathbb{R}^n)$. Then $\tilde{\phi}_{r,\epsilon}$ is a C^∞ periodic modulo $\mathbb{Z}^n$ function.

□

5 Proof of the Main result

Proof of Main results. By a change of variable we can suppose that $x(0) = \mathbb{0}$ and $f(z) \neq \mathbb{0}$ for all $z \in \mathbb{R}^n$. Let $(\epsilon_k)_k \subset (0, 1]$ be a sequence satisfying $\lim_{k\to\infty} \epsilon_k = 0$. For every $k \geq 1$ let $\gamma_k \in \mathbb{R}^n \to \mathbb{Q}^n/\{\mathbb{0}\}$ be a bijective function satisfying

$$\|y - \gamma_k(y)\| < \epsilon_k.$$

Since $\gamma_k(y) \in \mathbb{Q}^n/\{\mathbb{0}\}$, consider for every $y \in \mathbb{R}^n$ and every $k \geq 1$ the function $\phi_{\gamma_k(y)}$ satisfies the Main Proposition such that

$$\sup_{z\in\mathbb{R}^n} \|\phi_{\gamma_k(y)}(z) - \sum_{p\in 2\pi\mathbb{Z}^n,\ \langle\gamma_k(y),p\rangle\neq 0} \frac{a_p[H[\phi_{\gamma_k(y)}]]}{\langle\gamma_k(y), p\rangle}(\exp(i\langle z, p\rangle) - 1)\| < \frac{1}{k},$$

Define the recurrent sequence $(\rho_k)_k \subset \mathbb{R}^n$ as

$$\rho_0 = \mathbb{0}, \quad \rho_{k+1} := \sum_{p \in i2\pi\mathbb{Z}^n,\ \langle \gamma_k(\rho_k), p\rangle = 0} a_p[H[\phi_{\gamma_k(\rho_k)}]], \quad \forall k \geq 0.$$

Prove that the sequence $(\rho_k)_k$ is bounded. Let be $\psi_k : \mathbb{R} \to \mathbb{R}^n$ the function defined by $t \mapsto \psi_k(t) := \phi_{\gamma_k(\rho_k)}(\gamma_k(\rho_k)t)$. By Item 1 of the Main Proposition, we get

$$\sup_{t\in\mathbb{R}} \|\psi_k(t) - \sum_{p \in 2\pi\mathbb{Z}^n,\ \langle \gamma_k(\rho_k), p\rangle \neq 0} \frac{a_p[H[\phi_{\gamma_k(\rho_k)}]]}{\langle \gamma_k(\rho_k), p\rangle}(\exp(i\langle \gamma_k(\rho_k), p\rangle t) - 1)\| < \frac{1}{k},$$

since the sum is normally convergent, that implies

$$\sup_{t\in\mathbb{R}} \|\psi_k(t) - \int_0^t \sum_{p \in 2\pi\mathbb{Z}^n,\ \langle \gamma_k(\rho_k), p\rangle \neq 0} a_p[H[\phi_{\gamma_k(\rho_k)}]] \exp(i\langle \gamma_k(\rho_k), p\rangle s) ds\| < \frac{1}{k}.$$

By Equation 2, we have the following Fourier development

$$f(z + \phi_{\gamma_k(\rho_k)}(z)) = \sum_{p \in 2\pi\mathbb{Z}^n} a_p[H[\phi_{\gamma_k(\rho_k)}]] \exp(i\langle z, p\rangle),$$

implies

$$\sup_{t\in\mathbb{R}} \|\psi_k(t) - \int_0^t f(\gamma_k(\rho_k)s + \psi_k(s))ds - t\rho_{k+1}\| < \frac{1}{k}.$$

Since $\|\psi_k\|_\infty < \infty$ then

$$\|\rho_{k+1} - \lim_{t\to\infty} \frac{1}{t} \int_0^t f(\gamma_k(\rho_k)s + \psi_k(s))ds\| < \frac{1}{k}.$$

we deduce that $\limsup_{k\to\infty} \|\rho_k\| \leq \|f\|$. There exists $\rho \in \mathbb{R}^n$ and a subsequence $(\rho_{k_s})_s$ which converge to ρ. In order to simplify the notation, we suppose that $(\rho_k)_k$ which converge to ρ. Since $\epsilon_k \to 0$ then

$$\lim_{k\to\infty} \rho_k = \lim_{k\to\infty} \gamma_k(\rho_k) = \rho.$$

We have supposed on the first of this proof, that $f(z) \neq \mathbb{0}$ for every $z \in \mathbb{R}^n$, then $\rho \neq \mathbb{0}$. There exists $c > 0$ and $k_0 \geq 0$ such that

$$\tau(f, \gamma_k(\rho_k)) < c, \quad \forall k \geq k_0.$$

By the Main Proposition, we obtain

$$\sup_{k\geq k_0}\|\phi_{\gamma_k(\rho_k)}\|_\infty \ \leq 2\beta \sup_{k\geq k_0}\tau(f,\gamma_k(\rho_k)) < 2\beta c.$$

The functions

$$s \mapsto \gamma_k(\rho_k)s + \phi_{\gamma_k(\rho_k)}(\gamma_k(\rho_k)s),$$

is a Picard iteration of the differential equation

$$\dot{x} = f(x), \quad x(0) = \mathbb{0},$$

there exists a weakly almost periodic function $\psi_\rho : \mathbb{R} \to \mathbb{R}^n$ of slope ρ such that

$$\begin{aligned}\psi_\rho(t) &= \lim_{k\to\infty}\phi_{\gamma_k(\rho_k)}(\gamma_k(\rho_k)t)\\ &= \int_0^t \lim_{k\to\infty} f(\gamma_k(\rho_k)s + \phi_{\gamma_k(\rho_k)}(\gamma_k(\rho_k)s))ds - t\lim_{k\to\infty}\rho_k\\ &= \int_0^t f(\rho s + \psi_\rho(s))ds - t\rho, \ \forall t\in\mathbb{R}.\end{aligned}$$

By uniqueness of solution of differential equation, we have proved that

$$x(t) = \rho t + \psi_\rho(t), \quad \forall t \in \mathbb{R}.$$

□

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Appendix. A

Proof of Lemma 6. By hypothesis, for $q = 0$ we have

$$\sum_{p\in 2\pi\mathbb{Z}^n/\mathbb{0}} \|c_p\| < +\infty,$$

the series is normally convergent and we have

$$\forall p \in 2\pi\mathbb{Z}^n :\ g(\omega z)\exp(-i\langle z,p\rangle) = \sum_{q\in 2\pi\mathbb{Z}^n} c_p \exp(i\langle z, q-p\rangle), \quad c_p \in \mathbb{C}^n,$$

implies

$$\begin{aligned}\forall p \in 2\pi\mathbb{Z}^n :\ a_p[g_\omega] &= \int_0^1 \dots \int_0^1 g(\omega z)\exp(-i\langle z,p\rangle) z_1 \dots z_n \\ &= \sum_{q\in 2\pi\mathbb{Z}^n} c_p \int_0^1 \dots \int_0^1 \exp(i\langle z, q-p\rangle) z_1 \dots z_n = c_p.\end{aligned}$$

Now, prove that $g \in E_\omega(\mathbb{R}^n)$. Denote

$$\theta_p(z) := \exp(i\langle z,p\rangle),\ \forall z \in \mathbb{R}^n,$$

It is sufficient to prove that for every $q \geq 1$ we have

$$S_q := \sum_{p\in 2\pi\mathbb{Z}^n/\mathbb{0}} \|c_p\| \|d^q\theta_p\|_\infty < +\infty,$$

where $d^q g$ is q^{th} differential of g, the function $d^q g$ is defined by

$$d^q g(v_1,\dots,v_q) = \sum_{p\in 2\pi\mathbb{Z}^n} c_p d^q\theta_p(v_1,\dots,v_q),\ \forall v_j := (v_{j,k})_{k=1}^n \in \mathbb{R}^n,\ \forall j \in [\![1,q]\!].$$

We have

$$\forall j \in [\![1,q]\!], \quad \forall v_j := (v_{j,k})_{k=1}^n \in \mathbb{R}^n, \quad p := (p_j)_{j=1}^n,$$
$$d^q\theta_p(v_1,\dots,v_q) = \sum_{k_1=1}^{n} \dots \sum_{k_q=1}^{n} i^q p_{k_1}\dots p_{k_q} \exp(i\langle z,p\rangle) v_{1,k_1}\dots v_{q,k_q},$$

then

$$\|d^q\theta_p\|_\infty = \Big(\sum_{k=1}^{n} |p_k|\Big)^q \leq n^q \|p\|^q.$$

implies

$$\forall q \geq 1: \quad S_q \leq \sum_{p\in 2\pi\mathbb{Z}^n/\mathbb{0}} \|c_p\| n^q \|p\|^q = \frac{1}{2} n^q \|g\|_{1,q} < +\infty.$$

then $g \in E_\omega(\mathbb{R}^n)$. □

Appendix. B

Proof of Lemma 7 . By Equation 2, we have

$$g(z) = \sum_{p\in 2\pi\mathbb{Z}^n} a_p[g_\omega] \exp(i\frac{1}{\omega}\langle z,p\rangle).$$

Since $g(\mathbb{0}) = \mathbb{0}$, then

$$a_{\mathbb{0}}[g_\omega] = -\sum_{p\in 2\pi\mathbb{Z}^n/\mathbb{0}} a_p[g_\omega],$$

implies

$$\|a_{\mathbb{0}}[g_\omega]\| \leq \sum_{p\in 2\pi\mathbb{Z}^n/\mathbb{0}} \|a_p[g_\omega]\| = \frac{1}{2}\|g\|_{\omega,0}.$$

Since

$$\|g\|_\infty \leq \sum_{p\in 2\pi\mathbb{Z}^n} \|a_p[g_\omega]\|.$$

We deduce that

$$\|g\|_\infty \leq \|g\|_{\omega,0}.$$

□

Appendix. C

Proof of Lemma 8. Since $g : \mathbb{R}^n \to \mathbb{R}^n$ is a trigonometric polynomial, then it is C^∞ and there exists $m \in \mathbb{N}$ such that

$$g_\omega(z) = g(\frac{z}{\omega}) = \sum_{p\in 2\pi\mathbb{Z}^n,\ \|p\|\leq m} a_p[g]\exp(i\frac{1}{\omega}\langle z,p\rangle),$$

Denote

$$\theta_p(z) := \exp(i\frac{1}{\omega}\langle z,p\rangle),\ \forall z\in\mathbb{R}^n,$$

Denote $d^k g$ is k^{th} differential of g, which implies

$$\forall v_j := (v_{j,s})_{s=1}^n \in \mathbb{R}^n,\ \forall j\in[\![1,k]\!],$$
$$d^k g(v_1,\dots,v_k) = \sum_{p\in 2\pi\mathbb{Z}^n,\|p\|\leq m} a_p[g]d^k\theta_p(v_1,\dots,v_k).$$

We have

$$\forall j\in[\![1,k]\!],\quad \forall v_j := (v_{j,s})_{s=1}^n\in\mathbb{R}^n,\quad p := (p_j)_{j=1}^n \in 2\pi\mathbb{Z}^n,$$
$$d^l\theta_p(v_1,\dots,v_k) = \sum_{s_1=1}^n\dots\sum_{s_k=1}^n i^k\frac{p_{s_1}}{\omega}\dots\frac{p_{s_k}}{\omega}\exp(i\langle z,p\rangle)v_{1,s_1}\dots v_{k,s_k},$$

then

$$\forall p\in 2\pi\mathbb{Z}^n,\ \|p\|\leq m:\quad \|d^k\theta_p\|_\infty = \Big(\sum_{s=1}^n|\frac{p_s}{\omega}|\Big)^k \leq n^k\|\frac{m}{\omega}\|^k.$$

We obtain

$$\begin{aligned}\|d^k g\|_{\omega,q} &\leq 2n^k(\frac{m}{\omega})^k \sum_{p\in 2\pi\mathbb{Z}^n,\ \|p\|\leq m}\|a_p[g]\|\|\frac{p}{\omega}\|^q\\ &\leq 2n^k(\frac{m}{\omega})^{k+q}\sum_{p\in 2\pi\mathbb{Z}^n,\ \|p\|\leq m}\|a_p[g]\| = 2n^k(\frac{m}{\omega})^{k+q}\|g\|_{\omega,0}.\end{aligned}$$

It is sufficient to choose $\beta := 2\max\{\|g\|_{\omega,0}, m\}$. □

Appendix. D

Lemma 15. *Let* $\omega\in\mathbb{N}^*$ *and* $\Big(h_j\in E_\omega(\mathbb{R}^n)\Big)_{j=1}^k$ *. Then*

$$\forall q\in\mathbb{N},\ \forall k\in\mathbb{N}^*:\ \|\Pi_{j=1}^k h_j\|_{\omega,q} \leq \frac{(k\omega^{k-1})^q}{2^{k-1}}\Pi_{j=1}^k[\|2a_{\mathbb{0}}[h_{j,\omega}]\| + \|h_j\|_{\omega,q}].$$

Proof of Lemma 15 . Since $(h_j)_{j=1}^k \subset E_\omega(\mathbb{R}^n)$, by Equation 2 we can write

$$h_j(z) = \sum_{p_j \in 2\pi\mathbb{Z}^n} a_{p_j}[h_j] \exp(i\frac{1}{\omega}\langle z, p_j\rangle).$$

By definition of the norm then

$$\|h_j\|_{\omega,q} = 2 \sum_{p_j \in 2\pi\mathbb{Z}^n/\{0\}} \|a_{p_j}[h_j]\| \|\frac{p_j}{\omega}\|^q. \tag{9}$$

We have

$$\begin{aligned}\Pi_{j=1}^k h_j(z) &= \sum_{p_1,\dots,p_k \in 2\pi\mathbb{Z}^n} \Pi_{j=1}^k a_{p_j}[h_j] \exp(i\frac{1}{\omega}\langle z, \sum_{j=1}^k p_j\rangle) \\ &= \sum_{v \in 2\pi\mathbb{Z}^n} \sum_{\sum_{j=1}^k p_j = v} \Pi_{j=1}^k a_{p_j}[h_j] \exp(i\frac{1}{\omega}\langle z, v\rangle).\end{aligned}$$

Then

$$\begin{aligned}\|\Pi_{j=1}^k h_j\|_{\omega,q} &= 2 \sum_{v \in 2\pi\mathbb{Z}^n/\mathbb{0}} \| \sum_{\sum_{j=1}^k p_j = v} \Pi_{j=1}^k a_{p_j}[h_j]\| \|\frac{v}{\omega}\|^q \\ &\leq 2 \sum_{v \in 2\pi\mathbb{Z}^n} \| \sum_{\sum_{j=1}^k p_j = v} \Pi_{j=1}^k a_{p_j}[h_j]\| \|\frac{v}{\omega}\|^q.\end{aligned}$$

By the triangular inequality we obtain

$$\|\Pi_{j=1}^k h_j\|_{\omega,q} \leq 2 \sum_{p_1,\dots,p_k \in 2\pi\mathbb{Z}^n} \|\Pi_{j=1}^k a_{p_j}[h_j]\| \|\frac{1}{\omega}\sum_{j=1}^k p_j\|^q.$$

Since

$$\|\sum_{j=1}^k p_j\| \leq k\Pi_{\|p_j\|\neq 0}\|p_j\|, \quad \forall p_j, \in 2\pi\mathbb{Z}^n,$$

then

$$\|\frac{1}{\omega}\sum_{j=1}^k p_j\|^q \leq k^q \omega^{(k-1)q} \Pi_{\|p_j\|\neq 0}\|\frac{p_j}{\omega}\|^q, \quad \forall p_j, \in 2\pi\mathbb{Z}^n,$$

implies

$$\|\Pi_{j=1}^{k} h_j\|_{\omega,q} \leq 2k^q \omega^{(k-1)q} \sum_{p_1,\ldots,p_k \in 2\pi\mathbb{Z}^n} \Big(\Pi_{\|p_j\|\neq 0} \|\frac{p_j}{\omega}\|^q\Big)\Big(\Pi_{j=1}^{k} \|a_{p_j}[h_j]\|\Big)$$

$$= 2k^q \omega^{(k-1)q} \sum_{p_1,\ldots,p_k \in 2\pi\mathbb{Z}^n} \Big(\Pi_{\|p_j\|=0} \|a_{p_j}[h_j]\|\Big) \Pi_{\|p_j\|\neq 0} \Big[\|\frac{p_j}{\omega}\|^q \|a_{p_j}[h_j]\|\Big].$$

By Equation (9), we obtain finally

$$\|\Pi_{j=1}^{k} h_j\|_{\omega,q} \leq \frac{1}{2^{k-1}} k^q \omega^{(k-1)q} \Pi_{j=1}^{k} \Big[2\Big(\|a_0[h_j]\| + \sum_{p_j \in 2\pi\mathbb{Z}^n/\{0\}} \|\frac{p_j}{\omega}\|^q \|a_{p_j}[h_j]\|\Big)\Big].$$

□