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Reduced dimension and Rotation vector formula of ordinary differential equation

W. Oukil¹, Ph. Thieullen² and A. Kessi³

1. Department of Mathematics and Computer Science, Médéa University,
Pôle Urbain -26000 Médéa, Algeria.
2. Institute of Mathematics of Bordeaux, Bordeaux University,
351, Cours de la Libération - F 33 405 Talence, France.
3. Faculty of Mathematics, University of Houari Boumediene,
BP 32 El Alia 16111, Bab Ezzouar, Algiers, Algeria.

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Abstract

The *leader trajectory* function defined in this article is an approximated solution of a differential equation. It is defined by some independent one-dimensional differential equations. The generalized main result of this article asserts that if the *leader trajectory* exists then it is at finite distance from the solution of the system. However, the differential equation can be controlled by a reduced one-dimensional differential equations. The application of the generalized main result is to control the trajectory of the periodic systems. We prove that for any *periodic* system and any initial condition there exists a *leader trajectory* which is a linear function of the time variable. Otherwise, we find an exact *Rotation vector formula* which is the relation between the *rotation vector* and the initial condition. In addition, we find easily a necessary and sufficient condition for the existence of a locally constant rotation vector under perturbation of the system, known by the the Arnold Tongue. The mean result answer to many problems in natural sciences.

Keywords: Differential equations, bounded solution, periodic system, rotation vector.

1 Introduction and main results

One important application of the result is to control the trajectory of the periodic systems as cited in the abstract; The applications of the periodic differential equations are various. The issue is to estimate the value of the *rotation vector*. The list of the related works is very large; for some biological works see [8, 44]. The mean result answer to many problems because we find an explicit relation between the rotation vector and the initial condition of the orbit. We study in this article the following system

$$\dot{x} = f(x), \quad t \in \mathbb{R}, \quad x = (x_i)_{i=1}^n, \quad x(0) = x_0, \quad x_0 \in \mathbb{R}^n. \quad (1)$$

where $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a C^2 function. The function $x(t) = (x_i(t))_{i=1}^n$ is the state of the system. Notice that any non-autonomous system can be written as an autonomous system. We introduce in the present Section some notations, the definition of the leader trajectory and the main results. The Section 2 is the ingredients to study the linearized of system (1) which has the form of the Riccati differential equation. In the rest of Sections we prove the main results

1.1 Periodic differential equations

Many mathematical models are defined by the periodic systems [3, 5, 1]. For example, in biological sciences the components of the rotation vector are called the *frequency* [38, 13, 8]. When the periodic system is stable, in the sens of the *Lyapunov stability*, the rotation vector exists [3, 5, 1, 4].

Definition 1 (Periodic function). Let be $q, p \in \mathbb{N}^*$ and let $g = (g_i)_1^p : \mathbb{R}^q \rightarrow \mathbb{R}^p$ be a function. We say that g is *periodic* if it is periodic relative to each variable. Otherwise,

$$g_i(y_1, \dots, y_j + 1, \dots, y_q) = g_i(y), \quad \forall i = \overline{1, q}, \forall j = \overline{1, q}, \forall y = (y_i)_{i=1}^q \in \mathbb{R}^q.$$

In this article, we use the usual norm on \mathbb{R}^q , $q \in \mathbb{N}^*$, defined by $\|z\| = \max_{1 \leq i \leq q} |z_i|$ for all $z := (z_i)_{i=1}^q \in \mathbb{R}^q$. We also denote $\|\cdot\|$ the associated matrix norm.

Definition 2 (The rotation vector). Let be $q \in \mathbb{N}^*$ and $\lambda \in \mathbb{R}^q$. Let $g : \mathbb{R} \rightarrow \mathbb{R}^q$ be a continuous function. We say that λ is the *rotation vector* of the function g if

$$\sup_{t \in \mathbb{R}} \|g(t) - \lambda t\| < +\infty.$$

Notice that if the rotation vector exists then it is unique. To formulate the main results let us introduce the following notations; Let $p, q \in \mathbb{N}^*$, for every $q \times p$ matrix $M = [m_{i,j}]_{i,j}$ we denote

$$\sigma(M) := \sum_{1 \leq i \leq q} \sum_{1 \leq j \leq p} m_{i,j}. \quad (2)$$

Let be $u \in \mathbb{R}^q$, we denote $\text{Diag}(u)$ the diagonal matrix of diagonal u . Let be $1 \leq i \leq q$, define the diagonal matrix $I_{i,q}$ as

$$I_{i,q} = \text{Diag}(v), \quad v = (v_j)_{j=1}^q : v_i = -1 \text{ and } v_j = 1, \quad \forall 1 \leq j \neq i \leq q. \quad (3)$$

Denote by I_q the $q \times q$ identity matrix and put $I_{0,q} = I_q$. The particularity of the matrix $I_{i,q}$ is the relation $2y_i = \sigma(I_{0,q}y) - \sigma(I_{i,q}y)$ for any vector $(y_i)_{i=1}^q$. Let $g : \mathbb{R}^q \rightarrow \mathbb{R}^q$ be a C^2 function we denote dg and d^2g the first and the second differential of g respectively.

Main Result. *Let be $x_0 \in \mathbb{R}^n$ and let x be the solution of the Equation (1). Suppose that f is periodic, then x admits a rotation vector $\rho := (\rho_j)_{j=1}^n \in \mathbb{R}^n$ which is the unique solution of the following Rotation vector formula*

$$\lim_{t \rightarrow +\infty} \frac{1}{t} \left[\Psi_0[f_{x_0}](\rho, t) - \Psi_i[f_{x_0}](\rho, t) \right] = 0, \quad \forall i = \overline{1, n}, \quad (4)$$

where the functions $\Psi_i[f_{x_0}]$ are defined by

$$\begin{aligned} \Psi_i[f_{x_0}](\rho, t) &:= \int_0^t A_i(\rho, s) \exp \left(\int_s^t \sigma \left(I_{i,q} df(\nu\rho + x_0) I_{i,q} \right) d\nu \right) ds, \\ A_i(\rho, s) &:= \sigma \left(I_{i,q} (\rho - f(s\rho + x_0)) \right). \end{aligned}$$

1.2 Generalized Main result

Define the leader trajectory function.

Definition 3. [Leader trajectory] Let be $q \in \mathbb{N}^*$, let $g : \mathbb{R}^q \rightarrow \mathbb{R}^q$ and $\mu : \mathbb{R} \rightarrow \mathbb{R}^q$ be a C^1 functions. We say that μ is a *leader trajectory of the function g* if the following hypotheses are satisfied

- $\sup \left\{ \left| \int_0^t \sigma \left(dg(\mu(\nu)) \right) d\nu \right|, \quad t \in \mathbb{R} \right\} < +\infty.$
- $\forall i = \overline{1, q}, \exists \tau_i \in \{-1, 1\}: \sup_{t \geq 0} \int_0^t \tau_i \sigma \left(I_{i,q} dg(\mu(\tau_i \nu)) I_{i,q} \right) d\nu < +\infty.$

- $\forall i = \overline{1, q}$: $\sup_{t \in \mathbb{R}} |\psi_0(\tau_i t) - \psi_i(\tau_i t)| < +\infty$.
where the function $\psi_i : \mathbb{R} \rightarrow \mathbb{R}$ satisfies the following one-dimensional differential equations

$$\begin{aligned} \dot{\psi}_i &= \sigma\left(I_{i,q}\dot{\mu}(t) - I_{i,q}g(\mu(t))\right) + \sigma\left(I_{i,q}dg(\mu(t))I_{i,q}\right)\psi_i, \quad i = \overline{0, q}, \\ \psi_i(0) &= 0, \quad t \in \mathbb{R}. \end{aligned}$$

Generalized Main Result. *Let be $x_0 \in \mathbb{R}^n$ and let x be the solution of the Equation (1). Suppose that*

$$\|df\|_\infty + \|d^2f\|_\infty < +\infty.$$

Let $\mu = (\mu_i)_{i=1}^n : \mathbb{R} \rightarrow \mathbb{R}^n$ be a leader trajectory of f such that $\mu(0) = x_0$. Then there exists $D > 0$ such that

$$\|\mu(t) - x(t)\| \leq D, \quad \forall t \in \mathbb{R}. \quad (5)$$

In order to prove the Generalized mains result, we use the following linearized system of (1): Let $\mu = (\mu_i)_{i=1}^n : \mathbb{R} \rightarrow \mathbb{R}^n$ be a C^1 function such that $\mu(0) = x_0$. Consider the change of variables

$$\eta(t) = \mu(t) - \Phi^t(x_0), \quad t \in \mathbb{R}, \quad \eta := (\eta_i)_{i=1}^n,$$

where Φ^t is the flow of the system (1). Use Taylor formula to obtain the following linearized system

$$\dot{\eta} = \dot{\mu}(t) - f(\mu(t)) + df(\mu(t))\eta + \eta^T \zeta_\mu(t)\eta, \quad \|\eta(0)\| = 0, \quad t \in \mathbb{R}, \quad (6)$$

where $\zeta_\mu : \mathbb{R} \times \mathbb{R}^{n \times n \times n}$ is a continuous function such that $\|\zeta_\mu\|_\infty < \|d^2f\|_\infty$ and where η^T is the transpose of the function η . Equation (6) take the form of Riccati differential equation. The aim is to prove that the solution $\eta : \mathbb{R} \rightarrow \mathbb{R}^q$ is uniformly bounded on \mathbb{R} .

2 Riccati differential equation

We denote v^T the transpose of a vector v and for any $q \in \mathbb{N}^*$ we denote

$$\mathbf{1}_q := (1, \dots, 1)^T \in \mathbb{R}^q \quad \text{and} \quad \mathbf{0}_q := (0, \dots, 0)^T \in \mathbb{R}^q.$$

In this Section, we study a generalized Riccati differential equation. Let $\mathcal{A} : \mathbb{R} \rightarrow \mathbb{R}^q$, $\mathcal{B} : \mathbb{R} \rightarrow \mathbb{R}^{q \times q}$ and $\mathcal{H} : \mathbb{R} \rightarrow \mathbb{R}^{q \times q \times q}$ be a continuous functions such that

$$\exists \gamma < 1 : \max \left\{ \|\mathcal{H}\|_\infty, \|\mathcal{B}\|_\infty \right\} \leq \gamma. \quad (7)$$

Consider the following Riccati differential equation

$$\dot{y} = \mathcal{A}(t) + \mathcal{B}(t)y + y^T \mathcal{H}(t)y, \quad y(0) = \mathbb{0}_q, \quad y = (y_i)_{i=1}^q, \quad t \in \mathbb{R}. \quad (8)$$

Recall that the notation $\sigma(\cdot)$ and the diagonal matrix $I_{i,q}$ are defined by Equations (2) and (3) respectively. Consider the following hypotheses [H1]

$$\sup \left\{ \left| \int_0^t \sigma(\mathcal{B}(\nu)) d\nu \right|, \quad t \in \mathbb{R} \right\} < +\infty,$$

and for every $i = \overline{1, q}$ there exists $\tau_i \in \{-1, 1\}$ such that

$$\sup \left\{ \tau_i \int_0^t \sigma(I_{i,q} \mathcal{B}(\tau_i \nu) I_{i,q}) d\nu, \quad t \geq 0 \right\} < +\infty,$$

[H2] The following one-dimensional differential equations

$$\dot{\psi}_i = \sigma(I_{i,q} \mathcal{A}(t)) + \sigma(I_{i,q} \mathcal{B}(t) I_{i,q}) \psi_i, \quad t \geq 0, \quad (9)$$

admit a solution $\psi_i : \mathbb{R}_+ \rightarrow \mathbb{R}$ of initial condition $\psi_i(0) = 0$ such that

$$\sup_{t \in \mathbb{R}} |\psi_0(\tau_i t) - \psi_i(\tau_i t)| < +\infty.$$

The following theorem is the main ingredient used to prove the main results.

Theorem 4. *Let y be the solution of the differential Equation (8) and let be $\gamma \in (0, 1)$ as defined by Equation (7). Suppose that Hypotheses [H1] and [H2] are satisfied. Then there exists $\gamma_* > 0$ such that for all $\gamma \in (0, \gamma_*)$ there exists $D_\gamma > 0$ such that the solution y of Equation (8) satisfies $\sup_{t \in \mathbb{R}} \|y(t)\| < D_\gamma$.*

The strategy it to use an appropriate change of variable in (8) and use the differential equation comparison. In order to simplify, we introduce the following notation. Define the cone V_+^q of \mathbb{R}^q by

$$V_+^q := \{z = (z_i)_{i=1}^q \in \mathbb{R}^q : z_i > 0, \quad \forall i = \overline{1, q}\}. \quad (10)$$

We consider the partial order \succeq on \mathbb{R}^q defined by

$$\forall z \in \mathbb{R}^q : \quad z \succ \mathbb{0}_q \iff z \in V_+^q \quad \text{and} \quad z \succeq \mathbb{0}_q \iff z \in \overline{V_+^q}.$$

Define the sets \mathcal{K}_q^+ and \mathcal{K}_q of the $q \times q$ diagonal matrix by

$$\mathcal{K}_q^+ = \{I_{i,q}, \quad i = \overline{0, q}\} \quad \text{and} \quad \mathcal{K}_q = \{-I : I \in \mathcal{K}_q^+\} \cup \mathcal{K}_q^+. \quad (11)$$

We have the following lemma

Lemma 5. *Let y be the solution of the differential Equation (8). Let be $\gamma > 0$ given by Equation (7). Let be $T > 0$ and put $U = [-T, T]$. For every $I \in \mathcal{K}_q$ and $\tau \in \{-1, 1\}$ denote*

$$\delta_I(t) := \exp(qt) \left[\psi_I(\tau t) - \sigma(Iy(\tau t)) \right] \quad (12)$$

where $\psi_I : \mathbb{R} \rightarrow \mathbb{R}$ satisfies

$$\dot{\psi}_I = \sigma(I\mathcal{A}(t)) + \tau\sigma(I\mathcal{B}(t)I)\psi_I, \quad t \in \mathbb{R}. \quad (13)$$

Let be $D > 0$ and suppose that $\sup_{t \in U} \|y(t)\| < D/\gamma$, then

$$\forall t \in \mathbb{R}, \forall I \in \mathcal{K}_q : \psi_I(\tau t) - \sigma(Iy(\tau t)) = \frac{D^2}{\gamma(1-\gamma)} \implies \dot{\delta}_I(t) > 0.$$

Proof. Denote $\tilde{I}_q := [c_{i,j}]_{1 \leq i,j \leq q}$ be the $q \times q$ matrix such that $c_{i,j} = 1$ for all $i, j = \overline{1, q}$. Let be $I \in \mathcal{K}_q$ and Consider the change of variable

$$z_I(\tau, t) := \exp(t\tilde{I})Iy(\tau t), \quad t \in U. \quad (14)$$

and put $\tilde{y}_I(t) := Iy(\tau t)$. Since the inverse matrix of I is itself the system (8) becomes

$$\dot{\tilde{y}}_I = \tau I\mathcal{A}(\tau t) + \tau I\mathcal{B}(\tau t)I\tilde{y}_I(t) + \tau Iy^T(\tau t)\mathcal{H}(\tau t)y(\tau t), \quad \forall t \in U.$$

That follows

$$\dot{z}_I(\tau, t) = \exp(t\tilde{I}) \left[\tau I\mathcal{A}(\tau t) + [\tilde{I} + \tau I\mathcal{B}(\tau t)I]\tilde{y}_I + \tau y^T(\tau t)\mathcal{H}(\tau t)y(\tau t) \right]. \quad (15)$$

Remark that $\tilde{I}_q^2 = q\tilde{I}_q$. We have the relation

$$\exp(t\tilde{I}_q) = I_q + \frac{1}{q}(\exp(qt) - 1)\tilde{I}_q \text{ and } \|\exp(t\tilde{I}_q)\| \leq \exp(qt), \quad \forall t \geq 0, \quad (16)$$

By hypothesis $\sup_{t \in U} \|\gamma y(t)\| < D$ and $\|\mathcal{H}\|_\infty \leq \gamma$ then

$$\|\gamma \exp(t\tilde{I})Iy^T(\tau t)\mathcal{H}(\tau t)y(\tau)\| < \exp(qt)D^2, \quad \forall t \in U.$$

From (17), we have

$$\dot{z}_I(\tau, t) \prec \gamma \exp(t\tilde{I}) \left[\tau I\mathcal{A}(\tau t) + [\tilde{I} + \tau I\mathcal{B}(t)I]\tilde{y}_I \right] + \frac{1}{\gamma} \exp(qt)D^2 \mathbf{1}_q. \quad (17)$$

Suppose that $\delta_I(t) = D^2/(\gamma(1-\gamma))$ then $\delta_I(t) > 0$. Use the fact that for every $c = (c_i)_{i=1}^q \in \mathbb{R}^q$ and $v = (v_i)_{i=1}^q \in \mathbb{R}^q$ such that $\sigma(v) \neq 0$ there exists $\tilde{c} \in \mathbb{R}$ that satisfies $|\tilde{c}| \leq \|c\|$ and such that

$$\sum_{j=1}^q c_j v_j = \tilde{c} \sigma(v).$$

The above vector v play the following role

$$v_i = \frac{1}{q} \psi_I(\tau t) - \tilde{y}_j(t).$$

Use the hypothesis hypothesis $\|\mathcal{B}(t)\| < \gamma$, to obtain

$$\begin{aligned} -\tau I \mathcal{B}(t) I \left(\frac{1}{q} \psi_I(\tau t) - \tilde{y}_j(t) \right) &\prec \gamma \delta_I(t) \mathbb{1}_q, \quad \forall t \in U, \\ -\left[\tilde{I} + \tau I \mathcal{B}(t) I \right] \left(\frac{1}{q} \psi_I(\tau t) - \tilde{y}_j(t) \right) &\prec -(1-\gamma) \delta_I(t) \mathbb{1}_q, \quad \forall t \in U. \end{aligned}$$

Since $\exp(t\tilde{I})V_q^+ \subset V_q^+$, then Equation (17) implies that

$$\begin{aligned} \dot{z}_I(\tau, t) &\prec \tau \exp(t\tilde{I}) I \mathcal{A}(\tau t) + \frac{1}{q} \exp(t\tilde{I}) [\tilde{I} + \tau I \mathcal{B}(\tau t) I] \psi_I(\tau t) \\ &\quad - (1-\gamma) \exp(t\tilde{I}) \delta(t) \mathbb{1}_q + \frac{1}{\gamma} \exp(qt) D^2 \mathbb{1}_q, \quad \forall t \in U. \end{aligned}$$

By Equation (16), we have

$$\sigma(I \mathcal{B}(\tau t) I) = \exp(-qt) \sigma(\exp(t\tilde{I}) I \mathcal{B}(\tau t) I) \text{ and } \exp(t\tilde{I}) \tilde{I} = \exp(qt) \mathbb{1}_q.$$

By addition, we get

$$\frac{d}{dt} [\exp(qt) \psi_I(\tau t) - \sigma(z_I(\tau, t))] < -q(1-\gamma) \exp(qt) \delta(t) + q \frac{1}{\gamma} \exp(qt) D^2,$$

We have proved $\delta_I(t) = D^2/(\gamma(1-\gamma))$ implies that $\dot{\delta}_I(t) > 0$. □

Recall that the diagonal matrix $I_{i,q}$ define the sets \mathcal{K}_q^+ and \mathcal{K}_q in Equation (11) and recall that I_q is the $q \times q$ identity matrix.

Lemma 6. *Let $v : \mathbb{R} \rightarrow \mathbb{R}^q$ be a function . Let $\psi : \mathcal{K}_q \times \{-1, 1\} \times \mathbb{R}_+ \rightarrow \mathbb{R}$ be a functions such that*

- For all $(I, \tau, t) \in \mathcal{K}_q \times \{-1, 1\} \times \mathbb{R}_+$

$$\psi(-I, \tau, t) = -\psi(I, \tau, t). \quad (18)$$

- There exists $\theta, \alpha, T > 0$ such that for all $I \in \mathcal{K}_q$ there exists $\tau_I \in \{-1, 1\}$ such that $\tau_I = \tau_{-I}$ and such that

$$\theta + \psi(I, \tau_I, t) - \sigma(Iv(\tau_I t)) > \alpha > \psi(I, \tau_I, -t) + \sigma(Iv(-\tau_I t)), \quad \forall t \in (0, T).$$

- For all $I_0 \in \{I_q, -I_q\}$ and all $\tau \in \{-1, 1\}$

$$\theta + \psi(I_0, \tau, t) - \sigma(I_0 v(\tau t)) > \alpha > \psi(I_0, \tau, -t) + \sigma(I_0 v(-\tau t)), \quad \forall t \in (0, T).$$

Then,

$$\|v(t)\| \leq \theta + \alpha + \frac{1}{2}\beta_T, \quad \forall t \in [-T, T],$$

where

$$\beta_T := \sup \left\{ \left| \psi(I_q, \tau_I, t) - \psi(I, \tau_I, t) \right| : t \in [-T, T], \quad I \in \mathcal{K}_q^+ \right\}.$$

Proof. Use the fact $\tau_I = \tau_{-I}$ and use the Equation (18) to obtain

$$\begin{aligned} \theta - \psi(I, \tau_I, t) + \sigma(Iv(\tau_I t)) &> \alpha > -\psi(I, \tau_I, -t) - \sigma(Iv(-\tau_I t)), \\ \theta - \psi(I_0, \tau, t) + \sigma(I_0 v(\tau t)) &> \alpha > -\psi(I_0, \tau, -t) - \sigma(I_0 v(-\tau t)). \end{aligned}$$

By addition we get:

$$\begin{aligned} 2(\theta - \alpha) + \psi(I_0, \tau_I, t) + \psi(I, \tau_I, t) &> [\sigma(I_0 v(\tau_I t)) + \sigma(Iv(\tau_I t))], \\ 2\alpha - \psi(I, \tau_I, -t) - \psi(I_0, \tau_I, -t) &> \sigma(I_0 v(-\tau_I t)) + \sigma(Iv(-\tau_I t)). \end{aligned}$$

For $I_0 = I_q$ and $I = -I_{i,q}$ we have

$$\begin{aligned} 2(\theta - \alpha) + \psi(I_q, \tau, t) - \psi(I_{i,q}, \tau_I, t) - \alpha &> \sigma(I_q v(\tau t)) - \sigma(I_{i,q} v(\tau_I t)), \\ \alpha - \psi(I_q, \tau, -t) + \psi(I_{i,q}, \tau_I, -t) &> \sigma(I_q v(-\tau t)) - \sigma(I_{i,q} v(-\tau_I t)). \end{aligned}$$

For $I_0 = -I_q$ and $I = I_{i,q}$ we have

$$\begin{aligned} 2(\theta - \alpha) - \psi(I_q, \tau, t) + \psi(I_{i,q}, \tau_I, t) - \alpha &> -[\sigma(I_q v(\tau t)) - \sigma(I_{i,q} v(\tau_I t))], \\ \alpha + \psi(I_q, \tau, -t) - \psi(I, \tau_I, -t) &> -\sigma(I_q v(-\tau t)) + \sigma(I_{i,q} v(-\tau_I t)). \end{aligned}$$

use the fact that for every $u = (u_i)_{i=1}^q \in \mathbb{R}^q$ we have

$$u_i = \frac{1}{2}[\sigma(I_q u) - \sigma(I_{i,q} u)], \quad \forall i = \overline{1, q}.$$

Then for every $i = \overline{1, q}$ and all $t \in (0, T)$ we have:

$$\begin{aligned} \left| v_i(\tau_I t) + \frac{1}{2} \left[\psi(I_q, \tau_i, t) - \psi(I_{i,q}, \tau_i, t) \right] \right| &< \theta + \alpha, \\ \left| v_i(-\tau_I t) - \frac{1}{2} \left[\psi(I_q, \tau_i, -t) - \psi(I_{i,q}, \tau_i, -t) \right] \right| &< \theta + \alpha. \end{aligned}$$

Finally, by definition of the constants β_T in the present Lemma, we get

$$\|v(t)\| \leq \theta + \alpha + \frac{1}{2}\beta_T, \quad \forall t \in [-T, T].$$

□

Proof of Theorem 4.

For all $\alpha \in \mathbb{R}$ and $i = \overline{0, q}$ let be $\psi_i(t, \alpha)$ be the solutions of the Equation (9) of initial condition $\psi_i(0, \alpha) = \alpha$. Let y be the solution of the differential Equation (8) and put

$$\bar{\delta}_i(t, \alpha) := \psi_i(\tau_i t, \alpha) + \sigma(I_{i,q} y(\tau_i t)) \text{ and } \underline{\delta}_i(\tau_i t, \alpha) := \psi_i(t, \alpha) - \sigma(I_{i,q} y(\tau_i t)).$$

Let be $D > 0$ and put $\alpha_* = D^2/(\gamma(1 - \gamma))$. Since $y(0) = 0_q$ then $\bar{\delta}_i(0, \alpha_*) = \underline{\delta}_i(0, \alpha_*) = \alpha_*$. By Lemma 5 we have

$$\dot{\bar{\delta}}_i(0, \alpha_*) > 0 \text{ and } \dot{\underline{\delta}}_i(0, \alpha_*) > 0 \quad \forall i = \overline{0, q}.$$

Since $\bar{\delta}_i(0, 0) = \underline{\delta}_i(0, 0) = 0$, then there exists $\epsilon > 0$ such that

$$\begin{aligned} \bar{\delta}_i(t, \alpha_*) &> \alpha_* > \bar{\delta}_i(-t, 0) \quad \forall t \in (0, \epsilon), \quad \forall i = \overline{0, q}, \\ \underline{\delta}_i(t, \alpha_*) &> \alpha_* > \underline{\delta}_i(-t, 0) \quad \forall t \in (0, \epsilon), \quad \forall i = \overline{0, q}. \end{aligned}$$

Define

$$\begin{aligned} \bar{T} &= \sup\{t \geq 0 : \bar{\delta}_i(s, \alpha_*) > \alpha_* > \bar{\delta}_i(-s, 0), \quad \forall s \in (0, t), \quad \forall i = \overline{0, q}\}, \\ \underline{T} &= \sup\{t \geq 0 : \underline{\delta}_i(s, \alpha_*) > \alpha_* > \underline{\delta}_i(-s, 0), \quad \forall s \in (0, t), \quad \forall i = \overline{0, q}\} \end{aligned}$$

Our strategy is to prove that $\bar{T} = \underline{T} = +\infty$. By contradiction, suppose that $\bar{T} = +\infty$ or $\underline{T} = +\infty$, then there exists $i \in \{0, q\}$ such that one of the following cases is true

$$\begin{aligned} \bar{\delta}_i(\bar{T}, \alpha_*) &= \alpha_*, & \bar{\delta}_i(-\bar{T}, 0) &= \alpha_*, \\ \underline{\delta}_i(\underline{T}, \alpha_*) &= \alpha_*, & \underline{\delta}_i(-\underline{T}, 0) &= \alpha_*. \end{aligned}$$

Then \overline{T} and \underline{T} satisfies

$$\frac{d}{dt} \left[\exp(qt) \overline{\delta}_i(\overline{T}, \alpha_*) \right] \leq 0 \quad \text{and} \quad \frac{d}{dt} \left[\exp(qt) \overline{\delta}_i(-\overline{T}, 0) \right] \leq 0, \quad (19)$$

$$\frac{d}{dt} \left[\exp(qt) \underline{\delta}_i(\underline{T}, \alpha_*) \right] \leq 0 \quad \text{and} \quad \frac{d}{dt} \left[\exp(qt) \underline{\delta}_i(-\underline{T}, 0) \right] \leq 0. \quad (20)$$

Let be $T_* = \min\{\underline{T}, \overline{T}\}$, remark that

$$\psi_i(t, \alpha) = \alpha \exp \left(\int_0^t \sigma(I_{i,q} \mathcal{B}(\nu) I_{i,q}) d\nu \right) + \psi_i(t, 0), \quad \forall t \in \mathbb{R}, \quad \forall I \in \mathcal{K}_q.$$

By hypothesis [H1] there exists $\omega > 0$ such that for all $i = \overline{1, q}$ we have

$$\begin{aligned} \sup \left\{ \tau_i \int_0^t \sigma(\mathcal{B}(\tau_i \nu)) d\nu, \quad \forall t \in \mathbb{R} \right\} &< \omega, \\ \sup \left\{ \tau_i \int_0^t \sigma(I_{i,q} \mathcal{B}(\tau_i \nu) I_{i,q}) d\nu, \quad \forall t \geq 0 \right\} &< \omega 0. \end{aligned}$$

Then for every $i = \overline{1, q}$ we obtain the following estimation

$$\begin{aligned} \psi_i(\tau_i t, \alpha) &\leq \exp(\omega) \alpha + \psi_i(\tau_i t, 0), \quad \forall t \in (0, T_*), \\ \psi_0(\tau_i t, \alpha) &\leq \exp(\omega) \alpha + \psi_0(\tau_i t, 0), \quad \forall t \in (0, T_*), \\ \psi_0(-\tau_i t, \alpha) &\leq \exp(\omega) \alpha + \psi_0(-\tau_i t, 0), \quad \forall t \in (0, T_*). \end{aligned}$$

Put

$$\psi(I_{i,q}, \tau_i, t) := \psi_i(\tau_i t, 0), \quad \forall i = \overline{1, q}.$$

By definition of the function $\psi_i(t, 0)$ we have $\psi(-I_{i,q}, \tau, t) = -\psi(I_{i,q}, \tau, t)$. By definition of T_* we have: for all $I \in \mathcal{K}_q$ there exists τ_I such that $\tau_{I_{i,q}} = \tau_{-I_{i,q}} = \tau_i$ and such that for all $t \in (0, T_*)$ we have

$$\exp(\omega) \alpha_* + \psi(I, \tau_I, t) + \sigma(Iy(\tau_I t)) > \alpha_* > \psi(I, \tau_I, -t) + \sigma(Iy(\tau_I t))$$

and for all $\tau \in \{-1, 1\}$ and $I \in \{I_q, -I_q\}$ we have

$$\exp(\omega) \alpha_* + \psi(I, \tau, t) + \sigma(Iy(\tau t)) > \alpha_* > \psi(I, \tau, -t) + \sigma(Iy(\tau t)).$$

where we recall that $\alpha_* = D^2/(\gamma(1 - \gamma))$. By Hypothesis [H2] there exists $\beta > 0$ such that

$$\sup_{t \in \mathbb{R}} |\psi_0(\tau_i t) - \psi_i(\tau_i t)| < \beta, \quad \forall i = \overline{1, q}.$$

By Lemma 6 we have

$$\|\gamma y(t)\| < (\exp(\omega) + 1)\alpha_* + \beta, \quad \forall [-T_*, T_*]. \quad (21)$$

There exists $\gamma_* > 0$ such that for all $\gamma \in (0, \gamma_*)$ there exists $D_\gamma > 0$ such that

$$\frac{\exp(\omega) + 1}{\gamma(1 - \gamma)} D_\gamma^2 + \beta < D_\gamma, \quad (22)$$

Choose $\alpha_* = D_\gamma^2 / (\gamma(1 - \gamma))$ and use equation (21) to get

$$\|\gamma y(t)\| < \frac{\exp(\omega) + 1}{\gamma(1 - \gamma)} D_\gamma^2 + \beta < D_\gamma, \quad \forall [-T_*, T_*].$$

Then $\|\gamma y(t)\| < D_\gamma$ for every $t \in [-T_*, T_*]$. In particular $\|\gamma y(T_*)\| < D_\gamma$ and $\|\gamma y(-T_*)\| < D_\gamma$. By Lemma 5 and Equation (19), we have

$$\begin{aligned} \frac{d}{dt} \left[\exp(qt) \bar{\delta}_i(T_*, \alpha_*) \right] &> 0 \quad \text{and} \quad \frac{d}{dt} \left[\exp(qt) \bar{\delta}_i(-T_*, 0) \right] > 0, \\ \frac{d}{dt} \left[\exp(qt) \underline{\delta}_i(T_*, \alpha_*) \right] &> 0 \quad \text{and} \quad \frac{d}{dt} \left[\exp(qt) \underline{\delta}_i(-T_*, 0) \right] > 0. \end{aligned}$$

We have obtain a contradiction with Equation (19). Then $T_* = +\infty$ and we have

$$\begin{aligned} \bar{\delta}_i(t, \alpha_*) &> \alpha_* > \bar{\delta}_i(-t, 0) \quad \forall t \geq 0, \quad \forall i = \overline{0, q}, \\ \underline{\delta}_i(t, \alpha_*) &> \alpha_* > \underline{\delta}_i(-t, 0) \quad \forall t \geq 0 \quad \forall i = \overline{0, q}. \end{aligned}$$

Finally, use again Equation (22) and Lemma 6 to deduce that

$$\|\gamma y(t)\| < \frac{\exp(\omega) + 1}{\gamma(1 - \gamma)} D_\gamma^2 + \beta < D_\gamma, \quad \forall t \in \mathbb{R}.$$

□

3 Proof of the Generalized main result

Proof of the Generalized main result.

Consider the Equation (1). By hypothesis of the Generalized main result there exist L such that

$$\max\{\|df\|_\infty, \|d^2f\|_\infty\} < L$$

Let be $\epsilon > 0$ and consider the change of variable

$$\eta(t) = \epsilon^2 \left[\mu(\epsilon t) - \Phi^{\epsilon t}(x_0) \right], \quad t \in \mathbb{R}, \quad \eta := (\eta_i)_{i=1}^n,$$

where $\Phi^t := (\Phi_i^t)_{i=1}^n$ is the flow of the system (1). Use Taylor formula

$$\dot{\eta} = \epsilon^3 [\dot{\mu}(\epsilon t) - f(\mu(\epsilon t))] + \epsilon^2 df(\mu(\epsilon t))\eta + \eta^T \zeta_\mu(t)\eta, \quad \eta(0) = \mathbb{0}_n, \quad t \in \mathbb{R},$$

where the function $\zeta_\mu : \mathbb{R} \times \mathbb{R}^{n \times n \times n}$ is a continuous function and satisfies $\|\zeta_\mu\|_\infty \leq \|d^2 f\|_\infty < \epsilon L$. By hypothesis the function $\mu : \mathbb{R} \rightarrow \mathbb{R}^n$ is a leader trajectory of f such that $\mu(0) = x_0$. Thanks to Theorem 4 there exists $\gamma_* > 0$ such that for all $\epsilon L < \gamma_*$ there exists $D_{\epsilon, L}$ such that

$$\|\mu(s) - \Phi^s(x_0)\| = \frac{1}{\gamma} \|\eta(t)\| < \frac{1}{\epsilon^2} D_{\epsilon, L}, \quad \forall s \in \mathbb{R}.$$

□

4 Leader trajectory and Rotation vector formula

We show in this Section that for any periodic function g there exists a constant vector $\rho \in \mathbb{R}^q$ such that the function $t \mapsto t\mathbb{1}_q$ is a leader trajectory of the function $t \mapsto I_\rho^{-1}g(\rho t)$. In order to prove the result let us introduce the following notations; Let $g = (g_i)_{i=1}^q : \mathbb{R}^q \rightarrow \mathbb{R}^q$ be a C^1 periodic function. Let $\Psi_i[g] : \mathbb{R}^q \times \mathbb{R} \rightarrow \mathbb{R}$ be the functions defined by

$$\begin{aligned} \Psi_i[g](z, t) &:= \int_0^t A_i(z, s) \exp \left(\int_s^t \sigma \left(I_{i,q} I_z^{-1} dg(\nu z) I_z I_{i,q} \right) d\nu \right) ds, \\ A_i(z, s) &:= \sigma \left(I_{i,q} (\mathbb{1} - I_z^{-1} g(sz)) \right), \end{aligned} \quad (23)$$

where I_z is the diagonal matrix of diagonal z . For every $i = \overline{0, q}$ the function $\Psi_i[g](z, t)$ is the solution of the following one-dimensional differential equation

$$\begin{aligned} \dot{\psi} &:= \sigma \left(I_{i,q} (\mathbb{1}_q - I_z^{-1} g(tz)) \right) + \sigma \left(I_{i,q} I_z^{-1} dg(tz) I_z I_{i,q} \right) \psi \\ \psi(0) &= 0, \quad t \in \mathbb{R}. \end{aligned} \quad (24)$$

Proposition 7. *Let $g = (g_i)_{i=1}^q : \mathbb{R}^q \rightarrow \mathbb{R}^q$ be a C^1 periodic function such*

$$\exists \beta, \gamma > 0 : \inf \{g_i(z), z \in \mathbb{R}^q, i = \overline{1, q}\} > 1 + \beta \quad \text{and} \quad \|dg\|_\infty < \gamma.$$

Then there exists $\gamma_* > 0$ such that for all $\gamma \in (0, \gamma_*)$ there exists $\rho \in \mathbb{R}^q$ such that the function $t \mapsto t\mathbb{1}_q$ is a leader trajectory of the function $t \mapsto I_\rho^{-1}g(\rho t)$. Further, ρ is solution of the following Equation

$$\lim_{t \rightarrow +\infty} \frac{1}{t} \left[\Psi_0[g](\rho, t) - \Psi_i[g](\rho, t) \right] = 0, \quad \forall i = \overline{0, q}.$$

For the proof of the following Lemma we refer to [3].

Lemma 8. *Let be $q \in \mathbb{N}^*$ and $g, h : \mathbb{R}^q \rightarrow \mathbb{R}$ be a periodic function. Let be $z \in \mathbb{R}^q$ and suppose that Suppose that*

$$\lim_{t \rightarrow +\infty} \frac{1}{t} \int_0^t g(zs) ds \leq 0,$$

then there exists $\lambda_z \in \mathbb{R}$ and a bounded function $p_z : \mathbb{R} \rightarrow \mathbb{R}$ such that the following differential equation

$$\dot{\psi} = h(zt) + g(zt)\psi, \quad \psi(0) = 0,$$

admits a solution $\psi : \mathbb{R} \rightarrow \mathbb{R}$ of the form

$$\psi(t) = \lambda_z t + p_z(t), \quad \forall t \in \mathbb{R}.$$

In particular, the Lemma is true when $g(z) = 0$ for all $z \in \mathbb{R}^q$. Further there exists $\omega > 0$ such that

$$\|p_z\| < \omega \|g\|_\infty, \quad \forall z = (z_i)_{i=1}^q, \quad z_i \geq 1, \quad \forall i = \overline{1, q}.$$

Proof of the Proposition 7. By using Lemma 8, for every $i = \overline{0, q}$ let $\tau_i : \mathbb{R}^q \rightarrow \{-1, 1\}$ defined by

$$\tau_i(z) = -\text{Sign}(\lambda_{i,z}), \quad \lambda_{i,z} := \lim_{t \rightarrow +\infty} \frac{1}{t} \int_0^t \sigma \left(I_{i,q} I_z^{-1} dg(\nu z) I_z I_{i,q} \right) d\nu, \quad (25)$$

where the function $\text{Sign} : \mathbb{R} \rightarrow \{-1, 1\}$ is defined by $\text{Sign}(t) = 1$ if $t \geq 0$ and $\text{Sign}(t) = -1$ if $t < 0$. Remark that for all $\tau \in \{-1, 1\}$ we have

$$\begin{aligned} \sigma \left(I_z^{-1} dg(\tau tz) I_z \right) &= \tau \frac{d}{dt} \left[I_z^{-1} g(\tau tz) \right], \\ \sigma \left(I_{i,q} I_z^{-1} dg(\tau tz) I_{i,q} I_z \right) &= \tau_i(z) \frac{d}{dt} \left[I_z^{-1} g(\tau tz) \right] - 2 \frac{\partial}{\partial x_i} g(\tau tz). \end{aligned}$$

By definition of the coefficients $\tau_i(z)$ in Equation (25) and by Lemma 8, there exists $\omega > 0$ such that

$$\begin{aligned} \int_0^t \sigma\left(I_z^{-1}dg(\tau\nu z)I_z\right)d\nu &< \gamma\omega, \quad \forall z \in \mathbb{R}^q, \quad t \in \mathbb{R}, \\ \int_0^t \tau_i(z)\sigma\left(I_{i,q}I_z^{-1}dg(\tau_i(z)\nu z)I_{i,q}I_z\right)d\nu &< \gamma\omega, \quad \forall z \in \mathbb{R}^q, \quad \forall t \geq 0. \end{aligned} \quad (26)$$

In order to prove that the function $t \mapsto t\mathbb{1}_q$ is a leader trajectory of the function $t \mapsto I_\rho^{-1}g(\rho t)$, we must prove that

$$\sup_{t \in \mathbb{R}} \left| \Psi_0[g](\rho, \tau_i(\rho)t) - \Psi_i[g](\rho, \tau_i(\rho)t) \right| < +\infty, \quad \forall i = \overline{0, q}.$$

Again, by Lemma 8, it is sufficient to prove that

$$\lim_{t \rightarrow +\infty} \frac{1}{t} \left[\Psi_0[g](\rho, t) - \Psi_i[g](\rho, t) \right] = 0, \quad \forall i = \overline{0, q}. \quad (27)$$

In order to simplify, for every $i = \overline{0, q}$, $z \in \mathbb{R}^q$ and $t, s \geq 0$ denote

$$\begin{aligned} \theta_i(z, t, s) &:= \exp\left(\gamma\tau_i(z) \int_s^t \sigma\left(I_{i,q}I_z^{-1}dg(\tau_i(z)\nu z)I_zI_{i,q}\right)d\nu\right), \\ \theta_0^i(z, t, s) &:= \exp\left(\gamma\tau_i(z) \int_s^t \sigma\left(I_z^{-1}dg(\tau_i(z)\nu z)I_z\right)d\nu\right), \\ \underline{\theta}_i(z, t, s) &:= \theta_0^i(z, t, s) - \theta_i(z, t, s), \\ \Theta_i(z, t) &:= \int_0^t \theta_0^i(z, t, s) + \frac{1}{2}\underline{\theta}_i(z, t, s)\sigma\left(I_{i,q}(\mathbb{1}_q - I_z^{-1}g(\tau_i(z)sz))\right)ds. \end{aligned}$$

For every fixed $k \in \mathbb{N}$, define the function $\Gamma_k = (\Gamma_{i,k})_{i=1}^q : \mathbb{R}^q \rightarrow \mathbb{R}^q$ by

$$\Gamma_{i,k}(z) := \frac{1}{\Theta_i(z, k)} \int_0^k \theta_0^i(z, k, s)g_i(\tau_i(z)sz)ds, \quad z \in \mathbb{R}^q.$$

Equation (27) is satisfied if there exists $\rho \in \mathbb{R}^q$ and a sequence $u_k \rightarrow +\infty$ such that $\lim_{k \rightarrow +\infty} \Gamma_{u_k}(\rho) = \rho$. We have

$$\Theta_i(z, k)\Gamma_{i,k}(z) - \int_0^k \theta_0^i(z, k, s)g_i(\tau_i(z)sz)ds = 0.$$

By the Mean value Theorem, for all $z \in \mathbb{R}^q$ and all $k \in \mathbb{N}$ there exists $s_{i,k}^z \geq 0$ such that

$$\theta_0^i(z, k, s_{i,k}^z) + \frac{1}{2}\underline{\theta}_i(z, k, s_{i,k}^z)\sigma\left(I_{i,q}(\mathbb{1}_q - I_z^{-1}g(s_{i,k}^z\tau_i(z)z))\right)\Gamma_{i,k}(z)$$

$$= \theta_0^i(z, k, s_{i,k}^z) g_i(s_{i,k}^z \tau_i(z) z).$$

That follows

$$\Gamma_{i,k}(z) := \frac{g_i(s_{i,k}^z \tau_i(z) z)}{1 + \zeta_i(z, k)}, \quad (28)$$

where

$$\begin{aligned} \zeta_i(z, k) &= \frac{1}{2} \frac{\theta_i(z, k, s_{i,k}^z)}{\theta_0^i(z, k, s_{i,k}^z)} \sigma \left(I_{i,q}(\mathbb{1}_q - I_z^{-1} g(s_{i,k}^z \tau_i(z) z)) \right) \\ &= \frac{1}{2} \left[1 - \frac{\theta_i(z, k, s_{i,k}^z)}{\theta_0^i(z, k, s_{i,k}^z)} \right] \sigma \left(I_{i,q}(\mathbb{1}_q - I_z^{-1} g(s_{i,k}^z \tau_i(z) z)) \right). \end{aligned} \quad (29)$$

Let be $L > 1$ and denote

$$\mathcal{V}_L = \left\{ z = (z_i)_{i=1}^q \in \mathbb{R}^q : \|z\| \leq (L-1)\|g\|_\infty \text{ and } z_i > 1, \forall i = \overline{1, q} \right\}.$$

By Equations (26), we have

$$\begin{aligned} \theta_i(z, t, s) &< \exp(\gamma\omega), \quad \forall i = \overline{1, q}, \quad \forall t \geq s \geq 0, \quad \forall z \in \mathcal{V}_L, \\ \exp(-\gamma\omega) &< \theta_0^i(z, t, s) < \exp(\gamma\omega), \quad \forall i = \overline{1, q}, \quad \forall t \geq s \geq 0, \quad \forall z \in \mathcal{V}_L. \end{aligned}$$

We have for all $t \geq 0$ and all $z \in \mathcal{V}$ the following estimation

$$|\zeta_i(z, t)| \leq \frac{1}{2} q [\exp(2\gamma\omega) - 1] [\|z\| + \|g\|_\infty] \leq \frac{1}{2} q L [\exp(2\omega) - 1] \|g\|_\infty.$$

Again, by the hypotheses of the present Proposition, there exists $\beta > 0$ such that

$$\inf\{g_i(z), \quad z \in \mathbb{R}^q, \quad i = \overline{1, q}\} > 1 + \beta$$

There exists $\gamma_* > 0$ such that for all $\gamma \in (0, \gamma_*)$ there exists $L_\gamma > 0$ such that

$$L_\gamma > \frac{2}{2 - qL_\gamma(\exp(2\gamma\omega) - 1)\|g\|_\infty} \text{ and } \frac{2(1 + \beta)}{2 + qL_\gamma(\exp(2\gamma\omega) - 1)\|g\|_\infty} > 1$$

By Equations (28) and (29), for a fixed $\gamma \in (0, \gamma_*)$ and $L_\gamma > 1$, the compact and convex set \mathcal{V}_{L_γ} satisfies

$$\Gamma_k(\mathcal{V}_{L_\gamma}) \subset \mathcal{V}_{L_\gamma}, \quad \forall k \in \mathbb{N}.$$

and the function Γ_k is C^1 on \mathcal{V}_{L_γ} . By Brouwer fixed-point theorem, for all $k \in \mathbb{N}$ there exists $\rho_k \in \mathcal{V}_{L_\gamma}$ such that $\Gamma_k(\rho_k) = \rho_k$. There exists $\rho \in \mathcal{V}_{L_\gamma}$ and there exists a sequence $(u_k)_{k \in \mathbb{N}} \subset \mathbb{N}$ such that

$$\lim_{k \rightarrow +\infty} \rho_{u_k} = \rho \text{ and } \lim_{k \rightarrow +\infty} \frac{1}{u_k} \Gamma_{u_k}(\rho, u_k) = \rho.$$

Otherwise,

$$\lim_{k \rightarrow +\infty} \frac{1}{u_k} [\Psi_0[g](\rho, u_k) - \Psi_i[g](\rho, u_k)] = 0.$$

Hence, Equation (27) is satisfied. By Equation (26) of this proof, we deduce that the the function $t \mapsto t\mathbb{1}_q$ is a leader trajectory of the function $t \mapsto I_\rho^{-1}g(\rho t)$.

□

5 Proof of the Main result

Proof of the Main Result. Let x be the solution of the system 1 and suppose that f is periodic. By periodicity, there exists $\omega > 0$ such that

$$\max \left\{ \|f\|_\infty, \|df\|_\infty, \|d^2f\|_\infty \right\} < \omega. \quad (30)$$

Let be $c, \gamma > 0$, and consider the following change of variable

$$\tilde{x}(t) := ct\mathbb{1}_n + x(\gamma t) - x_0, \quad \forall t \in \mathbb{R}.$$

System 1, becomes

$$\dot{\tilde{x}} = c + \gamma \tilde{f}_{x_0}(\tilde{x}, t), \quad \tilde{x}(0) = \mathbb{0}_q, \quad t \in \mathbb{R}.$$

$$\tilde{f}_{x_0}(z, z_{n+1}) = c + \gamma f(z + x_0 - cz_{n+1}\mathbb{1}_n), \quad \forall z = (z_i)_{i=1}^{n+1}.$$

By Equation (30) we have

$$\begin{aligned} \forall (z, t) \in \mathbb{R}^n \times \mathbb{R}: \quad & \max \left\{ \left\| \frac{d}{dz} \tilde{f}_{x_0}(z, t) \right\|, \left\| \frac{d^2}{dz^2} \tilde{f}_{x_0}(z, t) \right\| \right\} < \gamma\omega, \\ & \tilde{f}_{x_0}(z, z_{n+1}) > c - \gamma\omega, \quad \forall z = (z_i)_{i=1}^{n+1}. \end{aligned}$$

Let be $\beta > 0$ fixed then for all $c > 1 + \beta + \gamma\omega$ we have

$$\tilde{f}_{x_0}(z, z_{n+1}) > 1 + \beta, \quad \forall z = (z_i)_{i=1}^{n+1}. \quad (31)$$

Thanks to the Proposition 7, there exist $\gamma > 0$ and $\rho_* \in \mathbb{R}^n$ such that the function $\mu(t) := t\mathbb{1}_n$ is a leader trajectory of the function $I_{\rho_*}^{-1}\tilde{f}_{x_0}$. Since $\tilde{x}_0 = \mathbb{0}_n$ and by the Generalized main result we have

$$\sup_{t \in \mathbb{R}} \|t\mathbb{1}_n - I_{\rho_*}^{-1}\tilde{x}(t)\| < +\infty,$$

by definition of \tilde{x} we have

$$\sup_{t \in \mathbb{R}} \left\| \frac{1}{\gamma} [(\rho_* - c)t \mathbb{1}_n - x_0 + x(t)] \right\| < +\infty,$$

Otherwise, the solution x of the Equation (1) admits a rotation vector $\rho := (\rho_* - c)/\gamma$. Again by Proposition 7 and by the uniqueness of the rotation vector the vector we deduce that ρ_* is the unique solution of the following equation

$$\lim_{t \rightarrow +\infty} \frac{1}{t} \left[\Psi_0[\tilde{f}_{x_0}](\rho_*, t) - \Psi_i[\tilde{f}_{x_0}](\rho_*, t) \right] = 0, \quad \forall i = \overline{1, n}$$

where the functions $\Psi_i[\tilde{f}_{x_0}]$ are defined in Equation (23) of the previous Section. Since $\rho_* = \rho\gamma + c$ and by using the change of variable $t \rightarrow \gamma t$ then it is equivalent

$$\lim_{t \rightarrow +\infty} \frac{1}{t} \left[\Psi_0[f_{x_0}](\rho_*, c, t) - \Psi_i[f_{x_0}](\rho_*, c, it) \right] = 0, \quad \forall i = \overline{1, n},$$

where

$$\begin{aligned} \Psi_i[f_{x_0}](\rho, c, t) &:= \int_0^t A_i(\rho, s) \exp \left(\int_s^t \sigma \left(I_{i,q} I_{\rho_*}^{-1} df(\nu\rho + x_0) I_{\rho_*} I_{i,q} \right) d\nu \right) ds, \\ A_i(\rho, c, s) &:= \sigma \left(I_{i,q} I_{\rho_*}^{-1} (\rho \mathbb{1} - f(s\rho + x_0)) \right), \end{aligned}$$

By Equation (31) the coefficient c is arbitrary in $(1 + \beta + \gamma\omega, +\infty)$, since ρ is unique, we get

$$\lim_{c \rightarrow \infty} \lim_{t \rightarrow +\infty} \frac{c}{t} \left[\Psi_0[\tilde{f}_{x_0}](\rho_*, c, t) - \Psi_i[\tilde{f}_{x_0}](\rho_*, c, t) \right] = 0, \quad \forall i = \overline{1, n}.$$

Finally, by Lemma 8 of the previous Section, we deduce that ρ is the unique solution of the following Rotation vector formula

$$\lim_{t \rightarrow +\infty} \frac{1}{t} \left[\Psi_0[f_{x_0}](\rho, t) - \Psi_i[f_{x_0}](\rho, t) \right] = 0, \quad \forall i = \overline{1, n},$$

where

$$\begin{aligned} \Psi_i[f_{x_0}](\rho, t) &:= \int_0^t A_i(\rho, s) \exp \left(\int_s^t \sigma \left(I_{i,q} df(\nu\rho + x_0) I_{i,q} \right) d\nu \right) ds, \\ A_i(\rho, s) &:= \sigma \left(I_{i,q} (\rho - f(s\rho + x_0)) \right). \end{aligned}$$

□

6 Conclusion and open problem

We have proved that for any periodic field f and any initial condition the solution of the system admits a unique rotation vector which is the solution of the Rotation vector formula (4). We conjecture that the solution x of initial condition x_0 and rotation vector ρ_x is periodic on the torus if and only if there exists $T > 0$ such that $T\rho_x \in \mathbb{Q}^n$ and $\tilde{f}_0(\rho, T) - \tilde{f}_i(\rho, T) = 0$ for every $i = \overline{1, q}$. Consider the perturbed system

$$\dot{x} = f(x) + \zeta(t), \quad t \in \mathbb{R}, \quad x(0) = x_0 \in \mathbb{R}^n,$$

where $\zeta : \mathbb{R} \rightarrow \mathbb{R}^n$ is an integrable function. By the definition of the diagonal matrix $I \in \mathcal{K}_n^+$ we deduce that if f is periodic then the rotation vector is locally constant for a small perturbation if and only if

$$\sup_{t \in \mathbb{R}} \left\| \int_0^t \exp \left(\int_s^t \sigma \left(Idf(\rho\nu + x_0)I \right) d\nu \right) I \zeta(s) ds \right\| < +\infty, \quad \forall I \in \mathcal{K}_n^+,$$

which gives information about the Arnold Tongue. The leader trajectory defined in this article is an approximated solution.

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