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Ordinary differential equations defined by a trigonometric polynomial field: Behavior of the solutions

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Abstract

We consider the ordinary differential equations defined by a trigonometric polynomial field, we prove that any solution x admits a *rotation vector* $\rho \in \mathbb{R}^n$. More precisely, the function $t \mapsto x(t) - \rho t$ is bounded on time and it is a *weak almost periodic* function of *slope* ρ .

Keywords: Periodic system, differential equation, rotation vector, rotation number, almost-periodic function, trigonometric polynomial.

AMS subject classifications: 34D05, 37B65, 34C15.

1 Introduction

In this article, we study the asymptotic behavior of solutions for ordinary differential equations (ODE) defined by a trigonometric polynomial field. The idea comes from the scalar case, where in this case H. Poincaré defined the *rotation number* for circle homeomorphisms [5]. The simple example is a scalar differential equation

$$\dot{x} = f(x), \quad x(0) \in \mathbb{R}, \quad t \in \mathbb{R},$$

where $f : \mathbb{R} \rightarrow \mathbb{R}$ is lipschitz, 1-periodic and $t \mapsto x(t)$ is the state of the system. There exists a rotation number $\lambda \in \mathbb{R}$ for which the function $t \mapsto x(t) - \lambda t$ is bounded (periodic). We know that any non-autonomous

system can be written as an autonomous system. Our result is a generalization of this asymptotic behavior to any dimension. In this case, λ is a vector and called a *rotation vector* or *rotation set* as it is defined in [4]. Under some assumptions of stability [[7], [2]] proved the existence of the rotation vector. Some biological works use the ODE defined by a trigonometric polynomial field and study the rotation vector components as in [[1], [3], [6], [8]]. Our contribution to this biological works has two key points, the mathematical proof of existence of the rotation vector and the study of the behavior of solutions.

2 Definition and Main result

We study in this article the following system

$$\dot{x} = f(x), \quad t \in \mathbb{R}, \quad x(0) = x_0 \in \mathbb{R}^n,$$

where $t \mapsto x(t) := (x_j(t))_{j=1}^n$ is the state of the system and $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a trigonometric polynomial in the following sense

Definition 1. [Trigonometric polynomial function] A function $g : \mathbb{R}^n \rightarrow \mathbb{R}$ is called a *trigonometric polynomial* if there exists a finite sequence $(c_p)_{p \in 2\pi\mathbb{Z}^n} \subset \mathbb{C}$ such that

$$\forall x \in \mathbb{R}^n : \quad g(x) = \sum_{p \in 2\pi\mathbb{Z}^n} c_p \exp(i\langle x, p \rangle),$$

where $\langle \cdot, \cdot \rangle$ is the usual scalar product on \mathbb{R}^n . A function $g : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a *trigonometric polynomial* if each component is a trigonometric polynomial function.

To formulate the Mains results let us introduce the following definitions. We use the usual norm $\|y\| := \max_{1 \leq j \leq n} \|y_j\|$ for every $y := (y_j)_{j=1}^n \in \mathbb{C}^n$.

Definition 2. [Rotation vector] Let $\lambda \in \mathbb{R}^n$ and $\phi : \mathbb{R} \rightarrow \mathbb{R}^n$ be a function. We say that ϕ admits λ as the *rotation vector* if

$$\sup_{t \in \mathbb{R}} \|\phi(t) - \lambda t\| < \infty.$$

For more information about the behavior of solutions, we introduce the following definitions.

Definition 3. [Periodic modulo \mathbb{Z}^n function] A function $g : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is called *periodic modulo \mathbb{Z}^n* , if

$$g(z_1 + k_1, \dots, z_n + k_n) = g(z_1, \dots, z_n), \quad \forall (k_j)_{j=1}^n \in \mathbb{Z}^n, \quad \forall (z_j)_{j=1}^n \in \mathbb{R}^n.$$

Definition 4. [Weakly almost-periodic function] Let be $r \in \mathbb{R}^n$. A function $h : \mathbb{R} \rightarrow \mathbb{R}^n$ is *weakly almost periodic of slope r* if it is C^∞ and if there exists a uniformly bounded sequence for the sup-norm of C^∞ functions $(g_k : \mathbb{R} \rightarrow \mathbb{R}^n)_{k \in \mathbb{N}}$ that are periodic modulo \mathbb{Z}^n and there exists a sequence $(r_k)_{k \in \mathbb{N}} \subset \mathbb{Q}^n$ such that

$$\lim_{k \rightarrow \infty} r_k = r \quad \text{and} \quad \forall t > 0 : \lim_{k \rightarrow \infty} \sup_{s \in [-t, t]} \|g_k(r_k s) - h(s)\| = 0.$$

We call the sequence $(g_k)_k$ the \mathbb{Z}^n -*periodic sequence of the function h* .

Remark 5. Remark that for every $k \in \mathbb{N}$ the function $s \mapsto g_k(r_k s)$ is a periodic function.

Main Result. *Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a trigonometric polynomial function. For every $x_0 \in \mathbb{R}^n$ the unique solution $x : \mathbb{R} \rightarrow \mathbb{R}^n$ of the differential equation*

$$\dot{x} = f(x), \quad t \in \mathbb{R}, \quad x(0) = x_0, \quad (1)$$

admits a rotation vector $\rho \in \mathbb{R}^n$. In addition, the function

$$t \mapsto x(t) - \rho t,$$

is weakly almost periodic of slope ρ .

3 Space of C^∞ periodic modulo \mathbb{Z}^n functions

We define in this Section the space and the norm used to prove the Main result. The proofs of Lemmas for this Section are left in Appendix. In order to use the Fourier development, let us introduce the following notation.

Notation 6. For every continuous function $g : \mathbb{R}^n \rightarrow \mathbb{R}^n$ and every $p \in 2\pi\mathbb{Z}^n$ we denote $a_p[g] \in \mathbb{C}^n$ the following limit if it exists

$$a_p[g] := \lim_{t \rightarrow +\infty} \frac{1}{(2t)^n} \int_{-t}^t \dots \int_{-t}^t g(z) \exp(-i\langle z, p \rangle) dz_1 \dots dz_n.$$

In this Section and the Section (4), for every function $g : \mathbb{R}^n \rightarrow \mathbb{R}^n$ and every $\alpha \in \mathbb{R}$, we denote g_α the function defined as $g_\alpha(z) := g(\alpha z)$ for all $z \in \mathbb{R}^n$. The following constant ω will be used as change of variable in order to find a contraction in Lemmas 15 and 16 of Section 4. For every $\omega \in \mathbb{N}^*$, we denote $E_\omega(\mathbb{R}^n)$ the set of C^∞ function $g : \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that g_ω is a periodic modulo \mathbb{Z}^n function. We remark, for every $\omega \in \mathbb{N}^*$ and $g \in E_\omega(\mathbb{R}^n)$ that

$$a_p[g_\omega] = \int_0^1 \dots \int_0^1 g_\omega(z) \exp(-i\langle z, p \rangle) dz_1 \dots dz_n, \quad (2)$$

which is the Fourier coefficient of the function g_ω . Since $g \in E_\omega(\mathbb{R}^n)$ is C^∞ , by Dirichlet Theorem,

$$g(z) = \sum_{p \in 2\pi\mathbb{Z}^n} a_p[g_\omega] \exp(i\frac{1}{\omega}\langle z, p \rangle), \quad (3)$$

$$\forall q \geq 0 : \sum_{p \in 2\pi\mathbb{Z}^n} \|a_p[g_\omega]\| \|\frac{p}{\omega}\|^q < +\infty.$$

We are now in position to define the following seminorm in $E_\omega(\mathbb{R}^n)$: Let $\omega \in \mathbb{N}^*$ and $g \in E_\omega(\mathbb{R}^n)$, we denote for every $\omega \in \mathbb{N}^*$ and $q \geq 0$

$$\|g\|_{\omega, q} := 2 \sum_{p \in 2\pi\mathbb{Z}^n / 0} \|a_p[g_\omega]\| \|\frac{p}{\omega}\|^q,$$

where $0 := (0, \dots, 0) \in \mathbb{R}^n$ and where we recall that $\|y\| := \max_{1 \leq j \leq n} \|y_j\|$ for every $y := (y_j)_{j=1}^n \in \mathbb{C}^n$. We prove in the following Lemma that a periodic modulo \mathbb{Z}^n function g is C^∞ if it is *uniformly bounded for the seminorm*, i.e

$$\forall q \geq 0 : \|g\|_{\omega, q} < +\infty,$$

In other words, the set $E_\omega(\mathbb{R}^n)$ is include in the set of the periodic modulo \mathbb{Z}^n functions uniformly bounded for the seminorm.

Lemma 7. *Let $\omega \in \mathbb{N}^*$. Let $(c_p)_p$ be a complex-valued family such that*

$$\forall q \geq 0 : \sum_{p \in 2\pi\mathbb{Z}^n / 0} \|c_p\| \|\frac{p}{\omega}\|^q < +\infty.$$

Then the following series is normally convergent

$$g(z) := \sum_{p \in 2\pi\mathbb{Z}^n} c_p \exp(i\frac{1}{\omega}\langle z, p \rangle), \quad c_p \in \mathbb{C}^n,$$

and $c_p = a_p[g_\omega]$ for every $p \in 2\pi\mathbb{Z}^n$. Further, $g \in E_\omega(\mathbb{R}^n)$.

Proof. Appendix. A □

In the following Lemma we prove that the seminorm $\|\cdot\|_{\omega,0}$ is a norm on the space $\{g \in E_{\omega}(\mathbb{R}^n) : g(0) = 0\}$ and we compare it to the uniform norm topology.

Lemma 8. *Let be $\omega \in \mathbb{N}^*$ and $g \in E_{\omega}(\mathbb{R}^n)$ such that $g(0) = 0$ then*

$$\|a_{\mathbb{0}}[g_{\omega}]\| \leq \frac{1}{2}\|g\|_{\omega,0} \quad \text{and} \quad \|g\|_{\infty} \leq \|g\|_{\omega,0}.$$

Proof. Appendix. B □

We denote $d^k g$ the k^{th} differential of a function $g : \mathbb{R}^n \rightarrow \mathbb{R}^n$. The following Lemma gives an upper-bound of the quantity $\|d^k g\|_{\omega,q}$ when g is a trigonometric polynomial. We recall that $g_{\frac{\perp}{\omega}}(z) := g(\frac{z}{\omega})$.

Lemma 9. *Let $g : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a trigonometric polynomial function. Then there exists $\beta := \beta(g) > 0$ such that for every $\omega \in \mathbb{N}^*$ we have*

$$\|d^k g_{\frac{\perp}{\omega}}\|_{\omega,q} < n^k \beta \left(\frac{\beta}{\omega}\right)^{q+k}, \quad \forall q, k \geq 0.$$

Proof. Appendix. C □

We end this Section by the following inequality.

Lemma 10. *Let $\omega \in \mathbb{N}^*$ and $(h_j \in E_{\omega}(\mathbb{R}^n))_{j=1}^k$. Then*

$$\forall q \in \mathbb{N}, \forall k \in \mathbb{N}^* : \|\Pi_{j=1}^k h_j\|_{\omega,q} \leq \frac{(k\omega^{k-1})^q}{2^{k-1}} \Pi_{j=1}^k [\|2a_{\mathbb{0}}[h_{j,\omega}]\| + \|h_j\|_{\omega,q}].$$

Proof. Appendix. D □

4 Main proposition

The Main result affirms that the solution x of Equation (1) is a sum of a linear part and a bounded part. The strategy to prove the Main result is to approximate the bounded part of x by a \mathbb{Z}^n -periodic sequence. Using the Fourier development and Equation (3), remark, for every C^{∞} periodic modulo \mathbb{Z}^n function g that

$$f(z + g(z)) = \sum_{p \in 2\pi\mathbb{Z}^n} a_p[H[g]] \exp(i\langle z, p \rangle), \quad H[g](z) := f(z + g(z)),$$

under some convergence assumption of the series, by integration we get

$$\begin{aligned} \forall v \in \mathbb{R}^n : \quad & \int_0^t f(vs + g(vs))ds = t \sum_{p \in 2\pi\mathbb{Z}^n, \langle v, p \rangle = 0} a_p[H[g]] \\ & + \sum_{p \in 2\pi\mathbb{Z}^n, \langle v, p \rangle \neq 0} \frac{a_p[H[g]]}{i\langle v, p \rangle} \left(\exp(i\langle v, p \rangle t) - 1 \right). \end{aligned}$$

The last term of the right member of the last equality will play the role of \mathbb{Z}^n -periodic sequence of the bounded part of the solution x of Equation (1). In order to find an upper-bound of the bounded part, let us introduce the following notations.

Notation 11. Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a trigonometric polynomial. We denote the finite subset $\Lambda_f \subset 2\pi\mathbb{Z}^n$ as

$$\Lambda_f := \{p \in 2\pi\mathbb{Z}^n : \|a_p[f]\| \neq 0\},$$

and we denote

$$|\Lambda_f| := \max\{\|p\|, \quad p \in \Lambda_f\}.$$

Let be $y \in \mathbb{R}^n/\{0\}$. Define,

$$\Lambda(f, y) := \{p \in 2\pi\mathbb{Z}^n : \|p\| \leq 2\pi + |\Lambda_f|, \langle y, p \rangle \neq 0\},$$

Remark that $\Lambda(f, y) \neq \emptyset$. We denote

$$\tau(f, y) := \max \left\{ \frac{1}{|\langle y, p \rangle|} : p \in \Lambda(f, y) \right\}.$$

Let be $y \in \mathbb{Q}^n/\{0\}$, we denote

$$\tau(y) = \max \left\{ \frac{1}{|\langle y, p \rangle|} : p \in 2\pi\mathbb{Z}^n, \langle y, p \rangle \neq 0 \right\}.$$

We denote β the constant $\beta(f)$ of the function f defined in Lemma 9.

In the following Proposition we prove that the bounded part of the solution x of Equation (1) can be approximated by a \mathbb{Z}^n -periodic functions and we find an appropriate upper-bound.

Proposition 12. *[Main proposition] Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a trigonometric polynomial function. Then for every $r \in \mathbb{Q}^n/\{0\}$ and every $\epsilon > 0$*

there exists a C^∞ periodic modulo \mathbb{Z}^n function $\phi_{r,\epsilon} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that $\|\phi_{r,\epsilon}\|_\infty < 2\beta\tau(f, r)$ and such that

$$\sup_{z \in \mathbb{R}^n} \|\phi_{r,\epsilon}(z) - \sum_{p \in 2\pi\mathbb{Z}^n, \langle r,p \rangle \neq 0} \frac{a_p[H[\phi_{r,\epsilon}]]}{i\langle r,p \rangle} (\exp(i\langle z,p \rangle) - 1)\| < \epsilon,$$

where

$$H[\phi_{r,\epsilon}](z) := f(z + \phi_{r,\epsilon}(z)), \quad \forall z \in \mathbb{R}^n.$$

As is state in the above Section, the following constant ω is used as change of variable in order to find a contraction.

Definition 13. For every $r \in \mathbb{R}^n$ and every $\omega \in \mathbb{N}^*$, define the set $K_{r,\omega}$ as $g \in K_{r,\omega}$ if

- there exists a complex-valued family $(c_p)_{p \in 2\pi\mathbb{Z}^n}$ such that

$$g(z) = \sum_{p \in 2\pi\mathbb{Z}^n} c_p (\exp(i\frac{1}{\omega}\langle z,p \rangle) - 1), \quad \forall z \in \mathbb{R}^n,$$

- $\|g\|_{\omega,0} \leq 2\beta\tau(f, r)$,
- $\|g\|_{\omega,q} < \infty$ for every $q \geq 1$.

Lemma 14. The set $K_{r,\omega}$ is a nonempty subset of $E_\omega(\mathbb{R}^n)$.

Proof. The set $K_{r,\omega} \neq \emptyset$ because it contains the function $z \mapsto g(z) = 0$. By definition of $K_{r,\omega}$ and by Lemma 7 the function g is C^∞ . \square

For every $r \in \mathbb{R}^n$, for every $\omega \in \mathbb{N}^*$, and every $g \in K_{r,\omega}$ let be $\Psi[r,\omega,g]$ the function defined by the following series in its convergence domain

$$\forall z \in \mathbb{R}^n : \quad \Psi[r,\omega,g](z) := \sum_{p \in 2\pi\mathbb{Z}^n, \langle r,p \rangle \neq 0} \frac{a_p[H[\omega,g]]}{i\langle r,p \rangle} (\exp(i\frac{1}{\omega}\langle z,p \rangle) - 1),$$

where

$$H[\omega,g](z) := f\left(z + g(\omega z)\right).$$

Since f is a real polynomial trigonometric function then $\Psi[r,\omega,g](z) \in \mathbb{R}^n$ for every $z \in \mathbb{R}^n$ such that the series converge. In the following Lemma we prove that $K_{r,\omega}$ is invariant under the operator $\Psi[r,\omega, \cdot]$. We deduce that $\Psi[r,\omega,g](z)$ is defined for every $g \in K_{r,\omega}$ and $z \in \mathbb{R}^n$.

Lemma 15. *Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a trigonometric polynomial function. For every $r \in \mathbb{Q}^n / \{0\}$, there exists $\omega_r > 0$ such that for every $\omega > \omega_r$ we have*

$$g \in K_{r,\omega} \implies \Psi[r, \omega, g] \in K_{r,\omega}.$$

In addition, $\Psi[r, \omega, g]$ is defined for every $z \in \mathbb{R}^n$.

Proof. Prove that

$$\forall g \in K_{r,\omega} : \quad \|\Psi[r, \omega, g]\|_{\omega,q} < \infty, \quad \forall q \geq 0.$$

Let be $g \in K_{r,\omega}$ and denote

$$\tilde{H}[\omega, g](z) := f\left(z + g(\omega z)\right) - f(z).$$

We have

$$a_p[H[\omega, g]] = a_p[f] + a_p[\tilde{H}[\omega, g]].$$

By definition of $\Psi[r, \omega, \cdot]$ for every $g \in K_{r,\omega}$,

$$\begin{aligned} \Psi[r, \omega, g](z) &= \sum_{p \in 2\pi\mathbb{Z}^n, \langle r, p \rangle \neq 0} \frac{a_p[f]}{i\langle r, p \rangle} (\exp(i\frac{1}{\omega}\langle z, p \rangle) - 1) \\ &+ \sum_{p \in 2\pi\mathbb{Z}^n, \langle r, p \rangle \neq 0} \frac{a_p[\tilde{H}[\omega, g]]}{i\langle r, p \rangle} (\exp(i\frac{1}{\omega}\langle z, p \rangle) - 1) \end{aligned}$$

Recall that $|\Lambda_f|$ is defined on the Notations 11: For every $p \in 2\pi\mathbb{Z}^n$ such that $\langle r, p \rangle \neq 0$ and such that $\|p\| > |\Lambda_f|$ we have $a_p[f] = 0$. By definition of $\tau(f, r)$ and $\tau(r)$ in Notation 11, we get

$$\begin{aligned} \|\Psi[r, \omega, g]\|_{\omega,q} &\leq 2\tau(f, r) \sum_{p \in 2\pi\mathbb{Z}^n, \langle r, p \rangle \neq 0} \|a_p[f]\| \left\| \frac{p}{\omega} \right\|^q \\ &+ 2\tau(r) \sum_{p \in 2\pi\mathbb{Z}^n, \langle r, p \rangle \neq 0} \|a_p[\tilde{H}[\omega, g]]\| \left\| \frac{p}{\omega} \right\|^q. \end{aligned}$$

By definition of the seminorm,

$$\|\Psi[r, \omega, g]\|_{\omega,q} \leq \tau(f, r) \|f_{\frac{1}{\omega}}\|_{\omega,q} + \tau(r) \left\| \left(\tilde{H}[\omega, g] \right)_{\frac{1}{\omega}} \right\|_{\omega,q}. \quad (4)$$

By Lemma 9, we have

$$\|f_{\frac{1}{\omega}}\|_{\omega,q} \leq \frac{\beta^{q+1}}{\omega^q}, \quad \forall q \geq 0. \quad (5)$$

Now, estimate the quantity $\|(\tilde{H}[\omega, g])_{\frac{1}{\omega}}\|_{\omega, q}$. By definition,

$$\left(\tilde{H}[\omega, g]\right)_{\frac{1}{\omega}}(z) = f\left(\frac{1}{\omega}z + g(z)\right) - f\left(\frac{1}{\omega}z\right) = \sum_{k=1}^{\infty} \frac{d^k f\left(\frac{z}{\omega}\right)}{k!} (g(z))^{(k)},$$

where

$$d^k f\left(\frac{z}{\omega}\right)(g(z))^{(k)} := \sum_{i_1, \dots, i_k=1}^n \frac{\partial f}{\partial z_{i_1} \dots \partial z_{i_k}}\left(\frac{z}{\omega}\right) g_{i_1}(z) \dots g_{i_k}(z).$$

Since f is polynomial trigonometric function, then

$$\forall z \in \mathbb{R}^n : f_{\frac{1}{\omega}}(z) = f\left(\frac{z}{\omega}\right) = a_0[f] + \sum_{p \in 2\pi\mathbb{Z}^n, p \neq 0} a_p[f] \exp\left(i\frac{1}{\omega}\langle z, p \rangle\right),$$

then for every $k \geq 1$ we get

$$\begin{aligned} d^k f_{\frac{1}{\omega}}(z) &= \sum_{p \in 2\pi\mathbb{Z}^n, p \neq 0} a_p[f] d^k \left[\exp\left(i\frac{1}{\omega}\langle z, p \rangle\right) \right] \\ &= \sum_{p \in 2\pi\mathbb{Z}^n, p \neq 0} \left(a_p[f] \sum_{s_1=1}^n \dots \sum_{s_k=1}^n i^k \frac{p_{s_1}}{\omega} \dots \frac{p_{s_k}}{\omega} \right) \exp\left(i\frac{1}{\omega}\langle z, p \rangle\right), \end{aligned}$$

By Notation 6

$$a_p[d^k f_{\frac{1}{\omega}}] := \lim_{t \rightarrow +\infty} \frac{1}{(2t)^n} \sum_{p \in 2\pi\mathbb{Z}^n, p \neq 0} \left(a_p[f] \sum_{s_1=1}^n \dots \sum_{s_k=1}^n i^k \frac{p_{s_1}}{\omega} \dots \frac{p_{s_k}}{\omega} \right) \theta(\omega, -t, t),$$

where

$$\forall t \in \mathbb{R} : \theta(\omega, t, -t) := \int_{-t}^t \dots \int_{-t}^t \exp\left(i\frac{1}{\omega}\langle z, p \rangle\right) dz_1 \dots dz_n.$$

Since $p \neq 0$, then

$$\lim_{t \rightarrow +\infty} \frac{1}{(2t)^n} \theta(\omega, t, -t) = 0,$$

we deduce that

$$\|a_0[d^k f_{\frac{1}{\omega}}]\| = 0, \quad \forall k \geq 1.$$

Since $g \in K_{r, \omega}$ then $g(0) = 0$. Thanks to Lemma 8 we obtain

$$\|a_0[g]\| \leq \|g\|_{\omega, 0} \leq \alpha_0 := 2\beta\tau(f, r).$$

By Lemma 10, we have for all $q \geq 0$

$$\begin{aligned} \left\| \left(\tilde{H}[\omega, g] \right)_{\frac{1}{\omega}} \right\|_{\omega, q} &\leq \sum_{k=1}^{\infty} \frac{1}{k!} \left\| (d^k f_{\frac{1}{\omega}})(g)^{(k)} \right\|_{\omega, q} \\ &\leq \sum_{k=1}^{\infty} \frac{(k+1)^q \omega^{kq}}{k! 2^k} \left\| d^k f_{\frac{1}{\omega}} \right\|_{\omega, q} (\|g\|_{\omega, q} + \alpha_0)^k. \end{aligned}$$

By Lemma 9, we find

$$\forall q \geq 0 : \quad \left\| \left(\tilde{H}[\omega, g] \right)_{\frac{1}{\omega}} \right\|_{\omega, q} \leq \sum_{k=1}^{\infty} \frac{\exp(qk) \omega^{kq}}{k! 2^k} n^k \frac{\beta^{k+q+1}}{\omega^{k+q}} [\alpha_0 + \|g\|_{\omega, q}]^k.$$

By hypothesis $g \in K_{r, \omega}$, then

$$\forall q \geq 0, \exists \alpha_q > 0 : \quad \|g\|_{\omega, q} \leq \alpha_q \text{ where } \alpha_0 := 2\beta\tau(f, r).$$

we deduce that for all $q \geq 0$,

$$\begin{aligned} \left\| \left(\tilde{H}[\omega, g] \right)_{\frac{1}{\omega}} \right\|_{\omega, q} &\leq \sum_{k=1}^{\infty} \frac{\exp(qk) \omega^{kq}}{k! 2^k} n^k \frac{\beta^{k+q+1}}{\omega^{k+q}} (\alpha_0 + \alpha_q)^k \\ &= \frac{\beta^{q+1}}{\omega^q} \sum_{k=1}^{\infty} \frac{1}{k!} \left(n \exp(q) \omega^q \frac{\beta}{2\omega} (\alpha_0 + \alpha_q) \right)^k \\ &\leq \frac{\beta^{q+1}}{\omega^q} \left[\exp \left(n \exp(q) \omega^q \frac{\beta}{2\omega} (\alpha_0 + \alpha_q) \right) - 1 \right] < \infty. \end{aligned}$$

By Equations (5) and (4), we obtain

$$\forall g \in K_{r, \omega} : \quad \|\Psi[r, \omega, g]\|_{\omega, q} < \infty, \quad \forall q \geq 0.$$

Choose $\omega > \omega_r > 0$, where $\omega_r \in \mathbb{N}^*$ satisfies

$$\beta \left[\exp \left(n \frac{\beta}{\omega_r} \alpha_0 \right) - 1 \right] < \frac{\beta\tau(f, r)}{\tau(r)}.$$

We obtain

$$\forall \omega > \omega_r : \quad \left\| \left(\tilde{H}[\omega, g] \right)_{\frac{1}{\omega}} \right\|_{\omega, 0} < \frac{\beta\tau(f, r)}{\tau(r)}. \quad (6)$$

Replace both Equations (5) and (6) on Equation (4), we obtain

$$\forall \omega > \omega_r : \quad \|\Psi[r, \omega, g]\|_{\omega, 0} < \beta\tau(f, r) + \beta\tau(f, r) = 2\beta\tau(f, r) = \alpha_0.$$

□

Lemma 16. *Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a trigonometric polynomial function. For every $r \in \mathbb{Q}^n / \{0\}$ there exists $\omega := \omega(r) > 0$ such that for every $\epsilon > 0$, there exists $\phi_{r,\omega,\epsilon} \in K_{r,\omega}$ satisfying*

$$\|\phi_{r,\omega,\epsilon} - \Psi[r, \omega, \phi_{r,\omega,\epsilon}]\|_\infty < \epsilon.$$

Proof. Let be $\omega > \beta$ and $h, g \in K_{r,\omega}$. For every fixed $s \in [0, 1]$, define the function $V_s \in K_{r,\omega}$ as

$$V_s(z) := sh(z) + (1-s)g(z), \quad \forall z \in \mathbb{Z}^n.$$

$$\begin{aligned} \forall z \in \mathbb{R}^n : \quad \Psi[r, \omega, h](z) - \Psi[r, \omega, g](z) &= \Psi[r, \omega, V_1(z)] - \Psi[r, \omega, V_0(z)] \\ &= \int_0^1 \frac{d}{ds} \Psi[r, \omega, V_s(z)] ds. \end{aligned}$$

By definition of Ψ we get

$$\begin{aligned} &\Psi[r, \omega, h](z) - \Psi[r, \omega, g](z) \\ &= \int_0^1 \sum_{p \in 2\pi\mathbb{Z}^n, \langle r, p \rangle \neq 0} \frac{\frac{d}{ds} a_p[H[\omega, V_s]]}{i\langle r, p \rangle} (\exp(i\frac{1}{\omega}\langle z, p \rangle) - 1) ds. \end{aligned}$$

We have

$$\frac{d}{ds} a_p[H[\omega, V_s]] = a_p[\frac{d}{ds} H[\omega, V_s]], \quad H[\omega, V_s](z) := f\left(z + V_s(\omega z)\right).$$

Since $\frac{d}{ds} V_s = h - g$, then

$$\begin{aligned} \frac{d}{ds} H[\omega, V_s](z) &= df\left(z + V_s(\omega z)\right) \frac{d}{ds} V_s(\omega z) \\ &= df\left(z + V_s(\omega z)\right) [h - g](\omega z). \end{aligned}$$

Then

$$a_p[\frac{d}{ds} H[\omega, V_s]] = a_p\left[\phi[\omega, V_s](h - g)_\omega\right],$$

where

$$\phi[\omega, V_s](z) := df\left(z + V_s(\omega z)\right). \quad (7)$$

For every fixed $s \in [0, 1]$ we have $\phi[\omega, V_s] \in K_{r,\omega}$. By Lemma 10, for every fixed $s \in [0, 1]$ we have

$$\left\| \left(\phi[\omega, V_s](h - g)_\omega \right)_{\frac{1}{\omega}} \right\|_{\omega,0} \leq \left\| \left(\phi[\omega, V_s] \right)_{\frac{1}{\omega}} \right\|_{\omega,0} \|h - g\|_{\omega,0}.$$

Then

$$\|\Psi[r, \omega, h] - \Psi[r, \omega, g]\|_{\omega, 0} \leq \tau(r) \left(\sup_{s \in [0, 1]} \left\| \left(\phi[\omega, V_s] \right)_{\frac{1}{\omega}} \right\|_{\omega, 0} \right) \|h - g\|_{\omega, 0}. \quad (8)$$

Prove that there exists $\omega_r > 0$ such that for every $\omega > \omega_r$ we have

$$\forall g, h \in K_{r, \omega} : \quad \tau(r) \left(\sup_{s \in [0, 1]} \left\| \left(\phi[\omega, V_s] \right)_{\frac{1}{\omega}} \right\|_{\omega, 0} \right) < \frac{1}{2}. \quad (9)$$

As in Proof of Lemma 15: By Equation (7) we have for every $s \in [0, 1]$,

$$\left(\phi[\omega, V_s] \right)_{\frac{1}{\omega}} = \sum_{k=0}^{\infty} \frac{d^{k+1} f_{\frac{1}{\omega}}(z)}{k!} \left(V_s(s) \right)^{(k)},$$

By Lemmas 10;

$$\forall s \in [0, 1] : \left\| \left(\phi[\omega, V_s] \right)_{\frac{1}{\omega}} \right\|_{\omega, 0} \leq \sum_{k=0}^{\infty} \frac{1}{k! 2^k} \|d^{k+1} f_{\frac{1}{\omega}}\|_{\omega, 0} \|V_s\|_{\omega, 0}^{2k}.$$

By hypothesis $g, h \in K_{r, \omega}$, by consequence

$$\sup_{s \in [0, 1]} \|V_s\|_{\omega, 0} \leq \|g\|_{\omega, 0} + \|h\|_{\omega, 0} \leq 4\beta\tau(f, r).$$

Using Lemma 9, we get

$$\begin{aligned} \sup_{s \in [0, 1]} \left\| \left(\phi[\omega, V_s] \right)_{\frac{1}{\omega}} \right\|_{\omega, 0} &\leq \sum_{k=0}^{\infty} \frac{1}{k! 2^k} \frac{n^{k+1} \beta^{k+2}}{\omega^{k+1}} (4\beta\tau(f, r))^k \\ &= \frac{n\beta^2}{\omega} \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{2n\beta^2}{\omega} \beta\tau(f, r) \right)^k \\ &= \frac{n\beta^2}{\omega} \exp \left(\frac{2n\beta^2}{\omega} \beta\tau(f, r) \right). \end{aligned}$$

Choose $\omega_r > 0$ large such that

$$\tau(r) \frac{n\beta^2}{\omega_r} \exp \left(\frac{2n\beta^2}{\omega_r} \beta\tau(f, r) \right) < \frac{1}{2}.$$

We have proved Equation (9). Thanks to Equation (8), for every $\omega > \omega_r$ we have

$$\|\Psi[r, \omega, h] - \Psi[r, \omega, g]\|_{\omega, 0} \leq \frac{1}{2} \|h - g\|_{\omega, 0}.$$

Now, choose $\omega > 0$ fixed and large. Let be $g \in K_{r,\omega}$, by the last inequality, for every $\epsilon > 0$ there exists $k_\epsilon \geq 0$ such that

$$\|\Psi^{k_\epsilon+1}[r, \omega, g] - \Psi^{k_\epsilon}[r, \omega, g]\|_{\omega,0} < \epsilon,$$

Denote

$$\phi_{r,\omega,\epsilon} := \Psi^{k_\epsilon}[r, \omega, g].$$

By Lemma 15 we have $\phi_{r,\omega,\epsilon} \in K_{r,\omega}$. By Lemma 8 we obtain

$$\|\Psi[r, \omega, \phi_{r,\omega,\epsilon}] - \phi_{r,\omega,\epsilon}\|_\infty < \epsilon,$$

□

Proof of Proposition 12. Let be $r \in \mathbb{Q}^n \setminus \{0\}$. By Lemma 16, there exists $\omega := \omega_r > 0$ such that for every $\epsilon > 0$ there exists $\phi_{r,\omega,\epsilon} \in K_{r,\omega}$ satisfying

$$\|\Psi[r, \omega, \phi_{r,\omega,\epsilon}] - \phi_{r,\omega,\epsilon}\|_\infty < \epsilon. \quad (10)$$

Define the functions,

$$\tilde{\phi}_{r,\epsilon}(z) := \phi_{r,\omega,\epsilon}(\omega z), \text{ and } H[g](z) := f(z + g(z)).$$

We recall that,

$$H[\omega, \phi_{r,\omega,\epsilon}](z) = f\left(z + \phi_{r,\omega,\epsilon}(\omega z)\right).$$

By Equation (2),

$$a_p[H[\omega, \phi_{r,\omega,\epsilon}]] = a_p[H[\tilde{\phi}_{r,\epsilon}]],$$

Using the definition of $\Psi[r, \omega, \tilde{\phi}_{r,\epsilon}]$ and replace on Equation (10), the function $\tilde{\phi}_{r,\epsilon}$ satisfies

$$\sup_{z \in \mathbb{R}^n} \|\tilde{\phi}_{r,\epsilon}(z) - \sum_{p \in i2\pi\mathbb{Z}^n, \langle r,p \rangle \neq 0} \frac{a_p[H[\tilde{\phi}_{r,\epsilon}]]}{i\langle r,p \rangle} (\exp(i\langle z,p \rangle) - 1)\| < \epsilon.$$

By Lemma 16, $\phi_{r,\omega,\epsilon} \in K_{r,\omega}$ then

$$\|\tilde{\phi}_{r,\epsilon}\|_\infty \leq \|\phi_{r,\omega,\epsilon}\|_\infty < 2\beta\tau(f, r).$$

By Lemma 14 the set $K_{r,\omega}$ is a subset of $E_\omega(\mathbb{R}^n)$. Then $\tilde{\phi}_{r,\epsilon}$ is a C^∞ periodic modulo \mathbb{Z}^n function.

□

5 Proof of the Main result

Proof of Main results. Consider the System (1) where f is a polynomial function. There exists $q \in \mathbb{N}^*$ such that $\|f\|_\infty < q$. Use the change of variables

$$x_q(t) = x(t) - x_0 + qt, \quad \forall t \in \mathbb{R},$$

we get

$$\frac{d}{dt}x_q(t) = f(x_q(t) + x_0 - qt\mathbb{1}) + q, \quad t \in \mathbb{R}, \quad x_q(0) = 0. \quad (11)$$

where $\mathbb{1} := (1, \dots, 1) \in \mathbb{R}^n$. Now, transform the last system to an autonomous systems. Define the functions $x_{n+1} : \mathbb{R} \rightarrow \mathbb{R}$ as the identity function: $x_{n+1}(t) := t$ for every $t \in \mathbb{R}$, the system (11) can be written as

$$\begin{aligned} \dot{x}_q &= f(x_q + x_0 - qx_{n+1}\mathbb{1}) + q, \quad t \in \mathbb{R}, \quad x_q(0) = 0, \\ \dot{x}_{n+1} &= 1 \quad t \in \mathbb{R}, \quad x_{n+1}(0) = 0, \end{aligned}$$

in other words,

$$\dot{\tilde{x}} = f_q(\tilde{x}), \quad t \in \mathbb{R}, \quad \tilde{x} = (x_q, x_{n+1}), \quad \tilde{x}(0) = (0, 0),$$

where $f_q : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^{n+1}$ satisfies

$$f_q(\tilde{z}) := \left(f(z + x_0 - qz_{n+1}\mathbb{1}), 1 \right), \quad \forall \tilde{z} := (z, z_{n+1}) \in \mathbb{R}^{n+1}.$$

Since $q \in \mathbb{N}^*$ then f_q is a polynomial trigonometric function. In addition, $\min_{z \in \mathbb{R}^{n+1}} f_q(z) \geq \min\{q - \|f\|_\infty, 1\} > 0$. Without loss of generality, we consider the system (1) by supposing that $x(0) = 0$ and $f(z) \neq 0$ for all $z \in \mathbb{R}^n$. Let $(\epsilon_k)_k \subset (0, 1]$ be a sequence satisfying $\lim_{k \rightarrow \infty} \epsilon_k = 0$. For every $k \geq 1$ let $\gamma_k \in \mathbb{R}^n \rightarrow \mathbb{Q}^n / \{0\}$ be a function satisfying-

$$\forall y \in \mathbb{R}^n : \quad \|y - \gamma_k(y)\| < \epsilon_k.$$

We have $\gamma_k(y) \in \mathbb{Q}^n / \{0\}$. For every $y \in \mathbb{R}^n$ and every $k \geq 1$ consider the function $\phi_{\gamma_k(y)}$ satisfying the Main Proposition such that

$$\sup_{z \in \mathbb{R}^n} \|\phi_{\gamma_k(y)}(z) - \sum_{p \in 2\pi\mathbb{Z}^n, \langle \gamma_k(y), p \rangle \neq 0} \frac{a_p[H[\phi_{\gamma_k(y)}]]}{i\langle \gamma_k(y), p \rangle} (\exp(i\langle z, p \rangle) - 1)\| < \frac{1}{k},$$

Define the recurrent sequence $(\rho_k)_k \subset \mathbb{R}^n$ as

$$\rho_0 = 0, \quad \rho_{k+1} := \sum_{p \in i2\pi\mathbb{Z}^n, \langle \gamma_k(\rho_k), p \rangle = 0} a_p[H[\phi_{\gamma_k(\rho_k)}]], \quad \forall k \geq 0.$$

Prove that the sequence $(\rho_k)_k$ is bounded. Let be $\psi_k : \mathbb{R} \rightarrow \mathbb{R}^n$ the function defined by $t \mapsto \psi_k(t) := \phi_{\gamma_k(\rho_k)}(\gamma_k(\rho_k)t)$. By the Main Proposition, we get

$$\sup_{t \in \mathbb{R}} \|\psi_k(t) - \sum_{p \in 2\pi\mathbb{Z}^n, \langle \gamma_k(\rho_k), p \rangle \neq 0} \frac{a_p[H[\phi_{\gamma_k(\rho_k)}]]}{i \langle \gamma_k(\rho_k), p \rangle} (\exp(i \langle \gamma_k(\rho_k), p \rangle t) - 1)\| < \frac{1}{k},$$

since the sum is normally convergent, that implies

$$\sup_{t \in \mathbb{R}} \|\psi_k(t) - \int_0^t \sum_{p \in 2\pi\mathbb{Z}^n, \langle \gamma_k(\rho_k), p \rangle \neq 0} a_p[H[\phi_{\gamma_k(\rho_k)}]] \exp(i \langle \gamma_k(\rho_k), p \rangle s) ds\| < \frac{1}{k}.$$

By Equation (3), we have the following Fourier development

$$f(z + \phi_{\gamma_k(\rho_k)}(z)) = \sum_{p \in 2\pi\mathbb{Z}^n} a_p[H[\phi_{\gamma_k(\rho_k)}]] \exp(i \langle z, p \rangle),$$

then

$$\sup_{t \in \mathbb{R}} \|\psi_k(t) - \int_0^t f(\gamma_k(\rho_k)s + \psi_k(s)) ds - t\rho_{k+1}\| < \frac{1}{k}. \quad (12)$$

Since $\|\psi_k\|_\infty < \infty$ then

$$\|\rho_{k+1} - \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t f(\gamma_k(\rho_k)s + \psi_k(s)) ds\| = 0.$$

we deduce that $\limsup_{k \rightarrow \infty} \|\rho_k\| \leq \|f\|$. There exists $\rho \in \mathbb{R}^n$ and a subsequence $(\rho_{k_s})_s$ which converge to ρ . In order to simplify the notation, we suppose that $(\rho_k)_k$ converge to ρ . Since $\epsilon_k \rightarrow 0$ then

$$\lim_{k \rightarrow \infty} \rho_k = \lim_{k \rightarrow \infty} \gamma_k(\rho_k) = \rho.$$

We have supposed that $f(z) \neq 0$ for every $z \in \mathbb{R}^n$, then $\rho \neq 0$. There exists $c > 0$ and $k_0 \geq 0$ such that

$$\tau(f, \gamma_k(\rho_k)) < c, \quad \forall k \geq k_0.$$

By the Main Proposition, we obtain

$$\sup_{k \geq k_0} \|\phi_{\gamma_k(\rho_k)}\|_\infty \leq 2\beta \sup_{k \geq k_0} \tau(f, \gamma_k(\rho_k)) < 2\beta c.$$

Now, prove that the sequence functions (ψ_k) converge uniformly on every interval $[0, T]$. Since f is a polynomial trigonometric function, then there

exist $\eta > 0$ such that f is uniformly η -Lipschitz function. For every $T > 0$ we have

$$\begin{aligned} \sup_{t \in [0, T]} \|\psi_{k_2}(t) - \psi_{k_1}(t)\| &= \sup_{t \in [0, T]} \|\exp(2\eta t) \exp(-2\eta t) \psi_{k_2}(t) - \psi_{k_1}(t)\| \\ &\leq \exp(2\eta T) \|\psi_{k_2} - \psi_{k_1}\|_T, \quad \forall k_1, k_2 \in \mathbb{N}, \end{aligned}$$

where

$$\|\psi_{k_2} - \psi_{k_1}\|_T := \sup_{t \in [0, T]} \|\exp(-2\eta t) [\psi_{k_2}(t) - \psi_{k_1}(t)]\|.$$

It is sufficient to prove that

$$\lim_{k_2, k_1 \rightarrow +\infty} \|\psi_{k_2} - \psi_{k_1}\|_T = 0.$$

By Equation (12)

$$\begin{aligned} \|\psi_{k_2} - \psi_{k_1}\|_T &\leq \eta \sup_{t \in [0, T]} \left[\exp(-2\eta t) \int_0^t s \|\gamma_{k_2}(\rho_{k_2}) - \gamma_{k_1}(\rho_{k_1})\| ds \right] \\ &\quad + \eta \sup_{t \in [0, T]} \left[\exp(-2\eta t) \int_0^t \|\psi_{k_2}(s) - \psi_{k_1}(s)\| ds \right] \\ &\quad + \sup_{t \in [0, T]} \left[\exp(-2\eta t) t \|\rho_{k_2+1} - \rho_{k_1+1}\| + \frac{1}{k_2} + \frac{1}{k_1} \right] \\ &\leq \eta T^2 \|\gamma_{k_2}(\rho_{k_2}) - \gamma_{k_1}(\rho_{k_1})\| + \frac{1}{2} \|\psi_{k_2} - \psi_{k_1}\|_T \\ &\quad + T \|\rho_{k_2+1} - \rho_{k_1+1}\| + \frac{1}{k_2} + \frac{1}{k_1}. \end{aligned}$$

Then

$$\begin{aligned} \frac{1}{2} \|\psi_{k_2} - \psi_{k_1}\|_T &\leq \eta T^2 \|\gamma_{k_2}(\rho_{k_2}) - \gamma_{k_1}(\rho_{k_1})\| \\ &\quad + T \|\rho_{k_2+1} - \rho_{k_1+1}\| + \frac{1}{k_2} + \frac{1}{k_1} \rightarrow 0, \quad \text{when } k_2, k_1 \rightarrow +\infty. \end{aligned}$$

We deduce that the sequence function $(\psi_k)_k$ is a Picard iteration for the solution of the differential equation

$$\dot{x} = f(x), \quad x(0) = 0,$$

there exists a weakly almost periodic function $\psi_\rho^* : \mathbb{R} \rightarrow \mathbb{R}^n$ of slope ρ such that

$$\begin{aligned} \psi_\rho^*(t) &= \lim_{k \rightarrow \infty} \psi_k(t) = \int_0^t \lim_{k \rightarrow \infty} f(\gamma_k(\rho_k)s + \psi_k(t)) ds - t \lim_{k \rightarrow \infty} \rho_k \\ &= \int_0^t f(\rho s + \psi_\rho^*(s)) ds - t\rho, \quad \forall t \in \mathbb{R}. \end{aligned}$$

By uniqueness of solution of differential equation, we have proved that

$$x(t) = \rho t + \psi_\rho^*(t), \quad \forall t \in \mathbb{R}.$$

□

6 Conclusion

We have proved that any solution x of ODE defined by a trigonometric polynomial field can be approximated by a sequence functions $t \mapsto \rho_k t + \phi_k(t)$ where $(\rho_k)_k \subset \mathbb{Q}^n$ and converge to the rotation vector of x . The functions $\psi_k : \mathbb{R} \rightarrow \mathbb{R}^n$ are periodic on t and uniformly bounded.

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Appendix. A

Proof of Lemma 7. By hypothesis, for $q = 0$ we have

$$\sum_{p \in 2\pi\mathbb{Z}^n/0} \|c_p\| < +\infty,$$

the series is normally convergent and we have

$$\forall p \in 2\pi\mathbb{Z}^n : g(\omega z) \exp(-i\langle z, p \rangle) = \sum_{q \in 2\pi\mathbb{Z}^n} c_q \exp(i\langle z, q - p \rangle), \quad c_q \in \mathbb{C}^n,$$

implies

$$\begin{aligned} a_p[g_\omega] &= \int_0^1 \cdots \int_0^1 g(\omega z) \exp(-i\langle z, p \rangle) dz_1 \cdots dz_n \\ &= \sum_{q \in 2\pi\mathbb{Z}^n} c_q \int_0^1 \cdots \int_0^1 \exp(i\langle z, q - p \rangle) dz_1 \cdots dz_n = c_p. \end{aligned} \quad (13)$$

Now, prove that $g \in E_\omega(\mathbb{R}^n)$. Denote

$$\theta_p(z) := \exp(i\frac{1}{\omega}\langle z, p \rangle), \quad \forall z \in \mathbb{R}^n,$$

It is sufficient to prove that for every $q \geq 1$ we have

$$S_q := \sum_{p \in 2\pi\mathbb{Z}^n/0} \|c_p\| \|d^q \theta_p\|_\infty < +\infty,$$

where $d^q g$ is q^{th} differential of g . The function $d^q g$ is defined as

$$d^q g = \sum_{p \in 2\pi\mathbb{Z}^n} c_p d^q \theta_p.$$

We have

$$\begin{aligned} \forall p &:= (p_j)_{j=1}^n \in 2\pi\mathbb{Z}^n : \\ d^q \theta_p &= \sum_{k_1=1}^n \cdots \sum_{k_q=1}^n i^q \frac{1}{\omega^q} p_{k_1} \cdots p_{k_q} \exp(i\langle z, p \rangle), \end{aligned}$$

By consequence,

$$\|d^q \theta_p\|_\infty = \left(\sum_{k=1}^n \left| \frac{p_k}{\omega} \right| \right)^q \leq n^q \left(\frac{\|p\|}{\omega} \right)^q.$$

Thanks to Equation (13), we get

$$\begin{aligned} \forall q \geq 1: \quad S_q &\leq n^q \sum_{p \in 2\pi\mathbb{Z}^n / 0} \|c_p\| \left(\frac{\|p\|}{\omega} \right)^q = n^q \sum_{p \in 2\pi\mathbb{Z}^n / 0} \|a_p[g_\omega]\| \left(\frac{\|p\|}{\omega} \right)^q \\ &= \frac{1}{2} n^q \|g\|_{\omega, q} < +\infty, \end{aligned}$$

which implies that $g \in E_\omega(\mathbb{R}^n)$. \square

Appendix. B

Proof of Lemma 8. By Equation (3), we have

$$g(z) = \sum_{p \in 2\pi\mathbb{Z}^n} a_p[g_\omega] \exp(i \frac{1}{\omega} \langle z, p \rangle).$$

Since $g(0) = 0$, then

$$a_0[g_\omega] = - \sum_{p \in 2\pi\mathbb{Z}^n / 0} a_p[g_\omega],$$

implies

$$\|a_0[g_\omega]\| \leq \sum_{p \in 2\pi\mathbb{Z}^n / 0} \|a_p[g_\omega]\| = \frac{1}{2} \|g\|_{\omega, 0}.$$

Since

$$\|g\|_\infty \leq \sum_{p \in 2\pi\mathbb{Z}^n} \|a_p[g_\omega]\|.$$

We deduce that

$$\|g\|_\infty \leq \|g\|_{\omega, 0}.$$

\square

Appendix. C

Proof of Lemma 9. Since $g : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a trigonometric polynomial, then it is C^∞ and there exists $m \in \mathbb{N}$ such that

$$g(z) = \sum_{p \in 2\pi\mathbb{Z}^n, \|p\| \leq m} a_p[g] \exp(i \langle z, p \rangle), \quad \forall z \in \mathbb{R}^n.$$

Implies,

$$g_{\perp}^{\omega}(z) = g\left(\frac{z}{\omega}\right) = \sum_{p \in 2\pi\mathbb{Z}^n, \|p\| \leq m} a_p[g] \exp\left(i\frac{1}{\omega}\langle z, p \rangle\right),$$

Denote

$$\theta_p(z) := \exp\left(i\frac{1}{\omega}\langle z, p \rangle\right), \quad \forall z \in \mathbb{R}^n,$$

Denote $d^k g$ is k^{th} differential of g , which implies

$$d^k g_{\perp}^{\omega}(z) = \sum_{p \in 2\pi\mathbb{Z}^n, \|p\| \leq m} a_p[g] d^k \theta_p.$$

We have

$$\begin{aligned} \forall p &:= (p_j)_{j=1}^n \in 2\pi\mathbb{Z}^n : \\ d^k \theta_p &= \sum_{s_1=1}^n \dots \sum_{s_k=1}^n i^k \frac{p_{s_1}}{\omega} \dots \frac{p_{s_k}}{\omega} \exp\left(i\frac{1}{\omega}\langle z, p \rangle\right), \end{aligned}$$

then

$$\forall p \in 2\pi\mathbb{Z}^n, \|p\| \leq m : \quad \|d^k \theta_p\|_{\infty} = \left(\sum_{s=1}^n \left| \frac{p_s}{\omega} \right| \right)^k \leq n^k \left\| \frac{p}{\omega} \right\|^k.$$

Since $\left(g_{\perp}^{\omega}\right)_{\omega} = g$, we obtain

$$\begin{aligned} \|d^k g_{\perp}^{\omega}\|_{\omega, q} &\leq 2n^k \left(\frac{m}{\omega}\right)^k \sum_{p \in 2\pi\mathbb{Z}^n, \|p\| \leq m} \|a_p[g]\| \left\| \frac{p}{\omega} \right\|^q \\ &\leq 2n^k \left(\frac{m}{\omega}\right)^{k+q} \left(\sum_{p \in 2\pi\mathbb{Z}^n, \|p\| \leq m} \|a_p[g]\| \right). \end{aligned}$$

It is sufficient to choose $\beta := 2 \max\{\sum_{p \in 2\pi\mathbb{Z}^n, \|p\| \leq m} \|a_p[g]\|, m\}$. \square

Appendix. D

Proof of Lemma 10 . Since $(h_j)_{j=1}^k \subset E_{\omega}(\mathbb{R}^n)$, by Equation (3) we can write

$$h_j(z) = \sum_{p_j \in 2\pi\mathbb{Z}^n} a_{p_j}[h_{j,\omega}] \exp\left(i\frac{1}{\omega}\langle z, p_j \rangle\right).$$

By definition of the seminorm,

$$\|h_j\|_{\omega,q} = 2 \sum_{p_j \in 2\pi\mathbb{Z}^n / \{0\}} \|a_{p_j}[h_{j,\omega}]\| \left\| \frac{p_j}{\omega} \right\|^q. \quad (14)$$

We have

$$\begin{aligned} \Pi_{j=1}^k h_j(z) &= \sum_{p_1, \dots, p_k \in 2\pi\mathbb{Z}^n} \Pi_{j=1}^k a_{p_j}[h_{j,\omega}] \exp\left(i \frac{1}{\omega} \langle z, \sum_{j=1}^k p_j \rangle\right) \\ &= \sum_{v \in 2\pi\mathbb{Z}^n} \sum_{\sum_{j=1}^k p_j = v} \Pi_{j=1}^k a_{p_j}[h_{j,\omega}] \exp\left(i \frac{1}{\omega} \langle z, v \rangle\right). \end{aligned}$$

Then

$$\begin{aligned} \|\Pi_{j=1}^k h_j\|_{\omega,q} &= 2 \sum_{v \in 2\pi\mathbb{Z}^n / 0} \left\| \sum_{\sum_{j=1}^k p_j = v} \Pi_{j=1}^k a_{p_j}[h_{j,\omega}] \right\| \left\| \frac{v}{\omega} \right\|^q \\ &\leq 2 \sum_{v \in 2\pi\mathbb{Z}^n} \left\| \sum_{\sum_{j=1}^k p_j = v} \Pi_{j=1}^k a_{p_j}[h_{j,\omega}] \right\| \left\| \frac{v}{\omega} \right\|^q. \end{aligned}$$

Using the triangular inequality, we obtain

$$\|\Pi_{j=1}^k h_j\|_{\omega,q} \leq 2 \sum_{p_1, \dots, p_k \in 2\pi\mathbb{Z}^n} \left\| \Pi_{j=1}^k a_{p_j}[h_{j,\omega}] \right\| \left\| \frac{1}{\omega} \sum_{j=1}^k p_j \right\|^q.$$

Since

$$\left\| \sum_{j=1}^k p_j \right\| \leq k \Pi_{\|p_j\| \neq 0} \|p_j\|, \quad \forall p_j \in 2\pi\mathbb{Z}^n,$$

then

$$\left\| \frac{1}{\omega} \sum_{j=1}^k p_j \right\|^q \leq k^q \omega^{(k-1)q} \Pi_{\|p_j\| \neq 0} \left\| \frac{p_j}{\omega} \right\|^q, \quad \forall p_j \in 2\pi\mathbb{Z}^n.$$

We deduce that

$$\begin{aligned} \|\Pi_{j=1}^k h_j\|_{\omega,q} &\leq 2k^q \omega^{(k-1)q} \sum_{p_1, \dots, p_k \in 2\pi\mathbb{Z}^n} \left(\Pi_{\|p_j\| \neq 0} \left\| \frac{p_j}{\omega} \right\|^q \right) \left(\Pi_{j=1}^k \|a_{p_j}[h_{j,\omega}]\| \right) \\ &= 2k^q \omega^{(k-1)q} \sum_{p_1, \dots, p_k \in 2\pi\mathbb{Z}^n} \left(\Pi_{\|p_j\| = 0} \|a_{p_j}[h_{j,\omega}]\| \right) \Pi_{\|p_j\| \neq 0} \left[\left\| \frac{p_j}{\omega} \right\|^q \|a_{p_j}[h_{j,\omega}]\| \right]. \end{aligned}$$

By Equation (14),

$$\|\Pi_{j=1}^k h_j\|_{\omega, q} \leq \frac{1}{2^{k-1}} k^q \omega^{(k-1)q} \Pi_{j=1}^k \left[2 \left(\|a_0[h_{j, \omega}]\| + \sum_{p_j \in 2\pi\mathbb{Z}^n / \{0\}} \left\| \frac{p_j}{\omega} \right\|^q \|a_{p_j}[h_{j, \omega}]\| \right) \right].$$

□