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SEMI-CLASSICAL RESOLVENT ESTIMATES FOR SHORT-RANGE L^∞ POTENTIALS. II

GEORGI VODEV

ABSTRACT. We prove semi-classical resolvent estimates for real-valued potentials $V \in L^\infty(\mathbb{R}^n)$, $n \geq 3$, of the form $V = V_L + V_S$, where V_L is a long-range potential which is C^1 with respect to the radial variable, while V_S is a short-range potential satisfying $V_S(x) = \mathcal{O}(\langle x \rangle^{-\delta})$ with $\delta > 1$.

Key words: Schrödinger operator, resolvent estimates, short-range potentials.

1. INTRODUCTION AND STATEMENT OF RESULTS

The goal of this paper is to extend the semi-classical resolvent estimates obtained recently in [7], [9] and [11] to a larger class of potentials. We are going to study the resolvent of the Schrödinger operator

$$P(h) = -h^2 \Delta + V(x)$$

where $0 < h \ll 1$ is a semi-classical parameter, Δ is the negative Laplacian in \mathbb{R}^n , $n \geq 3$, and $V \in L^\infty(\mathbb{R}^n)$ is a real-valued potential of the form $V = V_L + V_S$, where $V_L \in C^1([r_0, +\infty))$ with respect to the radial variable $r = |x|$, $r_0 > 0$ being some constant, is a long-range potential, while V_S is a short-range potential satisfying

$$(1.1) \quad |V_S(x)| \leq C_1(|x| + 1)^{-\delta}$$

with some constants $C_1 > 0$ and $\delta > 1$. We suppose that there exists a decreasing function $p(r) > 0$, $p(r) \rightarrow 0$ as $r \rightarrow \infty$, such that

$$(1.2) \quad V_L(x) \leq p(|x|) \quad \text{for } |x| \geq r_0.$$

We also suppose that

$$(1.3) \quad \partial_r V_L(x) \leq C_2(|x| + 1)^{-\beta} \quad \text{for } |x| \geq r_0$$

with some constants $C_2 > 0$ and $\beta > 1$. As in [11] we introduce the quantity

$$g_s^\pm(h, \theta) := \log \left\| (|x| + 1)^{-s} (P(h) - E \pm i\theta)^{-1} (|x| + 1)^{-s} \right\|_{L^2 \rightarrow L^2}$$

where $L^2 := L^2(\mathbb{R}^n)$, $0 < \theta < 1$, $s > 1/2$ is independent of h and $E > 0$ is a fixed energy level independent of h . Our first result is the following

Theorem 1.1. *Suppose the conditions (1.1), (1.2) and (1.3) fulfilled with δ and β satisfying the condition*

$$(1.4) \quad \delta > 3, \quad \beta > 3.$$

Then there exist constants $C > 0$ and $h_0 > 0$ independent of h and θ but depending on s and E such that for all $0 < h \leq h_0$ we have the bound

$$(1.5) \quad g_s^\pm(h, \theta) \leq Ch^{-4/3} \log(h^{-1}).$$

If the conditions (1.1), (1.2) and (1.3) are fulfilled with δ and β satisfying the condition

$$(1.6) \quad \delta > 3, \quad \beta = 3,$$

then we have the bound

$$(1.7) \quad g_s^\pm(h, \theta) \leq Ch^{-4/3} (\log(h^{-1}))^{3/2} (\log \log(h^{-1}))^{-1}.$$

When $V_S \equiv 0$ and V_L satisfying conditions similar to (1.2) and (1.3), it is proved in [4] when $n \geq 3$ and in [8] when $n = 2$ that

$$(1.8) \quad g_s^\pm(h, \theta) \leq Ch^{-1}$$

with some constant $C > 0$ independent of h and θ . Previously, the bound (1.8) was proved for smooth potentials in [2] and an analog of (1.8) for Hölder potentials was proved in [10]. A high-frequency analog of (1.8) on Riemannian manifolds was also proved in [1] and [3]. When $V_L \equiv 0$ and V_S satisfying the condition (1.1) with $\delta > 3$, the bound (1.5) has been recently proved in [11]. Previously, (1.5) was proved in [7] and [9] for real-valued compactly supported L^∞ potentials. When $n = 1$ it was shown in [6] that we have the better bound (1.8) instead of (1.5). The method we use to prove Theorem 1.1 also allows us to get resolvent bounds when the conditions (1.4) and (1.6) are not satisfied, which however are much weaker than the bound (1.5). More precisely, we have the following

Theorem 1.2. *Suppose the conditions (1.1), (1.2) and (1.3) fulfilled with δ and β satisfying either the condition*

$$(1.9) \quad 1 < \delta \leq 3, \quad \beta > 1,$$

or the condition

$$(1.10) \quad \delta > 3, \quad 1 < \beta < 3.$$

Then, there exist constants $C > 0$ and $h_0 > 0$ independent of h and θ but depending on s and E such that for all $0 < h \leq h_0$ we have the bounds

$$(1.11) \quad g_s^\pm(h, \theta) \leq Ch^{-\frac{2}{3}-m_1} (\log(h^{-1}))^\nu$$

if (1.9) holds, and

$$(1.12) \quad g_s^\pm(h, \theta) \leq Ch^{-\frac{4}{3}-\frac{1}{2}(3-\beta)m_2} (\log(h^{-1}))^\nu$$

if (1.10) holds, where

$$m_1 = \max \left\{ \frac{7}{3(\delta-1)}, \frac{4}{3(\beta-1)} \right\} \geq \frac{7}{6},$$

$$m_2 = \max \left\{ \frac{1}{\delta-\beta}, \frac{4}{3(\beta-1)} \right\} > \frac{2}{3},$$

$$\nu = \max \left\{ \frac{1}{\delta-1}, \frac{1}{\beta-1} \right\}.$$

Clearly, this theorem implies the following

Corollary 1.3. *Suppose that $V_L \equiv 0$ and let $V = V_S$ satisfy the condition (1.1) with $1 < \delta \leq 3$. Then, there exist constants $C > 0$ and $h_0 > 0$ independent of h and θ but depending on s and E such that for all $0 < h \leq h_0$ we have the bound*

$$(1.13) \quad g_s^\pm(h, \theta) \leq Ch^{-\frac{2\delta+5}{3(\delta-1)}} (\log(h^{-1}))^{\frac{1}{\delta-1}}.$$

To prove the above theorems we follow the same strategy as in [11] which in turn is inspired by the paper [9]. It consists of using Carleman estimates with phase and weight functions, denoted by φ and μ below, depending only on the radial variable r and the parameter h , which have very weak regularity. It turns out that it suffices to choose φ belonging only to C^1 and μ only continuous. Thus we get derivatives φ'' and μ' belonging to L^∞ , which proves sufficient for the Carleman estimates to hold. Note that higher derivatives of φ and μ are not involved in the proof of the Carleman estimates (see the proof of Theorem 3.1 below). In order to be able to prove the Carleman estimates the functions φ and μ must satisfy some conditions (see the inequalities (2.3) and (2.9) below). On the other hand, to get as good resolvent bounds as possible we are looking for a phase function φ such that $\max \varphi$ is as small as possible. The construction of such phase and weight functions is carried out in Section 2 following that one in [11]. However, here the construction is more complicated due to the more general class the potential belongs to.

It is not clear if the bounds (1.5), (1.7), (1.11) and (1.12) are optimal for L^∞ potentials. In any case, they seem hard to improve unless one manages to construct a better phase function. By contrast, the optimality of the bound (1.8) for smooth potentials is well known (e.g. see [5]).

2. THE CONSTRUCTION OF THE PHASE AND WEIGHT FUNCTIONS REVISITED

We will follow closely the construction in Section 2 of [11] making some suitable modifications in order to adapt it to the more general class of potentials we consider in the present paper. We will first construct the weight function μ as follows:

$$\mu(r) = \begin{cases} (r+1)^{2k} - 1 & \text{for } 0 \leq r \leq a, \\ (a+1)^{2k} - 1 + (a+1)^{-2s+1} - (r+1)^{-2s+1} & \text{for } r \geq a, \end{cases}$$

where $a = h^{-m}$ with

$$m = \begin{cases} m_0 + \epsilon \lambda T_0 & \text{if (1.4) holds,} \\ m_0 + \epsilon \lambda / 2 + \epsilon T_1 & \text{if (1.6) holds,} \\ m_1 + \nu \epsilon \lambda + \epsilon T_2 & \text{if (1.9) holds,} \\ m_2 + \nu \epsilon \lambda + \epsilon T_3 & \text{if (1.10) holds,} \end{cases}$$

where $\epsilon = (\log \frac{1}{h})^{-1}$, $\lambda = \log \log \frac{1}{h}$, $m_0 = \max \left\{ \frac{2}{3}, \frac{1}{\delta-3} \right\}$, m_1 , m_2 and ν are as in Theorem 1.2, while $T_j > 0$, $j = 0, 1, 2, 3$, are parameters independent of h to be fixed in the proof of Lemma 2.3. Furthermore,

$$k = \begin{cases} 1 - \epsilon & \text{if (1.4) holds,} \\ 1 - \frac{\epsilon \lambda}{2m_0} - \epsilon t_1 & \text{if (1.6) holds,} \\ \frac{2}{3m_1} - \frac{2\nu \epsilon \lambda}{3m_1^2} - \epsilon t_2 & \text{if (1.9) holds,} \\ \frac{\beta-1}{2} - \frac{(\beta-1)\nu \epsilon \lambda}{2m_2} - \epsilon t_3 & \text{if (1.10) holds,} \end{cases}$$

and

$$(2.1) \quad s = \frac{1 + \epsilon}{2}$$

where $t_j > 1$, $j = 1, 2, 3$, are parameters independent of h to be fixed in the proof of Lemma 2.3. Clearly, the first derivative (in sense of distributions) of μ satisfies

$$(2.2) \quad \mu'(r) = \begin{cases} 2k(r+1)^{2k-1} & \text{for } 0 \leq r < a, \\ (2s-1)(r+1)^{-2s} & \text{for } r > a. \end{cases}$$

The following properties of the functions μ and μ' are essential to prove the Carleman estimates in the next section.

Lemma 2.1. *For all $r > 0$, $r \neq a$, we have the inequalities*

$$(2.3) \quad 2r^{-1}\mu(r) - \mu'(r) \geq 0,$$

$$(2.4) \quad \mu'(r) \geq \epsilon(r+1)^{-2s},$$

$$(2.5) \quad \frac{\mu(r)}{\mu'(r)} \lesssim \epsilon^{-1}a^{2k}(r+1)^{2s},$$

$$(2.6) \quad \frac{\mu(r)^2}{\mu'(r)} \lesssim \epsilon^{-1}a^{4k}(r+1)^{2s}.$$

Proof. It is easy to see that for $r < a$ (2.3) follows from the inequality

$$f(r) := 1 + (1-k)r - (r+1)^{1-2k} \geq 0$$

for all $r \geq 0$ and $0 \leq k \leq 1$. It is obvious for $1/2 \leq k \leq 1$, while for $0 \leq k < 1/2$ we have

$$f'(r) = 1 - k - (1-2k)(r+1)^{-2k} \geq k \geq 0.$$

Hence in this case the function f is increasing, which implies $f(r) \geq f(0) = 0$ as desired. For $r > a$ the left-hand side of (2.3) is bounded from below by

$$2r^{-1}((a+1)^{2k} - 1 - s) > 0$$

provided a is taken large enough. The lower bound (2.4) is an immediate consequence of (2.1) and (2.2), while the bounds (2.5) and (2.6) follow from (2.4) and the fact that $\mu = \mathcal{O}(a^{2k})$. \square

We now turn to the construction of the phase function $\varphi \in C^1([0, +\infty))$ such that $\varphi(0) = 0$ and $\varphi(r) > 0$ for $r > 0$. We define the first derivative of φ by

$$\varphi'(r) = \begin{cases} \tau(r+1)^{-k} - \tau(a+1)^{-k} & \text{for } 0 \leq r \leq a, \\ 0 & \text{for } r \geq a, \end{cases}$$

where

$$(2.7) \quad \tau = \tau_0 h^{-1/3}$$

with some parameter $\tau_0 \gg 1$ independent of h to be fixed in Lemma 2.3 below. Clearly, the first derivative of φ' satisfies

$$\varphi''(r) = \begin{cases} -k\tau(r+1)^{-k-1} & \text{for } 0 \leq r < a, \\ 0 & \text{for } r > a. \end{cases}$$

Lemma 2.2. *For all $r \geq 0$ we have the bounds*

$$(2.8) \quad h^{-1}\varphi(r) \lesssim \begin{cases} h^{-4/3} \log(h^{-1}) & \text{if (1.4) holds,} \\ h^{-4/3} (\log(h^{-1}))^{3/2} (\log \log(h^{-1}))^{-1} & \text{if (1.6) holds,} \\ h^{-\frac{2}{3}-m_1} (\log(h^{-1}))^\nu & \text{if (1.9) holds,} \\ h^{-\frac{4}{3}-\frac{1}{2}(3-\beta)m_2} (\log(h^{-1}))^\nu & \text{if (1.10) holds.} \end{cases}$$

Proof. Since $k < 1$ we have

$$\max \varphi = \int_0^a \varphi'(r) dr \leq \tau \int_0^a (r+1)^{-k} dr \leq \frac{\tau}{1-k} (a+1)^{1-k} \lesssim (1-k)^{-1} h^{-\frac{1}{3}-m(1-k)}$$

and

$$(1-k)^{-1} \lesssim \begin{cases} \epsilon^{-1} & \text{if (1.4) holds,} \\ (\epsilon\lambda)^{-1} & \text{if (1.6) holds,} \\ 1 & \text{if (1.9) holds,} \\ 1 & \text{if (1.10) holds.} \end{cases}$$

Since

$$m(1-k) = \begin{cases} \mathcal{O}(\epsilon) & \text{if (1.4) holds,} \\ \epsilon\lambda/2 + \mathcal{O}(\epsilon) & \text{if (1.6) holds,} \\ m_1 - \frac{2}{3} + \nu\epsilon\lambda + \mathcal{O}(\epsilon) & \text{if (1.9) holds,} \\ \frac{1}{2}(3-\beta)m_2 + \nu\epsilon\lambda + \mathcal{O}(\epsilon) & \text{if (1.10) holds,} \end{cases}$$

and taking into account that $h^{-\epsilon\lambda} = \epsilon^{-1}$ and $h^{-\epsilon} = e$, we get (2.8) from the above bounds. \square

Let $\phi \in C_0^\infty([1, 2])$, $\phi \geq 0$, be a real-valued function independent of h such that $\int_{-\infty}^\infty \phi(\sigma) d\sigma = 1$. Given a parameter $b \gg r_0$ to be fixed in the proof of Theorem 3.1 below, independent of h , set

$$\psi_b(r) = b^{-1} \int_r^\infty \phi(\sigma/b) d\sigma.$$

Clearly, we have $0 \leq \psi_b \leq 1$ and $\psi_b(r) = 1$ for $r \leq b$, $\psi_b(r) = 0$ for $r \geq 2b$. For $r > 0$, $r \neq a$, set

$$A(r) = (\mu\varphi'^2)'(r)$$

and

$$B(r) = \frac{3(\mu(r)(h^{-1}C_1(r+1)^{-\delta} + h^{-1}Q_b\psi_b(r) + |\varphi''(r)|))^2}{h^{-1}\varphi'(r)\mu(r) + \mu'(r) + \mu(r)(1-\psi_b(r))C_2(r+1)^{-\beta}}$$

where $Q_b \geq 0$ is some constant depending only on b . The following lemma will play a crucial role in the proof of the Carleman estimates in the next section.

Lemma 2.3. *There exist constants $b_0 = b_0(E) > 0$, $\tau_0 = \tau_0(b, E) > 0$ and $h_0 = h_0(b, E) > 0$ so that for τ satisfying (2.7) and for all $b \geq b_0$, $0 < h \leq h_0$ we have the inequality*

$$(2.9) \quad A(r) - B(r) \geq -\frac{E}{2}\mu'(r)$$

for all $r > 0$, $r \neq a$.

Proof. For $r < a$ we have

$$\begin{aligned} A(r) &= -(\varphi'^2)'(r) + \tau^2 \partial_r \left(1 - (r+1)^k (a+1)^{-k} \right)^2 \\ &= -2\varphi'(r)\varphi''(r) - 2k\tau^2 (r+1)^{k-1} (a+1)^{-k} \left(1 - (r+1)^k (a+1)^{-k} \right) \\ &\geq 2k\tau (r+1)^{-k-1} \varphi'(r) - 2k\tau^2 (r+1)^{k-1} (a+1)^{-k} \\ &\geq 2k\tau (r+1)^{-k-1} \varphi'(r) - \tau^2 a^{-k} \mu'(r). \end{aligned}$$

Taking into account the definition of the parameters a and τ we conclude

$$(2.10) \quad A(r) \geq 2k\tau (r+1)^{-k-1} \varphi'(r) - \mathcal{O}(h^{km-2/3})\mu'(r)$$

for all $r < a$. Observe now that if (1.4) holds, we have

$$km - 2/3 = m_0 - 2/3 + \epsilon\lambda T_0 - \mathcal{O}(\epsilon) \geq \epsilon\lambda$$

provided we take $T_0 \geq 2$ and λ big enough. If (1.6) holds, we have

$$\begin{aligned} km - 2/3 &= \left(1 - \frac{\epsilon\lambda}{2m_0} - \epsilon t_1\right) (m_0 + \epsilon\lambda/2 + \epsilon T_1) - 2/3 \\ &= m_0 - 2/3 + \epsilon(T_1 - m_0 t_1) - \mathcal{O}(\epsilon^2 \lambda^2) \geq \epsilon m_0 t_1 \end{aligned}$$

provided we take $T_1 = 3m_0 t_1$. If (1.9) holds, we have

$$\begin{aligned} km - 2/3 &= \left(\frac{2}{3m_1} - \frac{2\nu\epsilon\lambda}{3m_1^2} - \epsilon t_2\right) (m_1 + \nu\epsilon\lambda + \epsilon T_2) - 2/3 \\ &= \left(\frac{2T_2}{3m_1} - m_1 t_2\right) \epsilon - \mathcal{O}(\epsilon^2 \lambda^2) \geq \epsilon m_1 t_2 \end{aligned}$$

provided we take $T_2 = 6m_1^2 t_2$. If (1.10) holds, we have

$$\begin{aligned} km - 2/3 &= \left(\frac{\beta - 1}{2} - \frac{(\beta - 1)\nu\epsilon\lambda}{2m_2} - \epsilon t_3\right) (m_2 + \nu\epsilon\lambda + \epsilon T_3) - 2/3 \\ &= \frac{(\beta - 1)m_2}{2} - \frac{2}{3} + \left(\frac{(\beta - 1)T_3}{2} - m_2 t_3\right) \epsilon - \mathcal{O}(\epsilon^2 \lambda^2) \geq \epsilon m_2 t_3 \end{aligned}$$

provided we take $T_3 = \frac{6m_2 t_3}{\beta - 1}$. Using that $h^{\epsilon\lambda} = \epsilon$, $h^\epsilon = e^{-1}$, we conclude that

$$(2.11) \quad h^{km-2/3} \leq \begin{cases} \epsilon & \text{if (1.4) holds,} \\ e^{-m_0 t_1} & \text{if (1.6) holds,} \\ e^{-m_1 t_2} & \text{if (1.9) holds,} \\ e^{-m_2 t_3} & \text{if (1.10) holds.} \end{cases}$$

Taking ϵ small enough and t_1, t_2, t_3 big enough, we obtain from (2.10) and (2.11) that in all cases we have the estimate

$$(2.12) \quad A(r) \geq 2k\tau(r+1)^{-k-1}\varphi'(r) - \frac{E}{4}\mu'(r)$$

for all $r < a$. We will now bound the function B from above. Note that taking h small enough we can arrange that $2b < a/2$. Let first $0 < r \leq \frac{a}{2}$. Since in this case we have

$$\varphi'(r) \geq \tilde{C}\tau(r+1)^{-k}$$

with some constant $\tilde{C} > 0$, we obtain

$$\begin{aligned} B(r) &\lesssim \frac{\mu(r) \left(h^{-2}\tilde{Q}_b(r+1)^{-2\delta} + \varphi''(r)^2 \right)}{h^{-1}\varphi'(r)} + \mu(r)(1 - \psi_b(r))(r+1)^{-\beta} \\ &\lesssim \tilde{Q}_b(\tau h)^{-1} \frac{\mu(r)(r+1)^{1+k-2\delta}}{\varphi'(r)^2} \tau(r+1)^{-k-1}\varphi'(r) + h \frac{\mu(r)\varphi''(r)^2}{\mu'(r)\varphi'(r)} \mu'(r) \\ &\quad + (1 - \psi_b(r))(r+1)^{2k-\beta} \\ &\lesssim \tilde{Q}_b \tau^{-3} h^{-1} (r+1)^{1+5k-2\delta} \tau(r+1)^{-k-1}\varphi'(r) + \tau h \mu'(r) \\ &\quad + (1 - \psi_b(r))(r+1)^{1-\beta} \mu'(r) \\ &\lesssim \tilde{Q}_b \tau_0^{-3} \tau(r+1)^{-k-1}\varphi'(r) + (\tau_0 h^{2/3} + b^{-\beta+1}) \mu'(r) \end{aligned}$$

where $\tilde{Q}_b > 0$ is some constant depending only on b and we have used that $k < (2\delta - 1)/5$ in all three cases. Taking h small enough, depending on τ_0 , and b big enough, independent of h and τ_0 , we get the bound

$$(2.13) \quad B(r) \leq C\tilde{Q}_b\tau_0^{-3}\tau(r+1)^{-k-1}\varphi'(r) + \frac{E}{4}\mu'(r)$$

with some constant $C > 0$. In this case we get (2.9) from (2.12) and (2.13) by taking τ_0 big enough depending on b and C but independent of h .

Let now $\frac{a}{2} < r < a$. Then we have the bound

$$\begin{aligned} B(r) &\lesssim \left(\frac{\mu(r)}{\mu'(r)}\right)^2 \left(h^{-1}(r+1)^{-\delta} + |\varphi''(r)|\right)^2 \mu'(r) + (r+1)^{-\beta+1}\mu'(r) \\ &\lesssim \left(h^{-2}(r+1)^{2-2\delta} + \tau^2(r+1)^{-2k}\right) \mu'(r) + a^{-\beta+1}\mu'(r) \\ &\lesssim \left(h^{-2}a^{2-2\delta} + \tau^2a^{-2k}\right) \mu'(r) + a^{-\beta+1}\mu'(r) \\ &\lesssim \left(h^{2m(\delta-1)-2} + h^{2km-2/3} + h^{m(\beta-1)}\right) \mu'(r) \leq \frac{E}{4}\mu'(r) \end{aligned}$$

provided h is taken small enough. Again, this bound together with (2.12) imply (2.9).

It remains to consider the case $r > a$. Using (2.5) we get

$$\begin{aligned} B(r) &\lesssim \frac{(\mu(r)(h^{-1}(r+1)^{-\delta}))^2}{\mu'(r)} + (r+1)^{-\beta}\mu(r) \\ &\lesssim \epsilon^{-2}h^{-2}a^{4k}(r+1)^{4s-2\delta}\mu'(r) + \epsilon^{-1}a^{2k}(r+1)^{2s-\beta}\mu'(r) \\ &\lesssim \left(\epsilon^{-2}h^{-2}a^{4k+4s-2\delta} + \epsilon^{-1}a^{2k+2s-\beta}\right) \mu'(r) \\ &\lesssim \left(h^{2m(\delta-2k-2s)-2-2\epsilon\lambda} + h^{m(\beta-2k-2s)-\epsilon\lambda}\right) \mu'(r) \end{aligned}$$

where we have used that $\epsilon^{-1} = h^{-\epsilon\lambda}$. When (1.4) holds we have

$$\begin{aligned} m(\delta - 2k - 2s) - 1 - \epsilon\lambda &= (m_0 + \epsilon\lambda T_0)(\delta - 3 + \epsilon) - 1 - \epsilon\lambda \\ &\geq m_0(\delta - 3) - 1 + \epsilon\lambda((\delta - 3)T_0 - 1) \geq \epsilon\lambda \end{aligned}$$

if we take $T_0 \geq \frac{2}{\delta-3}$, and

$$m(\beta - 2k - 2s) - \epsilon\lambda = m_0(\beta - 3) - \mathcal{O}(\epsilon\lambda) \geq m_0(\beta - 3)/2.$$

When (1.6) holds we have

$$\begin{aligned} &m(\delta - 2k - 2s) - 1 - \epsilon\lambda \\ &= (m_0 + \epsilon\lambda/2 + \epsilon T_1) \left(\delta - 3 + \frac{\epsilon\lambda}{m_0} + \epsilon(2t_1 - 1) \right) - 1 - \epsilon\lambda \\ &= m_0(\delta - 3) - 1 + (\delta - 3)\epsilon\lambda/2 - \mathcal{O}(\epsilon^2\lambda^2) \\ &\geq (\delta - 3)\epsilon\lambda/2 - \mathcal{O}(\epsilon^2\lambda^2) \geq (\delta - 3)\epsilon\lambda/3 \end{aligned}$$

and

$$\begin{aligned} &m(\beta - 2k - 2s) - \epsilon\lambda \\ &= (m_0 + \epsilon\lambda + \epsilon T_1) \left(\frac{\epsilon\lambda}{m_0} + \epsilon(2t_1 - 1) \right) - \epsilon\lambda \geq \epsilon t_1 m_0. \end{aligned}$$

When (1.9) holds we have

$$m(\delta - 2k - 2s) - 1 - \epsilon\lambda$$

$$\begin{aligned}
&= (m_1 + \nu\epsilon\lambda + \epsilon T_2) \left(\delta - 1 - \frac{4}{3m_1} + \frac{4\nu\epsilon\lambda}{3m_1^2} + \epsilon(2t_2 - 1) \right) - 1 - \epsilon\lambda \\
&= m_1(\delta - 1) - \frac{7}{3} + ((\delta - 1)\nu - 1)\epsilon\lambda + \epsilon((\delta - 1)T_2 + m_1(2t_2 - 1)) - \mathcal{O}(\epsilon^2\lambda^2) \geq \epsilon m_1 t_2
\end{aligned}$$

and

$$\begin{aligned}
& m(\beta - 2k - 2s) - \epsilon\lambda \\
&= (m_1 + \nu\epsilon\lambda + \epsilon T_2) \left(\beta - 1 - \frac{4}{3m_1} + \frac{4\nu\epsilon\lambda}{3m_1^2} + \epsilon(2t_2 - 1) \right) - \epsilon\lambda \\
&= m_1(\beta - 1) - \frac{4}{3} + ((\beta - 1)\nu - 1)\epsilon\lambda + \epsilon((\beta - 1)T_2 + m_1(2t_2 - 1)) - \mathcal{O}(\epsilon^2\lambda^2) \geq \epsilon m_1 t_2.
\end{aligned}$$

When (1.10) holds we have

$$\begin{aligned}
& m(\delta - 2k - 2s) - 1 - \epsilon\lambda \\
&= (m_2 + \nu\epsilon\lambda + \epsilon T_3) \left(\delta - \beta + \frac{(\beta - 1)\nu\epsilon\lambda}{m_2} + \epsilon(2t_3 - 1) \right) - 1 - \epsilon\lambda \\
&= m_2(\delta - \beta) - 1 + ((\delta - 1)\nu - 1)\epsilon\lambda + \epsilon((\delta - \beta)T_3 + m_2(2t_3 - 1)) - \mathcal{O}(\epsilon^2\lambda^2) \geq \epsilon m_2 t_3
\end{aligned}$$

and

$$\begin{aligned}
& m(\beta - 2k - 2s) - \epsilon\lambda \\
&= (m_2 + \nu\epsilon\lambda + \epsilon T_3) \left(\frac{(\beta - 1)\nu\epsilon\lambda}{m_2} + \epsilon(2t_3 - 1) \right) - \epsilon\lambda \\
&= ((\beta - 1)\nu - 1)\epsilon\lambda + \epsilon m_2(2t_3 - 1) - \mathcal{O}(\epsilon^2\lambda^2) \geq \epsilon m_2 t_3.
\end{aligned}$$

We conclude from the above inequalities that

$$(2.14) \quad h^{2m(\delta-2k-2s)-2-2\epsilon\lambda} + h^{m(\beta-2k-2s)-\epsilon\lambda} \leq \begin{cases} \epsilon^2 + h^\gamma & \text{if (1.4) holds,} \\ \epsilon^\gamma + e^{-m_0 t_1} & \text{if (1.6) holds,} \\ 2e^{-m_1 t_2} & \text{if (1.9) holds,} \\ 2e^{-m_2 t_3} & \text{if (1.10) holds,} \end{cases}$$

with some constant $\gamma > 0$ independent of h . It follows from (2.14) that taking h small enough and t_1, t_2 and t_3 large enough, independent of h , we can arrange the bound

$$(2.15) \quad B(r) \leq \frac{E}{2} \mu'(r).$$

Since in this case $A(r) = 0$, the bound (2.15) clearly implies (2.9). \square

3. CARLEMAN ESTIMATES

In this section we will prove the following

Theorem 3.1. *Suppose (1.1), (1.2) and (1.3) fulfilled and let s satisfy (2.1). Then, for all functions $f \in H^2(\mathbb{R}^n)$ such that $(|x| + 1)^s(P(h) - E \pm i\theta)f \in L^2$ and for all $0 < h \leq h_0$, $0 < \theta \leq \epsilon h a^{-2k}$, we have the estimate*

$$\begin{aligned}
(3.1) \quad & \|(|x| + 1)^{-s} e^{\varphi/h} f\|_{L^2} \leq C a^{2k} (\epsilon h)^{-1} \|(|x| + 1)^s e^{\varphi/h} (P(h) - E \pm i\theta) f\|_{L^2} \\
& + C a^k \tau \left(\frac{\theta}{\epsilon h} \right)^{1/2} \|e^{\varphi/h} f\|_{L^2}
\end{aligned}$$

with a constant $C > 0$ independent of h, θ and f .

Proof. We will adapt the proof of Theorem 3.1 of [11] to this more general case. We pass to the polar coordinates $(r, w) \in \mathbb{R}^+ \times \mathbb{S}^{n-1}$, $r = |x|$, $w = x/|x|$, and recall that $L^2(\mathbb{R}^n) = L^2(\mathbb{R}^+ \times \mathbb{S}^{n-1}, r^{n-1} dr dw)$. In what follows we denote by $\|\cdot\|$ and $\langle \cdot, \cdot \rangle$ the norm and the scalar product in $L^2(\mathbb{S}^{n-1})$. We will make use of the identity

$$(3.2) \quad r^{(n-1)/2} \Delta r^{-(n-1)/2} = \partial_r^2 + \frac{\tilde{\Delta}_w}{r^2}$$

where $\tilde{\Delta}_w = \Delta_w - \frac{1}{4}(n-1)(n-3)$ and Δ_w denotes the negative Laplace-Beltrami operator on \mathbb{S}^{n-1} . Set $u = r^{(n-1)/2} e^{\varphi/h} f$ and

$$\begin{aligned} \mathcal{P}^\pm(h) &= r^{(n-1)/2} (P(h) - E \pm i\theta) r^{-(n-1)/2}, \\ \mathcal{P}_\varphi^\pm(h) &= e^{\varphi/h} \mathcal{P}^\pm(h) e^{-\varphi/h}. \end{aligned}$$

Using (3.2) we can write the operator $\mathcal{P}^\pm(h)$ in the coordinates (r, w) as follows

$$\mathcal{P}^\pm(h) = \mathcal{D}_r^2 + \frac{\Lambda_w}{r^2} - E \pm i\theta + V$$

where we have put $\mathcal{D}_r = -ih\partial_r$ and $\Lambda_w = -h^2 \tilde{\Delta}_w$. Since the function φ depends only on the variable r , this implies

$$\mathcal{P}_\varphi^\pm(h) = \mathcal{D}_r^2 + \frac{\Lambda_w}{r^2} - E \pm i\theta - \varphi'^2 + h\varphi'' + 2i\varphi' \mathcal{D}_r + V.$$

We now write $V = \tilde{V}_S + \tilde{V}_L$ with

$$\tilde{V}_S(x) = V_S(x) + \psi_b(|x|) V_L(x)$$

and

$$\tilde{V}_L(x) = (1 - \psi_b(|x|)) V_L(x).$$

For $r > 0$, $r \neq a$, introduce the function

$$F(r) = -\langle (r^{-2} \Lambda_w - E - \varphi'(r)^2 + \tilde{V}_L(r, \cdot)) u(r, \cdot), u(r, \cdot) \rangle + \|\mathcal{D}_r u(r, \cdot)\|^2$$

where $\tilde{V}_L(r, w) := \tilde{V}_L(rw)$. It is easy to check that its first derivative is given by

$$\begin{aligned} F'(r) &= \frac{2}{r} \langle r^{-2} \Lambda_w u(r, \cdot), u(r, \cdot) \rangle + ((\varphi')^2 - \tilde{V}_L)' \|u(r, \cdot)\|^2 \\ &\quad - 2h^{-1} \text{Im} \langle \mathcal{P}_\varphi^\pm(h) u(r, \cdot), \mathcal{D}_r u(r, \cdot) \rangle \\ &\quad \pm 2\theta h^{-1} \text{Re} \langle u(r, \cdot), \mathcal{D}_r u(r, \cdot) \rangle + 4h^{-1} \varphi' \|\mathcal{D}_r u(r, \cdot)\|^2 \\ &\quad + 2h^{-1} \text{Im} \langle (\tilde{V}_S + h\varphi'') u(r, \cdot), \mathcal{D}_r u(r, \cdot) \rangle. \end{aligned}$$

Thus, if μ is the function defined in the previous section, we obtain the identity

$$\begin{aligned} \mu' F + \mu F' &= (2r^{-1} \mu - \mu') \langle r^{-2} \Lambda_w u(r, \cdot), u(r, \cdot) \rangle + (E\mu' + (\mu(\varphi')^2 - \mu \tilde{V}_L)') \|u(r, \cdot)\|^2 \\ &\quad - 2h^{-1} \mu \text{Im} \langle \mathcal{P}_\varphi^\pm(h) u(r, \cdot), \mathcal{D}_r u(r, \cdot) \rangle \\ &\quad \pm 2\theta h^{-1} \mu \text{Re} \langle u(r, \cdot), \mathcal{D}_r u(r, \cdot) \rangle + (\mu' + 4h^{-1} \varphi' \mu) \|\mathcal{D}_r u(r, \cdot)\|^2 \\ &\quad + 2h^{-1} \mu \text{Im} \langle (\tilde{V}_S + h\varphi'') u(r, \cdot), \mathcal{D}_r u(r, \cdot) \rangle. \end{aligned}$$

Using that $\Lambda_w \geq 0$ together with (2.3) we get the inequality

$$\begin{aligned} \mu' F + \mu F' &\geq (E\mu' + (\mu(\varphi')^2 - \mu \tilde{V}_L)') \|u(r, \cdot)\|^2 + (\mu' + 4h^{-1} \varphi' \mu) \|\mathcal{D}_r u(r, \cdot)\|^2 \\ &\quad - \frac{3h^{-2} \mu^2}{\mu'} \|\mathcal{P}_\varphi^\pm(h) u(r, \cdot)\|^2 - \frac{\mu'}{3} \|\mathcal{D}_r u(r, \cdot)\|^2 \\ &\quad - \theta h^{-1} \mu (\|u(r, \cdot)\|^2 + \|\mathcal{D}_r u(r, \cdot)\|^2) \end{aligned}$$

$$-3h^{-2}\mu^2(\mu' + 4h^{-1}\varphi'\mu)^{-1}\|(\tilde{V}_S + h\varphi'')u(r, \cdot)\|^2 - \frac{1}{3}(\mu' + 4h^{-1}\varphi'\mu)\|\mathcal{D}_r u(r, \cdot)\|^2.$$

In view of the assumptions (1.2) and (1.3) we have

$$\begin{aligned} (\mu\tilde{V}_L)' &= \mu'\tilde{V}_L + \mu\tilde{V}_L' = \mu'(1 - \psi_b)V_L - \mu\psi_b'V_L + \mu(1 - \psi_b)V_L' \\ &\leq \mu'(1 - \psi_b)p(r) + \mu b^{-1}\phi(r/b)p(r) + \mu(1 - \psi_b)C_2(r+1)^{-\beta} \\ &\leq \mu'(1 - \psi_b)p(b) + \mathcal{O}(r)b^{-1}\phi(r/b)p(b)\mu' + \mu(1 - \psi_b)C_2(r+1)^{-\beta} \\ &\leq \mathcal{O}(1)p(b)\mu' + \mu(1 - \psi_b)C_2(r+1)^{-\beta} \leq \frac{E}{3}\mu' + \mu(1 - \psi_b)C_2(r+1)^{-\beta} \end{aligned}$$

provided b is taken large enough. Observe also that the assumption (1.1) yields

$$|\tilde{V}_S| \leq |V_S| + \psi_b|V_L| \leq C_1(r+1)^{-\delta} + Q_b\psi_b$$

where $Q_b = \sup_{|x| \leq 2b} |V_L(x)|$. Combining the above inequalities we get

$$\begin{aligned} \mu'F + \mu F' &\geq \left(\frac{2E}{3}\mu' + (\mu(\varphi')^2)' \right) \|u(r, \cdot)\|^2 \\ &- \left(3\mu^2(\mu' + h^{-1}\varphi'\mu)^{-1}(h^{-1}C_1(r+1)^{-\delta} + h^{-1}Q_b\psi_b + |\varphi''|)^2 + \mu(1 - \psi_b)C_2(r+1)^{-\beta} \right) \|u(r, \cdot)\|^2 \\ &- \frac{3h^{-2}\mu^2}{\mu'} \|\mathcal{P}_\varphi^\pm(h)u(r, \cdot)\|^2 - \theta h^{-1}\mu (\|u(r, \cdot)\|^2 + \|\mathcal{D}_r u(r, \cdot)\|^2) \\ &= \left(\frac{2E}{3}\mu' + A(r) - B(r) \right) \|u(r, \cdot)\|^2 \\ &- \frac{3h^{-2}\mu^2}{\mu'} \|\mathcal{P}_\varphi^\pm(h)u(r, \cdot)\|^2 - \theta h^{-1}\mu (\|u(r, \cdot)\|^2 + \|\mathcal{D}_r u(r, \cdot)\|^2). \end{aligned}$$

Now we use Lemma 2.3 to conclude that

$$\begin{aligned} \mu'F + \mu F' &\geq \frac{E}{6}\mu' \|u(r, \cdot)\|^2 - \frac{3h^{-2}\mu^2}{\mu'} \|\mathcal{P}_\varphi^\pm(h)u(r, \cdot)\|^2 \\ &- \theta h^{-1}\mu (\|u(r, \cdot)\|^2 + \|\mathcal{D}_r u(r, \cdot)\|^2). \end{aligned}$$

We integrate this inequality with respect to r and use that, since $\mu(0) = 0$, we have

$$\int_0^\infty (\mu'F + \mu F') dr = 0.$$

Thus we obtain the estimate

$$\begin{aligned} \frac{E}{6} \int_0^\infty \mu' \|u(r, \cdot)\|^2 dr &\leq 3h^{-2} \int_0^\infty \frac{\mu^2}{\mu'} \|\mathcal{P}_\varphi^\pm(h)u(r, \cdot)\|^2 dr \\ (3.3) \quad &+ \theta h^{-1} \int_0^\infty \mu (\|u(r, \cdot)\|^2 + \|\mathcal{D}_r u(r, \cdot)\|^2) dr. \end{aligned}$$

Using that $\mu = \mathcal{O}(a^{2k})$ together with (2.4) and (2.6) we get from (3.3)

$$\begin{aligned} \int_0^\infty (r+1)^{-2s} \|u(r, \cdot)\|^2 dr &\leq Ca^{4k}(\epsilon h)^{-2} \int_0^\infty (r+1)^{2s} \|\mathcal{P}_\varphi^\pm(h)u(r, \cdot)\|^2 dr \\ (3.4) \quad &+ C\theta(\epsilon h)^{-1}a^{2k} \int_0^\infty (\|u(r, \cdot)\|^2 + \|\mathcal{D}_r u(r, \cdot)\|^2) dr \end{aligned}$$

with some constant $C > 0$ independent of h and θ . On the other hand, we have the identity

$$\operatorname{Re} \int_0^\infty \langle 2i\varphi' \mathcal{D}_r u(r, \cdot), u(r, \cdot) \rangle dr = \int_0^\infty h\varphi'' \|u(r, \cdot)\|^2 dr$$

and hence

$$\begin{aligned} \operatorname{Re} \int_0^\infty \langle \mathcal{P}_\varphi^\pm(h)u(r, \cdot), u(r, \cdot) \rangle dr &= \int_0^\infty \|\mathcal{D}_r u(r, \cdot)\|^2 dr + \int_0^\infty \langle r^{-2}\Lambda_w u(r, \cdot), u(r, \cdot) \rangle dr \\ &\quad - \int_0^\infty (E + \varphi'^2) \|u(r, \cdot)\|^2 dr + \int_0^\infty \langle Vu(r, \cdot), u(r, \cdot) \rangle dr. \end{aligned}$$

This implies

$$(3.5) \quad \begin{aligned} &\int_0^\infty \|\mathcal{D}_r u(r, \cdot)\|^2 dr \leq \mathcal{O}(\tau^2) \int_0^\infty \|u(r, \cdot)\|^2 dr \\ &+ \gamma \int_0^\infty (r+1)^{-2s} \|u(r, \cdot)\|^2 dr + \gamma^{-1} \int_0^\infty (r+1)^{2s} \|\mathcal{P}_\varphi^\pm(h)u(r, \cdot)\|^2 dr \end{aligned}$$

for every $\gamma > 0$. We take now γ small enough, independent of h , and recall that $\theta(\epsilon h)^{-1}a^{2k} \leq 1$. Thus, combining the estimates (3.4) and (3.5), we get

$$(3.6) \quad \begin{aligned} &\int_0^\infty (r+1)^{-2s} \|u(r, \cdot)\|^2 dr \leq Ca^{4k}(\epsilon h)^{-2} \int_0^\infty (r+1)^{2s} \|\mathcal{P}_\varphi^\pm(h)u(r, \cdot)\|^2 dr \\ &+ C\theta(\epsilon h)^{-1}a^{2k}\tau^2 \int_0^\infty \|u(r, \cdot)\|^2 dr \end{aligned}$$

with a new constant $C > 0$ independent of h and θ . Clearly, the estimate (3.6) implies (3.1). \square

4. RESOLVENT ESTIMATES

Theorems 1.1 and 1.2 can be obtained from Theorem 3.1 in the same way as in Section 4 of [11]. Here we will sketch the proof for the sake of completeness. Observe that it follows from the estimate (3.1) and Lemma 2.2 that for $0 < h \ll 1$, $0 < \theta \leq \epsilon ha^{-2k}$ and s satisfying (2.1) we have the estimate

$$(4.1) \quad \|(|x|+1)^{-s}f\|_{L^2} \leq M\|(|x|+1)^s(P(h) - E \pm i\theta)f\|_{L^2} + M\theta^{1/2}\|f\|_{L^2}$$

where

$$M = \begin{cases} Ch^{-4/3} \log(h^{-1}) & \text{if (1.4) holds,} \\ Ch^{-4/3}(\log(h^{-1}))^{3/2}(\log \log(h^{-1}))^{-1} & \text{if (1.6) holds,} \\ Ch^{-\frac{2}{3}-m_1}(\log(h^{-1}))^\nu & \text{if (1.9) holds,} \\ Ch^{-\frac{4}{3}-\frac{1}{2}(3-\beta)m_2}(\log(h^{-1}))^\nu & \text{if (1.10) holds,} \end{cases}$$

with a constant $C > 0$ independent of h and θ . On the other hand, since the operator $P(h)$ is symmetric, we have

$$(4.2) \quad \begin{aligned} &\theta\|f\|_{L^2}^2 = \pm \operatorname{Im} \langle (P(h) - E \pm i\theta)f, f \rangle_{L^2} \\ &\leq (2M)^{-2} \|(|x|+1)^{-s}f\|_{L^2}^2 + (2M)^2 \|(|x|+1)^s(P(h) - E \pm i\theta)f\|_{L^2}^2. \end{aligned}$$

We rewrite (4.2) in the form

$$(4.3) \quad M\theta^{1/2}\|f\|_{L^2} \leq \frac{1}{2} \|(|x|+1)^{-s}f\|_{L^2} + 2M^2 \|(|x|+1)^s(P(h) - E \pm i\theta)f\|_{L^2}.$$

We now combine (4.1) and (4.3) to get

$$(4.4) \quad \|(|x|+1)^{-s}f\|_{L^2} \leq 4M^2 \|(|x|+1)^s(P(h) - E \pm i\theta)f\|_{L^2}.$$

It follows from (4.4) that the resolvent estimate

$$(4.5) \quad \|(|x| + 1)^{-s}(P(h) - E \pm i\theta)^{-1}(|x| + 1)^{-s}\|_{L^2 \rightarrow L^2} \leq 4M^2$$

holds for all $0 < h \ll 1$, $0 < \theta \leq \epsilon h a^{-2k}$ and s satisfying (2.1). On the other hand, for $\theta \geq \epsilon h a^{-2k}$ the estimate (4.5) holds in a trivial way. Indeed, in this case, since the operator $P(h)$ is symmetric, the norm of the resolvent is upper bounded by $\theta^{-1} = \mathcal{O}(h^{-2km-2})$. Finally, observe that if (4.5) holds for s satisfying (2.1), it holds for all $s > 1/2$ independent of h . Indeed, given an arbitrary $s' > 1/2$ independent of h , we can arrange by taking h small enough that s defined by (2.1) is less than s' . Therefore the bound (4.5) holds with s replaced by s' as desired.

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