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Trotter–Kato Product Formulae in Dixmier Ideal



Valentin A. Zagrebnov

On the occasion of the 100th birthday of Tosio Kato

Abstract It is shown that for a certain class of the Kato functions, the Trotter–Kato product formulae converge in Dixmier ideal $\mathcal{C}_{1,\infty}$ in topology, which is defined by the $\|\cdot\|_{1,\infty}$ -norm. Moreover, the rate of convergence in this topology inherits the error-bound estimate for the corresponding operator-norm convergence.

1 Preliminaries. Symmetrically-Normed Ideals

Let \mathcal{H} be a separable Hilbert space. For the first time, the Trotter–Kato product formulae in Dixmier ideal $\mathcal{C}_{1,\infty}(\mathcal{H})$ were shortly discussed in conclusion of the paper [19]. This remark was a programme addressed to extension of results, which were known for the von Neumann–Schatten ideals $\mathcal{C}_p(\mathcal{H})$, $p \geq 1$ since [14, 24].

Note that a subtle point of this programme is the question about the rate of convergence in the corresponding topology. Since the limit of the Trotter–Kato product formula is a strongly continuous semigroup, for the von Neumann–Schatten ideals, this topology is the trace norm $\|\cdot\|_1$ on the trace-class ideal $\mathcal{C}_1(\mathcal{H})$. In this case, the limit is a Gibbs semigroup [25].

For self-adjoint Gibbs semigroups, the rate of convergence was estimated for the first time in [7, 9]. The authors considered the case of the Gibbs–Schrödinger semigroups. They scrutinised, in these papers, a dependence of the rate of conver-

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gence for the (exponential) Trotter formula on the smoothness of the potential in the Schrödinger generator.

The first abstract result in this direction was due to [19]. In this paper, a general scheme of *lifting* the operator-norm rate convergence for the Trotter–Kato product formulae was proposed and advocated for estimating the rate of the trace-norm convergence. This scheme was then improved and extended in [2] to the case of *nonself-adjoint* Gibbs semigroups.

The aim of the present note is to elucidate the question about the existence of other than the von Neumann–Schatten proper two-sided ideals $\mathfrak{I}(\mathcal{H})$ of $\mathcal{L}(\mathcal{H})$ and then to prove the (non-exponential) Trotter–Kato product formula in topology of these ideals together with an estimate of the corresponding rate of convergence. Here, a particular case of the Dixmier ideal $\mathcal{C}_{1,\infty}(\mathcal{H})$ [4, 6] is considered. To specify this ideal we recall in Section 2, the notion of *singular* trace and then of the Dixmier trace [3, 5] in Section 3. Main results about the Trotter–Kato product formulae in the Dixmier ideal $\mathcal{C}_{1,\infty}(\mathcal{H})$ are collected in Section 4. There the arguments based on the lifting scheme [19] (Theorem 5.1) are refined for proving the Trotter–Kato product formulae convergence in the $\|\cdot\|_{1,\infty}$ -topology with the rate, which is *inherited* from the operator-norm convergence.

To this end, in the rest of the present section, we recall an important auxiliary tool: the concept of *symmetrically-normed* ideals, see e.g. [8, 22].

Let $c_0 \subset l^\infty(\mathbb{N})$ be the subspace of bounded sequences $\xi = \{\xi_j\}_{j=1}^\infty \in l^\infty(\mathbb{N})$ of real numbers, which tend to zero. We denote by c_f the subspace of c_0 consisting of all sequences with *finite* number of non-zero terms (*finite sequences*).

Definition 1 A real-valued function $\phi : \xi \mapsto \phi(\xi)$ defined on c_f is called a *norming function* if it has the following properties:

$$\phi(\xi) > 0, \quad \forall \xi \in c_f, \quad \xi \neq 0, \quad (1.1)$$

$$\phi(\alpha\xi) = |\alpha|\phi(\xi), \quad \forall \xi \in c_f, \quad \forall \alpha \in \mathbb{R}, \quad (1.2)$$

$$\phi(\xi + \eta) \leq \phi(\xi) + \phi(\eta), \quad \forall \xi, \eta \in c_f, \quad (1.3)$$

$$\phi(1, 0, \dots) = 1. \quad (1.4)$$

A norming function ϕ is called to be *symmetric* if it has the additional property

$$\phi(\xi_1, \xi_2, \dots, \xi_n, 0, 0, \dots) = \phi(|\xi_{j_1}|, |\xi_{j_2}|, \dots, |\xi_{j_n}|, 0, 0, \dots) \quad (1.5)$$

for any $\xi \in c_f$ and any permutation j_1, j_2, \dots, j_n of integers $1, 2, \dots, n$.

It turns out that for any *symmetric norming* function and for any elements ξ, η from the positive *cone* c^+ of non-negative, non-increasing sequences such that $\xi, \eta \in c_f$ obey $\xi_1 \geq \xi_2 \geq \dots \geq 0, \eta_1 \geq \eta_2 \geq \dots \geq 0$ and

$$\sum_{j=1}^n \xi_j \leq \sum_{j=1}^n \eta_j, \quad n = 1, 2, \dots, \quad (1.6)$$

one gets the Ky Fan inequality [8] (Sect. 3, §3) :

$$\phi(\xi) \leq \phi(\eta). \quad (1.7)$$

Moreover, (1.7) together with the properties, (1.1), (1.2) and (1.4) imply

$$\xi_1 \leq \phi(\xi) \leq \sum_{j=1}^{\infty} \xi_j, \quad \xi \in c_f^+ := c_f \cap c^+. \quad (1.8)$$

Note that the left- and right-hand sides of (1.8) are the simplest examples of symmetric norming functions on domain c_f^+ :

$$\phi_{\infty}(\xi) := \xi_1 \quad \text{and} \quad \phi_1(\xi) := \sum_{j=1}^{\infty} \xi_j. \quad (1.9)$$

By Definition 1, the observations (1.8) and (1.9) yield

$$\begin{aligned} \phi_{\infty}(\xi) &:= \max_{j \geq 1} |\xi_j|, \quad \phi_1(\xi) := \sum_{j=1}^{\infty} |\xi_j|, \\ \phi_{\infty}(\xi) &\leq \phi(\xi) \leq \phi_1(\xi), \quad \text{for all } \xi \in c_f. \end{aligned} \quad (1.10)$$

We denote by $\xi^* := \{\xi_1^*, \xi_2^*, \dots\}$ a decreasing *rearrangement*: $\xi_1^* = \sup_{j \geq 1} |\xi_j|$, $\xi_1^* + \xi_2^* = \sup_{i \neq j} \{|\xi_i| + |\xi_j|\}$, \dots , of the sequence of absolute values $\{|\xi_n|\}_{n \geq 1}$, i.e. $\xi_1^* \geq \xi_2^* \geq \dots$. Then, $\xi \in c_f$ implies $\xi^* \in c_f$ and by (1.5) one obtains also that

$$\phi(\xi) = \phi(\xi^*), \quad \xi \in c_f. \quad (1.11)$$

Therefore, any symmetric norming function ϕ is uniquely defined by its values on the positive cone c^+ .

Now, let $\xi = \{\xi_1, \xi_2, \dots\} \in c_0$. We define

$$\xi^{(n)} := \{\xi_1, \xi_2, \dots, \xi_n, 0, 0, \dots\} \in c_f.$$

Then, if ϕ is a symmetric norming function, we set

$$c_{\phi} := \{\xi \in c_0 : \sup_n \phi(\xi^{(n)}) < +\infty\}. \quad (1.12)$$

Therefore, one gets

$$c_f \subseteq c_{\phi} \subseteq c_0.$$

Note that by (1.5)–(1.7) and (1.12), one gets

$$\phi(\xi^{(n)}) \leq \phi(\xi^{(n+1)}) \leq \sup_n \phi(\xi^{(n)}), \text{ for any } \xi \in c_\phi.$$

Then, the limit

$$\phi(\xi) := \lim_{n \rightarrow \infty} \phi(\xi^{(n)}), \quad \xi \in c_\phi,$$

exists and $\phi(\xi) = \sup_n \phi(\xi^{(n)})$, i.e. the symmetric norming function ϕ is a *normal* functional on the set c_ϕ (1.12), which is a linear space over \mathbb{R} .

By (1.3) and (1.10) one also gets that any symmetric norming function is *continuous* on c_f :

$$|\phi(\xi) - \phi(\eta)| \leq \phi(\xi - \eta) \leq \phi_1(\xi - \eta), \quad \forall \xi, \eta \in c_f.$$

Suppose that X is a compact operator, i.e. $X \in \mathcal{C}_\infty(\mathcal{H})$. Then by

$$s(X) := \{s_1(X), s_2(X), \dots\},$$

we denote the sequence of *singular values* of X counting multiplicities. We always assume that

$$s_1(X) \geq s_2(X) \geq \dots \geq s_n(X) \geq \dots$$

To define *symmetrically-normed* ideals of the compact operators $\mathcal{C}_\infty(\mathcal{H})$, we introduce the notion of a symmetric norm.

Definition 2 Let \mathfrak{I} be a two-sided ideal of $\mathcal{C}_\infty(\mathcal{H})$. A functional $\|\cdot\|_{sym} : \mathfrak{I} \rightarrow \mathbb{R}_0^+$ is called a *symmetric norm* if besides the usual properties of the operator norm $\|\cdot\|$:

$$\begin{aligned} \|X\|_{sym} &> 0, \quad \forall X \in \mathfrak{I}, \quad X \neq 0, \\ \|\alpha X\|_{sym} &= |\alpha| \|X\|_{sym}, \quad \forall X \in \mathfrak{I}, \quad \forall \alpha \in \mathbb{C}, \\ \|X + Y\|_{sym} &\leq \|X\|_{sym} + \|Y\|_{sym}, \quad \forall X, Y \in \mathfrak{I}, \end{aligned}$$

it verifies the following additional properties:

$$\|AXB\|_{sym} \leq \|A\| \|X\|_{sym} \|B\|, \quad X \in \mathfrak{I}, \quad A, B \in \mathcal{L}(\mathcal{H}), \quad (1.13)$$

$$\|\alpha X\|_{sym} = |\alpha| \|X\| = |\alpha| s_1(X), \text{ for any one-rank operator } X \in \mathfrak{I}. \quad (1.14)$$

If the condition (1.13) is replaced by

$$\begin{aligned} \|UX\|_{sym} &= \|XU\|_{sym} = \|X\|_{sym}, \quad X \in \mathfrak{I}, \\ &\text{for any unitary operator } U \text{ on } \mathcal{H}, \end{aligned} \quad (1.15)$$

then, instead of the symmetric norm, one gets definition of an *invariant norm* $\|\cdot\|_{inv}$.

First, we note that the ordinary operator norm $\|\cdot\|$ on any ideal $\mathfrak{I} \subseteq \mathcal{C}_\infty(\mathcal{H})$ is evidently a symmetric norm as well as an invariant norm.

Second, we observe that, in fact, every symmetric norm is invariant. Indeed, for any unitary operators U and V one gets by (1.13) that

$$\|UXV\|_{sym} \leq \|X\|_{sym}, \quad X \in \mathfrak{J}. \quad (1.16)$$

Since $X = U^{-1}UXVV^{-1}$, we also get $\|X\|_{sym} \leq \|UXV\|_{sym}$, which together with (1.16) yield (1.15).

Third, we claim that $\|X\|_{sym} = \|X^*\|_{sym}$. To this aim, let $X = U|X|$ be the polar representation of the operator $X \in \mathfrak{J}$. Since $U^*X = |X|$, then by (1.13), we obtain $\|X\|_{sym} = \||X|\|_{sym}$. The same line of reasoning applied to the adjoint operator $X^* = |X|U^*$ yields $\|X^*\|_{sym} = \||X|\|_{sym}$, which proves the claim.

Now, we can apply the concept of the symmetric norming functions to describe the symmetrically-normed ideals of the unital algebra of bounded operators $\mathcal{L}(\mathcal{H})$, or in general, the symmetrically-normed ideals generated by symmetric norming functions. Recall that any *proper* two-sided ideal $\mathfrak{J}(\mathcal{H})$ of $\mathcal{L}(\mathcal{H})$ is contained in compact operators $\mathcal{C}_\infty(\mathcal{H})$ and contains the set $\mathcal{K}(\mathcal{H})$ of finite-rank operators, see e.g. [20, 22]:

$$\mathcal{K}(\mathcal{H}) \subseteq \mathfrak{J}(\mathcal{H}) \subseteq \mathcal{C}_\infty(\mathcal{H}). \quad (1.17)$$

To clarify the relation between symmetric norming functions and the symmetrically-normed ideals, we mention that there is an obvious one-to-one correspondence between functions ϕ (Definition 1) on the cone c^+ and the symmetric norms $\|\cdot\|_{sym}$ on $\mathcal{K}(\mathcal{H})$. To proceed with a general setting, one needs definition of the following relation.

Definition 3 Let c_ϕ be the set of vectors (1.12) generated by a symmetric norming function ϕ . We associate with c_ϕ a subset of compact operators

$$\mathcal{C}_\phi(\mathcal{H}) := \{X \in \mathcal{C}_\infty(\mathcal{H}) : s(X) \in c_\phi\}. \quad (1.18)$$

This definition implies that the set $\mathcal{C}_\phi(\mathcal{H})$ is a proper two-sided ideal of the algebra $\mathcal{L}(\mathcal{H})$ of all bounded operators on \mathcal{H} . Setting

$$\|X\|_\phi := \phi(s(X)), \quad X \in \mathcal{C}_\phi(\mathcal{H}), \quad (1.19)$$

one obtains a *symmetric norm*: $\|\cdot\|_{sym} = \|\cdot\|_\phi$, on the ideal $\mathfrak{J} = \mathcal{C}_\phi(\mathcal{H})$ (Definition 2) such that this symmetrically-normed ideal becomes a Banach space. Then, in accordance with (1.17) and (1.18), we obtain by (1.10) that

$$\mathcal{K}(\mathcal{H}) \subseteq \mathcal{C}_1(\mathcal{H}) \subseteq \mathcal{C}_\phi(\mathcal{H}) \subseteq \mathcal{C}_\infty(\mathcal{H}). \quad (1.20)$$

Here, the trace-class operators $\mathcal{C}_1(\mathcal{H}) := \mathcal{C}_{\phi_1}(\mathcal{H})$, where the symmetric norming function ϕ_1 is defined in (1.9), and

$$\|X\|_\phi \leq \|X\|_1, \quad X \in \mathcal{C}_1(\mathcal{H}).$$

Remark 1 By virtue of inequality (1.7) and by definition of symmetric norm (1.19), the so-called *dominance* property holds: if $X \in \mathcal{C}_\phi(\mathcal{H})$, $Y \in \mathcal{C}_\infty(\mathcal{H})$ and

$$\sum_{j=1}^n s_j(Y) \leq \sum_{j=1}^n s_j(X), \quad n = 1, 2, \dots,$$

then $Y \in \mathcal{C}_\phi(\mathcal{H})$ and $\|Y\|_\phi \leq \|X\|_\phi$.

Remark 2 To distinguish in (1.20) *nontrivial* ideals \mathcal{C}_ϕ one needs some criteria based on the properties of ϕ or $\|\cdot\|_\phi$. For example, any symmetric norming function (1.11) defined by

$$\phi^{(r)}(\xi) := \sum_{j=1}^r \xi_j^*, \quad \xi \in c_f,$$

generates for arbitrary fixed $r \in \mathbb{N}$, the symmetrically-normed ideals, which are trivial in the sense that all $\mathcal{C}_{\phi^{(r)}}(\mathcal{H}) = \mathcal{C}_\infty(\mathcal{H})$. Criterion for an operator A to belong to a nontrivial ideal \mathcal{C}_ϕ is

$$M = \sup_{m \geq 1} \|P_m A P_m\|_\phi < \infty, \quad (1.21)$$

where $\{P_m\}_{m \geq 1}$ is a monotonously increasing sequence of the finite-dimensional orthogonal projectors on \mathcal{H} strongly convergent to the identity operator [8]. Note that for $A \in \mathcal{C}_\infty$, the condition (1.21) is trivial.

We consider now a couple of examples to elucidate the concept of the symmetrically-normed ideals $\mathcal{C}_\phi(\mathcal{H})$ generated by the symmetric norming functions ϕ and the rôle of the functional *trace* on these ideals.

Example 1 The von Neumann–Schatten ideals $\mathcal{C}_p(\mathcal{H})$ [21]. These ideals of $\mathcal{C}_\infty(\mathcal{H})$ are generated by symmetric norming functions $\phi(\xi) := \|\xi\|_p$, where

$$\|\xi\|_p = \left(\sum_{j=1}^{\infty} |\xi_j|^p \right)^{1/p}, \quad \xi \in c_f,$$

for $1 \leq p < +\infty$, and by

$$\|\xi\|_\infty = \sup_j |\xi_j|, \quad \xi \in c_f,$$

for $p = +\infty$. Indeed, if we set $\{\xi_j^* := s_j(X)\}_{j \geq 1}$, for $X \in \mathcal{C}_\infty(\mathcal{H})$, then the symmetric norm $\|X\|_\phi = \|s(X)\|_p$ coincides with $\|X\|_p$ and the corresponding symmetrically-normed ideal $\mathcal{C}_\phi(\mathcal{H})$ is identical to the von Neumann–Schatten class $\mathcal{C}_p(\mathcal{H})$.

By definition, for any $X \in \mathcal{C}_p(\mathcal{H})$ the trace: $|X| \mapsto \text{Tr}|X| = \sum_{j \geq 1} s_j(X) \geq 0$. The trace norm $\|X\|_1 = \text{Tr}|X|$ is finite on the trace-class operators $\mathcal{C}_1(\mathcal{H})$ and it is

infinite for $X \in \mathcal{C}_{p>1}(\mathcal{H})$. We say that for $p > 1$, the von Neumann–Schatten ideals admit *no* trace, whereas for $p = 1$ the map: $X \mapsto \text{Tr } X$ exists and it is continuous in the $\|\cdot\|_1$ -topology.

Note that by virtue of the Tr-linearity the trace norm: $\mathcal{C}_{1,+}(\mathcal{H}) \ni X \mapsto \|X\|_1$ is linear on the positive cone $\mathcal{C}_{1,+}(\mathcal{H})$ of the trace-class operators.

Example 2 Now, we consider symmetrically-normed ideals $\mathcal{C}_\Pi(\mathcal{H})$. To this aim let $\Pi = \{\pi_j\}_{j=1}^\infty \in c^+$ be a non-increasing sequence of positive numbers with $\pi_1 = 1$. We associate with Π the function

$$\phi_\Pi(\xi) = \sup_n \left\{ \frac{1}{\sum_{j=1}^n \pi_j} \sum_{j=1}^n \xi_j^* \right\}, \quad \xi \in c_f. \quad (1.22)$$

It turns out that ϕ_Π is a symmetric norming function. Then, the corresponding (1.12) set c_{ϕ_Π} is defined by

$$c_{\phi_\Pi} := \left\{ \xi \in c_f : \sup_n \frac{1}{\sum_{j=1}^n \pi_j} \sum_{j=1}^n \xi_j^* < +\infty \right\}.$$

Hence, the two-sided symmetrically-normed ideal $\mathcal{C}_\Pi(\mathcal{H}) := \mathcal{C}_{\phi_\Pi}(\mathcal{H})$ generated by symmetric norming function (1.22) consists of all those compact operators X that

$$\|X\|_{\phi_\Pi} := \sup_n \frac{1}{\sum_{j=1}^n \pi_j} \sum_{j=1}^n s_j(X) < +\infty. \quad (1.23)$$

This equation defines a symmetric norm $\|X\|_{\text{sym}} = \|X\|_{\phi_\Pi}$ on the ideal $\mathcal{C}_\Pi(\mathcal{H})$, see Definition 2.

Now, let $\Pi = \{\pi_j\}_{j=1}^\infty$, with $\pi_1 = 1$, satisfy

$$\sum_{j=1}^\infty \pi_j = +\infty \quad \text{and} \quad \lim_{j \rightarrow \infty} \pi_j = 0. \quad (1.24)$$

Then, the ideal $\mathcal{C}_\Pi(\mathcal{H})$ is *nontrivial*: $\mathcal{C}_\Pi(\mathcal{H}) \neq \mathcal{C}_\infty(\mathcal{H})$ and $\mathcal{C}_\Pi(\mathcal{H}) \neq \mathcal{C}_1(\mathcal{H})$, see Remark 2, and one has

$$\mathcal{C}_1(\mathcal{H}) \subset \mathcal{C}_\Pi(\mathcal{H}) \subset \mathcal{C}_\infty(\mathcal{H}). \quad (1.25)$$

If in addition to (1.24), the sequence $\Pi = \{\pi_j\}_{j=1}^\infty$ is *regular*, i.e. it obeys

$$\sum_{j=1}^n \pi_j = O(n\pi_n), \quad n \rightarrow \infty, \quad (1.26)$$

then $X \in \mathcal{C}_\Pi(\mathcal{H})$, if and only if

$$s_n(X) = O(\pi_n), \quad n \rightarrow \infty, \quad (1.27)$$

cf. condition (1.21). On the other hand, the asymptotics

$$s_n(X) = o(\pi_n), \quad n \rightarrow \infty,$$

implies that X belongs to some ideal in between of $\mathcal{C}_1(\mathcal{H})$ and $\mathcal{C}_{1+\varepsilon}(\mathcal{H})$.

Remark 3 A natural choice of the sequence $\{\pi_j\}_{j=1}^\infty$ that satisfies (1.24) is $\pi_j = j^{-\alpha}$, $0 < \alpha \leq 1$. Note that if $0 < \alpha < 1$, then the sequence $\Pi = \{\pi_j\}_{j=1}^\infty$ satisfies (1.26), i.e. it is regular for $\varepsilon = 1 - \alpha$. Therefore, the two-sided symmetrically-normed ideal $\mathcal{C}_\Pi(\mathcal{H})$ generated by symmetric norming function (1.22) consists of all those compact operators X , which singular values obey (1.27):

$$s_n(X) = O(n^{-\alpha}), \quad 0 < \alpha < 1, \quad n \rightarrow \infty. \quad (1.28)$$

Let $\alpha = 1/p$, $p > 1$. Then, the corresponding to (1.28) symmetrically-normed ideal is defined by

$$\mathcal{C}_{p,\infty}(\mathcal{H}) := \{X \in \mathcal{C}_\infty(\mathcal{H}) : s_n(X) = O(n^{-1/p}), \quad p > 1\},$$

which is known as the *weak- \mathcal{C}_p* ideal [20, 22].

Whilst by virtue of (1.28) the weak- \mathcal{C}_p ideal admit no trace, definition (1.23) implies that for the regular case $p > 1$ a symmetric norm on $\mathcal{C}_{p,\infty}(\mathcal{H})$ is equivalent to

$$\|X\|_{p,\infty} = \sup_n \frac{1}{n^{1-1/p}} \sum_{j=1}^n s_j(X), \quad (1.29)$$

and $\mathcal{C}_1(\mathcal{H}) \subset \mathcal{C}_{p,\infty}(\mathcal{H}) \subset \mathcal{C}_\infty(\mathcal{H})$.

2 Singular Traces

Note that (1.29) implies: $\mathcal{C}_1(\mathcal{H}) \ni A \mapsto \|A\|_{p,\infty} < \infty$, but any related linear, positive and unitarily invariant functional (*trace*), which is nontrivial on the ideal $\mathcal{C}_{p,\infty}(\mathcal{H})$ is *zero* on the set of finite-rank operators $\mathcal{K}(\mathcal{H})$. We remind that these *non-normal* traces:

$$\mathrm{Tr}_\omega(X) := \omega(\{n^{-1+1/p} \sum_{j=1}^n s_j(X)\}_{n=1}^\infty), \quad (2.1)$$

are called *singular*, [5, 13]. Here, ω is an *appropriate* linear positive normalised functional (*state*) on the Banach space $l^\infty(\mathbb{N})$ of bounded sequences. Recall that the set of these states $\mathcal{S}(l^\infty(\mathbb{N})) \subset (l^\infty(\mathbb{N}))^*$, where $(l^\infty(\mathbb{N}))^*$ is dual of the Banach space $l^\infty(\mathbb{N})$. Note that the singular trace (2.1) is continuous in topology defined by the norm (1.29).

Remark 4 (a) The *weak- \mathcal{C}_p* ideal, which is defined for $p = 1$ by

$$\mathcal{C}_{1,\infty}(\mathcal{H}) := \{X \in \mathcal{C}_\infty(\mathcal{H}) : \sum_{j=1}^n s_j(X) = O(\ln(n)), n \rightarrow \infty\}, \quad (2.2)$$

has a special interest. Note that since $\Pi = \{j^{-1}\}_{j=1}^\infty$ does not satisfy (1.26), the characterisation $s_n(X) = O(n^{-1})$ is *not* true, see (1.27), (1.28). In this case, the equivalent norm can be defined on the ideal (2.2) as

$$\|X\|_{1,\infty} := \sup_{n \in \mathbb{N}} \frac{1}{1 + \ln(n)} \sum_{j=1}^n s_j(X). \quad (2.3)$$

By virtue of (1.25) and Remark 3 one gets that $\mathcal{C}_1(\mathcal{H}) \subset \mathcal{C}_{1,\infty}(\mathcal{H}) \subset \mathcal{C}_{1+\varepsilon}(\mathcal{H})$ for any $\varepsilon > 0$ and that $\mathcal{C}_1(\mathcal{H}) \ni A \mapsto \|A\|_{1,\infty} = \|A\|_1 \geq \|A\|_{1+\varepsilon}$.

(b) In contrast to the trace norm $\|\cdot\|_1$ on the positive cone $\mathcal{C}_{1,+}(\mathcal{H})$, see Example 1, the map $X \mapsto \|X\|_{1,\infty}$ on the positive cone $\mathcal{C}_{1,\infty,+}(\mathcal{H})$ is *not* linear although homogeneous. On the space $l^\infty(\mathbb{N})$, there exists a state $\omega \in \mathcal{S}(l^\infty(\mathbb{N}))$ such that the map

$$X \mapsto \text{Tr}_\omega(X) := \omega(\{(1 + \ln(n))^{-1} \sum_{j=1}^n s_j(X)\}_{n=1}^\infty), \quad (2.4)$$

is *linear* and verifies the properties of the (singular) *trace* for any $X \in \mathcal{C}_{1,\infty}(\mathcal{H})$. We construct ω in Section 3. This particular choice of the state ω defines the *Dixmier trace* on the space $\mathcal{C}_{1,\infty}(\mathcal{H})$, which is called, in turn, the *Dixmier ideal*, see e.g. [3, 4]. The Dixmier trace (2.4) is obviously continuous in topology defined by the norm (2.3). This last property is basic for discussion in Section 4 of the Trotter–Kato product formula in the $\|\cdot\|_{p,\infty}$ -topology, for $p \geq 1$.

Example 3 With non-increasing sequence of positive numbers $\pi = \{\pi_j\}_{j=1}^\infty, \pi_1 = 1$, one can associate the symmetric norming function ϕ_π given by

$$\phi_\pi(\xi) := \sum_{j=1}^\infty \pi_j \xi_j^*, \quad \xi \in c_f.$$

The corresponding symmetrically-normed ideal is denoted by $\mathcal{C}_\pi(\mathcal{H}) := \mathcal{C}_{\phi_\pi}(\mathcal{H})$.

If the sequence π satisfies (1.24), then ideal $\mathcal{C}_\pi(\mathcal{H})$ does not coincide neither with $\mathcal{C}_\infty(\mathcal{H})$ nor with $\mathcal{C}_1(\mathcal{H})$. If, in particular, $\pi_j = j^{-\alpha}$, $j = 1, 2, \dots$, for $0 < \alpha \leq 1$,

then the corresponding ideal is denoted by $\mathcal{C}_{\infty,p}(\mathcal{H})$, $p = 1/\alpha$. The norm on this ideal is given by

$$\|X\|_{\infty,p} := \sum_{j=1}^{\infty} j^{-1/p} s_j(X), \quad p \in [1, \infty).$$

The symmetrically-normed ideal $\mathcal{C}_{\infty,1}(\mathcal{H})$ is called the *Macaev ideal* [8]. It turns out that the Dixmier ideal $\mathcal{C}_{1,\infty}(\mathcal{H})$ is dual of the Macaev ideal: $\mathcal{C}_{1,\infty}(\mathcal{H}) = \mathcal{C}_{\infty,1}(\mathcal{H})^*$.

Proposition 2.1 *The space $\mathcal{C}_{1,\infty}(\mathcal{H})$ endowed with the norm $\|\cdot\|_{1,\infty}$ is a Banach space.*

The proof is quite standard although tedious and long. We address the readers to the corresponding references, e.g. [8].

Proposition 2.2 *The space $\mathcal{C}_{1,\infty}(\mathcal{H})$ endowed with the norm $\|\cdot\|_{1,\infty}$ is a Banach ideal in the algebra of bounded operators $\mathcal{L}(\mathcal{H})$.*

Proof To this end, it is sufficient to prove that if A and C are bounded operators, then $B \in \mathcal{C}_{1,\infty}(\mathcal{H})$ implies $ABC \in \mathcal{C}_{1,\infty}(\mathcal{H})$. Recall that singular values of the operator ABC verify the estimate $s_j(ABC) \leq \|A\| \|C\| s_j(B)$. By (2.3), it yields

$$\begin{aligned} \|ABC\|_{1,\infty} &= \sup_{n \in \mathbb{N}} \frac{1}{1 + \ln(n)} \sum_{j=1}^n s_j(ABC) \leq \\ &\|A\| \|C\| \sup_{n \in \mathbb{N}} \frac{1}{1 + \ln(n)} \sum_{j=1}^n s_j(B) = \|A\| \|C\| \|B\|_{1,\infty}, \end{aligned} \quad (2.5)$$

and consequently the proof of the assertion. \square

Recall that for any $A \in \mathcal{L}(\mathcal{H})$ and all $B \in \mathcal{C}_1(\mathcal{H})$ one can define a linear functional on $\mathcal{C}_1(\mathcal{H})$ given by $\text{Tr}_{\mathcal{H}}(AB)$. The set of these functionals $\{\text{Tr}_{\mathcal{H}}(A \cdot)\}_{A \in \mathcal{L}(\mathcal{H})}$ is just the dual space $\mathcal{C}_1(\mathcal{H})^*$ of $\mathcal{C}_1(\mathcal{H})$ with the operator-norm topology. In other words, $\mathcal{L}(\mathcal{H}) = \mathcal{C}_1(\mathcal{H})^*$, in the sense that the map $A \mapsto \text{Tr}_{\mathcal{H}}(A \cdot)$ is the isometric isomorphism of $\mathcal{L}(\mathcal{H})$ onto $\mathcal{C}_1(\mathcal{H})^*$.

With the help of the *duality* relation

$$\langle A|B \rangle := \text{Tr}_{\mathcal{H}}(AB), \quad (2.6)$$

one can also describe the space $\mathcal{C}_1(\mathcal{H})_*$, which is a *predual* of $\mathcal{C}_1(\mathcal{H})$, i.e. its dual $(\mathcal{C}_1(\mathcal{H})_*)^* = \mathcal{C}_1(\mathcal{H})$. To this aim for each fixed $B \in \mathcal{C}_1(\mathcal{H})$, we consider the functionals $A \mapsto \text{Tr}_{\mathcal{H}}(AB)$ on $\mathcal{L}(\mathcal{H})$. It is known that they are not *all* continuous linear functional on bounded operators $\mathcal{L}(\mathcal{H})$, i.e. $\mathcal{C}_1(\mathcal{H}) \subset \mathcal{L}(\mathcal{H})^*$, but they yield the entire dual only of compact operators, i.e. $\mathcal{C}_1(\mathcal{H}) = \mathcal{C}_{\infty}(\mathcal{H})^*$. Hence, $\mathcal{C}_1(\mathcal{H})_* = \mathcal{C}_{\infty}(\mathcal{H})$.

Now, we note that under duality relation (2.6), the Dixmier ideal $\mathcal{C}_{1,\infty}(\mathcal{H})$ is the dual of the Macaev ideal: $\mathcal{C}_{1,\infty}(\mathcal{H}) = \mathcal{C}_{\infty,1}(\mathcal{H})^*$, where

$$\mathcal{C}_{\infty,1}(\mathcal{H}) = \{A \in \mathcal{C}_{\infty}(\mathcal{H}) : \sum_{n \geq 1} \frac{1}{n} s_n(A) < \infty\}, \quad (2.7)$$

see Example 3. By the same duality relation and by similar calculations one also obtains that the *predual* of $\mathcal{C}_{\infty,1}(\mathcal{H})$ is the ideal $\mathcal{C}_{\infty,1}(\mathcal{H})_* = \mathcal{C}_{1,\infty}^{(0)}(\mathcal{H})$, defined by

$$\mathcal{C}_{1,\infty}^{(0)}(\mathcal{H}) := \{A \in \mathcal{C}_{\infty}(\mathcal{H}) : \sum_{j \geq 1}^n s_j(A) = o(\ln(n)), n \rightarrow \infty\}. \quad (2.8)$$

By virtue of (2.2) (see Remark 4), the ideal (2.8) is not self-dual since

$$\mathcal{C}_{1,\infty}^{(0)}(\mathcal{H})^{**} = \mathcal{C}_{1,\infty}(\mathcal{H}) \supset \mathcal{C}_{1,\infty}^{(0)}(\mathcal{H}).$$

The problem which has motivated construction of the Dixmier trace in [5] was related to the question of a general definition of the *trace*, i.e. a linear, positive and unitarily invariant functional on a *proper* Banach ideal $\mathfrak{I}(\mathcal{H})$ of the unital algebra of bounded operators $\mathcal{L}(\mathcal{H})$. Since any proper two-sided ideal $\mathfrak{I}(\mathcal{H})$ of $\mathcal{L}(\mathcal{H})$ is contained in compact operators $\mathcal{C}_{\infty}(\mathcal{H})$ and contains the set $\mathcal{K}(\mathcal{H})$ of finite-rank operators ((1.17), Section 1), *domain* of definition of the trace has to coincide with the ideal $\mathfrak{I}(\mathcal{H})$.

Remark 5 The *canonical* trace $\text{Tr}_{\mathcal{H}}(\cdot)$ is nontrivial only on domain, which is the trace-class ideal $\mathcal{C}_1(\mathcal{H})$, see Example 1. We recall that it is characterised by the property of *normality*: $\text{Tr}_{\mathcal{H}}(\sup_{\alpha} B_{\alpha}) = \sup_{\alpha} \text{Tr}_{\mathcal{H}}(B_{\alpha})$, for every directed increasing bounded family $\{B_{\alpha}\}_{\alpha \in \Delta}$ of positive operators from $\mathcal{C}_{1,+}(\mathcal{H})$.

Note that every nontrivial *normal* trace on $\mathcal{L}(\mathcal{H})$ is proportional to the canonical trace $\text{Tr}_{\mathcal{H}}(\cdot)$, see e.g. [6, 20]. Therefore, the Dixmier trace (2.4) : $\mathcal{C}_{1,\infty} \ni X \mapsto \text{Tr}_{\omega}(X)$, is *not* normal.

Definition 4 A *trace* on the proper Banach ideal $\mathfrak{I}(\mathcal{H}) \subset \mathcal{L}(\mathcal{H})$ is called *singular* if it vanishes on the set $\mathcal{K}(\mathcal{H})$.

Since a singular trace is defined up to trace-class operators $\mathcal{C}_1(\mathcal{H})$, then by Remark 5 it is obviously *not* normal.

3 Dixmier Trace

Recall that only the ideal of trace-class operators has the property that on its positive cone $\mathcal{C}_{1,+}(\mathcal{H}) := \{A \in \mathcal{C}_1(\mathcal{H}) : A \geq 0\}$ the trace norm is *linear* since $\|A + B\|_1 = \text{Tr}(A + B) = \text{Tr}(A) + \text{Tr}(B) = \|A\|_1 + \|B\|_1$ for $A, B \in \mathcal{C}_{1,+}(\mathcal{H})$, see

Example 1. Then, the uniqueness of the trace norm allows to extend the trace to the whole linear space $\mathcal{C}_1(\mathcal{H})$. Imitation of this idea *fails* for other symmetrically-normed ideals.

This problem motivates the Dixmier trace construction as a certain limiting procedure involving the $\|\cdot\|_{1,\infty}$ -norm. Let $\mathcal{C}_{1,\infty,+}(\mathcal{H})$ be a positive cone of the Dixmier ideal. One can try to construct on $\mathcal{C}_{1,\infty,+}(\mathcal{H})$ a *linear, positive and unitarily* invariant functional (called *trace* \mathcal{T}) via *extension* of the limit (called *Lim*) of the sequence of properly normalised finite sums of the operator X singular values:

$$\mathcal{T}(X) := \text{Lim}_{n \rightarrow \infty} \frac{1}{1 + \ln(n)} \sum_{j=1}^n s_j(X), \quad X \in \mathcal{C}_{1,\infty,+}(\mathcal{H}). \quad (3.1)$$

First, we note that since for any unitary $U : \mathcal{H} \rightarrow U$, the singular values of $X \in \mathcal{C}_\infty(\mathcal{H})$ are invariant: $s_j(X) = s_j(U X U^*)$, it is also true for the sequence

$$\sigma_n(X) := \sum_{j=1}^n s_j(X), \quad n \in \mathbb{N}. \quad (3.2)$$

Then, the *Lim* in (3.1) (if it exists) inherits the property of *unitarity*.

Now, we comment that positivity: $X \geq 0$ implies the positivity of eigenvalues $\{\lambda_j(X)\}_{j \geq 1}$ and consequently: $\lambda_j(X) = s_j(X)$. Therefore, $\sigma_n(X) \geq 0$ and the *Lim* in (3.1) is a *positive* mapping.

The next problem with the formula for $\mathcal{T}(X)$ is its *linearity*. To proceed, we recall that if $P : \mathcal{H} \rightarrow P(\mathcal{H})$ is an orthogonal projection on a finite-dimensional subspace with $\dim P(\mathcal{H}) = n$, then for any bounded operator $X \geq 0$ the (3.2) gives

$$\sigma_n(X) = \sup_P \{\text{Tr}_{\mathcal{H}}(X P) : \dim P(\mathcal{H}) = n\}. \quad (3.3)$$

As a corollary of (3.3), one obtains the Horn–Ky Fan inequality

$$\sigma_n(X + Y) \leq \sigma_n(X) + \sigma_n(Y), \quad n \in \mathbb{N}, \quad (3.4)$$

valid, in particular, for any pair of bounded *positive* compact operators X and Y . For $\dim P(\mathcal{H}) \leq 2n$, one similarly gets from (3.3) that

$$\sigma_{2n}(X + Y) \geq \sigma_n(X) + \sigma_n(Y), \quad n \in \mathbb{N}. \quad (3.5)$$

Motivated by (3.1), we now introduce

$$\mathcal{T}_n(X) := \frac{1}{1 + \ln(n)} \sigma_n(X), \quad X \in \mathcal{C}_{1,\infty,+}(\mathcal{H}), \quad (3.6)$$

and denote by $\text{Lim}\{\mathcal{T}_n(X)\}_{n \in \mathbb{N}} := \text{Lim}_{n \rightarrow \infty} \mathcal{T}_n(X)$ the right-hand side of the functional in (3.1). Note that by (3.6), the inequalities (3.4) and (3.5) yield for $n \in \mathbb{N}$

$$\mathcal{T}_n(X + Y) \leq \mathcal{T}_n(X) + \mathcal{T}_n(Y) \quad , \quad \frac{1 + \ln(2n)}{1 + \ln(n)} \mathcal{T}_{2n}(X + Y) \geq \mathcal{T}_n(X) + \mathcal{T}_n(Y). \quad (3.7)$$

Since the functional Lim includes the limit $n \rightarrow \infty$, the inequalities (3.7) would give a desired linearity of the trace \mathcal{T} :

$$\mathcal{T}(X + Y) = \mathcal{T}(X) + \mathcal{T}(Y) \quad , \quad (3.8)$$

if one proves that the $\text{Lim}_{n \rightarrow \infty}$ in (3.1) exists and finite for X, Y as well as for $X + Y$.

To this end, we note that if the right-hand side of (3.1) is finite, then one has (3.8), and hence the $\text{Lim}\{\mathcal{T}_n(X)\}_{n \in \mathbb{N}}$ is a positive linear map $\text{Lim} : l^\infty(\mathbb{N}) \rightarrow \mathbb{R}$, which defines a state $\omega \in \mathcal{S}(l^\infty(\mathbb{N}))$ on the Banach space of sequences $\{\mathcal{T}_n(\cdot)\}_{n \in \mathbb{N}} \in l^\infty(\mathbb{N})$. To make this definition more precise, we impose on the state ω the following conditions:

- (a) $\omega(\eta) \geq 0$, for $\forall \eta = \{\eta_n \geq 0\}_{n \in \mathbb{N}}$,
- (b) $\omega(\eta) = \text{Lim}\{\eta_n\}_{n \in \mathbb{N}} = \lim_{n \rightarrow \infty} \eta_n$, if $\{\eta_n \geq 0\}_{n \in \mathbb{N}}$ is convergent .

By virtue of (a) and (b), the definitions (3.1) and (3.6) imply that for $X, Y \in \mathcal{C}_{1,\infty,+}(\mathcal{H})$, one gets

$$\mathcal{T}(X) = \omega(\{\mathcal{T}_n(X)\}_{n \in \mathbb{N}}) = \lim_{n \rightarrow \infty} \mathcal{T}_n(X) \quad , \quad (3.9)$$

$$\mathcal{T}(Y) = \omega(\{\mathcal{T}_n(Y)\}_{n \in \mathbb{N}}) = \lim_{n \rightarrow \infty} \mathcal{T}_n(Y) \quad , \quad (3.10)$$

$$\mathcal{T}(X + Y) = \omega(\{\mathcal{T}_n(X + Y)\}_{n \in \mathbb{N}}) = \lim_{n \rightarrow \infty} \mathcal{T}_n(X + Y) \quad , \quad (3.11)$$

if the limits in the right-hand sides of (3.9)–(3.11) exist.

Now, to ensure (3.8) one has to select ω in such a way that it allows to restore the equality in (3.7), when $n \rightarrow \infty$. To this aim, we impose on the state ω the condition of *dilation* \mathcal{D}_2 -invariance.

Let $\mathcal{D}_2 : l^\infty(\mathbb{N}) \rightarrow l^\infty(\mathbb{N})$ be dilation mapping $\eta \mapsto \mathcal{D}_2(\eta)$:

$$\mathcal{D}_2 : (\eta_1, \eta_2, \dots, \eta_k, \dots) \rightarrow (\eta_1, \eta_1, \eta_2, \eta_2, \dots, \eta_k, \eta_k, \dots) \quad , \quad \forall \eta \in l^\infty(\mathbb{N}). \quad (3.12)$$

We say that ω is dilation \mathcal{D}_2 -invariant if for any $\eta \in l^\infty(\mathbb{N})$ it verifies the property

$$(c) \quad \omega(\eta) = \omega(\mathcal{D}_2(\eta)). \quad (3.13)$$

We shall discuss the question of *existence*, the dilation \mathfrak{D}_2 -invariant states (the *invariant means*), on the Banach space $l^\infty(\mathbb{N})$ in Remark 6.

Let $X, Y \in \mathcal{C}_{1,\infty,+}(\mathcal{H})$. Then, applying the property (c) to the sequence $\eta = \{\xi_{2n} := \mathcal{T}_{2n}(X + Y)\}_{n=1}^\infty$, we obtain

$$\omega(\eta) = \omega(\mathfrak{D}_2(\eta)) = \omega(\xi_2, \xi_2, \xi_4, \xi_4, \xi_6, \xi_6, \dots). \quad (3.14)$$

Note that if $\xi = \{\xi_n = \mathcal{T}_n(X + Y)\}_{n=1}^\infty$, then the difference of the sequences:

$$\mathfrak{D}_2(\eta) - \xi = (\xi_2, \xi_2, \xi_4, \xi_4, \xi_6, \xi_6, \dots) - (\xi_1, \xi_2, \xi_3, \xi_4, \xi_5, \xi_6, \dots),$$

converges to zero if $\xi_{2n} - \xi_{2n-1} \rightarrow 0$ as $n \rightarrow \infty$. Then, by virtue of (3.11) and (3.14), this would imply

$$\omega(\{\mathcal{T}_{2n}(X + Y)\}_{n \in \mathbb{N}}) = \omega(\mathfrak{D}_2(\{\mathcal{T}_{2n}(X + Y)\}_{n \in \mathbb{N}})) = \omega(\{\mathcal{T}_n(X + Y)\}_{n \in \mathbb{N}}),$$

or by (3.11): $\lim_{n \rightarrow \infty} \mathcal{T}_{2n}(X + Y) = \lim_{n \rightarrow \infty} \mathcal{T}_n(X + Y)$, which by estimates (3.7) would also yield

$$\lim_{n \rightarrow \infty} \mathcal{T}_n(X + Y) = \lim_{n \rightarrow \infty} \mathcal{T}_n(X) + \lim_{n \rightarrow \infty} \mathcal{T}_n(Y). \quad (3.15)$$

Now, summarising (3.9), (3.10), (3.11) and (3.15), we obtain the linearity (3.8) of the limiting functional \mathcal{T} on the positive cone $\mathcal{C}_{1,\infty,+}(\mathcal{H})$ if it is defined by the corresponding \mathfrak{D}_2 -invariant state ω or a dilation-invariant mean.

Therefore, to finish the proof of linearity it rests only to check that $\lim_{n \rightarrow \infty} (\xi_{2n} - \xi_{2n-1}) = 0$. To this end, we note that by definitions (3.2) and (3.6), one gets

$$\begin{aligned} \xi_{2n} - \xi_{2n-1} &= \left[\frac{1}{\ln(2n)} - \frac{1}{\ln(2n-1)} \right] \sigma_{2n-1}(X + Y) \\ &\quad + \frac{1}{\ln(2n)} s_{2n}(X + Y). \end{aligned} \quad (3.16)$$

Since $X, Y \in \mathcal{C}_{1,\infty,+}(\mathcal{H})$, we obtain that $\lim_{n \rightarrow \infty} s_{2n}(X + Y) = 0$ and that $\sigma_{2n-1}(X + Y) = O(\ln(2n-1))$. Then taking into account that $(1/\ln(2n) - 1/\ln(2n-1)) = o(1/\ln(2n-1))$ one gets that for $n \rightarrow \infty$ the right-hand side of (3.16) converges to zero.

To conclude our construction of the trace $\mathcal{T}(\cdot)$ we note that by linearity (3.8) one can uniquely extend this functional from the positive cone $\mathcal{C}_{1,\infty,+}(\mathcal{H})$ to the real subspace of the Banach space $\mathcal{C}_{1,\infty}(\mathcal{H})$, and finally to the entire ideal $\mathcal{C}_{1,\infty}(\mathcal{H})$.

Definition 5 The Dixmier trace $\text{Tr}_\omega(X)$ of the operator $X \in \mathcal{C}_{1,\infty,+}(\mathcal{H})$ is the value of the linear functional (3.1):

$$\text{Tr}_\omega(X) := \text{Lim}_{n \rightarrow \infty} \frac{\sigma_n(X)}{1 + \ln(n)}, \quad (3.17)$$

where $\text{Lim}_{n \rightarrow \infty}$ is defined by a dilation-invariant state $\omega \in \mathcal{S}(l^\infty(\mathbb{N}))$ on $l^\infty(\mathbb{N})$ that satisfies the properties (a), (b) and (c). Since any self-adjoint operator $X \in \mathcal{C}_{1,\infty}(\mathcal{H})$ has the representation: $X = X_+ - X_-$, where $X_\pm \in \mathcal{C}_{1,\infty,+}(\mathcal{H})$, one gets $\text{Tr}_\omega(X) = \text{Tr}_\omega(X_+) - \text{Tr}_\omega(X_-)$. Then, for arbitrary $Z \in \mathcal{C}_{1,\infty}(\mathcal{H})$, the Dixmier trace is $\text{Tr}_\omega(Z) = \text{Tr}_\omega(\text{Re}Z) + i\text{Tr}_\omega(\text{Im}Z)$.

Note that if $X \in \mathcal{C}_{1,\infty,+}(\mathcal{H})$, then definition (3.17) of $\text{Tr}_\omega(\cdot)$ together with definition of the norm $\|\cdot\|_{1,\infty}$ in (2.3) readily imply the estimate $\text{Tr}_\omega(X) \leq \|X\|_{1,\infty}$, which, in turn, yields the inequality for arbitrary Z from the Dixmier ideal $\mathcal{C}_{1,\infty}(\mathcal{H})$:

$$|\text{Tr}_\omega(Z)| \leq \|Z\|_{1,\infty}. \quad (3.18)$$

Remark 6 A decisive for construction of the Dixmier trace $\text{Tr}_\omega(\cdot)$ is the *existence* of the invariant mean $\omega \in \mathcal{S}(l^\infty(\mathbb{N})) \subset (l^\infty(\mathbb{N}))^*$. Here, the space $(l^\infty(\mathbb{N}))^*$ is *dual* to the Banach space of bounded sequences. Then, by the Banach–Alaoglu theorem, the convex set of states $\mathcal{S}(l^\infty(\mathbb{N}))$ is *compact* in $(l^\infty(\mathbb{N}))^*$ in the weak* topology. Now, for any $\phi \in \mathcal{S}(l^\infty(\mathbb{N}))$, the relation $\phi(\mathfrak{D}_2(\cdot)) =: (\mathfrak{D}_2^*\phi)(\cdot)$ defines the dual \mathfrak{D}_2^* -dilation on the set of states. By definition (3.12), this map is such that $\mathfrak{D}_2^* : \mathcal{S}(l^\infty(\mathbb{N})) \rightarrow \mathcal{S}(l^\infty(\mathbb{N}))$, as well as continuous and affine (in fact, linear). Then, by the Markov–Kakutani theorem, the dilation \mathfrak{D}_2^* has a fix point $\omega \in \mathcal{S}(l^\infty(\mathbb{N}))$: $\mathfrak{D}_2^*\omega = \omega$. This observation justifies the existence of the *invariant mean* (c) for \mathfrak{D}_2 -dilation.

Note that Remark 6 has a straightforward extension to any \mathfrak{D}_k -dilation for $k > 2$, which is defined similar to (3.12). Since dilations for different $k \geq 2$ *commute*, the extension of the Markov–Kakutani theorem yields that the commutative family $\mathcal{F} = \{\mathfrak{D}_k^*\}_{k \geq 2}$ has in $\mathcal{S}(l^\infty(\mathbb{N}))$ the common fix point $\omega = \mathfrak{D}_2^*\omega$. Therefore, Definition 5 of the Dixmier trace does not depend on the degree $k \geq 2$ of dilation \mathfrak{D}_k .

For more details about different constructions of *invariant means* and the corresponding Dixmier trace on $\mathcal{C}_{1,\infty}(\mathcal{H})$, see e.g., [3, 13].

Proposition 3.1 *The Dixmier trace has the following properties:*

- (a) *For any bounded operator $B \in \mathcal{L}(\mathcal{H})$ and $Z \in \mathcal{C}_{1,\infty}(\mathcal{H})$ one has $\text{Tr}_\omega(ZB) = \text{Tr}_\omega(BZ)$.*
- (b) *$\text{Tr}_\omega(C) = 0$ for any operator $C \in \mathcal{C}_1(\mathcal{H})$ from the trace-class ideal, which is the closure of finite-rank operators $\mathcal{K}(\mathcal{H})$ for the $\|\cdot\|_1$ -norm.*
- (c) *The Dixmier trace $\text{Tr}_\omega : \mathcal{C}_{1,\infty}(\mathcal{H}) \rightarrow \mathbb{C}$ is continuous in the $\|\cdot\|_{1,\infty}$ -norm.*

Proof (a) Since every operator $B \in \mathcal{L}(\mathcal{H})$ is a linear combination of four unitary operators, it is sufficient to prove the equality $\text{Tr}_\omega(ZU) = \text{Tr}_\omega(UZ)$ for a unitary operator U and moreover only for $Z \in \mathcal{C}_{1,\infty,+}(\mathcal{H})$. Then, the corresponding equality follows from the unitary invariance: $s_j(Z) = s_j(ZU) = s_j(UZ) = s_j(UZU^*)$ of singular values of the positive operator Z for all $j \geq 1$.

(b) Since $C \in \mathcal{C}_1(\mathcal{H})$ yields $\|C\|_1 < \infty$, definition (3.2) implies $\sigma_n(C) \leq \|C\|_1$ for any $n \geq 1$. Then, by Definition 5 one gets $\text{Tr}_\omega(C) = 0$. Proof of the last part of the statement is standard.

(c) Since the ideal $\mathcal{C}_{1,\infty}(\mathcal{H})$ is a Banach space and $\text{Tr}_\omega : \mathcal{C}_{1,\infty}(\mathcal{H}) \rightarrow \mathbb{C}$ a linear functional, it is sufficient to consider continuity at $X = 0$. Then, let the sequence $\{X_k\}_{k \geq 1} \subset \mathcal{C}_{1,\infty}(\mathcal{H})$ converges to $X = 0$ in $\|\cdot\|_{1,\infty}$ -topology, i.e. by (2.3)

$$\lim_{k \rightarrow \infty} \|X_k\|_{1,\infty} = \lim_{k \rightarrow \infty} \sup_{n \in \mathbb{N}} \frac{1}{1 + \ln(n)} \sigma_n(X_k) = 0. \quad (3.19)$$

Since (3.18) implies $|\text{Tr}_\omega(X_k)| \leq \|X_k\|_{1,\infty}$, the assertion follows from (3.19). \square

Therefore, the Dixmier construction gives an example of a *singular* trace in the sense of Definition 4.

4 Trotter–Kato Product Formulae in the Dixmier Ideal

Let $A \geq 0$ and $B \geq 0$ be two non-negative self-adjoint operators in a separable Hilbert space \mathcal{H} and let the subspace $\mathcal{H}_0 := \overline{\text{dom}(A^{1/2}) \cap \text{dom}(B^{1/2})}$. It may happen that $\text{dom}(A) \cap \text{dom}(B) = \{0\}$, but the form-sum of these operators: $H = A + B$ is well-defined in the subspace $\mathcal{H}_0 \subseteq \mathcal{H}$.

Kato proved in [12] that under these conditions the *Trotter product formula*

$$s - \lim_{n \rightarrow \infty} (e^{-tA/n} e^{-tB/n})^n = e^{-tH} P_0, \quad t \geq 0, \quad (4.1)$$

converges in the *strong* operator topology uniformly in $t \in [0, T]$, $0 < T < +\infty$, where P_0 denotes the orthogonal projection from \mathcal{H} onto \mathcal{H}_0 . Moreover, in [12] it was also shown that the product formula is true not only for the *exponential* function e^{-x} , $x \geq 0$, but for a whole class of Borel measurable functions $f(\cdot)$ and $g(\cdot)$, which are defined on $\mathbb{R}_0^+ := [0, \infty)$ and satisfy the conditions:

$$0 \leq f(x) \leq 1, \quad f(0) = 1, \quad f'(+0) = -1, \quad (4.2)$$

$$0 \leq g(x) \leq 1, \quad g(0) = 1, \quad g'(+0) = -1. \quad (4.3)$$

Namely, the main result of [12] says that besides (4.1) one also gets convergence

$$\tau - \lim_{n \rightarrow \infty} (f(tA/n)g(tB/n))^n = e^{-tH} P_0, \quad t \geq 0, \quad (4.4)$$

uniformly in $t \in [0, T]$, $0 < T < +\infty$, if topology $\tau = s$.

Product formulae of the type (4.4) are called the *Trotter–Kato product formulae* for functions (4.2), (4.3), which are called the *Kato functions* \mathcal{K} . Note that \mathcal{K} is closed with respect to the *products* of Kato functions.

For some particular classes of the Kato functions we refer to [15, 25]. In the following, it is useful to consider instead of $f(x)g(x)$ two Kato functions: $g(x/2)f(x)g(x/2)$ and $f(x/2)g(x)f(x/2)$ that produce the self-adjoint operator families

$$F(t) := g(tB/2)f(tA)g(tB/2) \text{ and } T(t) := f(tA/2)g(tB)f(tA/2), \quad t \geq 0. \quad (4.5)$$

Since [14] it is known that the *lifting* of the topology of convergence in (4.4) to the operator norm $\tau = \|\cdot\|$ needs more conditions on operators A and B as well as on the key Kato functions $f, g \in \mathcal{K}$. One finds a discussion and more references on this subject in [25]. Here, we quote a result that will be used below for the Trotter–Kato product formulae in the Dixmier ideal $\mathcal{C}_{1,\infty}(\mathcal{H})$.

Consider the class \mathcal{K}_β of Kato functions, which is defined in [10, 11] as given below:

- (i) Measurable functions $0 \leq h \leq 1$ on \mathbb{R}_0^+ , such that $h(0) = 1$ and $h'(+0) = -1$.
- (ii) For $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) < 1$, such that $h(s) \leq 1 - \delta(\varepsilon)$ for $s \geq \varepsilon$ and

$$[h]_\beta := \sup_{s>0} \frac{|h(s) - 1 + s|}{s^\beta} < \infty, \quad \text{for } 1 < \beta \leq 2.$$

The standard examples are $h(s) = e^{-s}$ and $h(s) = (1 + a^{-1}s)^{-a}$, $a > 0$.

Below we consider the class \mathcal{K}_β and a particular case of generators A and B , such that for the Trotter–Kato product formulae the estimate of the convergence rate is *optimal*.

Proposition 4.1 ([11]) *Let $f, g \in \mathcal{K}_\beta$ with $\beta = 2$, and let A, B be non-negative self-adjoint operators in \mathcal{H} such that the operator sum $C := A + B$ is self-adjoint on domain $\text{dom}(C) := \text{dom}(A) \cap \text{dom}(B)$. Then, the Trotter–Kato product formulae converge for $n \rightarrow \infty$ in the operator norm:*

$$\begin{aligned} \| [f(tA/n)g(tB/n)]^n - e^{-tC} \| &= O(n^{-1}), \quad \| [g(tB/n)f(tA/n)]^n - e^{-tC} \| = O(n^{-1}), \\ \| F(t/n)^n - e^{-tC} \| &= O(n^{-1}), \quad \| T(t/n)^n - e^{-tC} \| = O(n^{-1}). \end{aligned}$$

Note that corresponding to each formula, error bounds $O(n^{-1})$ are equal up to coefficients $\{\Gamma_j > 0\}_{j=1}^4$ and that each rate of convergence $\Gamma_j \varepsilon(n) = O(n^{-1})$, $j = 1, \dots, 4$, is optimal.

The first *lifting* lemma yields sufficient conditions that allow to strengthen the strong operator convergence to the $\|\cdot\|_\phi$ -norm convergence in the symmetrically-normed ideal $\mathcal{C}_\phi(\mathcal{H})$.

Lemma 4.2 *Let self-adjoint operators: $X \in \mathcal{C}_\phi(\mathcal{H})$, $Y \in \mathcal{C}_\infty(\mathcal{H})$ and $Z \in \mathcal{L}(\mathcal{H})$. If $\{Z(t)\}_{t \geq 0}$ is a family of self-adjoint bounded operators such that*

$$s - \lim_{t \rightarrow +0} Z(t) = Z, \quad (4.6)$$

then

$$\lim_{r \rightarrow \infty} \sup_{t \in [0, \tau]} \|(Z(t/r) - Z)YX\|_\phi = \lim_{r \rightarrow \infty} \sup_{t \in [0, \tau]} \|XY(Z(t/r) - Z)\|_\phi = 0, \quad (4.7)$$

for any $\tau \in (0, \infty)$.

Proof Note that (4.6) yields the strong operator convergence $s - \lim_{r \rightarrow \infty} Z(t/r) = Z$, uniformly in $t \in [0, \tau]$. Since $Y \in \mathcal{C}_\infty(\mathcal{H})$, this implies

$$\lim_{r \rightarrow \infty} \sup_{t \in [0, \tau]} \|(Z(t/r) - Z)Y\| = 0. \quad (4.8)$$

Since $\mathcal{C}_\phi(\mathcal{H})$ is a Banach space with symmetric norm (1.13) that verifies $\|ZX\|_\phi \leq \|Z\|\|X\|_\phi$, one gets the estimate

$$\|(Z(t/r) - Z)YX\|_\phi \leq \|(Z(t/r) - Z)Y\|\|X\|_\phi, \quad (4.9)$$

which together with (4.8) give the proof of (4.7). \square

The second *lifting* lemma allows to estimate the rate of convergence of the Trotter–Kato product formula in the norm (1.19) of symmetrically-normed ideal $\mathcal{C}_\phi(\mathcal{H})$ via the error bound $\varepsilon(n)$ in the operator norm due to Proposition 4.1.

Lemma 4.3 *Let A and B be non-negative self-adjoint operators on the separable Hilbert space \mathcal{H} that satisfy the conditions of Proposition 4.1. Let $f, g \in \mathcal{K}_2$ be such that $F(t_0) \in \mathcal{C}_\phi(\mathcal{H})$ for some $t_0 > 0$.*

If $\Gamma_{t_0}\varepsilon(n)$, $n \in \mathbb{N}$ is the operator-norm error bound away from $t_0 > 0$ of the Trotter–Kato product formula for $\{f(tA)g(tB)\}_{t \geq 0}$, then for some $\Gamma_{2t_0}^\phi > 0$ the function $\varepsilon_\phi(n) := \{\varepsilon([n/2]) + \varepsilon([(n+1)/2])\}$, $n \in \mathbb{N}$ defines the error bound away from $2t_0$ of the Trotter–Kato product formula in the ideal $\mathcal{C}_\phi(\mathcal{H})$:

$$\|[f(tA/n)g(tB/n)]^n - e^{-tC}\|_\phi = \Gamma_{2t_0}^\phi \varepsilon_\phi(n), \quad n \rightarrow \infty. \quad t \geq 2t_0. \quad (4.10)$$

Here $[x] := \max\{l \in \mathbb{N}_0 : l \leq x\}$, for $x \in \mathbb{R}_0^+$.

Proof To prove the assertion for the family $\{f(tA)g(tB)\}_{t \geq 0}$, we use decompositions $n = k + m$, $k \in \mathbb{N}$ and $m = 2, 3, \dots, n \geq 3$, for representation

$$\begin{aligned} (f(tA/n)g(tB/n))^n - e^{-tC} = \\ ((f(tA/n)g(tB/n))^k - e^{-ktC/n}) (f(tA/n)g(tB/n))^m \\ + e^{-ktC/n} ((f(tA/n)g(tB/n))^m - e^{-mtC/n}). \end{aligned} \quad (4.11)$$

Since by conditions of lemma $F(t_0) \in \mathcal{C}_\phi(\mathcal{H})$, definition (4.5) and representation $f(tA/n)g(tB/n))^m = f(tA/n)g(tB/n)^{1/2} F(t/n)^{m-1} g(tB)^{1/2}$ yield

$$\|(f(tA/n)g(tB/n))^m\|_\phi \leq \|F(t_0)\|_\phi, \quad (4.12)$$

for t such that $t_0 \leq (m-1)t/n \leq (m-1)t_0$ and $m-1 \geq 1$.

Note that for self-adjoint operators e^{-tC} and $F(t)$ by Araki's log-order inequality for compact operators [1], one gets for $kt/n \geq t_0$ the bound of $e^{-ktC/n}$ in the $\|\cdot\|_\phi$ -norm:

$$\|e^{-ktC/n}\|_\phi \leq \|F(t_0)\|_\phi. \quad (4.13)$$

Since by Definitions 2 and 3 the ideal $\mathcal{C}_\phi(\mathcal{H})$ is a Banach space, and from (4.11)–(4.13), we obtain the estimate

$$\begin{aligned} & \|(f(tA/n)g(tB/n))^n - e^{-tC}\|_\phi \leq \\ & \|F(t_0)\|_\phi \|(f(tA/n)g(tB/n))^k - e^{-ktC/n}\| \\ & + \|F(t_0)\|_\phi \|(f(tA/n)g(tB/n))^m - e^{-mtC/n}\|, \end{aligned} \quad (4.14)$$

for t such that $(1 + (k+1)/(m-1))t_0 \leq t \leq nt_0$, $m \geq 2$ and $t \geq (1 + m/k)t_0$.

Now, by conditions of lemma $\Gamma_{t_0}\varepsilon(\cdot)$ is the operator-norm error bound away from t_0 , for any interval $[a, b] \subseteq (t_0, +\infty)$. Then, there exists $n_0 \in \mathbb{N}$ such that

$$\|(f(tA/n)g(tB/n))^k - e^{-ktC/n}\| \leq \Gamma_{t_0}\varepsilon(k), \quad (4.15)$$

for $kt/n \in [a, b] \Leftrightarrow t \in [(1 + m/k)a, (1 + m/k)b]$ and

$$\|(f(tA/n)g(tB/n))^m - e^{-mtC/n}\| \leq \Gamma_{t_0}\varepsilon(m), \quad (4.16)$$

for $mt/n \in [a, b] \Leftrightarrow t \in [(1 + k/m)a, (1 + k/m)b]$ for all $n > n_0$.

Setting $m := [(n+1)/2]$ and $k = [n/2]$, $n \geq 3$, we satisfy $n = k + m$ and $m \geq 2$, as well as, $\lim_{n \rightarrow \infty} (k+1)/(m-1) = 1$, $\lim_{n \rightarrow \infty} m/k = 1$ and $\lim_{n \rightarrow \infty} k/m = 1$. Hence, for any interval $[\tau_0, \tau] \subseteq (2t_0, +\infty)$, we find that $[\tau_0, \tau] \subseteq [(1 + (k+1)/(m-1))t_0, nt_0]$ for sufficiently large n . Moreover, choosing $[\tau_0/2, \tau/2] \subseteq (a, b) \subseteq (t_0, +\infty)$, we satisfy $[\tau_0, \tau] \subseteq [(1 + m/k)a, (1 + m/k)b]$ and $[\tau_0, \tau] \subseteq [(1 + k/m)a, (1 + k/m)b]$ again for sufficiently large n .

Thus, for any interval $[\tau_0, \tau] \subseteq (2t_0, +\infty)$ there is $n_0 \in \mathbb{N}$ such that (4.14)–(4.16) hold for $t \in [\tau_0, \tau]$ and $n \geq n_0$. Therefore, (4.14) yields the estimate

$$\begin{aligned} & \|(f(tA/n)g(tB/n))^n - e^{-tC}\|_\phi \leq \\ & \Gamma_{t_0} \|F(t_0)\|_\phi \{\varepsilon([n/2]) + \varepsilon([(n+1)/2])\}, \end{aligned} \quad (4.17)$$

for $t \in [\tau_0, \tau] \subseteq (2t_0, +\infty)$ and $n \geq n_0$. Hence, $\Gamma_{2t_0}^\phi := \Gamma_{t_0} \|F(t_0)\|_\phi$ and $\Gamma_{2t_0}^\phi \varepsilon_\phi(\cdot)$ is an error bound in the Trotter–Kato product formula (4.10) away from $2t_0$ in $\mathcal{C}_\phi(\mathcal{H})$ for the family $\{f(tA)g(tB)\}_{t \geq 0}$.

The lifting Lemma 4.2 allows to extend the proofs for other approximants: $\{g(tB)f(tA)\}_{t \geq 0}$, $\{F(t)\}_{t \geq 0}$ and $\{T(t)\}_{t \geq 0}$. \square

Now, we apply Lemma 4.3 in Dixmier ideal $\mathcal{C}_\phi(\mathcal{H}) = \mathcal{C}_{1,\infty}(\mathcal{H})$. This concerns the norm convergence (4.10), but also the estimate of the convergence rate for Dixmier traces:

$$|\mathrm{Tr}_\omega(e^{-tC}) - \mathrm{Tr}_\omega(F(t/n)^n)| \leq \Gamma^\omega \varepsilon_\omega(n). \quad (4.18)$$

In fact, it is the same (up to Γ^ω) for all Trotter–Kato approximants: $\{T(t)\}_{t \geq 0}$, $\{f(t)g(t)\}_{t \geq 0}$ and $\{g(t)f(t)\}_{t \geq 0}$.

Indeed, since by inequality (3.18) and Lemma 4.3 for $t \in [\tau_0, \tau]$ and $n \geq n_0$, one has

$$|\mathrm{Tr}_\omega(e^{-tC}) - \mathrm{Tr}_\omega(F(t/n)^n)| \leq \|e^{-tC} - F(t/n)^n\|_{1,\infty} \leq \Gamma_{2t_0}^\phi \varepsilon_{1,\infty}(n), \quad (4.19)$$

we obtain for the rate in (4.18): $\varepsilon_\omega(\cdot) = \varepsilon_{1,\infty}(\cdot)$. Therefore, the estimate of the convergence rate for Dixmier traces (4.18) and for $\|\cdot\|_{1,\infty}$ -convergence in (4.19) are *entirely* defined by the operator-norm error bound $\varepsilon(\cdot)$ from Lemma 4.3 and have the following form:

$$\varepsilon_{1,\infty}(n) := \{\varepsilon([n/2]) + \varepsilon([(n+1)/2])\}, \quad n \in \mathbb{N}. \quad (4.20)$$

Note that for the particular case of Proposition 4.1, these arguments yield for (4.17) the explicit convergence rate asymptotics $O(n^{-1})$ for the Trotter–Kato formulae and consequently, the same asymptotics for convergence rates of the Trotter–Kato formulae for the Dixmier trace (4.18), (4.19).

Therefore, we proved in the Dixmier ideal $\mathcal{C}_{1,\infty}(\mathcal{H})$ the following assertion.

Theorem 1 *Let $f, g \in \mathcal{K}_\beta$ with $\beta = 2$, and let A, B be non-negative self-adjoint operators in \mathcal{H} such that the operator sum $C := A + B$ is self-adjoint on domain $\mathrm{dom}(C) := \mathrm{dom}(A) \cap \mathrm{dom}(B)$.*

If $F(t_0) \in \mathcal{C}_{1,\infty}(\mathcal{H})$ for some $t_0 > 0$, then the Trotter–Kato product formulae converge for $n \rightarrow \infty$ in the $\|\cdot\|_{1,\infty}$ -norm:

$$\begin{aligned} \| [f(tA/n)g(tB/n)]^n - e^{-tC} \|_{1,\infty} &= O(n^{-1}), \quad \| [g(tB/n)f(tA/n)]^n - e^{-tC} \|_{1,\infty} = O(n^{-1}), \\ \| F(t/n)^n - e^{-tC} \|_{1,\infty} &= O(n^{-1}), \quad \| T(t/n)^n - e^{-tC} \|_{1,\infty} = O(n^{-1}), \end{aligned}$$

away from $2t_0$. The rate $O(n^{-1})$ of convergence is optimal in the sense of [11].

By virtue of (4.19), the same asymptotics $O(n^{-1})$ of the convergence rate are valid for convergence of the Trotter–Kato formulae for the Dixmier trace:

$$\begin{aligned} |\mathrm{Tr}_\omega([f(tA/n)g(tB/n)]^n) - \mathrm{Tr}_\omega(e^{-tC})| &= O(n^{-1}), \\ |\mathrm{Tr}_\omega(F(t/n)^n) - \mathrm{Tr}_\omega(e^{-tC})| &= O(n^{-1}), \quad |\mathrm{Tr}_\omega(T(t/n)^n) - \mathrm{Tr}_\omega(e^{-tC})| = O(n^{-1}), \end{aligned}$$

away from $2t_0$.

Optimality of the estimates in Theorem 1 is a heritage of the optimality in Proposition 4.1. Recall that, in particular, this means that in contrast to the Lie product formula for

bounded generators A and B , the symmetrisation of approximants $\{f(t)g(t)\}_{t \geq 0}$, and $\{g(t)f(t)\}_{t \geq 0}$ by $\{F(t)\}_{t \geq 0}$ and $\{T(t)\}_{t \geq 0}$, does not yield (in general) the improvement of the convergence rate, see [11, 23] and discussion in [26].

We resume that the *lifting* Lemma 4.2 and 4.3 are a general method to study the convergence in symmetrically-normed ideals $\mathcal{C}_\phi(\mathcal{H})$ as soon as it is established in $\mathcal{L}(\mathcal{H})$ in the operator-norm topology. The crucial is to check that for any of the *key* Kato functions (e.g. for $\{F(t)\}_{t \geq 0}$) there exists $t_0 > 0$ such that $F(t)|_{t \geq t_0} \in \mathcal{C}_\phi(\mathcal{H})$. Sufficient conditions for that one can be found in [16–18] or in [25].

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