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Trotter-Kato product formulae in Dixmier ideal

On the occasion of the 100th birthday of Tosio Kato

Valentin A.Zagrebnov

Abstract It is shown that for a certain class of the Kato functions the Trotter-Kato product formulae converge in Dixmier ideal $\mathcal{C}_{1,\infty}$ in topology, which is defined by the $\|\cdot\|_{1,\infty}$ -norm. Moreover, the rate of convergence in this topology inherits the error-bound estimate for the corresponding operator-norm convergence.

1 Preliminaries. Symmetrically-normed ideals

Let \mathcal{H} be a separable Hilbert space. For the first time the Trotter-Kato product formulae in Dixmier ideal $\mathcal{C}_{1,\infty}(\mathcal{H})$, were shortly discussed in conclusion of the paper [19]. This remark was a program addressed to extension of results, which were known for the von Neumann-Schatten ideals $\mathcal{C}_p(\mathcal{H})$, $p \geq 1$ since [24], [14].

Note that a subtle point of this program is the question about the rate of convergence in the corresponding topology. Since the limit of the Trotter-Kato product formula is a strongly continuous semigroup, for the von Neumann-Schatten ideals this topology is the trace-norm $\|\cdot\|_1$ on the trace-class ideal $\mathcal{C}_1(\mathcal{H})$. In this case the limit is a Gibbs semigroup [25].

For self-adjoint Gibbs semigroups the rate of convergence was estimated for the first time in [7] and [9]. The authors considered the case of the Gibbs-Schrödinger semigroups. They scrutinised in these papers a dependence of the rate of convergence for the (exponential) Trotter formula on the smoothness of the potential in the Schrödinger generator.

The first abstract result in this direction was due to [19]. In this paper a general scheme of *lifting* the operator-norm rate convergence for the Trotter-Kato product formulae was proposed and advocated for estimation the rate of the trace-norm

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convergence. This scheme was then improved and extended in [2] to the case of *nonself-adjoint* Gibbs semigroups.

The aim of the present note is to elucidate the question about the existence of other than the von Neumann-Schatten proper two-sided ideals $\mathfrak{I}(\mathcal{H})$ of $\mathcal{L}(\mathcal{H})$ and then to prove the (non-exponential) Trotter-Kato product formula in topology of these ideals together with estimate of the corresponding rate of convergence. Here a particular case of the Dixmier ideal $\mathcal{C}_{1,\infty}(\mathcal{H})$ [6], [4], is considered. To specify this ideal we recall in Section 2 the notion of *singular* trace and then of the Dixmier trace [5], [3], in Section 3. Main results about the Trotter-Kato product formulae in the Dixmier ideal $\mathcal{C}_{1,\infty}(\mathcal{H})$ are collected in Section 4. There the arguments based on the lifting scheme [19] (Theorem 5.1) are refined for proving the Trotter-Kato product formulae convergence in the $\|\cdot\|_{1,\infty}$ -topology with the rate, which is *inherited* from the operator-norm convergence.

To this end, in the rest of the present section we recall an important auxiliary tool: the concept of *symmetrically-normed* ideals, see e.g. [8], [22].

Let $c_0 \subset l^\infty(\mathbb{N})$ be the subspace of bounded sequences $\xi = \{\xi_j\}_{j=1}^\infty \in l^\infty(\mathbb{N})$ of real numbers, which tend to *zero*. We denote by c_f the subspace of c_0 consisting of all sequences with *finite* number of non-zero terms (*finite sequences*).

Definition 1. A real-valued function $\phi : \xi \mapsto \phi(\xi)$ defined on c_f is called a *norming function* if it has the following properties:

$$\phi(\xi) > 0, \quad \forall \xi \in c_f, \quad \xi \neq 0, \quad (1.1)$$

$$\phi(\alpha\xi) = |\alpha|\phi(\xi), \quad \forall \xi \in c_f, \quad \forall \alpha \in \mathbb{R}, \quad (1.2)$$

$$\phi(\xi + \eta) \leq \phi(\xi) + \phi(\eta), \quad \forall \xi, \eta \in c_f, \quad (1.3)$$

$$\phi(1, 0, \dots) = 1. \quad (1.4)$$

A norming function ϕ is called to be *symmetric* if it has the additional property

$$\phi(\xi_1, \xi_2, \dots, \xi_n, 0, 0, \dots) = \phi(|\xi_{j_1}|, |\xi_{j_2}|, \dots, |\xi_{j_n}|, 0, 0, \dots) \quad (1.5)$$

for any $\xi \in c_f$ and any permutation j_1, j_2, \dots, j_n of integers $1, 2, \dots, n$.

It turns out that for any *symmetric norming* function ϕ and for any elements ξ, η from the positive *cone* c^+ of non-negative, non-increasing sequences such that $\xi, \eta \in c_f$ obey $\xi_1 \geq \xi_2 \geq \dots \geq 0, \eta_1 \geq \eta_2 \geq \dots \geq 0$ and

$$\sum_{j=1}^n \xi_j \leq \sum_{j=1}^n \eta_j, \quad n = 1, 2, \dots, \quad (1.6)$$

one gets the Ky Fan inequality [8] (Sec.3, §3) :

$$\phi(\xi) \leq \phi(\eta). \quad (1.7)$$

Moreover, (1.7) together with the properties (1.1), (1.2) and (1.4) yield inequalities

$$\xi_1 \leq \phi(\xi) \leq \sum_{j=1}^{\infty} \xi_j, \quad \xi \in c_f^+ := c_f \cap c^+. \quad (1.8)$$

Note that the left- and right-hand sides of (1.8) are the simplest examples of symmetric norming functions on domain c_f^+ :

$$\phi_{\infty}(\xi) := \xi_1 \quad \text{and} \quad \phi_1(\xi) := \sum_{j=1}^{\infty} \xi_j. \quad (1.9)$$

By Definition 1 the observations (1.8) and (1.9) yield

$$\begin{aligned} \phi_{\infty}(\xi) &:= \max_{j \geq 1} |\xi_j|, \quad \phi_1(\xi) := \sum_{j=1}^{\infty} |\xi_j|, \\ \phi_{\infty}(\xi) &\leq \phi(\xi) \leq \phi_1(\xi), \quad \text{for all } \xi \in c_f. \end{aligned} \quad (1.10)$$

We denote by $\xi^* := \{\xi_1^*, \xi_2^*, \dots\}$ a decreasing *rearrangement*: $\xi_1^* = \sup_{j \geq 1} |\xi_j|$, $\xi_1^* + \xi_2^* = \sup_{i \neq j} \{|\xi_i| + |\xi_j|\}$, \dots , of the sequence of absolute values $\{|\xi_n|\}_{n \geq 1}$, i.e., $\xi_1^* \geq \xi_2^* \geq \dots$. Then $\xi \in c_f$ implies $\xi^* \in c_f$ and by (1.5) one obtains also that

$$\phi(\xi) = \phi(\xi^*), \quad \xi \in c_f. \quad (1.11)$$

Therefore, any symmetric norming function ϕ is uniquely defined by its values on the positive cone c^+ .

Now, let $\xi = \{\xi_1, \xi_2, \dots\} \in c_0$. We define

$$\xi^{(n)} := \{\xi_1, \xi_2, \dots, \xi_n, 0, 0, \dots\} \in c_f.$$

Then if ϕ is a symmetric norming function, we define

$$c_{\phi} := \{\xi \in c_0 : \sup_n \phi(\xi^{(n)}) < +\infty\}. \quad (1.12)$$

Therefore, one gets

$$c_f \subseteq c_{\phi} \subseteq c_0 \subset l^{\infty}.$$

Note that by (1.5)-(1.7) and (1.12) one gets

$$\phi(\xi^{(n)}) \leq \phi(\xi^{(n+1)}) \leq \sup_n \phi(\xi^{(n)}), \quad \text{for any } \xi \in c_{\phi}.$$

Then the limit

$$\phi(\xi) := \lim_{n \rightarrow \infty} \phi(\xi^{(n)}), \quad \xi \in c_{\phi}, \quad (1.13)$$

exists and $\phi(\xi) = \sup_n \phi(\xi^{(n)})$, i.e. the symmetric norming function ϕ is a *normal* functional on the set c_{ϕ} (1.12), which is a linear space over \mathbb{R} .

By virtue of (1.3) and (1.10) one also gets that any symmetric norming function is *continuous* on c_f :

$$|\phi(\xi) - \phi(\eta)| \leq \phi(\xi - \eta) \leq \phi_1(\xi - \eta), \quad \forall \xi, \eta \in c_f.$$

Suppose that X is a compact operator, i.e. $X \in \mathcal{C}_\infty(\mathcal{H})$. Then we denote by

$$s(X) := \{s_1(X), s_2(X), \dots\},$$

the sequence of *singular values* of X counting multiplicities. We always assume that

$$s_1(X) \geq s_2(X) \geq \dots \geq s_n(X) \geq \dots$$

To define *symmetrically-normed* ideals of the compact operators $\mathcal{C}_\infty(\mathcal{H})$ we introduce the notion of a symmetric norm.

Definition 2. Let \mathfrak{J} be a two-sided ideal of $\mathcal{C}_\infty(\mathcal{H})$. A functional $\|\cdot\|_{sym} : \mathfrak{J} \rightarrow \mathbb{R}_0^+$ is called a *symmetric norm* if besides the usual properties of the operator norm $\|\cdot\|$:

$$\begin{aligned} \|X\|_{sym} &> 0, \quad \forall X \in \mathfrak{J}, \quad X \neq 0, \\ \|\alpha X\|_{sym} &= |\alpha| \|X\|_{sym}, \quad \forall X \in \mathfrak{J}, \quad \forall \alpha \in \mathbb{C}, \\ \|X + Y\|_{sym} &\leq \|X\|_{sym} + \|Y\|_{sym}, \quad \forall X, Y \in \mathfrak{J}, \end{aligned}$$

it verifies the following additional properties:

$$\|AXB\|_{sym} \leq \|A\| \|X\|_{sym} \|B\|, \quad X \in \mathfrak{J}, \quad A, B \in \mathcal{L}(\mathcal{H}), \quad (1.14)$$

$$\|\alpha X\|_{sym} = |\alpha| \|X\|_{sym} = |\alpha| s_1(X), \quad \text{for any rank - one operator } X \in \mathfrak{J}. \quad (1.15)$$

If the condition (1.14) is replaced by

$$\begin{aligned} \|UX\|_{sym} &= \|XU\|_{sym} = \|X\|_{sym}, \quad X \in \mathfrak{J}, \\ &\text{for any unitary operator } U \text{ on } \mathcal{H}, \end{aligned} \quad (1.16)$$

then, instead of the symmetric norm, one gets definition of *invariant norm* $\|\cdot\|_{inv}$.

First, we note that the ordinary operator norm $\|\cdot\|$ on any ideal $\mathfrak{J} \subseteq \mathcal{C}_\infty(\mathcal{H})$ is evidently a symmetric norm as well as an invariant norm.

Second, we observe that in fact, every symmetric norm is invariant. Indeed, for any unitary operators U and V one gets by (1.14) that

$$\|UXV\|_{sym} \leq \|X\|_{sym}, \quad X \in \mathfrak{J}. \quad (1.17)$$

Since $X = U^{-1}UXVV^{-1}$, we also get $\|X\|_{sym} \leq \|UXV\|_{sym}$, which together with (1.17) yield (1.16).

Third, we claim that $\|X\|_{sym} = \|X^*\|_{sym}$. Let $X = U|X|$ be the polar representation of the operator $X \in \mathfrak{J}$. Since $U^*X = |X|$, then by (1.16) we obtain $\|X\|_{sym} = \||X|\|_{sym}$. The same line of reasoning applied to the adjoint operator $X^* = |X|U^*$ yields $\|X^*\|_{sym} = \||X|\|_{sym}$, that proves the claim.

Now we can apply the concept of the symmetric norming functions to describe the symmetrically-normed ideals of the unital algebra of bounded operators $\mathcal{L}(\mathcal{H})$,

or in general, the symmetrically-normed ideals generated by symmetric norming functions. Recall that any *proper* two-sided ideal $\mathfrak{I}(\mathcal{H})$ of $\mathcal{L}(\mathcal{H})$ is contained in compact operators $\mathcal{C}_\infty(\mathcal{H})$ and contains the set $\mathcal{K}(\mathcal{H})$ of finite-rank operators, see e.g. [20], [22]:

$$\mathcal{K}(\mathcal{H}) \subseteq \mathfrak{I}(\mathcal{H}) \subseteq \mathcal{C}_\infty(\mathcal{H}) . \quad (1.18)$$

To clarify the relation between symmetric norming functions and the symmetrically-normed ideals we mention that there is an obvious one-to-one correspondence between functions ϕ (Definition 1) on the cone c^+ and the symmetric norms $\|\cdot\|_{sym}$ on $\mathcal{K}(\mathcal{H})$. To proceed with a general setting one needs definition of the following relation.

Definition 3. Let c_ϕ be the set of vectors (1.12) generated by a symmetric norming function ϕ . We associate with c_ϕ a subset of compact operators

$$\mathcal{C}_\phi(\mathcal{H}) := \{X \in \mathcal{C}_\infty(\mathcal{H}) : s(X) \in c_\phi\} . \quad (1.19)$$

This definition implies that the set $\mathcal{C}_\phi(\mathcal{H})$ is a proper two-sided ideal of the algebra $\mathcal{L}(\mathcal{H})$ of all bounded operators on \mathcal{H} . Setting, see (1.13),

$$\|X\|_\phi := \phi(s(X)) , \quad X \in \mathcal{C}_\phi(\mathcal{H}) , \quad (1.20)$$

one obtains the *symmetric norm*: $\|\cdot\|_{sym} = \|\cdot\|_\phi$, on the ideal $\mathfrak{I} = \mathcal{C}_\phi(\mathcal{H})$ (Definition 2) such that this symmetrically-normed ideal becomes a Banach space. Then in accordance with (1.18) and (1.19) we obtain by (1.10) that

$$\mathcal{K}(\mathcal{H}) \subseteq \mathcal{C}_1(\mathcal{H}) \subseteq \mathcal{C}_\phi(\mathcal{H}) \subseteq \mathcal{C}_\infty(\mathcal{H}) . \quad (1.21)$$

Here the trace-class operators $\mathcal{C}_1(\mathcal{H}) := \mathcal{C}_{\phi_1}(\mathcal{H})$, where the symmetric norming function ϕ_1 is defined in (1.9), and

$$\|X\|_\phi \leq \|X\|_1 , \quad X \in \mathcal{C}_1(\mathcal{H}) .$$

Remark 1. By virtue of inequality (1.7) and by definition of symmetric norm (1.20) the so-called *dominance* property holds: if $X \in \mathcal{C}_\phi(\mathcal{H})$, $Y \in \mathcal{C}_\infty(\mathcal{H})$ and

$$\sum_{j=1}^n s_j(Y) \leq \sum_{j=1}^n s_j(X) , \quad n = 1, 2, \dots ,$$

then $Y \in \mathcal{C}_\phi(\mathcal{H})$ and $\|Y\|_\phi \leq \|X\|_\phi$.

Remark 2. To distinguish in (1.21) *nontrivial* ideals \mathcal{C}_ϕ one needs some criteria based on the properties of ϕ or of the norm $\|\cdot\|_\phi$. For example, any symmetric norming function (1.11) defined by

$$\phi^{(r)}(\xi) := \sum_{j=1}^r \xi_j^* , \quad \xi \in c_f ,$$

generates for arbitrary fixed $r \in \mathbb{N}$ the symmetrically-normed ideals, which are trivial in the sense that $\mathcal{C}_{\phi(r)}(\mathcal{H}) = \mathcal{C}_\infty(\mathcal{H})$. Criterion for an operator A to belong to a nontrivial ideal \mathcal{C}_ϕ is

$$M = \sup_{m \geq 1} \|P_m A P_m\|_\phi < \infty, \quad (1.22)$$

where $\{P_m\}_{m \geq 1}$ is a monotonously increasing sequence of the finite-dimensional orthogonal projectors on \mathcal{H} strongly convergent to the identity operator [8]. Note that for $A \in \mathcal{C}_\infty$ the condition (1.22) is trivial.

We consider now a couple of examples to elucidate the concept of the symmetrically-normed ideals $\mathcal{C}_\phi(\mathcal{H})$ generated by the symmetric norming functions ϕ and the rôle of the functional *trace* on these ideals.

Example 1. The von Neumann-Schatten ideals $\mathcal{C}_p(\mathcal{H})$ [21]. These ideals of $\mathcal{C}_\infty(\mathcal{H})$ are generated by symmetric norming functions $\phi(\xi) := \|\xi\|_p$, where

$$\|\xi\|_p = \left(\sum_{j=1}^{\infty} |\xi_j|^p \right)^{1/p}, \quad \xi \in c_f,$$

for $1 \leq p < +\infty$, and by

$$\|\xi\|_\infty = \sup_j |\xi_j|, \quad \xi \in c_f,$$

for $p = +\infty$. Indeed, if we put $\{\xi_j^* := s_j(X)\}_{j \geq 1}$, for $X \in \mathcal{C}_\infty(\mathcal{H})$, then the symmetric norm $\|X\|_\phi = \|s(X)\|_p$ coincides with $\|X\|_p$ and the corresponding symmetrically-normed ideal $\mathcal{C}_\phi(\mathcal{H})$ is identical to the von Neumann-Schatten class $\mathcal{C}_p(\mathcal{H})$.

By definition, for any $X \in \mathcal{C}_p(\mathcal{H})$ the trace: $|X| \mapsto \text{Tr}|X| = \sum_{j \geq 1} s_j(X) \geq 0$. The trace norm $\|X\|_1 = \text{Tr}|X|$ is finite on the trace-class operators $\mathcal{C}_1(\mathcal{H})$ and it is *infinite* for $X \in \mathcal{C}_{p>1}(\mathcal{H})$. We say that for $p > 1$ the von Neumann-Schatten ideals admit *no* trace, whereas for $p = 1$ the map: $X \mapsto \text{Tr}X$, exists and it is continuous in the $\|\cdot\|_1$ -topology.

Note that by virtue of the Tr-linearity the trace norm: $\mathcal{C}_{1,+}(\mathcal{H}) \ni X \mapsto \|X\|_1$ is *linear* on the positive cone $\mathcal{C}_{1,+}(\mathcal{H})$ of the trace-class operators.

Example 2. Now we consider symmetrically-normed ideals $\mathcal{C}_\Pi(\mathcal{H})$. To this aim let $\Pi = \{\pi_j\}_{j=1}^\infty \in c^+$ be a non-increasing sequence of positive numbers with $\pi_1 = 1$. We associate with Π the function

$$\phi_\Pi(\xi) = \sup_n \left\{ \frac{1}{\sum_{j=1}^n \pi_j} \sum_{j=1}^n \xi_j^* \right\}, \quad \xi \in c_f. \quad (1.23)$$

It turns out that ϕ_Π is a symmetric norming function. Then the corresponding to (1.12) set c_{ϕ_Π} is defined by

$$c_{\phi_{\Pi}} := \left\{ \xi \in c_f : \sup_n \frac{1}{\sum_{j=1}^n \pi_j} \sum_{j=1}^n \xi_j^* < +\infty \right\}.$$

Hence, the two-sided symmetrically-normed ideal $\mathcal{C}_{\Pi}(\mathcal{H}) := \mathcal{C}_{\phi_{\Pi}}(\mathcal{H})$ generated by symmetric norming function (1.23) consists of all those compact operators X that

$$\|X\|_{\phi_{\Pi}} := \sup_n \frac{1}{\sum_{j=1}^n \pi_j} \sum_{j=1}^n s_j(X) < +\infty. \quad (1.24)$$

This equation defines a symmetric norm $\|X\|_{\text{sym}} = \|X\|_{\phi_{\Pi}}$ on the ideal $\mathcal{C}_{\Pi}(\mathcal{H})$, see Definition 2.

Now let $\Pi = \{\pi_j\}_{j=1}^{\infty}$, with $\pi_1 = 1$, satisfy

$$\sum_{j=1}^{\infty} \pi_j = +\infty \quad \text{and} \quad \lim_{j \rightarrow \infty} \pi_j = 0. \quad (1.25)$$

Then the ideal $\mathcal{C}_{\Pi}(\mathcal{H})$ is *nontrivial*: $\mathcal{C}_{\Pi}(\mathcal{H}) \neq \mathcal{C}_{\infty}(\mathcal{H})$ and $\mathcal{C}_{\Pi}(\mathcal{H}) \neq \mathcal{C}_1(\mathcal{H})$, see Remark 2, and one has

$$\mathcal{C}_1(\mathcal{H}) \subset \mathcal{C}_{\Pi}(\mathcal{H}) \subset \mathcal{C}_{\infty}(\mathcal{H}). \quad (1.26)$$

If in addition to (1.25) the sequence $\Pi = \{\pi_j\}_{j=1}^{\infty}$ is *regular*, i.e. it obeys

$$\sum_{j=1}^n \pi_j = O(n\pi_n), \quad n \rightarrow \infty, \quad (1.27)$$

then $X \in \mathcal{C}_{\Pi}(\mathcal{H})$ if and only if

$$s_n(X) = O(\pi_n), \quad n \rightarrow \infty, \quad (1.28)$$

cf. condition (1.22). On the other hand, the asymptotics

$$s_n(X) = o(\pi_n), \quad n \rightarrow \infty,$$

implies that X belongs to:

$$\mathcal{C}_{\Pi}^0(\mathcal{H}) := \{X \in \mathcal{C}_{\Pi}(\mathcal{H}) : \lim_{n \rightarrow \infty} \frac{1}{\sum_{j=1}^n \pi_j} \sum_{j=1}^n s_j(X) = 0\},$$

such that $\mathcal{C}_1(\mathcal{H}) \subset \mathcal{C}_{\Pi}^0(\mathcal{H}) \subset \mathcal{C}_{\Pi}(\mathcal{H})$.

Remark 3. A natural choice of the sequence $\{\pi_j\}_{j=1}^{\infty}$ that satisfies (1.25) is $\pi_j = j^{-\alpha}$, $0 < \alpha \leq 1$. Note that if $0 < \alpha < 1$, then the sequence $\Pi = \{\pi_j\}_{j=1}^{\infty}$ satisfies (1.27), i.e. it is regular for $\varepsilon = 1 - \alpha$. Therefore, the two-sided symmetrically-normed ideal $\mathcal{C}_{\Pi}(\mathcal{H})$ generated by symmetric norming function (1.23) consists of all those compact operators X , which singular values obey (1.28):

$$s_n(X) = O(n^{-\alpha}), \quad 0 < \alpha < 1, \quad n \rightarrow \infty. \quad (1.29)$$

Let $\alpha = 1/p$, $p > 1$. Then the corresponding to (1.29) symmetrically-normed ideal defined by

$$\mathcal{C}_{p,\infty}(\mathcal{H}) := \{X \in \mathcal{C}_\infty(\mathcal{H}) : s_n(X) = O(n^{-1/p}), p > 1\},$$

is known as the *weak- \mathcal{C}_p* ideal [20], [22].

Whilst by virtue of (1.29) the weak- \mathcal{C}_p ideal admit no trace, definition (1.24) implies that for the regular case $p > 1$ a symmetric norm on $\mathcal{C}_{p,\infty}(\mathcal{H})$ is equivalent to

$$\|X\|_{p,\infty} = \sup_n \frac{1}{n^{1-1/p}} \sum_{j=1}^n s_j(X), \quad (1.30)$$

and it is obvious that $\mathcal{C}_1(\mathcal{H}) \subset \mathcal{C}_{p,\infty}(\mathcal{H}) \subset \mathcal{C}_\infty(\mathcal{H})$. Taking into account the Hölder inequality one can to refine these inclusions for $1 \leq q \leq p$ as follows: $\mathcal{C}_1(\mathcal{H}) \subseteq \mathcal{C}_q(\mathcal{H}) \subseteq \mathcal{C}_{p,\infty}(\mathcal{H}) \subset \mathcal{C}_\infty(\mathcal{H})$.

2 Singular traces

Note that (1.30) implies: $\mathcal{C}_1(\mathcal{H}) \ni A \mapsto \|A\|_{p,\infty} < \infty$, but any related to the ideal $\mathcal{C}_{p,\infty}(\mathcal{H})$ linear, positive, and unitarily invariant functional (*trace*) is zero on the set of finite-rank operators $\mathcal{K}(\mathcal{H})$, or trivial. We remind that these not *normal* traces:

$$\mathrm{Tr}_\omega(X) := \omega\left(\left\{n^{-1+1/p} \sum_{j=1}^n s_j(X)\right\}_{n=1}^\infty\right), \quad (2.1)$$

are called *singular*, [5], [13]. Here ω is an *appropriate* linear positive normalised functional (*state*) on the Banach space $l^\infty(\mathbb{N})$ of bounded sequences. Recall that the set of the states $\mathcal{S}(l^\infty(\mathbb{N})) \subset (l^\infty(\mathbb{N}))^*$, where $(l^\infty(\mathbb{N}))^*$ is dual of the Banach space $l^\infty(\mathbb{N})$. The singular trace (2.1) is continuous in topology defined by the norm (1.30).

Remark 4. (a) The *weak- \mathcal{C}_p* ideal, which is defined for $p = 1$ by

$$\mathcal{C}_{1,\infty}(\mathcal{H}) := \{X \in \mathcal{C}_\infty(\mathcal{H}) : \sum_{j=1}^n s_j(X) = O(\ln(n)), n \rightarrow \infty\}, \quad (2.2)$$

has a special interest. Note that since $\Pi = \{j^{-1}\}_{j=1}^\infty$ does not satisfy (1.27), the characterisation $s_n(X) = O(n^{-1})$, is *not* true, see (1.28), (1.29). In this case the equivalent norm can be defined on the ideal (2.2) as

$$\|X\|_{1,\infty} := \sup_{n \in \mathbb{N}} \frac{1}{1 + \ln(n)} \sum_{j=1}^n s_j(X). \quad (2.3)$$

By, virtute of (1.26) and Remark 3 one gets that $\mathcal{C}_1(\mathcal{H}) \subset \mathcal{C}_{1,\infty}(\mathcal{H})$ and that $\mathcal{C}_1(\mathcal{H}) \ni A \mapsto \|A\|_{1,\infty} < \infty$.

(b) In contrast to linearity of the trace-norm $\|\cdot\|_1$ on the positive cone $\mathcal{C}_{1,+}(\mathcal{H})$, see Example 1, the map $X \mapsto \|X\|_{1,\infty}$ on the positive cone $\mathcal{C}_{1,\infty,+}(\mathcal{H})$ is *not* linear. Although this map is homogeneous: $\alpha A \mapsto \alpha \|A\|_{1,\infty}$, $\alpha \geq 0$, for $A, B \in \mathcal{C}_{1,\infty,+}(\mathcal{H})$ one gets that in general $\|A+B\|_{1,\infty} \neq \|A\|_{1,\infty} + \|B\|_{1,\infty}$.

But it is known that on the space $l^\infty(\mathbb{N})$ there exists a state $\omega \in \mathcal{S}(l^\infty(\mathbb{N}))$ such that the map

$$X \mapsto \text{Tr}_\omega(X) := \omega(\{(1 + \ln(n))^{-1} \sum_{j=1}^n s_j(X)\}_{n=1}^\infty), \quad (2.4)$$

is *linear* and verifies the properties of the (singular) *trace* for any $X \in \mathcal{C}_{1,\infty}(\mathcal{H})$. We construct ω in Section 3. This particular choice of the state ω defines the *Dixmier trace* on the space $\mathcal{C}_{1,\infty}(\mathcal{H})$, which is called, in turn, the *Dixmier ideal*, see e.g. [3], [4]. The Dixmier trace (2.4) is obviously continuous in topology defined by the norm (2.3). This last property is basic for discussion in Section 4 of the Trotter-Kato product formula in the $\|\cdot\|_{p,\infty}$ -topology, for $p \geq 1$.

Example 3. With non-increasing sequence of positive numbers $\pi = \{\pi_j\}_{j=1}^\infty$, $\pi_1 = 1$, one can associate the symmetric norming function ϕ_π given by

$$\phi_\pi(\xi) := \sum_{j=1}^\infty \pi_j \xi_j^*, \quad \xi \in c_f.$$

The corresponding symmetrically-normed ideal we denote by $\mathcal{C}_\pi(\mathcal{H}) := \mathcal{C}_{\phi_\pi}(\mathcal{H})$.

If the sequence π satisfies (1.25), then ideal $\mathcal{C}_\pi(\mathcal{H})$ does not coincide neither with $\mathcal{C}_\infty(\mathcal{H})$ nor with $\mathcal{C}_1(\mathcal{H})$. If, in particular, $\pi_j = j^{-\alpha}$, $j = 1, 2, \dots$, for $0 < \alpha \leq 1$, then the corresponding ideal is denoted by $\mathcal{C}_{\infty,p}(\mathcal{H})$, $p = 1/\alpha$. The norm on this ideal is given by

$$\|X\|_{\infty,p} := \sum_{j=1}^\infty j^{-1/p} s_j(X), \quad p \in [1, \infty).$$

The symmetrically-normed ideal $\mathcal{C}_{\infty,1}(\mathcal{H})$ is called the *Macaev ideal* [8]. It turns out that the Dixmier ideal $\mathcal{C}_{1,\infty}(\mathcal{H})$ is dual of the Macaev ideal: $\mathcal{C}_{1,\infty}(\mathcal{H}) = \mathcal{C}_{\infty,1}(\mathcal{H})^*$.

Proposition 2.1 *The space $\mathcal{C}_{1,\infty}(\mathcal{H})$ endowed by the norm $\|\cdot\|_{1,\infty}$ is a Banach space.*

The proof is quite standard although tedious and long. We address the readers to the corresponding references, e.g. [8].

Proposition 2.2 *The space $\mathcal{C}_{1,\infty}(\mathcal{H})$ endowed by the norm $\|\cdot\|_{1,\infty}$ is a Banach ideal in the algebra of bounded operators $\mathcal{L}(\mathcal{H})$.*

Proof. To this end it is sufficient to prove that if A and C are bounded operators, then $B \in \mathcal{C}_{1,\infty}(\mathcal{H})$ implies $ABC \in \mathcal{C}_{1,\infty}(\mathcal{H})$. Recall that singular values of the operator ABC verify the estimate $s_j(ABC) \leq \|A\| \|C\| s_j(B)$. By (2.3) it yields

$$\begin{aligned} \|ABC\|_{1,\infty} &= \sup_{n \in \mathbb{N}} \frac{1}{1 + \ln(n)} \sum_{j=1}^n s_j(ABC) \leq \\ &\|A\| \|C\| \sup_{n \in \mathbb{N}} \frac{1}{1 + \ln(n)} \sum_{j=1}^n s_j(B) = \|A\| \|C\| \|B\|_{1,\infty}, \end{aligned} \quad (2.5)$$

and consequently the proof of the assertion. \square

Recall that for any $A \in \mathcal{L}(\mathcal{H})$ and all $B \in \mathcal{C}_1(\mathcal{H})$ one can define a linear functional on $\mathcal{C}_1(\mathcal{H})$ given by $\text{Tr}_{\mathcal{H}}(AB)$. The set of these functionals $\{\text{Tr}_{\mathcal{H}}(A \cdot)\}_{A \in \mathcal{L}(\mathcal{H})}$ is just the dual space $\mathcal{C}_1(\mathcal{H})^*$ of $\mathcal{C}_1(\mathcal{H})$ with the operator-norm topology. In other words, $\mathcal{L}(\mathcal{H}) = \mathcal{C}_1(\mathcal{H})^*$, in the sense that the map $A \mapsto \text{Tr}_{\mathcal{H}}(A \cdot)$ is the isometric isomorphism of $\mathcal{L}(\mathcal{H})$ onto $\mathcal{C}_1(\mathcal{H})^*$.

With help of the *duality* relation

$$\langle A|B \rangle := \text{Tr}_{\mathcal{H}}(AB), \quad (2.6)$$

one can also describe the space $\mathcal{C}_1(\mathcal{H})_*$, which is a *predual* of $\mathcal{C}_1(\mathcal{H})$, i.e., its dual $(\mathcal{C}_1(\mathcal{H})_*)^* = \mathcal{C}_1(\mathcal{H})$. To this aim for each fixed $B \in \mathcal{C}_1(\mathcal{H})$ we consider the functionals $A \mapsto \text{Tr}_{\mathcal{H}}(AB)$ on $\mathcal{L}(\mathcal{H})$. It is known that they are not *all* continuous linear functional on bounded operators $\mathcal{L}(\mathcal{H})$, i.e., $\mathcal{C}_1(\mathcal{H}) \subset \mathcal{L}(\mathcal{H})^*$, but they yield the entire dual only of compact operators, i.e., $\mathcal{C}_1(\mathcal{H}) = \mathcal{C}_{\infty}(\mathcal{H})^*$. Hence, $\mathcal{C}_1(\mathcal{H})_* = \mathcal{C}_{\infty}(\mathcal{H})$.

Now we note that under duality relation (2.6) the Dixmier ideal $\mathcal{C}_{1,\infty}(\mathcal{H})$ is the dual of the Macaev ideal: $\mathcal{C}_{1,\infty}(\mathcal{H}) = \mathcal{C}_{\infty,1}(\mathcal{H})^*$, where

$$\mathcal{C}_{\infty,1}(\mathcal{H}) = \{A \in \mathcal{C}_{\infty}(\mathcal{H}) : \sum_{n \geq 1} \frac{1}{n} s_n(A) < \infty\}, \quad (2.7)$$

see Example 3. By the same duality relation and by similar calculations one also obtains that the *predual* of $\mathcal{C}_{\infty,1}(\mathcal{H})$ is the ideal $\mathcal{C}_{\infty,1}(\mathcal{H})_* = \mathcal{C}_{1,\infty}^{(0)}(\mathcal{H})$, defined by

$$\mathcal{C}_{1,\infty}^{(0)}(\mathcal{H}) := \{A \in \mathcal{C}_{\infty}(\mathcal{H}) : \sum_{j \geq 1} s_j(A) = o(\ln(n)), n \rightarrow \infty\}. \quad (2.8)$$

By virtue of (2.2) (see Remark 4) the ideal (2.8) is not self-dual since

$$\mathcal{C}_{1,\infty}^{(0)}(\mathcal{H})^{**} = \mathcal{C}_{1,\infty}(\mathcal{H}) \supset \mathcal{C}_{1,\infty}^{(0)}(\mathcal{H}).$$

The problem which has motivated construction of the Dixmier trace in [5] was related to the question of a general definition of the *trace*, i.e. a linear, positive, and unitarily invariant functional on a *proper* Banach ideal $\mathfrak{J}(\mathcal{H})$ of the unital algebra

of bounded operators $\mathcal{L}(\mathcal{H})$. Since any proper two-sided ideal $\mathfrak{J}(\mathcal{H})$ of $\mathcal{L}(\mathcal{H})$ is contained in compact operators $\mathcal{C}_\infty(\mathcal{H})$ and contains the set $\mathcal{H}(\mathcal{H})$ of finite-rank operators ((1.18), Section 1), *domain* of definition of the trace has to coincide with the ideal $\mathfrak{J}(\mathcal{H})$.

Remark 5. The *canonical* trace $\text{Tr}_{\mathcal{H}}(\cdot)$ is nontrivial only on domain, which is the trace-class ideal $\mathcal{C}_1(\mathcal{H})$, see Example 1. We recall that it is characterised by the property of *normality*: $\text{Tr}_{\mathcal{H}}(\sup_\alpha B_\alpha) = \sup_\alpha \text{Tr}_{\mathcal{H}}(B_\alpha)$, for every directed increasing bounded family $\{B_\alpha\}_{\alpha \in \Delta}$ of positive operators from $\mathcal{C}_{1,+}(\mathcal{H})$.

Note that every nontrivial *normal* trace on $\mathcal{L}(\mathcal{H})$ is proportional to the canonical trace $\text{Tr}_{\mathcal{H}}(\cdot)$, see e.g. [6], [20]. Therefore, the Dixmier trace (2.4) : $\mathcal{C}_{1,\infty} \ni X \mapsto \text{Tr}_\omega(X)$, is *not* normal.

Definition 4. A *trace* on the proper Banach ideal $\mathfrak{J}(\mathcal{H}) \subset \mathcal{L}(\mathcal{H})$ is called *singular* if it vanishes on the set $\mathcal{H}(\mathcal{H})$.

Since a singular trace is defined up to trace-class operators $\mathcal{C}_1(\mathcal{H})$, then by Remark 5 it is obviously *not* normal.

3 Dixmier trace

Recall that only the ideal of trace-class operators has the property that on its positive cone $\mathcal{C}_{1,+}(\mathcal{H}) := \{A \in \mathcal{C}_1(\mathcal{H}) : A \geq 0\}$ the trace-norm is *linear* since $\|A+B\|_1 = \text{Tr}(A+B) = \text{Tr}(A) + \text{Tr}(B) = \|A\|_1 + \|B\|_1$ for $A, B \in \mathcal{C}_{1,+}(\mathcal{H})$, see Example 1. Then the uniqueness of the trace-norm allows to extend the trace to the whole linear space $\mathcal{C}_1(\mathcal{H})$. Imitation of this idea *fails* for other symmetrically-normed ideals.

This problem motivates the Dixmier trace construction as a certain limiting procedure involving the $\|\cdot\|_{1,\infty}$ -norm. Let $\mathcal{C}_{1,\infty,+}(\mathcal{H})$ be a positive cone of the Dixmier ideal. One can try to construct on $\mathcal{C}_{1,\infty,+}(\mathcal{H})$ a *linear, positive, and unitarily invariant* functional (called *trace* \mathcal{T}) via *extension* of the limit (called *Lim*) of the sequence of properly normalised finite sums of the operator X singular values:

$$\mathcal{T}(X) := \text{Lim}_{n \rightarrow \infty} \frac{1}{1 + \ln(n)} \sum_{j=1}^n s_j(X), \quad X \in \mathcal{C}_{1,\infty,+}(\mathcal{H}). \quad (3.1)$$

First we note that since for any unitary $U : \mathcal{H} \rightarrow U$, the singular values of $X \in \mathcal{C}_\infty(\mathcal{H})$ are invariant: $s_j(X) = s_j(UXU^*)$, it is also true for the sequence

$$\sigma_n(X) := \sum_{j=1}^n s_j(X), \quad n \in \mathbb{N}. \quad (3.2)$$

Then the *Lim* in (3.1) (if it exists) inherits the property of *unitarity*.

Now we comment that positivity: $X \geq 0$, implies the positivity of eigenvalues $\{\lambda_j(X)\}_{j \geq 1}$ and consequently: $\lambda_j(X) = s_j(X)$. Therefore, $\sigma_n(X) \geq 0$ and the Lim in (3.1) is a *positive* mapping.

The next problem with the formula for $\mathcal{T}(X)$ is its *linearity*. To proceed we recall that if $P: \mathcal{H} \rightarrow P(\mathcal{H})$ is an orthogonal projection on a finite-dimensional subspace with $\dim P(\mathcal{H}) = n$, then for any bounded operator $X \geq 0$ the (3.2) gives

$$\sigma_n(X) = \sup_P \{ \text{Tr}_{\mathcal{H}}(XP) : \dim P(\mathcal{H}) = n \}. \quad (3.3)$$

As a corollary of (3.3) one obtains the Horn-Ky Fan inequality

$$\sigma_n(X+Y) \leq \sigma_n(X) + \sigma_n(Y), \quad n \in \mathbb{N}, \quad (3.4)$$

valid in particular for any pair of bounded *positive* compact operators X and Y . For $\dim P(\mathcal{H}) \leq 2n$ one similarly gets from (3.3) that

$$\sigma_{2n}(X+Y) \geq \sigma_n(X) + \sigma_n(Y), \quad n \in \mathbb{N}. \quad (3.5)$$

Motivated by (3.1) we now introduce

$$\mathcal{T}_n(X) := \frac{1}{1 + \ln(n)} \sigma_n(X), \quad X \in \mathcal{C}_{1,\infty,+}(\mathcal{H}), \quad (3.6)$$

and denote by $\text{Lim}\{\mathcal{T}_n(X)\}_{n \in \mathbb{N}} := \text{Lim}_{n \rightarrow \infty} \mathcal{T}_n(X)$ the right-hand side of the functional in (3.1). Note that by (3.6) the inequalities (3.4) and (3.5) yield for $n \in \mathbb{N}$

$$\mathcal{T}_n(X+Y) \leq \mathcal{T}_n(X) + \mathcal{T}_n(Y), \quad \frac{1 + \ln(2n)}{1 + \ln(n)} \mathcal{T}_{2n}(X+Y) \geq \mathcal{T}_n(X) + \mathcal{T}_n(Y) \quad (3.7)$$

Since the functional Lim includes the limit $n \rightarrow \infty$, the inequalities (3.7) *would* give a desired linearity of the trace \mathcal{T} :

$$\mathcal{T}(X+Y) = \mathcal{T}(X) + \mathcal{T}(Y), \quad (3.8)$$

if one proves that the $\text{Lim}_{n \rightarrow \infty}$ in (3.1) exists and finite for X, Y as well as for $X+Y$.

To this end we note that if the right-hand of (3.1) exists, then one obtains (3.8). Hence the $\text{Lim}\{\mathcal{T}_n(X)\}_{n \in \mathbb{N}}$ is a positive linear map $\text{Lim}: l^\infty(\mathbb{N}) \rightarrow \mathbb{R}$, which defines a *state* $\omega \in \mathcal{S}(l^\infty(\mathbb{N}))$ on the Banach space of sequences $\{\mathcal{T}_n(\cdot)\}_{n \in \mathbb{N}} \in l^\infty(\mathbb{N})$, such that $\mathcal{T}(X) = \omega(\{\mathcal{T}_n(X)\}_{n \in \mathbb{N}})$.

Remark 6. Scrutinising description of $\omega(\cdot)$, we infer that its values $\text{Lim}\{\mathcal{T}_n(X)\}_{n \in \mathbb{N}}$ are completely determined only by the "tail" behaviour of the sequences $\{\mathcal{T}_n(X)\}_{n \in \mathbb{N}}$ as it is defined by $\text{Lim}_{n \rightarrow \infty} \mathcal{T}_n(X)$. For example, one concludes that the state $\omega(\{\mathcal{T}_n(X)\}_{n \in \mathbb{N}}) = 0$ for the whole set c_0 of sequences: $\{\mathcal{T}_n(X)\}_{n \in \mathbb{N}} \in c_0$, which tend to zero. The same is also plausible for the non-zero converging limits.

To make this description more precise we impose on the state ω the following conditions:

- (a) $\omega(\eta) \geq 0$, for $\forall \eta = \{\eta_n \geq 0\}_{n \in \mathbb{N}}$,
 (b) $\omega(\eta) = \text{Lim}\{\eta_n\}_{n \in \mathbb{N}} = \lim_{n \rightarrow \infty} \eta_n$, if $\{\eta_n \geq 0\}_{n \in \mathbb{N}}$ is convergent.

By virtue of (a) and (b) the definitions (3.1) and (3.6) imply that for $X, Y \in \mathcal{C}_{1, \infty, +}(\mathcal{H})$ one gets

$$\mathcal{T}(X) = \omega(\{\mathcal{T}_n(X)\}_{n \in \mathbb{N}}) = \lim_{n \rightarrow \infty} \mathcal{T}_n(X), \quad (3.9)$$

$$\mathcal{T}(Y) = \omega(\{\mathcal{T}_n(Y)\}_{n \in \mathbb{N}}) = \lim_{n \rightarrow \infty} \mathcal{T}_n(Y), \quad (3.10)$$

$$\mathcal{T}(X+Y) = \omega(\{\mathcal{T}_n(X+Y)\}_{n \in \mathbb{N}}) = \lim_{n \rightarrow \infty} \mathcal{T}_n(X+Y), \quad (3.11)$$

if the limits in the right-hand sides of (3.9)-(3.11) exist.

Now, to ensure (3.8) one has to select ω in such a way that it allows to restore the equality in (3.7), when $n \rightarrow \infty$. To this aim we impose on the state ω the condition of *dilation* \mathcal{D}_2 -invariance.

Let $\mathcal{D}_2 : l^\infty(\mathbb{N}) \rightarrow l^\infty(\mathbb{N})$, be dilation mapping $\eta \mapsto \mathcal{D}_2(\eta)$:

$$\mathcal{D}_2 : (\eta_1, \eta_2, \dots, \eta_k, \dots) \rightarrow (\eta_1, \eta_1, \eta_2, \eta_2, \dots, \eta_k, \eta_k, \dots), \quad \forall \eta \in l^\infty(\mathbb{N}). \quad (3.12)$$

We say that ω is dilation \mathcal{D}_2 -invariant if for any $\eta \in l^\infty(\mathbb{N})$ it verifies the property

$$(c) \quad \omega(\eta) = \omega(\mathcal{D}_2(\eta)). \quad (3.13)$$

We shall discuss the question of *existence* the dilation \mathcal{D}_2 -invariant states (the *invariant means*) on the Banach space $l^\infty(\mathbb{N})$ in Remark 7.

Let $X, Y \in \mathcal{C}_{1, \infty, +}(\mathcal{H})$. Then applying the property (c) to the sequence $\eta = \{\xi_{2n} := \mathcal{T}_{2n}(X+Y)\}_{n=1}^\infty$, we obtain

$$\omega(\eta) = \omega(\mathcal{D}_2(\eta)) = \omega(\xi_2, \xi_2, \xi_4, \xi_4, \xi_6, \xi_6, \dots). \quad (3.14)$$

Note that if $\xi = \{\xi_n = \mathcal{T}_n(X+Y)\}_{n=1}^\infty$, then the difference of the sequences:

$$\mathcal{D}_2(\eta) - \xi = (\xi_2, \xi_2, \xi_4, \xi_4, \xi_6, \xi_6, \dots) - (\xi_1, \xi_2, \xi_3, \xi_4, \xi_5, \xi_6, \dots),$$

converges to *zero* if $\xi_{2n} - \xi_{2n-1} \rightarrow 0$ as $n \rightarrow \infty$. Then by virtue of (3.11) and (3.14) this would imply

$$\omega(\{\mathcal{T}_{2n}(X+Y)\}_{n \in \mathbb{N}}) = \omega(\mathcal{D}_2(\{\mathcal{T}_{2n}(X+Y)\}_{n \in \mathbb{N}})) = \omega(\{\mathcal{T}_n(X+Y)\}_{n \in \mathbb{N}}),$$

or by (3.11): $\lim_{n \rightarrow \infty} \mathcal{T}_{2n}(X+Y) = \lim_{n \rightarrow \infty} \mathcal{T}_n(X+Y)$, which by estimates (3.7) would also yield

$$\lim_{n \rightarrow \infty} \mathcal{T}_n(X+Y) = \lim_{n \rightarrow \infty} \mathcal{T}_n(X) + \lim_{n \rightarrow \infty} \mathcal{T}_n(Y). \quad (3.15)$$

Now, summarising (3.9), (3.10), (3.11) and (3.15) we obtain the linearity (3.8) of the limiting functional \mathcal{T} on the positive cone $\mathcal{C}_{1,\infty,+}(\mathcal{H})$ if it is defined by the corresponding \mathfrak{D}_2 -invariant state ω , or a dilation-invariant mean.

Therefore, to finish the proof of linearity it rests only to check that $\lim_{n \rightarrow \infty} (\xi_{2n} - \xi_{2n-1}) = 0$. To this end we note that by definitions (3.2) and (3.6) one gets

$$\begin{aligned} \xi_{2n} - \xi_{2n-1} &= \left[\frac{1}{\ln(2n)} - \frac{1}{\ln(2n-1)} \right] \sigma_{2n-1}(X+Y) \\ &\quad + \frac{1}{\ln(2n)} s_{2n}(X+Y). \end{aligned} \quad (3.16)$$

Since $X, Y \in \mathcal{C}_{1,\infty,+}(\mathcal{H})$, we obtain that $\lim_{n \rightarrow \infty} s_{2n}(X+Y) = 0$ and that $\sigma_{2n-1}(X+Y) = O(\ln(2n-1))$. Then taking into account that $(1/\ln(2n) - 1/\ln(2n-1)) = o(1/\ln(2n-1))$ one gets that for $n \rightarrow \infty$ the right-hand side of (3.16) converges to zero.

To conclude our construction of the trace $\mathcal{T}(\cdot)$ we note that by linearity (3.8) one can uniquely extend this functional from the positive cone $\mathcal{C}_{1,\infty,+}(\mathcal{H})$ to the real subspace of the Banach space $\mathcal{C}_{1,\infty}(\mathcal{H})$, and finally to the entire ideal $\mathcal{C}_{1,\infty}(\mathcal{H})$.

Definition 5. The *Dixmier trace* $\text{Tr}_\omega(X)$ of the operator $X \in \mathcal{C}_{1,\infty,+}(\mathcal{H})$ is the value of the linear functional (3.1):

$$\text{Tr}_\omega(X) := \text{Lim}_{n \rightarrow \infty} \frac{\sigma_n(X)}{1 + \ln(n)} = \omega(\{\mathcal{T}_n(X)\}_{n \in \mathbb{N}}), \quad (3.17)$$

where $\text{Lim}_{n \rightarrow \infty}$ is defined by a dilation-invariant state $\omega \in \mathcal{S}(l^\infty(\mathbb{N}))$ on $l^\infty(\mathbb{N})$, that satisfies the properties (a), (b), and (c). Since any self-adjoint operator $X \in \mathcal{C}_{1,\infty}(\mathcal{H})$ has the representation: $X = X_+ - X_-$, where $X_\pm \in \mathcal{C}_{1,\infty,+}(\mathcal{H})$, one gets $\text{Tr}_\omega(X) = \text{Tr}_\omega(X_+) - \text{Tr}_\omega(X_-)$. Then for arbitrary $Z \in \mathcal{C}_{1,\infty}(\mathcal{H})$ the Dixmier trace is $\text{Tr}_\omega(Z) = \text{Tr}_\omega(\text{Re}Z) + i\text{Tr}_\omega(\text{Im}Z)$.

Note that if $X \in \mathcal{C}_{1,\infty,+}(\mathcal{H})$, then definition (3.17) of $\text{Tr}_\omega(\cdot)$ together with definition of the norm $\|\cdot\|_{1,\infty}$ in (2.3), readily imply the estimate $\text{Tr}_\omega(X) \leq \|X\|_{1,\infty}$, which in turn yields the inequality for arbitrary Z from the Dixmier ideal $\mathcal{C}_{1,\infty}(\mathcal{H})$:

$$|\text{Tr}_\omega(Z)| \leq \|Z\|_{1,\infty}. \quad (3.18)$$

Remark 7. A decisive for construction of the Dixmier trace $\text{Tr}_\omega(\cdot)$ is the *existence* of the invariant mean $\omega \in \mathcal{S}(l^\infty(\mathbb{N})) \subset (l^\infty(\mathbb{N}))^*$. Here the space $(l^\infty(\mathbb{N}))^*$ is dual to the Banach space of bounded sequences. Then by the Banach-Alaoglu theorem the convex set of states $\mathcal{S}(l^\infty(\mathbb{N}))$ is *compact* in $(l^\infty(\mathbb{N}))^*$ in the weak*-topology. Now, for any $\phi \in \mathcal{S}(l^\infty(\mathbb{N}))$ the relation $\phi(\mathfrak{D}_2(\cdot)) =: (\mathfrak{D}_2^* \phi)(\cdot)$ defines the dual \mathfrak{D}_2^* -dilation on the set of states. By definition (3.12) this map is such that $\mathfrak{D}_2^* : \mathcal{S}(l^\infty(\mathbb{N})) \rightarrow \mathcal{S}(l^\infty(\mathbb{N}))$, as well as continuous and affine (in fact linear). Then by the Markov-Kakutani theorem the dilation \mathfrak{D}_2^* has a fix point $\omega \in \mathcal{S}(l^\infty(\mathbb{N})) : \mathfrak{D}_2^* \omega = \omega$. This observation justifies the existence of the *invariant mean* (c) for \mathfrak{D}_2 -dilation.

Note that Remark 7 has a straightforward extension to any \mathfrak{D}_k -dilation for $k > 2$, which is defined similar to (3.12). Since dilations for different $k \geq 2$ commute, the extension of the Markov-Kakutani theorem yields that the commutative family $\mathcal{F} = \{\mathfrak{D}_k^*\}_{k \geq 2}$ has in $\mathcal{S}(l^\infty(\mathbb{N}))$ the common fix point $\omega = \mathfrak{D}_2^* \omega$. Therefore, Definition 5 of the Dixmier trace does not depend on the degree $k \geq 2$ of dilation \mathfrak{D}_k .

For more details about different constructions of *invariant means* and the corresponding Dixmier trace on $\mathcal{C}_{1,\infty}(\mathcal{H})$, see, e.g., [3], [13].

Proposition 3.1 *The Dixmier trace has the following properties:*

- (a) For any bounded operator $B \in \mathcal{L}(\mathcal{H})$ and $Z \in \mathcal{C}_{1,\infty}(\mathcal{H})$ one has $\text{Tr}_\omega(ZB) = \text{Tr}_\omega(BZ)$.
- (b) $\text{Tr}_\omega(C) = 0$ for any operator $C \in \mathcal{C}_1(\mathcal{H})$ from the trace-class ideal, which is the closure of finite-rank operators $\mathcal{K}(\mathcal{H})$ for the $\|\cdot\|_1$ -norm.
- (c) The Dixmier trace $\text{Tr}_\omega : \mathcal{C}_{1,\infty}(\mathcal{H}) \rightarrow \mathbb{C}$, is continuous in the $\|\cdot\|_{1,\infty}$ -norm.

Proof. (a) Since every operator $B \in \mathcal{L}(\mathcal{H})$ is a linear combination of four unitary operators, it is sufficient to prove the equality $\text{Tr}_\omega(ZU) = \text{Tr}_\omega(UZ)$ for a unitary operator U and moreover only for $Z \in \mathcal{C}_{1,\infty,+}(\mathcal{H})$. Then the corresponding equality follows from the unitary invariance: $s_j(Z) = s_j(ZU) = s_j(UZ) = s_j(UZU^*)$, of singular values of the positive operator Z for all $j \geq 1$.

(b) Since $C \in \mathcal{C}_1(\mathcal{H})$ yields $\|C\|_1 < \infty$, definition (3.2) implies $\sigma_n(C) \leq \|C\|_1$ for any $n \geq 1$. Then by Definition 5 one gets $\text{Tr}_\omega(C) = 0$. Proof of the last part of the statement is standard.

(c) Since the ideal $\mathcal{C}_{1,\infty}(\mathcal{H})$ is a Banach space and $\text{Tr}_\omega : \mathcal{C}_{1,\infty}(\mathcal{H}) \rightarrow \mathbb{C}$ a linear functional it is sufficient to consider continuity at $X = 0$. Then let the sequence $\{X_k\}_{k \geq 1} \subset \mathcal{C}_{1,\infty}(\mathcal{H})$ converges to $X = 0$ in $\|\cdot\|_{1,\infty}$ -topology, i.e. by (2.3)

$$\lim_{k \rightarrow \infty} \|X_k\|_{1,\infty} = \lim_{k \rightarrow \infty} \sup_{n \in \mathbb{N}} \frac{1}{1 + \ln(n)} \sigma_n(X_k) = 0. \quad (3.19)$$

Since (3.18) implies $|\text{Tr}_\omega(X_k)| \leq \|X_k\|_{1,\infty}$, the assertion follows from (3.19). \square

Therefore, the Dixmier construction gives an example of a *singular* trace in the sense of Definition 4.

4 Trotter-Kato product formulae in the Dixmier ideal

Let $A \geq 0$ and $B \geq 0$ be two non-negative self-adjoint operators in a separable Hilbert space \mathcal{H} and let the subspace $\mathcal{H}_0 := \text{dom}(A^{1/2}) \cap \text{dom}(B^{1/2})$. It may happen that $\text{dom}(A) \cap \text{dom}(B) = \{0\}$, but the form-sum of these operators: $H = A \dot{+} B$, is well-defined in the subspace $\mathcal{H}_0 \subseteq \mathcal{H}$.

T. Kato proved in [12] that under these conditions the *Trotter product formula*

$$s - \lim_{n \rightarrow \infty} \left(e^{-tA/n} e^{-tB/n} \right)^n = e^{-tH} P_0, \quad t > 0, \quad (4.1)$$

converges in the *strong* operator topology *away from zero* (i.e., for $t \in \mathbb{R}^+$), and *locally uniformly* in $t \in \mathbb{R}^+$ (i.e. uniformly in $t \in [\varepsilon, T]$, for $0 < \varepsilon < T < +\infty$), to a *degenerate* semigroup $\{e^{-tH}P_0\}_{t>0}$. Here P_0 denotes the orthogonal projection from \mathcal{H} onto \mathcal{H}_0 .

Moreover, in [12] it was also shown that the product formula is true not only for the *exponential* function e^{-x} , $x \geq 0$, but for a whole class of Borel measurable functions $f(\cdot)$ and $g(\cdot)$, which are defined on $\mathbb{R}_0^+ := [0, \infty)$ and satisfy the conditions:

$$0 \leq f(x) \leq 1, \quad f(0) = 1, \quad f'(+0) = -1, \quad (4.2)$$

$$0 \leq g(x) \leq 1, \quad g(0) = 1, \quad g'(+0) = -1. \quad (4.3)$$

Namely, the main result of [12] says that besides (4.1) one also gets convergence

$$\tau - \lim_{n \rightarrow \infty} (f(tA/n)g(tB/n))^n = e^{-tH}P_0, \quad t > 0, \quad (4.4)$$

locally uniformly away from zero, if topology $\tau = s$.

Product formulae of the type (4.4) are called the *Trotter-Kato product formulae* for functions (4.2), (4.3), which are called the *Kato functions* \mathcal{K} . Note that \mathcal{K} is closed with respect to the *products* of Kato functions.

For some particular classes of the Kato functions we refer to [15], [25]. In the following it is useful to consider instead of $f(x)g(x)$ two Kato functions: $g(x/2)f(x)g(x/2)$ and $f(x/2)g(x)f(x/2)$, that produce the self-adjoint operator families

$$F(t) := g(tB/2)f(tA)g(tB/2) \text{ and } T(t) := f(tA/2)g(tB)f(tA/2), \quad t \geq 0. \quad (4.5)$$

Since [14] it is known, that the *lifting* of the topology of convergence in (4.4) to the *operator norm* $\tau = \|\cdot\|$ needs more conditions on operators A and B as well as on the key Kato functions $f, g \in \mathcal{K}$. One finds a discussion and more references on this subject in [25]. Here we quote a result that will be used below for the Trotter-Kato product formulae in the Dixmier ideal $\mathcal{C}_{1,\infty}(\mathcal{H})$.

Consider the class \mathcal{K}_β of Kato-functions, which is defined in [10], [11] as:

- (i) Measurable functions $0 \leq h \leq 1$ on \mathbb{R}_0^+ , such that $h(0) = 1$, and $h'(+0) = -1$.
- (ii) For $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) < 1$, such that $h(s) \leq 1 - \delta(\varepsilon)$ for $s \geq \varepsilon$, and

$$[h]_\beta := \sup_{s>0} \frac{|h(s) - 1 + s|}{s^\beta} < \infty, \quad \text{for } 1 < \beta \leq 2.$$

The standard examples are: $h(s) = e^{-s}$ and $h(s) = (1 + a^{-1}s)^{-a}$, $a > 0$.

Below we consider the class \mathcal{K}_β and a particular case of generators A and B , such that for the Trotter-Kato product formulae the estimate of the convergence rate is *optimal*.

Proposition 4.1 [11] *Let $f, g \in \mathcal{K}_\beta$ with $\beta = 2$, and let A, B be non-negative self-adjoint operators in \mathcal{H} such that the operator sum $C := A + B$ is self-adjoint on domain $\text{dom}(C) := \text{dom}(A) \cap \text{dom}(B)$. Then the Trotter-Kato product formulae con-*

verge for $n \rightarrow \infty$ in the operator norm:

$$\begin{aligned} \|[f(tA/n)g(tB/n)]^n - e^{-tC}\| &= O(n^{-1}), \quad \|[g(tB/n)f(tA/n)]^n - e^{-tC}\| = O(n^{-1}), \\ \|F(t/n)^n - e^{-tC}\| &= O(n^{-1}), \quad \|T(t/n)^n - e^{-tC}\| = O(n^{-1}). \end{aligned}$$

Note that for the corresponding to each formula error bounds $O(n^{-1})$ are equal up to coefficients $\{\Gamma_j > 0\}_{j=1}^4$ and that each rate of convergence $\Gamma_j \varepsilon(n) = O(n^{-1})$, $j = 1, \dots, 4$, is optimal.

The first *lifting* lemma yields sufficient conditions that allow to strengthen the strong operator convergence to the $\|\cdot\|_\phi$ -norm convergence in the the symmetrically-normed ideal $\mathcal{C}_\phi(\mathcal{H})$.

Lemma 4.2 *Let self-adjoint operators: $X \in \mathcal{C}_\phi(\mathcal{H})$, $Y \in \mathcal{C}_\infty(\mathcal{H})$ and $Z \in \mathcal{L}(\mathcal{H})$. If $\{Z(t)\}_{t \geq 0}$, is a family of self-adjoint bounded operators such that*

$$s - \lim_{t \rightarrow +0} Z(t) = Z, \quad (4.6)$$

then

$$\lim_{r \rightarrow \infty} \sup_{t \in [0, \tau]} \|(Z(t/r) - Z)YX\|_\phi = \lim_{r \rightarrow \infty} \sup_{t \in [0, \tau]} \|XY(Z(t/r) - Z)\|_\phi = 0, \quad (4.7)$$

for any $\tau \in (0, \infty)$.

Proof. Note that (4.6) yields the strong operator convergence $s - \lim_{r \rightarrow \infty} Z(t/r) = Z$, uniformly in $t \in [0, \tau]$. Since $Y \in \mathcal{C}_\infty(\mathcal{H})$, this implies

$$\lim_{r \rightarrow \infty} \sup_{t \in [0, \tau]} \|(Z(t/r) - Z)Y\| = 0. \quad (4.8)$$

Since $\mathcal{C}_\phi(\mathcal{H})$ is a Banach space with symmetric norm (1.14) that verifies $\|ZX\|_\phi \leq \|Z\| \|X\|_\phi$, one gets the estimate

$$\|(Z(t/r) - Z)YX\|_\phi \leq \|(Z(t/r) - Z)Y\| \|X\|_\phi, \quad (4.9)$$

which together with (4.8) give the prove of (4.7). \square

The second *lifting* lemma allows to estimate the rate of convergence of the Trotter-Kato product formula in the norm (1.20) of symmetrically-normed ideal $\mathcal{C}_\phi(\mathcal{H})$ via the error bound $\varepsilon(n)$ in the operator norm due to Proposition 4.1.

Lemma 4.3 *Let A and B be non-negative self-adjoint operators on the separable Hilbert space \mathcal{H} , that satisfy the conditions of Proposition 4.1. Let $f, g \in \mathcal{K}_2$ be such that $F(t_0) \in \mathcal{C}_\phi(\mathcal{H})$ for some $t_0 > 0$.*

If $\Gamma_0 \varepsilon(n)$, $n \in \mathbb{N}$, is the operator-norm error bound away from $t_0 > 0$ of the Trotter-Kato product formula for $\{f(tA)g(tB)\}_{t \geq 0}$, then for some $\Gamma_{2t_0}^\phi > 0$ the function $\varepsilon_\phi(n) := \{\varepsilon([n/2]) + \varepsilon([(n+1)/2])\}$, $n \in \mathbb{N}$, defines the error bound away from $2t_0$ of the Trotter-Kato product formula in the ideal $\mathcal{C}_\phi(\mathcal{H})$:

$$\| [f(tA/n)g(tB/n)]^n - e^{-tC} \|_\phi = \Gamma_{2t_0}^\phi \varepsilon_\phi(n), \quad n \rightarrow \infty, \quad t \geq 2t_0. \quad (4.10)$$

Here $[x] := \max\{l \in \mathbb{N}_0 : l \leq x\}$, for $x \in \mathbb{R}_0^+$.

Proof. To prove the assertion for the family $\{f(tA)g(tB)\}_{t \geq 0}$ we use decompositions $n = k + m$, $k \in \mathbb{N}$ and $m = 2, 3, \dots, n \geq 3$, for representation

$$\begin{aligned} & (f(tA/n)g(tB/n))^n - e^{-tC} = \\ & \left((f(tA/n)g(tB/n))^k - e^{-ktC/n} \right) (f(tA/n)g(tB/n))^m \\ & + e^{-ktC/n} \left((f(tA/n)g(tB/n))^m - e^{-mC/n} \right). \end{aligned} \quad (4.11)$$

Since by conditions of lemma $F(t_0) \in \mathcal{C}_\phi(\mathcal{A})$, definition (4.5) and representation $f(tA/n)g(tB/n)^m = f(tA/n)g(tB/n)^{1/2} F(t/n)^{m-1} g(tB/n)^{1/2}$ yield

$$\| (f(tA/n)g(tB/n))^m \|_\phi \leq \| F(t_0) \|_\phi, \quad (4.12)$$

for t such that $t_0 \leq (m-1)t/n \leq (m-1)t_0$ and $m-1 \geq 1$.

Note that for self-adjoint operators e^{-tC} and $F(t)$ by Araki's log-order inequality for compact operators [1] one gets for $kt/n \geq t_0$ the bound of $e^{-ktC/n}$ in the $\|\cdot\|_\phi$ -norm:

$$\| e^{-ktC/n} \|_\phi \leq \| F(t_0) \|_\phi. \quad (4.13)$$

Since by Definitions 2 and 3 the ideal $\mathcal{C}_\phi(\mathcal{A})$ is a Banach space, from (4.11)-(4.13) we obtain the estimate

$$\begin{aligned} & \| (f(tA/n)g(tB/n))^n - e^{-tC} \|_\phi \leq \\ & \| F(t_0) \|_\phi \| (f(tA/n)g(tB/n))^k - e^{-ktC/n} \| \\ & + \| F(t_0) \|_\phi \| (f(tA/n)g(tB/n))^m - e^{-mC/n} \|, \end{aligned} \quad (4.14)$$

for t such that: $(1 + (k+1)/(m-1))t_0 \leq t \leq nt_0$, $m \geq 2$ and $t \geq (1 + m/k)t_0$.

Now, by conditions of lemma $\Gamma_{t_0} \varepsilon(\cdot)$ is the operator-norm error bound away from t_0 , for any interval $[a, b] \subseteq (t_0, +\infty)$. Then there exists $n_0 \in \mathbb{N}$ such that

$$\| (f(tA/n)g(tB/n))^k - e^{-ktC/n} \| \leq \Gamma_{t_0} \varepsilon(k), \quad (4.15)$$

for $kt/n \in [a, b] \Leftrightarrow t \in [(1 + m/k)a, (1 + m/k)b]$ and

$$\| (f(tA/n)g(tB/n))^m - e^{-mC/n} \| \leq \Gamma_{t_0} \varepsilon(m), \quad (4.16)$$

for $mt/n \in [a, b] \Leftrightarrow t \in [(1 + k/m)a, (1 + k/m)b]$ for all $n > n_0$.

Setting $m := [(n+1)/2]$ and $k = [n/2]$, $n \geq 3$, we satisfy $n = k + m$ and $m \geq 2$, as well as, $\lim_{n \rightarrow \infty} (k+1)/(m-1) = 1$, $\lim_{n \rightarrow \infty} m/k = 1$ and $\lim_{n \rightarrow \infty} k/m = 1$. Hence, for any interval $[\tau_0, \tau] \subseteq (2t_0, +\infty)$ we find that $[\tau_0, \tau] \subseteq [(1 + (k+1)/(m-1))t_0, nt_0]$ for sufficiently large n . Moreover, choosing $[\tau_0/2, \tau/2] \subseteq (a, b) \subseteq (t_0, +\infty)$ we sat-

isfy $[\tau_0, \tau] \subseteq [(1+m/k)a, (1+m/k)b]$ and $[\tau_0, \tau] \subseteq [(1+k/m)a, (1+k/m)b]$ again for sufficiently large n .

Thus, for any interval $[\tau_0, \tau] \subseteq (2t_0, +\infty)$ there is $n_0 \in \mathbb{N}$ such that (4.14), (4.15) and (4.16) hold for $t \in [\tau_0, \tau]$ and $n \geq n_0$. Therefore, (4.14) yields the estimate

$$\begin{aligned} \|(f(tA/n)g(tB/n))^n - e^{-tC}\|_\phi &\leq \\ \Gamma_{t_0} \|F(t_0)\|_\phi \{\varepsilon([n/2]) + \varepsilon([(n+1)/2])\}, \end{aligned} \quad (4.17)$$

for $t \in [\tau_0, \tau] \subseteq (2t_0, +\infty)$ and $n \geq n_0$. Hence, $\Gamma_{2t_0}^\phi := \Gamma_{t_0} \|F(t_0)\|_\phi$ and $\Gamma_{2t_0}^\phi \varepsilon_\phi(\cdot)$ is an error bound in the Trotter-Kato product formula (4.10) away from $2t_0$ in $\mathcal{C}_\phi(\mathcal{H})$ for the family $\{f(tA)g(tB)\}_{t \geq 0}$.

The lifting Lemma 4.2 allows to extend the proofs for other approximants: $\{g(tB)f(tA)\}_{t \geq 0}$, $\{F(t)\}_{t \geq 0}$ and $\{T(t)\}_{t \geq 0}$. \square

Now we apply Lemma 4.3 in Dixmier ideal $\mathcal{C}_\phi(\mathcal{H}) = \mathcal{C}_{1,\infty}(\mathcal{H})$. This concerns the norm convergence (4.10), but also the estimate of the convergence rate for Dixmier traces:

$$|\mathrm{Tr}_\omega(e^{-tC}) - \mathrm{Tr}_\omega(F(t/n)^n)| \leq \Gamma^\omega \varepsilon_\omega(n). \quad (4.18)$$

In fact, it is the same (up to Γ^ω) for all Trotter-Kato approximants: $\{T(t)\}_{t \geq 0}$, $\{f(t)g(t)\}_{t \geq 0}$, and $\{g(t)f(t)\}_{t \geq 0}$.

Indeed, since by inequality (3.18) and Lemma 4.3 for $t \in [\tau_0, \tau]$ and $n \geq n_0$, one has

$$|\mathrm{Tr}_\omega(e^{-tC}) - \mathrm{Tr}_\omega(F(t/n)^n)| \leq \|e^{-tC} - F(t/n)^n\|_{1,\infty} \leq \Gamma_{2t_0}^\phi \varepsilon_{1,\infty}(n), \quad (4.19)$$

we obtain for the rate in (4.18): $\varepsilon_\omega(\cdot) = \varepsilon_{1,\infty}(\cdot)$. Therefore, the estimate of the convergence rate for Dixmier traces (4.18) and for $\|\cdot\|_{1,\infty}$ -convergence in (4.19) are *entirely* defined by the operator-norm error bound $\varepsilon(\cdot)$ from Lemma 4.3 and have the form:

$$\varepsilon_{1,\infty}(n) := \{\varepsilon([n/2]) + \varepsilon([(n+1)/2])\}, \quad n \in \mathbb{N}. \quad (4.20)$$

Note that for the particular case of Proposition 4.1, these arguments yield for (4.17) the explicit convergence rate asymptotics $O(n^{-1})$ for the Trotter-Kato formulae and consequently, the same asymptotics for convergence rates of the Trotter-Kato formulae for the Dixmier trace (4.18), (4.19).

Therefore, we proved in the Dixmier ideal $\mathcal{C}_{1,\infty}(\mathcal{H})$ the following assertion.

Theorem 1. *Let $f, g \in \mathcal{K}_\beta$ with $\beta = 2$, and let A, B be non-negative self-adjoint operators in \mathcal{H} such that the operator sum $C := A + B$ is self-adjoint on domain $\mathrm{dom}(C) := \mathrm{dom}(A) \cap \mathrm{dom}(B)$.*

If $F(t_0) \in \mathcal{C}_{1,\infty}(\mathcal{H})$ for some $t_0 > 0$, then the Trotter-Kato product formulae converge for $n \rightarrow \infty$ in the $\|\cdot\|_{1,\infty}$ -norm:

$$\begin{aligned} \|[f(tA/n)g(tB/n)]^n - e^{-tC}\|_{1,\infty} &= O(n^{-1}), \quad \|[g(tB/n)f(tA/n)]^n - e^{-tC}\|_{1,\infty} = O(n^{-1}), \\ \|F(t/n)^n - e^{-tC}\|_{1,\infty} &= O(n^{-1}), \quad \|T(t/n)^n - e^{-tC}\|_{1,\infty} = O(n^{-1}), \end{aligned}$$

away from $2t_0$. The rate $O(n^{-1})$ of convergence is optimal in the sense of [11].

By virtue of (4.19) the same asymptotics $O(n^{-1})$ of the convergence rate are valid for convergence the Trotter-Kato formulae for the Dixmier trace:

$$\begin{aligned} |\mathrm{Tr}_\omega([f(tA/n)g(tB/n)]^n) - \mathrm{Tr}_\omega(e^{-tC})| &= O(n^{-1}), \\ |\mathrm{Tr}_\omega([g(tB/n)f(tA/n)]^n) - \mathrm{Tr}_\omega(e^{-tC})| &= O(n^{-1}), \\ |\mathrm{Tr}_\omega(F(t/n)^n) - \mathrm{Tr}_\omega(e^{-tC})| &= O(n^{-1}), \quad |\mathrm{Tr}_\omega(T(t/n)^n) - \mathrm{Tr}_\omega(e^{-tC})| = O(n^{-1}), \end{aligned}$$

away from $2t_0$.

Optimality of the estimates in Theorem 1 is a heritage of the optimality in Proposition 4.1. Recall that in particular this means that in contrast to the Lie product formula for *bounded* generators A and B , the *symmetrisation* of approximants $\{f(t)g(t)\}_{t \geq 0}$, and $\{g(t)f(t)\}_{t \geq 0}$ by $\{F(t)\}_{t \geq 0}$ and $\{T(t)\}_{t \geq 0}$, does not yield (in general) the improvement of the convergence rate, see [11] and discussion in [26].

We resume that the *lifting* Lemmata 4.2 and 4.3 are a general method to study the convergence in symmetrically-normed ideals $\mathcal{C}_\phi(\mathcal{H})$ as soon as it is established in $\mathcal{L}(\mathcal{H})$ in the operator norm topology. The crucial is to check that for any of the *key* Kato functions (e.g. for $\{F(t)\}_{t \geq 0}$) there exists $t_0 > 0$ such that $F(t)|_{t \geq t_0} \in \mathcal{C}_\phi(\mathcal{H})$. Sufficient conditions for that one can find in [16]-[18], or in [25].

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