

LIMIT CYCLES STABILITY FOR COMPLEX SYSTEM

Lala H. Rajaoarisoa, Nacer K. M'Sirdi

*Laboratory of Systems and Information Sciences LSIS UMR CNRS 6168,
Domaine Universitaire. St Jerome, Av. Escadrille Normandie Niemen,
Marseille, France*

lala.rajaoarisoa@lsis.org, nacer.msirdi@lsis.org

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Abstract: Limit cycles are common in physical complex systems. However the nonsmooth dynamics of such systems makes stability analysis difficult. This paper gives an extension of Discrete Events System formalism to show the stability of Limit cycles using multiple Lyapunov function.

1 INTRODUCTION

Many real systems are characterized by interactions between continuous (smooth or none) dynamics and discrete events. Such systems are common across a diverse range of application areas. Examples include a robotic assembly (McCarragher96 and al.), manufacturing (Pettersson99) and power system (Hiskens00 and al.).

Dynamics of several physical systems can be well described by use of ordinary differential equations despite the fact that some times nonlinearities are introduced and can be taken into account. However, there is as of time this class of systems exhibit a periodic behavior. Discrete events, such as saturation limits, can act to trap evolving system state within a constrained region of state space.

Therefore even when the underlying continuous system are unstable, the discrete events can induce a stable limit set. Limit cycles (periodic behaviour) are created in this way. Limit cycles can be stable (attracting), unstable (repelling) or non-stable(saddle). The stability of periodic behaviour can be defined also as the stability of the fixed point.

So, in this paper we want to study the problems of stability of some class of hybrid systems and how to deal with observability and controllability of such systems. Despite an amount of theory and proposed methods to test observability, it seems to be rather difficult to find an optimal way and efficient rules which

cope with some class of hybrid non-linear systems. This is why we have chosen some simple mechanical systems with different phases when operating, or commutation of structures (contact and non contact situations).

2 MODEL

2.1 Background

We choose as illustrative example a mass spring system with a simple structure. But this system has a structure which is near to a vehicle one or a jumping robot.

Definition of a class of systems having variable structures, commutations in their dynamic behavior, non linearities (hard or smooth), non stationarity, varying parameters and other non standard features, is difficult to be done in general. So we can restrict our case to some simple situations with known involved physical phenomena.

The discrete events taken into account and the choice of their effect may be driven by some higher level or simply selected according of optimization of some criterion or performance index (Giambiasi02 and al.), (Allur94).

The switchings and commutations often appear abruptly but changes from one representation to an-

other one may be very smooth or not. In another hand we must note also that such systems representation is not unique and differences can appear between the behavioral representation and physical system description or modeling for diagnosis and control.

Discrete Events System (Zeigler76) defines a way to specify systems which states change either upon the reception of an input event or due to the expiration of a time delay.

The system equation can then be written in following form:

$$\begin{cases} \dot{x} = f_{m_i}(x, u, t) \\ m_i : \{S, I, O, \delta_{int}, \delta_{ext}, \lambda, t_{\alpha i}\} \end{cases} \quad (1)$$

where:

- x is the set of continuous states of system
- u is the set of input controls
- S is the set of sequential states
- I is the set of external events
- O is the set of internal events
- $\delta_{int} : S \rightarrow S$ is the internal state transition function
- $\delta_{ext} : S_- \times I \rightarrow S_+$ is the external state transition function
- $\lambda : S \rightarrow O$ is the output function
- $t_{\alpha i}$ is the time advance function

2.2 Limit Cycles Analysis

In the sequel, we adopted following definitions to approach the stability in dynamical of the system given by the relation (1) (MSirdi98 and al.). We considered the system behavioural in the vicinity of one cyclic trajectory in the goal to analysis the orbital stability.

Limit Cycles: for a second order system with state equation defined by $\dot{\chi} = f(\chi, t)$, we define a positive limited set invaring for a boundedness trajectory $\chi(t) (\|\chi(t)\|) < \mu, \forall t > 0$ by:

- $\{p \in \mathfrak{R}^n, \forall \xi > 0, \exists t_k$ such that $\|p - \chi(t_k)\| < \xi, \forall k \in N\}$ where t_k is the time sequential, with, $\lim_{k \rightarrow \infty} t_k = \infty$

Orbital Stability: Trajectory of the system in the phase plane \mathfrak{R}^2 is an orbit stable if:

- $\forall \xi > 0, \exists \rho > 0$ such that $\|x_0 - (z_0)\| < \rho \Rightarrow \inf \|x(t) - p\| < \xi, \forall t > t_0$ with $p \in \mathfrak{R}^n$

Asymptotic Orbital Stability: Trajectory of the system in the phase plane \mathfrak{R}^2 is an asymptotic orbit stable if: - $\|x_0 - (\cdot)\| < \rho \Rightarrow k \rightarrow \infty \inf \|x(t) - p\| = 0$ with $p \in \mathfrak{R}^n$

3 System Description

3.1 Mass Spring Model

Let us consider the mass-spring system of figure(4), with mass M , stiffness constant k and z_0 the original length of massless spring. The environment is assumed infinitely rigid: $k_e \gg k$. If this is not the case let k_r be the stiffness of the spring and k the equivalent stiffness of interaction with the ground ($k \cong \frac{k_r k_e}{k_r + k_e}$). The position of the mass M , in a frame attached to the ground, is noted z . The gravity constant is $g = 9.81 \text{ms}^{-2}$. Let $v_d = \dot{z}_d > 0$ be the lift off velocity and $v_c = \dot{z}_c < 0$ the touch down velocity.

The dynamic interaction with the ground is composed by two phases: flying and stance phases [16][17]. This system is composed by interconnection of three subsystems (mass, spring and ground) and energy evolutions are:

Potential (g) \rightarrow Kinetic \rightarrow potential accumulation \rightarrow potential restitution \rightarrow Kinetic and so on.

We assume the landing without rebounds and no energy loss. Note that the values used in simulations are estimations of equivalent coefficients for our robot SAP [15] ($M \cong 2.6 \text{kg}; k = 1100$).

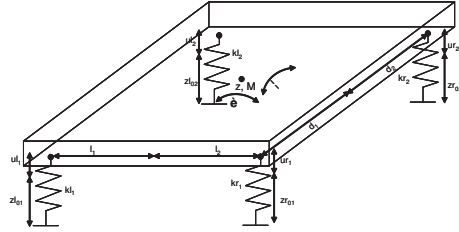


Figure 1: Mass Spring Model

- Contact Phases. In these phases the controls are active when the springs are in contact.

- The two springs are in contact: $\dot{x} = f_1(x, u, t) \quad t \in \mathbb{R}_+$ and $x(t) \in D_1 \subset \mathbb{R}^4$

$$\begin{cases} \dot{x}_4 = \ddot{z} = -\frac{k_l}{M}(z_l - z_{l0} - u_l) - \frac{k_r}{M}(z_r - z_{r0} - u_r) - g \\ \dot{x}_5 = \ddot{\theta} = \frac{lk_l}{I}(z_l - z_{l0} - u_l) - \frac{lk_r}{I}(z_r - z_{r0} - u_r) \\ \dot{x}_6 = \ddot{\phi} = \frac{lk_l}{T}(z_l - z_{l0} - u_l) + \frac{lk_r}{T}(z_r - z_{r0} - u_r) \end{cases} \quad (2)$$

- The right spring is in contact: $\dot{x} = f_2(x, u, t) \quad t \in \mathbb{R}_+$ and $x(t) \in D_2 \subset \mathbb{R}^4$

$$\begin{cases} \dot{x}_4 = \ddot{z} = -\frac{k_r}{M}(z_r - z_{r0} - u_r) - g \\ \dot{x}_5 = \ddot{\theta} = -\frac{lk_r}{I}(z_r - z_{r0} - u_r) \\ \dot{x}_6 = \ddot{\phi} = \frac{lk_r}{T}(z_r - z_{r0} - u_r) \end{cases} \quad (3)$$

- The left spring is in contact: $\dot{x} = f_3(x, u, t) \quad t \in \mathbb{R}_+$ and $x(t) \in D_3 \subset \mathbb{R}^4$

$$\begin{cases} \dot{x}_4 = \ddot{z} = -\frac{k_l}{M}(z_l - z_{l0} - u_l) - g \\ \dot{x}_5 = \dot{\theta} = \frac{lk_l}{I}(z_l - z_{l0} - u_l) \\ \dot{x}_6 = \dot{\phi} = \frac{lk_l}{I}(z_l - z_{l0} - u_l) \end{cases} \quad (4)$$

• Flying Phase: $\dot{x} = f_4(x, u, t) \quad t \in \mathbb{R}_+$ and $x(t) \in D_4 \subset \mathbb{R}^4$

$$\begin{cases} \dot{x}_1 = x_3 \\ \dot{x}_2 = x_4 \\ \dot{x}_4 = \ddot{z} = -g \\ \dot{x}_5 = \dot{\theta} = 0 \\ \dot{x}_6 = \dot{\phi} = 0 \end{cases} \quad (5)$$

z_r et z_l can be expressed in function of z if we assume θ and ϕ small:

$$\begin{cases} z_{l1} = z - l \sin \theta - d \sin \phi = z - l\theta - d\phi \\ z_{l2} = z + l \sin \theta - d \sin \phi = z + l\theta - d\phi \\ z_{r1} = z + l \sin \theta - d \sin \phi = z + l\theta - d\phi \\ z_{r2} = z + l \sin \theta + d \sin \phi = z + l\theta + d\phi \end{cases}$$

3.2 Switching control and supervisor model

As we have said above, our system is composed of two sub-system. One hand with continuous state and in other hand discrete behavioural state. For this last, we considerate the following phases (Figure2) :

- All springs are in contact with ground
- All springs are in flying phase
- Before Right-hand side Spring is in contact with ground
- Before Left-hand side Spring is in contact with ground
- Back Right-hand side Spring is in contact with ground
- Back Left-hand side Spring is in contact with ground
- Before and Back Right-hand side Spring is in contact with ground
- Before and Back Left-hand side Spring is in contact with ground

- The set of the output variables are gotten by the selector element device (Select1 and Select2). These selectors are associated respectively to Contact model and Flight model. Thus, we can write:

$$-O_c = Select_1 = Contact \ model = 1$$

$$-O_{r1C} = Select_2 = Right1Contact \ model = 2$$

$$-O_{l1C} = Select_3 = Left1Contact \ model = 3$$

$$-O_f = Select_4 = Flight \ model = 4$$

$$-O_{r2C} = Select_2 = Right2Contact \ model = 5$$

$$-O_{l2C} = Select_3 = Left2Contact \ model = 6$$

$$-O_{r1l1C} = Select_4 = Right1Left1Contact \ model = 7$$

$$-O_{r2l2C} = Select_4 = Right2Left2Contact \ model = 8$$

For this application let us consider the functions $\xi_r(z_r, u_r)$ and $\xi_l(z_l, u_l)$ as the external state transition function. So we can write:

$$\delta_{ext} : S_- \times I \rightarrow S_+ = \xi_\tau(z_\tau, u_\tau) \quad (6)$$

with $\tau = l, r$ which defined the Left or Right spring
This function can be equal to 0 in the flying phase and 1 when there is contact with the corresponding spring.

$$\xi_r(z_r, u_r) = \frac{1}{2}(1 - \text{sign}(z_r - z_0 - u_r)) \quad (7)$$

$$\xi_l(z_l, u_l) = \frac{1}{2}(1 - \text{sign}(z_l - z_0 - u_l)) \quad (8)$$

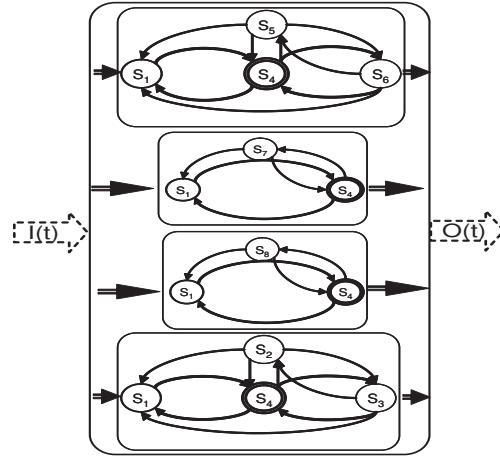


Figure 2: Discrete Event Device Model

We want give a periodic motion for our system i.e. to damp any rotational motion and maintain hopping along z axis. For obtain this goal, let us consider the following Lyapunov function (energy of the system):

$$V = \frac{1}{2}\dot{z}^2 + gz + \frac{I}{2M}\dot{\theta}^2 + \frac{I}{2M}\dot{\phi}^2 \quad (9)$$

$$V = V_{LC} + V_{T(1)} + V_{T(2)} \quad (10)$$

$$\text{whit} \equiv \begin{cases} V_{LC} = \frac{1}{2}\dot{z}^2 + gz \\ V_{T(1)} = \frac{I}{2M}\dot{\theta}^2 \\ V_{T(2)} = \frac{I}{2M}\dot{\phi}^2 \end{cases} \quad (11)$$

Energy is splitted in two parts: V_{LC} the energy corresponding to the desired periodic hopping motion and $V_{T(.)}$ the transverse motion energy. It is clear that one of this energy has to be regulated to some level and the other must be damped. The right an left control inputs are

$$\begin{cases} \xi_{r1}u_{r1} = \frac{1}{2}(u_{LC} - u_{T(1)} - u_{T(2)}) \\ \xi_{l1}u_{l1} = \frac{1}{2}(u_{LC} + u_{T(1)} - u_{T(2)}) \\ \xi_{r2}u_{r2} = \frac{1}{2}(u_{LC} - u_{T(1)} + u_{T(2)}) \\ \xi_{l2}u_{l2} = \frac{1}{2}(u_{LC} + u_{T(1)} + u_{T(2)}) \end{cases} \quad (12)$$

u_T is the control which has to damp transverse energy V_T and then rotational motions. This has as consequence to keep the system state in the plane (z, \dot{z}) with $\theta = \dot{\theta} = 0$ and $\phi = \dot{\phi} = 0$. The u_{LC} has to stabilise priodic cycle (cyclic motion). The two control inputs u_T and u_{LC} have to be applied in the time period where the corresponding spring is in contact with ground. This is made by displacement of the springs attach points u_r and u_l .

3.3 Convergence and stability of the periodic motion

The transverse motion and its energy V_T have to be damped. Let us use as Lypunov candidate function V_1

$$V_1 = \frac{1}{2}V_{T(1)}^2 + \frac{1}{2}V_{T(2)}^2 \quad (13)$$

Its time derivative is:

$$\dot{V}_1 = V_{T(1)}\dot{V}_{T(1)} + V_{T(2)}\dot{V}_{T(2)} \quad (14)$$

$$\dot{V}_{T(1)} = \frac{I}{M}\dot{\theta}\ddot{\theta} \quad (15)$$

$$\dot{V}_{T(2)} = \frac{I}{M}\dot{\phi}\ddot{\phi} \quad (16)$$

using expression (??) and (16), lead:

$$\begin{aligned} \dot{V}_T &= \frac{lk}{M}(\xi_l(z_l - z_{l0}) - \xi_r(z_r - z_{r0}) \\ &+ (\xi_r u_r - \xi_l u_l)\dot{\theta} + \frac{lk}{M}(\xi_l(z_l - z_{l0}) \\ &- \xi_r(z_r - z_{r0}) + (\xi_r u_r - \xi_l u_l)\dot{\phi} \end{aligned} \quad (17)$$

Substituting controls $\xi_r u_r, \xi_l u_l$ by equation (12) in \dot{V}_T , we have:

$$\begin{aligned} \dot{V}_T &= \frac{lk}{M}(\xi_l(z_l - z_{l0}) - \xi_r(z_r - z_{r0}) + u_T)\dot{\theta} \\ &+ \frac{lk}{M}(\xi_l(z_l - z_{l0}) - \xi_r(z_r - z_{r0}) + u_T)\dot{\phi} \end{aligned} \quad (20)$$

We propose a transverse control input u_T as follows:

$$\begin{aligned} u_T &= \xi_r u_r - \xi_l u_l = -\Gamma_{11}\Psi(V_{T(1)})\dot{\theta} - \Gamma_{12}\Psi(V_{T(2)})\dot{\phi} \\ &- \xi_l(z_l - z_{l0}) + \xi_r(z_r - z_{r0}) \end{aligned} \quad (21)$$

Ψ is a positive function and can be sign or saturation function (21). we then obtain:

$$\dot{V}_T = -\frac{lk}{M}\Gamma_{11}\Psi(V_{T(1)})\dot{\theta}^2 - \frac{lk}{M}\Gamma_{12}\Psi(V_{T(2)})\dot{\phi}^2 \quad (22)$$

$$\dot{V}_1 = V_{T(1)}\dot{V}_{T(1)} + V_{T(2)}\dot{V}_{T(2)} \quad (23)$$

$$= -\Gamma_{11}V_{T(1)}\Psi(V_{T(1)})\dot{\theta}^2 \quad (24)$$

$$- \Gamma_{12}V_{T(2)}\Psi(V_{T(2)})\dot{\phi}^2 \leq 0 \quad (25)$$

$V_{T(.)}\dot{V}_{T(.)}$ is negative then the transverse energy V_T converges to zero. We can conclude that

$$\forall \varepsilon_1 > 0, \exists t_1 \geq 0, \text{ such as } |V_T| < \varepsilon_1, \forall t > t_1$$

Convergence of V_{LC} desired reference V_{LC}^* . Let us choose another Lyapunov candidate function

$$V_2 = \frac{1}{2}V_T^2 + \frac{1}{2}(V_{LC} - V_{LC}^*)^2 \quad (26)$$

V_{LC}^* the constant reference energy is defined at the lft off pont z_d or at the maximal desired height z_{\max} :

$$V_{LC}^*(0, z_d) = \frac{1}{2}z_d^2 \quad \text{or} \quad V_{LC}^*(z_{\max}, 0) = gz_{\max}$$

when $t > t_1$ we have $(V_T, \dot{V}_T) = (0, 0)$, then

$$\dot{V}_2 = (V_{LC} - V_{LC}^*)\dot{V}_{LC} \quad \forall t > t_1$$

with

$$\dot{V}_{LC} = (\ddot{z} + g)\dot{z} \quad \forall t > t_1 \quad (27)$$

$$\dot{V}_{LC} = \left(-\xi_l \frac{k}{M}(z_l - z_{l0}) - \xi_r \frac{k}{M}(z_r - z_{r0}) \right. \quad (28)$$

$$\left. + \frac{k}{M}(\xi_r u_r + \xi_l u_l)\dot{z} \right) \quad (29)$$

$$\dot{V}_{LC} = \left(-\xi_l \frac{k}{M}(z_l - z_{l0}) - \xi_r \right. \quad (30)$$

$$\left. \frac{k}{M}(z_r - z_{r0}) + \frac{k}{M}u_{LC}\dot{z} \right) \quad (31)$$

we propose as control input :

$$u_{LC} = \xi_r u_r + \xi_l u_l = -\Gamma_2 \Psi(V - V_{LC}^*) \dot{z} + \xi_l (z_l - z_{l0}) + \xi_r (z_r - z_{r0}) \quad (32)$$

such as the derivative \dot{V}_{LC} will be negative:

$$\dot{V}_{LC} = -\frac{k}{M} \Gamma_2 \Psi(V - V_{LC}^*) \dot{z}^2 \quad (33)$$

Then

$$\begin{aligned} \dot{V}_2 &= (V_{LC} - V_{LC}^*) \dot{V}_{LC} \\ &= -\frac{k}{M} \Gamma_2 (V_{LC} - V_{LC}^*) \Psi(V_{LC} - V_{LC}^*) \dot{z}^2 \leq 0 \end{aligned} \quad (34)$$

The second Lyapunov function ensures global asymptotic convergence of the system trajectories z to the orbit Ω $\theta(t)$, $\phi(t)$ converge to zero with the following control:

$$\begin{cases} u_T = -\Gamma_{11} \Psi(V_{T(1)}) \dot{\theta} - \Gamma_{12} \Psi(V_{T(2)}) \dot{\phi} \\ -\xi_l (z_l - z_{l0}) + \xi_r (z_r - z_{r0}) \\ u_{LC} = -\Gamma_2 \Psi(V - V_{LC}^*) \dot{z} + \xi_l (z_l - z_{l0}) + \xi_r (z_r - z_{r0}) \end{cases} \quad (35)$$

4 EXPERIMENTAL RESULTS

– First test:

We want just to show firstly when $\theta_0 = 0rad$ and $\phi_0 = 0rad$ that our system can respect the limit cycle that we have imposed with the control considered. These conditions give for the simulation in figure(03) good results with the desired height $z_m = 3.5m$. The figure (04) illustrates the model commutation, here we have just commutation of model 1 to model 4, the transverse energy V_T and the cycle in vertical direction (V_{LC} goes to its imposed reference) and the Figure(05) shows the angle θ and ϕ equal to zero.

– Second test:

Now we consider that $\theta_0 = 0.5rad$, $\phi_0 = 0rad$. With the same conditions for simulation figure(06) shows the good results with the desired height $z_m = 3.5m$ obtain after few seconds of simulation, here $t = 1.5s$. Figure(07) illustrated the switching model result and the moment that corresponds to the necessary time interval to damp the transverse energy V_T and stabilize the cycle in vertical direction (V_{LC} . The Figure (08) shows the convergence of the angle θ .

– Third test:

For the sequel, we make one test whose the goal is to show the effectiveness of our supervision

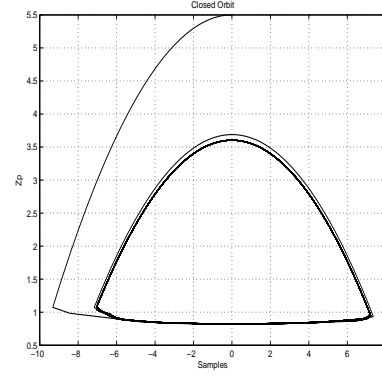


Figure 3: Cycle limite stable for $\theta_0 = 0rad$, $\phi_0 = 0rad$

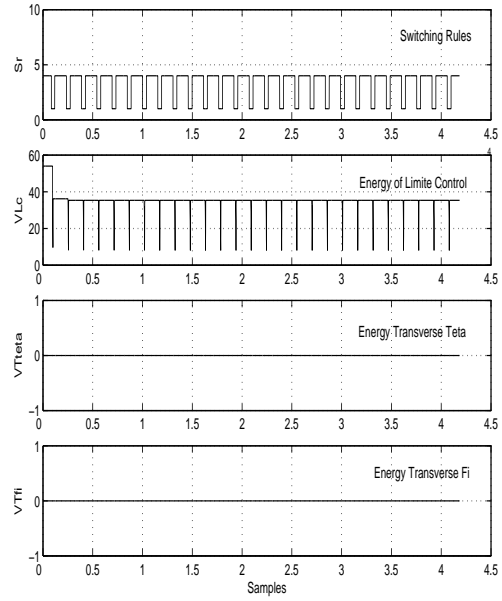


Figure 4: Energy of the System M when $\theta_0 = 0rad$, $\phi_0 = 0rad$

and commutation device. In this effect, let us $\theta_0 = 1rad$, $\phi_0 = 0.5rad$. As before, the simulation show us the good results with the desired height in figure(09) and in the Figure (10) we can see one more variation a both, of commutation models and the system Energy. The Figure(11) shows the convergence of the angle θ and the angle ϕ after few seconds of the simulation time.

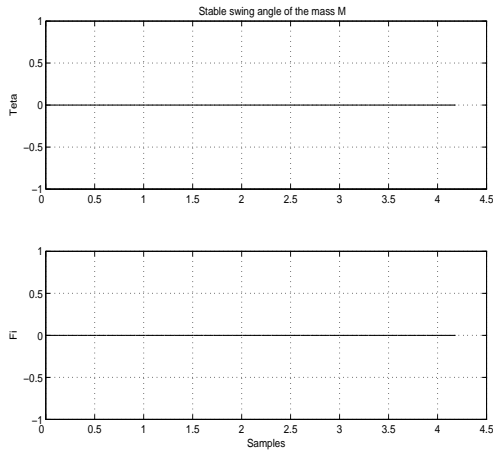


Figure 5: Stable swing angle of the mass M when $\theta_0 = 1.5rad$,

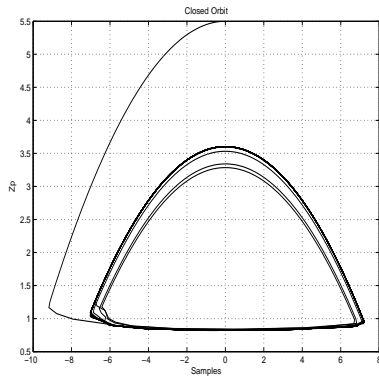


Figure 6: Cycle limite stable for $\theta_0 = 0.5rad$, $\phi_0 = 0rad$

5 CONCLUSION

We propose of this work is to define an approach to identify and then control and supervise such class of complex systems represented by switched models. The system is composed by different sub-models. Each model switches to another instantaneously when the thresholds that define some operating points or zone, is reached by the application of the external or internal transition function. In the goal to build the best prediction of system outputs, we have to get the best switching and supervision device depending on operating point, the behavior and environment. We have illstured the observability of this class of system with different phases when operating, or commutation of structures (contact and non contact situations).

The presented experimental results emphasize efficiency of this approach for modeling, behavior analysis and prediction for such class of complex systems. We have shown, in our first results using this ap-

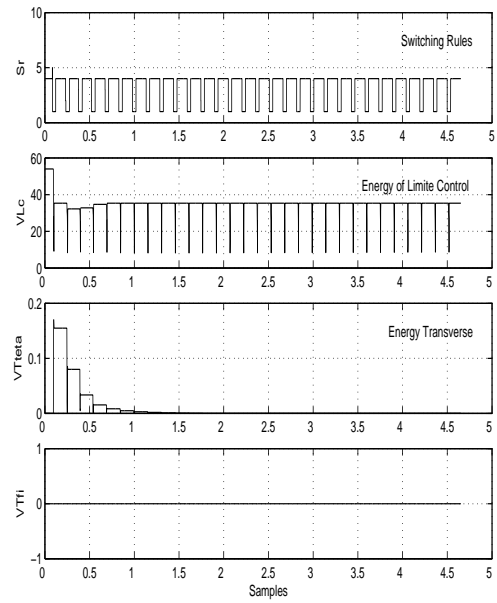


Figure 7: Energy of the System M when $\theta_0 = 0.5rad$, $\phi_0 = 0rad$

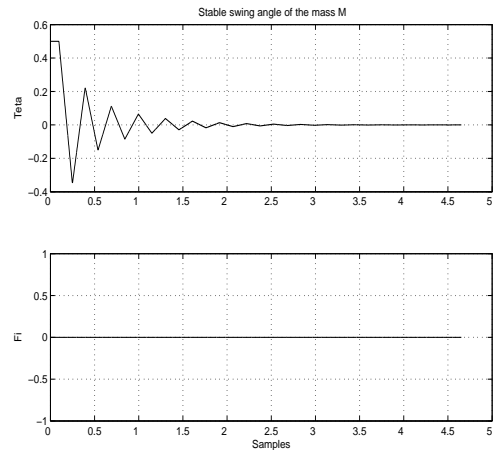


Figure 8: Stable swing angle of the mass M when $\theta_0 = 0.5rad$, $\phi_0 = 0rad$

proach, an important difference of performance of the prediction regard to the case when using fuzzy logic for estimation and supervision (Duplaix05).

In a future work, this approach will be used for modelling a vehicle in the goal to make a diagnosis, fault detection and monitoring. A diagnostic framework on this application will be considered to detect defaults and control the system.

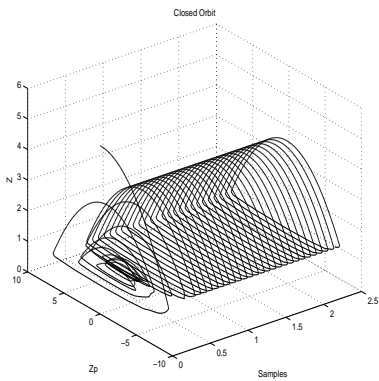


Figure 9: Cycle limite stable for $\theta_0 = 1rad, \phi_0 = 0.5rad$

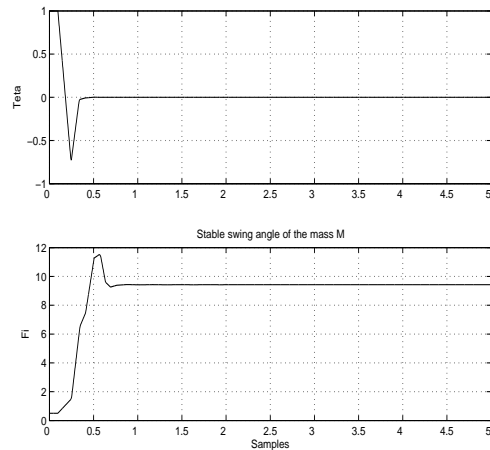


Figure 11: Stable swing angle of the mass M when $\theta_0 = 1rad, \phi_0 = 0.5rad$

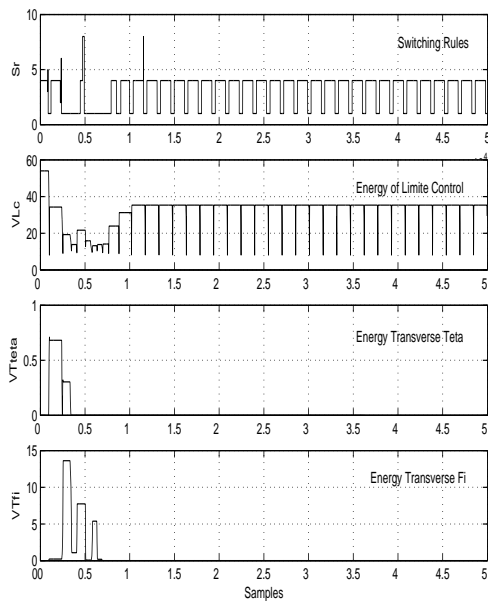


Figure 10: Energy of the System M when $\theta_0 = 1rad, \phi_0 = 0.5rad$

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