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A holomorphic functional calculus for finite families of commuting semigroups

Jean Esterle

December 29, 2018

Abstract

Let $A$ be a commutative Banach algebra such that $uA \neq \{0\}$ for $u \in A \setminus \{0\}$ which possesses dense principal ideals. The purpose of the paper is to give a general framework to define $F(-\lambda_1\Delta_{T_1}, \ldots, -\lambda_k\Delta_{T_k})$ where $F$ belongs to a natural class of holomorphic functions defined on suitable open subsets of $\mathbb{C}^k$ containing the "Arveson spectrum" of $(-\lambda_1\Delta_{T_1}, \ldots, -\lambda_k\Delta_{T_k})$, where $\Delta_{T_1}, \ldots, \Delta_{T_k}$ are the infinitesimal generators of commuting one-parameter semigroups of multipliers on $A$ belonging to one of the following classes:

1. The class of strongly continuous semigroups $T = (T(t)e^{ia_t})_{t > 0}$ such that $\cup_{t > 0} T(te^{ia_t})A$ is dense in $A$, where $a_t \in \mathbb{R}$.

2. The class of semigroups $T = (T(\zeta))_{\zeta \in S_{\alpha,\beta}}$ holomorphic on an open sector $S_{\alpha,\beta}$ such that $T(\zeta)A$ is dense in $A$ for some, or equivalently for all $\zeta \in S_{\alpha,\beta}$.

We use the notion of quasimultiplier, introduced in 1981 by the author at the Long Beach Conference on Banach algebras: the generators of the semigroups under consideration will be defined as quasimultipliers on $A$, and for $\zeta$ in the Arveson resolvent set $\sigma_{ar}(\Delta_T)$ the resolvent $(\Delta_T - \zeta I)^{-1}$ will be defined as a regular quasimultiplier on $A$, i.e. a quasimultiplier $S$ on $A$ such that $\sup_{n \geq 1} \lambda^n ||S^n u|| < +\infty$ for some $\lambda > 0$ and some $u$ generating a dense ideal of $A$ and belonging to the intersection of the domains of $S^n$, $n \geq 1$.

The first step consists in "normalizing" the Banach algebra $A$, i.e. continuously embedding $A$ in a Banach algebra $B$ having the same quasimultiplier algebra as $A$ but for which $\limsup_{t \to 0^+} ||T(te^{ia_t})||_{M(B)} < +\infty$ if $T$ belongs to the class (1), and for which $\limsup_{\zeta \to 0} ||T(\zeta)|| < +\infty$ for all pairs $(\alpha, \beta)$ such that $a < \alpha < \beta < b$ if $T$ belongs to the class (2). Iterating this procedure this allows to consider $(\lambda_j\Delta_{T_j} + \zeta I)^{-1}$ as an element of $M(B)$ for $\zeta \in Res_{ar}(-\lambda_j\Delta_{T_j})$, the "Arveson resolvent set" of $-\lambda_j\Delta_{T_j}$, and to use the standard integral 'resolvent formula' even if the given semigroups are not bounded near the origin.

A first approach to the functional calculus involves the dual $G_{a,b}$ of an algebra of fast decreasing functions, described in Appendix 2. Let $a = (a_1, \ldots, a_k)$, $b = (b_1, \ldots, b_k)$, with $a_j \leq b_j \leq a_j + \pi$, and denote by $M_{a,b}$ the set of families $(\alpha, \beta) = (\alpha_1, \beta_1), \ldots, (\alpha_k, \beta_k)$ such that
\[ \alpha_j = \beta_j = a_j \text{ if } a_j = b_j \text{ and such that } a_j < \alpha_j \leq \beta_j < b_j \text{ if } a_j < b_j. \]

Let \( \mathcal{U}_{\alpha, \beta} \) denote the class of all functions \( f : \mathbb{C} \to \mathbb{C} \) that are continuous on the product space \( \mathbb{S}_{\alpha, \beta} = \mathbb{B}_{\alpha, \beta} \) and converging to \( 0 \) at infinity such that the function \( \sigma \to f(\zeta_1, \ldots, \zeta_k) \) is holomorphic on \( S_{\alpha_j, \beta_j} \) whenever \( \alpha_j < \beta_j \). Elements of the dual \( \mathcal{U}_{\alpha, \beta}^\prime \) admit a "representing measure", and we describe in appendix 1 some certainly well-known ways to implement the duality between \( \mathcal{U}_{\alpha, \beta} \) and \( \mathcal{U}_{\alpha, \beta}^\prime \) and extend the action of elements of \( \mathcal{U}_{\alpha, \beta}^\prime \) to vector-valued analogs spaces \( \mathcal{U}_{\alpha, \beta}(X) \) and their "bounded" counterparts \( \mathcal{V}_{\alpha, \beta}(X) \) via representing measures and Cauchy and Fourier-Borel transforms.

In appendix 2 we introduce a natural algebra of fast decreasing functions, which is the intersection for \( (\alpha, \beta) \in M_{a,b} \) and \( z \in \mathbb{C}^k \) of all \( e_{-U \alpha, \beta} \), where \( e_{-U}(\zeta) = e^{\pi i \zeta^1 + \ldots + \pi i \zeta_k} \).

The dual \( \mathcal{G}_{a,b} \) of this algebra is an algebra with respect to convolution, this dual space is the union for \( z \in \mathbb{C}^k \) and \( (\alpha, \beta) \in M_{a,b} \) of the dual spaces \( e_{-U \alpha, \beta}^\prime \), and elements of these dual spaces act on the vector-valued spaces \( e_{-U \alpha, \beta}(X) \). This action can also be implemented via representing measures, Cauchy transforms and Fourier-Borel transforms as indicated in appendix 2. If \( \lambda_j S_{\alpha_j, \beta_j} \) is contained in the domain of definition of \( T_j \) for \( j \leq k \), this allows to define the action of \( \phi \in \mathcal{G}_{a,b} \) on \( T(\lambda) = (T_1(\lambda_1), \ldots, T_k(\lambda_k)) \) by using the formula

\[
< T(\lambda), \phi > u = < T_1(\lambda_1 \zeta_1) \ldots T_k(\lambda_k \zeta_k) u, \phi_1, \ldots, \phi_k > \quad (u \in \mathcal{B}),
\]

where \( \mathcal{B} \) denotes a normalization of the given commutative Banach algebra \( \mathcal{A} \) with respect to \( T = (T_1, \ldots, T_k) \), when \( \phi \in (e_{-U \alpha, \beta})^\prime \) and when \( \sup_{\zeta \in \mathbb{S}_{\alpha, \beta}} \| e^{\pi i T_1(\lambda_1 \zeta_1) + \ldots + \pi i T_k(\lambda_k \zeta_k)} \|_{\mathcal{M}(\mathcal{B})} < +\infty \).

For \( (\alpha, \beta) \in M_{a,b} \), set \( S_{\alpha, \beta}^\prime = \Pi_{1 \leq j \leq k} S_{\frac{\alpha_j - \beta_j}{2}, \frac{\beta_j - \alpha_j}{2}} \). An open set \( U = \Pi_{a \leq \lambda < b} U_j \) is said to be admissible with respect to \( (\alpha, \beta) \) if for every \( j \leq k \) the boundary \( \partial U_j \) is a piecewise \( C^1 \)-curve, if \( U + \epsilon < U \) for every \( \epsilon \in S_{\alpha, \beta} \) and if \( S_{\alpha, \beta} \setminus U \) is compact. Standard properties of the class \( H^1(U) \) of all holomorphic functions \( F \) on \( U \) such that \( \| F \|_{H^1(U)} := \sup_{\zeta \in S_{\alpha, \beta}} \int_{\epsilon + \partial U} |F(\sigma)| d\sigma \) are given in appendix 3 (when \( a_j = b_j \) for \( j \leq k \), this space is the usual Hardy space \( H^1 \) on a product of open half-planes).

The results of appendix 3 allow when an open set \( U \subset \mathbb{C}^k \) admissible with respect to \( (\alpha, \beta) \in M_{a,b} \) satisfies some more suitable admissibility conditions with respect to \( T = (T_1, \ldots, T_k) \) and \( \lambda \in \mathcal{U}_{(\alpha, \beta)} \) to define \( F(-\lambda_1 \Delta T_1, \ldots, -\lambda_k \Delta T_k) \) for \( F \in H^1(U) \) by using the formula

\[
F(-\lambda_1 \Delta T_1, \ldots, -\lambda_k \Delta T_k) = \frac{1}{(2\pi i)^k} \int_{\epsilon + \partial U} F(\zeta_1, \ldots, \zeta_k)(\lambda_1 \Delta T_1 + \zeta_1 I)^{-1} \ldots (\lambda_k \Delta T_k + \zeta_k I)^{-1} d\zeta_1 \ldots d\zeta_k,
\]

where \( \partial U \) denotes the "distinguished boundary of \( U \) and where \( \epsilon \in S_{\alpha, \beta} \) is chosen so that \( \epsilon + U \) still satisfies the required admissibility conditions with respect to \( T \) and \( \lambda \). Given \( T \) and \( \lambda \), this gives a family \( \mathcal{W}_{T, \lambda} \) of open sets stable under finite intersection and an algebra homomorphism \( F \to F(-\lambda_1 \Delta T_1, \ldots, -\lambda_k \Delta T_k) \) from \( \bigcup_{U \in \mathcal{W}_{T, \lambda}} H^1(U) \) into the multiplier algebra \( \mathcal{M}(\mathcal{B}) \subset \mathcal{Q}(\mathcal{M}(\mathcal{A})) \). This homomorphism extends in a natural
1 Introduction

The author observed in [14] that if a Banach algebra $A$ does not possess any nonzero idempotent then $\inf_{x \in A, x \neq 0} \|x^2 - x\| \geq 1/4$. If $x$ is quasinilpotent, and if $\|x\| \geq 1/2$, then $\|x\| > 1/4$. Concerning (nonzero) strongly continuous semigroups $T = (T(t))_{t \geq 0}$ of bounded operators on a Banach space $X$, these elementary considerations lead to the following results, obtained in 1987 by Mokhtari [24]

1. If $\limsup_{t \to 0^+} \|T(t) - T(2t)\| < 1/4$, then the generator of the semigroup is bounded, and so $\limsup \|T(t) - T(2t)\| = 0$.
2. If the semigroup is quasinilpotent, then $\|T(t) - T(2t)\| > 1/4$ when $t$ is sufficiently small.

If the semigroup is norm continuous, and if there exists a sequence $(t_n)_{n \geq 1}$ of positive real numbers such that $\lim_{n \to +\infty} t_n = 0$ and $\|T(t_n) - T(2t_n)\| < 1/4$, then the closed subalgebra $A_T$ of $B(X)$ generated by the semigroup possesses an exhaustive sequence of idempotents, i.e. there exists a sequence $(P_n)_{n \geq 1}$ of idempotents of $A_T$ such that for every compact set $K \subset A_T$ there exists $n_K > 0$ satisfying $\chi(P_n) = 1$ for $\chi \in K$, $n \geq n_K$. 

Keywords: analytic semigroup, infinitesimal generator, resolvent, Cauchy transform, Fourier-Borel transform, Laplace transform, holomorphic functional calculus, Cauchy theorem, Cauchy formula

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More sophisticated arguments allowed A. Mokhtari and the author to obtain in 2002 in [18] more general results valid for every integer \( p \geq 1 \).

These results led the author to consider in [15] the behavior of the distance \( \|T(s) - T(t)\| \) for \( s > t \) near 0. The following results were obtained in [15]:

1. If there exist for some \( \delta > 0 \) two continuous functions \( r \to t(r) \) and \( r \to s(r) \) on \([0, \delta[\), such that \( s(0) = 0 \) and such that \( 0 < t(r) < s(r) \) and \( \|T(t(r)) - T(s(r))\| < \frac{s(r) - t(r)}{t(r)} \) for \( r \in (0, \delta) \), then the generator of the semigroup is bounded, and so \( \|T(t) - T(s)\| \to 0 \) as \( 0 < t < s, s \to 0^+ \).

2. If the semigroup is quasinilpotent, there exists \( \delta > 0 \) such that \( \|T(t) - T(s)\| > (s - t)\frac{\delta}{t} \) for \( 0 < t < s \leq \delta \).

3. If the semigroup is norm continuous, and if there exists two sequences of positive real numbers such that \( 0 < t_n < s_n, \lim_{n \to +\infty} s_n = 0 \), and such that \( \|T(t_n) - T(s_n)\| < (s_n - t_n)\frac{\delta}{t_n s_n} \), then the closed subalgebra \( \mathcal{A}_T \) of \( B(X) \) generated by the semigroup possesses an exhaustive sequence of idempotents.

The quantities appearing in these statements are not mysterious: consider the Hilbert space \( L^2([0, 1]) \), and for \( t > 0 \) define \( T_0(t) : L^2([0, 1]) \to L^2([0, 1]) \) by the formula \( T_0(t)(f)(x) = x^t f(x) \) \((0 < x \leq 1)\). Then \( \|T_0(t) - T_0(s)\| = (s - t)\frac{\delta}{t} \).

This remark also shows that assertions (1) and (3) in these statements are sharp, and examples show that assertion (2) is also sharp.

One can consider \( T(t) \) as defined by the formula \( \int_0^{+\infty} T(x)d\delta_t(x) \), where \( \delta_t \) denotes the Dirac measure at \( t \). Heuristically, \( T(t) = e^{t\Delta_T} \), where \( \Delta_T \) denotes the generator of the semigroup, and since the Laplace transform of \( \delta_t \) is defined by the formula \( \mathcal{L}(\delta_t)(z) = \int_0^{+\infty} e^{-zt}d\delta_t(x) = e^{-zt} \), it is natural to write \( \mathcal{L}(\delta_t)(-\Delta_T) = T(t) \). More generally, if an entire function \( F \) has the form \( F = \mathcal{L}(\mu) \), where \( \mu \) is a measure supported by \([a, b[\), with \( 0 < a < b < +\infty \), we can set

\[
F(-\Delta_T) = \int_0^{+\infty} T(x)d\mu(x),
\]

and consider the behavior of the semigroup near 0 in this context.

I. Chalendar, J.R. Partington and the author used this point of view in [8]. Denote by \( \mathcal{M}_c(0, +\infty) \) the set of all measures \( \mu \) supported by some interval \([a, b[\), where \( 0 < a < b < +\infty \). For the sake of simplicity we restrict attention to statements analogous to assertion 2. The following result is proved in [8]:

Theorem: Let \( \mu \in \mathcal{M}_c(0, +\infty) \) be a nontrivial real measure such that \( \int_0^{+\infty} d\mu(t) = 0 \) and let \( T = (T(t))_{t \geq 0} \) be a quasinilpotent semigroup of bounded operators. Then there exists \( \delta > 0 \) such that \( \|F(-s\Delta_T)\| > \max_{x \geq 0} |F(x)| \) for \( 0 < s \leq \delta \).
When $\mu = \delta_1 - \delta_2$ this gives assertion 3 of Mokhtari's result, and when $\mu = \delta_1 - \delta_{p+1}$ this gives assertion 3 of the extension of Mokhtari's result given in [18] (but several variables extensions of this functional calculus would be needed in order to obtain extensions of the results of [15]).

This theory applies, for example, to quantities of the form $\|T(t) - 2T(2t) + T(3t)\|$, or Bochner integrals $\| \int_0^2 T(tx)dx - \int_2^3 T(tx)dx \|$, which are not accessible by the methods of [24] or [18]. Preliminary results concerning semigroups holomorphic in a sector were obtained by I. Chalendar, J.R. Partington and the author in [9].

More generally it would be interesting to obtain lower estimates as $(\lambda_1, \ldots, \lambda_k) \to (0, \ldots, 0)$ for quantities of the form $F(-\lambda_1 \Delta_T, \ldots, -\lambda_k \Delta_T)$ when the generator $\Delta_T$ of the semigroup is unbounded, and when $F$ is an analytic function of several complex variables defined and satisfying natural growth conditions on a suitable neighbourhood of $\sigma_{ar}(\Delta_T)$, where $\sigma_{ar}(\Delta_T)$ denotes the "Arveson spectrum" of the infinitesimal generator $\Delta_T$ of $T$. The purpose of the present paper is to pave the way to such a program by defining more generally $F(-\lambda_1 \Delta_T, \ldots, -\lambda_k \Delta_T)$ when $F$ belongs to a suitable class of holomorphic functions on some element of a family $\mathcal{W}_{T_1, \ldots, T_k, \lambda}$ of open sets, and where $(T_1, \ldots, T_k)$ denotes a finite family of commuting semigroups.

More precisely consider $a = (a_1, \ldots, a_k) \in \mathbb{R}^k, b = (b_1, \ldots, b_k) \in \mathbb{R}^k$ satisfying $a_j \leq b_j \leq a_j + \pi$ for $j \leq k$, and consider a commutative Banach algebra $\mathcal{A}$ such that $u\mathcal{A}$ is dense in $\mathcal{A}$ for some $u \in \mathcal{A}$ and such that $u\mathcal{A} \neq \{0\}$ for $u \in \mathcal{A} \setminus \{0\}$. This allows to consider the algebra $\mathcal{QM}(\mathcal{A})$ of all quasimultipliers on $\mathcal{A}$ and the algebra $\mathcal{QM}_r(\mathcal{A})$ of all regular quasimultipliers on $\mathcal{A}$ introduced by the author in [14], see section 2, and the usual algebra $\mathcal{M}(\mathcal{A})$ of all multipliers on $\mathcal{A}$ can be identified to the algebra of all quasimultipliers on $\mathcal{A}$ of domain equal to the whole of $\mathcal{A}$. We will be interested here in finite families $(T_1, \ldots, T_k)$ of commuting semigroups of multipliers on $\mathcal{A}$ satisfying the following conditions

- the semigroup $T_j$ is strongly continuous on $e^{ia_j}(0, +\infty)$, and $\cup_{t>0} T_j(e^{ia_j}t)\mathcal{A}$ is dense in $\mathcal{A}$ if $a_j = b_j$,
- the semigroup $T_j$ is holomorphic on the open sector $S_{a_j, b_j} := \{ z \in \mathbb{C} \setminus \{0\} \mid a_j < \arg(z) < b_j \}$ and $T_j(\zeta)\mathcal{A}$ is dense in $\mathcal{A}$ for some (or, equivalently, for all) $\zeta \in S_{a_j, b_j}$ if $a_j < b_j$.

The first step of the construction consists in obtaining a "normalization" $\mathcal{A}_T$ of the Banach algebra $\mathcal{A}$ with respect to a strongly continuous one-parameter semigroup $(T(t))_{t>0}$ of multipliers on $\mathcal{A}$. The idea behind this normalization process goes back to Feller [20], and we use for this the notion of "QM-homomorphism" between commutative Banach algebras introduced in section 2, which seems more appropriate than the related notion of "s-homomorphism" introduced by the author in [14]. Set $\omega_T = \|T(t)\|$ for $t > 0$. A slight improvement of a result proved by P. Koosis and the author in section 6 of [13] shows that the weighted convolution algebra $L^1(\mathbb{R}^+, \omega_T)$ possesses dense principal ideals, which allows to construct in section 3 a commutative Banach algebra $\mathcal{A}_T \subset \mathcal{QM}_r(\mathcal{A})$. 

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which contains $\mathcal{A}$ as a dense subalgebra and has dense principal ideals such that the injection $j : \mathcal{QM}(\mathcal{A}) \to \mathcal{QM}(\mathcal{A}_T)$ associated to the norm-decreasing inclusion map $j : \mathcal{A} \to \mathcal{A}_T$ is onto and such that $j(\mathcal{A}(\mathcal{A})) \subset \mathcal{M}(\mathcal{A}_T)$ for which $\limsup_{t \to +\infty} \|T(t)\|_{\mathcal{M}(\mathcal{A}_T)} < +\infty$. Set $\phi_T(f) = \int_0^{+\infty} f(t) T(t) \, dt$ for $f \in L^1(\mathbb{R}^+, \omega_T)$, where the Bochner integral is computed with respect to the strong operator topology on $\mathcal{M}(\mathcal{A})$, and denote by $\mathcal{I}_T$ the closed subalgebra of $\mathcal{M}(\mathcal{A})$ generated by $\phi_T(L^1(\mathbb{R}^+ \omega_T))$. In section 5 we give an interpretation of the generator $\Delta_T$ of the semigroup $T$ as a quasimultiplier on $\mathcal{I}_T$, and we define the "Arveson spectrum" $\sigma_{ar}(\Delta_T)$ to be the set $\{\chi(\Delta_T)\} \chi \in \mathcal{I}_T^\ast$, where $\chi$ denotes the unique extension to $\mathcal{QM}(\mathcal{I}_T)$ of a character $\chi$ on $\mathcal{I}_T$, with the convention $\sigma_{ar}(\Delta_T) = \emptyset$ if the "Arveson ideal" $\mathcal{I}_T$ is radical. The quasimultiplier $\Delta_T - M$ is invertible in $\mathcal{QM}(\mathcal{I}_T)$ and $(\Delta_T - M)^{-1} \in \mathcal{M}(\mathcal{A}_T) \subset \mathcal{QM}(\mathcal{A})$ if $\lambda \in \mathbb{C} \setminus \sigma_{ar}(\Delta_T)$, and we observe in section 6 that we have, for $\zeta > \limsup_{t \to +\infty} \frac{\|T(t)\|}{t}$,

$$(\Delta_T - \zeta \mathbf{1})^{-1} = -\int_0^{+\infty} e^{-\zeta t} T(t) \, dt \in \mathcal{QM}(\mathcal{A}_T) \subset \mathcal{QM}(\mathcal{A}),$$

which is the usual "resolvent formula" extended to strongly continuous semigroups not necessarily bounded near the origin.

In section 4 we construct a more sophisticated normalization of the Banach algebra $\mathcal{A}$ with respect to a semigroup $T = (T(\zeta))_{\zeta \in \mathcal{S}_{a,b}}$ which is holomorphic on an open sector $\mathcal{S}_{a,b}$, where $a < b \leq a + \pi$. In this case the normalization $\mathcal{A}_T$ of $\mathcal{A}$ with respect to the semigroup $T$ satisfies two more conditions

- $T(\zeta) u_{\mathcal{A}_T}$ is dense in $\mathcal{A}_T$ for $\zeta \in \mathcal{S}_{a,b}$ if $u_{\mathcal{A}}$ is dense in $\mathcal{A}$,
- $\limsup_{0 < \alpha \leq \beta < \rho(\zeta) \leq \pi} \|T(\zeta)\|_{\mathcal{M}(\mathcal{A}_T)} < +\infty$ for $a < \alpha < \beta < b$.

The generator of the holomorphic semigroup $T$ is interpreted as in [7] as a quasimultiplier on the closed subalgebra of $\mathcal{A}$ generated by the semigroup, which is equal to the Arveson ideal $\mathcal{I}_T$, where $\mathcal{I}_T$ denotes the restriction of $T$ to the half-line $(0, e^{-i\frac{\pi}{2}}, +\infty)$, and the resolvent $\zeta \to (\Delta_T - \zeta \mathbf{1})^{-1}$, which is defined and holomorphic outside a closed sector of the form $z + \mathcal{S}_{-i\alpha, i\beta}^{i\alpha, i\beta}$ is studied in section 7.

Consider again $a = (a_1, \ldots, a_k) \in \mathbb{R}^k, b = (b_1, \ldots, b_k) \in \mathbb{R}^k$ satisfying $a_j \leq b_j \leq a_j + \pi$ for $j \leq k$ and a finite family $T = (T_1, \ldots, T_k)$ of commuting semigroups of multipliers on $\mathcal{A}$ satisfying the conditions given above. By iterating the normalization process of $\mathcal{A}$ with respect to the family $T$, see definition 8.1, which is a commutative Banach algebra $\mathcal{B} \subset \mathcal{QM}(\mathcal{A})$ for which the injection $j : \mathcal{A} \to \mathcal{B}$ is norm-decreasing, has dense range and extends to a norm-decreasing homomorphism from $\mathcal{M}(\mathcal{A})$ into $\mathcal{M}(\mathcal{B})$, for which the natural embedding $j : \mathcal{QM}(\mathcal{A}) \to \mathcal{QM}(\mathcal{B})$ is onto, and for which $\limsup_{t \to 0+} \|T(t)h\|_{\mathcal{M}(\mathcal{B})} < +\infty$ if $a_j = b_j$, and for which $\limsup_{t \to 0+} \|T(\zeta)\| < +\infty$ for $a_j < \alpha_j \leq \beta_j < b_j$ if $a_j < b_j$. 

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Denote by $M_{a,b}$ the set of all pairs $(\alpha, \beta) \in \mathbb{R}^k \times \mathbb{R}^k$ such that $\alpha_j = \beta_j = a_j$ if $a_j = b_j$ and such that $a_j < \alpha_j \leq \beta_j < b_j$ if $a_j < b_j$. Let $W_{a,b}$ be the algebra of continuous functions $f$ on $\bigcup_{(\alpha, \beta) \in M_{a,b}} S_{\alpha, \beta} := \Pi_{1 \leq k} S_{\alpha_j, \beta_j}$ such that $e_z(\zeta)f(\zeta) \to 0$ as $|\zeta| \to 0$ in $\Pi_{1 \leq k} S_{\alpha_j, \beta_j}$ for every $z = (z_1, \ldots, z_k) \in \mathbb{C}^k$ and every $(\alpha, \beta) \in M_{a,b}$, and such that the maps $\zeta \mapsto f(\zeta, \zeta_{j-1}, \zeta_{j+1}, \ldots, \zeta_k)$ are holomorphic on $S_{\alpha_j, \beta_j}$ if $a_j < b_j$. For every element $\phi$ of the dual space $G_{a,b} = W_{a,b}'$ there exists $(\alpha, \beta) \in M_{a,b}$, $z \in \mathbb{C}^k$ and a measure $\nu$ of bounded variation on $S_{\alpha, \beta} := \Pi_{1 \leq k} S_{\alpha_j, \beta_j}$ such that

$$< f, \phi > = \int_{S_{\alpha, \beta}} e^{z\zeta}f(\zeta)d\nu(\zeta) \quad (f \in W_{a,b}),$$

and this formula allows to extend the action of $\phi$ to $e^{-z}V_{a,\beta}(X) \supset e^{-z}U_{a,\beta}(X)$, where $X$ denotes a separable Banach space and where $U_{a,\beta}(X)$ (resp. $V_{a,\beta}(X)$) denotes the algebra of continuous functions $f : S_{\alpha, \beta} \to X$ which converge to 0 as $\zeta \to \infty$ (resp. bounded continuous functions $f : S_{\alpha, \beta} \to X$) such that the maps $\zeta \to f(\zeta, \zeta_{j-1}, \zeta_{j+1}, \ldots, \zeta_k)$ are holomorphic on $S_{\alpha_j, \beta_j}$ when $a_j < b_j$.

Set $U_{a,\beta} := U_{a,\beta}(\mathbb{C})$. We describe in appendix 1 some certainly well-known ways to implement the action of $U'_{a,\beta}$ on $V_{a,\beta}(X)$ when $(\alpha, \beta) \in M_{a,b}$ by using Cauchy transforms and Fourier-Borel transforms, and these formulae are extended to the action of elements of $(e^{-z}U_{a,\beta})'$ to spaces $e^{-z}V_{a,\beta}(X)$ in appendix 2.

If $\phi \in \left( \cap_{\gamma \in \mathbb{C}^k} e^{-z}U_{a,\beta} \right)'$, define the domain $\text{Dom}(FB(\phi))$ of the Fourier-Borel transform $FB(\phi)$ of $\phi$ to be the set of all $z \in \mathbb{C}^k$ such that $\phi \in (e^{-z}U_{a,\beta})'$, and set $FB(\phi)(z) = < e^{-z}, \phi >$ for $z \in \text{Dom}(FB(\phi))$. One can also define in a natural way the Fourier-Borel transform of $f \in e^{-z}V_{a,\beta}(X)$. Let $\lambda \in \bigcup_{(\gamma, \delta) \in M_{a,\beta}} S_{\gamma, \delta}$, and set $T(\lambda)(\zeta) = T(\lambda_1 \zeta_1, \ldots, \lambda_k \zeta_k)$ for $\zeta \in S_{\alpha, \beta}$. If $\limsup_{|\zeta| \to \infty} |e^{-\zeta}||T(\lambda)(\zeta)|| < +\infty$, where $\partial S_{\alpha, \beta}$ denotes the "distinguished boundary" of $S_{\alpha, \beta}$, then

$$\sup_{\zeta \in S_{\alpha, \beta}} \|e^{z\zeta}T_1(\lambda_1 \zeta_1) \ldots T_k(\lambda_k \zeta_k)\|_{\mathcal{M}(B)} < +\infty,$$

and one can define the action of $\phi$ on $T(\lambda)$ by using the formula

$$< T(\lambda), \phi > u = < T(\lambda)u, \phi > = \int_{S_{\alpha, \beta}} e^{z\zeta}T(\lambda)(\zeta)u d\nu(\zeta) \quad (u \in \mathcal{B}),$$

where $\nu$ is a representing measure for $\phi e^{-z} : f \mapsto < e^{-z}f, \phi >$ $(f \in U_{a,\beta})$.

Then $< T, \phi > \in \mathcal{M}(\mathcal{B}) \subset \mathcal{Q}_{\mathcal{M}_{\nu}}(\mathcal{A})$.

The Fourier-Borel transform of $e_zT(\lambda)$ takes values in $\mathcal{M}(\mathcal{B})$ and extends analytically to $-\text{Res}_{ar}(\Delta T(\lambda)) := \Pi_{1 \leq j \leq k} (\mathcal{C} \setminus \sigma_{ar}(-\lambda_j \Delta T_j))$, which gives the formula

$$FB(e_zT(\lambda))((\zeta)) = (-1)^k \Pi_{1 \leq j \leq k} (\lambda_1 \Delta_1 + (z_1 + \zeta_1)I)^{-1} \ldots (\lambda_k \Delta_k - (z_k + \zeta_k)I)^{-1}.$$
Set $S^*_{\alpha,\beta} = \Pi_{j \leq k} S^{-\alpha_j, \beta_j}_j$, and set $W_n(\zeta) = \Pi_{1 \leq j \leq k} \frac{n^2}{(n + \zeta e^{i\theta_j})^2}$.

for $n \geq 1$, $\zeta = (\zeta_1, \ldots, \zeta_n) \in \overline{S^*_{\alpha,\beta}}$. The results of section 2 give for $u \in B$, if $z \in \text{Dom}(\phi)$, and if $\lim_{|\zeta| \to +\infty} |e^{-z_\zeta}| |T(\lambda)(\zeta)| < +\infty$, where $\partial S^*_{\alpha,\beta}$ denotes the "distinguished boundary" of $S^*_{\alpha,\beta}$,

$$< T(\lambda), \phi > u = \lim_{\epsilon \to (0, \ldots, 0, \ldots)} \left( \lim_{n \to +\infty} \frac{(-1)^k}{2\pi i} \int_{z + \partial S^*_{\alpha,\beta}} W_n(\sigma - z)FB(\phi)(\sigma)(\sigma_1 - \epsilon_1)I + \lambda_1 \Delta_T)^{-1} \ldots (\sigma_k - \epsilon_k)I + \lambda_k \Delta_T)^{-1} u d\sigma \right).$$

where $\partial S^*_{\alpha,\beta} := \Pi_{1 \leq j \leq k} \partial S_{\alpha_j,\beta_j}$ denotes the "distinguished boundary" of $\overline{S^*_{\alpha,\beta}}$, and where $\partial S^*_{\alpha,\beta}$ is oriented from $-ie^{i\theta_j} \infty$ to $ie^{i\theta_j} \infty$.

If, further, $\int_{z + \partial S^*_{\alpha,\beta}} \|FB(\phi)(\sigma)\| d\sigma < +\infty$, then we have, for $u \in B$,

$$< T_\lambda, \phi > u = \lim_{\epsilon \to (0, \ldots, 0, \ldots)} \int_{z + \partial S^*_{\alpha,\beta}} FB(\phi)(\sigma) (\lambda_1 - \epsilon_1)I + \lambda_1 \Delta_T)^{-1} \ldots (\lambda_k - \epsilon_k)I + \lambda_k \Delta_T)^{-1} u d\sigma.$$

Finally, if $z \in \text{Dom}(\phi)$, if $\int_{z + \partial S^*_{\alpha,\beta}} \|FB(\phi)(\sigma)\| d\sigma < +\infty$, and if $\lim_{\zeta \to \partial S^*_{\alpha,\beta}} |e^{-z_\zeta}| |T(\lambda)(\zeta)| = 0$, then we have, for $u \in B$,

$$< T(\lambda), \phi > u = (\frac{-1)^k}{2\pi i} \int_{z + \partial S^*_{\alpha,\beta}} FB(\phi)(\sigma) (\lambda_1 - \epsilon_1)I + \lambda_1 \Delta_T)^{-1} \ldots (\lambda_k - \epsilon_k)I + \lambda_k \Delta_T)^{-1} u d\sigma.$$

The convolution product of two elements of $(\epsilon - \mathcal{U}_{\alpha,\beta})'$ may be defined in a natural way, and if $\lambda, \phi_1, \phi_2$ satisfies the conditions above we have

$$< T(\lambda), \phi_1 * \phi_2 > = < T(\lambda), \phi_1 > < T(\lambda), \phi_2 >,$$

but there is no direct extension of this formula to the convolution product of two arbitrary elements of $\mathcal{H}_{\alpha,\beta}$, see the comments at the end of section 8.

In section 9 of the paper we introduce a class $\mathcal{U}$ of "admissible open sets" $U$, with piecewise $C^1$-boundary, of the form $(z + S^*_{\alpha,\beta}) \setminus K$, where $K$ is bounded and where $(\alpha, \beta) \in M_{\alpha,\beta}$. These open sets $U$ have the property that $U + \epsilon \subset \mathcal{U}$ for $\epsilon \in \overline{S^*_{\alpha,\beta}}$ and that $\overline{U} + \epsilon \subset \text{Res}_{\alpha,\beta}(-\lambda \Delta_T)$ for some $\epsilon \in \overline{S^*_{\alpha,\beta}}$. Also $R(-\lambda \Delta_T, \cdot) := (-\lambda \Delta_T - I)^{-1} \ldots (-\lambda \Delta_T - I)^{-1}$ is bounded on the distinguished boundary of $U + \epsilon$ for $\epsilon \in S^*_{\alpha,\beta}$ when $|\epsilon|$ is sufficiently small. Standard properties of the class $H^{(1)}(U)$ of all holomorphic functions $F$ on $U$ such that $\|F\|_{H^{(1)}(U)} := \sup_{\epsilon \in S^*_{\alpha,\beta}} \int_{z + \partial S^*_{\alpha,\beta}} |F(\sigma)| d\sigma < +\infty$ are given in appendix 3 (when $a_j = b_j$ for $j \leq k$, this space is the usual Hardy space $H^1$ on a product of open half-planes).
The results of appendix 3 allow when an open set $U \subset \mathbb{C}^k$ admissible with respect to $(\alpha, \beta) \in M_{a,b}$ satisfies some more suitable admissibility conditions with respect to $T = (T_1, \ldots, T_k)$ and $\lambda \in \mathcal{S}_{a,b}$ for some $(\gamma, \delta) \in M_{a-\alpha,b-\beta}$ to define $F(-\lambda_1 \Delta_{T_1}, \ldots, -\lambda_k \Delta_{T_k}) \in \mathcal{M}(B) \subset \mathcal{QM}(A)$ for $F \in H^{(1)}(U)$ by using the formula

$$F(-\lambda_1 \Delta_{T_1}, \ldots, -\lambda_k \Delta_{T_k}) = \frac{1}{(2\pi)^k} \int_{\partial U} F(\zeta_1, \ldots, \zeta_k)(\lambda_1 \Delta_{T_1} + \zeta_1 I)^{-1} \ldots (\lambda_k \Delta_{T_k} + \zeta_k I)^{-1} d\zeta_1 \ldots d\zeta_k,$$

where $\partial U$ denotes the distinguished boundary of $U$, where $\epsilon \in \mathcal{S}^*_{a,b}$ is chosen so that $\epsilon + U$ still satisfies the required admissibility conditions with respect to $T$ and $\lambda$.

Given $T$ and $\lambda \in \cup_{(\alpha, \beta) \in M_{a,b}} \mathcal{S}_{a-\alpha,b-\beta}$, denote by $\mathcal{W}_{T,\lambda}$ the family of all open sets $U \subset \mathbb{C}^k$ satisfying these admissibility conditions with respect to $T$ and $\lambda$. Then $\mathcal{W}_{T,\lambda}$ is stable under finite intersections, $\cup U \in \mathcal{W}_{T,\lambda} H^{(1)}(U)$ is stable under products and we have

$$(F_1 F_2)(-\lambda_1 \Delta_{T_1}, \ldots, -\lambda_k \Delta_{T_k}) = F_1(-\lambda_1 \Delta_{T_1}, \ldots, -\lambda_k \Delta_{T_k}) F_2(-\lambda_1 \Delta_{T_1}, \ldots, -\lambda_k \Delta_{T_k}).$$

This homomorphism extends in a natural way to a bounded algebra homomorphism from $\cup_{U \in \mathcal{W}_{T,\lambda}} H^{\infty}(U)$ into $\mathcal{QM}(B) = \mathcal{QM}(A)$, and we have, if $\phi \in \mathcal{F}_{a,b}$ for some $(\alpha, \beta) \in M_{a,b}$ such that $\lambda \in \mathcal{S}_{a,b}$ for some $(\gamma, \delta) \in M_{a-\alpha,b-\beta}$, and if $\lim_{|\xi| \to +\infty} \|e^{-z\xi}T_1(\lambda_1 \xi_1) \ldots T_k(\lambda_k \xi_k)\| = 0$ for some $z \in \text{Dom}(FB(\phi))$,

$$FB(\phi)(-\lambda_1 \Delta_{T_1}, \ldots, -\lambda_k \Delta_{T_k}) = < T(\lambda), \phi >,$$

so that $F(-\lambda_1 \Delta_{T_1}, \ldots, -\lambda_k \Delta_{T_k}) = T(\nu \lambda_j)$ if $F(\zeta) = e^{-\nu \zeta_j}$, where $\nu \lambda_j$ is in the domain of definition of $T_j$.

A function $F \in H^{\infty}(U)$ will be said to be strongly outer if there exists a sequence $(F_n)_{n \geq 1}$ of invertible elements of $H^{\infty}(U)$ such that $|F(\zeta)| \leq |F_n(\zeta)|$ and $\lim_{n \to +\infty} F(\zeta) F_n^{-1}(\zeta) = 1$ for $\zeta \in U$. If $U$ is admissible with respect to some $(\alpha, \beta) \in M_{a,b}$ then there is a conformal map $\theta$ from $\mathbb{D}^k$ onto $U$ and the map $F \to F \circ \theta$ is a bijection from the set of strongly outer bounded functions on $U$ onto the set of strongly outer bounded functions on $\mathbb{D}^k$. Every bounded outer function on the open unit disc $\mathbb{D}$ is strongly outer, but the class of strongly outer bounded functions on $\mathbb{D}^k$ is smaller than the usual class of bounded outer functions on $\mathbb{D}^k$ if $k \geq 2$. We then define the Smirnov class $\mathcal{S}(U)$ to be the class of those holomorphic functions $F$ on $U$ such that $FG \in H^{\infty}(U)$ for some strongly outer function $G \in H^{\infty}(U)$. The bounded algebra homomorphism $F \to F(-\lambda_1 \Delta_{T_1}, \ldots, -\lambda_k \Delta_{T_k})$ from $\cup U \in \mathcal{W}_{T,\lambda} \mathcal{S}(U)$ into $\mathcal{QM}(B) = \mathcal{QM}(A)$ extends to a bounded homomorphism from $\cup U \in \mathcal{W}_{T,\lambda} \mathcal{S}(U)$ into $\mathcal{QM}(B) = \mathcal{QM}(A)$. If $F : \zeta \to \zeta_j$, is the $j$-th coordinate projection then of course $F(-\lambda_1 \Delta_{T_1}, \ldots, -\lambda_k \Delta_{T_k}) = \lambda_j \Delta_{T_j}$.

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2 Quasimultipliers on weakly cancellative commutative Banach algebras with dense principal ideals

We will say that a Banach algebra $\mathcal{A}$ is weakly cancellative if $u\mathcal{A} \neq \{0\}$ for every $u \in \mathcal{A}\setminus\{0\}$. In the whole paper we will consider weakly cancellative commutative Banach algebras with dense principal ideals, i.e. weakly cancellative commutative Banach algebras such that the set $\Omega(\mathcal{A}) := \{u \in \mathcal{A} \mid [u\mathcal{A}]^\sim = \mathcal{A}\}$ is not empty.

A quasimultiplier on such an algebra $\mathcal{A} \neq \{0\}$ is a closed operator $S = S_{u/v} : \mathcal{D}_S \to \mathcal{A}$, where $u \in \mathcal{A}$, $v \in \Omega(\mathcal{A})$, where $\mathcal{D}_S := \{x \in \mathcal{A} \mid ux \in v\mathcal{A}\}$, and where $Sx$ is the unique $y \in \mathcal{A}$ such that $vy = ux$ for $x \in \mathcal{D}_S$. Let $QM(\mathcal{A})$ be the algebra of all quasimultipliers on $\mathcal{A}$. A set $U \subset QM(\mathcal{A})$ is said to be pseudobounded if $\sup_{S \in U} \|Su\| < +\infty$ for some $u \in \Omega(\mathcal{A}) \cap (\bigcap_{S \in U} \mathcal{D}(S))$ and a quasimultiplier $S \in QM(\mathcal{A})$ is said to be regular if the family $\{\lambda^n S^n\}_{n \geq 1}$ is pseudobounded for some $\lambda > 0$. The algebra of all regular quasimultipliers on $\mathcal{A}$ will be denoted by $QM_r(\mathcal{A})$. A multiplier on $\mathcal{A}$ is a bounded linear operator $S$ on $\mathcal{A}$ such that $S(uv) = (Su)v$ for $u \in \mathcal{A}, v \in \mathcal{A}$, and the multiplier algebra $\mathcal{M}(\mathcal{A})$ of all multipliers on $\mathcal{A}$, which is a closed subalgebra of $B(\mathcal{A})$, is also the algebra of all quasimultipliers on $\mathcal{A}$ such that $\mathcal{D}_S = \mathcal{A}$, and $\mathcal{M}(\mathcal{A}) \subset QM_r(\mathcal{A})$.

Also if $S = S_{u/v} \in QM(\mathcal{A})$, $w \in \mathcal{D}(S), R \in \mathcal{M}(\mathcal{A})$, then $u(Rw) = R(v(Sw)) = v(R(Sw))$, so $Rw \in \mathcal{D}(S)$, and we have

$$R(Sw) = S(Rw).$$

If $\mathcal{A}$ is unital then $\Omega(\mathcal{A}) = \mathcal{G}(\mathcal{A})$, where $\mathcal{G}(\mathcal{A})$ denotes the group of invertible elements of $\mathcal{A}$, and $QM(\mathcal{A}) = \mathcal{M}(\mathcal{A})$.

The following notion is slightly more flexible than the notion of $s$-homomorphism introduced by the author in [14].

**Definition 2.1**: Let $\mathcal{A}$ be a weakly cancellative commutative Banach algebra with dense principal ideals, and let $\mathcal{B}$ be a weakly cancellative Banach algebra. A homomorphism $\Phi : \mathcal{A} \to \mathcal{B}$ is said to be a $QM$-homomorphism if the following conditions are satisfied

(i) $\Phi$ is one-to-one, and $\Phi(\mathcal{A})$ is dense in $\mathcal{B}$.

(ii) $\Phi(u)\mathcal{B} \subset \Phi(\mathcal{A})$ for some $u \in \Omega(\mathcal{A})$.

If the conditions of definition 2.1 are satisfied, we will say that $\Phi$ is a $QM$-homomorphism with respect to $u$. Notice that $\Phi(u) \in \Omega(\mathcal{B})$, and so the existence of such an homomorphism implies that $\mathcal{B}$ is a weakly cancellative commutative Banach algebra with dense principal ideals. Notice also that condition (ii) shows that $\mathcal{B}$ may be identified to a subalgebra of $QM(\Phi(\mathcal{A})) = QM(\mathcal{A})$.

**Proposition 2.2** Let $\Phi : \mathcal{A} \to \mathcal{B}$ be a homomorphism between weakly cancellative commutative Banach algebras with dense principal ideals, and assume that $\Phi$ is a $QM$-homomorphism with respect to some $u_0 \in \Omega(\mathcal{A})$. 

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(i) There exists $M > 0$ such that $\|\Phi^{-1}(\Phi(\Phi(u)v))\| \leq M\|v\|$ for $v \in \mathcal{B}$.

(ii) $\Phi^{-1}(\Phi(\Phi(u)v)) \in \mathcal{B}$ for $v \in \mathcal{B}$.

(iii) Set $\Phi(S_{u/v}) = S_{\Phi(u)/\Phi(v)}$ for $S_{u/v} \in \mathcal{M}(\mathcal{A})$. Then $\Phi : \mathcal{M}(\mathcal{A}) \to \mathcal{M}(\mathcal{B})$ is a pseudobounded isomorphism, and $\Phi^{-1}(S_{u/v}) = S_{\Phi^{-1}(\Phi(u)v)/\Phi^{-1}(\Phi(u)v)}$ for $S_{u/v} \in \mathcal{M}(\mathcal{B})$.

Proof: (i) Set $\Psi(v) = \Phi^{-1}(\Phi(u)v)$ for $v \in \mathcal{B}$. If $\lim_{n \to +\infty} v_n = v \in \mathcal{B}$, and if $\lim_{n \to +\infty} \psi(v_n) = w \in \mathcal{A}$, then $\Phi(u)v = \Phi(w)$, so that $w = \Psi(v)$ and (i) follows from the closed graph theorem.

(ii) Let $v \in \Omega(\mathcal{B})$, and $(w_n)_{n \geq 1}$ be a sequence of elements of $\mathcal{A}$ such that $\lim_{n \to +\infty} v\Phi(w_n) = \Phi(u)$. Then $\lim_{n \to +\infty} \Phi^{-1}(\Phi(u)v)w_n = u_0 \in \Omega(\mathcal{A})$, and so $\Phi^{-1}(\Phi(u)v) \in \Omega(\mathcal{A})$.

(iii) Let $U \subset \mathcal{M}(\mathcal{A})$ be a pseudobounded set, and let $w \in \Omega(\mathcal{A}) \cap (\bigcap_{S \in \mathcal{U}} \mathcal{D}(S))$ be such that $\sup_{S \in \mathcal{U}} \|\Phi(S)\| < +\infty$. Then $\Phi(w) \in \Omega(\mathcal{B})$, and $\Phi(w) \in \Omega(\mathcal{A}) \cap (\bigcap_{S \in \mathcal{U}} \mathcal{D}(S))$.

Now set $\theta(S_{u/v}) = S_{\Phi^{-1}(\Phi(u))/\Phi^{-1}(\Phi(u))}$ for $S_{u/v} \in \mathcal{M}(\mathcal{B})$. It follows from (ii) that $\theta : \mathcal{M}(\mathcal{B}) \to \mathcal{M}(\mathcal{A})$ is well-defined. Let $U \subset \mathcal{M}(\mathcal{B})$ be pseudobounded, and let $w \in \Omega(\mathcal{B}) \cap (\bigcap_{S \in \mathcal{U}} \mathcal{D}(S))$ be such that $\sup_{S \in \mathcal{U}} \|S\| < +\infty$. We have, for $S = S_{u/v} \in U$,

$$u(S_{u/v}) = uw, u\Phi(u) = v\Phi(u),$$

$$\Phi^{-1}(u\Phi(u)) \Phi^{-1}(S_{u/v}w\Phi(u)) = \Phi^{-1}(v\Phi(u))\Phi^{-1}(w\Phi(u)).$$

So $\Phi^{-1}(w\Phi(u)) \in \mathcal{D}(\mathcal{S}(\Phi(u)))$, and $\theta(S) = \Phi^{-1}(w\Phi(u)) = \phi^{-1}(S_{u/v}w\Phi(u))$. Since $\sup_{S \in \mathcal{U}} \|S\| < +\infty$, this shows that $\theta : \mathcal{M}(\mathcal{B}) \to \mathcal{M}(\mathcal{A})$ is pseudobounded. We have, for $S = S_{u/v} \in \mathcal{M}(\mathcal{A})$,

$$\theta(S_{u/v}) = S_{\Phi^{-1}(\Phi(u))/\Phi^{-1}(\Phi(u))} = S_{u/v} = S.$$
(i) $B$ is a weakly cancellative commutative Banach algebra, and the inclusion map $j : A \to B$ is a $Q \mathcal{M}$-homomorphism with respect to $w$ for $w \in (A) \cap L$.

(ii) $\tilde{j} (\mathcal{M}(A)) \subset \mathcal{M}(B)$, and $\|\tilde{j}(R)\|_{\mathcal{M}(B)} \leq \|\mathcal{M}(A)\|$ for $R \in \mathcal{M}(A)$, where $\tilde{j} : Q \mathcal{M}(A) \to Q \mathcal{M}(B)$ is the pseudobounded isomorphism associated to $j$ in proposition 2.2(iii).

(iii) $S \in \mathcal{M}(B)$, and $\|S\|_{\mathcal{M}(B)} \leq \|S\|_{\mathcal{M}(L)} \leq 1$ for every $S \in U$.

Proof: The fact that $(L, \|\|_L)$ is a Banach space follows from a standard argument given in the proof of theorem 7.11 of [14]. Clearly, $L$ is an ideal of $A$, and it follows from the definition of $\|\|_L$ that $\|u\| \leq \|u\|_L$ for $u \in L$. We have, for $u \in L, v \in L$,

$$\|uv\|_L \leq \|u\|_L \|v\| \leq \|u\|_L \|v\|,$$

and so $(L, \|\|_L)$ is a Banach algebra. If $R \in \mathcal{M}(A), u \in L$, then it follows from (i) that $Ru \in \cap_{S \in U} S(L)$, and that we have

$$\sup_{S \in U} \|S(Ru)\| = \sup_{S \in U} \|R(Su)\| \leq \|R\|_{\mathcal{M}(A)} \|u\|_L.$$

Hence $Ru \in L$, and $\|Ru\|_L \leq \|R\|_{\mathcal{M}(A)} \|u\|_L$.

Now denote by $B$ the closure of $A$ in $(L, \|\|_L)$, and let $w \in \Omega(A) \cap L$. Since $L \subset B$, $B$ is weakly cancellative, and $B$ is commutative and has dense principal ideals since $j(A)$ is dense in $B$. Since $wB \subset L \subset A$, we see that the inclusion map $j : A \to B$ is a $Q \mathcal{M}$-homomorphism with respect to $w$, which proves (i).

Let $R \in \mathcal{M}(A)$, and denote by $R_1$ the restriction of $R$ to $L$. Then $R_1 \in \mathcal{M}(L)$, and $\|R_1\|_{\mathcal{M}(L)} \leq \|R\|_{\mathcal{M}(A)}$. Set $R_2u = R_1u$ for $u \in B$. Then $R_2u \in \mathcal{M}(B)$, and $\|R_2u\|_{\mathcal{M}(B)} \leq \|R_1\|_{\mathcal{M}(L)} \|u\|_{\mathcal{M}(L)} \leq \|R\|_{\mathcal{M}(A)} \|u\|_{\mathcal{M}(L)}$. Hence $R_2 \in \mathcal{M}(B)$, and $\|R_2\|_{\mathcal{M}(B)} \leq \|R\|_{\mathcal{M}(A)}$.

Now let $S_0 \in U$. we have, for $u \in L$, since $U$ is stable under products,

$$\|S_0u\|_L = \sup_{S \in U} \|S_0Su\| \leq \sup_{S \in U} \|Su\| = \|u\|_L,$$

and so $S_0u \in L$, and $\|S_0\|_{\mathcal{M}(L)} \leq 1$. This implies that $S_0(B) \subset B$, so that $S_0 \in \mathcal{M}(B)$, and $\|S\|_{\mathcal{M}(B)} \leq \|S_0\|_{\mathcal{M}(L)} \leq 1$, which proves (iii). \Box

We have the following very easy observation.

**Proposition 2.4**: Let $A_0, A_1$ and $A_2$ be weakly cancellative commutative Banach algebras, and assume that $\Phi_0 : A_0 \to A_1$ is a $Q \mathcal{M}$-homomorphism with respect to $u_0 \in \Omega(A_0)$ and that $\Phi_1 : A_1 \to A_2$ is a $Q \mathcal{M}$-homomorphism with respect to $u_1 \in \Omega(A_1)$. Then $\Phi_1 \circ \Phi_0 : A_0 \to A_2$ is a $Q \mathcal{M}$-homomorphism with respect to $\Phi_0^{-1}(\Phi_0(u_0)u_1)$.

Proof: The homomorphism $\Phi_1 \circ \Phi_0$ is one-to-one and has dense range, and it follows from proposition 2.2 (ii) that $\Phi_0^{-1}(\Phi_0(u_0)u_1) \in \Omega(A_0)$. Let $u \in A_2$, let $v \in A_2$ be such that $\Phi_1(v) = \Phi_1(u_1)u$, and let $w \in A_0$ be such that $\Phi_0(w) = \Phi_0(u_0)v$. Then $(\Phi_1 \circ \Phi_0)(\Phi_0^{-1}(\Phi_0(u_0)u_1)) u = (\Phi_1 \circ \Phi_0)(\Phi_0(u_0)\Phi_1(u_1)) u = u$.\[\Box\]
\[ \Phi_1(\Phi_0(u_0)v) = (\Phi_1 \circ \Phi_0)(w) \subset (\Phi_1 \circ \Phi_0)(A_0), \] and so \( \Phi_1 \circ \Phi_0 \) is a QM-homomorphism with respect to \( \Phi_0^{-1}(\Phi_0(u_0)u_1) \). □

We will denote by \( \hat{A} \) the space of all characters on a commutative Banach algebra \( A \), equipped with the Gelfand topology. Recall that \( A \) is said to be radical when \( \hat{A} = \emptyset \).

**Definition 2.5** Let \( A \) be a weakly cancellative commutative Banach algebra with dense principal ideals. For \( \chi \in A \), set \( \hat{\chi}(S) = \frac{\chi(S)}{\chi(0)} \), and for \( S \in QM(A) \), set \( \sigma_A(S) := \{ \hat{\chi}(S) \}_{\chi \in \hat{A}} \), with the convention \( \sigma_A(S) = \emptyset \) if \( A \) is radical.

 Clearly, \( \hat{\chi} \) is a character on \( QM(A) \) for \( \chi \in \hat{A} \), and the map \( \chi \to \hat{\chi}(S) \) is continuous on \( \hat{A} \) for \( S \in QM(A) \).

We will use the following result in the study of the resolvent of semigroups.

**Proposition 2.6** Let \( A \) be a weakly cancellative commutative Banach algebra with dense principal ideals, and let \( S \in QM(A) \). If \( \lambda_0 - S \) has an inverse \((\lambda_0I - S)^{-1} \) in \( QM(A) \) which belongs to \( A \) for some \( \lambda_0 \in \mathbb{C} \), where \( I \) denotes the unit element of \( M(A) \), then \( \sigma_A(S) \) is closed, \( \lambda I - S \) has an inverse \((\lambda I - S)^{-1} \) in \( QM(A) \) which belongs to \( A \) for every \( \lambda \in \mathbb{C} \setminus \sigma_A(S) \), and the \( A \)-valued map \( \lambda \to (\lambda I - S)^{-1} \) is holomorphic on \( \mathbb{C} \setminus \sigma_A(S) \).

**Proof:** If \( A \) is unital, then \( QM(A) = A \), and there is nothing to prove. So assume that \( A \) is not unital, and set \( \hat{A} := A \oplus CI \). Then \( \hat{A} = \{ \chi_0 \} \cup \{ \hat{\chi}, \chi \in \hat{A} \} \), where \( \chi_0(a + \lambda I) = \lambda \) for \( a \in A, \lambda \in \mathbb{C} \).

Set \( a = (\lambda_0I - S)^{-1} \in A \). Then \( \chi(a) = \hat{\chi}(a) = \frac{1}{\lambda_0 - \chi(0)} \), so that \( \hat{\chi}(S) = \lambda_0 - \frac{1}{\chi(a)} \) for \( \chi \in \hat{A} \). Since \( \sigma_A(a) \cup \{ 0 \} = \sigma_A \) is a compact subset of \( \mathbb{C} \), this shows that \( \sigma_A(S) \) is closed. We have, for \( \lambda \in \mathbb{C} \),

\[ \text{spec}_A(I + (\lambda - \lambda_0)a) = \{ 1 \} \cup \left\{ \frac{\lambda - \hat{\chi}(S)}{\lambda_0 - \hat{\chi}(S)} \right\}_{\chi \in \hat{A}}. \]

So \( I + (\lambda - \lambda_0)a \) is invertible in \( \hat{A} \) for \( \lambda \in \mathbb{C} \setminus \sigma_A(S) \), and the map \( \lambda \to a(I + (\lambda - \lambda_0)a)^{-1} \in A \) is holomorphic on \( \mathbb{C} \setminus \sigma_A(S) \). We have, for \( \lambda \in \mathbb{C} \setminus \sigma_A(S) \),

\[
(\lambda I - S)a(I + (\lambda - \lambda_0)a)^{-1} = ((\lambda - \lambda_0)I + (\lambda_0 - S))a(I + (\lambda - \lambda_0)a)^{-1} = (I + (\lambda - \lambda_0)a)(I + (\lambda - \lambda_0)a)^{-1} = I.
\]

Hence \( \lambda I - S \) has an inverse \((\lambda I - S)^{-1} \in A \) for \( \lambda \in \mathbb{C} \setminus \sigma_A(S) \), and the map \( \lambda \to (\lambda I - S)^{-1} = a(I + (\lambda - \lambda_0)a)^{-1} \) is holomorphic on \( \mathbb{C} \setminus \sigma_A(S) \). □
3 Normalization of a commutative Banach algebra with respect to a strongly continuous semigroup of multipliers

A semigroup \( T = (T(t))_{t \geq 0} \) of multipliers on a commutative Banach algebra \( A \) is said to be strongly continuous if the map \( t \mapsto T(t)u \) is continuous on \((0, +\infty)\) for every \( u \in A \). This implies that \( \sup_{t \leq \beta} \|T(t)\| < +\infty \) for \( 0 < \alpha \leq \beta < +\infty \), and so \( \|T(t)\|^\dagger \) has a limit \( \rho_T \) as \( t \to +\infty \), and \( \rho_T = \lim_{n \to +\infty} \|T(u)\|^\dagger \). In the remainder of the section \( T = (T(t))_{t > 0} \) will denote a strongly continuous group of multipliers on a weakly cancellative commutative Banach algebra \( A \) such that \( \cup_{t > 0} T(t)A \) is dense in \( A \). Hence if \( u \in \cap_{t > 0} \ker(T(t)) \), then \( uw = 0 \) for every \( v \in \cup_{t > 0} T(t)A \), so \( uA = \{0\} \) and \( u = 0 \).

Notice that in this situation if \( A \) has a unit element \( 1 \) then if we set \( \hat{T}(t) = T(t)1 \) then \( \hat{T} := (\hat{T}(t))_{t > 0} \) is a norm-continuous semigroup of elements of \( A \). Since \( \cup_{t > 0} T(t)A \) is dense in \( A \), \( T(t_0)1 \) is invertible in \( A \) for some \( t_0 > 0 \), and so \( \lim_{n \to +\infty} \|T(t_0)1 - 1\| = 0 \), which implies that the generator of \( \hat{T} \) is bounded. So there exist \( R \in \mathcal{M}(A) \approx A \) such that \( T(t) = e^{tR} \) for \( t > 0 \) if \( A \) is unital.

Let \( \omega \) be a positive measurable weight on \((0, +\infty)\). Recall that if \( \omega(s + t) \leq \omega(s)\omega(t) \) for \( s > 0, t > 0 \), then the space \( L^1_{\omega, R} (\mathbb{R}^+) \) of all \( \omega \)-integrable functions on \([0, +\infty)\), satisfying \( \|f\|_{\omega, R} := \int_0^{+\infty} |f(t)|\omega(t)dt < +\infty \), equipped with the norm \( \|\cdot\|_{\omega, R} \), is a Banach algebra with respect to convolution. Set \( \omega_T(t) = \|T(t)\| \) for \( t > 0 \). For \( f \in L^1_{\omega, R} (\mathbb{R}^+) \), define \( \Phi_T(f) \in \mathcal{M}(A) \) by the formula

\[
\Phi_T(f)u = \int_0^{+\infty} f(t)T(t)udt \quad (u \in A).
\]  

Denote by \( I_T \) the closure of \( \Phi_T(L^1_{\omega, R} (\mathbb{R}^+)) \) in \( \mathcal{M}(A) \). Let \((f_n)_{n \geq 1} \) be a Dirac sequence, i.e. a sequence \((f_n) \) of nonnegative integrable functions on \( \mathbb{R}^+ \) such that \( \int_0^{+\infty} f_n(t)dt = 1 \) and such that \( f_n(t) = 0 \) a.e. on \((\alpha_n, +\infty)\) with \( \lim_{n \to +\infty} \alpha_n = 0 \). Since the semigroup \( T \) is strongly continuous on \( A \), a standard argument shows that \( \lim_{n \to +\infty} \|\Phi_T f_n * \delta_0 - T(t)u\| = 0 \) for every \( t > 0 \) and every \( u \in A \). This shows that if \( v \in \Omega(I_T) \), and if \( w \in \Omega(A) \), then \( vw \in \Omega(A) \).

The following result is then a consequence of theorem 6.8 of [13] and of proposition 5.4 of [16].

**Lemma 3.1** There exists \( w \in \Omega(A) \) such that \( \limsup_{t \to 0^+} \|T(t)w\| < +\infty \).

**Proof:** Let \( \lambda > \log(\rho_T) \), and set \( \omega _\lambda (t) = e^{\lambda t} \sup_{s \geq t} e^{\lambda s}\|T(s)\| \) for \( t > 0 \). An extension to lower semicontinuous weights of theorem 6.8 of [13] given in [16] shows that \( \Omega(L^1_{\omega _\lambda, R} (\mathbb{R}^+)) \neq \emptyset \). It follows also from proposition 5.4 of [16] that \( \|\Phi_T(g)T(t)\| \leq e^{\lambda t}\|g\|_{L^1_{\omega _\lambda}} \) for every \( g \in L^1_{\omega _\lambda} (\mathbb{R}^+) \) and every \( t > 0 \), and that \( \Phi_T(g) \subset \Omega(I_T) \) for every \( g \in \Omega(L^1_{\omega _\lambda, R} (\mathbb{R}^+)) \). Hence if \( g \in \Omega(L^1_{\omega _\lambda, R} (\mathbb{R}^+)) \) and if \( v \in \Omega(A) \), then \( \Phi_T(g)v \in \Omega(A) \), and \( \limsup_{t \to 0^+} \|T(t)\Phi_T(g)v\| < +\infty \). □

The following result is a version specific to one-parameter semigroups of proposition 2.3.
Proposition 3.2 Let \( T := (T(t))_{t>0} \) be a strongly continuous semigroup of multipliers on a weakly cancellative commutative Banach algebra \( A \) with dense principal ideals such that \( \cup_{t>0} T(t)A \) is dense in \( A \). Let \( L_T := \{ u \in A \mid \limsup_{t \to 0^+} \| T(t)u \| < +\infty \} \supset \cup_{t>0} T(t)A \), choose \( \lambda > \log(\rho_T) \), set \( \| u \|_\lambda := \sup_{s \geq 0} e^{-\lambda s} \| T(s)u \| \) for \( u \in L_T \), with the convention \( T(0) = I \), and set \( \| R \|_{\lambda,op} := \sup\{ \| Ru \|_\lambda \mid u \in L_T, \| u \|_\lambda \leq 1 \} = \| R \|_{M(L_T)} \) for \( R \in M(L_T) \). Denote by \( A_T \) the closure of \( A \) in \( (M(L_T), \| . \|_{\lambda,op}) \).

Then \( (L_T, \| . \|_\lambda) \) is a Banach algebra, the norm topology on \( L_T \) does not depend on the choice of \( \lambda \), and the following properties hold

(i) The inclusion \( j : A \to A_T \) is a \( QM \)-homomorphism with respect to \( u \) for every \( \omega \in \Omega(A) \cap L_T \), the tautological map \( j : S_{u/v} \to S_{u/v} \) is a pseudobounded isomorphism from \( QM(A) \) onto \( QM(A_T) \) and if \( w \in \Omega(A) \cap L_T \), then \( j^{-1}(S) = S_{uw/vw} \) for \( S = S_{u/v} \in QM(A_T) \).

(ii) \( j(M(A)) \subset M(A_T) \), and \( \| R \|_{M(A_T)} \leq \| R \|_{M(A)} \) for \( R \in M(A) \).

(iii) \( \| T(t)u \|_{M(A_T)} \leq \| T(t)u \|_{\lambda,op} \leq e^\lambda t \) for \( t > 0 \), and \( \limsup_{t \to 0^+} \| T(t)u - u \|_{\lambda,op} = 0 \) for every \( u \in A_T \).

Proof: It follows from lemma 3.1 that the family \( U = \{ e^{-\lambda t} T(t) \}_{t>0} \) is pseudobounded for \( \lambda > \log(\rho_T) \). The fact that \( (L_T, \| . \|_\lambda) \) is a Banach algebra, and assertions (i) and (ii) follow from proposition 2.2 and proposition 2.3 applied to \( U \), and an elementary argument given in the proof of theorem 7.1 of [16] shows that there exists \( k > 0 \) and \( K > 0 \) such that \( k \| u \|_\lambda \leq \sup_{0 \leq t \leq 1} \| T(t)u \| \leq K \| u \|_\lambda \) for \( u \in L_T \), which shows that the norm topology on \( L_T \) does not depend on the choice of \( \lambda \).

It follows also from proposition 2.3 applied to \( U \) that \( \| T(t)u \|_{M(A_T)} \leq \| T(t)u \|_{\lambda,op} \leq e^\lambda t \) for \( t > 0 \), and \( \limsup_{t \to 0^+} \| T(t)u - u \|_{\lambda,op} = 0 \) for every \( u \in L_T \). Since \( \cup_{t>0} T(t)A \) is dense in \( A_T \), a standard density argument shows that \( \limsup_{t \to 0^+} \| T(t)u - u \|_{\lambda,op} = 0 \) for every \( u \in A_T \).

This suggests the following definition

Definition 3.3 Let \( A \) be a weakly cancellative commutative Banach algebra with dense principal ideals, let \( T := ((T(t))_{t>0} \) be a strongly continuous semigroup of multipliers on \( A \) such that \( T(t)A \) is dense in \( A \) for \( t > 0 \). A normalization \( B \) of \( A \) with respect to \( T \) is a subalgebra \( B \) of \( QM(A) \) which is a Banach algebra with respect to a norm \( \| . \|_B \) and satisfies the following conditions

(i) The inclusion map \( j : A \to B \) is a \( QM \)-homomorphism, and \( \| j(u) \|_B \leq \| u \|_A \) for \( u \in A \).

(ii) \( j(R) \subset M(B) \), and \( \| j(R) \|_{M(B)} \leq \| R \|_{M(A)} \) for \( R \in M(A) \), where \( j : QM(A) \to QM(B) \) is the pseudobounded isomorphism associated to \( j \) introduced in proposition 2.2 (ii).

(iii) \( \limsup_{t \to 0^+} \| j(T(t)) \|_{M(B)} < +\infty \).

For example the algebra \( A_T \) constructed in proposition 3.2 is a normalization of the given Banach algebra \( A \) with respect to the semigroup \( T \). Notice that if \( B \) is a normalization of \( A \) with respect to \( T \), the same density argument as above shows that \( \limsup_{t \to 0^+} \| T(t)u - u \|_B = 0 \) for every \( u \in B \).
4 Normalization of a commutative Banach algebra with respect to a holomorphic semigroup of multipliers

For $a < b \leq a + \pi$, denote by $S_{a,b}$ the open sector $\{ z \in \mathbb{C} \setminus \{0\} \mid a < \arg(z) < b \}$, with the convention $S_{a,a} = \{ re^{i\alpha} \mid r \geq 0 \}$. In this section we consider again a weakly cancellative commutative Banach algebra $\mathcal{A}$ with dense principal ideals and we consider a semigroup $T = (T(\zeta))_{\zeta \in S_{a,b}}$ of multipliers on $\mathcal{A}$ such that $\cup_{t \in S_{a,b}} T(t)\mathcal{A}$ is dense in $\mathcal{A}$ which is holomorphic on $S_{a,b}$, which implies that $T(\zeta)\mathcal{A}$ is dense in $\mathcal{A}$ for every $\zeta \in S_{a,b}$. So $T(\zeta)u \in \Omega(\mathcal{A})$ for every $\zeta \in S_{a,b}$ and every $u \in \Omega(\mathcal{A})$. We state as a lemma the following easy observations.

**Lemma 4.1** Let $u \in \mathcal{M}(\mathcal{A})$ such that $\limsup_{\zeta \to 0} \| T(\zeta)u \| < +\infty$, where $a < \alpha \leq \beta < b$.

(i) If $\lambda > \frac{1}{\cos(\frac{\beta - \alpha}{2})}$, then

$$\lim_{r \to +\infty} \log(\max\{\|T(re^{i\alpha})\|, \|T(re^{i\beta})\|\})$$

then

$$\sup_{\zeta \in S_{a,b}} \| e^{-\lambda \zeta} e^{-\frac{\alpha + \beta}{2}} T(\zeta)u \| < +\infty.$$

(ii) $\lim_{\lambda \to +\infty} \sup_{\zeta \in S_{a,b}} \left[ \sup_{\zeta \in S_{a,b}} \| e^{-\lambda \zeta} e^{-\frac{\alpha + \beta}{2}} T(\zeta)u \| \right] = \lim_{\zeta \to 0} \sup_{\zeta \in S_{a,b}} \| T(\zeta)u \|.$$

**Proof:** Set $\gamma := \frac{\beta - \alpha}{2}$. We have, for $r_1 \geq 0$, $r_2 \geq 0$,

$$\left\| e^{-\lambda \zeta} e^{-\frac{\alpha + \beta}{2}} (r_1 e^{i\alpha} + r_2 e^{i\beta})u \right\| \leq \inf \left[ e^{-\lambda \cos(\gamma) r_1} \| T(r_1 e^{i\alpha})u \|, e^{-\lambda \cos(\gamma) r_2} \| T(r_2 e^{i\beta})u \|, e^{-\lambda \cos(\gamma) r_1} \| T(r_1 e^{i\alpha}) \| e^{-\lambda \cos(\gamma) r_2} \| T(r_2 e^{i\beta})u \| \right].$$

Set $m := \limsup_{\zeta \to 0} \| T(t)u \|$, and let $\epsilon > 0$. There exists $\delta > 0$ such that $\| T(\zeta)u \| \leq m + \epsilon$ for $0 \leq |\zeta| \leq \delta$, $\zeta \in S_{a,b}$. We obtain, considering the cases $\sup(r_1, r_2) \leq \delta/2$, $\inf(r_1, r_2) \leq \delta/2$ and $\sup(r_1, r_2) \geq \delta/2$, and the case where $\inf(r_1, r_2) \geq \delta/2$,

$$\sup_{\zeta \in S_{a,b}} \| e^{-\lambda \zeta} e^{-\frac{\alpha + \beta}{2}} T(\zeta)u \| \leq \max \left[ (m + \epsilon) \sup_{|\zeta| \leq \delta/2} | e^{-\lambda \zeta} | \left( 1 + \sup_{r \geq \delta/2} e^{-\lambda \cos(\gamma) \max(\| T(re^{i\alpha})\|, \| T(re^{i\beta})\|)} \right) \right].$$

and (i) and (ii) follow from this inequality. □
We now use a construction proposed by I. Chalendar, J.R. Partington and the author in proposition 3.6 of [7] to associate to construct a \( \mathcal{QM} \)-homomorphism from \( \mathcal{A} \) into a weakly cancellative commutative Banach algebra \( \mathcal{B} \) such that \( \sup_{t \leq 1} \|T(t)\| < +\infty \) for every \( \alpha, \beta \) satisfying \( a < \alpha \leq \beta < b \).

The following result is more general than proposition 3.6 of [7].

**Proposition 4.2** Let \( \mathcal{A} \) be a weakly cancellative commutative Banach algebra with dense principal ideals, and let \( T = ((T(\zeta))_{\zeta \in \mathcal{S}_{a,b}} \) be a holomorphic semigroup of multipliers on \( \mathcal{A} \) such that \( T(\zeta)\mathcal{A} \) is dense in \( \mathcal{A} \) for \( \zeta \in \mathcal{S}_{a,b} \). Set \( \alpha_n = a + \frac{b-a}{2(n+1)} \), \( \beta_n = b - \frac{b-a}{2(n+1)} \) for \( n \geq 0 \), and let \( \mu = (\mu_n)_{n \geq 0} \) be a nondecreasing sequence of positive real numbers satisfying the following conditions

\[
\mu_n > \frac{1}{\cos \left( \frac{\beta_n - \alpha_n}{2} \right)} \lim_{r \to +\infty} \frac{\log(\max(\|T(r e^{i\alpha_n})\|, \|T(r e^{i\beta_n})\|))}{r} \quad (n \geq 0). \tag{3}
\]

\[
\sup_{\zeta \in \mathcal{S}_{a,b}} \|e^{-\mu_n \zeta} e^{-i\frac{\alpha_n + \beta_n}{2}} T(\zeta + 2^{-n} e^{i\frac{\alpha_n + \beta_n}{2}})\|_{\mathcal{M}(\mathcal{A})} \leq e^{2^{-n}(\mu_0 + 1)} \quad (n \geq 1). \tag{4}
\]

where \( \|\cdot\|_{\mathcal{M}(\mathcal{A})} \) is the norm on the normalization \( \mathcal{A}_{T_0} \) of \( \mathcal{A} \) with respect to the semigroup \( T_0 := (T(te^{i(\alpha_n + \beta_n)/2}))_{t > 0} \) associated to \( \mu_0 \) in theorem 2.2.

For \( n \geq 1 \) set \( V_n := \{e^{-\mu_n \zeta} e^{-i\frac{\alpha_n + \beta_n}{2}} T(\zeta)\} \subset \mathcal{S}_{a,b} \), set \( W_n := V_1 \ldots V_n \) and set \( W := \bigcup_{n \geq 1} W_n \), so that \( W \) is stable under products. Set \( \mathcal{L}_\mu := \{u \in \mathcal{A}_{T_0} \mid \sup_{w \in W} \|uw\|_{\mathcal{M}(\mathcal{A})} < +\infty \} \) and set \( \|u\|_\mu = \sup_{w \in W} \|uw\|_{\mathcal{M}(\mathcal{A})} \) for \( u \in \mathcal{L}_\mu \). Denote by \( \mathcal{A}_{\mu,T} \) the closure of \( \mathcal{A}_{T_0} \) in \( \mathcal{M}(\mathcal{L}_\mu) \). Then \( T(\zeta)u \in \mathcal{L}_\mu \) for \( \zeta \in \mathcal{S}_{a,b}, u \in \mathcal{A}_{\mu,T} \) is a Banach algebra, the inclusion map \( \hat{j} : \mathcal{A} \to \mathcal{A}_{\mu,T} \) is a \( \mathcal{QM} \) homomorphism with respect to \( T(\zeta)w^2 \) for \( w \in \Omega(\mathcal{A}) \), and we have the following properties

(i) The tautological map \( \hat{j} : S_{u,v} \to S_{u,v} \) is a pseudobounded isomorphism from \( \mathcal{QM}(\mathcal{A}) \) onto \( \mathcal{QM}(\mathcal{A}_{\mu,T}) \) and if \( \zeta \in \mathcal{S}_{a,b} \) then \( \hat{j}^{-1}(S) = S_{T(\zeta)w^2 u/T(\zeta)w^2 v} \) for \( S = S_{u,v} \in \mathcal{QM}(\mathcal{A}_{\mu,T}), w \in \Omega(\mathcal{A}) \).

(ii) \( \hat{j}(\mathcal{M}(\mathcal{A})) \subset \mathcal{M}(\mathcal{A}_{\mu,T}) \) and \( \|R\|_{\mathcal{M}(\mathcal{A})} \leq \|R\|_{\mathcal{M}(\mathcal{A})} \) for \( R \in \mathcal{M}(\mathcal{A}) \).

(iii) \( \|T(\zeta)\|_{\mathcal{M}(\mathcal{L}_\mu)} \leq \|T(\zeta)\|_{\mathcal{M}(\mathcal{A})} \leq e^{\mu_n \Re\Re(e^{-\frac{\alpha_n + \beta_n}{2}})} \) for \( \zeta \in \mathcal{S}_{a,b} \), \( n \geq 1 \).

(iv) If \( a < \alpha < \beta < b \), then we have, for \( v \in \mathcal{A}_{\mu,T} \),

\[
\lim_{n \to \infty} \sup_{\zeta \in \mathcal{S}_{a,b}} \|T(\zeta)v - v\|_{\mathcal{M}(\mathcal{A}_{\mu,T})} \leq \lim_{n \to \infty} \sup_{\zeta \in \mathcal{S}_{a,b}} \|T(\zeta)v - v\|_{\mathcal{M}(\mathcal{L}_\mu)} = 0.
\]

Proof: Since \( \|T_0(2^{-n})\|_{\mathcal{M}(\mathcal{L}_\mu)} \leq e^{2^{-n}\mu_0} \), the existence of a sequence \( (\mu_n)_{n \geq 1} \) satisfying the required conditions follows from the lemma.

Let \( t > 0 \), let \( n_0 \geq 2 \) be such that \( 2^{-n_0+1} < t \), and let \( v \in W \). Since \( 1 \in V_n \) for \( n \geq 1 \), we can assume that \( v = v_1 \ldots v_n \), where \( n \geq n_0 \) and where
\( v_j \in V_j \) for \( 1 \leq j \leq n \). Then \( \| T_0(2^{-j}v_j) \|_{\text{op}, \mu_0} \leq e^{2^{-j}(\mu_0+1)} \) for \( j \geq n_0 \), and so \( \| T_0(2^{-n_0+1} - 2^{-n})v_{n_0} \ldots v_n \|_{\text{op}, \mu_0} \leq e^{(2^{-n_0+1} - 2^{-n})(\mu_0+1)} \). We obtain

\[
\| T_0(2^{-n_0+1})v_{n_0} \ldots v_n \|_{\text{op}, \mu_0} \\
\leq \| T_0(2^{-n})\|_{\text{op}, \mu_0} T((2^{-n_0+1} - 2^{-n})v_{n_0} \ldots v_n \|_{\text{op}, \mu_0} \\
\leq e^{2^{-n_0+1}(\mu_0+1) - 2^{-n}} \leq e^{2^{-n_0+1}(\mu_0+1)}.
\]

Set \( r = \frac{t-2^{-n_0+1}}{n_0} \). It follows from lemma 4.1(i) that for every \( j \leq n_0 - 1 \) there exists \( k_j > 0 \) such that \( \sup_{v \in V_j} \| T(r)v \|_{\text{op}, \mu_0} \leq k_j \). This gives

\[
\| T_0(t)v \|_{\text{op}, \mu_0} \leq \| T_0(r)v_1 \|_{\text{op}, \mu_0} \ldots \| T_0(r)v_{n_0-1} \|_{\text{op}, \mu_0} \| T_0(2^{-n_0+1})v_{n_0} \ldots v_n \|_{\text{op}, \mu_0} \\
\leq k_1 \ldots k_{n_0-1} e^{2^{-n_0+1}(\mu_0+1)},
\]

and so \( T_0(t)A_{T_0} \subset L_\mu \). Now let \( \zeta \in S_{a,b} \). Then \( \zeta - te^{a+b} \in S_{a,b} \) for some \( t > 0 \), and so \( T(\zeta)A_{T_0} = T(\zeta - te^{(a+b)})T_0(t)A_{T_0} \subset T(\zeta - te^{(a+b)})L_\mu \subset L_\mu \).

Since \( W \) is stable under products, the fact that \( L_\mu \) is a Banach algebra from proposition 2.3, which also implies (ii) and (iii). Let \( \zeta_0 \in S_{a,b} \). Since (iv) holds for \( u \in T(\zeta_0)A_{\mu,T} \), and since \( T(\zeta_0)A_{\mu,T} \) is dense in \( (A_{\mu,T}, \| \|_{L_\mu}) \) (v) follows from (iii) by a standard density argument.

Let \( \zeta \in S_{a,b} \), and let \( w \in \Omega(A) \subset \Omega(A_{T_0}) \). Since \( T(\zeta/2)w \in \Omega(A) \), and since \( \lim \sup_{t\to a^+} \| T_0(t)T(\zeta)w \| < +\infty \), it follows from proposition 2.3 that the inclusion map \( j_0 : A \to A_{T_0} \) is a QM-homomorphism with respect to \( T(\zeta/2)w \). Since \( T(\zeta/2)w \in M_\mu \cap \Omega(A_{T_0}) \), it follows also from proposition 2.3 that the inclusion map \( j_1 : A_{T_0} \to A_{\mu,T} \) is a QM-homomorphism with respect to \( T(\zeta/2)w \), and it follows then from proposition 2.4 that the inclusion map \( j = j_1 \circ j_0 : A \to A_{\mu,T} \) is a QM homomorphism with respect to \( T(\zeta/2)wT(\zeta/2)w = T(\zeta)w^2 \). It follows then from proposition 2.2 that the tautological map \( j : S_{a,v} \to S_{a,v} \) is a pseudobounded homomorphism from \( \Omega(A) \) onto \( \Omega(A_{\mu,T}) \) and that \( j^{-1}(S_{a,v}) = S_{T(\zeta)w^2/T(\zeta)w^2} \) for \( S = S_{a,v} \in \Omega(A_{\mu,T}) \).

Since \( T(\zeta)A_{\mu,T} \subset T(\zeta/2)[T(\zeta/2)A_{\mu,T}] \subset T(\zeta)A_{T_0} \subset \Omega(A) \), and since \( T(\zeta)vA \) contains \( T(\zeta)vA \) which is dense in \( A \) for \( v \in \Omega(A_{\mu,T}) \), we have \( T(\zeta)v \in \Omega(A) \) for \( \zeta \in S_{a,b} \), \( v \in \Omega(A_{\mu,T}) \), and so \( j^{-1}(S) = S_{T(\zeta)w^2/T(\zeta)w^2} \) for \( S = S_{a,v} \in \Omega(A_{\mu,T}) \). □

We will use the following notion.

**Definition 4.3** Let \( A \) be a weakly commutative Banach algebra with dense principal ideals, and let \( T := (T(\zeta))_{\zeta \in S_{a,b}} \) be an analytic semigroup of multipliers on \( A \) such that \( T(\zeta)A \) is dense in \( A \) for \( \zeta \in S_{a,b} \). A normalization of the algebra \( A \) with respect to the semigroup \( T \) is a subalgebra \( B \) of \( \Omega(A) \) which is a Banach algebra with respect to a norm \( \| \|_{B} \) and satisfies the following conditions

(i) There exists \( u \in \Omega(A) \) such that the inclusion map \( j : A \to B \) is a QM-homomorphism with respect to \( T(\zeta)u \) for every \( \zeta \in S_{a,b} \), and \( \| j(u) \|_{B} \leq \| u \|_{A} \) for \( u \in A \).
(ii) \( \hat{j}(R) \subset \mathcal{M}(\mathcal{B}) \), and \( \|\hat{j}(R)\|_{\mathcal{M}(\mathcal{B})} \leq \|R\|_{\mathcal{M}(\mathcal{A})} \) for \( R \in \mathcal{M}(\mathcal{A}) \), where \( \hat{j} : \mathcal{Q}\mathcal{M}(\mathcal{A}) \rightarrow \mathcal{Q}\mathcal{M}(\mathcal{B}) \) is the pseudobounded isomorphism associated to \( j \) introduced in proposition 2.2 (ii).

(iii) \( \limsup_{t \to 0} \|\hat{j}(T(t))\|_{\mathcal{M}(\mathcal{B})} < +\infty \) for \( a < \alpha < \beta < b \).

If \( \mathcal{B} \) is a normalization of \( \mathcal{A} \) with respect to the holomorphic semigroup \( T = (T(\zeta))_{\zeta \in S_{a,b}} \), a standard density argument shows that if \( a < \alpha < \beta < b \) then \( \limsup_{\zeta \to 0} \|T(\zeta)u - u\|_{\mathcal{B}} = 0 \) for every \( u \in \mathcal{B} \).

Notice that the algebra \( \mathcal{A}_{0,T} = \mathcal{A}_{\mu_0,T} \) and its norm topology associated to the norm \( \|\| \mu_0,op \| \) discussed above do not depend on the choice of \( \mu_0 \). This is no longer the case for the Banach algebra \( \mathcal{A}_{\mu,T} \) and its norm topology, which may depend on the choice of the sequence \( \mu \). In order to get a more intrinsic renormalization one could consider the Fréchet algebra \( \mathcal{L} := \bigcap_{n \geq 1} \{ u \in \mathcal{A}_T \mid \sup_{\zeta \in S_{a,b}} \|T(\zeta)u\|_{\mu_0,op} < +\infty \} \), then consider the closed subalgebra \( \mathcal{U} \) of \( \mathcal{L} \) generated by the semigroup and introduce an intrinsic normalization of \( \mathcal{A}_T \) to be the closure of \( \mathcal{U} \) in \( \mathcal{M}(\mathcal{U}) \) with respect to the Mackey-convergence associated to a suitable notion of boundedness on subsets of \( \mathcal{M}(\mathcal{U}) \), but it seems more convenient to adopt the point of view used in proposition 4.2.

5 Generator of a strongly continuous semigroup of multipliers and Arveson spectrum

In this section we consider again a weakly cancellative commutative Banach algebra \( \mathcal{A} \) with dense principal ideals and a strongly continuous semigroup \( T = (T(t))_{t > 0} \) of multipliers on \( \mathcal{A} \) such that \( \cup_{t > 0} T(t) \mathcal{A} \) is dense in \( \mathcal{A} \). We set again \( \omega_T(t) = \|T(t)\| \) for \( t > 0 \). Denote by \( \mathcal{M}_{\omega_T} \) the space of all measures \( \nu \) on \( (0, +\infty) \) such that \( \int_0^{+\infty} \omega_T(t)|\nu|(t) < +\infty \), and for \( \nu \in \mathcal{M}_{\omega_T} \) define a \( \Phi_T : \mathcal{M}_{\omega_T} \rightarrow \mathcal{M}(\mathcal{A}) \) by the formula

\[
\Phi_T(\nu)u := \int_0^{+\infty} T(t)u\nu(t) \quad (u \in \mathcal{A}).
\]  

The Bochner integral is well-defined since the semigroup is strongly continuous, \( \mathcal{M}_{\omega_T} \) is a Banach algebra with respect to convolution of measures on the half-line, and we will identify again the space \( L^1_{\omega_T} \) of (classes of) measurable functions on \( [0, +\infty) \) satisfying \( \int_0^{+\infty} |f(t)||T(t)||dt < +\infty \) to the ideal of all \( \nu \in \mathcal{M}_{\omega_T} \) which are absolutely continuous with respect to Lebesgue measure. Denote by \( \mathcal{B}_T \) the closure of \( \phi_T(\mathcal{M}_\omega) \) in \( \mathcal{M}(\mathcal{A}) \), and denote again by \( \mathcal{I}_T \) the closure of \( \Phi_T(L^1_{\omega_T}) \) in \( \mathcal{M}(\mathcal{A}) \), so that the "Arveson ideal" \( \mathcal{I}_T \) is a closed ideal of \( \mathcal{B}_T \).

The idea of considering the generator of a semigroup as a quasimultiplier on some suitable Banach algebra goes back to [21] and [22] for groups of bounded
operators and, more generally, for groups of regular quasimultipliers. An obvious such interpretation was given by I. Chalendar, J. R. Partington and the author in [7] for analytic semigroups, and the author interpreted in section 8 of [16] the generator of a semigroup of bounded operators which is weakly continuous in the sense of Arveson [2] as a quasimultiplier on the corresponding Arveson ideal $\mathcal{I}_T$. Since in the present context $\Phi_T(\Omega(L^1_{\omega_T})) \subset \Omega(\mathcal{I}_T)$ and $uv \in \Omega(A)$ for $u \in \Omega(\mathcal{I}_T)$, $v \in \Omega(A)$, the map $j_T : S_{w/v} \to S_{uw/vw}$ is a pseudobounded homomorphism from $QM(I_T)$ into $QM(A)$ for every $w \in \Omega(A)$, and the definition of $j_T$ does not depend on the choice of $w$.

The generator $\Delta_{T,I_T}$ of $T$ considered as a strongly continuous semigroup of $T$ has been defined in [16], def. 8.1 by the formula

$$\Delta_{T,\mathcal{I}_T} = S_{-\Phi_T(f_0)/\Phi_T(f_0)};$$

where $f_0 \in C^1([0, +\infty)) \cap \Omega(L^1_{\omega_T})$ satisfies $f_0 = 0$, $f'_0 \in L^1_{\omega_T}$,

and an easy verification given in [16] shows that this definition does not depend on the choice of $f_0$. This suggests the following definition

**Definition 5.1**: The infinitesimal generator $\Delta_{T,A}$ of $T$ is the quasimultiplier on $A$ defined by the formula

$$\Delta_{T,A} = j_T(\Delta_{T,\mathcal{I}_T}) = S_{-\Phi_T(f_0)/\Phi_T(f_0)}u_0,$$

where $f_0 \in C^1([0, +\infty)) \cap \Omega(L^1_{\omega_T})$ satisfies $f_0 = 0$, $f'_0 \in L^1_{\omega_T}$, and where $u_0 \in \Omega(A)$.

Assume that $f_1$ and $u_1$ also satisfy the conditions of the definition, and set $f_2 = f_0 * f_1$, $u_2 = u_0 u_1$. Since $\Omega(L^1_{\omega_T})$ is stable under convolution, $f_2 \in \Omega(L^1_{\omega_T})$, and $f'_2 = f'_0 * f_1 = f_0 * f'_1 \in L^1_{\omega_T}$ is continuous. Also $f_2(0) = 0$, and we have

$$\Phi_T(f_2)u_2\Phi_T(f_0)u_0 = \Phi_T(f_0)\Phi_T(f_1)u_2\Phi_T(f_0)u_0 = \Phi_T(f_0)u_0\Phi_T(f_2)u_2,$$

and similarly $\Phi_T(f'_2)u_2\Phi_T(f_1)u_1 = \Phi_T(f'_1)u_1\Phi_T(f_2)u_2$, which shows that the definition of $\Delta_{T,A}$ does not depend on the choice of $f_0$ and $u_0$.

**Proposition 5.2** Let $T_1$ and $T_2$ be two semigroups satisfying the conditions of definition 5.1 with respect to $A$. If $T_1(t)T_2(t) = T_2(t)T_1(t)$ for $t > 0$, then

$$\Delta_{T_1T_2,A} = \Delta_{T_1,A} + \Delta_{T_2,A}.$$ 

Proof: Let $f_1$ and $f_2$ be two functions on $[0, +\infty)$ satisfying the conditions of definition 5.1 with respect to $T_1$ and $T_2$. Since $\omega_{T_1T_2}(t) \leq \omega_{T_1}(t)\omega_{T_2}(t)$ for $t > 0$, $f_1f_2$ satisfies the conditions of definition 5.1 with respect to $T_1T_2$, and it follows from Leibnitz rule and Fubini’s theorem that we have

$$\Phi_{T_1T_2}(f_1f_2) = \Phi_{T_1}(f_1)\Phi_{T_2}(f_2), \Phi_{T_1T_2}((f_1f_2)') = \Phi_{T_1}(f_1')\Phi_{T_2}(f_2) + \Phi_{T_1}(f_1)\Phi_{T_2}(f_2'),$$

and the results follows. □
We now give a link between the quasimultiplier approach and the classical approach based on the study of \(\frac{T(t)u-u}{t}\) as \(t \to 0^+\). A proof of the following folklore result is given for example in [16], lemma 8.4.

**Lemma 5.3** Let \(\omega\) be a lower semicontinuous submultiplicative weight on \((0, +\infty)\), and let \(f \in C^1([0, +\infty)) \cap L^1_\omega\). If \(f(0) = 0\), and if \(f' \in L^1_\omega\), then the Bochner integral \(\int_0^\infty (f' \ast \delta_t)ds\) is well-defined in \(L^1_\omega\) for \(t \geq 0\), and we have

\[
  f \ast \delta_t - f = -\int_0^t (f' \ast \delta_s)ds, \quad \text{and} \quad \lim_{t \to 0^+} \left\| \frac{f \ast \delta_t - f}{t} + f' \right\|_{L^1_\omega} = 0.
\]

It follows from the lemma that we have if \(f \in C^1([0, +\infty)) \cap L^1_\omega\), and if \(f' \in L^1_\omega\),

\[
  T(t)\Phi_T(f) - \Phi_T(f) = -\int_0^t T(s)\Phi_T(f')ds \quad (t \geq 0).
\]

**Proposition 5.4** (i) Let \(u \in \mathcal{A}\). If \(\lim_{t \to 0^+} \left\| \frac{T(t)u-u}{t} - v \right\| = 0\) for some \(v \in \mathcal{A}\), then \(u \in \mathcal{D}_{\Delta_{T,A}}\), and \(\Delta_{T,A}u = v\).

(ii) Conversely if \(\limsup_{t \to 0^+} \|T(t)\| < +\infty\), then \(\limsup_{t \to 0^+} \left\| \frac{T(t)u-u}{t} - \Delta_{T,A}u \right\| = 0\) for every \(u \in \mathcal{D}_{\Delta_{T,A}}\).

Proof: (i) If \(u \in \mathcal{A}\), and if \(\lim_{t \to 0^+} \left\| \frac{T(t)u-u}{t} - v \right\| = 0\) for some \(v \in \mathcal{U}_T\), let \(f_0 \in C^1([0, +\infty)) \cap \Omega(L^1_\omega)\) satisfying \(f_0 = 0\), \(f'_0 \in L^1_\omega\), and let \(u_0 \in \Omega(\mathcal{A})\). It follows from the lemma that we have, with respect to the norm topology on \(\mathcal{A}\),

\[
  -\Phi_T(f'_0)u_0u = \left[ \lim_{t \to 0^+} \frac{T(t)\Phi_T(f_0) - \Phi_T(f_0)}{t} \right]u_0u = \Phi_T(f_0)u_0 \left[ \lim_{t \to 0^+} \frac{T(t)u-u}{t} \right] = \Phi_T(f_0)u_0v,
\]

and so \(u \in \mathcal{D}_{\Delta_{T,A}}\), and \(\Delta_{T,A}u = v\).

(ii) Conversely assume that \(\limsup_{t \to 0^+} \|T(t)\| < +\infty\), let \(u \in \mathcal{D}_{\Delta_{T,A}}\), let \(f_0 \in C^1([0, +\infty)) \cap \Omega(L^1_\omega)\) such that \(f'_0 \in L^1_\omega\), and let \(u_0 \in \Omega(\mathcal{A})\). Let \(v = \Delta_{T,A}u\). It follows from (6) that we have, for \(t \geq 0\),

\[
  \Phi_T(f_0)u_0 \int_0^t T(s)uds = \int_0^t T(s)\Phi_T(f_0)u_0vds = -\int_0^t T(s)\phi_T(f'_0)u_0u\delta_tds
\]

\[
  = -\left[ \int_0^t T(s)\Phi_T(f'_0)ds \right]u_0u = [T(t)\Phi_T(f_0)u_0 - \Phi_T(f_0)u_0]u = \Phi_T(f_0)u_0(T(t)u-u).
\]

Since \(\Phi_T(f_0)u_0 \in \Omega(\mathcal{A})\), this shows that \(T(t)u-u = \int_0^t T(s)uds\), and so

\[
  \lim_{t \to 0^+} \left\| \frac{T(t)u-u}{t} - v \right\| = 0. \quad \square
\]

We now consider a normalization \(\mathcal{A}\) with respect to \(T\), see definition 3.3.
Proposition 5.5 Let $\mathcal{B}$ be a normalization of $\mathcal{A}$ with respect to $T$. Set $v_\lambda(t) = te^{-\lambda t}$ for $\lambda \in \mathbb{R}, t \geq 0$, and let $\lambda > \log(\rho_T)$.

(i) If $u \in \Omega(\mathcal{B})$, then

$$\Delta_{T,B} = -\mathcal{S} f_0^+ v'_\lambda(T(t)u)dt + f_0^+ v_\lambda(T(t)u)dt.$$

(ii) Let $\tilde{j} : \mathcal{Q} \mathcal{M}(\mathcal{A}) \to \mathcal{Q} \mathcal{M}(\mathcal{B})$ be the pseudobounded isomorphism given in proposition 3.2 (i). Then $\tilde{j}^{-1}(\Delta_{T,B}) = \Delta_{T,A}$. So if $u \in \Omega(\mathcal{A})$, and if $\limsup_{t \to 0^+} \|T(t)u\| < +\infty$, then

$$\Delta_{T,A} = -\mathcal{S} f_0^+ v'_\lambda(T(t)u)dt + f_0^+ v_\lambda(T(t)u)dt.$$

Proof: (i) Set $e_{\lambda}(t) = e^{\lambda t}$ for $\lambda \in \mathbb{R}, t \geq 0$. If $\lambda > \mu > \log(\rho_T)$, then $v_\lambda \in L^1_{\omega_T} \subset L^1_{\omega_T}$, where $\omega_T(t) = \|T(t)\|_{\mathcal{M}(\mathcal{A})}$, and $L^1_{\omega_T}$ is dense in $L^1_{\omega_T}$ since it contains the characteristic function of $[\alpha, \beta]$ for $0 < \alpha < \beta < +\infty$. It follows from Nyman’s theorem [25] about closed ideals of $L^1(\mathbb{R}^+)$ that $v_\lambda \in QM_0 \subset \Omega(L^1_{\omega_T})$ and so $v_\lambda \in \Omega(L^1_{\omega_T}) \subset \Omega(L^1_{\omega_T})$. So $v_\lambda$ and $u$ satisfy the conditions of definition 5.1 with respect to $T$ and $\mathcal{B}$, and (i) holds.

(ii) The map $\tilde{j}$ is the tautological map $S_{u/v} \to S_{u/v}$, where $u \in \mathcal{A} \subset \mathcal{B}$ and $v \in \Omega(\mathcal{A}) \subset \Omega(\mathcal{B})$. Now let $f_0 \in L^1(\mathbb{R}^+) \subset C^1([0, +\infty))$ satisfying definition 5.1 with respect to $T$ and $\mathcal{A}$ and let $u_0 \in \Omega(\mathcal{A})$. Since $\Omega(L^1_{\omega_T}) \subset \Omega(L^1_{\omega_T})$, and since $\Omega(\mathcal{A}) \subset \Omega(\mathcal{B})$, it follows from definition 5.1 that $\tilde{j}(\Delta_{T,A}) = \Delta_{T,B}$, and so $\tilde{j}^{-1}(\Delta_{T,B}) = \Delta_{T,A}$.

Let $u \in \Omega(\mathcal{A}) \subset \Omega(\mathcal{B})$, and assume that $\limsup_{t \to 0^+} \|T(t)u\| < +\infty$. Let $w \in \Omega(\mathcal{A})$ be such that $w \mathcal{B} \subset \mathcal{A}$. Since $v_\lambda \in \Omega(L^1_{\omega_T})$, we see as in the proof of lemma 3.1 that $f_0^+ v_\lambda(T(t)u)dt \in \Omega(\mathcal{B})$, and it follows from proposition 2.2 that $w f_0^+ v_\lambda(T(t)u)dt \in \Omega(\mathcal{A})$. Using the characterization of $\tilde{j}^{-1}$ given in proposition 2.2, we obtain

$$\Delta_{T,B} = -\mathcal{S} f_0^+ v'_\lambda(T(t)u)dt + f_0^+ v_\lambda(T(t)u)dt,$$

and

$$\Delta_{T,A} = \tilde{j}^{-1}(\Delta_{T,B}) = -\mathcal{S} w f_0^+ v'_\lambda(T(t)u)dt + f_0^+ v_\lambda(T(t)u)dt$$

$$= -\mathcal{S} f_0^+ v'_\lambda(T(t)u)dt + f_0^+ v_\lambda(T(t)u)dt.$$

$\square$

We will denote by $\mathcal{I}_T$ the space of characters of $\mathcal{I}_T$, equipped with the usual Gelfand topology. Notice that if $\chi \in \mathcal{I}_T$ then there exists a unique character $\tilde{\chi}$ on $\mathcal{Q} \mathcal{M}(\mathcal{I}_T)$ such that $\tilde{\chi}|_{\mathcal{I}_T} = \chi$, which is defined by the formula $\tilde{\chi}(S_{u/v}) = \frac{\chi(v)}{\chi(u)}$ for $u \in \mathcal{I}_T, v \in \Omega(\mathcal{I}_T)$.

Definition 5.6 Assume that $\mathcal{I}_T$ is not radical, and let $S \in \mathcal{Q} \mathcal{M}(\mathcal{I}_T)$. The Arveson spectrum $\sigma_{ar}(S)$ is defined by the formula

$$\sigma_{ar}(S) = \{\lambda = \tilde{\chi}(S) : \chi \in \mathcal{I}_T\}.$$
If $\nu$ is a measure on $[0, +\infty)$, the Laplace transform of $\nu$ is defined by the usual formula $L(\nu)(z) = \int_0^{+\infty} e^{-zt}d\nu(t)$ when $\int_0^{+\infty} e^{-Re(z)t}d\nu(t) < +\infty$.

We have the following easy observation.

**Proposition 5.7** Let $\nu \in \mathcal{M}_{\omega_T}$. Then we have, for $\chi \in \hat{\mathcal{I}}_T$,

$$\hat{\chi} \left( \int_0^{+\infty} T(t)d\nu(t) \right) = L(\nu)(-\hat{\chi}(\Delta_{T,I_T})).$$

Similarly we have, for $\nu \in \mathcal{M}_{\omega_T}, \chi \in \hat{\mathcal{I}}_T$,

$$\hat{\chi}(\Phi_T(\nu)) = \hat{\chi} \left( \int_0^{+\infty} T(t)d\nu(t) \right) = L(\nu)(-\hat{\chi}(\Delta_{T,I_T})) = L(\nu)(-\hat{\chi}(\Delta_{T,I_T})) = L(\nu)(-\hat{\chi}(\Delta_{T,I_T})).$$

In particular $\hat{\chi}(T(t)) = e^{\hat{\chi}(\Delta_{T,I_T})t}$ for $t > 0$.

**Proof:** If $\chi \in \hat{\mathcal{I}}_T$, then $\hat{\chi}|_{\mathcal{A}_T}$ is a character on $\mathcal{A}_T$, the map $t \to \hat{\chi}(T(t))$ is continuous on $(0, +\infty)$ and so there exists $\lambda \in \mathbb{C}$ such that $\hat{\chi}(T(t)) = e^{-\lambda t}$ for $t > 0$, and $|e^{-\lambda t}| \leq \|T(t)\|$, which shows that $Re(\lambda) \geq -\log(\rho_T)$.

Let $u \in \Omega(I_T)$, and let $\nu \in \mathcal{M}_{\omega_T}$. We have

$$\int_0^{+\infty} \chi(u)\hat{\chi} \left( \int_0^{+\infty} T(t)d\nu(t) \right) = \chi \left( u \int_0^{+\infty} T(t)d\nu(t) \right) = \chi \left( \int_0^{+\infty} T(t)d\nu(t) \right)$$

$$= \int_0^{+\infty} \chi(T(t)u)d\nu(t) = \chi(u) \int_0^{+\infty} e^{-\lambda t}d\nu(t) = \chi(u)L(\nu)(\lambda),$$

and so $\hat{\chi}(\Phi_T(\nu)) = L(\nu)(\lambda)$.

Let $f_0 \in C^1((0, +\infty)) \cap \Omega(I_T)$ such that $f_0(0) = 0$. We have

$$\lambda L(f_0)(\lambda) = L(f_0)(\lambda) = \chi(\Phi_T(f_0)) = -\hat{\chi}(\Delta_{T,I_T}) = L(\nu)(\lambda),$$

and so $\lambda = -\hat{\chi}(\Delta_{T,I_T})$, which proves (7), and formula (8) follows from a similar argument. In particular $\chi(T(t)) = L(\Delta_t)(-\hat{\chi}(\Delta_{T,I_T})) = e^{\hat{\chi}(\Delta_{T,I_T})t}$ for $t > 0$.

The following consequence of proposition 5.7 pertains to folklore.

**Corollary 5.8** Assume that $I_T$ is not radical. Then the map $\chi \to \hat{\chi}(\Delta_{T,I_T})$ is a homeomorphism from $\hat{\mathcal{I}}_T$ onto $\sigma_{ar}(\Delta_{T,I_T})$, and the set $\Lambda_t := \{ \lambda \in \sigma_{ar}(\Delta_{T,I_T}) \mid Re(\lambda) \leq t \}$ is compact for every $t \in \mathbb{R}$, so that $\sigma_{ar}(\Delta_{T,I_T})$ is closed.

**Proof:** Let $f_0 \in C^1((0, +\infty)) \cap \Omega(I_T)$ such that $f_0(0) = 0$. We have $\chi(\Phi_T(f_0)) \neq 0$ and $\hat{\chi}(\Delta_{T,I_T}) = -\frac{\chi(\Phi_T(f_0))}{\chi(\Phi_T(f_0))}$ for $\chi \in \hat{\mathcal{I}}_T$, and so the map $\chi \to \hat{\chi}(\Delta_{T,I_T})$ is continuous with respect to the Gelfand topology on $\hat{\mathcal{I}}_T$. 

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Conversely let \( f \in L^1_{\omega_T} \). It follows from proposition 5.6 that we have, for \( \chi \in \hat{I}_T \),

\[
\chi(\Phi_T(f)) = \mathcal{L}(f)(-\hat{\chi}((\Delta_{T,I_T}))).
\]

Since the set \( \{ u = \Phi_T(f): f \in L^1_{\omega_T} \} \) is dense in \( \hat{I}_T \), this shows that the map \( \chi \to \hat{\chi}((\Delta_{T,I_T})) \) is one-to-one on \( \hat{I}_T \), and that the inverse map \( \sigma_{ar}(\Delta_{T,I_T}) \to \hat{I}_T \) is continuous with respect to the Gelfand topology.

Now let \( t \in \mathbb{R} \), and set \( U_t := \{ \chi \in \hat{I}_T : \text{Re}(\chi((\Delta_{T,I_T}))) \leq t \} \). Then \( |\hat{\chi}(T(1))| \geq e^{-t} \) for \( \chi \in U_t \), and so 0 does not belong to the closure of \( U_t \) with respect to the weak* topology on the unit ball of the dual of \( I_T \). Since \( \hat{I}_T \cup \{0\} \) is compact with respect to this topology, \( U_t \) is a compact subset of \( \hat{I}_T \), and so the set \( \Lambda_t \) is compact, which implies that \( \sigma_{ar}(\Delta_{T,I_T}) = \bigcup_{n \geq 1} \Lambda_n \) is closed. \( \square \)

6 The resolvent

We now wish to discuss the resolvent of the generator of a strongly continuous semigroup \( T = (T(t))_{t \geq 0} \) of multipliers on \( A \), where \( A \) is a weakly cancellative commutative Banach algebra with dense principal ideals, and where \( \cup_{t \geq 0} T(t)A \) is dense in \( A \). From now on we will write \( \Delta_T = \Delta_{T,A} \) and we will denote by \( D_{\Delta_{T,A}} \) the domain of \( \Delta_T \) considered as a quasimultiplier on \( A \). The Arveson ideal \( I_T \) is as above the closed subalgebra of \( \mathcal{M}(A) \) generated by \( \Phi_T(L^1_{\omega_T}) \).

The Arveson resolvent set is defined by the formula \( \text{Res}_{ar}(\Delta_{T,I_T}) = \mathbb{C} \setminus \sigma_{ar}(\Delta_{T,I_T}) \), with the convention \( \sigma_{ar}(\Delta_{T,I_T}) = \emptyset \) if \( I_T \) is radical. The usual "resolvent formula," interpreted in terms of quasimultipliers, shows that \( \lambda - \Delta_{T,I_T} \in \mathcal{QM}(I_T) \) is invertible in \( \mathcal{QM}(I_T) \) and that its inverse \( (\lambda - \Delta_{T,I_T})^{-1} \) belongs to the Banach algebra \( I_{T,B} \subset \mathcal{Q}M_T(I_T) \) obtained by applying theorem 2.2 to \( I_T \) with respect to the semigroup \( T \), and that we have, for \( \text{Re}(\lambda) > \log(\rho_T) \),

\[
(\lambda I - \Delta_{T,I_T})^{-1} = \int_0^{+\infty} e^{-\lambda s}T(s)ds \in I_{T,B},
\]

where the Bochner integral is computed with respect to the strong operator topology on \( \mathcal{M}(I_{T,B}) \). Also the \( I_{T,B} \)-valued map \( \lambda \to (\lambda I - \Delta_{T,I_T})^{-1} \) is holomorphic on \( \text{Res}_{ar}(T,I_T) \). The details of the adaptation to the context of quasimultipliers of this classical part of semigroup theory are given in [16], proposition 10.2.

We now give a slightly more general version of this result, which applies in particular to the case where \( B \) is the normalization \( A_T \) of \( A \) with respect to the semigroup \( T \) introduced in proposition 2.2.

In the following we will identify the algebras \( \mathcal{QM}(A) \) and \( \mathcal{QM}(B) \) using the isomorphism \( j \) introduced in proposition 2.2 (iii) if \( B \) is a normalization of \( A \) with respect to \( T \). We set \( \Phi_{T,B}(\nu)u = \int_0^{+\infty} T(t)u\nu(t) \) for \( u \in B, \nu \in \mathcal{M}_{\omega_{T,B}} \), where \( \omega_{T,B}(t) = ||T(t)||_{\mathcal{M}(B)} \) for \( t > 0 \), and we denote by \( I_{T,B} \) the closure of \( \Phi_{T,B}(L^1_{\omega_{T,B}}) \) in \( \mathcal{M}(B) \).
Proposition 6.1 Let $\mathcal{A}$ be a weakly cancellative commutative Banach algebra with dense principal ideals, let $T = ((T(t))_{t>0}$ be a strongly continuous semigroup of multipliers on $\mathcal{A}$ such that $T(t)A$ is dense in $\mathcal{A}$ for $t > 0$, and let $B$ be a normalization of $\mathcal{A}$ with respect to $T$. Set $\text{Res}_{ar}(\Delta_T) = \text{Res}_{ar}(\Delta_{T,T}) = \mathbb{C} \setminus \sigma_{ar}(\Delta_{T,T})$.

The quasimultiplier $\lambda I - \Delta_T \in \mathcal{QM}(\mathcal{A})$ admits an inverse $(\lambda I - \Delta_T)^{-1} \in \mathcal{I}_{T,B} \subset \mathcal{M}(B) \subset \mathcal{QM}(\mathcal{A})$ for $\lambda \in \text{Res}_{ar}(\Delta_T)$, and the map $\lambda \rightarrow (\lambda I - \Delta_T)^{-1}$ is an holomorphic map from $\text{Res}_{ar}(\Delta_T)$ into $\mathcal{I}_{T,B}$. Moreover we have, for $Re(\lambda) > log(\rho_T)$,

$$(\lambda I - \Delta_T)^{-1} = \int_0^{+\infty} e^{-\lambda s} T(s) ds \in \mathcal{I}_{T,B},$$

where the Bochner integral is computed with respect to the strong operator topology on $\mathcal{M}(B)$, and $\| (\lambda I - \Delta_T)^{-1} \|_{\mathcal{M}(B)} \leq \int_0^{+\infty} e^{-Re(\lambda)t} \| T(t) \|_{\mathcal{M}(B)} dt.$

Proof: We could deduce this version of the resolvent formula from proposition 10.2 of [16], but we give a proof for the sake of completeness. Set again $e_\lambda(t) = e^{\lambda t}$ for $t \geq 0, \lambda \in \mathbb{C}$. Assume that $Re(\lambda) > log(\rho_T) \geq \lim_\lambda \rightarrow +\infty \frac{\log \| T(t) \|_{\mathcal{M}(B)}}{t}$, let $v \in \mathcal{I}_{T,B}$, and set $a = \Phi_{T,B}(e^{-\lambda})$. We have

$$av = \int_0^{+\infty} e^{-\lambda s} T(s) v ds, T(t)av - av = \int_0^{+\infty} e^{-\lambda s} T(s+t) v ds - \int_0^{+\infty} e^{-\lambda s} T(s) v ds$$

$$= e^{\lambda t} \int_0^{+\infty} e^{-\lambda s} T(s) v ds - \int_0^{+\infty} e^{-\lambda s} T(s) v ds = (e^{\lambda t} - 1) av - e^{\lambda t} \int_0^{t} e^{-\lambda s} T(s) v ds.$$

Since $\lim_{t \rightarrow 0} \| T(t) v - v \|_{\mathcal{I}_{T,B}} = 0$, we obtain

$$\lim_{t \rightarrow 0^+} \left\| \frac{T(t)av - av}{t} - \lambda av + v \right\|_{\mathcal{I}_{T,B}} = 0,$$

and so $av \in D_{\Delta_T, \mathcal{I}_{T,B}}$ and $\Delta_{T, \mathcal{I}_{T,B}}(av) = \lambda av - v$. This shows that $a \mathcal{I}_{T,B} \subset D_{\Delta_T, \mathcal{I}_{T,B}}$, and that $(\lambda I - \Delta_{T, \mathcal{I}_{T,B}}) av = v$ for every $v \in \mathcal{I}_{T,B}$. We have $\lambda I - \Delta_{T, \mathcal{I}_{T,B}} = S_{v/u}$, where $u \in \mathcal{I}_{T,B}, v \in \Omega(\mathcal{I}_{T,B})$, and we see that $ua = v$. Hence $u \in \Omega(\mathcal{I}_{T,B}), \lambda I - \Delta_{T, \mathcal{I}_{T,B}}$ is invertible in $\mathcal{QM}(\mathcal{I}_{T,B})$, and $(\lambda I - \Delta_{T, \mathcal{I}_{T,B}})^{-1} a = \Phi_{T,B}(e^{-\lambda}) = \int_0^{+\infty} e^{-\lambda t} T(t) dt \in \mathcal{I}_{T,B}$, where the Bochner integral is computed with respect to the strong operator topology on $\mathcal{M}(\mathcal{I}_{T,B})$.

Let $\chi \in \mathcal{I}_{T,B}$. Then $\chi \circ j \in \mathcal{I}_{T,B}$, and so $\sigma_{ar}(\Delta_{T, \mathcal{I}_{T,B}}) \subset \sigma_{ar}(\Delta_{T, \mathcal{I}_{T,B}})$. It follows then from proposition 2.6 that $\lambda I - \Delta_{T,B}$ has an inverse $(\lambda I - \Delta_{T,B})^{-1} \in \mathcal{I}_{T,B}$ in $\mathcal{QM}(\mathcal{I}_{T,B})$ for $\lambda \in \text{Res}_{ar}(\Delta_{T, \mathcal{I}_{T,B}})$ and that the $\mathcal{I}_{T,B}$-valued map $\lambda \rightarrow (\lambda I - \Delta_{T,B})^{-1}$ is holomorphic on $\text{Res}_{ar}(\Delta_{T,B})$.

Fix $u_0 \in \Omega(\mathcal{A}) \subset \Omega(\mathcal{B})$, and set $j_T(S) = S_{u_0/v_{u_0}}$ for $S = S_{u/v} \in \mathcal{QM}(\mathcal{I}_{T,B})$. Then $j_T : \mathcal{QM}(\mathcal{I}_{T,B}) \rightarrow \mathcal{QM}(\mathcal{B})$ is a pseudobounded homomorphism, and $j_T(\Delta_{T, \mathcal{I}_{T,B}}) = \Delta_{T,B}$. Identifying $\mathcal{I}_{T,B}$ to a subset of $\mathcal{QM}(\mathcal{I}_{T,B})$ as above in the obvious way, we see that the restriction of $j_T$ to $\mathcal{I}_{T,B}$ is the identity map, and so $\lambda - \Delta_T$ is invertible in $\mathcal{QM}(\mathcal{B})$ for $\lambda \in \text{Res}_{ar}(\Delta_T)$, $(\lambda I - \Delta_T)^{-1} = \lambda I - \Delta_T$. 25
\[(\lambda I - \Delta_T)\inver T \) and the \( I_{T,B} \) valued map \( \lambda \to (\lambda I - \Delta_T)^{-1} \) is holomorphic on \( \text{Res}_{a}(\Delta_T) \).

If \( \text{Re}(\lambda) > \log(\rho_T) \geq \lim_{t \to +\infty} \frac{\log\|T(t)\|_{\mathcal{M}(B)}}{t} \), then if \( u \in I_{T,B}, v \in B \), we have
\[
(\lambda I - \Delta_T)^{-1} uv = ((\lambda I - \Delta_T)^{-1} u)v = \int_{0}^{+\infty} e^{-\lambda t} T(t) uv dt.
\]

Since \( uv \in \Omega(B) \) for \( u \in \Omega(I_{T,B}), v \in \Omega(B) \), \( uv \) is dense in \( B \) for \( u \in \Omega(I_{T,B}) \), and we obtain \( (\lambda I - \Delta_T)^{-1} = \int_{0}^{+\infty} e^{-\lambda t} T(t) dt \in I_{T,B} \), where the Bochner integral is computed with respect to the strong operator topology on \( \mathcal{M}(B) \), so that \( \|((\lambda I - \Delta_T)^{-1})_{\mathcal{M}(B)} \| \leq \int_{0}^{+\infty} e^{-\text{Re}(\lambda)t} \|T(t)\|_{\mathcal{M}(B)} dt \).

If we consider \( \Delta_T \) as a quasimultiplier on \( B \), the fact that \( (\lambda I - \Delta_T)^{-1} \in \mathcal{M}(B) \) is the inverse of \( \lambda I - \Delta_T \) for \( \lambda \in \text{Res}(\Delta_T) \) means that \( (\lambda I - \Delta_T)^{-1} v \in D_{\Delta_T,B} \) and that \( (\lambda I - \Delta_T)((\lambda I - \Delta_T)^{-1} v) = v \) for every \( v \in B \), and that if \( w \in D_{\Delta_T,B} \), then \( (\lambda I - \Delta_T)^{-1}((\lambda I - \Delta_T)w) = w \). The situation is slightly more complicated if we consider \( \Delta_T \) as a quasimultiplier on \( A \) when \( \sup_{t \to +\infty} \|T(t)\| = +\infty \). In this case the domain \( D_{\Delta_T,A} \) of \( (\lambda I - \Delta_T)^{-1} \in \mathcal{Q}\mathcal{M}(A) \) is a proper subspace of \( A \) containing \( \mathcal{L}_T \cup \cup_{t \geq 0} T(t) A \), and we have \( (\lambda I - \Delta_T)^{-1} v \in D_{\Delta_T,A} \) and \( (\lambda I - \Delta_T)((\lambda I - \Delta_T)^{-1} v) = v \) for every \( v \in D_{\Delta_T,A} \). Also if \( w \in D_{\Delta_T,A} \), then \( (\lambda I - \Delta_T)w \in D_{\lambda I - \Delta_T}^{-1},A \), and we have \( (\lambda I - \Delta_T)^{-1}((\lambda I - \Delta_T)w) = w \).

In order to interpret \( (\lambda I - \Delta_T)^{-1} \) as a partially defined operator on \( A \) for \( \text{Re}(\lambda) > \log(\rho_T) \), we can use the formula
\[
(\lambda I - \Delta_T)^{-1} v = \int_{0}^{+\infty} e^{-\lambda t} T(t) v dt \quad (v \in \mathcal{L}_T),
\]

which defines a quasimultiplier on \( A \) if we apply it to some \( v \in \Omega(A) \) such that \( \sup_{t \to +\infty} \|T(t)u\| < +\infty \). The fact that this quasimultiplier is regular is not completely obvious but follows from the previous discussion since \( (\lambda I - \Delta_T)^{-1} \in \mathcal{M}(B) \subset \mathcal{Q}\mathcal{M}(A) \). Notice that since \( \cup_{t \geq 0} T(t) A \) is dense in \( \mathcal{L}T_{\Omega(I_{T,B})} \), \( (\lambda I - \Delta_T)^{-1} \) is characterized by the simpler formula
\[
(\lambda I - \Delta_T)^{-1} T(s) v = e^{\lambda s} \int_{s}^{+\infty} e^{-\lambda t} T(t) v dt \quad (s > 0, v \in A).
\]

7 The generator of a holomorphic semigroup and its resolvent

Let \( a < b \leq a + \pi \). In this section we consider a holomorphic semigroup \( T = (T(\zeta))_{\zeta \in S_{a,b}} \) of multipliers on a weakly cancellative commutative Banach algebra \( A \) having dense principal ideals such that \( T(\zeta)A \) is dense in \( A \) for some, or, equivalently, for every \( \zeta \in S_{a,b} \).

Denote by \( I_T \) the closed span of \( \{T(\zeta)\}_{\zeta \in S_{a,b}} \), which is equal to the closed span of \( \{T(\zeta)\}_{\zeta \in S_{a,b}} \) for \( \zeta \in S_{a,b} \). For \( \zeta \in S_{a,b} \), set \( T(\zeta) = (T(\zeta))_{t > 0} \), let \( \Phi_{T(\zeta)} : \)
\( \mathcal{M}_{\omega_T} \rightarrow \mathcal{M}(\mathcal{A}) \) be the homomorphism defined by (2). Set \( \omega_T(\zeta) = \|T(\zeta)\| \) for \( \zeta \in S_{a,b} \), denote by \( \mathcal{M}_{\omega_T}(S_{a,b}) \) the space of all measures \( \mu \) on \( S_{a,b} \) such that \( \|\mu\|_{\omega_T} := \int_{S_{a,b}} \omega_T(\zeta)d\mu(\zeta) < +\infty \), which is a Banach algebra with respect to convolution. The convolution algebra \( L^1_{\omega_T}(S_{a,b}) \) is defined in a similar way and will be identified to the closed ideal of \( \mathcal{M}_{\omega_T} \) consisting of measures which are absolutely continuous with respect to Lebesgue measure. Define \( \Phi_T : \mathcal{M}_{\omega_T} \rightarrow \mathcal{I}_T \subset \mathcal{M}(\mathcal{A}) \) by the formula

\[
\Phi_T(\mu) = \int_{S_{a,b}} T(\zeta)d\mu(\zeta),
\]

which is well-defined since the map \( \zeta \rightarrow T(\zeta) \) is continuous with respect to the norm topology on \( \mathcal{M}(\mathcal{A}) \) and since \( \mathcal{I}_T \) is separable.

Let \( \zeta \in S_{a,b} \). Since the semigroup \( T_\zeta \) is continuous with respect to the norm topology on \( \mathcal{M}(\mathcal{A}) \), a standard argument shows that we have, for every Dirac sequence \( (f_n)_{n \geq 1} \),

\[
\lim_{n \to +\infty} \sup_{s > 0} \left\| \int_0^{+\infty} (f_n * \delta_s)(t)T(t\zeta)dt - T(s\zeta) \right\| = 0,
\]

and so \( T(s\zeta) \in \mathcal{I}_T \) for every \( s > 0 \), which implies that \( \mathcal{I}_{T_\zeta} = \mathcal{I}_T \), and a similar argument shows that \( \mathcal{I}_T \) is the closure in \( (\mathcal{M}(\mathcal{A}), \|\cdot\|_{\mathcal{M}(\mathcal{A})}) \) of \( \Phi_T(\mathcal{M}_{\omega_T}(S_{a,b})) \), as well as the closure of \( \Phi_T(L^1_{\omega_T}(S_{a,b})) \) and the closure of \( \Phi_T(L^1_{\omega_T}(S_{a,b})) \) in \( (\mathcal{M}(\mathcal{A}), \|\cdot\|_{\mathcal{M}(\mathcal{A})}) \), and the notation \( \mathcal{I}_T \) is consistent with the notation used to denote the Arveson ideal associated to a strongly continuous semigroup of multipliers on the half-line.

The following interpretation of the generator of a holomorphic semigroup as a quasimultiplier follows the interpretation given in [7] in the case where \( \mathcal{A} = \mathcal{I}_T \).

**Proposition 7.1** Set

\[
\Delta_{T_\zeta, \mathcal{A}} := S_{T(\zeta_0)/T(\zeta_0)u_0} \in \mathcal{Q}\mathcal{M}(\mathcal{A}),
\]

where \( \zeta_0 \in S_{a,b}, u_0 \in \Omega(\mathcal{A}) \).

Then this definition does not depend on the choice of \( \zeta_0 \) and \( u_0 \), and we have, for \( \zeta \in S_{a,b} \),

\[
\Delta_{T_\zeta, \mathcal{A}} = \zeta \Delta_{T_\zeta, \mathcal{A}},
\]

where the generator \( \Delta_{T_\zeta, \mathcal{A}} \) of the semigroup \( T_\zeta \) is the quasimultiplier on \( \mathcal{A} \) introduced in definition 4.1.

Moreover if \( T_1 = (T_1(\zeta))_{\zeta \in S_{a,b}} \) and \( T_2 = (T_2(\zeta))_{\zeta \in S_{a,b}} \) are two holomorphic semigroups of multipliers on \( \mathcal{A} \) such that \( T_1(\zeta)\mathcal{A} \) and \( T_2(\zeta)\mathcal{A} \) are dense in \( \mathcal{A} \) and such that \( T_1(\zeta)T_2(\zeta) = T_2(\zeta)T_1(\zeta) \) for \( \zeta \in S_{a,b} \), then we have

\[
\Delta_{T_1T_2, \mathcal{A}} = \Delta_{T_1, \mathcal{A}} + \Delta_{T_2, \mathcal{A}}.
\]
Corollary 7.2

Let \( u \in \Omega(t) \) and \( B \) with respect to the semigroup \( T \),

and so the definition of \( \Delta_{T,A} \) does not depend on the choice of \( \zeta_0 \), and an easy argument given in the comments following definition 4.1 shows that this definition does not depend on the choice of \( u_0 \in \Omega(A) \) either.

Now let \( \zeta_0 \in S_{a,b} \), and \( f \in C^1([0, +\infty)) \cap \Omega(L^1_{\infty \cdot \zeta_0}) \) such that \( \int_{0}^{+\infty} |f(t)||T(t\zeta_0)||dt < \infty \) and \( \int_{0}^{+\infty} |f(t)||T(t\zeta_0)||dt < \infty \) satisfying \( f(0) = 0 \). We have, integrating by parts, since \( \lim_{p \to +\infty} |f(n_p)||T(n_p\zeta_0)| = 0 \) for some strictly increasing sequence \( (n_p)_{p \geq 1} \) of integers,

\[
T(\zeta_0) \int_{0}^{+\infty} f'(t)T(t\zeta_0)dt = \lim_{p \to +\infty} \int_{0}^{n_p} f'(t)T(\zeta_0 + t\zeta_0)dt
\]

\[
= \lim_{p \to +\infty} \left( |f(t)T(\zeta_0 + t\zeta_0)|^{n_p} - \zeta \int_{0}^{n_p} f(t)T'(\zeta_0 + t\zeta_0)dt \right)
\]

\[
= -\zeta T'(\zeta_0) \int_{0}^{+\infty} f(t)T(t\zeta_0)dt,
\]

and formula (11) follows since \( \left( \int_{0}^{+\infty} f(t)T(t\zeta_0)dt \right) u = \phi_{T\cdot\zeta_0}(f)u \in \Omega(A) \) for \( u \in \Omega(A) \). The last assertion follows immediately from the Leibnitz rule. \( \square \)

The following corollary follows then from proposition 5.4.

**Corollary 7.2**

(i) Let \( u \in A \), and let \( \zeta \in S_{a,b} \). If \( \lim_{t \to 0^+} \left\| \frac{T(t\zeta)u - u}{t} \right\| = 0 \) for some \( v \in A \), then \( u \in D_{\Delta_{T,A}} \), and \( \zeta \Delta_{T,A}u = v \).

(ii) Conversely if \( \limsup_{t \to 0^+} \left\| T(t\zeta) \right\| < +\infty \), then \( \limsup_{t \to 0^+} \left\| \frac{T(t\zeta)u - u}{t} - \zeta \Delta_{T,A}u \right\| = 0 \) for every \( u \in D_{\Delta_{T,A}} \).

In the remaining of the section we will denote by \( B \) a normalization of \( A \) with respect to the semigroup \( T \), see definition 4.3. Since \( \Omega(A) \) is isomorphic to \( \Omega(B) \), we can consider the generator \( \Delta_{T,A} \) as a quasimultiplier on \( B \), and it follows immediately from definition 7.1 that this quasimultiplier on \( B \) is the generator of the semigroup \( T \) considered as a semigroup of multipliers on \( B \). From now on we will thus set \( \Delta_T = \Delta_{T,A} = \Delta_{T,B} \). Applying corollary 7.2 to \( T \) and \( B \), we obtain

**Corollary 7.3**

(i) Let \( u \in B \) Then the following conditions imply each other

(i) There exists \( \zeta_0 \in S_{a,b} \) and \( v \in B \) such that \( \lim_{t \to 0^+} \left\| \frac{T(t\zeta_0)u - v}{t} \right\|_{B} = 0 \),

(ii) \( u \in D_{\Delta_{T,B}} \),

and in this situation \( \lim_{t \to 0^+} \left\| \frac{T(t\zeta_0)u - v}{t} - \zeta \Delta_{T,B}u \right\|_{B} = 0 \) for every \( \zeta \in S_{a,b} \).

Denote by \( \widehat{\mathcal{A}}_T \) the space of characters on \( \mathcal{A}_T \), equipped with the usual Gelfand topology. If \( \chi \in \widehat{\mathcal{A}}_T \), the map \( \zeta \to \chi(T(\zeta)) \) is holomorphic on \( S_{a,b} \), and so there
exists a unique complex number $c_\chi$ such that $\chi(T(\zeta)) = e^{tc_\chi}$ for $\zeta \in S_{a,b}$. We see as in section 5 that there exists a unique character $\tilde{\chi}$ on $QM(I_T)$ such that $\tilde{\chi}|_{I_T} = \chi$, and since $\Delta T_\zeta I_T = \zeta \Delta T_\zeta I_T$ it follows from proposition 5.7 and proposition 7.1 that $\chi(T(t\zeta)) = e^{\tilde{\chi}(\Delta T_\zeta I_T)} = e^{t\tilde{\chi}(\Delta T_\zeta I_T)}$ for $\zeta \in S_{a,b}$, $t > 0$, and so $c_\chi = \tilde{\chi}(\Delta T_\zeta I_T)$.

Since $\Delta T_a I_T = \zeta \Delta T_a I_T$ for $\zeta \in S_{a,b}$, we deduce from corollary 5.8 and proposition 6.1 the following result.

**Proposition 7.4** Let $T = (T(\zeta))_{\zeta \in S_{a,b}} \subset \mathcal{M}(A)$ be a holomorphic semigroup. Set $\sigma_{ar}(\Delta T I_T) = \{\tilde{\chi}(\Delta T I_T)\}_{\tilde{\chi} \in \tilde{\mathcal{F}}}$, with the convention $\sigma_{ar}(\Delta T I_T) = \emptyset$ if the semigroup is quasinilpotent, and set $Res_{ar}(\Delta T) = Res_{ar}(\Delta T I_T) = \mathbb{C} \setminus \sigma_{ar}(\Delta T I_T)$. Let $B$ be a normalization of $A$ with respect to the holomorphic semigroup $T$, and let $I_{T,B}$ be the closed subalgebra of $\mathcal{M}(B)$ generated by the semigroup.

(i) The set $\Lambda_{T,\zeta} := \{\lambda \in \sigma_{ar}(\Delta T I_T) \mid \Re(\lambda \zeta) \leq t\}$ is compact for $\zeta \in S_{a,b}, t \in \mathbb{R}$.

(ii) The quasimultiplier $\lambda I - \Delta T$ has an inverse $(\lambda I - \Delta T)^{-1} \in I_{T,B} \subset \mathcal{M}(B) \subset QM_r(A)$ for $\lambda \in Res_{ar}(\Delta T)$, and the map $\lambda \mapsto (\lambda I - \Delta T)^{-1}$ is a holomorphic map from $Res_{ar}(\Delta T)$ into $I_{T,B}$.

(iii) If $\zeta \in S_{a,b}$, then $\lambda \in Res_{ar}(\Delta T)$ for $\Re(\lambda \zeta) > \lim_{t \to +\infty} \frac{\log\|T(t \zeta)\|}{t}$, and we have

$$
(\lambda I - \Delta T)^{-1} = \int_{0}^{\infty} e^{-s \lambda} T(s) ds,
$$

so that

$$
\| (\lambda I - \Delta T)^{-1} \|_{\mathcal{M}(B)} \leq |\lambda| \int_{0}^{+\infty} e^{-t \Re(\lambda \zeta)} \| T(t \zeta) \|_{\mathcal{M}(B)} dt.
$$

Proof: (i) Let $\zeta \in S_{a,b}, t > 0$, and set $V = \{\lambda \in \zeta \sigma_{ar}(\Delta T B_\zeta) \mid \Re(\lambda) \leq t\} = \{\lambda \in \sigma_{ar}(\Delta T_\zeta I_T) \mid \Re(\lambda) \leq t\}$. It follows from corollary 4.7 that $V$ is compact, and so $\Lambda_{T,\zeta} = \zeta^{-1} V$ is compact.

(ii) Fix $\zeta_0 \in S_{a,b}$. We have $\lambda I - \Delta T = \lambda I - \zeta_0^{-1} \Delta T_{\zeta_0} = \zeta_0^{-1} (\lambda \zeta_0 I - \Delta T_{\zeta_0})$. If $\lambda \in Res_{ar}(\Delta T)$, then $\lambda I - \Delta T$ is invertible in $QM(A)$, and $(\lambda I - T)^{-1} = \zeta_0^{-1} (\lambda \zeta_0 I - \Delta T_{\zeta_0})^{-1} \in I_{T,B} \subset \mathcal{M}(B) \subset QM_r(A)$, since in this situation $\zeta_0^{-1} \in Res(\Delta T_{\zeta_0})$, and it follows also from proposition 6.1 that the $I_{T,B}$-valued map $\lambda \mapsto (\lambda I - T)^{-1} = \zeta_0^{-1} (\lambda \zeta_0 I - \Delta T_{\zeta_0})^{-1}$ is holomorphic on $Res_{ar}(\Delta T)$.

(iii) This follows from proposition 6.1 applied to $\lambda \zeta$ and $T_ \zeta$. □
8 Multivariable functional calculus for holomorphic semigroups associated to linear functionals

In the following definition, we write by convention \( T_j(0) = I \) for \( 1 \leq j \leq k \). Set \( \sigma_1 = \sigma_1 q_1 + \ldots + \sigma_k q_k \) for \( \sigma = (\sigma_1, \ldots, \sigma_k) \), where \( \zeta = (q_1, \ldots, q_k) \in \mathbb{C}^k \).

Let \( a = (a_1, \ldots, a_p) \in \mathbb{R}^k, b = (b_1, \ldots, b_p) \in \mathbb{R}^k \) such that \( a_j \leq b_j \leq a_j + \pi \) for \( 1 \leq j \leq k \). As in appendix 2, we set \( M_{a,b} = \{(\alpha, \beta) \in \mathbb{R}^k \times \mathbb{R}^k | a_j < \alpha_j \leq \beta_j < b_j \text{ if } a_j < b_j, \alpha_j = b_j \text{ if } a_j = b_j \} \).

**Definition 8.1:** Let \( a = (a_1, \ldots, a_p) \in \mathbb{R}^k, b = (b_1, \ldots, b_p) \in \mathbb{R}^k \) such that \( a_j \leq b_j \leq a_j + \pi \) for \( j \leq k \), let \( A \subseteq \mathbb{C}^k \) be a weakly cancellative commutative Banach algebra with dense principal ideals, and let \( T = (T_1, \ldots, T_k) \) be a family of semigroups of multipliers on \( A \) which possesses the following properties

\[
\begin{align*}
T_j &= (T_j(\zeta))_{\zeta \in (0, a_j, \infty)} \text{ is strongly continuous on } (0, e^{i\alpha_j}, \infty), \text{ and } \bigcup_{t > 0} T(te^{i\alpha_j})A \\
&\text{is dense in } A \text{ if } a_j = b_j, \\
T_j &= (T(\zeta))_{\zeta \in S_{a_j, b_j}} \text{ is holomorphic on } S_{a_j, b_j}, \text{ and } T(\zeta)A \text{ is dense in } A
\end{align*}
\]

for every \( \zeta \in S_{a_j, b_j} \) if \( a_j < b_j \).

For \( \zeta = (\zeta_1, \ldots, \zeta_k) \in \bigcup_{(\alpha, \beta) \in M_{a,b}} \mathcal{S}_{\alpha, \beta} \) set

\[
T(\zeta) = T_1(\zeta_1) \cdots T_k(\zeta_k).
\]

A subalgebra \( B \) of \( \mathcal{QM}(A) \) is said to be a normalization of \( A \) with respect to \( T \) if the following conditions are satisfied

(a) \( (B, \| \cdot \|_B) \) is a Banach algebra with respect to a norm \( \| \cdot \|_B \) satisfying \( \|u\|_B \leq \|u\|_A \) for \( u \in A \), and there exists a family \((w_1, \ldots, w_k)\) of elements of \( \Omega(A) \) such that the inclusion map \( j : A \to B \) is \( \mathcal{QM} \)-homomorphism with respect to \( T_1(\zeta_1) \cdots T_k(\zeta_k)w_1 \cdots w_k \) for every family \((\zeta_1, \ldots, \zeta_k)\) of complex numbers such that \( \zeta_j \in S_{a_j, b_j} \) if \( a_j < b_j \) and such that \( \zeta_j = 0 \), if \( a_j = b_j \).

(b) \( j(\mathcal{M}(A)) \subseteq \mathcal{M}(B) \), and \( \|j(R)\|_B \leq \|R\|_A \) for every \( R \in \mathcal{M}(A) \), where \( j : \mathcal{QM}(A) \to \mathcal{QM}(B) \) is the pseudobounded isomorphism associated to \( j \) in proposition 2.2 (ii).

(c) \( \limsup_{\zeta \to 0} \|T(\zeta)\|_B < +\infty \text{ for } a_j < b_j < \beta_j \text{ if } a_j < b_j, \text{ and } \limsup_{t \to 0^+} \|T(te^{i\alpha})\|_B < +\infty \text{ if } a_j = b_j. \)

It follows from proposition 3.2 and proposition 4.2 that there exists a normalization \( B_1 \) of \( A \) with respect to \( T_1 \). Also if \( B_m \) is a normalization of \( A \) with respect to \((T_1, \ldots, T_m)\) and if \( B_{m+1} \) is a normalization of \( B_m \) with respect to \( T_{m+1} \), it follows from proposition 2.4 and definitions 3.3 and 4.3 that \( B_{m+1} \) is a normalization of \( A \) with respect to \((T_1, \ldots, T_{m+1})\). It is thus immediate to construct a normalization of \( A \) with respect to \( T \) by a finite induction. Notice that if \( B \) is a normalization of \( A \) with respect to \( T \), then \( B \) is a normalization of \( A \) with respect to \( T_\sigma := (T(t\sigma))_{t > 0} \) for every \( \sigma \in \bigcup_{(\alpha, \beta) \in M_{a,b}} \mathcal{S}_{\alpha, \beta} \).
Since $\cup_{j \geq 0} \mathcal{T}(e^{i\alpha}) \mathcal{B}$ is dense in $\mathcal{B}$, when $a_j = b_j$, and since $T(\zeta)A$ is dense in $A$ for $\zeta \in S_{a_j, b_j}$ if $a_j < b_j$, it follows from condition (c) of definition 10.1 that the map $\zeta \mapsto T(\zeta)u_1 \ldots u_k$ is continuous on $S_{a,b}$ for $(\alpha, \beta) \in M_{a,b}$, $u_1, \ldots, u_k \in \mathcal{B}$.

Since $u_1 \ldots u_k \in \Omega(\mathcal{B})$ for $u_1, \ldots, u_k \in \Omega(\mathcal{B})$, it follows again from condition (c) of definition 10.1 that the map $\zeta \mapsto T(\zeta)u$ is continuous on $S_{a,b}$ for $(\alpha, \beta) \in M_{a,b}$ for every $u \in \mathcal{B}$. Let $(\alpha, \beta) \in M_{a,b}$ and assume that $a_j < b_j$. Since the semigroup $\mathcal{T}_j$ is holomorphic on $S_{a_j, b_j}$ the map $\eta \mapsto T(\zeta_1, \zeta_2, \ldots, \zeta_j, \zeta_{j+1})u(\zeta_1, \zeta_2, \ldots, \zeta_{j+1}, \zeta_k)$ is holomorphic on $S_{a_j, b_j}$ for every $(\zeta_1, \ldots, \zeta_j, \zeta_{j+1}, \zeta_k) \in \Pi_{1 \leq j \leq k} \mathcal{S}_{\alpha_j, \beta_j}$.

Notice that if $u \in \mathcal{B}$, where $\mathcal{B}$ is a normalization of $A$ with respect to $\mathcal{T}$, then the closed subspace $\mathcal{B}_{T,u}$ spanned by the set $\{T(\zeta)u \mid \zeta \in S_{a,b}\}$ is separable, and so the function $\zeta \mapsto T(\zeta)u$ takes its values in a closed separable subspace of $\mathcal{B}$.

With the convention $\mathcal{T}_j(0) = I$ for $1 \leq j \leq k$, we see that if $(\alpha, \beta) \in M_{a,b}$ and if $\lambda \in \cup_{(\gamma, \delta) \in M_{a,b}} \mathcal{S}_{\gamma, \delta}$ then $T(\lambda) : \zeta \mapsto T(\lambda \zeta) = (T(\lambda(\zeta_1), \ldots, T(\lambda(\zeta_k))$ is well-defined for $\zeta \in S_{a,b}$.

Proposition 8.2 Let $(\alpha, \beta) \in M_{a,b}$. For $\lambda \in \cup_{(\gamma, \delta) \in M_{a,b}} \mathcal{S}_{\gamma, \delta}$, denote by $N(\mathcal{T}, \lambda, \alpha, \beta)$ the set of all $z \in \mathbb{C}^k$ such that

$$\lim_{t \to +\infty} \sup_{t \leq \infty} |e^{t \zeta}e^{i\omega}||T_j(t \lambda_j e^{i\omega})|| < +\infty \quad \text{for } \alpha_j \leq \omega \leq \beta_j, 1 \leq j \leq k,$$

and denote by $N_0(\mathcal{T}, \lambda, \alpha, \beta)$ the set of all $z \in \mathbb{C}^k$ such that

$$\lim_{t \to +\infty} |e^{t \zeta}e^{i\omega}||T_j(t \lambda_j e^{i\omega})|| = 0 \quad \text{for } \alpha_j \leq \omega \leq \beta_j, 1 \leq j \leq k.$$

Then $z \in N(\mathcal{T}, \lambda, \alpha, \beta)$ if and only if $\limsup_{t \to +\infty} |e^{t \zeta}e^{i\omega}||T_j(t \lambda_j e^{i\omega})|| < +\infty$ and $\limsup_{t \to +\infty} |e^{t \zeta}e^{i\omega}||T_j(t \lambda_j e^{i\omega})|| < +\infty$ for $1 \leq j \leq k$.

Also $z \in N_0(\mathcal{T}, \lambda, \alpha, \beta)$ if and only if $Re(z_j e^{i\alpha_j}) < -\inf_{t \to +\infty} \frac{\log||T(t \lambda_j e^{i\alpha_j})||}{t}$ and $Re(z_j e^{i\beta_j}) < -\inf_{t \to +\infty} \frac{\log||T(t \lambda_j e^{i\beta_j})||}{t}$ for $1 \leq j \leq k$.

Proof: Let $j \leq k$ such that $\alpha_j < \beta_j$. If $\alpha_j \leq \omega \leq \beta_j$, there exists $r_0 > 0$ and $s_0 > 0$ such that $e^{i\omega} = r_0 e^{i\alpha_j} + s_0 e^{i\beta_j}$, and we have, for $z_j \in \mathbb{C}$,

$$|e^{t \zeta}e^{i\omega}||T_j(t \lambda_j e^{i\omega})|| \leq |e^{rt \zeta}e^{i\alpha_j}||T_j(r_0 t \lambda_j e^{i\alpha_j})||e^{st \zeta}e^{i\beta_j}||T_j(s_0 t \lambda_j e^{i\beta_j})||,$$  \hfill (14)

and we see that $z \in N(\mathcal{T}, \lambda, \alpha, \beta)$ if and only if $\limsup_{t \to +\infty} |e^{t \zeta}e^{i\alpha_j}||T_j(t \lambda_j e^{i\omega})|| < +\infty$ and $\limsup_{t \to +\infty} |e^{t \zeta}e^{i\beta_j}||T_j(t \lambda_j e^{i\omega})|| < +\infty$ for $1 \leq j \leq k$, which implies that $Re(z_j e^{i\alpha_j}) \leq -\inf_{t \to +\infty} \frac{\log||T(t \lambda_j e^{i\alpha_j})||}{t}$ and $Re(z_j e^{i\beta_j}) \leq -\inf_{t \to +\infty} \frac{\log||T(t \lambda_j e^{i\beta_j})||}{t}$ for $1 \leq j \leq k$. A similar argument shows that $z \in N_0(\mathcal{T}, \lambda, \alpha, \beta)$ if and only if $Re(z_j e^{i\alpha_j}) < -\inf_{t \to +\infty} \frac{\log||T(t \lambda_j e^{i\alpha_j})||}{t}$ and $Re(z_j e^{i\beta_j}) < -\inf_{t \to +\infty} \frac{\log||T(t \lambda_j e^{i\beta_j})||}{t}$, which implies that $Re(z_j e^{i\omega}) < -\inf_{t \to +\infty} \frac{\log||T(t \lambda_j e^{i\omega})||}{t}$ for $\alpha_j \leq \omega \leq \beta_j$, $1 \leq j \leq k$. \hfill $\square$
Notice that it follows from (16) and (17) that \( N(T, \lambda, \alpha, \beta) - \Theta_{a, \beta} \subset N(T, \lambda, \alpha, \beta) \) and that \( N(T, \lambda, \alpha, \beta) - S_{a, \beta} \subset N(0, T, \lambda, \alpha, \beta) \).

Set again \( e_z(\zeta) = e^{\zeta} \) for \( z \in \mathbb{C}^k, \zeta \in \mathbb{C}^k \). If \( B \) is a normalization of \( A \) with respect to \( T \), then \( \sup_{|\zeta| \leq 1} \| T(\zeta) \|_{M(B)} < +\infty \) for \((\alpha, \beta) \in M_{a,b} \), and it follows from (42) that \( \sup_{\zeta \in S_{a, \beta}} |e_z(\zeta)||T(\lambda \zeta)|_{M(B)} < +\infty \) for \( z \in N(T, \alpha, \beta, \lambda) \) and that \( \lim_{z \to +\infty} |e_z(\zeta)||T(\lambda \zeta)|_{M(B)} = 0 \) if \( z \in N(0, T, \lambda, \alpha, \beta) \) when \( \lambda \in \cup_{(\gamma, \delta) \in M_{a-b-\delta}} S_{\gamma, \delta} \). With the notations of appendices 1 and 2, we obtain the following result.

**Proposition 8.3** Let \((\alpha, \beta) \in M_{a, b}, \) and let \( \lambda \in \cup_{(\gamma, \delta) \in M_{a-b-\delta}} S_{\gamma, \delta} \).

(i) If \( z \in N(T, \lambda, \alpha, \beta) \), then \( e_z T(\lambda) u_{\alpha, \beta} \in \mathcal{V}_{\alpha, \beta}(B) \), \( \zeta_j - z_j \in \text{Res}_{ar}(\lambda_j \Delta T_j) \) for \( \zeta \in S_{a, b}^* \), \( u \in \mathcal{B}, 1 \leq j \leq k \), and we have

\[
\mathcal{F}B(e_z T(\lambda) u_{\alpha, \beta})(\zeta) = (-1)^k((z_1 - \zeta_1) I + \lambda_1 \Delta T_1)^{-1} \cdots ((z_k - \zeta_k) I + \lambda_k \Delta T_k)^{-1} u.
\]

(ii) If \( z \in N(0, T, \lambda, \alpha, \beta) \), then \( e_z T(\lambda) u_{\alpha, \beta} \in \mathcal{U}_{\alpha, \beta}(B) \), \( \mathcal{F}B(e_z T(\lambda) u_{\alpha, \beta}) \) has a continuous extension to \( S_{a, \beta}^* \), \( z_j + \zeta_j \in \text{Res}_{ar}(\lambda_j \Delta j) \) for \( 1 \leq j \leq k \), and we have, for \( \zeta \in S_{a, b}, u \in \mathcal{B} \),

\[
\mathcal{F}B(e_z T(\lambda) u_{\alpha, \beta})(\zeta) = (-1)^k((z_1 - \zeta_1) I + \lambda_1 \Delta T_1)^{-1} \cdots ((z_k - \zeta_k) I + \lambda_k \Delta T_k)^{-1} u.
\]

Proof: It follows from the discussion above that \( e_z T(\lambda) u_{\alpha, \beta} \in \mathcal{V}_{\alpha, \beta}(B) \) if \( z \in N(T, \lambda, \alpha, \beta) \), and that \( e_z T(\lambda) u_{\alpha, \beta} \in \mathcal{U}_{\alpha, \beta}(B) \) if \( z \in N(0, T, \lambda, \alpha, \beta) \). Let \( z \in N(T, \lambda, \alpha, \beta) \), and let \( u \in \mathcal{B} \). It follows from definition 10.3 (iii) that we have, for \( \zeta \in S_{a, b}, \)

\[
\mathcal{F}B(e_z T(\lambda) u_{\alpha, \beta})(\zeta)
= \int_0^{e^{i\omega_1}} \cdots \int_0^{e^{i\omega_k}} e^{(z_1 - \zeta_1)\omega_1 + \cdots + (z_k - \zeta_k)\omega_k} T_1(\lambda_1 \sigma_1) \cdots T_k(\lambda_k \sigma_k) u d\sigma_1 \cdots d\sigma_k,
\]

where \( \alpha_j \leq \omega_j \leq \beta_j \) and where \( \text{Re}(\zeta_j e^{i\omega_j}) > 0 \) for \( 1 \leq j \leq k \).

Since \( \text{Re}((\zeta_j - z_j) e^{i\omega_j}) > \lim_{t \to +\infty} \frac{\log(\|T(t, \omega_j^*)\|)}{t} \), it follows from proposition 6.1 and proposition 7.4 that \( \zeta_j - z_j \in \text{Res}_{ar}(\lambda_j \Delta T_j) \) for \( j \leq k \), and that we have, for \( v \in \mathcal{B}, \)

\[
\int_0^{e^{i\omega_j}} e^{(z_j - \zeta_j)\omega_j} T(\lambda \sigma_j) v d\sigma_j = -((z_j - \zeta_j) I + \lambda_j \Delta T_j)^{-1} v.
\]

Using Fubini's theorem, we obtain

\[
\mathcal{F}B(e_z T(\lambda) u_{\alpha, \beta})(\zeta)
\]

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Let \( a, b \in R^k \), let \( T = (T_1, \ldots, T_k) \) be a family of semi-groups of multipliers on \( A \) satisfying the conditions of definition 8.1, let \( B \) be a normalization of \( A \) with respect to \( T \), let \( (\alpha, \beta) \in M_{a,b} \), and let \( \lambda \in \cup_{(\gamma, \delta) \in M_{a-b}} S_{\alpha, \beta} \), with the convention \( S_{\alpha, \beta} = \{ (1) \} \) and \( S_{\alpha, \beta} = \{ (2) \} \). For \( (\alpha, \beta) \in M_{a,b} \), \( z^{(1)} \in \mathbb{C}^k \), and \( z^{(2)} \in \mathbb{C}^k \), we define as in definition 11.1 sup\( (z^{(1)}, z^{(2)}) \) to be the set of all \( z \in \mathbb{C}^k \) such that \( z + \overline{S}_{\alpha, \beta} \supset (z^{(1)} + \overline{S}_{\alpha, \beta}) \cap (z^{(2)} + \overline{S}_{\alpha, \beta}) \), so that sup\( (z^{(1)}, z^{(2)}) \) is a singleton if \( a_j < b_j \) for \( j \leq k \).

**Lemma 8.5** If \( \phi_1 \in F_{a, \beta}, \phi_2 \in F_{a, \beta} \), and if \( N(T, \lambda, \alpha, \beta) \cap \text{Dom}(F_{a, \beta}) \) \( \neq \emptyset \), \( N(T, \lambda, \alpha, \beta) \cap \text{Dom}(F_{a, \beta}) \) \( \neq \emptyset \), then sup\( (z^{(1)}, z^{(2)}) \) is a singleton if \( a_j < b_j \) for \( j \leq k \).

**Theorem 8.6** Let \( A \) be a weakly cancellative commutative Banach algebra with dense principal ideals, let \( a, b \in R^k \), let \( T = (T_1, \ldots, T_k) \) be a family of semi-groups of multipliers on \( A \) satisfying the conditions of definition 8.1, let \( B \) be a normalization of \( A \) with respect to \( T \), let \( (\alpha, \beta) \in M_{a,b} \), and let \( \lambda \in \cup_{(\gamma, \delta) \in M_{a-b}} S_{\gamma, \delta} \).
If $N(T, \lambda, \alpha, \beta) \cap \text{Dom}(\mathcal{B}(\phi)) \neq \emptyset$ for some $\phi \in \mathcal{F}_{\alpha,\beta}$, then the following properties hold

(i) $< T_{(\lambda)}, \phi > \in \mathcal{M}(\mathcal{B}) \subset \mathcal{QM}_{\tau}(\mathcal{A})$, and we have, for $z \in N(T, \lambda, \alpha, \beta) \cap \text{Dom}(\mathcal{B}(\phi))$, if $\nu$ is a $z$-representative measure for $\phi$,

$$< T_{(\lambda)}, \phi > = \int_{\mathcal{F}_{\alpha,\beta}} e^{zT}(\lambda \zeta)d\nu(\zeta),$$

where the Bochner integral is computed with respect to the strong operator topology on $\mathcal{M}(\mathcal{B})$, and if $\chi$ is a character on $\mathcal{A}$, then we have

$$\tilde{\chi}(< T_{(\lambda)}, \phi >) = \mathcal{B}(\phi)(-\lambda_1 \tilde{\chi}(\Delta T), \ldots, -\tilde{\chi}(\Delta T_k)),$$

where $\tilde{\chi}$ denotes the unique character on $\mathcal{Q.M}(\mathcal{A})$ such that $\tilde{\chi}_\mathcal{A} = \chi$.

(ii) 

$$\lim_{\substack{n \to (0, \ldots, 0) \in \mathcal{F}_{\alpha,\beta} \\ \epsilon \to (0, \ldots, 0) \in \mathcal{F}_{\alpha,\beta}}} \| < e^{-\epsilon} T_{(\lambda)}, \phi \ast \delta_\eta > - < T_{(\lambda)}, \phi >\|_B = 0 \quad \text{for} \quad u \in \mathcal{B}.$$

(iii) If $\alpha_j > \beta_j < \alpha_j + \pi$ for $1 \leq j \leq k$, then we have, for $\eta \in S_{\alpha,\beta}$, $\epsilon \in S_{\alpha,\beta}^*$,

$$< e^{-\epsilon} T_{(\lambda)}, \phi \ast \delta_\eta > = e^{-\epsilon \eta} \int_{S_{\alpha,\beta}^*} e^{(z-\epsilon)\sigma} C_z(\phi)(\sigma - \eta) T(\lambda \sigma) d\sigma$$

where the Bochner integral is computed with respect to the strong operator topology on $\mathcal{M}(\mathcal{B})$.

(iv) In the general case, set $W_n(\zeta) = \Pi_{1 \leq j \leq k} \left( \frac{n^2}{n + \zeta_j e^{-\frac{\alpha_j + \beta_j}{2}}} \right)$ for $n \geq 1$, $\zeta = (\zeta_1, \ldots, \zeta_n) \in S_{\alpha,\beta}$. Then we have

$$< T_{(\lambda)}, \phi > = \lim_{\epsilon \to 0} \left( \lim_{n \to +\infty} \frac{(-1)^k}{(2i\pi)^k} \int_{z + \delta S_{\alpha,\beta}^*} W_n(\sigma - z) \mathcal{B}(\phi)(\sigma)((\sigma_1 - \epsilon_1)I + \lambda_1 \Delta T)^{-1} \ldots ((\sigma_k - \epsilon_k)I + \lambda_k \Delta T_k)^{-1} d\sigma \right),$$

where the Bochner integral is computed with respect to the norm topology on $\mathcal{M}(\mathcal{B})$.

(v) If, further, $\int_{z + \delta S_{\alpha,\beta}^*} |\mathcal{B}(\phi)(\sigma)||d\sigma| < +\infty$, then we have

$$< T_{(\lambda)}, \phi > = \lim_{\epsilon \to 0} < e^{-\epsilon} T_{(\lambda)}, \phi > = \lim_{\epsilon \to 0} \frac{(-1)^k}{(2i\pi)^k} \int_{z + \delta S_{\alpha,\beta}^*} \mathcal{B}(\phi)(\sigma)((\sigma_1 - \epsilon_1)I + \lambda_1 \Delta T)^{-1} \ldots ((\sigma_k - \epsilon_k)I + \lambda_k \Delta T_k)^{-1} d\sigma,$$

where the Bochner integral is computed with respect to the norm topology on $\mathcal{M}(\mathcal{B})$. 

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(vi) If the condition of (v) is satisfied for some \( z \in N_0(T, \lambda, \alpha, \beta) \cap \text{Dom}(FB(\phi)) \), then we have

\[
<T_\lambda, \phi> = \frac{(-1)^k}{(2\pi)^k} \int_{\mathbb{Z} + \partial S_{\alpha,\beta}} FB(\phi)(\sigma)((\sigma_1 I + \lambda_1 \Delta_T)^{-1} \ldots (\sigma_k I + \lambda_k \Delta_T)^{-1} d\sigma.
\]

(vii) If \( \phi_1 \in F_{\alpha,\beta}, \phi_2 \in F_{\alpha,\beta} \), and if \( N(T, \lambda, \alpha, \beta) \cap \text{Dom}(FB(\phi_1)) \neq \emptyset \) and \( N(T, \lambda, \alpha, \beta) \cap \text{Dom}(FB(\phi_2)) \neq \emptyset \), then \( N(T, \lambda, \alpha, \beta) \cap \text{Dom}(FB(\phi_1 * \phi_2)) \neq \emptyset \), and

\[
<T_\lambda, \phi_1 * \phi_2> = <T_\lambda, \phi_1> <T_\lambda, \phi_2>.
\]

Proof: (i) Let \( z \in N(T, \lambda, \alpha, \beta) \cap \text{Dom}(FB(\phi)) \), and set \( m = \sup_{\zeta \in S_{\alpha,\beta}} |e^{z\zeta}||T_\lambda(\zeta)||_{M(B)} < +\infty \). We have

\[
\| <T_\lambda, \phi> u \| \leq m \| e^{z\zeta} u \|, \]

and so \( <T_\lambda, \phi > \in \mathcal{M}(B) \). The integral formula in (i) follows then immediately from the definition given in proposition 10.2 and from definition 11.3.

Assume that \( A \) is not radical, let \( \chi \) be a character on \( A \), and let \( \tilde{\chi} \) be the unique character on \( \mathcal{Q}M(A) \) such that \( \tilde{\chi}(u) = \chi(u) \) for every \( u \in A \). Set \( f_n(t) = 0 \) if \( 0 \leq t < \frac{1}{n+1} \), or if \( t > \frac{1}{n} \), and \( f_n(t) = n(n+1) \) if \( \frac{1}{n+1} \leq t \leq \frac{1}{n} \), and let \( \zeta \) be an element of the domain of definition of \( T_j \). Set \( T_{j,\zeta} := (T_j(t)\zeta)_{t>0} \).

Then \( (f_n)_{n \geq 1} \subset L^1_{\omega_{j,\zeta}}(\mathbb{R}^+) \) is a Dirac sequence, and since the map \( t \to T_j(t)\zeta u \) is continuous on \( (0, +\infty) \), a standard argument shows that we have, for \( s > 0, u \in A \),

\[
\lim_{n \to +\infty} \sup_{n>0} \left\| \Phi_{T_{j,\zeta}}(f_n)T_j(s \zeta)u - T_j(s \zeta)u \right\| = \lim_{n \to +\infty} \sup_{n>0} \left\| \int_{0}^{+\infty} (f_n * \delta_\zeta)(t)T_j(t)\zeta u dt - T_j(s \zeta)u \right\| = 0.
\]

Since \( \cup_{T_j \in T_j} \zeta T_j(t)\zeta(A) \) is dense in \( A \), there exist \( n \geq 1 \) such that \( \tilde{\chi}(\Phi_{T_{j,\zeta}}(f_n)) \neq 0 \), and the restriction of \( \tilde{\chi} \) to the Arveson ideal \( I_{T_{j,\zeta}} \) is a character on \( I_{T_{j,\zeta}} \). It follows then from proposition 5.7 and proposition 7.1 that \( \tilde{\chi}(T_j(t)\zeta) = e^{t\tilde{\chi}(\Delta_T \zeta)} \) for \( t > 0 \), and so \( \tilde{\chi}(T_j(\zeta)) = e^{\tilde{\chi}(\Delta_T \zeta)} \) for every \( \zeta \) in the domain of definition of \( T_j \). Let \( u \in M(B) \). By continuity, we see that \( \tilde{\chi}(T_j(\zeta) u) = e^{\tilde{\chi}(\Delta_T \zeta)} \tilde{\chi}(u) \) for every \( \zeta \in \mathfrak{S}_{\alpha,\beta}^\lambda \). Set \( \lambda_1 \tilde{\chi}(\Delta_T) = \lambda_1 \tilde{\chi}(\Delta_T) + \ldots + \lambda_k \tilde{\chi}(\Delta_T) \). Consider again \( z \in N(T, \lambda, \alpha, \beta) \cap \text{Dom}(FB(\phi)) \). Since \( FB(\phi e_{-z})(\zeta) = <e_{-z}, \phi e_{-z} >= <e_{-z}, \phi = FB(\phi)(\zeta + z) \) for \( \zeta \in \mathfrak{S}_{\alpha,\beta}^\lambda \), we obtain

\[
\tilde{\chi}(T_\lambda, \phi>)\tilde{\chi}(u) = \tilde{\chi} \left( \int_{\mathfrak{S}_{\alpha,\beta}} e^{z \zeta} T(\lambda \zeta) u d\nu(\zeta) \right) = \int_{\mathfrak{S}_{\alpha,\beta}} e^{z \zeta} \tilde{\chi}(T(\lambda \zeta)) \tilde{\chi}(u) d\nu(\zeta)
\]
\[
\left(\int_{S_{\alpha,\beta}} e^{zT} e^{\lambda_z \Delta_T} d\nu(t)\right) \chi(u) = \mathcal{F}(\phi e_z)(-\lambda z \Delta_T - z) \chi(u) = \mathcal{F}(\phi)(-\lambda \Delta_T) \chi(u),
\]

which concludes the proof of (i), since \( \chi(u) \neq 0 \).

(ii) Let \( u \in \mathcal{B} \), and set \( f(\zeta) = T(\lambda \zeta)u \) for \( \zeta \in S_{\alpha,\beta} \). Using definition 11.3, we see that (ii) follows from (34) applied to \( f \).

(iii) Define \( f \) as above. We have, for \( \epsilon \in S_{\alpha,\beta}, \eta \in S_{\alpha,\beta}, u \in \mathcal{B} \),

\[
<e^{-T(\lambda)}, \phi \ast \delta_\eta > u =<e^{-\epsilon f}, \phi \ast \delta_\eta > =< (e^{-\epsilon f})_\eta f_\eta, \phi > = e^{-\epsilon} f_\eta, \phi >, 
\]

so (iii) follows from (37).

(iv) The result follows from proposition 11.9 (i) applied to \( T(\lambda)u|_{S_{\alpha,\beta}}, z \) and \( \phi \) for \( u \in \mathcal{B} \).

(v) The result follows from proposition 11.9 (ii) applied to \( T(\lambda)u|_{S_{\alpha,\beta}}, z \) and \( \phi \) for \( u \in \mathcal{B} \).

(vi) Now assume that the condition of (v) is satisfied for some \( z \in N_0(T,\lambda,\alpha,\beta) \cap \text{Dom}(\mathcal{F}(\phi)) \). Then there exists \( \epsilon \in S_{\alpha,\beta}^* \) such that \( z + \epsilon \in N_0(T,\lambda,\alpha,\beta) \), and \( z + \epsilon \in \text{Dom}(\mathcal{F}(\phi_\epsilon)) \). We have

\[
\int_{\partial S_{\alpha,\beta}} e^{Re(\sigma)} \| T(\lambda)(\sigma) \|_{\mathcal{M}(\mathcal{B})} d\sigma \\
\leq \left( \sup_{\zeta \in S_{\alpha,\beta}} e^{Re(\epsilon + \zeta)} \| T(\lambda)(\zeta) \|_{\mathcal{M}(\mathcal{B})} \right) \int_{\partial S_{\alpha,\beta}} e^{-Re(\sigma)} |d\sigma| < +\infty, 
\]

and (vi) follows from proposition 11.9 (iii) applied to \( T(\lambda)u|_{S_{\alpha,\beta}}, z \) and \( \phi \) for \( u \in \mathcal{B} \).

(vii) Now assume that \( \phi_1 \in \mathcal{F}_{\alpha,\beta}, \phi_2 \in \mathcal{F}_{\alpha,\beta} \) satisfy the hypothesis of (vi) with respect to \( T \) and \( \lambda \), and let \( z^{(1)} \in N(T,\lambda,\alpha,\beta) \cap \text{Dom}(\mathcal{F}(\phi_1)) \) and \( z^{(2)} \in N(T,\lambda,\alpha,\beta) \cap \text{Dom}(\mathcal{F}(\phi_2)) \). Set \( z = \sup(z^{(1)}, z^{(2)}) \). It follows from lemma 8.5 that \( z \in N(T,\lambda,\alpha,\beta) \cap \text{Dom}(\mathcal{F}(\phi_1)) \cap \text{Dom}(\mathcal{F}(\phi_2)) \subset N(T,\lambda,\alpha,\beta) \cap \text{Dom}(\mathcal{F}(\phi_1 \ast \phi_2)) \). Let \( \nu_1 \) be a \( z \)-representative measure for \( \phi_1 \) and let \( \nu_2 \) be a \( z \)-representative measure for \( \phi_2 \). Then \( \nu_1 \ast \nu_2 \) is a \( z \)-representative measure for \( \phi_1 \ast \phi_2 \), and we have, for \( u \in \mathcal{B} \),

\[
< T(\lambda), \phi_1 \ast \phi_2 > u = \int_{S_{\alpha,\beta}} e^{zT(\lambda \zeta)u} d(\nu_1 \ast \nu_2)(\zeta) \\
= \int_{S_{\alpha,\beta} \times S_{\alpha,\beta}} e^{z(\zeta_1 + \zeta_2)} T(z(\zeta_1 + \zeta_2))u d\nu_1(\zeta_1) d\nu_2(\zeta_2) \\
\left[ \int_{S_{\alpha,\beta}} e^{z\zeta_1} T(\lambda \zeta_1) d\nu_1(\zeta_1) \right] \left[ \int_{S_{\alpha,\beta}} e^{z\zeta_2} T(\lambda \zeta_2) d\nu_2(\zeta_2) \right] \\
\]
which proves (vii). □

Let $G_{a,b} = \cup_{(\alpha,\beta) \in M_{a,b}} \mathcal{F}_{\alpha,\beta}$ be the dual space introduced in definition 11.2, which is an algebra with respect to convolution according to proposition 11.13. If $\phi_1 \in F_{\alpha(1),\beta(1)}$ and if $\phi_1 \in F_{\alpha(2),\beta(2)}$, where $(\alpha(1),\beta(1)) \in M_{a,b}, (\alpha(2),\beta(2)) \in M_{a,b}$, then $\phi_1 * \phi_2$ is well-defined but in general the fact that $N(T, \lambda, \alpha(1), \beta(1)) \cap \text{Dom}(\phi_1) \neq \emptyset$ and $N(T, \lambda, \alpha(2), \beta(2)) \cap \text{Dom}(\phi_2) \neq \emptyset$ does not seem to imply that $N(T, \lambda, \inf(\alpha(1), \alpha(2)), \sup(\beta(1), \beta(2))) \cap \text{Dom}(\phi_1 * \phi_2) \neq \emptyset$, which prevents from obtaining a direct extension of (vi) to the case where $\phi_1 \in G_{a,b}, \phi_2 \in G_{a,b}$. This difficulty will be circumvented in the next section by using Fourier-Borel transforms.

9 Multivariable functional calculus for holomorphic semigroups associated to holomorphic functions of several complex variables

In the following definition, the generator $\Delta_{T_j}$ of the strongly continuous semigroup $T_j$ and its Arveson spectrum $\sigma_{ar}(\Delta_{T_j})$ are defined according to section 5 if $a_j = b_j$, and the generator $\Delta_{T_j}$ of the holomorphic semigroup $T_j$ and its Arveson spectrum $\sigma_{ar}(\Delta_{T_j})$ are defined according to section 7 if $a_j < b_j$.

Definition 9.1 Let $a = (a_1, \ldots, a_p) \in \mathbb{R}^k, b = (b_1, \ldots, b_p) \in \mathbb{R}^k$ such that $a_j \leq b_j \leq a_j + \pi$ for $j \leq k$, let $A$ be a weakly cancellative commutative Banach algebra having dense principal ideals, and let $T = (T_1, \ldots, T_k)$ be a family of semigroups of multipliers on $A$ satisfying the conditions of definition 8.1. Let $(\alpha, \beta) \in M_{a,b}$ and let $\lambda \in \cup_{(\gamma, \delta) \in M_{a,b}, \beta - \alpha - \delta} \mathbb{S}_{\gamma, \delta}$.

An open set $U \subset \mathbb{C}^k$ is said to be admissible with respect to $(T, \lambda, \alpha, \beta)$ if

$U = \Pi_{1 \leq j \leq k} U_j$ where the open sets $U_j \subset \mathbb{C}$ satisfy the following conditions for some $z = (z_1, \ldots, z_k) \in N_0(T, \alpha, \beta, \lambda)$

(i) $U_j + \mathbb{S}_{\alpha_j, \beta_j} \subset U_j$

(ii) $U_j \subset z_j + S_{\alpha_j, \beta_j}^*$ and $\partial U_j = (z_j + e(-\alpha_j, -\beta_j), \infty, z_j + e(-\alpha_j, \beta_j), s_{0,j}) \cup \gamma([0, 1]) \cup (z_j + e(\alpha_j, -\beta_j), s_{1,j}, z_j + e(\alpha_j, \beta_j), \infty)$, where $s_{0,j} \geq 0$, $s_{1,j} \geq 0$, and $\gamma : [0, 1] \rightarrow z_j + \mathbb{S}_{\alpha_j, \beta_j}^* \setminus (e(-\alpha_j, -\beta_j), \infty, e(-\alpha_j, \beta_j), s_{0,j}) \cup (e(\alpha_j, -\beta_j), s_{1,j}, e(\alpha_j, \beta_j), \infty)$ is a one-to-one piecewise-$C^1$ curve such that $\gamma(0) = e(-\alpha_j, -\beta_j), s_{0,j}$ and $\gamma_1 = e(\alpha_j, -\beta_j), s_{1,j}$.

(iii) $\lambda_j \sigma_{ar}(\Delta_{T_j}) = \sigma_{ar}(\Delta_{T_j}(\lambda_j)) \subset U_j$.

Conditions (i) and (ii) mean that $U$ is admissible with respect to $(\alpha, \beta)$ in the sense of definition 12.1 and that some, hence all elements $z \in \mathbb{C}^k$ with respect to which $U$ satisfies condition (ii) of definition 12.1 belong to $N_0(T, \alpha, \beta, \lambda)$. Hence $U_j$ is a open half-plane if $a_j = \beta_j$, and the geometric considerations about $\partial U_j$ made in section 12 when $a_j < \beta_j$ apply.
For $\alpha = (\alpha_1, \ldots, \alpha_k) \in \mathbb{R}^k$, $\beta = (\beta_1, \ldots, \beta_k) \in \mathbb{R}^k$, we will use as in appendix 3 the obvious conventions

$$\inf(\alpha, \beta) = (\inf(\alpha_1, \beta_1), \ldots, \inf(\alpha_k, \beta_k)), \sup(\alpha, \beta) = (\sup(\alpha_1, \beta_1), \ldots, \sup(\alpha_k, \beta_k)).$$

**Proposition 9.2** If $U^{(1)}$ is admissible with respect to $(T, \lambda, \alpha^{(1)}, \beta^{(1)})$ and if $U^{(2)}$ is admissible with respect to $(T, \lambda, \alpha^{(2)}, \beta^{(2)})$, then $U^{(1)} \cap U^{(2)}$ is admissible with respect to $(T, \lambda, \inf(\alpha^{(1)}, \alpha^{(2)}), \sup(\beta^{(1)}, \beta^{(2)}))$.

Proof: Set $\alpha^{(3)} = \inf(\alpha^{(1)}, \alpha^{(2)}), \beta^{(3)} = \sup(\beta^{(1)}, \beta^{(2)})$, and set $U^{(3)} = U^{(1)} \cap U^{(2)}$. Then $\bigcup_{(\gamma, \delta) \in M_{a-a}(\lambda_{-\beta}(\alpha^{(3)}, \beta^{(3)}))} S_{\gamma, \delta} \cap \bigcup_{(\gamma, \delta) \in M_{a-a}(\lambda_{-\beta}(\alpha^{(3)}, \beta^{(3)}))} S_{\gamma, \delta} \subseteq \bigcup_{(\gamma, \delta) \in M_{a-a}(\lambda_{-\beta}(\alpha^{(3)}, \beta^{(3)}))} S_{\gamma, \delta}$, so it makes sense to check whether $U^{(1)} \cap U^{(2)}$ is admissible with respect to $(T, \lambda, \alpha^{(3)}, \beta^{(3)})$. The fact that $U^{(3)}$ satisfies (i) and (ii) follows from proposition 12.2, and the fact that $U^{(3)}$ satisfies (iii) is obvious. □

If an open set $U \subset \mathbb{C}^k$ is admissible with respect to $(T, \lambda, \alpha, \beta)$, we denote as in section 12 by $H^{(1)}(U)$ the set of all holomorphic functions $F$ on $U$ satisfying the condition

$$\|F\|_{H^{(1)}(U)} := \sup_{e \in S_{a, \beta}} \int_{\sigma \in \partial U^+} |F(\sigma)| |d\sigma| < +\infty.$$

Notice that $\bigcup_{(\gamma, \delta) \in M_{a, b}} \bigcup_{(\lambda, \alpha, \beta) \in M_{a, b} \setminus M_{a, b}} S_{\gamma, \delta} = \bigcup_{(\gamma, \delta) \in M_{a, b} \setminus M_{a, b}} S_{\gamma, \delta}$. The inclusion $\bigcup_{(\gamma, \delta) \in M_{a, b} \setminus M_{a, b}} S_{\gamma, \delta} \subseteq \bigcup_{(\gamma, \delta) \in M_{a, b} \setminus M_{a, b}} S_{\gamma, \delta}$ for $(\alpha, \beta) \in M_{a, b}$. Conversely assume that $\lambda \in S_{a, \alpha, b, \beta}$ for some $(\alpha, \beta) \in M_{a, b}$. If $a_j = b_j$ then $a_j = \alpha_j = \beta_j = b_j$, and so $\lambda_j$ is a nonnegative real number. In this situation set $a_j = b_j, \gamma_j = \delta_j = 0$. If $a_j < b_j$, then $a_j < \alpha_j \leq \beta_j < b_j$, and $a_j - \alpha_j \leq \arg(\lambda_j) < b_j - \beta_j$ if $\lambda_j \neq 0$. In this situation set $a_j = -\alpha_j, \gamma_j = \alpha_j - \beta_j, \delta_j = b_j - \beta_j + b_j^2$. Then $(\alpha', \beta') \in M_{a, b}, (\gamma, \delta) \in M_{a, \alpha', b, -\beta'}$, and $\lambda \in S_{\gamma, \delta}$, which concludes the proof of the reverse inclusion.

**Corollary 9.3** For $\lambda \in \bigcup_{(\alpha, \beta) \in M_{a, b}} S_{a, \alpha, b, \beta} = \bigcup_{(\alpha, \beta) \in M_{a, b}} \bigcup_{(\gamma, \delta) \in M_{a, b}} S_{\gamma, \delta}$, denote by $N_\lambda$ the set of all $(\alpha, \beta) \in M_{a, b}$ such that $\lambda \in \bigcup_{(\gamma, \delta) \in M_{a, b}} S_{\gamma, \delta}$, and denote by $W_{T, \lambda}$ the set of all open sets $U \subset \mathbb{C}^k$ which are admissible with respect to $(T, \lambda, \alpha, \beta)$ for some $(\alpha, \beta) \in N_\lambda$.

Then $W_{T, \lambda}$ is stable under finite intersections, and $\bigcap_{U \in W_{T, \lambda}} H^{(1)}(U)$ is stable under products.

Proof: The first assertion follows from the proposition and the second assertion follows from the fact that the restriction of $F \in H^{(1)}(U)$ is bounded on $U + \epsilon$ if $U$ is admissible with respect to $(\alpha, \beta) \in M_{a, b}$ and if $\epsilon \in S_{a, \beta}$, see corollary 12.4. □

A set $E \subset \bigcup_{U \in W_{T, \lambda}} H^{(1)}(U)$ will be said to be bounded if there exists $U \in W_{T, \lambda}$ such that $E \subset H^{(1)}(U)$ and such that $\sup_{F \in E} \|F\|_{H^{(1)}(U)} < +\infty$, and bounded subsets of $\bigcup_{U \in W_{T, \lambda}} H^{(1)}(U)$ are defined in a similar way. A homomorphism $\phi : \bigcup_{U \in W_{T, \lambda}} H^{(1)}(U) \rightarrow \mathcal{M}(B)$ will be said to be bounded if $\phi(E)$ is
bounded for every bounded subset $E$ of $\cup_{U \in W_{T,\lambda}} H^{(1)}(U)$, and a homomorphism $\phi : \cup_{U \in W_{T,\lambda}} H^{\infty}(U) \to QM_r(\mathcal{B}) = QM_r(\mathcal{A})$ will be said to be bounded if $\phi(E)$ is pseudobounded for every bounded subset $E$ of $\cup_{U \in W_{T,\lambda}} H^{\infty}(U)$.

Similarly let $S(U)$ be the Smirnov class on $U \in W_{T,\lambda}$ introduced in definition 12.6. A set $E \subseteq \cup_{U \in W_{T,\lambda}} S(U)$ will be said to be bounded if there exists $U \in W_{T,\lambda}$ such that $E \subseteq S(U)$ and such that $\sup_{F \in E} \|FG\|_{H^{\infty}(U)} < +\infty$ for some strongly outer function $G \in H^{\infty}(U)$, and a homomorphism $\phi : \cup_{U \in W_{T,\lambda}} S(U) \to QM(\mathcal{B}) = QM(\mathcal{A})$ will be said to be bounded if $\phi(E)$ is pseudobounded for every bounded subset $E$ of $\cup_{U \in W_{T,\lambda}} S(U)$.

Let $U = \Pi_{j \leq k} U_j \in W_{T,\lambda}$, and let $(\alpha, \beta) \in N_\lambda$ such that $U$ is admissible with respect to $(T, \lambda, \alpha, \beta)$. Let $\partial U_j$ be oriented from $e^{\frac{\pi}{2} - \alpha_j} \infty$ to $e^{\frac{\pi}{2} - \beta_j} \infty$. This gives an orientation on the distinguished boundary $\partial U = \Pi_{j \leq k} \partial U_j$ of $U$, to be used in the following theorem.

**Theorem 9.4** Let $a = (a_1, \ldots, a_p) \in \mathbb{R}^k$, $b = (b_1, \ldots, b_p) \in \mathbb{R}^k$ such that $a_j \leq b_j \leq a_j + \pi$ for $j \leq k$, let $\mathcal{A}$ be a weakly cancellative commutative Banach algebra with dense principal ideals, let $T = (T_1, \ldots, T_k)$ be a family of semigroups of multipliers on $\mathcal{A}$ satisfying the conditions of definition 8.1 with respect to $(a, b)$ and $\mathcal{A}$ and let $\mathcal{B}$ be a normalization of $\mathcal{A}$ with respect to $T$.

(i) For $\lambda \in \cup_{(\alpha, \beta) \in N_\lambda} \mathcal{S}_{\alpha, \beta}$, $U \in W_{T,\lambda}$, $F \in H^{(1)}(U)$, set
\[
F(-\lambda_1 T_1, \ldots, -\lambda_k T_k) = \frac{1}{(2i\pi)^k} \int_{\partial U + \epsilon} F(\zeta_1, \ldots, \zeta_k) (\lambda_1 \Delta T_1 + \zeta_1 I)^{-1} \ldots (\lambda_k \Delta T_k + \zeta_k I)^{-1} d\zeta_1 \ldots d\zeta_k,
\]
where $U$ is admissible to respect to $(T, \lambda, \alpha, \beta)$, with $(\alpha, \beta) \in N_\lambda$, and where $\epsilon \in S_{\alpha, \beta}$ is such that $U + \epsilon \in W_{T,\lambda}$. Then this definition does not depend on the choice of $U$ and $\epsilon$, and the map $F \mapsto F(-\lambda_1 T_1, \ldots, -\lambda_k T_k)$ is a bounded algebra homomorphism from $\cup_{U \in W_{T,\lambda}} H^{(1)}(U)$ into $\mathcal{M}(\mathcal{B}) \subseteq \mathcal{M}(\mathcal{A})$.

(ii) For every $U \in W_{T,\lambda}$, there exists $G \in H^{(1)}(U) \cap H^{\infty}(U)$ such that $G(-\lambda_1 T_1, \ldots, -\lambda_k T_k)(\mathcal{B})$ is dense in $\mathcal{B}$, and for every $F \in H^{\infty}(U)$ there exists a unique $R_F \in \mathcal{M}(\mathcal{B}) = \mathcal{M}(\mathcal{A})$ satisfying
\[
R_F G(-\lambda_1 T_1, \ldots, -\lambda_k T_k) = (FG)(-\lambda_1 T_1, \ldots, -\lambda_k T_k) \quad (G \in H^{(1)}(U)).
\]
The definition of $R_F$ does not depend on the choice of $U$, and if we set $F(-\lambda_1 T_1, \ldots, -\lambda_k T_k) = R_F$ the definition of $F(-\lambda_1 T_1, \ldots, -\lambda_k T_k)$ agrees with the definition given in (i) if $F \in \cup_{U \in W_{T,\lambda}} H^{(1)}(U)$, and the map $F \mapsto F(-\lambda_1 T_1, \ldots, -\lambda_k T_k)$ is a bounded homomorphism from $\cup_{U \in W_{T,\lambda}} H^{(1)}(U)$ into $\mathcal{M}(\mathcal{B}) = \mathcal{M}(\mathcal{A})$.

(iii) If $(\alpha, \beta) \in N_\lambda$, if $\phi \in \mathcal{S}_{\alpha, \beta}$, and if $\mathcal{N}_0(T, \lambda, \alpha, \beta) \cap \text{Dom}(FB(\phi)) \neq \emptyset$, then
\[
FB(\phi)(-\lambda_1 T_1, \ldots, -\lambda_k T_k) = \left\langle T(\lambda), \phi \right\rangle.
\]
In particular if $F(\zeta) = e^{\nu \zeta_j}$ for some $\nu \in \mathbb{C}$ such that $\nu \lambda_j \in \cup_{(\gamma, \delta) \in M_{\alpha, \beta}} \mathcal{S}_{\gamma, \delta}$ then $F(-\lambda_1 T_1, \ldots, -\lambda_k T_k) = T_j(\nu \lambda_j)$.\]
(iv) If \((\alpha, \beta) \in \mathcal{N}_\lambda\), if \(\phi \in \mathcal{F}_{\alpha, \beta}\), and if \(N(T, \lambda, \alpha, \beta) \cap \text{Dom}(FB(\phi)) \neq \emptyset\), then

\[
< T(\lambda), \phi > u = \lim_{\epsilon \to (0, \ldots, 0)} F(\phi)(-\lambda_1 \Delta T_1 + \epsilon_1 I, \ldots, -\lambda_k \Delta T_k + \epsilon_k I)u \quad (u \in \mathcal{B}).
\]

(v) If \(U \in W_{T, \lambda}\), and if \(F \in H^\infty(U)\) is strongly outer on \(U\), then there exists \(u \in \Omega(\mathcal{B}) \cap \text{Dom}(F(-\lambda_1 \Delta T_1, \ldots, -\lambda_k \Delta T_k))\) such that \(F(-\lambda_1 \Delta T_1, \ldots, -\lambda_k \Delta T_k)u \in \Omega(\mathcal{B})\).

(vi) For every \(U \in W_{T, \lambda}\) and every \(F \in S(U)\) there exists a unique \(R_F \in \mathcal{Q}\mathcal{M}(\mathcal{B}) = \mathcal{Q}\mathcal{M}(\mathcal{A})\) satisfying \(R_F G(-\lambda_1 T_1, \ldots, -\lambda_k T_k) = (FG)(-\lambda_1 T_1, \ldots, -\lambda_k T_k)\) for every \(G \in H^\infty(U)\) such that \(FG \in H^\infty(U)\). The definition of \(R_F\) does not depend on the choice of \(U\), and if we set \(F(-\lambda_1 T_1, \ldots, -\lambda_k T_k) = R_F\) the definition of \(F(-\lambda_1 \Delta T_1, \ldots, -\lambda_k \Delta T_k)\) agrees with the definition given in (ii) if \(F \in \cup_{U \in W_{T, \lambda}} H^\infty(U)\), the map \(F \to F(-\lambda_1 T_1, \ldots, -\lambda_k T_k)\) is a bounded homomorphism from \(\cup_{U \in W_{T, \lambda}} S(U)\) into \(\mathcal{Q}\mathcal{M}(\mathcal{B}) = \mathcal{Q}\mathcal{M}(\mathcal{A})\), and we have, for \(\chi \in \hat{\mathcal{A}}\),

\[
\hat{\chi}(F(-\lambda_1 T_1, \ldots, -\lambda_k T_k)) = F(-\lambda_1 \hat{\chi}(\Delta T_1), \ldots, -\lambda_k \hat{\chi}(\Delta T_k)) \quad (F \in \cup_{U \in W_{T, \lambda}} S(U)),
\]

where \(\hat{\chi}\) is the character on \(\mathcal{Q}\mathcal{M}(\mathcal{A})\) such that \(\hat{\chi}_|\mathcal{A} = \chi\).

(vii) If \(F(\zeta_1, \ldots, \zeta_k) = -\zeta_j\) then \(F(-\lambda_1 \Delta_1, \ldots, -\lambda_k \Delta_k) = \lambda_j \Delta_{T_j}\).

Proof: In the following we will use the notations \(d\zeta = d\zeta_1 \ldots d\zeta_k\), \(\lambda \Delta_T = (\lambda_1 \Delta T_1, \ldots, \lambda_k \Delta T_k)\), \(R(-\lambda \Delta_T, \zeta) = (-1)^k (\lambda_1 \Delta T_1 + \zeta_1 I)^{-1} \ldots (\lambda_k \Delta T_k + \zeta_k I)^{-1}\)

for \(\zeta = (\zeta_1, \ldots, \zeta_k) \in \text{Res}_{ar}(\lambda \Delta_T) := -\Pi_{j=1}^k \text{Res}_{ar}(\Delta_{T_j}(\lambda_j))\). With these notations, the formula given in (i) takes the form

\[
F(-\lambda \Delta_T) = \frac{(-1)^k}{(2\pi i)^k} \int_{\partial U^* \epsilon} F(\zeta) R(-\lambda \Delta_T, \zeta) d\zeta.
\]

Clearly, \(F(-\lambda \Delta_T) \in \mathcal{M}(\mathcal{B}) \subset \mathcal{Q}\mathcal{M}_{r}(\mathcal{A})\). Let \(U, U' \in W_{T, \lambda}\), let \((\alpha, \beta)\) and \((\alpha', \beta')\) be the elements of \(M_{\alpha, \beta}\) associated to \(U\) and \(U'\) and let \(\epsilon \in S_{\alpha, \beta}^r\) and \(\epsilon' \in S_{\alpha', \beta'}^r\) such that \(U + \epsilon \in W_{T, \lambda}\) and \(U' + \epsilon' \in W_{T, \lambda}\).

Set \(V = U + \epsilon, V' = U' + \epsilon', V'' = V \cap V'\). Then the function \(G : \zeta \to F(\zeta) R(-\lambda \Delta_T, \zeta)\) is holomorphic on a neighborhood of \(\overline{V \setminus V''}\), and it follows from (43) that there exists \(M > 0\) such that \(|G(\zeta)| \leq M\) for \(\zeta \in \overline{V \setminus V''}\). The open sets \(V = \Pi_{j \leq k} V_j\) and \(V'' = \Pi_{j \leq k} V''_j\) have the form \((z + S_{\alpha, \beta}^r) \setminus \overline{K}\) and \((z'' + S_{\alpha', \beta'}^r) \setminus \overline{K''}\) where \(K\) and \(K''\) are compact subsets of \(\mathbb{C}^k\), and where \(\alpha'' = \text{inf}(\alpha, \alpha')\) and \(\beta'' = \text{sup}(\beta, \beta')\). Choose \(\epsilon'' \in S_{\alpha'', \beta'', \beta''}\), and denote by \(V_{L, \epsilon}\) the intersection of \(V_k \setminus V_{l, \epsilon}\) with the strip having for boundaries the lines \(D^1_L = Le^{i(\pi + \beta_k)} + \Re \epsilon''\) and \(D^2_L = Le^{i(\pi + \beta_k)} + \Re \epsilon''\). Set \(W_{n, j}(\zeta_j) = \frac{n^2}{(\alpha + (\zeta_j - z_j) i)^{\epsilon''(\alpha + z_j, \beta')}}\), and set

\[
W_n(\zeta) = W_{n, 1}(\zeta_1) \ldots W_{n, k}(\zeta_k).\]
It follows from Cauchy’s theorem that we have, when \( L \) is sufficiently large

\[
0 = \int_{\Pi \leq k - 1} \partial V \left[ \int_{\partial V_{L,k}} W_n(\zeta)G(\zeta)d\zeta \right] d\zeta_1 \ldots d\zeta_{k-1}.
\]

We have, for \( s = 1, 2, \)

\[
\left\| \int_{(\Pi \leq k - 1) \times (\partial V_{L,D}^*)} W_n(\zeta)G(\zeta)d\zeta \right\| \leq M \left[ \Pi_{\leq k - 1} \int_{\partial V} |W_n,\beta| |d\zeta| \right] \int_{\partial V_{L,D}^*} |W_n,\kappa| d\zeta_k,
\]

and so \( \lim_{L \to +\infty} \int_{(\Pi \leq k - 1) \times (\partial V_{L,D}^*)} W_n(\zeta)G(\zeta)d\zeta = 0. \) We obtain

\[
\int_{\partial V} W_n(\zeta)G(\zeta)d\zeta = \int_{\Pi \leq k - 1} \partial V \times \partial V_\epsilon \ W_n(\zeta)G(\zeta)d\zeta.
\]

It follows then from the Lebesgue dominated convergence theorem that we have

\[
\int_{\partial V} G(\zeta)d\zeta = \int_{\Pi \leq k - 1} \partial V \times \partial V_\epsilon \ G(\zeta)d\zeta.
\]

Using the same argument and a finite induction, we obtain

\[
\int_{\partial V} G(\zeta)d\zeta = \int_{\partial V'} \ G(\zeta)d\zeta.
\]

Similarly \( \int_{\partial V} G(\zeta)d\zeta = \int_{\partial V'} G(\zeta)d\zeta \), which shows that the definition of

\[ F(-\lambda_1 \Delta T_1, \ldots, -\lambda_k \Delta T_k) \]

does not depend on the choice of \( U \) and \( \epsilon \).

Now let \( F \in \cup_{U \in W_{T,\lambda}} H^{(1)}(U) \), let \( G \in \cup_{U \in W_{T,\lambda}} H^{(1)}(U) \). There exists \( U \in W_{T,\lambda} \) such that \( F|_U \in H^{(1)}(U) \) and \( G|_U \in H^{(1)}(U) \). Choose \( \epsilon \in S_{\alpha,\beta} \), where \( (\alpha, \beta) \) is the element of \( M_{n,b} \) associated to \( U \), such that \( U + \epsilon \in W_{T,\lambda} \), and set \( V = U + \frac{\epsilon}{2}, V' = U + \epsilon \). For \( M \subset \{1, \ldots, k\} \), denote by \( |M| \) the cardinal of \( M \). Then \( \{1, \ldots, k\} \setminus M = 2^k - |M| \). Since \( (\lambda_j \Delta T_j + \zeta_j I)^{-1}(\lambda_j \Delta T_j + \sigma_j I)^{-1} = \frac{1}{\sigma_j - \zeta_j} (\lambda_j \Delta T_j + \zeta_j I)^{-1} - (\lambda_j \Delta T_j + \sigma_j I)^{-1} \), we have

\[
F(-\lambda \Delta T)G(-\lambda \Delta T) = \frac{1}{(2\pi)^{2k}} \int_{\partial V \times \partial V'} F(\zeta)G(\sigma)R(-\lambda \Delta T, \zeta)R(-\lambda \Delta T, \sigma)d\zeta d\sigma
\]

\[
= \frac{1}{(2\pi)^{2k}} \sum_{M \subset \{1, \ldots, k\}} L_M,
\]

where

\[
L_M := (-1)^{|M|} \int_{\partial V \times \partial V'} \frac{1}{\sigma_1 - \zeta_1} \ldots \frac{1}{\sigma_k - \zeta_k} F(\zeta)G(\sigma) \prod_{j \in M}(\lambda_j \Delta T_j + \zeta_j I)^{-1} \prod_{j' \not\in M}(\lambda_j \Delta T_j + \sigma_j I)^{-1} d\zeta d\sigma.
\]
Assume that $M \neq \emptyset$, and set \( W_{n,M}((\sigma_j)_{j \in M}) = \Pi_{j \in M} \left( \frac{n^2}{(n+1)(\sigma_j - \zeta_j)} \right) \),

where \( \zeta \in \mathbb{C}^k \) is chosen so that \( \zeta + S^*_{\alpha,\beta} \supset U \). It follows from corollary 12.4 that \( G \) is bounded on \( V \), and so the function \( (\sigma_j)_{j \in M} \rightarrow W_{n,M}((\sigma_j)_{j \in M}) \Pi_{j \in M} \frac{1}{\sigma_j - \zeta_j} G(\sigma) \) belongs to \( H^{(1)}(\Pi_{j \in M} V_j + \frac{1}{2}(\sigma_j)_{j \in M}) \) for every \( (\sigma_j)_{j \notin M} \) and every \( \zeta \in \partial V \). Since the open set \( \Pi_{j \in M} V_j \) is admissible with respect to the family \( \{(\alpha_j, \beta_j)\}_{j \in M} \), it follows from theorem 12.5 that we have, for every \( (\sigma_j)_{j \notin M} \) and every \( \zeta \in \partial V \)

\[
\int_{\Pi_{j \in M} \partial V_j} W_{n,M}((\sigma_j)_{j \in M}) \Pi_{j \in M} \frac{1}{\sigma_j - \zeta_j} G(\sigma) \Pi_{j \in M} d\sigma_j = 0.
\]

Set \( P(\zeta, (\sigma_j)_{j \notin M}) := \int_{\Pi_{j \in M} \partial V_j} \Pi_{j \in M} \frac{1}{\sigma_j - \zeta_j} G(\sigma) \Pi_{j \in M} d\sigma_j \).

It follows then from the Lebesgue dominated convergence theorem that we have, for \( (\sigma_j)_{j \notin M} \in \Pi_{j \notin M} \partial V_j \) and \( \zeta \in \partial V \)

\[
P(\zeta, (\sigma_j)_{j \notin M}) = 0,
\]

and so

\[
(-1)^{|M|} L_M = \int_{\partial V \times (\Pi_{j \notin M} \partial V_j)} \Pi_{j \notin M} \frac{1}{\sigma_j - \zeta_j} F(\zeta, (\sigma_j)_{j \notin M}) \Pi_{j \in M} (\lambda_j \Delta r_j + \zeta_I)^{-1} \Pi_{j \in M} (\lambda_j \Delta r_j + \sigma_j I)^{-1} d(\Pi_{j \notin M} \partial \sigma_j).
\]

We obtain

\[
F(-\lambda \Delta r_j) G(-\lambda \Delta r_j) = \frac{1}{(2\pi)^{2k}} F_0
\]

\[
= \frac{1}{(2\pi)^{2k}} \int_{\partial V} \left[ \int_{\partial V} \frac{F(\zeta)}{(\sigma_j - \zeta_j) \ldots (\sigma_k - \zeta_k)} d\zeta \right] G(\sigma)(\lambda_1 \Delta_1 + \sigma_1 I)^{-1}(\lambda_k \Delta_k + \sigma_k I)^{-1} d\sigma
\]

\[
= \frac{1}{(2\pi)^{k}} \int_{\partial V} F(\sigma) G(\sigma)(\lambda_1 \Delta_1 + \sigma_1 I)^{-1}(\lambda_k \Delta_k + \sigma_k I)^{-1} d\sigma = (FG)(-\lambda \Delta r),
\]

and so the map \( F \rightarrow F(-\lambda \Delta r) \) is an algebra homomorphism from \( \cup U \in W_{T,\lambda} H^{(1)}(U) \) into \( \mathcal{M}(B) \).

Let \( E \) be a bounded subset of \( \cup U \in W_{T,\lambda} H^{(1)}(U) \), let \( U \in W_{T,\lambda} \) such that \( E \) is a bounded subset of \( H^{(1)}(U) \), let \( (\alpha, \beta) \) be the element of \( M_{n,b} \) associated to \( U \), and let \( \epsilon \in S_{n}^{*} \) be such that \( U + \epsilon \in W_{T,\lambda} \). Set \( K = \sup_{\zeta \in \partial U + \epsilon} \| R(-\lambda \Delta r, \zeta) \|_{\mathcal{M}(B)} \).

We have

\[
\sup_{F \in E} \| F(-\lambda \Delta r) \|_{\mathcal{M}(B)} \leq \frac{K}{(2\pi)^{k}} \sup_{F \in E} \| F \|_{H^{(1)}(U)} < +\infty,
\]

which shows that the map \( F \rightarrow F(-\lambda \Delta r) \) is a bounded homomorphism from \( \cup U \in W_{T,\lambda} H^{(1)}(U) \) into \( \mathcal{M}(B) \subset \mathcal{Q} \mathcal{M}_{r}(\mathcal{A}) \).
(ii) Let $U \in \mathcal{W}_{T, \lambda}$, and let $(\alpha, \beta) \in M_{a, b}$ and $z \in \mathbb{C}^k$ be such that $U \subset z + S_{\alpha, \beta}^k$ and $(z + S_{\alpha, \beta}^k) \setminus U$ is bounded. For $j \leq k$, set

$$s_j = 1 + \sup \left( \lim_{t \to +\infty} \log \left( \left\| T \left( t \lambda_j e^{i \frac{\alpha_j + \beta_j}{2}} \right) \right\|, -\Re(z_j e^{i \frac{\alpha_j + \beta_j}{2}}) \right) \right).$$

Set $\tilde{T}_j(t) = T(t \lambda_j e^{i \frac{\alpha_j + \beta_j}{2}})$ for $t > 0$, with the convention $\tilde{T}_j(0) = I$, and set, for $f \in \cap_{\zeta \in \mathbb{C}^k} e^{-\zeta U_{a, \beta}}$,

$$< f, \phi > = \int_{(\mathbb{R}^+)^k} f(t_1 e^{i \frac{\alpha_1 + \beta_1}{2}}, \ldots, t_k e^{i \frac{\alpha_k + \beta_k}{2}}) e^{-s_1 t_1 - \ldots - s_k t_k} dt_1 \ldots dt_k.$$

Then $z \in \text{Dom}(\mathcal{F}B(\phi))$, and we have, for $\zeta \in \text{Dom}(\mathcal{F}B(\phi))$,

$$\mathcal{F}B(\phi)(\zeta) = \int_{(\mathbb{R}^+)^k} e^{-t_1 \zeta_1 e^{i \frac{\alpha_1 + \beta_1}{2}} - \ldots - t_k \zeta_k e^{i \frac{\alpha_k + \beta_k}{2}}} e^{-s_1 t_1 - \ldots - s_k t_k} dt_1 \ldots dt_k$$

$$= \frac{1}{(\zeta_1 e^{i \frac{\alpha_1 + \beta_1}{2}} + s_1) \ldots (\zeta_k e^{i \frac{\alpha_k + \beta_k}{2}} + s_k)},$$

$$< T_{(\lambda)}, \phi > = \int_{(\mathbb{R}^+)^k} T_1(t_1 \lambda_1 e^{i \frac{\alpha_1 + \beta_1}{2}}) \ldots T_k(t_k \lambda_k e^{i \frac{\alpha_k + \beta_k}{2}}) e^{-s_1 t_1 - \ldots - s_k t_k} dt_1 \ldots dt_k$$

$$= \left[ \int_0^{+\infty} \tilde{T}_1(t_1) e^{-s_1 t_1} dt_1 \right] \ldots \left[ \int_0^{+\infty} \tilde{T}_k(t_1) e^{-s_k t_k} dt_k \right],$$

where the Bochner integrals are computed with respect to the strong operator topology on $\mathcal{M}(B)$.

It follows from the observations in section 5 that $\left( \int_0^{+\infty} \tilde{T}_j(t_1) e^{-s_1 t_1} dt_1 \right)(B)$ is dense in $B$ for $1 \leq j \leq k$, and so $< T_{(\lambda)}, \phi > (B)$ is dense in $B$. Now set $\phi_1 = \phi * \phi$. It follows from theorem 8.6 that we have

$$< T_{(\lambda)}, \phi_1 >=< T_{\lambda}, \phi >^2,$$

and so $< T_{(\lambda)}, \phi_1 > (B)$ is dense in $B$.

Set $F = \mathcal{F}B(\phi_1) = \mathcal{F}B(\phi)^2$. Then $F \in H^{(1)}(U - \epsilon) \cap H^{\infty}(U - \epsilon)$ for some $\epsilon \in S_{\alpha, \beta}^k$, and we have, using assertion (vi) of theorem 8.6

$$F(-\lambda \Delta_T) = \frac{(-1)^k}{(2\pi)^k} \int_{z + \partial S_{\alpha, \beta}^k} F(\zeta)(\lambda_1 \Delta_T + \sigma_1)^{-1} \ldots (\lambda_k \Delta_T + \sigma_k)^{-1} d\sigma_1 \ldots d\sigma_k$$

$$=< T_{(\lambda)}, \phi_1 >,$$
which shows that $F(-\lambda \Delta_T)(\mathcal{B})$ is dense in $\mathcal{B}$.

Now consider again $U \in \mathcal{W}_{T, \lambda}$, and let $F \in H^\infty(U)$. Let $G_0 \in H^{(1)}(U)$ be such that $G_0(-\lambda \Delta_T)(\mathcal{B})$ is dense in $\mathcal{B}$, and let $u \in \Omega(\mathcal{B})$. Then $G_0(-\lambda \Delta_T)u \in \Omega(\mathcal{B})$, $FG_0 \in H^{(1)}(U)$, and so there exists a unique quasimultiplier $R_F \in \mathcal{Q}\mathcal{M}_r(\mathcal{B}) = \mathcal{Q}\mathcal{M}_r(A)$ such that $R_FG_0(-\lambda \Delta_T)u = (FG_0)(-\lambda \Delta_T)u$, and $R_F = F(-\lambda \Delta_T)$ if $F \in H^{(1)}(U)$.

Let $U' \in \mathcal{W}_{T, \lambda}$, and let $G \in H^{(1)}(U')$. We have

$$R_FG(-\lambda \Delta_T)G_0(-\lambda \Delta_T) = R_FG_0(-\lambda \Delta_T)G(-\lambda \Delta_T) = (FG_0)(-\lambda \Delta_T)G(-\lambda \Delta_T) = (FG_0)(-\lambda \Delta_T)G(-\lambda \Delta_T),$$

and so $R_FG(-\lambda \Delta_T) = (FG)(-\lambda \Delta_T)$, which shows that the definition of $R_F$ does not depend on the choice of $U$. The map $F \rightarrow R_F$ is clearly linear. Now let $F_1 \in \cup_{U \in \mathcal{W}_{T, \lambda}} H^\infty(U)$, let $F_2 \in \cup_{U \in \mathcal{W}_{T, \lambda}} H^\infty(U)$, and let $G \in \cup_{U \in \mathcal{W}_{T, \lambda}} H^{(1)}(U)$ such that $G(-\lambda \Delta_T)(\mathcal{B})$ is dense in $\mathcal{B}$. We have

$$R_{F_1,F_2}G^2(-\lambda \Delta_T) = (F_1F_2G^2)(-\lambda \Delta_T) = (F_1G)(-\lambda \Delta_T)(F_2G)(-\lambda \Delta_T) = R_{F_1,F_2}G^2(-\lambda \Delta_T),$$

and so $R_{F_1,F_2} = R_{F_1}, R_{F_2}$, since $G^2(-\lambda \Delta_T)\mathcal{B}$ is dense in $\mathcal{B}$.

Now let $\mathcal{E}$ be a bounded family of elements of $\cup_{U \in \mathcal{W}_{T, \lambda}} H^\infty(U)$. There exists $U \in \mathcal{W}_{T, \lambda}$ and $M > 0$ such that $F \in H^\infty(U)$ and $\|F\|_{H^\infty(U)} \leq M$ for every $F \in \mathcal{E}$. Let $G \in H^{(1)}(U)$ such that $G(-\lambda \Delta_T)(\mathcal{B})$ is dense in $\mathcal{B}$. Then the family $\{FG\}_{F \in \mathcal{E}}$ is bounded in $H^{(1)}(U)$, and it follows from (i) that there exists $u \in \Omega(\mathcal{B})$ such that $\sup_{F \in \mathcal{E}} \|(FG)(-\lambda \Delta_T)u\|_{\mathcal{B}} < +\infty$.

We obtain

$$\sup_{F \in \mathcal{E}} \|R_FG(-\lambda \Delta_T)u\|_{\mathcal{B}} = \sup_{F \in \mathcal{E}} \|(FG)(-\lambda \Delta_T)u\|_{\mathcal{B}} < +\infty,$$

and so the family $\{R_F\}_{F \in \mathcal{E}}$ is pseudobounded in $\mathcal{Q}\mathcal{M}(\mathcal{B}) = \mathcal{Q}\mathcal{M}(A)$ since $G(-\lambda \Delta_T)u \in \Omega(\mathcal{B})$. Since the family $\{\lambda^{-n}F^n\}_{n \geq 1}$ is bounded in $H^\infty(U)$ for $F \in H^\infty(U), \lambda > (1+\|F\|_{H^\infty(U)})^{-1}$, this shows that $R_F \in \mathcal{Q}\mathcal{M}_r(\mathcal{B}) = \mathcal{Q}\mathcal{M}_r(A)$ for $F \in \cup_{U \in \mathcal{F}} H^\infty(U)$, and that the map $F \rightarrow R_F$ is a bounded algebra homomorphism from $\cup_{U \in \mathcal{F}} H^\infty(U)$ into $\mathcal{Q}\mathcal{M}_r(\mathcal{B}) = \mathcal{Q}\mathcal{M}_r(A)$, which concludes the proof of (ii).

(iii) Let $(\alpha, \beta) \in \mathcal{N}_\lambda$, let $\phi \in \mathcal{F}_{\alpha, \beta}$, assume that $N_0(T, \lambda, \alpha, \beta) \cap \text{Dom}(\mathcal{F}\mathcal{B}(\phi)) \neq \emptyset$, and let $z \in N_0(T, \lambda, \alpha, \beta) \cap \text{Dom}(\mathcal{F}\mathcal{B}(\phi))$. Then $z + S_{\alpha, \beta}^*$ is admissible with respect to $(T, \lambda, \alpha, \beta)$. As in the proof of (ii) we can construct $\phi_1 \in \mathcal{F}_{\alpha, \beta}$ having the following properties

- $z \in N_0(T, \lambda, \alpha, \beta) \cap \text{Dom}(\mathcal{F}\mathcal{B}(\phi_1))$,
- $G := \mathcal{F}\mathcal{B}(\phi_1) \in H^{(1)}(z + S_{\alpha, \beta}^*) \cap H^\infty(z + S_{\alpha, \beta}^*)$,
- $<T(\lambda), \phi_1> = G(-\lambda \Delta_T)$, and $G(-\lambda \Delta_T)(\mathcal{B})$ is dense in $\mathcal{B}$.
Let $\epsilon \in S_{\alpha,\beta}^*$ be such that $z+\epsilon+S_{\alpha,\beta}^*$ is admissible with respect to $(T, \lambda, \alpha, \beta)$. It follows from assertions (v) and (vi) of theorem 8.6 and from (i) and (ii) that we have

$$<T(\lambda), \phi > FB(\phi_1)(-\lambda \Delta_T) = <T \lambda_j, \phi_1 > <T(\lambda), \phi_1 > = <T(\lambda), \phi * \phi_1 >$$

$$= \int_{z+\epsilon+S_{\alpha,\beta}^*} FB(\phi_1)(\sigma) FB(\phi_1)(\sigma)(\lambda_1 \Delta T_1 + \sigma_1 I)^{-1} \ldots (\lambda_k \Delta T_k + \sigma_k I)^{-1} d\sigma_1 \ldots d\sigma_k$$

$$= (FB(\phi)FB(\phi_1))(-\lambda \Delta_T) = FB(\phi)(-\lambda \Delta_T)FB(\phi_1)(-\lambda \Delta_T),$$

and so $<T(\lambda), \phi > = FB(\phi)(-\lambda \Delta_T)$ since $FB(\phi_1)(-\lambda \Delta_T)(B)$ is dense in $B$.

Now let $\nu \in \mathbb{C}$ such that $\nu \lambda_j \in \cup_{(\gamma_j, \delta_j) \in M_{a_j,b_j}} S_{\gamma_j,\delta_j}$, and let $z_j = (\nu_j, I, \ldots, \nu_j, k)$ be the $k$-tuple defined by the conditions $\nu_j = 0$ if $s \neq j$, $\nu_{j,j} = \nu$. There exists $(\gamma_j, \delta_j) \in N_\lambda\nu$ such that $\nu \in S_{\gamma_j,\delta_j}$, and there exists $(\alpha, \beta) \in N_\lambda\nu$ such that $\alpha_j = \gamma_j$ and $\beta_j = \delta_j$. Set $F(\zeta) = e^{-\nu \zeta}$ for $\zeta \in \mathbb{C}^k$, and set $<f, \phi > = f(\nu_j)$ for $f \in \cap_{z \in \mathbb{C}^k} e^{-\nu_z} M$, and we have, for $\zeta \in \mathbb{C}^k$:

$$FB(\phi)(\zeta) = <e_{-\zeta}, \phi > = e^{-\nu \zeta} = e^{-\nu \zeta},$$

and so $F = FB(\phi)$. Let $z \in N_0(T, \lambda, \alpha, \beta) = N_0(T, \lambda, \alpha, \beta) \cap Dom(FB(\phi))$. Let $\delta_{\nu_j}$ be the Dirac measure at $\nu_j$. Since $e_{-\nu_z} \delta_{\nu_j}$ is a representing measure for $\phi e_{-z}$ we have

$$F(-\lambda \Delta_T) = <T(\lambda), \phi > T_j(\nu \lambda_j),$$

which concludes the proof of (iii).

(iv) Let $(\alpha, \beta) \in N_\lambda\nu$, let $\phi \in F_{\alpha,\beta}$, and assume that $N(T, \lambda, \alpha, \beta) \cap Dom(FB(\phi)) \neq \emptyset$. Set $e_{-\epsilon} T = (e_{-\epsilon_j} T_1, \ldots, e_{-\epsilon_k} T_k)$. Then $N(T, \lambda, \alpha, \beta) \subset N(e_{-\epsilon} T, \lambda, \alpha, \beta)$ for $\epsilon \in S_{\alpha,\beta}^*$, and it follows from theorem 8.6 (ii) and from (iii) that we have, for $u \in B$:

$$<T(\lambda), \phi > u = \lim_{\epsilon \to 0, \ldots, \epsilon} <e_{-\epsilon} T(\lambda), \phi > u$$

$$= \lim_{\epsilon \to 0, \ldots, \epsilon} FB(\phi)(e_{-\epsilon} T(\lambda)) u = \lim_{\epsilon \to 0, \ldots, \epsilon} FB(\phi)(-\lambda_1 T_1 + \epsilon_1 I, \ldots, -\lambda_k T_k + \epsilon_k I) u,$$

which concludes the proof of (iv).

(v) Let $U \in W_{T,\lambda}$, let $F \in H^\infty(U)$ be strongly outer, and let $(F_n)_{n \geq 1}$ be a sequence of invertible elements of $H^\infty(U)$ satisfying the conditions of definition 12.6 with respect to $F$. It follows from (ii) that there exists $G \in H^{(1)}(U) \cap H^\infty(U)$ such that $G(-\lambda \Delta_T)(B)$ is dense in $B$. Let $(\alpha, \beta) \in M_{a,b}$ and $z \in N_0(T, \lambda, \alpha, \beta)$ such that $U \subset z + S_{\alpha,\beta}^*$ and such that $(z + S_{\alpha,\beta}^*) \setminus U$ is bounded. There exists $\epsilon \in \mathbb{C}^k$ such that $U + \epsilon \subset U$ is admissible with respect to $(T, \lambda, \alpha, \beta)$ and we have

$$F(-\lambda \Delta_T) F_n^{-1} (-\lambda \Delta_T) G^2(-\lambda \Delta_T)$$

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\[
\frac{1}{(2\pi i)^k} \int_{\epsilon+i\partial U} F(\sigma)F_n^{-1}(\sigma)G^2(\sigma)(\lambda_1 \Delta T_1 + \sigma_1 I)^{-1} \cdots (\lambda_k \Delta T_k + \sigma_k I)^{-1} d\sigma_1 \cdots d\sigma_k,
\]
and it follows from the Lebesgue dominated convergence theorem that
\[
\lim_{n \to +\infty} \|F(-\lambda \Delta T)F_n^{-1}(-\lambda \Delta T)G^2(-\lambda \Delta T) - G^2(-\lambda \Delta T)\|_{\mathcal{M}(B)} = 0.
\]

Let \( u \in \Omega(B) \). Then \( G(-\lambda \Delta T)u \in Dom(F(-\lambda \Delta T)) \cap \Omega(B) \). Set \( u_n = F_n^{-1}(-\lambda \Delta T)G(-\lambda \Delta T)u \in B \). We have
\[
G(-\lambda \Delta T)^2 u^2 = \lim_{n \to +\infty} F(-\lambda \Delta T)G(-\lambda \Delta T)u_n.
\]
Since \( G(-\lambda \Delta T)^2 u^2 \in \Omega(B) \), this shows that \( F(-\lambda \Delta T)G(-\lambda \Delta T)u \in \Omega(B) \), which proves (v).

(vi) Let \( U \in \mathcal{W}_{T,\lambda} \), let \( F \in \mathcal{S}(U) \), let \( G_0 \in H^\infty(U) \) be a strongly outer function such that \( FG_0 \in H^\infty(U) \), and let \( u \in Dom(G_0(-\lambda \Delta T)) \) such that \( G_0(-\lambda \Delta T)u \in \Omega(B) \). Let \( v \in \Omega(B) \cap Dom(FG_0(-\lambda \Delta T)) \). There exists a unique \( R_F \in \mathcal{Q}(\mathcal{M}(B) = \mathcal{Q}(\mathcal{M}(A)) \) satisfying the equation
\[
(FG_0)(-\lambda \Delta T)uv = R_FG_0(-\lambda \Delta T)uv,
\]
and we have
\[
(FG_0)(-\lambda \Delta T) = R_FG_0(-\lambda \Delta T),
\]
so that \( R_F = F(-\lambda \Delta T) \) if \( F \in H^\infty(U) \).

Let \( G \in \cup_{\lambda \in \mathcal{W}_{T,\lambda}} H^\infty(V) \) such that \( FG \in H^\infty(W) \) for some \( W \in \mathcal{W}_{T,\lambda} \), and let \( w \in \Omega(B) \cap Dom(G(-\lambda \Delta T)) \). We have
\[
((FG)(-\lambda \Delta T)vw)G_0(-\lambda \Delta T)u = (FG_0)(-\lambda \Delta T)G(-\lambda \Delta T)uvw
\]
\[
= R_FG_0(-\lambda \Delta T)G(-\lambda \Delta T)uvw = (R_FG(-\lambda \Delta T)vw)G_0(-\lambda \Delta T)u.
\]

Since \( vw \ (G_0(-\lambda \Delta T)u) \in \Omega(B) \), this shows that \( (FG)(-\lambda \Delta T) = R_FG(-\lambda \Delta T) \). So if we set \( F(-\lambda \Delta T) = R_F \), we obtain \( F(-\lambda \Delta T)G(-\lambda \Delta T) = (FG)(-\lambda \Delta T) \) for every \( F \in \cup_{\lambda \in \mathcal{W}_{T,\lambda}} \mathcal{S}(U) \) and for every \( G \in \cup_{\lambda \in \mathcal{W}_{T,\lambda}} H^\infty(U) \) such that \( FG \in \cup_{\lambda \in \mathcal{W}_{T,\lambda}} H^\infty(U) \). The map \( F \to F(-\lambda \Delta T) \) is clearly linear. Now let \( F_1 \in \cup_{\lambda \in \mathcal{W}_{T,\lambda}} \mathcal{S}(U), F_2 \in \cup_{\lambda \in \mathcal{W}_{T,\lambda}} \mathcal{S}(U), \) and let \( G_1 \in \cup_{\lambda \in \mathcal{W}_{T,\lambda}} H^\infty(U) \) and \( G_2 \in \cup_{\lambda \in \mathcal{W}_{T,\lambda}} H^\infty(U) \) be strongly outer functions such that \( F_1G_1 \in \cup_{\lambda \in \mathcal{W}_{T,\lambda}} H^\infty(U) \) and \( F_2G_2 \in \cup_{\lambda \in \mathcal{W}_{T,\lambda}} H^\infty(U) \). We have
\[
(F_1F_2)(-\lambda \Delta T)G_1(-\lambda \Delta T)G_2(-\lambda \Delta T) = (F_1F_2G_1G_2)(-\lambda \Delta T)
\]
\[
= (F_1G_1)(-\lambda \Delta T)(F_2G_2)(-\lambda \Delta T) = F_1(-\lambda \Delta T)F_2(-\lambda \Delta T)G_1(-\lambda \Delta T)G_2(-\lambda \Delta T),
\]
and so \( (F_1F_2)(-\lambda \Delta T) = F_1(-\lambda \Delta T)F_2(-\lambda \Delta T) \) since \( Dom(G_1(-\lambda \Delta T)) \cap \Omega(B) \neq \emptyset \) and \( Dom(G_2(-\lambda \Delta T)) \cap \Omega(B) \neq \emptyset \), and the map \( F \to F(-\lambda \Delta T) \) is an algebra homomorphism from \( \cup_{\lambda \in \mathcal{W}_{T,\lambda}} \mathcal{S}(U) \) into \( \mathcal{Q}(\mathcal{M}(B) = \mathcal{Q}(\mathcal{M}(A)) \).

Now let \( \mathcal{E} \) be a bounded family of elements of \( \cup_{\lambda \in \mathcal{W}_{T,\lambda}} \mathcal{S}(U) \). There exists \( U \in \mathcal{W}_{T,\lambda} \) and a strongly outer function \( G \in H^\infty(U) \) such that \( FG \in H^\infty(U) \) for every \( F \in \cup_{\lambda \in \mathcal{W}_{T,\lambda}} \mathcal{S}(U) \) and for every \( G \in \cup_{\lambda \in \mathcal{W}_{T,\lambda}} H^\infty(U) \) such that \( FG \in \cup_{\lambda \in \mathcal{W}_{T,\lambda}} H^\infty(U) \). The map \( F \to F(-\lambda \Delta T) \) is clearly linear. Now let \( F_1 \in \cup_{\lambda \in \mathcal{W}_{T,\lambda}} \mathcal{S}(U), F_2 \in \cup_{\lambda \in \mathcal{W}_{T,\lambda}} \mathcal{S}(U), \) and let \( G_1 \in \cup_{\lambda \in \mathcal{W}_{T,\lambda}} H^\infty(U) \) and \( G_2 \in \cup_{\lambda \in \mathcal{W}_{T,\lambda}} H^\infty(U) \) be strongly outer functions such that \( F_1G_1 \in \cup_{\lambda \in \mathcal{W}_{T,\lambda}} H^\infty(U) \) and \( F_2G_2 \in \cup_{\lambda \in \mathcal{W}_{T,\lambda}} H^\infty(U) \). We have
\[
(F_1F_2)(-\lambda \Delta T)G_1(-\lambda \Delta T)G_2(-\lambda \Delta T) = (F_1F_2G_1G_2)(-\lambda \Delta T)
\]
\[
= (F_1G_1)(-\lambda \Delta T)(F_2G_2)(-\lambda \Delta T) = F_1(-\lambda \Delta T)F_2(-\lambda \Delta T)G_1(-\lambda \Delta T)G_2(-\lambda \Delta T),
\]
and so \( (F_1F_2)(-\lambda \Delta T) = F_1(-\lambda \Delta T)F_2(-\lambda \Delta T) \) since \( Dom(G_1(-\lambda \Delta T)) \cap \Omega(B) \neq \emptyset \) and \( Dom(G_2(-\lambda \Delta T)) \cap \Omega(B) \neq \emptyset \), and the map \( F \to F(-\lambda \Delta T) \) is an algebra homomorphism from \( \cup_{\lambda \in \mathcal{W}_{T,\lambda}} \mathcal{S}(U) \) into \( \mathcal{Q}(\mathcal{M}(B) = \mathcal{Q}(\mathcal{M}(A)) \).
$H^\infty(U)$ for every $F \in \mathcal{E}$ and such that $\sup_{F \in \mathcal{E}} \|FG\|_{H^\infty(U)} < +\infty$. So the family $\{FG(-\lambda \Delta_T)\}_{F \in \mathcal{E}}$ is a pseudobounded family of elements of $\mathcal{QM}(B) = \mathcal{QM}(\mathcal{A})$, and there exists $u \in \Omega(B) \cap (\cap_{F \in \mathcal{E}} \text{Dom}(FG(-\lambda \Delta_T)))$ such that $\sup_{F \in \mathcal{E}} \|(FG(-\lambda \Delta_T)u)\|_B < +\infty$. Let $v \in \text{Dom}(G(-\lambda \Delta_T)) \cap \Omega(B)$, and set $w = G(-\lambda \Delta_T)uv$. Then $w \in \Omega(B) \cap (\cap_{F \in \mathcal{E}} \text{Dom}(F(-\lambda \Delta_T)))$

$$\sup_{F \in \mathcal{E}} \|F(-\lambda \Delta_T)w\|_B = \sup_{F \in \mathcal{E}} \|(F(-\lambda \Delta_T)G(-\lambda \Delta_T)uv\|_B$$

$$\leq \sup_{F \in \mathcal{E}} \|\|(FG(-\lambda \Delta_T)u\|_B\|v\|_B < +\infty,$$

and so the family $\{F(-\lambda \Delta_T)\}_{F \in \mathcal{E}}$ is pseudobounded in $\mathcal{QM}(\mathcal{B}) = \mathcal{QM}(\mathcal{A})$, and the map $F \rightarrow F(-\lambda \Delta_T)$ is a bounded algebra homomorphism from $\cup_{U \in \mathcal{W}_{T,\lambda}} \mathcal{S}(U)$ into $\mathcal{QM}(\mathcal{B}) = \mathcal{QM}(\mathcal{A})$, which concludes the proof of (vi).

Now assume that $\mathcal{A}$ is not radical, let $\chi \in \hat{\mathcal{A}}$, and let $\tilde{\chi}$ be the unique character on $\mathcal{QM}(\mathcal{A})$ such that $\tilde{\chi}(u) = \chi(u)$ for every $u \in \mathcal{A}$.

Let $F \in H^{(1)}(U)$, where $U \in \mathcal{W}_{T,\lambda}$, let $(\alpha, \beta)$ be the element of $M_{a,b}$ associated to $U$, and let $\epsilon \in S_{a,b}$ be such that $U + \epsilon$ is admissible with respect to $(T, \lambda, \alpha, \beta)$. Since Bochner integrals commute with linear functionals, we have

$$\tilde{\chi}(F(-\lambda_1 \Delta_{T_1}, \ldots, -\lambda_k \Delta_{T_k})) = \frac{1}{(2\pi)^k} \int_{\partial U + \epsilon} \cdots \frac{d\zeta}{(\lambda_1 \tilde{\chi}(\Delta_{T_1}) + \zeta_1 I)^{-1} \cdots (\lambda_k \tilde{\chi}(\Delta_{T_k}) + \zeta_k I)^{-1}} d\zeta_1 \cdots d\zeta_k.$$

Since $U + \epsilon$ is admissible with respect to $(T, \lambda, \alpha, \beta)$, $(-\lambda_1 \tilde{\chi}(\Delta_{T_1}), \ldots, -\lambda_k \tilde{\chi}(\Delta_{T_k})) \in U + \epsilon$, and it follows from theorem 12.5 that we have

$$\tilde{\chi}(F(-\lambda_1 \Delta_{T_1}, \ldots, -\lambda_k \Delta_{T_k})) = F(-\lambda_1 \tilde{\chi}(\Delta_{T_1}), \ldots, -\lambda_k \tilde{\chi}(\Delta_{T_k})).$$

Now let $F \in H^\infty(U)$, where $U \in \mathcal{W}_{T,\lambda}$, and let $G \in H^{(1)}(U)$ such that $G(-\lambda \Delta_T)(B)$ is dense in $B$. Then $\tilde{\chi}(G(-\lambda \Delta_T)) \neq 0$, and we have

$$\tilde{\chi}(G(-\lambda \Delta_T)) = \frac{\tilde{\chi}((FG)(-\lambda \Delta_T))}{\tilde{\chi}(G(-\lambda \Delta_T))} = \frac{(FG)(-\lambda \tilde{\chi}(\Delta_T))}{G(-\lambda \tilde{\chi}(\Delta_T))} = F(-\lambda \tilde{\chi}(\Delta_T)).$$

Finally let $F \in \mathcal{S}(U)$, where $U \in \mathcal{W}_{T,\lambda}$, and let $G \in H^\infty(U)$ be a strongly outer function such that $FG \in H^\infty(U)$. It follows from (v) that $G(-\lambda \Delta_T)u \in \Omega(B)$ for some $u \in B$, and so $\tilde{\chi}(G(-\lambda \Delta_T)) \neq 0$. The same argument as above shows then that $\tilde{\chi}(F(-\lambda \Delta_T)) = F(-\lambda \tilde{\chi}(\Delta_T))$, which concludes the proof of (vi).

(vii) Set $F(\zeta_1, \ldots, \zeta_k) = -\zeta_j$, choose $\nu_0 > \nu_1 > \lim_{t \to +\infty} \frac{\log\|T(t\lambda_j)\|}{t}$, and set again $v_{\nu_0}(t) = te^{-nt}$. It follows from proposition 12.8(ii) that $F \in \mathcal{S}(U)$ for every $U \in \mathcal{W}_{T,\lambda}$, and it follows from proposition 5.5(i) that we have

$$\lambda_j \Delta_{T_j} \int_{[0,\infty)} v_{\nu_0}(t)T_j(t\lambda_j)dt = -\int_0^{+\infty} v_{\nu_0}'(t)T_j(t\lambda_j)dt,$$

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where the Bochner integrals are computed with respect to the strong operator topology on $\mathcal{M}(B)$.

Now choose $(\alpha, \beta) \in \mathcal{N}_\lambda$, and set, for $f \in \cap_{z \in \mathbb{C}} e^{-z} \mathcal{U}_{\alpha, \beta}$,

$$
< f, \phi_0 > = \int_{[0, +\infty)^k} f(0, \ldots, 0, t_j, 0, \ldots, 0) v_{\nu_0}(t_j) dt_j,
$$

$$
< f, \phi_1 > = \int_{[0, +\infty)^k} f(0, \ldots, 0, t_j, 0, \ldots, 0) v'_{\nu_0}(t_j) dt_j.
$$

Then $\phi_0 \in \mathcal{F}_{\alpha, \beta}$, $\phi_1 \in \mathcal{F}_{\alpha, \beta}$, $-\nu_1 \lambda_j + S^*_{\alpha, \beta} \in \mathcal{N}_0(T, \lambda, \alpha, \beta) \cap \text{Dom}(FB(\phi_0)) \cap \text{Dom}(FB(\phi_1))$, and it follows from (iii) that we have

$$
\int_{[0, +\infty)^n} v_{\nu_0}(t_j) T_j(t_{\lambda_j}) dt_j = < T_{\lambda}, \phi_0 > = FB(\phi_0)(-\lambda \Delta_T),
$$

$$
\int_{[0, +\infty)^n} v'_{\nu_0}(t_j) T_j(t_{\lambda_j}) dt_j = < T_{\lambda}, \phi_1 > = FB(\phi_1)(-\lambda \Delta_T).
$$

But $FB(\phi_0)(\zeta) = \frac{1}{v_{\nu_0 + \zeta_j}}, FB(\phi_1)(\zeta) = \frac{\zeta_j}{v_{\nu_0 + \zeta_j}} = -F(\zeta)FB(\phi_0)(\zeta)$, which gives

$$
\lambda_j \Delta_T \int_{[0, +\infty)^n} v_{\nu_0}(t_j) T_j(t_{\lambda_j}) dt = F(-\lambda \Delta_T) \int_{[0, +\infty)^n} v_{\nu_0}(t_j) T_j(t_{\lambda_j}) dt,
$$

and so $F(-\lambda \Delta_T) = \lambda_j \Delta_T$, since $(\int_{[0, +\infty)^n} v_{\nu_0}(t_j) T_j(t_{\lambda_j}) dt) \langle B \rangle$ is dense in $B$, as observed in section 5. □

10 Appendix 1: Fourier-Borel and Cauchy transforms

In this section we present some certainly well-known results about Fourier-Borel and Cauchy transforms of linear functionals on some spaces of holomorphic functions on sectors.

For $\alpha < \beta \leq \alpha + \pi$ denote as usual by $\overline{S}_{\alpha, \beta}$ the closure of the open sector $S_{\alpha, \beta}$, and set by convention $\overline{S}_{\alpha, \alpha} := \{te^{i\alpha} \}_{t \geq 0}$.

We set

$$
S^*_{\alpha, \beta} = S_{-\pi/2-\alpha, \pi/2-\beta}, \overline{S}_{\alpha, \beta} = \overline{S}_{-\pi/2-\alpha, \pi/2-\beta}, \quad (15)
$$

Notice that $S^*_{\alpha, \alpha+\pi} = \emptyset$, while $\overline{S}_{\alpha, \alpha+\pi} = \overline{S}_{-\pi/2-\alpha, -\pi/2-\alpha} = \{te^{-i\alpha} \}_{t \geq 0}$

Now assume that $\alpha \leq \beta < \alpha + \pi$. Let $\lambda = |\lambda|e^{i\omega} \in \overline{S}_{\alpha, \beta}$ and let $\zeta = |\zeta|e^{i\theta} \in \overline{S}_{\alpha, \beta}$, with $-\frac{\pi}{2} - \alpha \leq \omega \leq \frac{\pi}{2} - \beta$, $\alpha \leq \theta \leq \beta$. We have $-\frac{\pi}{2} \leq \omega + \theta \leq \frac{\pi}{2}$, $|e^{-\lambda \zeta}| = e^{-|\lambda| |\zeta| \cos(\omega + \theta)}$, and we obtain

$$
|e^{-\lambda \zeta}| < 1 \quad (\lambda \in S^*_{\alpha, \beta}, \zeta \in \overline{S}_{\alpha, \beta} \setminus \{0\}), \quad (16)
$$

$$
|e^{-\lambda \zeta}| \leq 1 \quad (\lambda \in \overline{S}_{\alpha, \beta}, \zeta \in \overline{S}_{\alpha, \beta}). \quad (17)
$$
**Definition 10.1** Let \( \alpha = (\alpha_1, \ldots, \alpha_k), \beta = (\beta_1, \ldots, \beta_k) \in \mathbb{R}^k \) such that \( \alpha_j \leq \beta_j < \alpha_j + \pi \) for \( 1 \leq j \leq k \). Set \( \overline{S}_{\alpha,\beta} := \prod_{j=1}^k \overline{S}_{\alpha_j,\beta_j}, \overline{S}_{\alpha,\beta}^* := \prod_{j=1}^k \overline{S}_{\alpha_j,\beta_j}^* \). If, further, \( \alpha_j < \beta_j \) for \( 1 \leq j \leq k \), set \( S_{\alpha,\beta} := \prod_{j=1}^k S_{\alpha_j,\beta_j} \).

Let \( X \) be a Banach space. We denote by \( \mathcal{U}_{\alpha,\beta}(X) \) the set of all continuous \( X \)-valued functions \( f \) on \( \overline{S}_{\alpha,\beta} \) satisfying \( \lim_{z \to k} \| f(z) \|_X = 0 \) such that the map \( \zeta \to f(\zeta_1, \ldots, \zeta_{j-1}, \zeta_j, \zeta_{j+1}, \ldots, \zeta_k) \) is holomorphic on \( S_{\alpha_j,\beta_j} \) for every \( (\zeta_1, \ldots, \zeta_{j-1}, \zeta_j, \zeta_{j+1}, \ldots, \zeta_k) \in \prod_{\leq j \leq k} \overline{S}_{\alpha_j,\beta_j} \) when \( \alpha_j < \beta_j \).

Similarly we denote by \( \mathcal{V}_{\alpha,\beta}(X) \) the set of all continuous bounded \( X \)-valued functions \( f \) on \( \overline{S}_{\alpha,\beta} \) such that the map \( \zeta \to f(\zeta_1, \ldots, \zeta_{j-1}, \zeta_j, \zeta_{j+1}, \ldots, \zeta_k) \) is holomorphic on \( S_{\alpha_j,\beta_j} \) for every \( (\zeta_1, \ldots, \zeta_{j-1}, \zeta_j, \zeta_{j+1}, \ldots, \zeta_k) \in \prod_{\leq j \leq k} \overline{S}_{\alpha_j,\beta_j} \) when \( \alpha_j < \beta_j \). The spaces \( \mathcal{U}_{\alpha,\beta}(X) \) and \( \mathcal{V}_{\alpha,\beta}(X) \) are equipped with the norm \( \| f \|_\infty = \sup_{z \in \overline{S}_{\alpha,\beta}} \| f(z) \|_X \), and we will write \( \mathcal{U}_{\alpha,\beta} := \mathcal{U}_{\alpha,\beta}(\mathbb{C}), \mathcal{V}_{\alpha,\beta} := \mathcal{V}_{\alpha,\beta}(\mathbb{C}) \).

A representing measure for \( \phi \in \mathcal{U}_{\alpha,\beta} \) is a bounded variation \( \nu \) on \( \overline{S}_{\alpha,\beta} \) satisfying
\[
< f, \phi > = \int_{\overline{S}_{\alpha,\beta}} f(\zeta) d\nu(\zeta) \quad (f \in \mathcal{U}_{\alpha,\beta}). \tag{18}
\]

Set \( I := \{ j \leq k \mid \alpha_j = \beta_j \}, J := \{ j \leq k \mid \alpha_j < \beta_j \} \). Since separate holomorphy with respect to each of the variables \( z_j, j \in J \) implies holomorphy with respect to \( z_J = (z_j)_{j \in J} \), the map \( \zeta \to f(z_1, z_J) \) is holomorphic on \( \Pi_{j \in J} \overline{S}_{\alpha_j,\beta_j} \) for every \( z_1 \in \Pi_{j \notin J} \overline{S}_{\alpha_j,\beta_j} \).

For \( z = (z_1, \ldots, z_k), \zeta = (\zeta_1, \ldots, \zeta_k) \in \mathbb{C}^k \), set again \( e_\zeta(z) = e^{z_1 \zeta_1 + \cdots + z_k \zeta_k} \). Also set, if \( X \) is a separable Banach space, and if \( \alpha = (\alpha_1, \ldots, \alpha_k) \) and \( \beta = (\beta_1, \ldots, \beta_k) \) satisfy the conditions above,
\[
\mathcal{U}_{\alpha,\beta}^*(X) = \mathcal{U}_{(-\pi/2-\alpha_1, \ldots, -\pi/2-\alpha_k), (\pi/2-\beta_1, \ldots, \pi/2-\beta_k)}(X), \tag{19}
\]
\[
\mathcal{V}_{\alpha,\beta}^*(X) = \mathcal{V}_{(-\pi/2-\alpha_1, \ldots, -\pi/2-\alpha_k), (\pi/2-\beta_1, \ldots, \pi/2-\beta_k)}(X), \tag{20}
\]
with the conventions \( \mathcal{U}_{\alpha,\beta}^* = \mathcal{U}_{\alpha,\beta}^*(\mathbb{C}) \), \( \mathcal{V}_{\alpha,\beta}^* = \mathcal{V}_{\alpha,\beta}^*(\mathbb{C}) \).

**Proposition 10.2** Let \( \phi \in \mathcal{U}_{\alpha,\beta}^* \), and let \( X \) be a separable Banach space. Set, for \( f \in \mathcal{V}_{\alpha,\beta}(X) \),
\[
< f, \phi > = \int_{\overline{S}_{\alpha,\beta}} f(\zeta) d\nu(\zeta),
\]
where \( \nu \) is a representing measure for \( \phi \). Then this definition does not depend on the choice of \( \nu \), and we have
\[
< f, \phi > = \lim_{\epsilon \to 0} < e^{-\epsilon f}, \phi >. \tag{21}
\]
Proof: It follows from (16) and (17) that \( e^{-\epsilon}f \in \mathcal{U}_{\alpha,\beta}(X) \) for \( f \in \mathcal{V}_{\alpha,\beta}(X), \epsilon \in \mathcal{S}_{\alpha,\beta}^\ast \). If \( f \in \mathcal{U}_{\alpha,\beta}(X) \), then we have, for \( l \in \mathcal{U}_{\alpha,\beta}(X)' \),

\[
< \int_{\mathcal{S}_{\alpha,\beta}} f(\zeta) d\nu(\zeta), l > = \int_{\mathcal{S}_{\alpha,\beta}} < f(\zeta), l > d\nu(\zeta) = < < f(\zeta), l >, \phi >, 
\]

which shows that the definition of \( < f, \phi > \) does not depend on the choice of \( \nu \). Now if \( f \in \mathcal{V}_{\alpha,\beta}(X) \), it follows from the Lebesgue dominated convergence theorem that we have

\[
\int_{\mathcal{S}_{\alpha,\beta}} f(\zeta) d\nu(\zeta) = \lim_{\epsilon \to 0} \int_{\mathcal{S}_{\alpha,\beta}} e^{-\epsilon}f(\zeta) d\nu(\zeta) = \lim_{\epsilon \to 0} < e^{-\epsilon}f, \phi >, 
\]

and we see again that the definition of \( < f, \phi > \) does not depend on the choice of the measure \( \nu \). □

We now introduce the classical notions of Cauchy transforms and Fourier-Borel transforms.

**Definition 10.3** Let \( \phi \in \mathcal{U}_{\alpha,\beta}' \), and let \( f \in \mathcal{V}_{\alpha,\beta}(X) \).

(i) The Fourier-Borel transform of \( \phi \) is defined on \( \mathcal{S}_{\alpha,\beta}^\ast \) by the formula

\[
\mathcal{FB}(\phi)(\lambda) = < e^{-\lambda}, \phi > \quad (\lambda \in \mathcal{S}_{\alpha,\beta}^\ast). 
\]

(ii) The Cauchy transform of \( \phi \) is defined on \( \Pi_{1 \leq j \leq k}(\mathbb{C} \setminus \mathcal{S}_{\alpha_j,\beta_j}) \) by the formula

\[
\mathcal{C}(\phi)(\lambda) = \frac{1}{(2\pi)^k} \frac{1}{\zeta_1 - \lambda_1} \cdots \frac{1}{\zeta_k - \lambda_k} < \phi_{\zeta_1,\ldots,\zeta_k} > 
\]

\[
(\lambda = (\lambda_1, \ldots, \lambda_k) \in \Pi_{1 \leq j \leq k}(\mathbb{C} \setminus \mathcal{S}_{\alpha_j,\beta_j})). 
\]

(iii) The Fourier-Borel transform of \( f \) is defined on \( \Pi_{1 \leq j \leq k}(\mathbb{C} \setminus \mathcal{S}_{\alpha_j,\beta_j}) \) by the formula

\[
\mathcal{FB}(f)(\lambda) = \int e^{i\omega} < e^{-\lambda}, f > d\zeta 
\]

\[
:= \int_{e^{i\omega_1}} e^{-\lambda_1 \zeta_1} \cdots \int_{e^{i\omega_k}} e^{-\lambda_k \zeta_k} f(\zeta_1, \ldots, \zeta_k) d\zeta_1 \cdots d\zeta_k 
\]

\[
(\lambda = (\lambda_1, \ldots, \lambda_k) \in \Pi_{1 \leq j \leq k}(\mathbb{C} \setminus \mathcal{S}_{\alpha_j,\beta_j})), 
\]

where \( \alpha_j \leq \omega_j \leq \beta_j \) and where \( \Re(\lambda_j e^{i\omega_j}) > 0 \) for \( 1 \leq j \leq k \).
It follows from these definitions that \( C(\phi) \) is holomorphic on \( \Pi_{1 \leq j \leq k}(\mathbb{C} \setminus \overline{S}_{\alpha_j, \beta_j}) \) for \( \phi \in \mathcal{U}'_{\alpha, \beta} \), and that \( \mathcal{F}B(f) \) is holomorphic on \( \Pi_{1 \leq j \leq k}(\mathbb{C} \setminus -\overline{S}_{\alpha_j, \beta_j}) \) for \( f \in \mathcal{V}_{\alpha, \beta}(X) \). Also using proposition 10.2 we see that \( \mathcal{F}B(\phi) \in \mathcal{V}_{\alpha, \beta}^* := \mathcal{V}_{-\overline{\alpha}, -\overline{\beta}} \) for \( \phi \in \mathcal{U}'_{\alpha, \beta} \).

**Proposition 10.4** Let \( \phi \in \mathcal{U}'_{\alpha, \beta} \). For \( j \leq k \), set \( I_{\eta,j} = (\pi/2 - \eta, \pi/2 - \beta_j) \) for \( \eta \in (\beta_j, \alpha_j + \pi] \), \( I_{\eta,j} = (-\pi/2 - \eta, \pi/2 - \beta_j) \) for \( \eta \in (\alpha_j + \pi, \beta_j + \pi] \), and set \( I_\eta = (-\pi/2 - \alpha_j, \pi/2 - \eta) \) for \( \eta \in (\beta_j + \pi, \alpha_j + 2\pi] \). Then \( I_{\eta,j} \subset [-\pi/2 - \alpha_j, \pi/2 - \beta_j] \), \( \cos(\eta + s) < 0 \) for \( s \in I_{\eta,j} \), and if \( \lambda = (\lambda_1, \ldots, \lambda_k) \in \Pi_{1 \leq j \leq k}(\mathbb{C} \setminus \overline{S}_{\alpha_j, \beta_j}) \), we have for \( \omega = (\omega_1, \ldots, \omega_k) \in \Pi_{1 \leq j \leq k}I_{\arg(\lambda_j), j} \),

\[
C(\phi)(\lambda) = \frac{1}{(2\pi)^k} \int_0^{e^{-i\omega}} \ldots \int_0^{e^{-i\omega}} e^{\lambda \sigma} \mathcal{F}B(\phi)(\sigma) d\sigma
:= \frac{1}{(2\pi)^k} \int_0^{e^{-i\omega_1}} \ldots \int_0^{e^{-i\omega_k}} e^{\lambda_1 \sigma_1 + \ldots + \lambda_k \sigma_k} \mathcal{F}B(\phi)(\sigma_1, \ldots, \sigma_k) d\sigma_1 \ldots d\sigma_k.
\]

(22)

**Proof:** It follows from the definition of \( I_{\eta,j} \) that \( I_{\eta,j} \subset [-\pi/2 - \alpha_j, \pi/2 - \beta_j] \). In the second case we have obviously \( \frac{\pi}{2} < \eta + s < \frac{3\pi}{2} \) for \( s \in I_{\eta,j} \). In the first case we have \( \frac{\pi}{2} < \eta + s < \frac{\pi}{2} + \eta - \beta_j \leq \frac{3\pi}{2} + \alpha_j - \beta_j < \frac{3\pi}{2} \) for \( \omega \in I_{\eta,j} \) and in the third case we have \( \frac{3\pi}{2} > \eta + s > \pi + \beta_j - \frac{\pi}{2} - \alpha_j > \frac{\pi}{2} \) for \( \omega \in I_{\eta,j} \). We thus see that \( \cos(\eta + s) < 0 \) for \( \eta \in (\beta_j + \pi, \alpha_j + 2\pi] \), \( \omega \in I_{\eta,j} \).

Now assume that \( \lambda \in \Pi_{1 \leq j \leq k}(\mathbb{C} \setminus \overline{S}_{\alpha_j, \beta_j}) \), let \( \eta_j \in (\beta_j, 2\pi + \alpha_j) \) be a determination of \( \arg(A_j) \), let \( \nu \) be a representing measure for \( \phi \) and let \( \omega \in \Pi_{1 \leq j \leq k}I_{\eta_j} \).

Then \( \mathcal{F}B(\phi) \) is bounded on \( \overline{S}_{-\overline{\alpha}, -\overline{\beta}} \), and since \( \cos(\eta_j + \omega_j) < 0 \) for \( j \leq k \), we have

\[
\frac{1}{(2\pi)^k} \int_0^{e^{-i\omega}} \ldots \int_0^{e^{-i\omega}} e^{\lambda \sigma} \mathcal{F}B(\phi)(\sigma) d\sigma = \frac{1}{(2\pi)^k} \int_0^{e^{-i\omega}} \ldots \int_0^{e^{-i\omega}} e^{\lambda \sigma} \left[ \int_{\overline{S}_{\alpha, \beta}} e^{-\sigma \zeta} d\nu(\zeta) \right] d\sigma
= \frac{1}{(2\pi)^k} \int_{\overline{S}_{\alpha, \beta}} \left[ \int_0^{e^{-i\omega}} e^{\sigma(\lambda - \zeta)} d\sigma \right] d\nu(\zeta) = \int_{\overline{S}_{\alpha, \beta}} \frac{1}{\lambda - \zeta} d\nu(\zeta) = C(\phi)(\lambda).
\]

\[\square\]

Now identify the space \( \mathcal{M}(\overline{S}_{\alpha, \beta}) \) of all measures of bounded variation on \( \overline{S}_{\alpha, \beta} \) to the dual space of the space \( C_0(\overline{S}_{\alpha, \beta}) \) of continuous functions on \( \overline{S}_{\alpha, \beta} \) vanishing at infinity via the Riesz representation theorem. The convolution product of two elements of \( \mathcal{M}(\overline{S}_{\alpha, \beta}) \) is defined by the usual formula

\[
\int_{\overline{S}_{\alpha, \beta}} f(\zeta) d(\nu_1 * \nu_2)(\zeta) := \int_{\overline{S}_{\alpha, \beta} \times \overline{S}_{\alpha, \beta}} f(\zeta + \zeta') d\nu_1(\zeta) d\nu_2(\zeta') \quad (f \in C_0(\overline{S}_{\alpha, \beta})).
\]

**Proposition 10.5** Let \( X \) be a separable Banach space.
(i) For \( f \in V_{\alpha,\beta}(X) \), \( \lambda \in S_{\alpha,\beta} \), set \( f_\lambda(\zeta) = f(\zeta + \lambda) \). Then \( f_\lambda \in V_{\alpha,\beta}(X) \) for \( f \in V_{\alpha,\beta}(X) \), \( f_\lambda \in U_{\alpha,\beta}(X) \) and the map \( \lambda \to f_\lambda \) belongs to \( U_{\alpha,\beta}(U_{\alpha,\beta}(X)) \) for \( f \in U_{\alpha,\beta}(X) \). Moreover if we set, for \( \phi \in U'_{\alpha,\beta} \),

\[ f_\phi(\lambda) = \langle f_\lambda, \phi \rangle, \]

then \( f_\phi \in V_{\alpha,\beta}(X) \) for \( f \in V_{\alpha,\beta}(X) \), and \( f_\phi \in U_{\alpha,\beta}(X) \) for \( f \in U_{\alpha,\beta}(X) \).

(ii) For \( \phi_1 \in U'_{\alpha,\beta} \), \( \phi_2 \in U'_{\alpha,\beta} \), set

\[ \langle f, \phi_1 \ast \phi_2 \rangle = \langle f, \phi_1 \rangle + \langle f, \phi_2 \rangle \quad (f \in U_{\alpha,\beta}). \]

Then \( \phi_1 \ast \phi_2 \in U'_{\alpha,\beta} \), \( \nu_1 \ast \nu_2 \) is a representing measure for \( \phi_1 \ast \phi_2 \) if \( \nu_1 \) is a representing measure for \( \phi_1 \) and if \( \nu_2 \) is a representing measure for \( \phi_2 \), and we have

\[ \langle f, \phi_1 \ast \phi_2 \rangle = \langle f, \phi_1 \rangle + \langle f, \phi_2 \rangle \quad (f \in V_{\alpha,\beta}(X)). \]

Proof: These results follow from standard easy verifications which are left to the reader. We will just prove the last formula. Let \( \phi_1 \in U'_{\alpha,\beta} \), \( \phi_2 \in U'_{\alpha,\beta} \).

We have, for \( z = (z_1, \ldots, z_k) \in S_{\alpha,\beta} \), \( \lambda = (\lambda_1, \ldots, \lambda_k) \in S_{\alpha,\beta} \), \( \zeta = (\zeta_1, \ldots, \zeta_k) \in S_{\alpha,\beta} \),

\[ (e^{-z})_\lambda(\zeta) = e^{-z_1(\lambda_1 + \zeta_1)} \cdots e^{-z_k(\lambda_k + \zeta_k)} = e^{-\lambda} e^{-z}(\zeta), \]

and so \( (e^{-z})_\lambda = e^{-z}(\lambda) e^{-z}, \)

\[ (e^{-z})_\phi = FB(\phi)(z) e^{-z}, \]

and

\[ FB(\phi_1 \ast \phi_2)(z) = \langle e^{-z}_\phi, \phi_2 \rangle = FB(\phi_1)(z) \langle e^{-z}_\phi, \phi_2 \rangle = FB(\phi_1)(z) FB(\phi_2)(z). \]

\[ \square \]

For \( \eta \in S_{\alpha,\beta} \), denote by \( \delta_\eta \) the Dirac measure at \( \eta \). We identify \( \delta_\eta \) to the linear functional \( f \to f(\eta) \) on \( U_{\alpha,\beta} \). With the above notations, we have, for \( f \in V_{\alpha,\beta}(X) \), \( \phi \in U_{\alpha,\beta} \),

\[ f_{\delta_\eta} = f_\eta, \]

\[ \langle f, \phi \ast \delta_\eta \rangle = \langle f, \phi \rangle. \]

If \( f \in U_{\alpha,\beta}(X) \), we have \( \lim_{\epsilon \to 0} \epsilon 0 \) \( \|f - f\|_\infty = 0 = \lim_{\eta \to 0} f_\eta - f \|_\infty \).

We obtain, since \( \|e^{-\epsilon f}\|_\infty \leq \|f\|_\infty \) for \( f \in U_{\alpha,\beta} \), \( \epsilon \in S_{\alpha,\beta} \),

\[ \lim_{\epsilon \to 0, \eta \to 0} \|e^{-\epsilon f_\eta} - f\|_\infty = 0 \quad (f \in U_{\alpha,\beta}(X)). \] (23)

Now let \( f \in V_{\alpha,\beta}(X) \), let \( \phi \in U'_{\alpha,\beta} \), and let \( \nu \) be a representative measure for \( \phi \). Since \( \langle e^{-\epsilon f_\eta}, \phi \rangle = \int_{S_{\alpha,\beta}} e^{-\epsilon \zeta} f(\zeta + \eta) d\nu(\zeta) \), and since \( \langle e^{-\epsilon f}, \phi \ast \delta_\eta \rangle = e^{-\epsilon \eta} \langle
\[ e^{-\epsilon f_{\eta}, \phi}, \] it follows from the Lebesgue dominated convergence theorem that we have

\[ \langle f, \phi \rangle = \lim_{\epsilon \rightarrow 0, \epsilon \in S_{\alpha,\beta}} < e^{-\epsilon f_{\eta}}, \phi \rangle = \lim_{\epsilon \rightarrow 0, \epsilon \in S_{\alpha,\beta}} < e^{-\epsilon f}, \phi * \delta_{\eta} \rangle \quad (f \in \mathcal{V}_{\alpha,\beta}(X), \phi \in \mathcal{U}'_{\alpha,\beta}). \] (24)

In the following we will denote by \( \tilde{\partial}S_{\alpha,\beta} = \Pi_{1 \leq j \leq k} \partial\overline{S}_{\alpha_j,\beta_j} \) the distinguished boundary of \( S_{\alpha,\beta} \), where \( \partial\overline{S}_{\alpha_j,\beta_j} = (e^{i\alpha_j}, \infty, 0] \cup [0, e^{i\beta_j}, \infty) \) is oriented from \( e^{i\alpha_j}, \infty \) towards \( e^{i\beta_j}, \infty \).

The following standard computations allow to compute in some cases \( \langle f, \phi \rangle \) by using the Cauchy transform when \( \alpha_j < \beta_j \) for \( j \leq k \).

**Proposition 10.6** Assume that \( \alpha_j < \beta_j < \alpha_j + \pi \) for \( 1 \leq j \leq k \), and let \( \phi \in \mathcal{U}'_{\alpha,\beta} \).

If \( f \in \mathcal{V}_{\alpha,\beta}(X) \), and if

\[ \int_{\tilde{\partial}S_{\alpha,\beta}} \| f(\sigma) \|_{X} d\sigma < +\infty, \]

then we have, for \( \eta \in S_{\alpha,\beta} \),

\[ \langle f_{\eta}, \phi \rangle = \langle f, \phi * \delta_{\eta} \rangle = \int_{\tilde{\partial}S_{\alpha,\beta}} C(\phi)(\sigma - \eta) f(\sigma) d\sigma. \] (25)

In particular we have, for \( f \in \mathcal{V}_{\alpha,\beta}(X) \), \( \epsilon \in S_{\alpha,\beta}^{*}, \eta \in S_{\alpha,\beta} \),

\[ e^{-\epsilon \eta} < e^{-\epsilon f_{\eta}}, \phi > = e^{-\epsilon f}, \phi * \delta_{\eta} > = \int_{\tilde{\partial}S_{\alpha,\beta}} e^{-\epsilon \sigma} C(\phi)(\sigma - \eta) f(\sigma) d\sigma \] (26)

**Proof:** Assume that \( f \in \mathcal{V}_{\alpha,\beta}(X) \) satisfies the condition \( \sup_{\sigma \in \overline{S}_{\alpha,\beta}} (1 + |\sigma|)^{2k} \| f(\sigma) \| < +\infty \).

Let \( \nu \in \mathcal{M}(S_{\alpha,\beta}) \) be a representing measure for \( \phi \). For \( R > 0, j \leq k \), we denote by \( \Gamma_{R,j} \) the Jordan curve \( \{ Re^{i\omega} \}_{\alpha_j \leq \omega \leq \beta_j} \cup [Re^{i\beta_j}, 0] \cup [0, Re^{i\alpha_j}] \), oriented counterclockwise.

We have, for \( \eta \in S_{\alpha,\beta}, \sigma \in \Pi_{1 \leq j \leq k} \partial S_{\alpha_j,\beta_j} \),

\[ |C(\phi)(\sigma - \eta)| \leq \frac{1}{(2\pi)^{k}} \| \phi \|_{\mathcal{U}'_{\alpha,\beta}} \Pi_{1 \leq j \leq k} \operatorname{dist}(\partial S_{\alpha_j,\beta_j} - \eta_j, \partial S_{\alpha_j,\beta_j})^{-1}. \]

It follows then from Fubini's theorem and Cauchy's formula that we have

\[ \int_{\tilde{\partial}S_{\alpha,\beta}} C(\phi)(\sigma - \eta) f(\sigma) d\sigma = \int_{S_{\alpha,\beta}} \left[ \frac{1}{2i\pi} \int_{\partial S_{\alpha,\beta}} \frac{f(\sigma)}{z - \sigma + \eta} d\sigma \right] d\nu(\zeta) \]

\[ = \int_{S_{\alpha,\beta}} \lim_{R \rightarrow +\infty} \frac{1}{(2\pi)^{k}} \left[ \int_{\Gamma_{R,1}} \cdots \int_{\Gamma_{R,k}} \frac{f(\sigma)}{(\sigma_1 - \zeta_1 - \eta_1) \cdots (\sigma_k - \zeta_k - \eta_k)} d\sigma \right] d\nu(\zeta). \]
Formula (26) follows from this equality applied to $e^{-t} f$. Taking the limit as $\epsilon \to 0, \epsilon \in S^\ast_{\alpha, \beta}$ in formula (26), we deduce formula (25) from the Lebesgue dominated convergence theorem. \(\square\)

The following result is indeed standard, but we give a proof for the convenience of the reader.

**Proposition 10.7** The linear span of the set $E_{\alpha, \beta} := \{f = e^{-\sigma} : \sigma \in \Pi_{j \leq k}(0, e^{-\frac{\alpha j + \beta j}{\pi}} \cdot \infty)\}$ is dense in $U_{\alpha, \beta}$, and the Fourier-Borel transform is one-to-one on $U^\ast_{\alpha, \beta}$.

**Proof:** Set $J_1 = \{j \in \{1, \ldots, k\} \mid \alpha_j = \beta_j\}$, set $J_2 := \{j \in \{1, \ldots, k\} \mid \alpha_j < \beta_j\}$, denote by $U_1$ the space of continuous functions on $\overline{S_1} = \Pi_{j \in J_1} S_{\alpha_j, \beta_j}$ vanishing at infinity, set $S_2 := \Pi_{j \in J_2} S_{\alpha_j, \beta_j}$, and denote by $U_2$ the space of continuous functions on $\overline{S_2}$ vanishing at infinity which satisfy the same analyticity condition as in definition 10.1 with respect to $\overline{S_2}$. Also set $E_1 := \{f = e^{-\sigma} : \sigma \in \Pi_{j \in J_1}(0, e^{-\frac{\alpha j + \beta j}{\pi}} \cdot \infty)\}$, and set $E_2 := \{f = e^{-\sigma} : \sigma \in \Pi_{j \in J_2}(0, e^{-\frac{\alpha j + \beta j}{\pi}} \cdot \infty)\}$.

Assume that $J_1 \neq \emptyset$. Then the complex algebra $\text{span}(E_1)$ is self-adjoint and separates the point on $U_1$, and it follows from the Stone-Weierstrass theorem applied to the one-point compactification of $S_1$ that $\text{span}(E_1) \oplus C.1$ is dense in $U_1 \oplus C.1$, which implies that $\text{span}(E_1)$ is dense in $U_1$ since $U_1$ is the kernel of a character on $U_1 \oplus C.1$.

Now assume that $J_2 \neq \emptyset$, set $S_2 := \Pi_{j \in J_2} S_{\alpha_j, \beta_j}$, and $\phi \in U_2$, and define the Cauchy transform and the Fourier-Borel transform of $\phi$ as in definition 10.3. Assume that $<f, \phi> = 0$ for $\phi \in E_2$. If $j \in J_2$, then $g = 0$ for every holomorphic function $g$ on $S_{\alpha_j, \beta_j}$ which vanishes on $\{0, e^{-\frac{\alpha j + \beta j}{\pi}} \cdot \infty\}$. An immediate finite induction shows then that $FB(\phi) = 0$ since $FB(\phi)$ is holomorphic on $S_2$. It follows then from proposition 10.4 that $C(\phi) = 0$, and it follows from (23) and (26) that $<f, \phi> = 0$ for every $f \in U_2$. Hence $\phi = 0$, which shows that $\text{span}(E_2)$ is dense in $U_2$. This shows that $\text{span}(E_{\alpha, \beta})$ is dense in $U_{\alpha, \beta}$ if $J_1 = \emptyset$ or if $J_2 = \emptyset$.

Now assume that $J_1 \neq \emptyset$ and $J_2 \neq \emptyset$, and denote by $E \subseteq U_{\alpha, \beta}$ the set of products $f = gh$, where $g \in U_1$ and $h \in U_2$. The space $U_1 = C_0(\overline{S_1})$ is a closed subspace of codimension one of $C(\overline{S_1} \cup \{\infty\})$. Since the space $C(K)$ has a Schauder basis for every compact space $K$, [3],[30], the space $U_1$ has a Schauder basis. Identifying the dual space of $U_1$ to the space of measures of bounded variation on $\overline{S_1}$, this means that there exists a sequence $(\nu_n)_{n \geq 1}$ of measures of bounded variation on $\overline{S_1}$ such that we have

$$g = \sum_{n=1}^{+\infty} \left( \int_{\overline{S_1}} g(\eta) d\nu_n(\eta) \right) g_n \quad (g \in U_1),$$

where the series is convergent in $(U_1, \|\cdot\|_{\infty})$.

Set $P_m(g) = \sum_{n=1}^{m} \left( \int_{\overline{S_1}} g(\eta) d\nu_n(\eta) \right) g_n$ for $g \in U_1, m \geq 1$. Then $P_m : U_1 \to U_1$ is a bounded linear operator, and $\limsup_{m \to +\infty} \|P_m(g)\| \leq \|g\| < +\infty$ for
every \( g \in U_1 \). It follows then from the Banach-Steinhaus theorem that there exists \( M > 0 \) such that \( \|P_n\|_{\mathcal{B}(U_1)} \leq M \) for \( m \geq 1 \), a standard property of Schauder bases in Banach spaces.

Now let \( \phi \in U_{\alpha,\beta}' \) such that \( < f, \phi > = 0 \) for \( f \in E \), let \( \nu \) be a representing measure for \( \phi \), and let \( f \in U_{\alpha,\beta} \). The function \( f_\xi = \eta \rightarrow f(\eta, \zeta) \) belongs to \( U_1 \) for \( \zeta \in \mathbb{S}_2 \), and a routine verification shows that the function \( h_n : \zeta \rightarrow \int_{\mathbb{S}_1} f_\xi(\sigma)d\nu_n(\sigma) = \int_{\mathbb{S}_1} f(\zeta, \eta)d\nu_n(\eta) \) belongs to \( U_2 \) for \( n \geq 1 \). Since the evaluation map \( g \rightarrow g(\eta) \) is continuous on \( U_1 \) for \( \eta \in \mathbb{S}_1 \), we obtain, for \( \eta \in \mathbb{S}_1, \zeta \in \mathbb{S}_2 \),

\[
  f(\eta, \zeta) = \lim_{m \to +\infty} \sum_{n=1}^{m} g_n(\eta)h_n(\zeta).
\]

We have, for \( m \geq 1, \eta \in \mathbb{S}_1, \zeta \in \mathbb{S}_2 \),

\[
  \left| \sum_{n=1}^{m} g_n(\eta)h_n(\zeta) \right| \leq \|P_m(f_\xi)\|_\infty \leq M\|f_\xi\|_\infty \leq M\|f\|_\infty.
\]

It follows then from the Lebesgue dominated convergence theorem that

\[
  \int_{\mathbb{S}_{\alpha,\beta}} f(\eta, \zeta)d\nu(\eta, \zeta) = \lim_{m \to +\infty} \sum_{n=1}^{m} \int_{\mathbb{S}_{\alpha,\beta}} g_n(\eta)h_n(\zeta)d\nu(\eta, \zeta) = 0.
\]

This shows that \( \text{span}(E) \) is dense in \( U_{\alpha,\beta} \). Since \( \text{span}(E_1) \) is dense in \( U_1 \) and \( \text{span}(E_2) \) is dense in \( U_2, \text{span}(E_{\alpha,\beta}) \) is dense in \( \text{span}(E) \), and so \( \text{span}(E_{\alpha,\beta}) \) is dense in \( U_{\alpha,\beta} \).

Now let \( \phi \in U_{\alpha,\beta}' \). If \( \mathcal{F}B(\phi) = 0 \), then \( < f, \phi > = 0 \) for every \( f \in U_{\alpha,\beta} \), and so \( \phi = 0 \) since \( \text{span}(E_{\alpha,\beta}) \) is dense in \( U_{\alpha,\beta} \), which shows that the Fourier-Borel transform is one-to-one on \( U_{\alpha,\beta} \).

\( \square \)

We will now give a way to compute \( < f, \phi > \) for \( f \in V_{\alpha,\beta}(X), \phi \in U_{\alpha,\beta}' \) by using Fourier-Borel transforms. For \( \sigma \in \Pi_{1 \leq j \leq k} \left( \mathbb{C} \setminus \overline{\mathbb{S}_{\alpha_j,\beta_j}} \right) \), define \( e_\sigma^* \in U_{\alpha,\beta}' \) by using the formula

\[
  < f, e_\sigma^* > = \mathcal{F}B(f)(-\sigma). \quad (27)
\]

Also for \( \phi \in U_{\alpha,\beta}' \), \( g \in U_{\alpha,\beta} \), define \( \phi g \in U_{\alpha,\beta}' \) by using the formula

\[
  < f, \phi g > = < f g, \phi > \quad (f \in U_{\alpha,\beta}).
\]

It follows from definition 10.3 that if \( \sigma = (\sigma_1, \ldots, \sigma_k) \in \Pi_{1 \leq j \leq k} \left( \mathbb{C} \setminus \overline{\mathbb{S}_{\alpha_j,\beta_j}} \right) \), we have, for \( f \in U_{\alpha,\beta} \),

\[
  < f, e_\sigma^* > = \int_{0}^{e^{i\omega} \cdot \infty} e^{\sigma \zeta} f(\zeta)d\zeta,
\]

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where $\omega = (\omega_1, \ldots, \omega_k)$ satisfies $\alpha_j \leq \omega_j \leq \beta_j, \text{Re}(\sigma_j \omega_j) < 0$ for $1 \leq j \leq k$, which gives

$$\|e^*_{\sigma}\|_{\infty} \leq \Pi_{1 \leq j \leq k} \int_0^{\infty} e^{t \text{Re}(\sigma_j \omega_j)} dt = \frac{1}{\Pi_{1 \leq j \leq k} (-\text{Re}(\sigma_j \omega_j))}.$$  

The same formula as above holds with the same $\omega$ to compute $<f, e^*_{\sigma'}>$ for $\sigma' \in \Pi_{1 \leq j \leq k} \left( \mathbb{C} \setminus \mathbb{S}_{\alpha_j, \beta_j} \right)$ when $|\sigma - \sigma'|$ is sufficiently small, and so the map $\sigma \to e^*_{\sigma} \in \mathcal{U}_{\alpha, \beta}$ is holomorphic on $\Pi_{1 \leq j \leq k} \left( \mathbb{C} \setminus \mathbb{S}_{\alpha_j, \beta_j} \right)$ since the map $\lambda \to e^{-\lambda} \in L^1(\mathbb{R}^+)$ is holomorphic on the open half-plane $P^+ := \{ \lambda \in \mathbb{C} \mid \text{Re}(\lambda) > 0 \}$.

Now let $\epsilon \in S^*_{\alpha, \beta}$ and let $\omega \in \Pi_{1 \leq j \leq k} [\alpha_j, \beta_j]$ such that $\text{Re}(\epsilon_j \omega_j) > 0$ for $j \leq k$. Then $\sigma - \epsilon \in \Pi_{1 \leq j \leq k} \left( \mathbb{C} \setminus \mathbb{S}_{\alpha_j, \beta_j} \right)$ for $\sigma \in \partial S^*_{\alpha, \beta}$, and $\text{Re}((\sigma_j - \epsilon_j) \omega_j) \leq -\text{Re}(\epsilon_j \omega_j) < 0$ for $1 \leq j \leq k$. We obtain

$$\|e^*_{\sigma - \epsilon}\|_{\infty} \leq \frac{1}{\Pi_{1 \leq j \leq k} \text{Re}(\epsilon_j \omega_j)},$$  

and so $\sup_{\sigma \in \partial S^*_{\alpha, \beta}} \|e^*_{\sigma - \epsilon}\|_{\infty} < +\infty$ ($\epsilon \in S^*_{\alpha, \beta}$).

We now give the following certainly well-known natural result.

**Proposition 10.8** Let $\phi \in \mathcal{U}'_{\alpha, \beta}$. Assume that

$$\int_{\partial S^*_{\alpha, \beta}} |\mathcal{F}(\phi)(\sigma)| d\sigma < +\infty.$$  

Then we have, for $\epsilon \in S^*_{\alpha, \beta}$,

$$\phi e^{-\epsilon} = \frac{1}{(2\pi)^k} \int_{\partial S^*_{\alpha, \beta}} \mathcal{F}(\phi)(\sigma) e^*_{\sigma - \epsilon} d\sigma,$$

where the Bochner integral is computed in $(\mathcal{U}'_{\alpha, \beta}, \|\cdot\|_{\infty})$, which gives, for $f \in \mathcal{V}_{\alpha, \beta}(X)$,

$$<f e^{-\epsilon}, \phi> = \frac{1}{(2\pi)^k} \int_{\partial S^*_{\alpha, \beta}} \mathcal{F}(\phi)(\sigma) \mathcal{F}(f)(-\sigma + \epsilon) d\sigma.$$  

(28)

**Proof:** Since the map $\sigma \to e^*_{\sigma - \epsilon} \in \mathcal{U}'_{\alpha, \beta}$ is continuous on $\partial S^*_{\alpha, \beta}$, and since $\sup_{\sigma \in \partial S^*_{\alpha, \beta}} \|e^*_{\sigma - \epsilon}\|_{\infty} < +\infty$, the Bochner integral $\int_{\partial S^*_{\alpha, \beta}} \mathcal{F}(\phi)(\sigma) e^*_{\sigma - \epsilon} d\sigma$ is well-defined in $(\mathcal{U}'_{\alpha, \beta}, \|\cdot\|_{\infty})$. Set $\phi_\epsilon := \frac{1}{(2\pi)^k} \int_{\partial S^*_{\alpha, \beta}} \mathcal{F}(\phi)(\sigma) e^*_{\sigma - \epsilon} d\sigma \in \mathcal{U}'_{\alpha, \beta}$. Since the map $\phi \to \mathcal{F}(\phi)(\zeta)$ is continuous on $\mathcal{U}'_{\alpha, \beta}$, we have, for $\zeta \in \mathbb{S}_{\alpha, \beta}$,

$$\mathcal{F}(\phi_\epsilon)(\zeta) = \frac{1}{(2\pi)^k} \int_{\partial S^*_{\alpha, \beta}} \mathcal{F}(\phi)(\sigma) \mathcal{F}(e^*_{\sigma - \epsilon})(\zeta) d\sigma$$

$$= \frac{1}{(2\pi)^k} \int_{\partial S^*_{\alpha, \beta}} \mathcal{F}(\phi)(\sigma) <e^{-\zeta}, e^*_{\sigma - \epsilon}> d\sigma.$$
It follows from definition (27) that \( < e_{-\zeta}, e_{\sigma-\epsilon}^* > = FB(e_{-\zeta})(\epsilon - \sigma) \). Let \( \omega \in \Pi_{1 \leq j \leq k}[\alpha_j, \beta_j] \) such that \( Re(e^{i\omega_j}) > 0 \) for \( j \leq k \). Since \( Re((\sigma_j - \epsilon_j)\omega_j) \leq -Re(\epsilon_j\omega_j) < 0 \) for \( 1 \leq j \leq k \), we have, for \( \sigma \in \partial S_{\alpha,\beta}^* \),

\[
FB(e_{-\zeta})(\epsilon - \sigma) = \int_0^{e^{i\omega_\infty}} e^{(\sigma-\epsilon)\eta}e_{-\zeta}(\eta)d\eta = \int_0^{e^{i\omega_\infty}} e^{(\sigma-\epsilon-\zeta)\eta}d\eta
\]

\[
= \prod_{1 \leq j \leq k}(\zeta_j + \epsilon - \sigma_j).
\]

Using the notation \( \frac{1}{\zeta + \epsilon - \sigma} := \frac{1}{\Pi_{1 \leq j \leq k}(\zeta_j + \epsilon - \sigma_j)} \), this gives

\[
FB(\phi_\epsilon)(\zeta) = \frac{1}{(2\pi)^k} \int_{\partial S_{\alpha,\beta}^*} \frac{FB(\phi)(\sigma)}{\zeta + \epsilon - \sigma} d\sigma.
\]

As in appendix 3, set \( W_{n,j}(\zeta) = \left( \frac{n^2}{n \epsilon + \zeta} \right)^j \) for \( n \geq 1 \), \( \zeta \in S_{\alpha,\beta}^* \), and set \( W_n(\zeta) = \Pi_{j \leq k} W_{n,j}(\zeta) \) for \( \zeta \in S_{\alpha,\beta}^* \). Then \( |W_{n,j}(\zeta_j)| \leq 1 \) for \( \zeta_j \in S_{\alpha,\beta}^* \), \( W_n(\zeta) \to 1 \) as \( n \to \infty \) uniformly on compact sets of \( S_{\alpha,\beta}^* \), and \( \lim_{\zeta \in S_{\alpha,\beta}^*} W_n(\zeta) = 0 \). The open set \( S_{\alpha,\beta}^* \) is admissible with respect to \((\alpha, \beta)\) in the sense of definition 12.1 and, since \( FB(\phi) \) is bounded on \( S_{\alpha,\beta}^* \), \( FB(\phi)W_n \in H^{(1)}(S_{\alpha,\beta}^*) \) for \( n \geq 1 \).

It follows then from theorem 12.5 that we have, for \( t \in (0,1), \zeta \in S_{\alpha,\beta}^* \),

\[
\frac{1}{(2\pi)^k} \int_{\partial S_{\alpha,\beta}^*} \frac{FB(\phi)(\sigma + te)W_n(\sigma + te)}{\zeta + (1-t)e - \sigma} d\sigma
\]

\[
= \frac{1}{(2\pi)^k} \int_{\partial S_{\alpha,\beta}^* + te} \frac{FB(\phi)(\sigma)W_n(\sigma)}{\zeta + \epsilon - \sigma} d\sigma = FB(\phi)(\zeta + \epsilon)W_n(\zeta + \epsilon),
\]

and it follows from the Lebesgue dominated convergence theorem that we have

\[
\frac{1}{(2\pi)^k} \int_{\partial S_{\alpha,\beta}^*} \frac{FB(\phi)(\sigma)W_n(\sigma)}{\zeta + \epsilon - \sigma} d\sigma = FB(\phi)(\zeta + \epsilon)W_n(\zeta + \epsilon).
\]

Taking the limit as \( n \to +\infty \), and using again the Lebesgue dominated convergence theorem, we obtain, for \( \zeta \in S_{\alpha,\beta}^* \),

\[
FB(\phi_\epsilon)(\zeta) = \frac{1}{(2\pi)^k} \int_{\partial S_{\alpha,\beta}^*} \frac{FB(\phi)(\sigma)}{\zeta + \epsilon - \sigma} d\sigma = FB(\phi)(\zeta + \epsilon)
\]

\[
= < e_{-\zeta}, e_{\epsilon}^* > = < e_{-\zeta}, \phi e_{-\epsilon} > = FB(\phi e_{-\epsilon})(\zeta),
\]

and it follows from the injectivity of the Fourier-Borel transform on \( U_{\alpha,\beta}' \) that \( \phi_\epsilon = \phi e_{-\epsilon} \).
This gives, for \( f \in \mathcal{V}_{\alpha,\beta}(X) \), since \( <f, e^*_{\sigma-\varepsilon}> = \mathcal{F}B(f)(-\sigma + \varepsilon) \) for \( \sigma \in \hat{S}^*_\alpha,\beta \),

\[
<fe^{-\varepsilon}, \phi> = <f, \phi e^{-\varepsilon}> = \frac{1}{(2\pi)^k} \int_{\hat{S}^*_\alpha,\beta} \mathcal{F}B(\phi)(\sigma) <f, e^*_{\sigma-\varepsilon}>
\]

\[
= \frac{1}{(2\pi)^k} \int_{\hat{S}^*_\alpha,\beta} \mathcal{F}B(\phi)(\sigma)\mathcal{F}B(f)(-\sigma + \varepsilon)d\sigma.
\]

\( \Box \)

For \( J \subset \{1, \ldots, k\} \), set \( P_{J,j} = \mathbb{C} \setminus \hat{S} - z_{\alpha,j}, \hat{S} - \alpha_j, \omega_{J,i} = \beta_j \) for \( j \in J, P_{J,j} = \mathbb{C} \setminus \hat{S} - z_{\alpha,j}, \hat{S} - \beta_j, \omega_{J,i} = \beta_j \) for \( j \in \{1, \ldots, k\} \setminus J \), and set \( P_J = \Pi_{1 \leq j \leq k}P_{J,j}, \omega_J = (\omega_{J,1}, \ldots, \omega_{J,k}) \). If \( f \in \mathcal{V}_{\alpha,\beta}(X) \), and if \( \int_{\hat{S}^*_\alpha,\beta} \|f(\xi)\|_X d\xi \) < \( +\infty \), then the formula \( \mathcal{F}B(f)(\sigma) = \int_0^\infty e^{-\xi} \mathcal{F}B(f)\xi d\sigma \) defines a continuous bounded extension of \( \mathcal{F}B(f) \) to \( P_J \). So in this situation \( \mathcal{F}B(f) \) has a continuous bounded extension to \( \bigcup_{J \subset \{1, \ldots, k\}} P_J = \Pi_{1 \leq j \leq k} \left( \mathbb{C} \setminus \hat{S}^*_\alpha,\beta,j \right) \). Applying formula (28) to the sequence \( (\epsilon_n) = \left( \frac{\pi}{n} \right) \) for some \( \epsilon \in S^*_\alpha,\beta \), we deduce from the Lebesgue dominated convergence theorem and from formula (23) the following result.

**Corollary 10.9** Let \( f \in \mathcal{V}_{\alpha,\beta}(X) \), and let \( \phi \in \mathcal{U}'_{\alpha,\beta} \). Assume that the following conditions are satisfied

\[
(i) \int_{\hat{S}^*_\alpha,\beta} \|f(\xi)\|_X d\xi < +\infty.
\]

\[
(ii) \int_{\hat{S}^*_\alpha,\beta} |\mathcal{F}B(\phi)(\sigma)| d\sigma < +\infty.
\]

Then

\[
< f, \phi > = \frac{1}{(2\pi)^k} \int_{\hat{S}^*_\alpha,\beta} \mathcal{F}B(\phi)(\sigma)\mathcal{F}B(f)(-\sigma)d\sigma. \tag{29}
\]

In the following we will denote by \( \hat{\nu} \) the functional \( f \to \int_{\hat{S}^*_\alpha,\beta} f(\xi)d\nu(\xi) \) for \( \nu \in \mathcal{M}(\hat{S}^*_\alpha,\beta) \). In order to give a way to compute \( <f, \phi> \) for \( \phi \in \mathcal{U}'_{\alpha,\beta}, f \in \mathcal{V}_{\alpha,\beta}(X) \), we will use the following easy observation.

**Proposition 10.10** Let \( \nu \) be a probability measure on \( \hat{S}^*_\alpha,\beta \), let \( R > 0 \), and let \( X \) be a separable Banach space. Set \( \nu_R(A) = \nu(RA) \) for every Borel set \( A \subset \hat{S}^*_\alpha,\beta \). Then \( \lim_{R \to +\infty} \|f_R - f\|_\infty = 0 \) for every \( f \in \mathcal{U}_{\alpha,\beta}(X) \).

**Proof:** Let \( f \in \mathcal{U}_{\alpha,\beta}(X) \). Then \( f \) is uniformly continuous on \( \hat{S}^*_\alpha,\beta \), and so for every \( \delta > 0 \) there exists \( r > 0 \) such that \( \|f(\xi + \eta) - f(\xi)\|_X < \delta \) for every \( \xi \in \hat{S}^*_\alpha,\beta \) and for every \( \eta \in \hat{S}^*_\alpha,\beta \cap \mathcal{B}(0, r) \). It follows from the Lebesgue dominated convergence theorem that \( \lim_{R \to +\infty} \nu_R(B(0, r)) = \lim_{R \to +\infty} \nu(B(0, rR)) = \nu(\hat{S}^*_\alpha,\beta) = 1 \). This gives

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\[
\lim_{R \to +\infty} \sup_{f \in \mathcal{B}(0,R)} \|f \|_\infty = \lim_{R \to +\infty} \sup_{f \in \mathcal{B}(0,R)} \left( \sup_{\zeta \in S_{\alpha,\beta}} \left\| \int_{S_{\alpha,\beta}} (f(\zeta + \eta) - f(\zeta))d\nu_R(\eta) \right\|_X \right)
\]

\[
\leq \lim_{R \to +\infty} \sup_{f \in \mathcal{B}(0,R)} \left( \sup_{\zeta \in S_{\alpha,\beta}} \int_{S_{\alpha,\beta}} (f(\zeta + \eta) - f(\zeta)) \|X\| \right)
\]

\[
+ 2\|f\|_{\infty} \lim_{R \to +\infty} \sup_{f \in \mathcal{B}(0,R)} \int_{S_{\alpha,\beta} \setminus (S_{\alpha,\beta} \cup B(0,R))} d\nu_R(\eta) \leq \delta.
\]

Hence \( \lim_{R \to +\infty} \|f_{\nu_R} - f\|_\infty = 0. \)

It follows from the definition of \( \nu_R \) that \( <f, \nu_R> = <f, \nu_t> \) for \( f \in \mathcal{V}_{\alpha,\beta}(X) \), where \( f_t(\zeta) = f(R^{-1}\zeta) \) (\( \zeta \in S_{\alpha,\beta} \)). In particular if \( \mathcal{F}(\nu_R) = \mathcal{F}(\nu_t) \) and \( (\nu_t)_R \ast (\nu_t)_R = (\nu_t \ast \nu_t)_R = (\nu_t \ast \nu_t)_R \) for \( R > 0 \) if \( \nu_t \) and \( \nu_t \) are two probability measures on \( S_{\alpha,\beta} \).

We deduce from proposition 10.8 and proposition 10.9 the following corollary, in which the sequence \( (W_n)_{n \geq 1} \) of functions on \( S_{\alpha,\beta} \) introduced in appendix 3 and used in the proof of proposition 10.8 allows to compute \( <f, \phi> \) for \( \phi \in \mathcal{U}_{\alpha,\beta}, f \in \mathcal{V}_{\alpha,\beta}(X) \) in the general case.

**Corollary 10.11** Set \( W_n(\zeta) = \Pi_{1 \leq j \leq k} \left( \frac{n^2}{n+\zeta e^{\alpha_j^\beta_j}} \right) \) for \( n \geq 1, \zeta = (\zeta_1, \ldots, \zeta_k) \in S_{\alpha,\beta} \). Then we have, for \( \phi \in \mathcal{U}_{\alpha,\beta}, f \in \mathcal{V}_{\alpha,\beta}(X) \),

\[
<f, \phi> = \lim_{n \to +\infty} \left( \lim_{n \to +\infty} \frac{1}{(2\pi)^k} \int_{\partial S_{\alpha,\beta}} W_n(\sigma) \mathcal{F}(\phi)(\sigma) \mathcal{F}(f)(\epsilon - \sigma) d\sigma \right).
\]

(30)

Proof: Define a measure \( \nu_0 \) on \( S_{\alpha,\beta} \) by using the formula

\[
<f, \nu_0> = \int_{[0, +\infty)^k} e^{-t_1 \ldots t_k} f(t_1 e^{\alpha_1^\beta_1}, \ldots, t_k e^{\alpha_k^\beta_k}) dt_1 \ldots dt_k \ (f \in \mathcal{C}(S_{\alpha,\beta})).
\]

Then \( \nu_0 \) and \( \nu = \nu_0 \ast \nu_0 \) are probability measures on \( S_{\alpha,\beta} \), and we have, for \( \zeta \in S_{\alpha,\beta} \),

\[
\mathcal{F}(\nu_0)(\zeta) = \int_{[0, +\infty)^k} e^{-t_1 \ldots t_k} e^{-t_1 \zeta e^{\alpha_1^\beta_1} - \ldots - t_k \zeta e^{\alpha_k^\beta_k}} dt_1 \ldots dt_k
\]

\[
= \Pi_{1 \leq j \leq k} \frac{1}{1 + \zeta e^{\alpha_j^\beta_j}}.
\]

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Hence $\mathcal{F}B(\tilde{\nu}) = \mathcal{F}B(\tilde{\nu}_n^2) = W_1$, and $\mathcal{F}B(\tilde{\nu}_n) = (W_1)_\alpha = W_\gamma$. It follows from (29) that we have, for $\epsilon \in S_{\alpha,\beta}$,

$$< fe_- \epsilon, \phi > = \lim_{n \to +\infty} < (fe_- \epsilon) \tilde{\nu}_n, \phi > = \lim_{n \to +\infty} < fe_- \epsilon, \phi * \tilde{\nu}_n >$$

$$= \lim_{n \to +\infty} \frac{1}{(2\pi)^k} \int_{\partial S_{\alpha,\beta}^*} \mathcal{F}B(\phi * \tilde{\nu}_n)(\sigma)\mathcal{F}B(f)(\epsilon - \sigma)d\sigma$$

$$= \lim_{n \to +\infty} \frac{1}{(2\pi)^k} \int_{\partial S_{\alpha,\beta}^*} W_n(\sigma)\mathcal{F}B(\phi)(\sigma)\mathcal{F}B(f)(\epsilon - \sigma)d\sigma,$$

and the result follows from the fact that $< f, \phi > = \lim_{\epsilon \to 0} < fe_- \epsilon, \phi >$.

\hspace{1cm} $\blacksquare$

11 Appendix 2: An algebra of fast-decreasing holomorphic functions on products of sectors and half-lines and its dual

In this section we will use the notations introduced in definition 4.1 for $\alpha = (\alpha_1, \ldots, \alpha_k) \in \mathbb{R}^k$ and $\beta = (\beta_1, \ldots, \beta_k) \in \mathbb{R}^k$ satisfying $\alpha_j \leq \beta_j < \alpha_j + \pi$ for $1 \leq j \leq k$. Notice that is $x \in \mathbb{C}, y \in \mathbb{C}$, there exists $z \in \mathbb{C}$ such that $(x + S_{\alpha_j,\beta_j}) \cap (y + S_{\alpha_j,\beta_j}) = z + S_{\alpha_j,\beta_j}$. Such a complex number $z$ is unique if $\alpha_j < \beta_j$. If $\alpha_j = \beta_j$, then $S_{\alpha_j,\beta_j} = S_{\alpha_j - \pi/2, \alpha_j + \pi/2}$ is a closed half-plane, the family $\{x + S_{\alpha_j,\beta_j} \}_{x \in \mathbb{C}}$ is linearly ordered with respect to inclusion and the condition $(x + S_{\alpha_j,\beta_j}) \cap (y + S_{\alpha_j,\beta_j}) = z + S_{\alpha_j,\beta_j}$ defines a real line of the form $z_0 + e^{i\alpha_j} \mathbb{R}$, where $z_0 \in \{x, y\}$.

The following partial preorder on $\mathbb{C}^k$ is the partial order associated to the cone $S_{\alpha,\beta}$ if $\alpha_j < \beta_j$ for $1 \leq j \leq k$.

**Definition 11.1** (i) For $z = (z_1, \ldots, z_k) \in \mathbb{C}^k$ and $z' = (z'_1, \ldots, z'_k) \in \mathbb{C}^k$, set $z \preceq z'$ if $z' \in z + S_{\alpha,\beta}$.

(ii) If $(z^{(j)})_{1 \leq j \leq m}$ is a finite family of elements of $\mathbb{C}^k$ denote by $\sup_{1 \leq j \leq m} z_j$ the set of all $z \in \mathbb{C}^k$ such that $\cap_{1 \leq j \leq m} (z^{(j)} + S_{\alpha,\beta}) = z + S_{\alpha,\beta}$.

For $z = (z_1, \ldots, z_k) \in \mathbb{C}^k$, set $e^z = (e^{z_1}, \ldots, e^{z_k})$, and denote again by $e^z : \mathbb{C}^k \to \mathbb{C}$ the map $(\zeta_1, \ldots, \zeta_k) \to e^{\zeta_1 + \ldots + \zeta_k}$.

It follows from (17) that $e^{-z} U_{\alpha,\beta} \subseteq e^{-z} U_{\alpha,\beta}$ if $z \preceq z'$.

For $f \in e^{-z} V_{\alpha,\beta}$, set $||f||_{e^{-z} V_{\alpha,\beta}} = ||e^z f||_\infty$, which defines a Banach space norm on $e^{-z} U_{\alpha,\beta}$ and $e^{-z} V_{\alpha,\beta}$.

**Proposition 11.2** (i) Set $\gamma_n = ne^{-i\frac{\pi \gamma}{2\pi}}$ for $n \geq 1$. Then the sequence $(\gamma_n)_{n \geq 1}$ is cofinal in $(\mathbb{C}^k, \preceq)$.
(ii) If \( z \leq z' \), then \( e^{-z}U_{\alpha,\beta} \) is a dense subset of \( (e^{-z}U_{\alpha,\beta}, \| \cdot \|_{e^{-z}U_{\alpha,\beta}}) \).

(iii) The set \( \cap_{z \in \mathbb{C}} e^{-z}U_{\alpha,\beta} \) is a dense ideal of \( U_{\alpha,\beta} \), which if a Fréchet algebra with respect to the family \( (\| \cdot \|_{e^{-\gamma}U_{\alpha,\beta}})_{n \geq 1} \).

(iv) If \( X \) is a separable Banach space, and if \( z \in \text{supp}_{1 \leq j \leq n} z^{(j)} \), then \( e^{-z}U_{\alpha,\beta}(X) = \bigcap_{1 \leq j \leq m} e^{-z^{(j)}}U_{\alpha,\beta}(X) = e^{-z}V_{\alpha,\beta}(X) = \bigcap_{1 \leq j \leq m} e^{-z^{(j)}}V_{\alpha,\beta}(X) \), and \( \| f \|_{e^{-z}V_{\alpha,\beta}} = \max_{1 \leq j \leq m} \| f \|_{e^{-z^{(j)}}V_{\alpha,\beta}} \) for \( f \in e^{-z}V_{\alpha,\beta} \).

Proof: (i) Let \( z = (z_1, \ldots, z_k) \in \mathbb{C} \), and let \( j \leq k \). Since \( (\frac{\pi}{2} - \beta_j) + (-\frac{\pi}{2} - \alpha_j) = -(\alpha_j + \beta_j), \) \( t_0 j e^{-i(\alpha_j + \beta_j) j} \in \partial(z_j + \sum_{i,j} \alpha_i j) \) for some \( t_0, j \in \mathbb{R} \), so \( t e^{-i(\alpha_j + \beta_j) j} \in \sum_{i,j} \alpha_i j \) for every \( t \geq t_0, j \), and (i) follows.

(ii) Assume that \( z \leq z' \). The fact that \( e^{-z}U_{\alpha,\beta} \subset e^{-z}U_{\alpha,\beta} \) follows from (16). Let \( z'' \in z' + S^*_\alpha,\beta \subset z + S^*_\alpha,\beta \). We have \( z'' = z + \rho e^{i\phi} \) where \( \rho > 0 \), and where \( \eta = (\eta_1, \ldots, \eta_k) \) satisfies \( -\frac{\pi}{2} - \alpha_j < \eta_j < \frac{\pi}{2} - \beta_j \) for \( j \leq k \).

The semigroup \( (e^{-re^{i\phi}z})_{t \geq 0} \) is analytic and bounded in the Banach algebra \( U_{\alpha,\beta} \), and \( \lim_{t \to 0^+} \| f - f e^{-re^{i\phi}z} \| = 0 \) for every \( f \in U_{\alpha,\beta} \).

It follows then from the analyticity of this semigroup that \( [e^{-re^{i\phi}z}]^{-1} = U_{\alpha,\beta} \). Hence \( e^{-z}U_{\alpha,\beta} \) is dense in \( e^{-z}U_{\alpha,\beta} \), which proves (ii) since \( e^{-z}U_{\alpha,\beta} \subset e^{-z}U_{\alpha,\beta} \).

(iii) Denote by \( \iota : z \mapsto f \) the inclusion map from \( e^{-z}U_{\alpha,\beta} \) into \( e^{-z}U_{\alpha,\beta} \) for \( z \leq z' \). Equipped with these maps, the family \( (e^{-z}U_{\alpha,\beta})_{z \in \mathbb{C}} \) is a projective system of Banach spaces, and we can identify \( \bigcap_{z \in \mathbb{C}} e^{-z}U_{\alpha,\beta}, (\| \cdot \|_{e^{-z}U_{\alpha,\beta}})_{z \in \mathbb{C}} \) to the inverse limit of this system, which defines a structure of complete locally convex topological space on \( \bigcap_{z \in \mathbb{C}} e^{-z}U_{\alpha,\beta}, (\| \cdot \|_{e^{-z}U_{\alpha,\beta}})_{z \in \mathbb{C}} \). It follows from (i) that the sequence \( (\| \cdot \|_{e^{-\gamma}U_{\alpha,\beta}})_{n \geq 1} \) of norms defines the same topology as the family \( (\| \cdot \|_{e^{-\gamma}U_{\alpha,\beta}})_{z \in \mathbb{C}} \) on \( \bigcap_{z \in \mathbb{C}} e^{-z}U_{\alpha,\beta}, \cap_{n \geq 1} e^{-\gamma}U_{\alpha,\beta} \), which defines a Fréchet algebra structure on \( \bigcap_{z \in \mathbb{C}} e^{-z}U_{\alpha,\beta} \).

It follows from (ii) that \( e^{-\gamma}U_{\alpha,\beta} \) is dense in \( e^{-\gamma}U_{\alpha,\beta} \) for \( n \geq 0 \), and a standard application of the Mittag-Leffler theorem of projective limits of complete metric spaces, see for example theorem 2.14 of [13], shows that \( \cap_{z \in \mathbb{C}} e^{-z}U_{\alpha,\beta} = \cap_{n \geq 1} e^{-\gamma}U_{\alpha,\beta} \) is dense in \( e^{-\gamma}U_{\alpha,\beta} = U_{\alpha,\beta} \).

(iv) Let \( z = (z_1, \ldots, z_k) \in \mathbb{C} \), let \( z' = (z'_1, \ldots, z'_k) \in \mathbb{C} \), and let \( z'' = z''_1, \ldots, z''_k \) in \( \sup(z, z') \). Then \( e^{-z''} \in \mathbb{V}_{\alpha,\beta}, e^{-z''} \in \mathbb{V}_{\alpha,\beta}, \| e^{-z''} \| \leq 1, \| e^{-z''} \| = 1 \), and so \( e^{-z''}U_{\alpha,\beta}(X) \subset e^{-z''}U_{\alpha,\beta}(X) \subset e^{-z''}U_{\alpha,\beta}(X) \subset e^{-z''}U_{\alpha,\beta}(X) \subset e^{-z''}U_{\alpha,\beta}(X) \), and max(\( \| f \|_{e^{-z''}U_{\alpha,\beta}(X)}, \| f \|_{e^{-z''}U_{\alpha,\beta}(X)} \) for \( f \in e^{-z''}U_{\alpha,\beta}(X) \). We claim that \( \| e^{z''} \| \leq \min(\| e^{z''} \|, \| e^{z''} \|) \) for \( \zeta \in \delta S_{\alpha,\beta,\beta,\zeta} \), or if \( z_j' \in S_{\zeta,\beta,\zeta} \), this is obviously true. Otherwise we have \( \alpha_j < \beta_j \) and, say, \( z'' = z_j + \rho e^{i(\alpha_j - \beta_j + \frac{\pi}{2})} \), with \( \rho > 0, \| e^{-z''} \| \leq 1 \). Let \( \zeta = e^{i\theta} \in S_{\alpha,\beta,\beta,\zeta} \), where \( \rho, \theta \in [\alpha, \beta] \). We have \( \text{Re}(z'' - z) \zeta = \rho \cos(\theta - \alpha_j), \) \( \text{Im}(z'' - z) \zeta = \rho \sin(\theta - \alpha_j) \), and \( \rho \cos(\theta - \beta_j), \rho \sin(\theta - \beta_j) \zeta = \rho \cos(\theta - \beta_j + \frac{\pi}{2}) \). So \( \| e^{z''} \| \leq \| e^{z''} \| \leq \| e^{z''} \| \leq \| e^{z''} \| \leq \| e^{z''} \| \) if \( \theta = \alpha_j \), and \( \| e^{z''} \| \leq \| e^{z''} \| \leq \| e^{z''} \| \) if \( \theta = \beta_j \), which proves the claim.

We now use the Phragmén-Lindelöf principle. Let \( s \subset \cup_{1 \leq j \leq k} (1, \beta_j, \infty) \) and for \( 1 \leq j \leq k \) let \( \zeta_j^* \) be a continuous determination of the s-power of \( \zeta \).
on $S_{\alpha_j, \beta_j}$ which is holomorphic on $S_{\alpha_j, \beta_j}$ when $\alpha_j < \beta_j$. Set $\zeta^s = \zeta^1 \cdots \zeta^k$ for $\zeta \in S_{\alpha, \beta}$. Let $f \in e_{-\zeta} V_{\alpha, \beta}(X) \cap e_{-\zeta'} V_{\alpha, \beta}(X)$, and let $\epsilon > 0$. Set $g_\epsilon(s) = e^{-\epsilon s} f(\zeta)$ for $\zeta \in S_{\alpha, \beta}$. It follows from the claim that $g_\epsilon \in U_{\alpha, \beta}(X)$, and it follows from the maximum modulus principle that there exists $\zeta_0 \in \partial S_{\alpha, \beta}$ such that $\|g_\epsilon\|_{U_{\alpha, \beta}(X)} = g_\epsilon(\zeta_0)$ for every $\epsilon > 0$ and $\zeta \in S_{\alpha, \beta}$.

Now let $f \in e_{-\zeta} V_{\alpha, \beta}(X) \cap e_{-\zeta'} V_{\alpha, \beta}(X)$. Then $f \in e_{-\zeta} V_{\alpha, \beta}$, and $\lim_{\epsilon \to 0} \|e^{-\epsilon} g(\zeta)\| = 0$. The Banach algebra $U_{\alpha, \beta}$ possesses a bounded approximate identity $(g_n)_{n \geq 1}$, one can take for example $g_n(\zeta_1, \ldots, \zeta_n) = \Pi_{1 \leq j \leq k} \frac{n \zeta_j}{n \zeta_j + e^{-\epsilon \zeta_j}}$. We have

$$\lim_{n \to +\infty} \|e^{-\epsilon} g_n - e^{-\epsilon} f\| = \lim_{n \to +\infty} \max_{\zeta \in S_{\alpha, \beta}} \|e^{-\epsilon} f(\zeta) g_n(\zeta) - e^{-\epsilon} f(\zeta)\| = 0,$$

and so $e^{-\epsilon} f \in U_{\alpha, \beta}$ since $U_{\alpha, \beta}$ is a closed subalgebra of $V_{\alpha, \beta}$. This concludes the proof of (iv) when $m = 2$. The general case follows by an immediate induction, since $\sup(\zeta, z(0)) = \sup_{1 \leq j \leq k} z(j)$ for every $\zeta \in \sup_{1 \leq j \leq k} z(j)$ if $(z^1, \ldots, z^0)$ is a finite family of elements of $\mathbb{C}^k$.

Notice that assertions (ii) and (iii) of the proposition do not extend to the case where $\beta_j = \alpha_j + \pi$ for some $j \leq k$. It suffices to consider the case where $\alpha_j = \frac{\pi}{2}$, $\beta_j = \frac{\pi}{2}$. Set $\lambda_{j,t}(s) = (\lambda_{j,t})_{1 \leq s \leq k}$, where $\lambda_{s,t} = 0$ for $s \neq j$ and $\lambda_{j,t} = t$. Then the map $f \to e_{-\lambda_{j,t}} f$ is an isometry on $U_{\alpha, \beta}$ for every $t \geq 0$ and $\cap_{t \geq 0} u_{\lambda_{j,t}} U_{\alpha, \beta} = \{0\}$ since the zero function is the only bounded holomorphic function $f$ on the right-hand open half-plane satisfying $\lim_{t \to +\infty} |e^{it} f(t)| = 0$ for every $t > 0$.

Let $i_{\zeta} : f \to f$ be the inclusion map from $\cap_{z \in \mathbb{C} e_{-\zeta} U_{\alpha, \beta}}$ into $e_{-\zeta} U_{\alpha, \beta}$. Since $i_{\zeta}$ has dense range, the map $i_{\zeta}^* : \phi \to \phi_{|z \in \mathbb{C} e_{-\zeta} U_{\alpha, \beta}}$ is a one-to-one map from $(e_{-\zeta} U_{\alpha, \beta})'$ into $\cap_{z \in \mathbb{C} e_{-\zeta} U_{\alpha, \beta}}'$, which allows to identify $(e_{-\zeta} U_{\alpha, \beta})'$ to a subset of $(\cap_{z \in \mathbb{C} e_{-\zeta} U_{\alpha, \beta}}')'$, so that we have

$$\langle n \in \mathbb{C} e_{-\zeta} U_{\alpha, \beta} \rangle' = \cup_{n \geq 1} (e_{-\zeta} U_{\alpha, \beta})' = \cup_{n \geq 1} (e_{-\zeta} U_{\alpha, \beta})'.$$

**Definition 11.3** Set $F_{\alpha, \beta} := (\cap_{z \in \mathbb{C} e_{-\zeta} U_{\alpha, \beta}}')$. Let $\phi \in F_{\alpha, \beta}$, and let $X$ be a separable Banach space.

(i) The domain of the Fourier-Borel transform of $\phi$ is defined by the formula

$$\text{Dom}(\mathcal{F}B(\phi)) := \{ z \in \mathbb{C}^k \mid \phi \in (e_{-\zeta} U_{\alpha, \beta})' \}.$$

(ii) For $z \in \text{Dom}(\mathcal{F}B(\phi))$ the functional $\phi_{-z} \in U_{\alpha, \beta}'$ is defined by the formula

$$\langle f, \phi_{-z} \rangle = \langle e_{-z} f, \phi \rangle \quad (f \in U_{\alpha, \beta}).$$

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and \( < g, \phi > \) is defined for \( g \in e_{-z}V_{\alpha,\beta}(X) \) by the formula
\[
< g, \phi > = \int e_{-z}g \, \phi e_{-z} \mathrm{d}v.
\]

(iii) The Fourier-Borel transform of \( \phi \) is defined for \( z \in \text{Dom}(FB(\phi)) \) by the formula
\[
FB(\phi)(z) = < e_{-z}, \phi >.
\]

(iv) The \( z \)-Cauchy transform of \( \phi \) is defined on \( C^k \setminus \overline{S_{\alpha,\beta}} \) for \( z \in \text{Dom}(FB(\phi)) \) by the formula
\[
C_z(\phi) = C(\phi e_{-z}).
\]

(v) If \( z \in \text{Dom}(FB(\phi)) \) a measure \( \nu \) of bounded variation on \( \overline{S_{\alpha,\beta}} \) is said to be a \( z \)-representing measure for \( \phi \) if \( \nu \) is a representing measure for \( \phi e_{-z} \).

Since the map \( \zeta \rightarrow e_{-\zeta} \) is holomorphic on \( S^*_{\alpha,\beta} \), the map \( z \rightarrow e_{-z} \) is a holomorphic map from \( \lambda + S^*_{\alpha,\beta} \) into \( e_{-z}M_{\alpha,\beta} \) for every \( \lambda \in \text{Dom}(FB(\phi)) \), and so \( FB(\phi) \) is holomorphic on the interior of \( \text{Dom}(FB(\phi)) \) for \( \phi \in F_{\alpha,\beta} \). Also the \( z \)-Cauchy transform \( C_z(\phi) \) is holomorphic on \( C \setminus \overline{S_{\alpha,\beta}} \) for every \( z \in \text{Dom}(FB(\phi)) \). Notice also that if \( \phi \in U_{\alpha,\beta} \), then \( \overline{S_{\alpha,\beta}} \subset \text{Dom}(FB(\phi)) \) and so the function \( FB(\phi) \) defined above is an extension to \( \text{Dom}(FB(\phi)) \) of the Fourier-Borel transform already introduced in definition 10.3 on \( \overline{S_{\alpha,\beta}} \).

Now let \( z \in \text{Dom}(FB(\phi)), z' \in \text{Dom}(FB(\phi)) \) and assume that \( g \in e_{-z}V_{\alpha,\beta}(X) \cap e_{-z'}V_{\alpha,\beta}(X) \). Let \( z'' = \text{sup}(z, z') \subset \text{Dom}(FB(\phi)) \). Then \( g \in e_{-z''}V_{\alpha,\beta}(X) \). Let \( \nu \) be a \( z \)-representative measure for \( \phi \). We have, for \( h \in \cap_{\lambda \in C^k} e_{-\lambda}M_{\alpha,\beta} \), since \( e_{-z} = e_{-z''}e_{z''-z} \),
\[
< h, \phi > = \int_{S_{\alpha,\beta}} e_{z''}(\zeta) h(\zeta) d\nu(\zeta) = \int_{S_{\alpha,\beta}} e_{z''}(\zeta) h(\zeta) e_{z''-z''}(\zeta) d\nu(\zeta).
\]

Since \( e_{z''} \nu \) is a measure of bounded variation on \( S_{\alpha,\beta} \), \( e_{z-z''} \nu \) is a \( z'' \)-representative measure for \( \phi \). Similarly if \( \nu' \) is a \( z' \)-representative measure for \( \phi \) then \( e_{z-z''} \nu' \) is a \( z'' \)-representative measure for \( \phi \), and we have
\[
\int_{S_{\alpha,\beta}} e_{z''}(\zeta) g(\zeta) d\nu(\zeta) = \int_{S_{\alpha,\beta}} e_{z''}(\zeta) g(\zeta) e_{z-z''}(\zeta) d\nu(\zeta)
\]
\[
= \int_{S_{\alpha,\beta}} e_{z''}(\zeta) g(\zeta) e_{z'-z''}(\zeta) d\nu'(\zeta) = \int_{S_{\alpha,\beta}} e_{z'}(\zeta) g(\zeta) d\nu'(\zeta),
\]

which shows that the definition of \( < g, \phi > \) does not depend on the choice of \( z \in \text{Dom}(FB(\phi)) \) such that \( g \in e_{-z}V_{\alpha,\beta}(X) \).

**Proposition 11.4** Let \( \phi \in F_{\alpha,\beta} \).

(i) The set \( \text{Dom}(FB(\phi)) \) is connected.

(ii) \( z + \overline{S_{\alpha,\beta}} \subset \text{Dom}(FB(\phi)) \), and \( FB(\phi) \) is continuous on \( z + \overline{S_{\alpha,\beta}} \) and holomorphic on \( z + S^*_{\alpha,\beta} \) for every \( z \in \text{Dom}(FB(\phi)) \).
Proof: (i) The fact that \( Dom(\mathcal{F}B(\phi)) \) is connected follows from the fact that the arcwise connected set \((z_1 + \mathcal{S}^*_\alpha,\beta) \cup (z_2 + \mathcal{S}^*_\alpha,\beta) \) is contained in \( Dom(\mathcal{F}B(\phi)) \) for \( z_1 \in Dom(\mathcal{F}B(\phi)); z_2 \in Dom(\mathcal{F}B(\phi)) \).

(ii) Let \( z \in Dom(\mathcal{F}B(\phi)) \). It follows from (16) that \( z + \mathcal{S}^*_\alpha,\beta \subset Dom(\mathcal{F}B(\phi)) \) and so \( \mathcal{F}B(\phi) \) is holomorphic on the open set \( z + \mathcal{S}^*_\alpha,\beta \subset Dom(\mathcal{F}B(\phi)) \). Let \( \nu \) be a measure of bounded variation on \( \mathcal{S}^*_\alpha,\beta \) which is \( z \)-representing measure for \( \phi \). We have, for \( \eta \in \mathcal{S}^*_\alpha,\beta \),

\[
\mathcal{F}B(\phi)(z + \eta) = < e^{-z-\eta}, \phi > = < e^{-\eta}, \phi e^{-z} > \int_{\mathcal{S}^*_\alpha,\beta} e^{-\eta \zeta} d\nu(\zeta),
\]

and the continuity of \( \mathcal{F}B(\phi) \) on \( z + \mathcal{S}^*_\alpha,\beta \) follows from the Lebesgue dominated convergence theorem.

Notice that \( Dom(\mathcal{F}B(\phi)) \) is not closed in general: for example if we set \(< f, \phi > = \int_{S^*_\alpha,\beta} \zeta f(\zeta) d\mu(\zeta) \) for \( f \in \cap_{z \in \mathcal{C}^k} e^{-z} \mathcal{U}_{\alpha,\beta} \), where \( m \) denotes the Lebesgue measure on \( \mathcal{C} \), then \( t \in Dom(\mathcal{F}B(\phi)) \) for every \( t > 0 \), but \( 0 \notin Dom(\mathcal{F}B(\phi)) \). Notice also that if \( \nu \) is a measure supported by a compact subset of \( S_{\alpha,\beta} \), and if we set \(< f, \phi > := \int_{\alpha,\beta} f(\zeta) d\nu(\zeta) \) for \( f \in \cap_{z \in \mathcal{C}^k} e^{-z} \mathcal{U}_{\alpha,\beta} \), then \( \phi \in \cap_{z \in \mathcal{C}^k} (e^{-z} \mathcal{U}_{\alpha,\beta})' \), so that \( Dom(\mathcal{F}B(\phi)) = \mathbb{C}^k \), and \( \mathcal{F}B(\phi) \) is the entire function defined on \( \mathbb{C}^k \) by the formula

\[
\mathcal{F}B(\phi)(z) = \int_{S_{\alpha,\beta}} e^{-z \zeta} d\nu(\zeta).
\]

We now introduce the convolution product of elements of \( \mathcal{F}_{\alpha,\beta} \). If \( \phi \in \mathcal{F}_{\alpha,\beta}, f \in \cap_{z \in \mathcal{C}^k} e^{-z} \mathcal{U}_{\alpha,\beta} \), \( \lambda \in \mathcal{S}_{\alpha,\beta} \), set again \( f_\lambda(\zeta) = f(\zeta + \lambda) \) for \( \zeta \in \mathcal{S}_{\alpha,\beta} \).

Then \( f_\lambda \in \cap_{z \in \mathcal{C}^k} e^{-z} \mathcal{U}_{\alpha,\beta} \), and we can compute \(< f_\lambda, \phi > \). The map \( \lambda \to f_\lambda \) is a continuous map from \( \mathcal{S}_{\alpha,\beta} \) into the Fréchet algebra \( \cap_{z \in \mathcal{C}^k} e^{-z} \mathcal{U}_{\alpha,\beta} \) which is holomorphic on \( S_{\alpha,\beta} \). We obtain

**Lemma 11.5** Let \( \phi \in \mathcal{F}_{\alpha,\beta} \). Then the function \( f_\phi : \lambda \to < f_\lambda, \phi > \) belongs to \( \cap_{z \in \mathcal{C}^k} e^{-z} \mathcal{U}_{\alpha,\beta} \) for every \( f \in \cap_{z \in \mathcal{C}^k} e^{-z} \mathcal{U}_{\alpha,\beta} \), and the linear map \( f \to f_\phi \) is continuous on \( \cap_{z \in \mathcal{C}^k} e^{-z} \mathcal{U}_{\alpha,\beta} \).

Proof: Let \( f \in \cap_{z \in \mathcal{C}^k} e^{-z} \mathcal{U}_{\alpha,\beta} \), let \( z_0 \in Dom(\mathcal{F}B(\phi)) \), let \( \nu \) be a \( z_0 \)-representing measure for \( \phi \) on \( \mathcal{S}_{\alpha,\beta} \), and let \( z \in \mathcal{C}^k \). Let \( z_1 \in \sup(z_0, z) \), so that \( (z_0 + \mathcal{S}_{\alpha,\beta}) \cap (z + \mathcal{S}_{\alpha,\beta}) = z_1 + \mathcal{S}_{\alpha,\beta} \), and set \( \eta_0 = z_1 - z_0, \eta = z_1 - z \). We have, for \( \lambda \in \mathcal{S}_{\alpha,\beta} \),

\[
e^{z_\lambda} < f_\lambda, \phi > = \int_{\mathcal{S}_{\alpha,\beta}} e^{z_\lambda + \eta \zeta} f(\zeta + \lambda) d\nu(\zeta) = \int_{\mathcal{S}_{\alpha,\beta}} e^{-\eta_0 \zeta + \eta \zeta} e^{z_\lambda + \eta_0 \zeta} f(\zeta + \lambda) d\nu(\zeta).
\]

Since \( |e^{-\eta_\lambda - \eta_0 \zeta} f(\zeta + \lambda)| \leq ||e^{z_\lambda} f||_\infty \), it follows from Lebesgue’s dominated convergence theorem that \( \lim_{\lambda \to +\infty} |e^{z_\lambda} < f_\lambda, \phi > | = 0 \), and so
$f_\phi \in \cap_{z \in \mathbb{C}} e^{-z} U_{\alpha, \beta}$. Also $\|e_2 f_\phi\|_\infty \leq \|e_2 f\|_\infty \int_{S_{\alpha, \beta}} d|\nu|(\zeta)$, which shows that the map $f \to f_\phi$ is continuous on $\cap_{z \in \mathbb{C}} e^{-z} U_{\alpha, \beta}$. □

Notice that it follows from the Hahn-Banach theorem that given $\phi \in (e^{-z} U_{\alpha, \beta})'$ there exists a $z_0$-representing measure $\nu$ for $\phi$ such that $\int_{S_{\alpha, \beta}} d|\nu|(\zeta) = \|\phi\|_{(e^{-z_0} U_{\alpha, \beta})'}$.

The calculation above shows then that we have, for $z \in \mathbb{C}$, $f \in \cap_{z \in \mathbb{C}} e^{-z} U_{\alpha, \beta}$, $\phi \in F_{\alpha, \beta}$, $z_0 \in \text{Dom}(F_{\alpha, \beta})$, $z_1 \in \text{sup}(z_0, z)$,

$$\|e_2 f_\phi\|_\infty \leq \|e_2 f\|_\infty \|\phi\|_{(e^{-z_0} U_{\alpha, \beta})'}.$$  

(32)

**Proposition 11.6** For $\phi_1 \in F_{\alpha, \beta}, \phi_2 \in F_{\alpha, \beta}$, define the convolution product $\phi_1 \ast \phi_2 \in F_{\alpha, \beta}$ by the formula

$$< f, \phi_1 \ast \phi_2 > = < f \phi_1, \phi_2 > \text{ } (f \in \cap_{z \in \mathbb{C}} e^{-z} U_{\alpha, \beta}).$$

Then sup$(z_1, z_2) \subset \text{Dom}(F_{\alpha, \beta}(\phi_1 \ast \phi_2))$ for $z_1 \in \text{Dom}(F_{\alpha, \beta}(\phi_1))$, $z_2 \in \text{Dom}(F_{\alpha, \beta}(\phi_2))$, and we have, for $z \in \text{sup}(z_1, z_2)$,

$$\|\phi_1 \ast \phi_2\|_{(e^{-z} U_{\alpha, \beta})'} \leq \|\phi_1\|_{(e^{-z_0} U_{\alpha, \beta})'} \|\phi_2\|_{(e^{-z_0} U_{\alpha, \beta})'}.$$ 

More generally Dom$(F_{\alpha, \beta}(\phi_1)) \cap \text{Dom}(F_{\alpha, \beta}(\phi_2)) \subset \text{Dom}(F_{\alpha, \beta}(\phi_1 \ast \phi_2))$, and if $z \in \text{Dom}(F_{\alpha, \beta}(\phi_1)) \cap \text{Dom}(F_{\alpha, \beta}(\phi_2))$ then $\phi_1 \ast \phi_2 e^{-z} = (\phi_1 e^{-z}) \ast (\phi_2 e^{-z})$, so that $\nu_1 \ast \nu_2$ is a $z$-representative measure for $\phi_1 \ast \phi_2$ if $\nu_1$ is a $z$-representative measure for $\phi_1$ and if $\nu_2$ is a $z$-representative measure for $\nu_2$, and we have

$$F_{\alpha, \beta}(\phi_1 \ast \phi_2)(z) = F_{\alpha, \beta}(\phi_1)(z) F_{\alpha, \beta}(\phi_2)(z) \text{ } (z \in \text{Dom}(F_{\alpha, \beta}(\phi_1)) \cap \text{Dom}(F_{\alpha, \beta}(\phi_2)).$$

Proof: Let $z_1 \in \text{Dom}(F_{\alpha, \beta}(\phi_1))$, let $z_2 \in \text{Dom}(F_{\alpha, \beta}(\phi_2))$, and let $z \in \text{sup}(z_1, z_2)$. It follows from (32) that we have, for $f \in \cap_{z \in \mathbb{C}} e^{-z} U_{\alpha, \beta}$,

$$< f, \phi_1 \ast \phi_2 > = < f \phi_1, \phi_2 > \leq \|e_2 f\|_\infty \|\phi_1\|_{(e^{-z_0} U_{\alpha, \beta})'} \|\phi_2\|_{(e^{-z_0} U_{\alpha, \beta})'}.$$ 

Hence $\phi_1 \ast \phi_2 \in F_{\alpha, \beta}$, sup$(z_1, z_2) \subset \text{Dom}(F_{\alpha, \beta}(\phi_1 \ast \phi_2))$, and $\|\phi_1 \ast \phi_2\|_{(e^{-z} U_{\alpha, \beta})'} \leq \|\phi_1\|_{(e^{-z_0} U_{\alpha, \beta})'} \|\phi_2\|_{(e^{-z_0} U_{\alpha, \beta})'}$ for $z \in \text{sup}(z_1, z_2)$.

Let $z \in \text{Dom}(F_{\alpha, \beta}(\phi_1)) \cap \text{Dom}(F_{\alpha, \beta}(\phi_2))$. Then $z \in \text{sup}(z_1, z_2) \subset \text{Dom}(F_{\alpha, \beta}(\phi_1 \ast \phi_2))$.

Let $\nu_1$ be a $z$-representative measure for $\phi_1$ and let $\nu_2$ be a $z$-representative measure for $\phi_2$. We have, for $f \in \cap_{z \in \mathbb{C}} e^{-z} U_{\alpha, \beta}$,

$$< f, \phi_1 \ast \phi_2 > = < f \phi_1, \phi_2 > = \int_{S_{\alpha, \beta}} e^{z\lambda} f_\phi(\lambda) d\nu_2(\lambda)$$

$$= \int_{S_{\alpha, \beta}} \int_{S_{\alpha, \beta}} e^{z\lambda} f(\lambda + \zeta) d\nu_1(\lambda) e^{z\zeta} d\nu_2(\lambda)$$

$$= \int \int_{S_{\alpha, \beta} \times S_{\alpha, \beta}} e^{z(\lambda + \zeta)} f(\zeta + \lambda) d\nu_1(\zeta) d\nu_2(\lambda) = \int_{S_{\alpha, \beta}} e^{z\lambda} d(\nu_1 \ast \nu_2)(s),$$
and so \( \nu_1 \ast \nu_2 \) is a \( z \)-representative measure for \((\phi_1 \ast \phi_2)e_{-z}\), which means that \( \nu_1 \ast \nu_2 \) is a \( z \)-representative measure for \( \phi_1 \ast \phi_2 \). Since \( \nu_1 \) is a representative measure for \( \phi_1 e_{-z} \), and since \( \phi_2 \) is a representative measure for \( \phi_2 e_{-z} \), it follows from proposition 10.5 (ii) that \((\phi_1 \ast \phi_2)e_{-z} = (\phi_1 e_{-z}) \ast (\phi_2 e_{-z})\).

It follows also from proposition 10.5(ii) that

\[
\mathcal{F}B((\phi_1 \ast \phi_2)(z)) = \mathcal{F}B((\phi_1 \ast \phi_2)e_{-z})(1) = \mathcal{F}B((\phi_1 e_{-z}) \ast (\phi_2 e_{-z}))(1)
\]

\[
= \mathcal{F}B(\phi_1 e_{-z})(1) \mathcal{F}B(\phi_2 e_{-z})(1) = \mathcal{F}B(\phi_1)(z) \mathcal{F}B(\phi_2)(z).
\]

□

Using proposition 10.4, we obtain the following link between \( z \)-Cauchy transforms and Fourier-Borel transforms of elements of \( \mathcal{F}_{\alpha,\beta} \).

**Proposition 11.7** Let \( \phi \in \mathcal{F}_{\alpha,\beta} \). For \( j \leq k \), set \( I_{\eta,j} = (\frac{\eta}{2} - \eta, \frac{\eta}{2} - \eta) \) for \( \eta \in (\beta_j, \alpha_j + \pi) \), \( I_{\eta,j} = (\frac{\eta}{2} - \alpha_j, \frac{\eta}{2} - \alpha_j) \) for \( \eta \in (\alpha_j + \pi, \beta_j + \pi) \), and set \( I_{\eta} = (\frac{\eta}{2} - \alpha_j, \frac{\eta}{2} - \alpha_j) \) for \( \eta \in (\beta_j + \pi, \alpha_j + 2\pi) \). Then \( I_{\eta,j} \subset [-\frac{\pi}{2} - \alpha_j, \frac{\pi}{2} - \alpha_j] \), \( \cos(\eta + s) < 0 \) for \( s \in I_{\eta,j} \), and if \( \lambda = (\lambda_1, \ldots, \lambda_k) \in \mathbb{C}^k \setminus S_{\alpha,\beta}^\ast \), we have for \( \omega = (\omega_1, \ldots, \omega_k) \in \Pi_{1 \leq j \leq k} I_{\arg(\lambda_j),j}, z \in \text{Dom}(\mathcal{F}B(\phi)) \),

\[
C_z(\phi)(\lambda) = \frac{1}{(2\pi)^k} \int_0^{\infty} e^{\lambda z} \mathcal{F}B(\phi)(\sigma + z) d\sigma
\]

\[
:= \frac{1}{(2\pi)^k} \int_0^{\infty} \cdots \int_0^{\infty} e^{\lambda z} \mathcal{F}B(\phi)(\sigma + z) d\sigma. \tag{33}
\]

**Proof:** We have \( C_z(\phi) = \mathcal{C}(\phi e_{-z}) \), and, for \( \sigma \in S_{\alpha,\beta}^\ast \),

\[
\mathcal{F}B(\phi)(\sigma + z) = < e_{-\sigma} e_{-z}, \phi >= < e_{-\sigma} e_{-z}, \phi > = < e_{-\sigma}, \phi e_{-z} > = \mathcal{F}B(\phi e_{-z})(\sigma).
\]

Applying formula (22) to \( \phi e_{-z} \), we obtain (33). □

Let \( X \) be a separable Banach space. For \( \eta \in S_{\alpha,\beta}^\ast \), \( z \in \mathbb{C}^k \), \( f \in e_{-z} \mathcal{V}_{\alpha,\beta}(X) \), set \( f_\eta(\zeta) = f(\zeta + \eta) \) (\( \zeta \in S_{\alpha,\beta}^\ast \)). If \( \phi \in \mathcal{F}_{\alpha,\beta} \), and if \( z \in \text{Dom}(\mathcal{F}B(\phi)) \), we have

\[
< f, \phi \ast \delta_\eta >= < e_z f, (\phi \ast \delta_\eta)e_{-z} >= < e_z f, (\phi e_{-z}) \ast (\delta_\eta e_{-z}) >= e^{-z\eta} < e_z f, (\phi e_{-z}) \ast \delta_\eta > = e^{-z\eta} < e_z f_\eta, (\phi e_{-z}) >= < e_z f_\eta, \phi e_{-z} >= < f_\eta, \phi >.
\]

We also have, for \( f \in e_{-z} \mathcal{U}_{\alpha,\beta}(X) \),

\[
\lim_{\eta \to 0} \| f_\eta - f \|_{e_{-z} \mathcal{U}_{\alpha,\beta}(X)} = \lim_{\eta \to 0} \sup_{\zeta \in S_{\alpha,\beta}^\ast} \| e^{z\zeta} f(\zeta + \eta) - e^{z\zeta} f(\zeta) \|_\infty
\]

\[
\leq \lim_{\eta \to 0} \left( \| (e_z f)_\eta - e_z f \|_\infty + |1 - e^{-z\eta}| \| (e_z f)_\eta \|_\infty \right) = 0,
\]

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and so, since \((e^{-\epsilon}f)_{\eta} = e^{-\epsilon\eta}e^{-\epsilon}f_{\eta}\),

\[
\lim_{\eta \to 0, \eta \in S_{\alpha,\beta}} \|(e^{-\epsilon}f)_{\eta} - f\|_{S_{\alpha,\beta}} = 0 \quad (f \in e^{-z}U_{\alpha,\beta}(X), z \in \mathbb{C}^k)
\]  

(34)

Now let \(f \in e^{-z}V_{\alpha,\beta}(X)\), and let \(\phi \in (e^{-z}U_{\alpha,\beta})'\). If \(\nu\) is a \(z\)-representative measure for \(\phi\), we have, for \(\eta \in S_{\alpha,\beta}, \epsilon \in S_{\alpha,\beta}^*\),

\[
< (e^{-\epsilon}f)_{\eta}, \phi > = e^{-\epsilon\eta} < e^{-\epsilon}f_{\eta}, \phi > = e^{-(\epsilon+z)\eta} \int_{\partial S_{\alpha,\beta}} e^{-z(\sigma - \epsilon \eta)} f(\sigma) d\sigma.
\]

and it follows from the Lebesgue dominated convergence theorem that we have

\[
\lim_{\eta \to 0, \eta \in S_{\alpha,\beta}} \| < (e^{-\epsilon}f)_{\eta}, \phi > - < f, \phi > \|_X = 0 \quad (f \in e^{-z}V_{\alpha,\beta}(X), z \in \mathbb{C}^k)
\]  

(35)

The following consequence of proposition 10.6 allow to compute in some cases \(< f, \phi >\) for \(\phi \in U'_{\alpha,\beta}, f \in V_{\alpha,\beta}(X), z \in Dom(FB(\phi))\) by using the \(z\)-Cauchy transform.

**Proposition 11.8** Assume that \(\alpha_j < \beta_j < \alpha_j + \pi\) for \(1 \leq j \leq k\), let \(\phi \in F_{\alpha,\beta}\), let \(z \in Dom(FB(\phi))\), and let \(X\) be a separable Banach space.

If \(f \in e^{-z}V_{\alpha,\beta}(X)\), and if

\[
\int_{\partial S_{\alpha,\beta}} e^{Re(z\sigma)} \|f(\sigma)\|_X d\sigma < +\infty,
\]

then we have, for \(\eta \in S_{\alpha,\beta}, \phi \in S^*_{\alpha,\beta}, \eta \in S_{\alpha,\beta}\),

\[
< f_{\eta}, \phi > = < f, \phi \ast \delta_\eta > = \int_{\partial S_{\alpha,\beta}} e^{(\sigma - \eta)\sigma} C_z(\phi)(\sigma - \eta) f(\sigma) d\sigma.
\]  

(36)

In particular we have, for \(f \in V_{\alpha,\beta}(X), \epsilon \in S^*_{\alpha,\beta}, \eta \in S_{\alpha,\beta}\),

\[
e^{-\epsilon\eta} < e^{-\epsilon}f_{\eta}, \phi > = < e^{-\epsilon}f, \phi \ast \delta_\eta > = \int_{\partial S_{\alpha,\beta}} e^{(z - \epsilon)(\sigma - \eta)} C_z(\phi)(\sigma - \eta) f(\sigma) d\sigma.
\]  

(37)
Proof: Assume that \( f \in e_{-z}V_{\alpha,\beta}(X) \) satisfies the condition \( \int_{\partial S_{\alpha,\beta}} |f(\sigma)| d\sigma < +\infty \). We have, for \( \eta \in S_{\alpha,\beta}, \epsilon \in S_{\alpha,\beta}^* \),

\[
< f_\eta, \phi > = < e_\xi f_\eta, \phi e_\xi > = e^{-\eta} < (e_\xi f)_\eta, \phi e_\xi >, \quad e^{-\epsilon} f_\eta = e^\eta (e^{-\epsilon} f)_\eta
\]

so (36) follows from (25) applied to \( e_\xi f \) and \( \phi e_\xi \), and (37) follows from (36) applied to \( e^{-\epsilon} f \). □

For \( z \in \mathbb{C}^k, f \in e_{-z}H_\infty(S_{\alpha,\beta}, X) \), define the Fourier-Borel transform of \( f \) for \( \zeta = (\zeta_1, \ldots, \zeta_k) \in \Pi_{1 \leq j \leq k} \left( \mathbb{C} \setminus (z_j - \overline{S}_{\alpha_j, \beta_j}) \right) \) by the formula

\[
FB(f)(\zeta) = FB(e_\xi f)(z + \zeta) = \int_0^{e^{i\omega \cdot \infty}} \int_0^{e^{i\omega \cdot k, \infty}} e^{-\zeta \sigma_1 \cdots \zeta_k} f(\sigma_1, \ldots, \sigma_k) d\sigma_1 \cdots d\sigma_k,
\]

where \( \alpha_j \leq \omega_j \leq \beta_j \) and where \( \text{Re}((z_j + \zeta_j)e^{i\omega_j}) > 0 \) for \( 1 \leq j \leq k \).

The following consequences of proposition 10.8, corollary 10.9 and corollary 10.11 allow to interpret the action of \( \phi \in F_{\alpha,\beta} \) on \( e_{-z}U_{\alpha,\beta} \) for \( z \in \text{Dom}(FB(\phi)) \) in terms of Fourier-Borel transforms.

**Proposition 11.9** Let \( \phi \in F_{\alpha,\beta} \), let \( z = (z_1, \ldots, z_k) \in \text{Dom}(FB(\phi)) \), and let \( f \in e_{-z}V_{\alpha,\beta}(X) \). Set again \( W_n(\zeta) = \Pi_{1 \leq j \leq k} \left( \frac{n^2}{n + \zeta e^{-i\omega_j}} \right) \) for \( \zeta = (\zeta_1, \ldots, \zeta_k) \in S_{\alpha,\beta}^*, n \geq 1 \). Then

(i) \(< f, \phi > = \lim_{\epsilon \to S_{\alpha,\beta}} \left( \lim_{n \to +\infty} \frac{1}{(2\pi)^k} \int_{z + \partial S^*_{\alpha,\beta}} W_n(\sigma - z) \cdot \phi(\sigma) \cdot FB(f)(-\sigma + \epsilon) d\sigma \right) \).

(ii) If, further, \( \int_{\partial S^*_{\alpha,\beta}} |FB(\phi)(\sigma)||d\sigma| < +\infty, \) then we have, for \( \epsilon \in S_{\alpha,\beta}^* \),

\[
< e^{-\epsilon} f, \phi > = \frac{1}{(2\pi)^k} \int_{z + \partial S^*_{\alpha,\beta}} FB(\phi)(\sigma) FB(f)(-\sigma + \epsilon) d\sigma,
\]

and so

\[
< f, \phi > = \lim_{\epsilon \to S_{\alpha,\beta}} \frac{1}{(2\pi)^k} \int_{z + \partial S^*_{\alpha,\beta}} FB(\phi)(\sigma) FB(f)(-\sigma + \epsilon) d\sigma.
\]

If, further, \( \int_{\partial S^*_{\alpha,\beta}} e^{\text{Re}(\sigma)} \| f(\sigma) \| d\sigma < +\infty, \) then

(iii) \(< f, \phi > = \frac{1}{(2\pi)^k} \int_{z + \partial S^*_{\alpha,\beta}} FB(\phi)(\sigma) FB(f)(-\sigma) d\sigma \).

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Proof: We have $< f, \phi > = < e_z f, \phi e^{-z} >$. Since $\mathcal{F}B(e_z f)(-\zeta - z) = \mathcal{F}B(e_z f)(-\zeta)$ for $\zeta \in \Gamma \subseteq \mathbb{C}$, and since $\mathcal{F}B(\phi e^{-z})(\zeta) = < e^{-\zeta - z}, \phi >$, it follows from corollary 10.11 that we have

$$
< f, \phi > = < e_z f, \phi e^{-z} > = \lim_{\epsilon \to 0} \left( \frac{1}{(2\pi)^k} \int_{\partial S^*_k} W_n(\zeta) \mathcal{F}B(\phi e^{-z})(\zeta) \mathcal{F}B(e_z f)(\epsilon - \zeta) d\zeta \right),
$$

and we obtain (i) by using the change of variables $\sigma = z + \zeta$ for $\zeta \in \partial S^*_k$. Using the same change of variables we deduce (ii) from proposition 10.8 and (iii) from corollary 10.9. □

**Lemma 11.10** Let $\alpha = (\alpha_1, \ldots, \alpha_k) \in \mathbb{R}^k$, $\alpha' = (\alpha'_1, \ldots, \alpha'_k) \in \mathbb{R}^k$, and assume that $\alpha'_j \leq \alpha_j \leq \beta_j \leq \beta'_j < \alpha'_j + \pi$ for $j \leq k$. Then $\bigcap_{\zeta \in \mathbb{C}} e_{-z} \mathcal{U}_{\alpha', \beta'}$ is dense in $\bigcap_{\zeta \in \mathbb{C}} e_{-z} \mathcal{U}_{\alpha, \beta}$.

**Proof:** Let $\phi \in \mathcal{F}_{\alpha, \beta}$, and assume that $< f, \phi > = 0$ for every $f \in \bigcap_{\zeta \in \mathbb{C}} e_{-z} \mathcal{U}_{\alpha', \beta'}$. Let $z \in \text{Dom}(\mathcal{F}B(\phi))$. Then $\mathcal{F}B(\phi)(z + \zeta) = 0$ for every $\zeta \in \mathbb{C}$, $\mathcal{F}B(\phi)$ is connected, we have $\mathcal{F}B(\phi) = 0$. Hence $\phi = 0$, since the Fourier-Borel transform is one-to-one on $\mathcal{F}_{\alpha, \beta}$. □

We can thus identify $\mathcal{F}_{\alpha, \beta}$ to a subset of $\mathcal{F}_{\alpha', \beta'}$ for $\alpha'_j \leq \alpha_j \leq \beta_j \leq \beta'_j < \alpha'_j + \pi$ for $j \leq k$.

A standard application of the Mittag-Leffler theorem of projective limits of complete metric spaces, see for example [13], theorem 2.14, shows that we have the following result, where as before $M_{a,b} = \{ (\alpha, \beta) \in \mathbb{R}^k \times \mathbb{R}^k | a_j < \alpha_j \leq \beta_j < b_j \text{ if } a_j < b_j, \alpha_j = \beta_j = a_j \text{ if } a_j = b_j \}$.

**Proposition 11.11** Let $a = (a_1, \ldots, a_k) \in \mathbb{R}^k$, $b = (b_1, \ldots, b_k) \in \mathbb{R}^k$ such that $a_j \leq b_j \leq a_j + \pi$ for $j \leq k$. Then $\bigcap_{\alpha', \beta'} \bigcap_{a, b, c} e_{-z} \mathcal{U}_{\alpha', \beta'}$ is dense in $e_{-z} \mathcal{U}_{\alpha, \beta}$ for every $z \in \mathbb{C}$ and every $\alpha, \beta \in M_{a, b}$.

Let $(a, b) \in \mathbb{R}^k \times \mathbb{R}^k$ be as above, and denote by $\Delta_{a,b}$ the set of all triples $(\alpha, \beta, z)$ where $(\alpha, \beta) \in M_{a, b}$ and $z \in \mathbb{C}^k$. Denote by $\leq$ the product partial order on $\mathbb{R}^k$ associated to the usual order on $\mathbb{R}$. If $(\alpha, \beta, z) \in \Delta_{a,b}$, $(\alpha', \beta', z') \in \Delta_{a,b}$, set $(\alpha, \beta, z) \leq (\alpha', \beta', z')$ if $\alpha' \leq \alpha, \beta \leq \beta'$ and $z' \in z + \mathcal{S}^{\alpha', \beta'}$. For every finite family $F = \{ (\alpha^{(i)}, \beta^{(i)}, z^{(i)}) \}_{1 \leq i \leq m}$ of elements of $\Delta_{a,b}$, set

$$
sup(F) = \{ \inf_{1 \leq i \leq m} \alpha^{(i)} \} \times \{ \sup_{1 \leq i \leq m} \beta^{(i)} \} \times \sup_{1 \leq i \leq m} z^{(i)},
$$

where $sup_{1 \leq i \leq m} z^{(i)}$ denotes the set of all $z \in \mathbb{C}^k$ satisfying the condition

$$
z + \mathcal{S}^{\inf_{1 \leq i \leq m} \alpha^{(i)}, \sup_{1 \leq i \leq m} \beta^{(i)}} = \bigcap_{1 \leq i \leq m} \left( z^{(i)} + \mathcal{S}^{\inf_{1 \leq i \leq k} \alpha^{(i)}, \sup_{1 \leq i \leq k} \beta^{(i)}} \right),
$$

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so that \( \sup_{1 \leq i \leq m} z(i) \), is the set introduced in definition 9.1(ii) when \( \alpha = \inf_{1 \leq i \leq m} \alpha(i) \) and \( \beta = \sup_{1 \leq i \leq m} \beta(i) \). Notice that \( \sup_{1 \leq i \leq m} z(i) \), is a singleton if \( \inf_{1 \leq i \leq m} \alpha(i) < \sup_{1 \leq i \leq m} \beta(i) \), for \( 1 \leq j \leq k \).

It follows from the proposition that we can identify the dual of the projective limit \( \cap_{(\alpha,\beta,\pi) \in \Delta, \varepsilon - \pi U_{\alpha,\beta}} \) to the inductive limit \( \cup_{(\alpha,\beta,\pi) \in \Delta, \varepsilon - \pi U_{\alpha,\beta}} \). This suggests the following definition.

**Definition 11.12** Let \( a = (a_1, \ldots, a_k) \in \mathbb{R}^k, b = (b_1, \ldots, b_k) \in \mathbb{R}^k \) such that \( a_j \leq b_j \leq a_j + \pi \) for \( j \leq k \). Set

\[
G_{a,b} = (\cap_{(\alpha,\beta,\pi) \in \Delta_{\alpha,\beta}} e^{-\pi U_{\alpha,\beta}})' = \cup_{(\alpha,\beta,\pi) \in \Delta_{\alpha,\beta}} (e^{-\pi U_{\alpha,\beta}})'.
\]

For \( \phi \in G_{a,b} \), set \( \text{dom}(\phi) = \{ (\alpha, \beta) \in \Delta_{\alpha,\beta} | \phi \in (e^{-\pi U_{\alpha,\beta}}) \}' \).

We thus see that the inductive limit \( G_{a,b} = \cup_{(\alpha,\beta) \in M_{a,b}} F_{\alpha,\beta} \) is an associative unital pseudo-Banach algebra with respect to the convolution product introduced above on the spaces \( F_{\alpha,\beta} \). A subset \( V \) of \( G_{a,b} \) is bounded if and only if there exists \( (\alpha, \beta) \in M_{a,b} \) and \( z \in \mathbb{C}^k \) such that \( V \) is a bounded subset of \( (e^{-\pi U_{\alpha,\beta}})' \).

The proof of the following proposition is left to the reader.

**Proposition 11.13** Let \( \phi \in G_{a,b} \), and let \( (\alpha, \beta, z) \in \text{dom}(\phi) \). Then \( (\alpha', \beta', z') \in \text{dom}(\phi) \) if \( (\alpha, \beta, z) \preceq (\alpha', \beta', z') \). In particular if \( (\phi_j)_{1 \leq j \leq m} \) is a finite family of elements of \( G_{a,b} \), and if \( (\alpha(j), \beta(j), z(j)) \in \text{dom}(\phi_j) \) for \( 1 \leq j \leq m \), then \( \sup_{1 \leq j \leq m} (\alpha(j), \beta(j), z(j)) \subset \cap_{1 \leq j \leq m} \text{dom}(\phi_j) \subset \text{dom}(\phi_1 \cdot \ldots \cdot \phi_m) \).

### 12 Appendix 3: Holomorphic functions on admissible open sets

**Definition 12.1** Let \( a = (a_1, \ldots, a_p) \in \mathbb{R}^k, b = (b_1, \ldots, b_p) \in \mathbb{R}^k \) such that \( a_j \leq b_j \leq a_j + \pi \) for \( j \leq k \).

An open set \( U \subset \mathbb{C}^k \) is said to be admissible with respect to \( (\alpha, \beta) \in M_{a,b} \) if \( U = \Pi_{1 \leq j \leq k} U_j \), where the open sets \( U_j \subset \mathbb{C} \) satisfy the following conditions for some \( z = (z_1, \ldots, z_k) \in \mathbb{C}^k \),

(i) \( U_j + \sum_{\alpha, \beta} S_{\alpha_j, \beta_j} \subset U_j \)

(ii) \( U_j \subset z_j + S_{\alpha_j, \beta_j} \), and \( \partial U_j - z_j = (e^{(-\frac{\pi}{2} - \alpha_j)\varepsilon, \infty}, e^{(-\alpha_j - \frac{\pi}{2})\varepsilon, s_{0,j})} \cup \theta_j([0,1]) \cup (e^{(-\frac{\pi}{2} - \beta_j)\varepsilon, s_{1,j}} e^{(-\alpha_j - \frac{\pi}{2})\varepsilon, s_{0,j}}, e^{(-\alpha_j - \frac{\pi}{2})\varepsilon, s_{1,j}}) \cup (e^{(-\frac{\pi}{2} - \beta_j)\varepsilon, \infty}, e^{(-\alpha_j - \frac{\pi}{2})\varepsilon, s_{0,j}}, e^{(-\alpha_j - \frac{\pi}{2})\varepsilon, s_{1,j}}) \) is a one-to-one piecewise-\( C^1 \) curve such that \( \theta_j(0) = e^{(-\alpha_j - \frac{\pi}{2})\varepsilon, s_{0,j}} \) and \( \theta_j(1) = e^{(-\frac{\pi}{2} - \beta_j)\varepsilon, s_{1,j}} \).

If \( U \) is an admissible open set with respect to some \( (\alpha, \beta) \in M_{a,b} \), \( H^{(1)}(U) \) denotes the space of all functions \( F \) holomorphic on \( U \) such that \( \|F\|_{H^{(1)}(U)} := \sup_{c \in S_{\alpha, \beta}} \int_{\partial U + c} |F(\sigma)| \, |d\sigma| < +\infty \).
For example if $\alpha_j = \beta_j$ conditions then conditions (i) and (ii) are satisfied if an only if $U_j$ is a half-plane of the form $\{z_j \in \mathbb{C} \mid Re(z_j e^{\alpha_j}) > \lambda\}$ for some $\lambda \in \mathbb{R}$.

If $\alpha_j < \beta_j$, define $\tilde{x}_j = \tilde{x}_j(\zeta_j)$ and $\tilde{y}_j = \tilde{y}_j(\zeta_j)$ for $\zeta_j \in \mathbb{C}$ by the formula

$$\zeta_j = z_j + \tilde{x}_j e^{(-\tilde{x}_j)e^{\alpha_j}} + \tilde{y}_j e^{(-\tilde{y}_j)e^{\beta_j}}.$$  \hspace{1cm} (39)

Notice that if $\zeta_j \in U_j$, and if $\tilde{x}_j(\zeta_j') \geq \tilde{x}_j(\zeta_j)$ and $\tilde{y}_j(\zeta_j') \geq \tilde{y}_j(\zeta_j)$ then $\zeta_j' \in z_j + \mathbb{R}_{\alpha_j,\beta_j} U_j$. This shows that there exists $t_{j,0} \in [0, s_{j,0}]$ and $t_{j,1} \in [0, s_{j,1}]$ and continuous piecewise $C^1$-functions $f_j$ and $g_j$ defined respectively on $[0, t_{j,0}]$ and $[0, t_{j,1}]$ such that

$$U_j - z_j = \{\zeta_j \in \mathbb{S}_{\alpha_j,\beta_j}^* \mid \tilde{x}_j(\zeta_j) \in (0, t_{j,0}], \tilde{y}_j(\zeta_j) > f_j(\tilde{x}_j(\zeta_j))\} \cup \{\zeta_j \in \mathbb{S}_{\alpha_j,\beta_j}^* \mid \tilde{x}_j(\zeta_j) > t_{j,0}\}$$

$$= \{\zeta_j \in \mathbb{S}_{\alpha_j,\beta_j}^* \mid \tilde{y}_j(\zeta_j) \in (0, t_{j,1}], \tilde{x}_j(\zeta_j) > g_j(\tilde{y}_j(\zeta_j))\} \cup \{\zeta_j \in \mathbb{S}_{\alpha_j,\beta_j}^* \mid \tilde{y}_j(\zeta_j) > t_{j,1}\}.$$  

We have $f_j(0) = t_{j,1}, f_j(t_{j,0}) = 0, g_j(0) = t_{j,1}, g_j(t_{j,1}) = 0, f_j$ and $g_j$ are strictly decreasing and $f_j = g_j^{-1}$ if $t_{j,s} > 0$ for some, hence for all $s \in \{1, 2\}$.

For $\alpha = (\alpha_1, \ldots, \alpha_k) \in \mathbb{R}^k, \beta = (\beta_1, \ldots, \beta_k) \in \mathbb{R}^k$, we will use the obvious conventions

$$\inf(\alpha, \beta) = (\inf(\alpha_1, \beta_1), \ldots, \inf(\alpha_k, \beta_k)), \sup(\alpha, \beta) = (\sup(\alpha_1, \beta_1), \ldots, \sup(\alpha_k, \beta_k)).$$

Clearly, $(\inf(\alpha^{(1)}, \alpha^{(2)}), \sup(\beta^{(1)}, \beta^{(2)}) \in M_{a,b}$ if $(\alpha^{(1)}, \beta^{(1)}) \in M_{a,b}$ and $(\alpha^{(2)}, \beta^{(2)}) \in M_{a,b}$.

**Proposition 12.2** If $U^{(1)}$ is admissible with respect to $(\alpha^{(1)}, \beta^{(1)}) \in M_{a,b}$ and if $U^{(2)}$ is admissible with respect to $(\alpha^{(2)}, \beta^{(2)}) \in M_{a,b}$, then $U^{(1)} \cap U^{(2)}$ is admissible with respect to $(\inf(\alpha^{(1)}, \alpha^{(2)}), \sup(\beta^{(1)}, \beta^{(2)}))$.

Set $\alpha^{(3)} = \inf(\alpha^{(1)}, \alpha^{(2)}), \beta^{(3)} = \sup(\beta^{(1)}, \beta^{(2)}), and set U^{(3)} = U^{(1)} \cap U^{(2)}$. The fact that $U^{(3)}$ satisfies (i) follows from the fact that $\mathbb{S}_{\alpha^{(3)},\beta^{(3)}} = \mathbb{S}_{\alpha^{(1)},\beta^{(1)}}^* \cap \mathbb{S}_{\alpha^{(2)},\beta^{(2)}}^*$. The fact that $U^{(3)}$ satisfies (ii) follows easily from the fact that $\mathbb{S}_{\alpha^{(1)},\beta^{(1)}}^* \cap \mathbb{S}_{\alpha^{(2)},\beta^{(2)}}^*$ is itself admissible with respect to $(\alpha^{(3)}, \beta^{(3)})$ if $U^{(1)}$ satisfies definition 12.1 with respect to $z^{(1)}$ and if $U^{(2)}$ satisfies definition 12.1 with respect to $z^{(2)}$. \hspace{1cm} \square

**Lemma 12.3** Let $U$ be an admissible open set with respect to some $(\alpha, \beta) \in M_{a,b}$, and let $F \in H^{(1)}(U)$.

(i) We have, for $\epsilon = (\epsilon_1, \ldots, \epsilon_k) \in \mathbb{S}_{\alpha,\beta}^*$,

$$\int_{\Pi_{j \leq k}(U_j \setminus (\mathbb{S}_{\alpha,\beta}^* + \epsilon_j))} |F(\zeta)зам(\zeta)| dm(\zeta) \leq |\epsilon_1| \cdots |\epsilon_k| \|F\|_{H^{(1)}(U)}.$$

where $m$ denotes the Lebesgue measure on $\mathbb{C}^k \approx \mathbb{R}^{2k}$.

(ii) We have, for $\zeta \in U$,
\[|F(\zeta)| \leq \frac{2^k}{\pi k \cos \left(\frac{\beta_1 - \alpha_1}{2}\right) \ldots \cos \left(\frac{\beta_k - \alpha_k}{2}\right)} \prod_{j=1}^{k} \frac{\text{dist}(\zeta_j, \partial S^*_{\alpha_j, \beta_j}) \ldots \text{dist}(\zeta_k, \partial S^*_{\alpha_k, \beta_k})}{\|F\|_{H^1(U)}}.\]

Proof: (i) Let \( F \in H^1(U) \), let \( \epsilon = (\epsilon_1, \ldots, \epsilon_k) \in S^*_{\alpha, \beta} \), for \( j \leq k \) let \( \gamma_j \in (-\frac{\pi}{2} - \alpha_j, \frac{\pi}{2} - \beta_j) \) be a determination of \( \arg(\epsilon_j) \), and set \( r_j = |\epsilon_j| > 0 \).

Set \( U_{j,1} = z_j + t_{j,0}e^{(-\frac{\pi}{2} - \alpha_j)i} + S_{\frac{\pi}{2} - \alpha_j - \gamma_j} \), \( U_{j,2} = z_j + t_{j,1}e^{(\frac{\pi}{2} - \beta_j)i} + S_{\gamma_j, \frac{\pi}{2} - \beta_j} \), and \( U_{j,3} = z_j + \cup_{p>0} \left( p e^{\epsilon_j i} + \partial U_j \cap S^*_{\alpha_j, \beta_j} \right) \), with the convention \( U_{j,3} = \emptyset \) if \( t_{j,0} = t_{j,1} = 0 \). Also for \( \zeta_j \in \C \) set \( x_j = \Re(\zeta_j), y_j = \Im(\zeta_j) \).

For \( t_j < 0, 0 < \rho_j < r_j \), set \( \zeta_j = \zeta_j(\rho_j, t_j) = \rho_j e^{i\gamma_j} + (t_{j,0} - t_j)e^{-i(\frac{\pi}{2} - \alpha_j)} \). This gives a parametrization of \( U_{j,1} \setminus (U_{j,1} + \epsilon_j) \), and we have

\[ dx_j dy_j = \begin{vmatrix}
\cos(\gamma_j) & \sin(\gamma_j) \\
\sin(\gamma_j) & \cos(\gamma_j)
\end{vmatrix} \, dp_j dt_j = \cos(\alpha_j + \gamma_j) dp_j dt_j. \]

Similarly for \( t_j > t_{j,1}, 0 < \rho_j < r_j \), set \( \zeta_j = \zeta_j(\rho_j, t_j) = \rho_j e^{i\gamma_j} + t_j e^{i(\frac{\pi}{2} - \beta_j)}. \) This gives a parametrization of \( U_{j,2} \setminus (U_{j,2} + \epsilon_j) \), and we have

\[ dx_j dy_j = \begin{vmatrix}
\cos(\gamma_j) & \sin(\gamma_j) \\
\sin(\gamma_j) & \cos(\gamma_j)
\end{vmatrix} \, dt_j dp_j = \cos(\beta_j + \gamma_j) dp_j dt_j. \]

Now assume that \( U_{j,3} \neq \emptyset \), so that \( t_{j,0} > 0 \) and \( t_{j,1} > 0 \). For \( 0 < t_j < t_{j,1}, 0 < \rho_j < r_j \) set \( \zeta_j = \zeta_j(\rho_j, t_j) = \rho_j e^{i\gamma_j} + g_j(t_j)e^{i(\frac{\pi}{2} - \alpha_j)} + t_j e^{i(\frac{\pi}{2} - \beta_j)i} \). This gives a parametrization of \( U_{j,3} \setminus (U_{j,3} + \epsilon_j) \), and we have

\[ dx_j dy_j = \begin{vmatrix}
\cos(\gamma_j) & -g_j'(t) \sin(\alpha_j) + \sin(\beta_j) \\
\sin(\gamma_j) & -g_j'(t) \cos(\alpha_j) + \cos(\beta_j)
\end{vmatrix} \, dp_j dt_j \\
= (\cos(\beta_j + \gamma_j) - g_j'(t) \cos(\alpha_j + \gamma_j)) dp_j dt_j. \]

We have \( 0 < \cos(\alpha_j + \gamma_j) < 1, 0 < \cos(\alpha_j + \gamma_j) < 1, g_j'(t_j) < 0 \), and using the Cauchy-Schwarz inequality, we obtain

\[ 0 < \cos(\beta_j + \gamma_j) - g_j'(t) \cos(\alpha_j + \gamma_j) \]
\[ = \cos(\gamma_j)(\cos(\beta_j) - g_j'(t) \cos(\alpha_j)) - \sin(\gamma_j)(\sin(\beta_j) - g_j'(t) \sin(\alpha_j)) \]
\[ \leq \sqrt{(\cos(\beta_j) - g_j'(t) \cos(\alpha_j))^2 + (\sin(\beta_j) - g_j'(t) \sin(\alpha_j))^2} \]
\[ = \sqrt{1 - 2g_j'(t) \cos(\beta_j - \alpha_j) + g_j'(t)^2}. \]

On the other hand we have

\[ \left| \frac{\partial \zeta_j}{\partial t_j}(\rho_j, t_j) \right|^2 = \left( g_j'(t) e^{(-\frac{\pi}{2} - \alpha_j)i} + e^{(\frac{\pi}{2} - \beta_j)i} \right) \left( g_j'(t) e^{(\frac{\pi}{2} + \alpha_j)i} + e^{(-\frac{\pi}{2} + \beta_j)i} \right) \]
\[ = 1 - 2g_j'(t) \cos(\beta_j - \alpha_j) + g_j'(t)^2. \]
The boundary \( \partial U_j + \rho_j e^{i\nu} \) being oriented from \( e^{(-\frac{\pi}{2} - \alpha_j)i} \) to \( e^{(\frac{\pi}{2} - \beta_j)i} \), we obtain

\[
\int_{(0,r_1) \times \ldots \times (0,r_k)} \left| F(\zeta) \right| dm(\zeta) \\
\leq \int_{(0,r_1) \times \ldots \times (0,r_k)} \left[ \int_{\partial U_j + \rho_j e^{i\nu}} \left| F(\sigma_j) \right| |d\sigma_1| \ldots |d\sigma_k| \right] dp_1 \ldots dp_k \\
\leq r_1 \ldots r_k \| F \|_{H^1(U)},
\]

which proves (i).

(ii) Let \( F \in H^1(U) \), let \( \zeta \in U \), set \( r_j = \text{dist}(\zeta_j, \partial U_j) \), set \( r = (r_1, \ldots, r_k) \), and set \( B(\zeta, r) = \Pi_{j \leq k} B(\zeta_j, r_j) \). Using Cauchy’s formula and polar coordinates, we obtain the standard formula

\[
F(\zeta) = \frac{1}{|B(\zeta, r)|} \int_{B(\zeta, r)} F(\eta) dm(\eta), \quad (40)
\]

where \( |B(\zeta, r)| = \pi^k r_1^2 \ldots r_k^2 \) denotes the Lebesgue measure of \( B(\zeta, r) \).

Denote by \( u_j \) the orthogonal projection of \( \zeta_j \) on the real line \( z_j + \mathbb{R} e^{(-\frac{\pi}{2} - \alpha_j)i} \), denote by \( v_j \) the orthogonal projection of \( \zeta_j \) on the real line \( z_j + \mathbb{R} e^{(\frac{\pi}{2} - \beta_j)i} \), and let \( w_j \in \{ u_j, v_j \} \) be such that \( |\zeta_j - w_j| = \min(|\zeta_j - u_j|, |\zeta_j - v_j|) \). An easy topological argument shows that \( w_j \in \partial S_{\alpha_j, \beta_j} \), so that \( |\zeta_j - w_j| = \text{dist}(\zeta_j, \partial S_{\alpha_j, \beta_j}) \geq \text{dist}(\zeta_j, U_j) = r_j \). For \( \lambda \in \mathbb{R} \), we have \( \zeta_j \notin S_{\alpha_j, \beta_j} = z_j + 2(\zeta_j - w_j) + \lambda i(\zeta_j - w_j) \subset U_j + 2(\zeta_j - w_j) + \lambda i(\zeta_j - w_j) \). If \( \pi - \beta_j + \alpha_j > \frac{\pi}{2} \), then \( \zeta_j - w_j \in S_{\alpha_j, \beta_j} \). If \( \pi - \beta_j + \alpha_j \leq \frac{\pi}{2} \), then we can choose \( \lambda \in \mathbb{R} \) such that \( |\zeta_j - w_j| + \lambda i(\zeta_j - w_j) \in S_{\alpha_j, \beta_j} \) and such that \( |\zeta_j - w_j| = \frac{|\zeta_j - w_j|}{\cos \left( \frac{\beta_j - \alpha_j}{2} \right)} \). So there exists in both cases \( \epsilon_j \in S_{\alpha_j, \beta_j}^* \) such that \( \zeta_j \notin S_{\alpha_j, \beta_j} \) and \( |\epsilon_j| = \frac{2 \text{dist}(\zeta_j, \partial S_{\alpha_j, \beta_j}^*)}{\cos \left( \frac{\beta_j - \alpha_j}{2} \right)} \).

Using (40) and (i), we obtain

\[
|F(\zeta)| \leq \frac{1}{\pi^k r_1^2 \ldots r_k^2} \int_{(0,r_1) \times \ldots \times (0,r_k)} |F(\eta)| dm(\eta) \\
\leq \frac{2^k}{\pi \cos \left( \frac{\beta_j - \alpha_j}{2} \right) \ldots \cos \left( \frac{\beta_j - \alpha_j}{2} \right)} \left[ \text{dist}(\zeta_1, \partial U_1) \ldots \text{dist}(\zeta_k, \partial U_k) \right]^2 \| F \|_{H^1(U)},
\]

which proves (ii).

\[\square\]

**Corollary 12.4** \( (H^1(U), \| \cdot \|_{H^1(U)}) \) is a Banach space, \( F_{U+\epsilon} \) is bounded on \( U + \epsilon \), and \( \lim_{\text{dist}(\zeta, \partial U) \to \infty} F(\zeta) = 0 \) for every \( F \in H^1(U) \) and every \( \epsilon \in S_{\alpha, \beta}^* \).

Proof: It follows from (ii) that for every compact set \( K \subset U \) there exists \( m_K > 0 \) such that \( \max_{\zeta \in K} |F(\zeta)| \leq m_K \| F \|_{H^1(U)} \). So every Cauchy sequence
\((F_n)_{n \geq 1}\) in \((H^1(U), \| \cdot \|_{H^1(U)})\) is a normal family which converges uniformly on every compact subset of \(U\) to a holomorphic function \(F : U \to \mathbb{C}\). Since
\[
\int_{\partial U + \epsilon} |F(\sigma) - F_n(\sigma)| \, d\sigma = \lim_{R \to +\infty} \int_{B(0,R)^c} (\partial U + \epsilon) |F(\sigma) - F_n(\sigma)| \, d\sigma,
\]
and that \(F_n\) converges uniformly to \(F\) on \(U\) as \(n \to \infty\). An easy argument using this shows that \(F \in H^1(U)\) and that \(\lim_{n \to +\infty} \|F - F_n\|_{H^1(U)} = 0\).

Now let \(\epsilon > 0\). There exists \(m_j > 0\) such that \(\text{dist}(\zeta_j, \partial U) \geq m_j \text{dist}(\zeta_j, \partial S_{\alpha,\beta}^k)\) for every \(\zeta_j \in \overline{U}_j + \epsilon,\) which gives, for \(\zeta \in U + \epsilon\),
\[
|F(\zeta)| \leq \frac{2^k}{\pi^k m_1 \ldots m_k \cos \left( \frac{\beta_j - \alpha_1}{2} \right) \ldots \cos \left( \frac{\beta_k - \alpha_k}{2} \right) \text{dist}(\zeta_1, \partial U_1) \ldots \text{dist}(\zeta_k, \partial U_k)} \|F\|_{H^1(U)}.
\]

Since \(\inf_{\zeta_j \in U_j + \epsilon_j} \text{dist}(\zeta_j, \partial U_j) > 0\) for \(j \leq k\), this shows that \(F\) is bounded on \(U + \epsilon\), and that \(\lim_{\zeta \to \partial U + \infty} F(\zeta) = 0\). \(\Box\)

**Theorem 12.5** Let \(U\) be an admissible open set with respect to some \((\alpha, \beta)\) in \(M_{a,b}\), and let \(F \in H^1(U)\). Then
\[
\int_{\partial U + \epsilon} F(\sigma) \, d\sigma = 0 \text{ for every } \epsilon \in S_{\alpha,\beta}^*, \text{ and for every } \zeta \in U + \epsilon,
\]
where \(\partial U_j\) is oriented from \(e^{i(\frac{\pi}{2} - \alpha_j)\cdot \infty}\) to \(e^{i(\frac{\pi}{2} - \beta_j)\cdot \infty}\) for \(j \leq k\).

**Proof:** Let \(z \in \mathbb{C}^k\) satisfying the conditions of definition 12.1 with respect to \(U\), let \(\epsilon \in S_{\alpha,\beta}^*,\) let \(L > 1\) such that \((z_j + e^{i\alpha_j}, \infty, z_j + Le^{i\alpha_j}) \subset \partial U_j\) and \([z_j + Le^{i\beta_j}, z_j + e^{i\beta_j}, \infty) \subset \partial U_j\) for \(j \leq k\), and let \(M > 1\). Set
\[
\Gamma_{L,j,1} = (\epsilon_j + \partial U_j) \cap \{(z_j + e^{i\alpha_j}, z_j + \epsilon_j + e^{i\alpha_j}, \infty) \cup (z_j + \epsilon_j + Le^{i\beta_j}, z_j + \epsilon_j + e^{i\beta_j}, \infty)\},
\]
\[
\Gamma_{L,j,2} = [z_j + \epsilon_j + Le^{i\beta_j}, z_j + Le_j + e^{i\beta_j}],
\]
\[
\Gamma_{L,j,3} = (Le_j + \partial U_j) \cap \{(z_j + Le_j + Le^{i\alpha_j}, \infty, z_j + Le_j + Le^{i\alpha_j}) \cup (z_j + Le_j + Le^{i\beta_j}, z_j + \epsilon_j + Le^{i\beta_j}, \infty)\},
\]
\[
\Gamma_{L,j,4} = [z_j + Le_j + Le^{i\alpha_j}, z_j + \epsilon_j + Le^{i\alpha_j}].
\]
where the Jordan curve \(\Gamma_{L,j}\) is oriented clockwise.

For \(n \geq 1, \zeta_j \in S_{\alpha,\beta}^*,\) set \(W_j,n(\zeta_j) = \frac{n^2}{n + e^{\frac{\pi}{2} \zeta_j}},\) and set \(W_n(\zeta) = \Pi_{j \leq k} W_{n,j}(\zeta_j)\) for \(\zeta \in S_{\alpha,\beta}^*\). Then \(|W_{n,j}(\zeta_j)| \leq 1\) for \(\zeta_j \in S_{\alpha,\beta}^*\), \(W_n(\zeta) \to 1\) as \(n \to \infty\) uniformly on compact sets of \(S_{\alpha,\beta}^*\), and \(\lim_{\zeta \to \infty} W_n(\zeta) = 0\).

Denote by \(V_{L,j}\) the interior of \(\Gamma_{L,j}\) and set \(V_L = \Pi_{j \leq k} V_{L,j}\). If \(\zeta \in V_L\), it follows from Cauchy’s theorem that we have
\[
\sum_{j \in \{1,2,3,4\}} \int_{\Pi_{j \leq k} \Gamma_{L,j}(U)} W_n(\sigma - z - \epsilon) F(\sigma) \, d\sigma = \int_{\partial V_L} W_n(\sigma - z - \epsilon) F(\sigma) \, d\sigma = 0.
\]

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Set \( l_0(j) = 1 \) for \( j \leq k \). It follows from the corollary that there exists \( M > 0 \) such that \( |F(\zeta)| \leq M \) for \( \zeta \in \overline{U} + \epsilon \), and there exists \( R_n > 0 \) such that
\[
\int_{\Gamma_{L,j}} |W_n(\sigma_j - z_j - \epsilon_j)| \, d\sigma_j \leq R_n \quad \text{for every } L.
\]
Also \( \lim_{L \to +\infty} \int_{\Gamma_{L,j,s}} |W_n(\sigma_j - z_j - \epsilon_j)| \, d\sigma_j = 0 \) for \( s \geq 2, j \leq k \).

Let \( l \neq l_0 \), and let \( j_1 \leq k \) such that \( j_1 \geq 2 \). We have
\[
\lim_{L \to +\infty} \sup_{l \leq k} \left| \int_{\Pi_{j \leq k} \Gamma_{L,j,j_1}} |W_n(\sigma - \epsilon) F(\sigma)| \, d\sigma \right| \leq M R_n^{-1} \int_{\Gamma_{L,j,j_1}} |W_n(\zeta_j - z_j - \epsilon_j)| \, d\sigma_j = 0.
\]
This gives
\[
\int_{\partial U + \epsilon} W_n(\sigma - \epsilon) F(\sigma) \, d\sigma = \lim_{L \to +\infty} \int_{\Pi_{j \leq k} \Gamma_{L,j,j_1}} W_n(\sigma - \epsilon) F(\sigma) \, d\sigma
\]
\[
= \lim_{L \to +\infty} \sum_{l \in \{1,2,3,4\}^k} \int_{\Pi_{j \leq k} \Gamma_{L,j,l_1}} W_n(\sigma - \epsilon) F(\sigma) \, d\sigma = 0.
\]
It follows then from the Lebesgue dominated convergence theorem that
\[
\int_{\partial U + \epsilon} F(\sigma) \, d\sigma = 0.
\]
Similarly, applying Cauchy’s formula when \( \zeta \in U + \epsilon \) is contained in \( V_L \), we obtain
\[
\frac{1}{(2\pi i)^k} \int_{\partial U + \epsilon} \frac{W_n(\sigma - \epsilon) F(\sigma)}{(\zeta_1 - \sigma_1) \ldots (\zeta_k - \sigma_k)} \, d\sigma = \lim_{L \to +\infty} \frac{1}{(2\pi i)^k} \int_{\Gamma_{L,j,j_1}} \frac{W_n(\sigma - \epsilon) F(\sigma)}{(\zeta_1 - \sigma_1) \ldots (\zeta_k - \sigma_k)} \, d\sigma
\]
\[
= \lim_{L \to +\infty} \frac{1}{(2\pi i)^k} \sum_{l \in \{1,2,3,4\}^k} \int_{\Pi_{j \leq k} \Gamma_{L,j,j_1}} \frac{W_n(\sigma - \epsilon) F(\sigma)}{(\zeta_1 - \sigma_1) \ldots (\zeta_k - \sigma_k)} \, d\sigma
\]
\[
= \lim_{L \to +\infty} \frac{1}{(2\pi i)^k} \int_{\partial V_L} \frac{W_n(\sigma - \epsilon) F(\sigma)}{(\zeta_1 - \sigma_1) \ldots (\zeta_k - \sigma_k)} \, d\sigma = W_n(\zeta - \epsilon) F(\zeta).
\]
It follows then again from the Lebesgue dominated convergence theorem that we have
\[
F(\zeta) = \lim_{n \to +\infty} W_n(\zeta - \epsilon) F(\zeta) = \lim_{n \to +\infty} \frac{1}{(2\pi i)^k} \int_{\partial U + \epsilon} \frac{W_n(\sigma - \epsilon) F(\sigma)}{(\zeta_1 - \sigma_1) \ldots (\zeta_k - \sigma_k)} \, d\sigma
\]
\[
= \frac{1}{(2\pi i)^k} \int_{\partial U + \epsilon} \frac{F(\sigma)}{(\zeta_1 - \sigma_1) \ldots (\zeta_k - \sigma_k)} \, d\sigma.
\]
\[
\Box
\]
Let \( \zeta \in U \), and let \( \epsilon \in S_{\alpha,\beta}^* \). It follows from the theorem that there exists \( \rho > 0 \) such that \( |F(\zeta)| \leq \frac{1}{(2\pi i)^k} \|F\|_{H^{(1)}_\alpha} \|\Pi_{j \leq k} \text{dist}(\zeta, \partial U + \epsilon)\|_t \) for \( t \in [0, \rho] \). Since
\[ \lim_{t \to 0^+} \text{dist}(\zeta_j, \partial U_j + t\epsilon_j) = \lim_{t \to 0^+} \text{dist}(\zeta_j - t\epsilon_j, \partial U_j) = \text{dist}(\zeta_j, \partial U_j), \]

we obtain, for \( F \in H^{(1)}(U) \), \( \zeta \in U \),
\[
|F(\zeta)| \leq \frac{1}{(2\pi)^k} \prod_{j \leq k} \text{dist}(\zeta_j, \partial U_j) \|F\|_{H^{(1)}(U)}
\]
which improves inequality (ii) of lemma 12.3.

If \( \alpha_j = \beta_j \) for \( j \leq k \), then every \((\alpha, \beta)\) admissible open set \( U \) is a product of open half-planes and the space \( H^{(1)}(U) \) is the usual Hardy space \( H^1(U) \). The standard conformal mappings of the open unit disc \( D \) onto half-planes induce an isometry from the Hardy space \( H^1(D^k) \) onto \( H^1(U) \). It follows then from standard results about \( H^1(D^k) \), see theorems 3.3.3 and 3.3.4 of [28], that \( F \) admits a.e. a nontangential limit \( F^* \) on \( \partial U \), and that \( \lim_{n \to 0} |\int_{\partial U} |F^*(\sigma) - F(\sigma + \epsilon)||d\sigma| = 0 \). This gives the formula
\[
F(\zeta) = \frac{1}{(2\pi)^k} \int_{\partial U} \frac{F^*(\sigma)d\sigma}{(\zeta_1 - \sigma_1)\ldots(\zeta_k - \sigma_k)} \quad \text{for every } \zeta \in U. \quad (42)
\]

We did not investigate whether such nontangential limits of \( F \) on \( \partial U \) exist in the general case.

Recall that the Smirnov class \( \mathcal{N}^+(P^+) \) on the right-hand open half-plane \( P^+ \) consists in those functions \( F \) holomorphic on \( P^+ \) which can be written under the form \( F = G/H \) where \( G \in H^\infty(P^+) \) and where \( H \in H^\infty(P^+) \) is outer, which means that we have, for \( \Re(\zeta) > 0 \),
\[
F(\zeta) = \exp \left( \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{1 - iy\zeta}{(\zeta - iy)(1 + y^2)} \log |F^*(iy)| dy \right),
\]
where \( F^*(iy) = \lim_{x \to 0^+} F(x + iy) \) if defined a.e. on the vertical axis and satisfies \( \int_{-\infty}^{+\infty} \frac{\log |F^*(iy)|}{1 + y^2} dy < +\infty \).

Set, for \( \Re(\zeta) > 0 \),
\[
F_n(\zeta) = \exp \left( \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{1 - iy\zeta}{(\zeta - iy)(1 + y^2)} \sup(\log |F^*(iy)|, -n) dy \right).
\]

It follows from the positivity of the Poisson kernel on the real line that
\[ |F(\zeta)| \leq |F_n(\zeta)| \]
and that \( \lim_{n \to +\infty} F_n(\zeta) = F(\zeta) \) for \( \Re(\zeta) > 0 \). Also the nontangential limit \( F_n^*(iy) \) of \( F \) at \( iy \) exists a.e. on the imaginary axis and \( |F_n^*(iy)| = \sup(e^{-n} \cdot |F^*(iy)|) \) a.e., which shows that \( \sup_{\zeta \in P^+} |F_n(\zeta)| = \sup_{\zeta \in P^+} |F(\zeta)| \) when \( n \) is sufficiently large. Hence \( \lim_{n \to +\infty} F(\zeta) F_n^{-1}(\zeta) = 1 \) for \( \zeta \in P^+ \).

This suggests the following notion;

**Definition 12.6** Let \( U \subset \mathbb{C}^k \) be a connected open set. A holomorphic function \( F \in H^\infty(U) \) is said to be strongly outer on \( U \) if there exists a sequence \( (F_n)_{n \geq 1} \) of invertible elements of \( H^\infty(U) \) satisfying the following conditions
(i) \( |F(\zeta)| \leq |F_n(\zeta)| \quad (\zeta \in U, n \geq 1) \),
(ii) \( \lim_{n \to +\infty} F(\zeta) F_n^{-1}(\zeta) = 1 \quad (\zeta \in U) \).
The Smirnov class $S(U)$ consists of those holomorphic functions $F$ on $U$ such that $FG \in H^\infty(U)$ for some strongly outer function $G \in H^\infty(U)$.

It follows from (ii) that $F(\zeta) \neq 0$ for every $\zeta \in U$ if $F$ is strongly outer on $U$, and $F|_V$ is strongly outer on $V$ if $V \subset U$. Similarly if $F \in S(U)$ then $F|_V \in S(V)$.

Also it follows immediately from the definition that the set of bounded strongly outer functions on $U$ is stable under products, and that if there is a conformal mapping $\theta$ from an open set $V \subset \mathbb{C}^k$ onto $U$ then $F \in H^\infty(U)$ is strongly outer on $U$ if and only if $F \circ \theta$ is strongly outer on $V$, and if $G$ is holomorphic on $U$ then $G \in S(U)$ if and only $F \circ \theta \in S(V)$.

Now let $(\alpha, \beta) \in M_{\alpha, \beta}$ and let $U = \Pi_{j \leq k} U_j$ be an admissible open set with respect to $(\alpha, \beta)$. Then each set $U_j$ is conformally equivalent to the open unit disc $\mathbb{D}$, and so there exists a conformal mapping $\theta$ from $\mathbb{D}^k$ onto $U$, and the study of the class of bounded strongly outer functions on $U$ (resp. the Smirnov class on $U$) reduces to the study of the class bounded strongly outer functions (resp. the Smirnov class) on $\mathbb{D}^k$.

Let $F \in H^\infty(D^k)$ be strongly outer, and let $(F_n)_{n \geq 1}$ be a sequence of invertible elements of $H^\infty(\mathbb{D}^k)$ satisfying the conditions of definition 12.6 with respect to $F$. Denote by $\mathbb{T} = \partial D$ the unit circle. Then $H^\infty(\mathbb{D}^k)$ can be identified to a $w^*$-closed subspace of $L^\infty(T^k)$ with respect to the $w^*$-topology $\sigma(L^1(T^k), L^\infty(T^k))$.

Let $L \in H^\infty(\mathbb{D}^k)$ be a $w^*$-cluster point of the sequence $(FF_n^{-1})_{n \geq 1}$. Since the map $G \to G(\zeta)$ is $w^*$-continuous on $H^\infty(\mathbb{D}^k)$ for $\zeta \in \mathbb{D}^k$, $L = 1$, and so $FH^\infty(\mathbb{D}^k)$ is $w^*$-dense in $H^\infty$. When $k = 1$, this implies as well-known that $F$ is outer, and the argument used for the half-plane shows that, conversely, every bounded outer function on $\mathbb{D}$ is strongly outer, and so $S(\mathbb{D}) = N^+(\mathbb{D})$.

Recall that a function $G \in H^\infty(\mathbb{D}^k)$ is said to be outer if $\log|G(0, \ldots, 0)| = \frac{1}{(2\pi)^k} \int_{\mathbb{T}^k} \log|G(e^{it_1}, \ldots, e^{it_k})| dt_1 \ldots dt_k$, where $G(e^{it_1}, \ldots, e^{it_k})$ denotes a.e. the nontangential limit of $G$ at $(e^{it_1}, \ldots, e^{it_k})$, see [28], definition 4.4.3, and $G$ is outer if and only if almost every slice function $G_\omega$ is outer on $\mathbb{D}$, where $G_\omega(\zeta) = G(\omega \zeta)$ for $\omega \in \mathbb{T}^k, \zeta \in \mathbb{D}$, see [28], lemma 4.4.4. If follows from definition 12.6 that every slice function $F_\omega$ is strongly outer on $\mathbb{D}$ if $F \in H^\infty(\mathbb{D}^k)$ is strongly outer on $\mathbb{D}^k$, and so every strongly outer bounded function on $\mathbb{D}^k$ is outer. It follows from an example from [28] that the converse is false if $k \geq 2$.

**Proposition 12.7** Let $k \geq 2$, and set $F(\zeta_1, \ldots, \zeta_k) = e^{\frac{\zeta_1 + \zeta_2 + \cdots + \zeta_k}{2}}$ for $(\zeta_1, \ldots, \zeta_k) \in \mathbb{D}^k$. Then $F$ is outer on $\mathbb{D}^k$, but $F$ is not strongly outer on $\mathbb{D}^k$.

**Proof:** Set $f(\zeta) = e^{\frac{\zeta}{2\pi}}$ for $\zeta \in \mathbb{D}$. Then $f \in H^\infty(\mathbb{D})$ is a singular inner function. Since $f(\zeta) \neq 0$ for $\zeta \in \mathbb{D}$, it follows from [28], lemma 4.4.4b that the function $\hat{f} : (\zeta_1, \zeta_2) \to f(\frac{\zeta_1 + \zeta_2}{2}, \frac{\zeta_1 + \zeta_2}{2}) = e^{\frac{\zeta_1 + \zeta_2 + \cdots + \zeta_k}{2}}$ is outer on $\mathbb{D}^2$. Hence we have

$$\log|F(0, \ldots, 0)| = \log|\hat{f}(0, 0)| = \frac{1}{(2\pi)^2} \int_{\mathbb{T}^2} \hat{f}(e^{it_1}, e^{it_2}) dt_1 dt_2$$

$$= \frac{1}{(2\pi)^k} \int_{\mathbb{T}^k} F(e^{it_1}, \ldots, e^{it_k}) dt_1 \ldots dt_k,$$

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and so $F$ is outer on $\mathbb{D}^k$.

Now set $\omega = (1, \ldots, 1)$. Then $F_\omega = f$ is not outer on $\mathbb{D}$, and so $F$ is not strongly outer on $\mathbb{D}^k$. □

The fact that some bounded outer functions on $\mathbb{D}$ are not strongly outer is not surprising: The Poisson integral of a real valued integrable function on $\mathbb{T}^k$ is the real part of some holomorphic function on $\mathbb{D}^k$ if an only if its Fourier coefficients vanish on $\mathbb{Z}^k \setminus (\mathbb{Z}^+)^k \cup (\mathbb{Z}^-)^k$, see [28], theorem 2.4.1, and so the construction of the sequence $(F_n)_{n \geq 1}$ satisfying the conditions of definition 12.6 with respect to a bounded outer function $F$ on $\mathbb{D}^k$ breaks down when $k \geq 2$.

We conclude this appendix with the following trivial observations.

**Proposition 12.8** Let $U = \Pi_{j \leq k} U_j \subset \mathbb{C}^k$ be an admissible open set with respect to some $(\alpha, \beta) \in M_{a,b}$.

(i) Let $\theta_j : U_j \to \mathbb{D}$ be a conformal map and let $\pi_j : (\zeta_1, \ldots, \zeta_k) \to \zeta_j$ be the $j$-th coordinate projection. If $f \in H^\infty(\mathbb{D})$ is outer, then $f \circ \theta_j \circ \pi_j$ is strongly outer on $U$.

(ii) The Smirnov class $\mathcal{S}(U)$ contains all holomorphic functions on $U$ having polynomial growth at infinity.

Proof: (i) Since $f$ is strongly outer on $D$, there exists a sequence $(f_n)_{n \geq 1}$ of invertible elements of $H^\infty(D)$ satisfying the conditions of definition 12.6 with respect to $f$. Then the sequence $(f_n \circ \theta_j \circ \pi_j)_{n \geq 1}$ satisfies the conditions of definition 12.6 with respect to $f \circ \theta_j \circ \pi_j$, and so $f \circ \theta_j \circ \pi_j$ is strongly outer on $U$.

(ii) For $j \leq k$ there exists $\gamma_j \in [-\pi, \pi)$ and $m_j \in \mathbb{R}$ such that open set $U_j$ is contained in the open half plane $P_j := \{ \zeta_j \in \mathbb{C} \mid Re(\zeta_j e^{i\gamma_j}) \geq m_j \}$. The function $\sigma \to \frac{1-\sigma}{2}$ is outer on $\mathbb{D}$, since $|\frac{1-\sigma}{2}| \leq |\frac{1+\sqrt{1-4\sigma}}{2}|$ for $\sigma \in \mathbb{D}$, and the function $\zeta_j \to \frac{\zeta_j e^{i\gamma_j} - m_j - 1}{\zeta_j e^{i\gamma_j} - m_j + 1}$ maps conformally $U_j$ onto $\mathbb{D}$. Set $F_j(\zeta_1, \ldots, \zeta_k) = \frac{1 - \zeta_j e^{i\gamma_j} - m_j - 1}{\zeta_j e^{i\gamma_j} - m_j + 1} \frac{2}{\zeta_j e^{i\gamma_j} - m_j + 1}$. It follows from (i) that $F_j$ is strongly outer on $\Pi_{j \leq k} P_j$, hence strongly outer on $U$.

Now assume that a function $F$ holomorphic on $U$ has polynomial growth at infinity. Then there exists $p \geq 1$ such that $F \Pi_{j \leq k} f_j^p$ is bounded on $U$, and so $F \in \mathcal{S}(U)$. □

**References**


[27] F. Ricci, Hardy spaces in one variable, homepage.sns.it/fricci/papers/hardy.pdf

