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Recover Dynamic Utility from Monotonic Characteristic/Extremal Processes. ^{*}

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Abstract

In the present paper, we are interested in the *forward-looking* inverse problem, where the observable are a so-called characteristic process \mathbf{X}^c and an initial utility function $U(0, \cdot) = u(\cdot)$. The recovery process is a dynamic (eventually random) utility performance \mathbf{U} . The main result is a necessary and sufficient condition for the existence of a utility performance process \mathbf{U} satisfying $U(t, X_t^c(x))$ is a martingale for any initial starting point x . Examples of applications are developed in the last section to support our approach in the special case of finance and economics: the first example concerns an aggregation problem, the second one a Markov equilibrium.

1 Introduction

In the real world, decision making under uncertainty is often viewed as an optimization problem under choice criterium, and most theories focus on the derivation of the "optimal decision" and its out-comes. The available observed data are the result of the decision process and its dynamics over the time. but poor information is available on the criterium yielding to these observed data. Economics is a typical example of this difficulty. The standard assumption that all participants in an economic system are utility maximizers likens economic modeling to optimal problem; stability is obtained at the equilibrium, where the system may be described in terms of the behavior of the strategic representative agent. In the applications, it is not clear how the preferences of this representative agent are chosen; often taken as an input of the problem, (chosen for instance by some "central planner"), we show in the last section of this paper, that the choice can be done only in a very limited utility family.

^{*}Key Words. inverse thinking, the utility recovery problem, revealed preference/utility, characteristic process, consistent utility, progressive utility, portfolio optimization, duality, economic equilibrium, aggregated utility.

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But, the increasing distance between the consequences of this theory and the economic reality created new incentives for different approaches. One of them is the evolutionary economics by W.B.Arthur [Art99], studying economies as complex evolutionary systems, where the agents try to predict the out-comes of their actions, and how the market would be modified by their decisions. In this forward-looking view point, the agents also need to adjust their (random) preferences over time. Following J.Gomez-Ramirez [GR13], this complexity suggests the use of an "inverse thinking" approach. The *forward modeling* allows anticipations on the future values of observables, and the inverse problem uses those predictions to infer the values of the parameters that characterize the system. Thus, the robustness of the method is obtained from a family of forward model solutions, consistent with the data rather than one prediction.

In this paper, we limit ourselves to a simple inverse problem, standard in Economics and Finance, called the *revealed preference* problem, and formulated as follows:

Pioneered by the economist P.A.Samuelson in [Sam38, Sam48], the theory is based on the idea that the preferences of consumers are revealed in their purchasing behavior. As explained in J.P.Chambers and F.Echenique book [CE16], "Revealed preference theory" has several interpretations in economics, but the central theme common to all interpretations is what economic models say about the observable world, a crucial step in reverse problem. An example is *the theory of recoverability*, a very active field in choice theory in the 1980's. Solving the recovery problem from the observation of one trajectory of the observed wealth process in the Merton framework (under strong additional assumptions), P.H.Dybvig and L.C.Rogers in [DR97] observed that "*recoverability results provide a guide to what can and cannot be learned from different types of data for deterministic performance criterium*". Closed to these problems is the neo-classical equilibrium characterization, of which we have described the importance and the complexity; as explained in the last section, this complex task is considerably simplified by a forward recovery point of view.

By definition, in the (forward-looking) revealed preference problem, the observable are a so-called characteristic process \mathbf{X}^c and an initial utility function $U(0, \cdot) = u(\cdot)$, and the recovery process is a dynamic (eventually random) utility performance \mathbf{U} . To define the "model" which underlies this inverse problem, we try to reproduce the properties of the stochastic value function of a portfolio optimization problem in finance. In particular, the utility (value function) process \mathbf{U} is a strictly concave stochastic family, evaluated at the observed (also called characteristic) process $\mathbf{X}^c = \{X_t^c(x)\}$ considered as optimal, implying that the optimal preferences $\{U(t, X_t^c(x))\}$ are martingale (constant in mean). Given the concavity of the criterium, tools of convex analysis play a large role, in particular the Fenchel-Legendre duality between the concave utility $U(t, x)$ and its Fenchel convex transform $\tilde{U}(t, y)$. These processes are linked by the "Master equation" based on the marginal utility $U_z(t, z)$, i.e. $U(t, z) - zU_z(t, z) = \tilde{U}(t, U_z(t, z))$. Moreover, as it is classical in convex analysis, the primal problem is equivalent to the dual problem with characteristic process c , if and only $X_t^c(x)$ and $Y_t(u_x(x))$ are orthogonal in the martingale sense

($X_t^c(x) Y_t(u_x(x))$) is a martingale). The pathwise condition " $Y_t(y) := U_x(t, X_t^c(u_z^{-1}(y)))$ ", true at the optimum in optimization problem, is one of major requirements of the problem, called *the first order condition*.

Along this paper, no regularity in time (of Itô's process type for example) on the characteristic process or dynamic utility is assumed. In return, monotony assumptions with respect of their initial condition are introduced. For instance, the functions $(x \rightarrow X_t^c(x)), (z \rightarrow -U_x(t, z)), (y \rightarrow Y_t(y))$ are supposed to be increasing. The first order condition $U_z(t, X_t^c(x)) = Y_t(u_z(x))$ associated with the monotony assumption induces a one to one correspondance between the revealed marginal utility $U_z(t, z)$ and the adjoint process $Y_t(u_z(z))$, given by $U_z(t, z) = Y_t(u_z(\mathcal{X}_t^c(z)))$ where \mathcal{X}^c is the inverse of X^c . A change of variable gives a way, via a Stieltjes integral, to study the martingale properties of $U(t, X_t^c(x)) = \int_0^x Y_t(u_z(z)) d_z X_t^c(z)$. It is the more technical part of the Section 2.

The inverse problem admits many solutions parametrized by the family $(\{Y_t(y)\})$ of "admissible" monotonic adjoint processes satisfying different orthogonality assumptions. Additional constraints imply a complex dependence between the initial utility $u(z)$, the admissible processes $(X_t^c(x), Y_t(y))$. Section 3 illustrates these results through some examples based on the aggregation of forward utilities and the solution of the equilibrium model of H.He and H.Leland [HL93] in complete market, with Itô's processes and Markov constraints. The revealed point of view allows us to solve completely the problem, in showing that the pricing kernel is necessarily a geometrical Brownian motion (GBM), and that dual and primal utilities are mixture of power utilities. In this framework, equilibrium can only be obtained in a too simplified economy.

2 The forward recovery problem

As explained in the introduction, recovery problems arise usually with optimization problems in economics or finance. In [CE16], C.P. Chambers observes:

"We can never see a utility function, but what we might be able to see are demand observations at a finite list of prices."

On the basis of such an analysis, the question which obviously arises is whether or not there has been a utility function which could generate these observations by a process of maximization. In a stochastic control setting, the question becomes: "what is the backward value function of the problem, with these observed data as optimal process?". Moreover, in real world, people consider, for simplicity, a Markov point of view for which the value function is a real function, solution of some backward non linear PDE's, which will be hard to solve. The forward point of view gives more flexibility in the resolution of the problem, for which constraints may be added progressively. Moreover, it is similar to the recent point of view of statistical learning.

2.1 Dynamic utility

We start by reminding some definition and properties of static or dynamic utility criterion.

A dynamic utility should represent, possibly changing over time, individual preferences of an agent starting with a today's specification of his utility, $U(0, z) = u(z)$. The preferences are affected over time by the available information represented by the filtration $(\mathcal{F}_t)_{t \geq 0}$ defined on the probability space $(\Omega, \mathbb{P}, \mathcal{F})$. The filtered probability space $(\Omega, \mathbb{P}, (\mathcal{F}_t)_{t \geq 0})$ is assumed to satisfy usual conditions of right continuity and completeness. The filtration \mathcal{F}_0 is not necessarily assumed to be trivial so that the initial condition $U(0, z)$ is not necessarily a deterministic function. On the space $(\Omega \times \mathbb{R}^+)$, the σ -fields \mathcal{O} of optional processes or \mathcal{P} of predictable processes are generated by the families of adapted, respectively right-continuous or left-continuous processes.

2.1.1 Deterministic utility function

In economics and finance, the standard notion of utility function, used for example as performance measure in portfolio optimization, refers to a concave function u on \mathbb{R}^+ , positive, increasing, normalized by $u(0) = 0$, whose range is \mathbb{R}^+ ($u(+\infty) = \infty$). An important role is played by its derivative u_z , also called *marginal utility*, which is continuous, positive and decreasing on $]0, +\infty[$, with range $]0, +\infty[$, satisfying the Inada's conditions $u_z(+\infty) = 0$ and $u_z(0) = +\infty$.

As usual, the dual problem highlights some other aspects of the optimization problem. It is based on the *Fenchel-Legendre convex conjugate function* $\tilde{u}(y)$ defined through the system (u, \tilde{u}) ,

$$\text{(Main equation)} \quad \begin{cases} \tilde{u}(y) = \sup_{z > 0} (u(z) - yz), & u(z) = \inf_{y > 0} (\tilde{u}(y) + yz). \end{cases} \quad (2.1)$$

$$\begin{cases} u(z) - z u_z(z) = \tilde{u}(u_z(z)), & u(-\tilde{u}_y(y)) + y \tilde{u}_y(y) = \tilde{u}(y). \end{cases} \quad (2.2)$$

In particular, since $\tilde{u}(y) > 0$, $u(z) > z u_z(z) > 0$ and $z u_z(z) \rightarrow u(0) = 0$. The range of the decreasing function $\tilde{u}(y)$ is $]0, +\infty[$ since $\tilde{u}(y) \rightarrow \infty$ when $y \rightarrow 0$ (since $\sup_{z > 0} u(z) = +\infty$) and $\tilde{u}(y) \rightarrow 0$ when $y \rightarrow +\infty$ ($u(0) = 0$ and $z u_z(z) > 0$). In Economics or Finance, utility functions are often assumed to be of class $\mathcal{C}^2([0, \infty[)$.

POWER UTILITY A typical example is the family of *power utility* $u^{(\alpha)}(z) = \frac{z^{1-\alpha}}{1-\alpha}$ for $\alpha \in [0, 1[$ and $u(z) = (\ln z)^+$ for $\alpha = 1$. The coefficient α is the *relative risk aversion* of the agent defined by the ratio $\gamma(z) = -z u_{zz}(z) / u_z(z)$.

Put $\beta = 1/\alpha > 1$. The Fenchel conjugate is the power function $\tilde{u}^{(\beta)}(y) = y^{1-\beta} / (\beta - 1)$. The risk tolerance is $\tilde{\gamma}(y) = -y \tilde{u}_{yy}(y) / \tilde{u}_y(y) = 1/\gamma(-\tilde{u}_y(y))$.

2.1.2 Dynamic Utilities and their Fenchel-Legendre conjugates

A dynamic utility process \mathbf{U} (in short dynamic utility) may be interpreted as a collection of random utility functions $\{U(t, \omega, x)\}$ whose temporal evolution is "updated" over the time in accordance with the new information (\mathcal{F}_t) from an initial utility value $u(z) = U(0, z)$, eventually random if \mathcal{F}_0 is not trivial.

Definition 2.1. A dynamic utility \mathbf{U} is a family of optional processes $\{U(t, z), z \in \mathbb{R}^+\}$ (also called optional random field) such that \mathbb{P} .a.s., for every $t \geq 0$, the function $(z \rightarrow U(t, z))$ is a standard utility function with $U(t, 0) = 0$.

- Its marginal utility \mathbf{U}_z is the decreasing optional random field $\{U_z(t, z)\}$.
- Its conjugate utility $\tilde{\mathbf{U}}$ is the convex optional random field defined by:

$$\tilde{U}(t, y) = \sup_{x>0} (U(t, x) - yx).$$

Equation (2.1) becomes the Master equation,

$$\mathbb{P}.a.s., \quad U(t, z) - zU_z(t, z) = \tilde{U}(t, U_z(t, z)) \text{ and } \tilde{U}_y(t, U_z(t, z)) = -z. \quad (2.3)$$

2.2 The forward recovery problem

In this paper, we address the question of the recoverability based on a forward reconstruction of the random dynamic utility from an observed process; the flexibility introduced by this forward point of view, and the random character of the utility allows us to relax some constraints imposed on the observed (also called characteristic) process. To define the "model" underlying this inverse problem, we try to reproduce the properties of the stochastic value function \mathbf{V} of a portfolio optimization problem in finance. In particular, the value process \mathbf{V} is a strictly concave stochastic family, parametrized by a number $z \in \mathbb{R}^+$ ($z \mapsto V(t, z)$), and *martingale* along the "optimal" process $\mathbf{X}^* = \{X_t^*(x)\}$. Moreover, the process $\{V_z(t, X_t^*(x)) = Y_t^*(V_z(0, z))\}$ is the "optimal" process of the dual problem.

To simplify as much as possible the recovery problem, we eliminate any reference to optimization problem, taking into account only specific dynamic relationships between "optimal" observed process and its value function. As inverse problem, the dynamic utility \mathbf{U} to be recovered from the observable process \mathbf{X}^c , requires the choice of an additional process, the so-called *adjoint process* $\mathbf{Y} = \{Y_t(y)\}$, used to guarantee a kind of *first-order condition*, $Y_t(u_z(x)) = U_z(t, X_t^c(x))$.

2.2.1 Dynamic utility generated by two monotonic processes

The first step is to define the family of utility models $U(t, z)$ consistent with the observed data $(u(z), \mathbf{X}^c)$, from a process $\{X_t^c(x)\}$ observed for different initial conditions x . Monotony and concavity of $z \rightarrow U(t, z)$ cannot be satisfied without some regularity of $x \rightarrow X_t^c(x)$.

The data $\mathbf{X} = \{X_t^c(x)\}$ is assumed to be an optional random field, increasing in x with range $[0, \infty)$. Then $x \rightarrow X_t^c(x)$ is continuous in x , and admits an optional increasing inverse flow $\mathcal{X}^c(t, z)$. The class of x -increasing optional processes with range $[0, \infty)$ is denoted \mathfrak{J} .

The *first-order condition* ($Y_t(u_z(x)) = U_z(t, X_t^c(x))$) imposes the same increasing assumption on the family of positive *adjoint processes* $\mathbf{Y} = \{Y_t(y)\}$, with the additional assumption that the function $z \rightarrow Y_t(u_z(\mathcal{X}_t(z)))$ is integrable in a neighborhood of 0, (the problem coming from

the condition $u_z(0) = +\infty$). The class of admissible adjoint processes is denoted $\mathfrak{J}(X^c, u)$. Surprisingly, this monotony property is rarely highlighted in economics and finance although it is satisfied by all known solutions.

Definition 2.2 (Compatible utility). *Let $(\mathbf{X}^c, \mathcal{X}^c) \in \mathfrak{J} \times \mathfrak{J}$ be an increasing observable process and its inverse, and u the initial utility.*

A dynamic utility \mathbf{U} is said to be compatible with (X^c, u) if and only if there exists an admissible adjoint process, $\mathbf{Y} \in \mathfrak{J}(X^c, u)$ satisfying the "first order condition":

$$U_z(t, z) = Y_t(u_x(\mathcal{X}_t^c(z))) \text{ or } U(t, X_t^c(x)) = \int_0^x Y_t(u_x(z)) d_z X_t^c(z). \quad (2.4)$$

The class of (X^c, u) -compatible dynamic utilities is denoted $\mathfrak{U}(X^c, u)$.

EXAMPLE: In a Markov framework, it is sometimes assumed that the adjoint process $\{Y_t(u_x(x))\}$ is a deterministic function of the characteristic process, $Y_t(u_x(x)) = W(t, X_t^c(x))$, where $W(t, z)$ is a z -decreasing function, with range $(0, \infty]$. Then, if W is integrable in 0, the function $U(t, x) = \int_0^x W(t, z) dz$ is a compatible utility with respect to $X_t^c(x)$ and $u_x(x) = W(0, x) := w(x)$. (See Section 3).

Observe that, given (\mathbf{X}^c, u) , there is a one to one correspondence between the classes of compatible utilities $\mathfrak{U}(X^c, u)$ and the admissible adjoint processes $\mathfrak{J}(X^c, u)$. As mentioned in the introduction and in the literature, the inverse problem admits many solutions parametrized by the family $\mathfrak{J}(X^c, u)$ of "admissible" monotonic processes satisfying this condition.

2.2.2 Martingale constraints and revealed utility

In the sequel, we will often encounter properties as "the product of two processes is martingale". So, it is convenient to introduce the following definition, frequently used for martingale processes, a little less often for general processes.

Definition 2.3 (Orthogonality). *Two optional processes $\{\Phi_t\}$, $\{\Psi_t\}$ whose product $\{\Phi_t \cdot \Psi_t\}$ is a martingale are said to be orthogonal. When the product is only a supermartingale, they are said to be sub-orthogonal.*

The class of compatible utilities is too large to characterize a "good choice" of X^c -utility criterium, and additional conditions must be introduced. We continue to be inspired by the properties of the value function of concave control problems. As in dynamic optimization, the Bellman's principle states that the best choice X_t^c today is also the best one in the future and verifies some "temporal stationarity" along of $U(t, z)$, that is $\mathbb{E}[U(\tau, X_\tau^c(x))] = u(x)$ for any bounded stopping time τ (such that the random variable $U(\tau, X_\tau^c(x))$ is integrable). In terms of paths, the process $\{U(t, X_t^c)\}$ is said to be a *strong martingale*, but most of the times for simplicity we simply refer to "martingale" property.

By means of a change of variable $x \rightarrow X_t^c(x)$, we have made the "curve" $\{U(t, X_t^c(x))\}$ constant in expectation, hence the name of characteristic process for \mathbf{X}^c . Definition 2.2 must be modified by the introduction of this new constraint.

Definition 2.4 (Revealed utility). *A compatible dynamic utility $\mathbf{U} \in \mathfrak{U}(X^c, u)$ is said to be a (X^c, u) -revealed dynamic utility if and only if:*

$$\forall x \in (0, \infty), \quad U(t, X_t^c(x)) \text{ is a positive martingale.} \quad (2.5)$$

In terms of adjoint process $\mathbf{Y} \in \mathfrak{J}(X^c, u)$, the condition " $\int_0^x Y_t(u_x(z)) d_z X_t^c(z)$ is a martingale" cannot be simplified in general.

In addition, for any revealed dynamic utility $\mathbf{U} \in \mathfrak{U}_{\mathfrak{M}}(X^c, u)$, the martingale property of the dual dynamic utility $(\tilde{U}(t, Y_t(y)))$ is satisfied if and only if:

$$\forall x \in (0, \infty), \quad X_t^c(x) Y_t(u_x(x)) \text{ is a positive martingale.} \quad (2.6)$$

This last condition which is equivalent to the orthogonality of $\{X_t^c(x) Y_t(u_x(x))\}$, obvious from the Master equation (2.3) is satisfied in most of optimization problems in economics and finance.

Sometimes it is simpler to solve the dual problem, whose observable are the adjoint processes of the primal problem, and the dual initial utility. For instance, in a complete market there is only one adjoint (also called pricing kernel) process, and the dual utility is easy to recover. The role of the processes X and Y are exchanged, and the dual problem would be formulated as:

Definition 2.5 (Revealed dual utility). *Let $\{Y_t^c(y)\}$ be an optional increasing process with range $(0, \infty)$ whose inverse is denoted \mathcal{Y}^c . Let \tilde{u} be a dual utility (decreasing convex function), whose increasing derivative \tilde{u}_y varies from $-\infty$ to 0.*

(i) *An increasing positive process X is said in the class $\tilde{\mathfrak{J}}(Y^c, \tilde{u})$ if the map $y \mapsto X_t(-\tilde{u}_y(\mathcal{Y}_t^c(y)))$ is integrable near ∞ .*

(ii) *A dynamic dual utility \tilde{U} is said to be compatible with the pair (Y^c, \tilde{u}) and denoted in $\tilde{\mathfrak{U}}(Y^c, \tilde{u})$ if $\exists X \in \tilde{\mathfrak{J}}(Y^c, \tilde{u})$ s.t. $\tilde{U}_y(Y_t^c(y)) = X_t(-\tilde{u}_y(y))$.*

(iii) *$\tilde{U} \in \tilde{\mathfrak{U}}(Y^c, \tilde{u})$ is said to be a revealed dual utility if and only if there exists $X \in \tilde{\mathfrak{J}}(Y^c, \tilde{u})$ such that the integral $\int_y^{+\infty} X_t(-\tilde{u}_y(z)) d_z Y_t^c(z)$ is a well defined positive martingale.*

2.3 Examples of Recovery Problem

2.3.1 Linear Characteristic process

The following simple examples give a nice illustration of the recovery problem, by specifying the property of the observed process. The case of constant characteristic process (corresponding to "to do nothing in finance") is very illustrative. The case of linear characteristic process is the most frequently used in economics.

Proposition 2.1. (i) Assume the characteristic process to be constant $X_t^c(z) = z$. A compatible utility \mathbf{U} is a dynamic utility $\{U(t, z)\}$ whose one of the adjoint processes $\{Y_t(u_z)\}$ is its marginal utility process $\{U_z(t, z)\}$.

A revealed utility is a martingale utility, if and only if the marginal utility is also a martingale.

(ii) Assume the characteristic process to be linear in x , $X_t^c(x) = xX_t$. The recovery problem has a solution if and only if there exists an adjoint process $\mathbf{Y} \in \mathfrak{I}(xX, u)$ such that for any y , $X_t Y_t(y)$ is a martingale. Then, $\{Y_t(y)\}$ is a characteristic process for the dual conjugate utility process.

Proof. (i) Assume the utility process $\{U(t, x)\}$ to be a martingale. By the Master equation (2.3), $zU_z(t, z) \leq U(t, z) \leq U(t, z_{\max})$. By Lebesgue's derivative theorem, the martingale property can be extended to the derivative random field $\{U_z(t, z)\}$. Conversely, if the x -decreasing process $\{U_z(t, x)\}$ is a martingale, by Fubini's theorem $\{U(t, x) - U(t, x_0)\}$ is also a martingale, with expectation $u(x) - u(x_0)$. Thanks to the monotony assumption, $U(t, x_0)$ decreases to 0 when x_0 goes to 0, and the martingale property passes to the limit. Then, $\{U(t, x)\}$ is a martingale.

(ii) If $X^c(t, x) = xX_t$, the change of numeraire $x \rightarrow x/X_t$ yields to a new forward utility process $U^X(t, x) = U(t, xX_t)$ which is a martingale, with characteristic process x . The previous characterization of the forward martingale utility imposes that $U_z^X(t, x) = X_t U_z(t, xX_t) = X_t Y_t(u_z(x))$ is a martingale. Moreover since $(xX_t)Y_t(u_z(x))$ is a martingale, $\tilde{U}(t, Y_t(u_z(x)))$ is a martingale and the adjoint process $\{Y_t(u_z(x))\}$ is a characteristic process for the dual utility process. \square

2.3.2 Differentiable characteristic process

The linear framework is a particular case of a differentiable characteristic process \mathbf{X}^c with derivative \mathbf{X}_x^c ($X_x^c(t, 0) = 1$). The orthogonality condition imposed in the linear case is extended into the orthogonality of the processes $\mathbf{X}_x^c(\cdot, x)$ and $\mathbf{Y}(\cdot, u_z(x))$. If in addition the characteristic process \mathbf{X}^c is x -convex, this condition is necessary.

Proposition 2.2. Let $\mathbf{U} \in \mathfrak{U}(X^c, u)$ be a dynamic utility with adjoint process \mathbf{Y} , whose characteristic process \mathbf{X}^c is x -differentiable with derivative $\{X_x^c(t, x)\}$.

(i) If the characteristic process is convex ($x \rightarrow X_x^c(t, x)$ positive increasing), then \mathbf{U} is a revealed utility if and only if $\{X_x^c(t, x)Y_t(u_z(x))\}$ is a martingale for any x .

(ii) In the general case, the condition is only sufficient; if $\{X_x^c(t, x)Y_t(u_z(x))\}$ is a martingale then $\{U(t, X_t^c(x))\}$ is a martingale.

The necessary condition holds true only if the Lebesgue derivative theorem can be applied, that is true for convex process; but observe that by definition the recovery problem is based only on the sufficient condition.

Proof. (i) This result can be viewed as a particular case of the previous example, since when the characteristic process is convex and differentiable in x , ($x \rightarrow X_x^c(t, x)$ increasing in x), for

any dynamic utility in $\mathfrak{U}(X^c, u)$, the random field $\{U(t, X_t^c(x))\}$ is x -concave. The new random field $\{U^X(t, z) = U(t, X_t^c(z))\}$ is a martingale dynamic utility, studied in Proposition 2.1; the equivalence between the martingale property of the utility and that of its derivative implies the equivalence of the martingale property of $\{U(t, X_t^c(x))\}$ and that of $\{Y_t(u_z(z))X_x^c(t, z)\}$.

(ii) The proof of the sufficient condition is close to the one of Proposition 2.1. Consider the primitive process $\Psi^X(t, x_0, x) := \int_{x_0}^x Y_t(u_z(z))X_x^c(t, z)dz$, which is a martingale with expectation $u(x) - u(x_0)$, by the positive Fubini's theorem. As above, by monotony and positivity, this property goes to the limit when $x_0 \rightarrow 0$, and $\Psi^X(t, x) = \int_0^x X_x(t, z)Y_t(u_z(z))dz$ is a well-defined martingale. The random field \mathbf{U} defined by $U(t, x) := \Psi^X(t, \mathcal{X}^c(t, x))$ is a revealed dynamic utility. \square

2.4 Main result

When the characteristic process is not differentiable, we have to reformulate the "first order condition" in a more global formulation.

2.4.1 Necessary and sufficient condition for recovery problem

In the general case, we use the rate of variation in place of the derivative, and the following representation, for $z' > z > 0$,

$$\begin{cases} U(t, z') - U(t, z) = (z' - z)U_z(t, \xi_t(z, z')), & \xi_t(z, z') \in (z, z') \\ \xi_t(z, z') = (U_z)^{-1}(t, \Delta U(t, (z, z'))), & \Delta U(t, (z, z')) = \frac{U(t, z') - U(t, z)}{z' - z}. \end{cases} \quad (2.7)$$

Next result provides a necessary condition for the forward recovery problem, when the sufficient condition is developed in the next theorem.

Proposition 2.3 (Necessary Condition). *Let $\mathbf{U} \in \mathfrak{U}(X^c, u)$ be a dynamic utility with adjoint process Y (i.e., $U_z(t, X_t^c(z)) = Y_t(u_z(z))$). If \mathbf{U} is a revealed utility, then for any $x' > x > 0$ there exists an optional process $\psi_t(x, x') \in [x, x']$ such that $\{(X_t^c(x') - X_t^c(x))Y_t(u_z(\psi_t(x, x')))\}$ is a martingale.*

Proof. The idea of the proof is a simple consequence of the decomposition given in equation (2.7) with $x < x'$, applied to $(z = X_t^c(x), z' = X_t^c(x'))$, which ensures the existence of an optional process $\xi_t(z, z') \in [X_t(x), X_t(x')]$ such that,

$$U(t, X_t^c(x')) - U(t, X_t^c(x)) = (X_t^c(x') - X_t^c(x))U_z(t, \xi_t(z, z'))$$

By a change of variable, $\xi_t(z, z')$ can be sent into $\psi_t(x, x')$ that belongs to the interval (x, x') by the formula $\xi_t(z, z') = X_t(\psi_t(x, x'))$. So, $U_z(t, \xi_t(z, z')) = U_z(t, X_t(\psi_t(x, x'))) = Y_t(u_z(\psi_t(x, x')))$. Since by assumption, $U(t, X_t^c(x')) - U(t, X_t^c(x))$ is a martingale, equal to $(X_t^c(x') - X_t^c(x))Y_t(u_z(\psi_t(x, x')))$, this last quantity is also a martingale, which is the required property. \square

Our main result is that this necessary condition is also sufficient. The argument uses the approximation of the Stieltjes integral $\int_{x_0}^x Y_t(u_x(z))d_z X_t^c(z)$ defined on a compact interval $[x_0, x]$ with the help of Darboux sums obtained as follows: we start with a partition of the interval $[x_0, x]$ into N subintervals $[z_n, z_{n+1}]$ where the mesh approaches zero, and we consider the following sequences

$$S_t^N(x_0, x) = \sum_{n=0}^{N-1} Y_t(u_z(\bar{z}_t^n))(X_t^c(z_{n+1}) - X_t^c(z_n)), \quad (2.8)$$

where \bar{z}_t^n is a random process in the interval $[z_n, z_{n+1}]$. Given the continuity in z of $z \rightarrow Y_t(u_z(z))$ and $z \rightarrow X_t^c(z)$, the Darboux theorem states that all the Darboux sums converge to the Stieltjes integral when the mesh goes to 0.

Theorem 2.4. *Let $\mathbf{U} \in \mathfrak{U}(X^c, u)$ be a dynamic utility with adjoint process Y (i.e., $U_z(t, X_t^c(z)) = Y_t(u_z(z))$). Assume that for any (x, x') , $x < x'$, there exists an optional process $\psi_t(x, x')$ taking values in the interval (x, x') , such that the process $(X_t^c(x') - X_t^c(x))Y_t(u_x(\psi_t(x, x')))$ is a martingale. Then, $\{U(t, X_t^c(x))\}$ is a martingale for any $x > 0$ and \mathbf{U} is a revealed utility.*

Proof. [Theorem 2.4] We use the approximations based on Darboux sums centered around the processes $\bar{z}^n(t) = \psi_t(z_n, z_{n+1})$. By assumption, these Darboux approximations $S_t^N(x_0, x)$ are finite sum of positive martingales, and then also positive martingales. By the positive Fubini theorem, we can interchange limit and expectation so that the martingale property is preserved and $\int_{x_0}^x Y_t^c(u_z(z))d_z X_t^c(x)$ is a martingale, with expectation $\int_{x_0}^x u_z(z)dz = u(x) - u(x_0)$. Once again, by monotony, the random variables $\int_{x_0}^x Y_t^c(u_z(z))d_z X_t^c(x)$ go to a limit with finite expectation. So, the Stieltjes integral is well-defined up to 0 and $\Psi_0^X(t, x) = \int_0^x Y_t^c(u_z(z))d_z X_t^c(x)$ is a martingale. So, the revealed utility process is given by $U(t, x) = \int_0^x Y_t^c(u_z(X^c(t, z)))dz$. \square

The primal and dual problems being similar, the same reasoning and proofs remain valid when studying the recovery dual problem, the following result is obvious from the above one.

Corollary 2.5. *Let $\tilde{\mathbf{U}} \in \tilde{\mathfrak{U}}(Y^c, \tilde{u})$ with the associated primal process denoted X (i.e., $\tilde{U}_y(Y_t^c(y)) = X_t(-\tilde{u}_y(y))$). $\tilde{\mathbf{U}}$ is a revealed dual utility if and only if for any $y < y'$ there exists an optional process $\phi_t(y, y')$ taking values in the interval (y, y') , such that the process $(Y_t^c(y') - Y_t^c(y))X_t(-\tilde{u}_y(\phi_t(y, y')))$ is a martingale.*

2.4.2 Supermartingale conditions

In the main result, the existence of a process $\psi_t(z, z')$ can be difficult to establish. The proof based on Darboux sums suggest to relax the martingale assumptions in the *only sufficient condition*, that the common Darboux sums, $S_t^{N,up}(x_0, x) = \sum_{n=0}^{N-1} Y_t(u_z(z_n))(X_t^c(z_{n+1}) - X_t^c(z_n))$ and $S_t^{N,down}(x_0, x) = \sum_{n=0}^{N-1} Y_t(u_z(z_{n+1}))(X_t^c(z_{n+1}) - X_t^c(z_n))$ are respectively supermartingales and submartingales. Since the both sequences have the same limit, this limit is expected to be a martingale.

Theorem 2.6. *Let $\mathbf{U} \in \mathfrak{U}(X^c, u)$ be dynamic utility with adjoint process Y . Assume that for $x' > x$, the positive process $\{Y_t(u_z(x))(X_t^c(x') - X_t^c(x))\}$ is a supermartingale and that $\{Y_t(u_z(x'))(X_t^c(x') - X_t^c(x))\}$ is a submartingale, then the dynamic utility \mathbf{U} is a revealed utility.*

Proof. Let $0 < x_0 < x$ and consider a partition of the interval $[x_0, x]$ into N subintervals $[z_n, z_{n+1}]$ where the mesh approaches zero. We approach the integral $\int_{x_0}^x Y_t(u_z(z))d_z X_t^c(z)$ by above, respectively by below, by the Darboux sums $S_t^{N,up}(x_0, x)$, respectively by $S_t^{N,down}(x_0, x)$. Thanks to the monotony of the processes \mathbf{Y} and \mathbf{X}^c , the Darboux sums $S_t^{N,down}(x_0, x)$ and $S_t^{N,up}(x_0, x)$ are bounded above, and converge a.s. to the Stieltjes integral $\int_{x_0}^x Y_t(u_z(z))d_z X_t^c(z)$.

Furthermore, by assumption, the sum $S_t^{N,up}(x_0, x)$ is a positive supermartingale, while the sum $S_t^{N,down}(x_0, x)$ is a positive submartingale.

By the positive Fubini theorem, for fixed $x_0 > 0$, one can interchange the $\lim_{N \rightarrow \infty}$ and the expectation to justify that the sub- and super- martingale properties are preserved at the limit. So, $\int_{x_0}^x Y_t(u_z(z))d_z X_t^c(x)$ is a martingale, with expectation $\int_{x_0}^x u_z(z)dz = u(x) - u(x_0)$.

Once again, by monotony, the integrals $\int_{x_0}^x Y_t(u_z(z))d_z X_t^c(x)$ go to a limit with finite expectation. So, the Stieltjes integral is well-defined up to 0 and $\{\Psi^X(t, x) = \int_0^x Y_t(u_z(z))d_z X_t^c(z)\}$ is a martingale; that implies that the dynamic utility $\{U(t, x) = \Psi^X(t, \mathcal{X}^c(t, x))\}$ is a revealed dynamic utility. \square

2.4.3 Forward starting dates and time consistency property

Definition 2.2 of the revealed utility \mathbf{U} is forward in time since $U(t, \cdot)$ is characterized from its initial condition u . In many situations, especially in optimization issues, the problem is formulated in a backward way, from the horizon T of the problem, from a given final utility $U(T, \cdot)$. The question is to characterize the value function, sometimes called *indirect utility*, from the dynamic programming principle.

The forward version of this problem requires to define a family of processes, starting from x at any time s , together with a dynamic time-consistency constraint. At the end, the problem is to generalize the stochastic representation of the deterministic function $u_z(x) = \mathcal{Y}_t(U_z(t, X_t^c(x)))$ where $\mathcal{Y}(t, z)$ is the inverse flow of $Y_t(y)$.

The data are still a characteristic process \mathbf{X}^c starting from x at time 0, with inverse process $\{\mathcal{X}^c(t, z)\}$, an utility function u , and an adjoint process $\{Y_t(u_z(x))\}$. Now, the problem is to build a revealed dynamic utility with respect to the filtration $(\mathcal{F}_t)_{t \geq s}$ starting for (x, s) .

DEFINITION OF THE FORWARD CHARACTERISTIC PROCESSES: Let us define the new families of characteristic processes, from the old processes \mathbf{X}^c and \mathbf{Y} and their inverses \mathcal{X}^c and \mathcal{Y} as follows, for $s < t$,

$$\begin{cases} X_t^c(s, x) = X_t^c(\mathcal{X}^c(s, x)), & \text{and } X_t^c(x) := X_t^c(s, X_s^c(x)) \\ Y_t(s, y) := Y_t(\mathcal{Y}(s, y)), & \text{and } Y_t(s, y) := Y_t(\mathcal{Y}(s, y)). \end{cases} \quad (2.9)$$

The starting value, at time s , of a time-consistent "revealed" dynamic utility is $U(s, x)$.

Proposition 2.7 (Backward Formulation). *Under assumptions of Theorem 2.4, the revealed dynamic utility \mathbf{U} , generated by $(\mathbf{X}, \mathbf{Y}, u)$, satisfies the consistency property, given in the backward formulation by*

$$U(0, x) = u(x) = \int_0^x \mathcal{Y}_T(U_z(t, X_t^c(z))) dz, \quad \forall t \geq 0.$$

This representation holds also for any times $(s, t), t \geq s$,

$$U(s, x) = \int_0^x \mathcal{Y}_t(s, U_z(t, X_t^c(s, z))) dz. \quad (2.10)$$

By the same manner, the dual dynamic revealed utility satisfies the backward representation

$$\tilde{U}(s, x) = \int_y^\infty \mathcal{X}_t^c(s, -\tilde{U}_y(t, Y_t(s, z))) dz. \quad (2.11)$$

Usually the value function is written as a conditional expectation of the final value of the problem, whereas here we have a pathwise representation.

Proof. Using the semigroup property for \mathbf{X}^c and \mathbf{Y} , $X_t^c(x) = X_t^c(s, X_s^c(x))$ and $Y_t(u_z(x)) = Y_t(s, Y_s(u_z(z)))$, we verify the first order conditions for the new delayed processes, for $t \geq s$,

$$U_z(t, X_t^c(x)) = U_z(t, X_t^c(s, X_s^c(x))) = Y_t(s, Y_s(u_z(z))).$$

Recall the identity $U_z(t, x) = Y_t(u_z(\mathcal{X}_t^c(x)))$, where \mathcal{X}^c is the inverse of \mathbf{X}^c . Then, the same dynamic identity holds true

$$U_z(t, X_t^c(s, x)) = Y_t(s, Y_s(u_z(\mathcal{X}(s, x)))) = Y_t(s, U_z(s, x))$$

and the backward formulation becomes clear $\mathcal{Y}_t(s, U_z(t, X_t^c(s, x))) = U_z(s, x)$. \square

This way to recover marginal dynamic utility from two monotonic processes has been highlighted for the first time in the classical portfolio optimization problem with dynamic forward utility in Itô's framework in M.Mrad & N.El Karoui (2013) [EKM13]. The solution is based on stochastic analysis using very technical arguments.

Link with optimization problem: The time-consistency property is important when optimizing the expected utility over a class of processes \mathcal{X} . With this condition and under the assumption that the class \mathcal{X} is rich enough, we deduce by the dynamic programming principle that $\forall X \in \mathcal{X}$, the process $V(t, X_t)$ is necessarily a supermartingale and martingale along the optimal (characteristic) trajectory, this property is called \mathcal{X} -consistency. In the case of financial semimartingale market, one can see in a previous works [EKM13, EKH18a] that revealed utility implies \mathcal{X} -consistency (\mathcal{X} is not necessarily unique). This is not true in all generalities, but in the financial market framework, the property of strong orthogonality between wealth and state price is satisfied.

2.5 Orthogonality Conditions and Risk aversion

2.5.1 Orthogonality conditions: definitions and remarks

In Definition 2.3, we have introduced a martingale condition satisfied by the product of two stochastic processes and said that the two processes are orthogonal. In particular, the processes $\{X_t^c(x), Y_t(u_x)\}$ or $\{X_x^c(t, x), Y_t(u_x)\}$ are often supposed to be orthogonal. This notion of orthogonality usually concerns the product of two martingales $\{M_t, N_t\}$, for which it is not important to precise the initial condition, because $\{M_t \cdot N_t\}$ is a martingale, if and only if $\{(M_t - M_0) \cdot (N_t - N_0)\}$ is a martingale. When the orthogonality condition is applied to general optional processes, called $\{\Phi_t(x), \Psi_t(y)\}$ in all generality, the difference is important because the martingale property of $\{\Phi_t(x)\Psi_t(y)\}$ is no more equivalent to the martingale property of $\{(\Phi_t(x) - x)(\Psi_t(y) - y)\}$. So it is important to clarify the role of the initial condition in the definition of the orthogonality.

DIFFERENT NOTIONS OF ORTHOGONALITY: In the linear example, the orthogonality condition introduced in Proposition 2.1 between the processes $\{x\bar{X}_t, Y_t(u_x(x))\}$ is the key of the martingale property of the revealed utility. But the role of initial conditions of \mathbf{X}^c and \mathbf{Y} is hidden by the linearity in x of \mathbf{X} . In particular, in this example, the processes $\Phi_t(x) = X_t^c(x)$ and $\Psi_t(y) = Y_t(y)$ are orthogonal for any $x > 0$ and $y > 0$, the random fields (Φ, Ψ) are said to be *strongly orthogonal*. In the case of differentiable characteristic processes, the orthogonality condition introduced in Proposition 2.2, called u_x -orthogonality, concerns the processes $(\Phi_t(x) = X_x^c(t, x))$ and $(\Psi_t(u_x(x)) = Y_t(u_x(x)))$, with linked initial conditions $y = u_x(x)$.

Definition 2.6. Let (Φ, Ψ) be two optional positive monotonic random fields, and $v(z) = -u_z(z)$ be a increasing function with range $[0, \infty]$. Then, we are concerned by the following different notions of orthogonality:

$$\left\{ \begin{array}{l} \text{strong orthogonality : } \{\Psi_t(y)\Phi_t(z)\} \text{ is a martingale } \forall (z, y > 0). \\ v\text{-weak orthogonality : } \{\Psi_t(v(x))\Phi_t(x)\} \text{ is a martingale for any } x > 0. \\ v\text{-sub orthogonality : } \forall x' > x, \{\Psi_t(v(x))(\Phi_t(x') - \Phi_t(x))\} \text{ is a supermartingale} \\ \text{and } \{\Psi_t(v(x'))(\Phi_t(x') - \Phi_t(x))\} \text{ is a positive submartingale.} \end{array} \right. \quad (2.12)$$

Obviously the strong orthogonality implies all the other forms of orthogonality.

In the recovery problem, we are concerned with the monotonic process $\mathbf{X}^c (= \Phi)$ and the adjoint process $\mathbf{Y} (= \Psi)$, satisfying different notions of orthogonality expected to imply the martingale property of $U(t, X_t^c(x))$. From previous results, the *strong orthogonality* implies the martingale property of the two processes $U(t, X_t^c)$ and $\tilde{U}(t, Y_t)$. The *suborthogonality* implies only the martingale property of $U(t, X_t^c)$, but not implies necessarily the *weak orthogonality*. On another side, the *weak orthogonality* condition between \mathbf{X}^c and \mathbf{Y} is not sufficient to obtain martingales properties for $U(t, X_t^c)$ or $\tilde{U}(t, Y_t)$. The sufficient condition is the u_z -weak orthogonality

of \mathbf{Y} and \mathbf{X}_x^c . The weak-orthogonality, which may seem unusual, allows to recover complex dependency between characteristics process or adjoint process and the initial utility.

2.5.2 Risk Aversion Coefficient

In this paragraph, we paid attention to relative risk aversion coefficient of the recovered utility process. It will be shown how the initial relative risk aversion noted here $\gamma(z) = -zu_{zz}(z)/u_z(z)$ is diffused over time in a consistent manner, when the weak optimality holds true between $(\mathbf{X}^c, \mathbf{Y})$ and between $(\mathbf{X}_x^c, \mathbf{Y})$. Since the risk aversion coefficient is associated with regular utility function, we assume as much regularity as necessary on the process $x \rightarrow X_t^c(x)$ and $y \rightarrow Y_t(y)$.

Proposition 2.8 (Risk aversion coefficient).

(i) The relative risk coefficient $\Gamma(t, z) = -zU_{zz}(t, z)/U_z(t, z)$ of compatible utility process \mathbf{U} verifies

$$\Gamma(t, X_t^c(x)) \frac{X_x^c(t, x)}{X_t^c(x)} = -\frac{\partial_x(Y_t(u_z(x)))}{Y_t(u_z(x))} \quad (2.13)$$

When the adjoint process Y is linear in y , $\Gamma(t, X_t^c(x)) \frac{xX_x^c(t, x)}{X_t^c(x)} = \gamma(x)$.

If the two processes X^c and Y are linear in their initial conditions, then the risk aversion is constant in time $\Gamma(t, X_t^c(x)) = -\frac{xu_{zz}(x)}{u_z(x)} = \gamma(x)$.

(ii) For revealed utility, with weak orthogonality between $(X_t^c(x)$ and $X_x^c(t, x))$ and $Y_t(u_z(x))$, the following martingale properties hold true:

- The process $\partial_x(Y_t(u_z(x)))X_t^c(x)$ is a martingale.
- The process $\Gamma(t, X_t^c(x))Y_t(u_z(x))X_x^c(t, x)$ is a martingale.

Proof. (Sketch of proof without index c .)

(i) By definition, the relative risk aversion along the characteristic process X is

$$\Gamma(t, X_t(x)) = -X_t(x)U_{zz}(t, X_t(x))/U_{zz}(t, X_t(x)).$$

It follows, differentiating the first order condition $U_z(t, X_t(x)) = Y_t(u_z(x))$ that $X_x(t, x)U_{zz}(t, X_t(x)) = \partial_x(Y_t(u_z(x)))$. Thus

$$\Gamma(t, X_t(x)) = -\frac{\partial_x(Y_t(u_z(x)))}{Y_t(u_z(x))} \frac{X_t(x)}{X_x(t, x)}.$$

Moreover, if Y is linear in y that is $Y_t(y) = y\bar{Y}_t$, then

$$\frac{\partial_x(Y_t(u_z(x)))}{Y_t(u_z(x))} = \frac{u_{zz}(x)}{u_z(x)}, \text{ and } \Gamma(t, X_t(x)) \frac{xX_x(t, x)}{X_t(x)} = \gamma(x).$$

Finally $X_t(x) = x\bar{X}_t$ implies $\frac{xX_x(t, x)}{X_t(x)} = 1$. This achieves the proof of (i).

(ii) By assumption $M_t(x) = Y_t(u_z(x))X_t(x)$, and $N_t(x) = Y_t(u_z(x))X_x(t, x)$ are regular martingales. Since $\partial_x M_t(x) = \partial_x(Y_t(u_z(x)))X_t(x) + N_t(x)$, the process $\partial_x(Y_t(u_z(x)))X_t(x)$ is also a martingale starting from xu_{xx} .

For risk aversion point of view, the process $\Gamma(t, X_t(x))N_t(x)$ is a martingale. \square

3 Applications

The purpose of this section is to illustrate the previous results by two applications in finance and economics, the first concerns an aggregation problem, the second a Markov equilibrium, following [HL93].

3.1 Aggregated portfolio

Consider a group of agents who invest in a financial market according to their own preferences. Our aim is to characterize a representative agent of the aggregated population and his representative preference. The agents are classified into classes represented by their forward dynamic utility U^θ and their "optimal" wealth (\mathbf{X}^θ, x) , in the sense where $U^\theta(t, X_t^\theta(x))$ is a martingale. In a financial market, all admissible portfolios are strongly orthogonal to a family of adjoint processes (also known as state price processes). Also it is reasonable to assume that,

For any (θ, θ') , the processes $(\mathbf{X}^\theta, \mathbf{Y}^{\theta'})$ are strongly orthogonal.

The weight of the different classes is quantified by a finite positive measure $m(d\theta)$ on a metric space Θ . The aggregation can be applied at different granularity levels: one can aggregate each agent individually, or aggregate different classes of agents having the same preferences and the same strategy inside the class (for example θ may be interpreted as the risk aversion parameter of the class and $m(d\theta)$ the proportion of this class among the whole).

3.1.1 Aggregated universe

To define a characteristic process for the aggregated economy with (global) wealth x , we need to know how the global wealth is shared between the different agents, that is to know the functions $(x^\theta(x))$ such that $x = \int x^\theta(x) m(d\theta)$. To guarantee the monotony of the various aggregated processes, the functions $x \rightarrow x^\theta(x)$ are assumed increasing and differentiable, with range $(0, \infty)$. The simplest example is the linear case as in [EKHM18b], where $x^\theta(x) = \alpha^\theta x$ with $\int \alpha^\theta m(d\theta) = 1$.

Then, the aggregated characteristic process is defined from the same representation,

$$X_t^{(m)}(x) = \int_{\Theta} X_t^\theta(x^\theta(x)) m(d\theta)$$

is an increasing process with range $[0, \infty)$.

The next step is to define the initial (deterministic) utility of the representative agent of the aggregated economy. As usual in convex analysis, the aggregation concerns the marginal utilities and not the utilities themselves. So, under the assumption that $u_z^\theta(x^\theta(z))$ is z -integrable in $z = 0$, the initial marginal utility of the aggregated problem is defined by

$$u_z^{(\mu)}(z) = \int_{\Theta} u_z^\theta(x^\theta(z)) m(d\theta) = \int_{\Theta} y^\theta(u_z(z)) m(d\theta). \quad (3.1)$$

Motivated by this construction, a natural choice of the increasing adjoint process and the marginal utility dynamics \mathbf{U}_z associated with it is

$$\begin{cases} X_t^{(\mu)}(x) = \int X_t^\theta(x^\theta(x)) m(d\theta) & (3.2) \\ Y_t^{(\mu)}(u_z(x)) := \int_\Theta Y_t^\theta(u_z^\theta(x^\theta(x))) m(d\theta) = \int Y_t^\theta(y^\theta(u_z(x))) m(d\theta) & (3.3) \\ U_z^{(\mu)}(t, X_t^{(\mu)}(x)) = \int_\Theta U_z^\theta(t, X_t^\theta(x^\theta(x))) m(d\theta) & (3.4) \end{cases}$$

The dynamic utility $\mathbf{U}^{(\mu)}$, which is compatible with $(X^{(\mu)}, Y_t^{(\mu)}, u^{(\mu)})$, is expected to be a *revealed utility*, such that $\{U(t, X_t^c(x))\}$ is a martingale. To satisfy this property, we have seen that it is sufficient to verify the strong orthogonality of the processes $(\mathbf{X}^{(\mu)}, \mathbf{Y}^{(\mu)})$ which will be obtained for instance from the strong orthogonality of the family $(\mathbf{X}^\theta, \mathbf{Y}^{\theta'})$.

Proposition 3.1. *Consider the characteristic processes of the aggregated economy $(\mathbf{X}^{(\mu)}, \mathbf{Y}^{(\mu)}, u)$ defined by Equations (3.3),(3.1),(3.4). Assume that :*

for any (θ, θ', x, y) , $X_t^\theta(x) Y_t^{\theta'}(y)$ is a martingale.

Then, the processes $X^{(\mu)}(x)$ and $Y^{(\mu)}(y)$ are strongly orthogonal and the utility $U^{(\mu)}(t, z)$ (3.4) is a revealed utility with characteristic process X^θ .

Proof. From previous results, it is sufficient to show that $(\mathbf{X}^{(\mu)}, \mathbf{Y}^{(\mu)c})$ are strongly orthogonal, or equivalently that $X_t^{(\mu)}(x) Y_t^{(\mu)}(y)$ is a martingale. By (positif) Fubini's Theorem, and the definition of $\mathbf{X}^{(\mu)}$, and $\mathbf{Y}^{(\mu)}$,

$$X_t^c(x) Y_t^c(y) = \iint m(d\theta) \mu(d\theta') X_t^\theta(x^\theta(x)) Y_t^{\theta'}(y^{\theta'}(y)).$$

The martingale property of $X_t^\theta(z) Y_t^{\theta'}(y)$ is extended to the product $X_t^c(x) Y_t^c(y)$, once again thanks to positif Fubini theorem. \square

Links with the Pareto wealth allocation We come back to the definition of the initial marginal utility from the wealth sharing functions $\{x^\theta(x)\}$. The question of the "optimal" choice of these functions is associated with the following optimization problem, known as the Pareto allocation problem: "Find the best allocation $\{x^\theta(x)\}$ such that $\int_\Theta x^\theta(x) m(d\theta) = x$, maximizing the sup-convolution criterium $\sup \int_\Theta u^\theta(x^\theta(x)) m(d\theta)$ ". An optimal solution $\{x^{\theta,*}(x)\}$, (if there exists), must satisfy the first order condition on the marginal rate of substitution, $(\forall x, \forall \theta, u_z^\theta(x^{\theta,*}(x)) = u_z(x))$. This optimal choice is Pareto optimal in the sense that the wealth is allocated in the most efficient manner, but this choice does not imply equality or fairness. The extension to dynamic processes and forward utility is discussed in [EKM17]

3.2 Economic Equilibrium

The neo-classical equilibrium theory, whose the two pillars are the representative agent, and the individual rationality, is a natural field for the application of the forward point of view. As in S.Wang [Wan93], A. Bick [Bic90] or H.He et H.Leland [HL93], the standard problem is to study the optimal portfolio, selected by a rational agent optimizing its expected utility at a given horizon $J(T, x)$ of its wealth. When the market is complete, the problem is easy to solve by duality, simplified by the uniqueness of the pricing kernel $\{Y_t\}$ (adjoint process in our framework). At time T , the optimal wealth $X_T^*(x)$ is deduced from the pricing kernel \mathbf{Y} , by the *first order condition* $J_x(T, X_T^*(x)) = c(x)Y_T$, relation equivalent to $X_T^*(x) = J_z^{-1}(T, c(x)Y_T) = -\tilde{J}_y(T, c(x)Y_T)$. The Lagrange multiplier $c(x)$ is adjusted by the initial wealth constraint $\mathbb{E}[Y_T X_T^*(x)] = \mathbb{E}[Y_T J_z^{-1}(T, c(x)Y_T)] = x$.

By the no arbitrage principle, the optimal portfolio process $\{X_t^{*,T}(x)\}$ is the replicating portfolio of $X_T^*(x)$, and the conditional "value function" $\{\mathcal{U}_t^{T,x}\}$ is the martingale $\mathbb{E}[J(T, J_z^{-1}(T, c(x)Y_T)) | \mathcal{F}_t]$. The Markov case will be detailed below.

The equilibrium policy is a priori independent of any maturity horizon T so that the first order condition $J_x(t, X_t^*(x)) = c(x)Y_t$ holds at any time; this point of view is close to that developed by S.Nadtochiy and M.Tehranchi [NT17] using a forward point of view in optimization problem.

Our approach is different, because formulated as a *revealed problem* in a complete financial market, without reference to any optimization problem. The data are, the unique market pricing kernel or adjoint process $\{Y_t(y)\}$ that is by definition strongly orthogonal to the risky asset and then to the equilibrium portfolio $\{X_t^e(x)\}$. The problem is to give conditions on the data $(\mathbf{X}^e, \mathbf{Y})$ that guarantee the existence of a *revealed dynamic utility* satisfying the first order condition. The main difference with the recovery problem defined above is that the adjoint process is given, but not the initial utility. A priori, it is not imposed that the revealed utility is the value function of a representative agent. This point of view is applied to the one-dimensional *Markov* framework as in [HL93] and yields to a complete characterization of the pair (equilibrium strategy and equilibrium preferences). Geometrical Brownian strategies coupled with power utility are the elementary solutions of the problem. All other solutions are obtained by aggregation of heterogeneous risk averse agents and are coherent with the theory of representative agent.

3.2.1 Equilibrium in He and Leland's framework

In [HL93], the authors examine the equilibrium strategy in a financial world which is dynamically complete, with only one risky asset and a bond whose price (and then the spot rate $\{r_t\}$) is exogenous and deterministic. The *asset price* \mathbf{S} follows a (one dimensional) stochastic differential equation (SDE) driven by a one dimensional Brownian motion \mathbf{W} , whose the drift (productivity) coefficient is $\mu(t, z)$ and the volatility coefficient is $\sigma(t, z)$. The *pricing kernel* (or adjoint process)

$\{yY_t^S\}$ is also a diffusion process whose volatility is the market risk premium $\{\eta^S\}$,

$$\begin{cases} dS_t = S_t(\mu(t, S_t)dt + \sigma(t, S_t)dW_t) = S_t(r_t dt + \sigma(t, S_t)(dW_t + \eta(t, S_t)dt), & S_0 = z & (3.5) \\ \eta_t^S = \eta(t, S_t) = (\mu(t, S_t) - r_t)/\sigma(t, S_t), & & (3.6) \\ dY_t^S = -Y_t^S(r_t dt + \eta(t, S_t) dW_t), & Y_0^S = 1. & (3.7) \end{cases}$$

By construction, the pair $\{(S_t(z), yY_t^S)\}$ is a Markov diffusion, the product $\{S_t(z)Y_t^S\}$ is a martingale. Moreover, under regularity assumptions (in time and space) on the coefficients $(\mu(t, z), \sigma(t, z))$, the process $z \rightarrow S_t(z)$ is increasing and z -differentiable.

Equilibrium problem as forward revealed utility problem In [HL93], the economy is in equilibrium, that is the total wealth is invested in the risky asset at each moment, and this regardless of the global market-utility. With the notations of the recovery problem, the characteristic process, here denoted $\{X_t^e(x)\}$ is the increasing risky asset, $x = z = S_0$, $X_t^e(x) = S_t(x)$. The *equilibrium revealed problem* consists in finding conditions on the coefficients of the processes $\{(S_t(x), yY_t^S)\}$ for the existence of a *deterministic time-dependent regular utility* $V^e(t, z)$ (\mathcal{C}^3 -regular in z) satisfying:

- The first order condition, $V_x^e(t, S_t(x)) = v_x^e(x)Y_t^S$, $V_x^e(0, x) = v_x^e(x)$.
- The martingale property of $\{V^e(t, S_t(x))\}$.

The first constraint imposes $v_x^e(x)Y_t^S$ being a regular decreasing function of $S_t(x)$, that is $Y_t^S = V_x^e(t, S_t(x))/v_x^e(x)$. Itô's formula gives the necessary closed relations between the Y^S -coefficients and the S -coefficients, and the V^e -derivatives as in [HL93]. But, the same formulation holds for the dual problem, easier to solve in complete market. The existence of $V^e(t, z)$ is equivalent to the existence of a dual conjugate utility $\tilde{V}^e(t, y)$ such that

$$S_t(x(y)) = -\tilde{V}_y^e(t, yY_t^S), \quad -\tilde{V}_y^e(0, y) = -\tilde{v}_y^e(y) = x(y) = (v^e(\cdot))^{-1}(y) \quad (3.8)$$

Plugging this identity in the Y^S -differential dynamics (3.7), the process $yY_t^S := Y_t^e(y)$ appears as solution of a SDE with return $-r_t$ and volatility $-\zeta(t, y) = -\eta(t, -\tilde{V}_y^e(t, y))$, starting from y . From the dual point of view, the revealed constraint is equivalent to the martingale property of the two process $\{\tilde{V}^e(t, Y_t^e(y))\}$ and $\{-Y_t^e(y)\tilde{V}_y^e(t, Y_t^e(y))\}$.

3.2.2 The paradigm of the geometrical Brownian motion in economy inequilibrium

We are looking for an economy, whose the pricing kernel is solution of the SDE(3.7).

$$dY_t^e(y) = Y_t^e(y)[-r_t dt - \zeta(t, Y_t^e(y))dW_t], \quad Y_0^e = y.$$

We first assume the existence of a revealed dual utility $\Phi(t, y)$ such that the two processes $\Phi(t, Y_t^e)$ and $-Y_t^e\Phi_y(t, Y_t^e)$ ($X_t^e = -\Phi_y(t, Y_t^e)$) are positive martingales. These conditions are very strong since the only solution $\{Y^e(t, y)\}$ to this problem is the geometrical Brownian motion ($GBM(-r_t, -\zeta_t)$) (where we assume for any time t , $\int_0^t (r_s + \zeta_s^2)ds < +\infty$).

Theorem 3.2. *A necessary condition for the economy to be in equilibrium is that the pricing kernel $Y^e(t, y)$ is a geometrical Brownian motion (GBM)*

$$Y_t^e(y) := yY_t^e = y \exp \left(- \int_0^t r_s ds - \int_0^t \zeta_s dW_s - \frac{1}{2} \int_0^t \zeta_s^2 ds \right) \quad (3.9)$$

Moreover, any dual revealed utility is solution of the following PDE,

$$\partial_t \Phi(t, y) + \frac{1}{2} y^2 \zeta_t^2 \Phi_{yy}(t, y) - y r_t \Phi_y(t, y) = 0 \quad (3.10)$$

A power dual utility $\Phi^{(\beta)}(t, y) = \tilde{H}(t, \beta) \frac{y^{1-\beta}}{\beta-1}$ is solution of this PDE if and only if $\beta > 1$ and $\tilde{H}(t, \beta) = \exp(-(\beta-1) \int_0^t (r_s + \frac{1}{2} \beta \zeta_s^2) ds)$.

Proof. The previous martingale conditions are equivalent to the PDE's system

$$\begin{cases} \partial_t \Phi(t, y) + [\frac{1}{2} (y\zeta)^2 \Phi_{yy} - y r_t \Phi_y](t, y) = 0, \\ [\partial_t (y\Phi_y) + \frac{1}{2} (y\zeta)^2 \partial_{yy} (y\Phi_y) - y r_t \partial_y (y\Phi_y)](t, y) = 0. \\ \partial_t \Phi_y(t, y) + [\frac{1}{2} (y\zeta)^2 \Phi_{yyy} + \partial_y (\frac{1}{2} (y\zeta)^2) \Phi_{yy} - r_t \partial_y (y\Phi_y)](t, y) = 0. \end{cases} \quad (3.11)$$

The last line is the derivative of the first with respect to y . But since $\partial_y (y\Phi_y)(t, y) = (\Phi_y + y\Phi_{yy})(t, y)$, then $\partial_{yy} (y\Phi_y)(t, y) = (2\Phi_{yy} + y\Phi_{yyy})(t, y)$, what implied that the second line is proportional to y . After simplification by y , and making the difference with the third line, it becomes that $\frac{1}{2} \partial_y (y\zeta)^2(t, y) - y\zeta^2(t, y) = 0$. An immediate consequence is that $\zeta(t, y)$ does not depend on y .

The function $\Phi^{(\beta)}(t, y) = \tilde{H}(t, \beta) \frac{y^{1-\beta}}{\beta-1}$ is solution of the PDE (3.14) iff \tilde{H} is a decreasing solution of the linear differential equation $\partial_t \tilde{H}(t, \beta) + \tilde{H}(t, \beta) (\beta-1) (r_t + \frac{1}{2} \beta \zeta_t^2) = 0$. Moreover, the power function is convex iff $\beta > 1$. \square

HJB equation for the primal utility. The primal market is defined by the utility function V associated with Φ and by the risky asset $\{S_t^e(x)\}$ defined by the first order condition, $S_t^e(x(y)) = -\Phi_y(t, y Y_t^S)$. If $\Phi^{(\beta)}(t, y)$ is a power function, then the associated utility $V^{(\alpha)}$ is also a power function, and the risky asset $\{S_t^{(\alpha)}\}$ is a GBM($r_t, \beta\zeta_t$),

$$V^{(\alpha)}(t, x) = H(t, \alpha) \frac{x^{1-\alpha}}{1-\alpha}, \quad \alpha = 1/\beta \quad \text{and} \quad H(t, \alpha) = (\tilde{H}(t, \beta))^{-1/\beta} \quad (3.13)$$

In general case, $V(t, x)$ is solution of an optimization problem in x i.e., by the conjugation relation $V(t, x) = \inf_{y>0} \{\Phi(t, y) + xy\} = \Phi(t, V_x(t, x)) + xV_x(t, x)$, Equation (2.1), the envelop theorem states that the partial derivatives with respect to the parameter t verifies $\partial_t V(t, x) = \partial_t \Phi(t, \cdot)(V_x(t, x))$, easy also by hand. Thus, the PDE satisfied by the primal utility V is obtained essentially by change of variables technique on the dual PDE, which we recall here

$$\partial_t \Phi(t, y) + \frac{1}{2} y^2 \zeta_t^2 \Phi_{yy}(t, y) - y r_t \Phi_y(t, y) = 0 \quad (3.14)$$

Proposition 3.3. *Let us consider the function V whose the derivative is the inverse of $-\Phi_y$. Then, this function satisfied the non linear PDE*

$$\partial_t V(t, x) - \frac{1}{2} \frac{V_z^2(t, x)}{V_{zz}(t, x)} \zeta_t^2 + r_t x V_z(t, x) = 0 \quad (3.15)$$

This PDE can be rewritten as an optimization program, of HJB-type,

$$\partial_t V(t, x) + \sup_{\kappa \in \mathbb{R}} \left\{ \frac{1}{2} x^2 V_{zz}(t, x) \kappa^2 + x V_z(t, x) \kappa \zeta_t \right\} + x V_z r_t = 0 \quad (3.16)$$

where the supremum is achieved on $\kappa_t^* = \sigma(t, x) := \zeta_t \frac{-V_z(t, x)}{x V_{zz}(t, x)} = \frac{\zeta_t}{\gamma^V(t, x)}$, where γ^V is the risk aversion coefficient of V .

Proof. As mentioned above, we use change of variable techniques: we replace y by $V_z(t, x)$ in the dual PDE (3.14). The identity $\partial_t V(t, x) = \partial_t \Phi(t, \cdot)(V_z(t, x))$, (Envelope Theorem), combined with identities $\Phi_y(t, V_z(t, x)) = -x$ and $\Phi_{yy}(t, V_z(t, x)) V_{zz}(t, x) = -1$, since $\Phi_y(t, \cdot)$ is the inverse function of $-V_z(t, \cdot)$, lead without much more effort to the primal PDE

$$\partial_t V(t, x) - \frac{1}{2} \frac{V_z^2(t, x)}{V_{zz}(t, x)} \zeta_t^2 + r_t x V_z(t, x) = 0$$

This non linear PDE is well-known to be the value function of some optimal control problem; to give a short proof, let us introduce the quadratic polynomial in κ , $Q(\alpha, \beta)(\kappa_t) = \frac{1}{2} \alpha \kappa_t^2 + \beta \kappa_t$ where $\alpha = x^2 V_{zz}(t, x) < 0$ and $\beta = x V_z(t, x) \zeta_t$. By definition, this quadratic form is maximal at $\kappa_t^* = -\frac{\beta}{\alpha} = -\frac{V_x \zeta_t}{x V_{zz}}(t, x)$ and the maximum is

$$\max_{\kappa} Q(\alpha, \beta)(\kappa) = -\frac{1}{2} \frac{\beta^2}{\alpha} = -\frac{1}{2} \frac{V_z^2 \zeta_t^2}{V_{zz}}(t, x)$$

This term is exactly the non linear part of the PDE (3.15). Observe that the optimal parameter $\kappa_t^* = -\frac{\beta}{\alpha} = -\frac{V_x \zeta_t}{x V_{zz}}(t, x)$ is proportional to the inverse of the *relative risk aversion coefficient* of the V function, introduced in Section 2.1.1, and denoted $\gamma^V(t, x)$. So HJB equation can be rewritten as the supremum of linear operators, and

$$\partial_t V(t, x) + \sup_{\kappa \in \mathbb{R}} \left\{ \frac{1}{2} x^2 V_{zz}(t, x) \kappa^2 + x V_z(t, x) \kappa \zeta_t \right\} + x V_z r_t = 0 \quad (3.17)$$

At the optimum the PDE is linearized by putting $\sigma(t, x) = \kappa^*(t, x)$ and $\sigma(t, x) \gamma^V(t, x) = \zeta_t$. \square

Description of an economy in equilibrium Assume the economy to be in equilibrium, with a revealed utility function $V(t, x)$ satisfying the non-linear PDE (3.15). The pricing kernel is a GBM($-r_t, -\zeta_t$) with deterministic volatility ($-\zeta_t$ in the dual point of view, $-\eta_t$ in the primal point of view) ($\eta_t \equiv \zeta_t$). Then, we have the main theorem:

Theorem 3.4. (i) *In this economic equilibrium, the volatility $\sigma(t, x)$ of the risky asset $S_t^e(x)$ is linked to the relative risk aversion coefficient $\gamma^V(t, x)$ by the relation $\sigma(t, x)\gamma^V(t, x) = \eta_t$.*

For power utility (Equation (3.13)) with relative risk aversion α , the equilibrium risky asset $X_t^{(\alpha)}(x)$ is linear in x i.e. $\{X_t^{(\alpha)}(x) = xX_t^{(\alpha)}\}$ and is a GBM($r_t, \eta_t/\alpha$).

(ii) *Consider the class \mathcal{X} of self-financing portfolios indexed by κ_t , with dynamics is*

$$dX_t^\kappa = X_t^\kappa [r_t dt + \kappa_t (dW_t + \eta_t dt)], \quad (3.18)$$

Moreover, the risky asset $X_t^e = S_t^e$ is optimal for the V -utility. This property can be interpreted as the existence of representative agent.

Proof. The proof stems quite simply from Proposition 3.3. Using the notation $L^{(\kappa)}V(t, x) = \partial_t V(t, x) + \frac{1}{2}x^2 V_{zz}(t, x)\kappa^2 + xV_z(t, x)\kappa\zeta_t + xV_z r_t$, for any $X^\kappa \in \mathcal{X}$, by Itô's formula

$$dV(t, X_t^\kappa) = L^{(\kappa)}V(t, X_t^\kappa)dt + X_t^\kappa V_z(t, X_t^\kappa)dW_t.$$

Note that the drift term is dominated by $\sup_{\kappa \in \mathbb{R}} L^{(\kappa)}V(t, x) \equiv 0$ from Equation (3.16). Then, for any $X^\kappa \in \mathcal{X}$, $V(t, X_t^\kappa)$ is a supermartingale and a martingale for $X_t^e = S_t^e$ with the optimal choice $\kappa_t^* = \sigma(t, x) := \zeta_t \frac{-V_z(t, x)}{xV_{zz}(t, x)} = \frac{\zeta_t}{\gamma^V(t, x)}$ (from Proposition 3.3), that is $\sigma(t, x)\gamma^V(t, x) = \zeta_t$ since $(\eta_t \equiv \zeta_t)$. So, a representative agent maximizes his expected utility at any times. \square

Remark: In [HL93], He and Leland studied the PDE satisfied by the ratio $f(t, x)$ of risk premium/volatility, given here by the relative risk aversion coefficient $\gamma^V(t, x)$. The PDE is very complicated and it was not clear that the risk premium is necessarily deterministic.

3.2.3 Pareto optimality at the equilibrium

Now we are concerned by the description of all equilibrium (primal or dual) utilities, and not only the power utilities. The proof is based on the description of the solutions of the "pricing" PDE (3.14) of the GBM($-r_t, -\zeta_t$) pricing kernel.

Aggregated power dual utility paradigm Since the PDE (3.11) is linear, we can reach the ideas of Section 3.1 and generate a more general class of solutions by aggregating the above dual power utilities in function of their risk tolerance coefficient β . But, a natural question is : *"Are there other solutions to this equilibrium problem than aggregated power functions? "*

The answer, given in the following theorem, is negative. In other words, the set of revealed dual equilibrium utilities is entirely described by the aggregation of dual power utilities. Similar questions were encountered by different authors in other frameworks, (T. Zariphopoulou and ali.[MZ10], M. Tehranchi and ali.[BRT09]), who were the first to use the Widder theorem, characterizing the positive space-time harmonic functions of the Brownian motion.

Theorem 3.5 (Widder 1963). *A function $\Psi : (0, \infty) \times \mathbb{R} \rightarrow \mathbb{R}$ is a positive classical solution to the heat equation, $\Psi_t(t, z) + \frac{1}{2}\Psi_{zz}(t, z) = 0$, if and only if it can be represented as*

$$\Psi(t, z) = \int_{\mathbb{R}} e^{\beta z - \frac{1}{2}\beta^2 t} m(d\beta), \quad (3.19)$$

where m is a Borel measure such that the above integral is finite for all $(t, z) \in (0, \infty) \times \mathbb{R}$.

The "revealed dual utility" version of this theorem is the following,

Theorem 3.6. *A regular function Φ is a revealed dual utility if and only if there exists a positive Borel measure $\mu(d\beta)$ on $(1, \infty)$ such that $\int_1^\infty \frac{y^{1-\beta}}{\beta-1} \mu(d\beta) < \infty$, such that*

$$\Phi^{(\mu)}(t, y) = \int_1^\infty \tilde{H}(t, \beta) \frac{y^{1-\beta}}{\beta-1} \mu(d\beta) = \int_1^\infty \Phi^{(\beta)}(t, y) \mu(d\beta) \quad (3.20)$$

where $\tilde{H}(t, \beta) = \exp(-(\beta-1) \int_0^t (r_s + \frac{1}{2}\beta \zeta_s^2) ds)$.

$\Phi^{(\mu)}$ is a solution of the PDE (3.14) whose the initial dual utility is necessarily an aggregate dual power utility.

Proof. The proof is based on the representation of the geometrical Brownian motion as a time-dependent function of a change of time Brownian motion. In addition of the square integrability of ζ , assume $\zeta_t > 0$ a.s. After a classical change of time driven by the inverse of the increasing process $A_t = \int_0^t \zeta_s^2 ds$, we can assume for simplicity that $\zeta_t \equiv 1$ for any t , and drop out ζ in the PDE (3.11) and replacing r_t by $\tilde{r}_t = r_t/\zeta_t^2$. The process $\{\tilde{Y}_t^e(y)\}$ is a time depending function of the Brownian motion $\{\tilde{W}_t = -W_t\}$, $\tilde{Y}_t^e(e^z) = \exp[z + \int_0^t -(\tilde{r}_s + 1/2)ds + \tilde{W}_t]$. Then, the martingale $\{\Phi(t, \tilde{Y}_t^e(e^z))\}$ is a function of the Brownian motion $\{\tilde{W}_t(z) = z + \tilde{W}_t\}$, $\Phi(t, \tilde{Y}_t^e(e^z)) = \Psi(t, \tilde{W}_t(z))$, where the function $\Psi(t, z) = \Phi(t, e^{-\int_0^t (\tilde{r}_s + 1/2)ds} e^z)$. The martingale property of $\{\Psi(t, \tilde{W}_t(z))\}$ implies that $\Psi(t, z)$ is a space-time harmonic solution of the heat equation $\partial_t \Psi(t, z) + \frac{1}{2}\Psi_{zz}(t, z) = 0$. By the Widder theorem, there exists a positive measure m such that $\Psi(t, z) = \Phi(t, e^{-\int_0^t (\tilde{r}_s + 1/2)ds} e^z) = \int_{\mathbb{R}} e^{[(1-\beta)z - \frac{1}{2}(1-\beta)^2 t]} m(d\beta)$, with $\beta > 1$. The proof of Theorem 3.6 when $\tilde{\zeta} = 1$ is achieved by taking the inverse change of variable $z = \ln(y) + \int_0^t (\tilde{r}_s + 1/2)ds$, and $\mu(d\beta) = (\beta-1)m(d\beta)$; the general case is attained after taking the inverse of the change of time. Monotony and convexity implies that $\beta > 1$. \square

Pareto equilibrium of aggregate economy with risk averse heterogeneous agents

The question is now to precise the additional properties of such economy given that the dual utility is an aggregation of dual power utility, (since we already known that all the expected properties hold true). The study of Section 3.1.1 helps us to precise the problem in more general context. Moreover, since our framework the heterogeneous agents have the same pricing kernel, we can refer to the notion of Pareto optimality defined at the end of Section 3.1.1, relative to the optimal allocation of the wealth between the different agents.

Characterization of the aggregated primal equilibrium utility. In the sequel, in the dual problem we use the parametrization $(\beta, \mu(d\beta))$ and in the primal problem the parametrization $(\alpha, \nu(d\alpha))$ with $\alpha = 1/\beta$ and $\nu(d\alpha)$ is the transport of $\mu(d\beta)$ by the function $\alpha = 1/\beta$ ($\int_0^1 k(\alpha)\nu(d\alpha) = \int_1^\infty k(1/\beta)\mu(d\beta)$).

When the dual revealed utility is the β -power function $\Phi^{(\beta)}(t, y) = \tilde{H}(t, \beta) \frac{y^{1-\beta}}{(\beta-1)}$, the primal revealed utility is also a power function $V^{(\alpha)}(t, x) = H(t, \alpha) \frac{x^{1-\alpha}}{(1-\alpha)}$. Moreover, the equilibrium portfolio is a *geometrical Brownian motion* (GBM) equal to $X_t^{(\alpha)}(x) = x \Phi_y^{(\beta)}(t, Y_t^e)$ by the first order condition.

In the general case of mixture of powers, the primal point of view is less explicit. Nevertheless, we will show that the primal utility function can be characterized as the value function of a Pareto optimal allocation (introduced at the end of Section 3.1.1).

First, we show this property for the primal utility at time 0, for which $\phi^{(\beta)}(y) = \frac{y^{1-\beta}}{\beta-1}$ and $\phi^{(\beta)}(y) + y\phi_y^{(\beta)}(y) = \beta\phi^{(\beta)}(y) = v^{(\alpha)}(-\phi_y^{(\beta)}(y)) := v^{(\alpha)}(x^{(\alpha)}(y))$, by the master equation. When considering the mixture of power dual utilities $\phi^{(\mu)}(y) = \int_1^\infty \phi^{(\beta)}(y)\mu(d\beta)$, some other quantities are also obtained by mixture, for example $\phi^{(\mu)}(y) + y\phi_y^{(\mu)}(y) = \int_1^\infty \beta\phi^{(\beta)}(y)\mu(d\beta)$. For the primal utility function $v^{(\nu)}(x)$, the property fails in general, but since the left side of the previous formula is the function $v^{(\nu)}(-\phi_y^{(\mu)}(y))$, we have that the mixture representatin still holds along the characteristic function $x^{(\nu)}(y) = -\phi_y^{(\mu)}(y)$.

Lemma 3.7. *Let us consider the aggregated economy at time 0, whose the dual utility $\phi^{(\mu)}$ is a mixture of dual power utilities. Then, the primal utility $v^{(\nu)}(x)$ read along the wealth $x^{(\nu)}(y) = -\phi_y^{(\mu)}(y)$ is still a mixture, and*

$$\begin{cases} \phi^{(\mu)}(y) = \int_1^\infty \frac{y^{1-\beta}}{\beta-1} \mu(d\beta), & x^{(\nu)}(y) = \int_0^1 x^{(\alpha)}(y) \nu(d\alpha) \\ v^{(\nu)}(x^{(\nu)}(y)) = \int_1^\infty \beta\phi^{(\beta)}(y) \mu(d\beta) = \int_0^1 v^{(\alpha)}(x^{(\alpha)}(y)) \nu(d\alpha). \end{cases} \quad (3.21)$$

$$\quad \quad \quad \begin{cases} v^{(\nu)}(x^{(\nu)}(y)) = \int_1^\infty \beta\phi^{(\beta)}(y) \mu(d\beta) = \int_0^1 v^{(\alpha)}(x^{(\alpha)}(y)) \nu(d\alpha). \end{cases} \quad (3.22)$$

Furthermore, the utility $v^{(\nu)}$ is the sup-convolution of power utilities $v^{(\alpha)}$

$$v^{(\nu)}(x) = \sup \left\{ \int_0^1 v^{(\alpha)}(z_\alpha) \nu(d\alpha); \int_0^1 z_\alpha \nu(d\alpha) = x \right\} \quad (3.23)$$

The "Pareto" supremum is achieved at the family $\{z^\alpha(x) := x^{(\alpha)}(v_z^{(\nu)}(x)), \alpha\}$.

Proof. The identity (3.22) is a simple consequence of the previous identities based on the master equation and the relation $\beta\phi^{(\beta)}(y) = v^{(\alpha)}(x^{(\alpha)}(y))$.

The identification of $v^{(\nu)}(x)$ as the solution of a sup convolution problem is based on the Fenchel transform,

$$\phi^{(\beta)}(y) := \max_{x>0} \{v^{(\alpha)}(x) - xy\} = v^{(\alpha)}(x^{(\alpha)}(y)) - yx^{(\alpha)}(y) \geq v^{(\alpha)}(z_\alpha) - z_\alpha y, \quad (z_\alpha > 0)$$

that is, for any $z_\alpha > 0$,

$$v^{(\alpha)}(x^{(\alpha)}(y)) \geq v^{(\alpha)}(z_\alpha) + y(x^{(\alpha)}(y) - z_\alpha).$$

Integrating in α and using Equation (3.22) implies that for any $z_\alpha > 0$.

$$v^{(\nu)}(x^{(\nu)}(y)) \geq \int_0^1 v^{(\alpha)}(z_\alpha) \nu(d\alpha) + y \int_0^1 (x^{(\alpha)}(y) - z_\alpha) \nu(d\alpha),$$

Then, taking the maximum over the class $\mathcal{Z}^{(\nu)} := \{z_\alpha : \int_0^1 z_\alpha \nu(d\alpha) = x^{(\nu)}(y)\}$, containing $(x^{(\alpha)}(y))_\alpha$ implies

$$v^{(\nu)}(x^{(\nu)}(y)) = \int_0^1 v^{(\alpha)}(x^{(\alpha)}(y)) \nu(d\alpha) = \max_{\mathcal{Z}^{(\nu)}} \left\{ \int_0^1 v^{(\alpha)}(z_\alpha) \nu(d\alpha) \right\},$$

To conclude, it suffices to replace y by $v_z^{(\nu)}(x)$. \square

Mainly due to the fact that we aggregate utility functions which are time-separable, this Pareto-optimal initial allocation is propagated with the same rule than at time 0, using an aggregating measure evolving randomly with the time. Indeed, using the same notations of the previous lemma, the first order condition allows to write the equilibrium risky asset as a mixture of GBM.

$$\text{(Mixing)} \begin{cases} X_t^{(\nu)}(x^{(\nu)}(y)) = \int_1^\infty y^{-\beta} \tilde{H}(t, \beta) (Y_t^e)^{-\beta} \mu(d\beta) = \int_1^\infty x^{(\alpha)}(y) X_t^{(\alpha)} \nu(d\alpha). \\ V^{(\nu)}(t, X_t^{(\nu)}(x^{(\nu)}(y))) = \int_0^1 V^{(\alpha)}(t, x^{(\alpha)}(y) X_t^{(\alpha)}) \nu(d\alpha), \end{cases} \quad (3.24)$$

where the last identity is easily obtained from optimality and from the Master equation. Starting from (3.24) and applying the same proof as in the previous lemma, we get the following result.

Theorem 3.8. *Let $V^{(\nu)}$ be the utility of a representative agent. A economic equilibrium holds if and only if there exists a positive Borel measure ν on $(0, 1)$ such that,*

(i) *The utility process $V^{(\nu)}$ is given as the sup-convolution:*

$$V^{(\nu)}(t, x) = \sup \left\{ \int_0^1 V^{(\alpha)}(t, z_\alpha) \nu(d\alpha); \int_0^1 z_\alpha \nu(d\alpha) = x \right\}$$

The supremum is achieved at the family $\{x^\alpha(t, x) := (V_z^{(\alpha)})^{-1}(t, V_z^{(\nu)}(t, x)), \alpha\}$ satisfying the condition $\int_0^1 x^\alpha(t, x) \nu(d\alpha) = x$.

(ii) **ECONOMIC INTERPRETATION:** *Assume the initial wealth optimally Pareto allocated, then at any time the allocation generated by the individual optimal wealth processes $X^{(\alpha)}(t, x^{(\alpha)}(x))$ is Pareto optimal for the aggregated utility $V^{(\nu)}(t, X_t^{(\nu)}(x))$ and the optimal wealth at time t .*

Proof. Using the notation $x^{(\nu)}(y) = -\Phi_y^{(\mu)}(y) = x$ in equation (3.24), follows

$$V^{(\nu)}(t, X_t^{(\nu)}(x)) = \int_0^1 V^{(\alpha)}(t, x^{(\alpha)}(v_z^{(\nu)}(x)) X_t^{(\alpha)}) \nu(d\alpha).$$

As $X_t^{(\nu)}(x) = -\Phi_y^{(\mu)}(t, v_z^{(\nu)}(x)Y_t)$, it is a monotonic function in x with inverse $(v_z^{(\nu)})^{-1}(V_z^{(\nu)}(t, x)/Y_t)$, we obtain

$$V^{(\nu)}(t, x) = \int_0^1 V^{(\alpha)}(t, x^{(\alpha)}(V_z^{(\nu)}(t, x)/Y_t)X_t^{(\alpha)})\nu(d\alpha).$$

Now, as $x^{(\alpha)}(y)X_t^{(\alpha)} = -\Phi_y^\beta(t, yY_t)$ ($\beta = 1/\alpha$), the quantity $x^{(\alpha)}(V_z^{(\nu)}(t, x)/Y_t)X_t^{(\alpha)}$ is equal to $-\Phi_y^\beta(t, V_z^{(\nu)}(t, x))$. In other words,

$$V^{(\nu)}(t, x) = \int_0^1 V^{(\alpha)}(t, -\Phi_y^\beta(t, V_z^{(\nu)}(t, x)))\nu(d\alpha).$$

Moreover, identical reasoning as in the proof of the previous result, using the inequality

$$V^{(\alpha)}(t, -\Phi_y^\beta(t, y)) \geq V^{(\alpha)}(t, z_\alpha) + (\Phi_y^\beta(t, y) - z_\alpha)y, \quad \forall z_\alpha > 0, \quad (3.25)$$

integrating and replace y by $V_z^{(\nu)}(t, x)$, follows

$$V^{(\nu)}(t, x) \geq \sup\left\{\int_0^1 V^{(\alpha)}(t, z_\alpha)\nu(d\alpha) + V_z^{(\nu)}(t, x) \int_0^1 (\Phi_y^\beta(t, V_z^{(\nu)}(t, x)) - z_\alpha)\nu(d\alpha)\right\}, \quad (3.26)$$

with equality iff $\int_0^1 \Phi_y^{1/\alpha}(t, V_z^{(\nu)}(t, x))\nu(d\alpha) = \int_0^1 z_\alpha\nu(d\alpha)$, in this case the supremum is achieved at $z_\alpha = -\Phi_y^\beta(t, V_z^{(\nu)}(t, x)) = (V_z^{(\alpha)})^{-1}(t, V_z^{(\nu)}(t, x))$. But, using the definition of $\Phi^{(\mu)}$ as a mixture, one can observe that the integral $\int_0^1 \Phi_y^{1/\alpha}(t, V_z^{(\nu)}(t, x))\nu(d\alpha)$ is equal to x . This achieves the proof. \square

Conclusion: In this work we have provided a necessary and sufficient condition for the existence of solution to the general problem of revealed utility, using very basic tools of analysis and the theory of integration. We have made almost no assumptions about the regularity of the processes in time t and the constructed utilities are only differentiable in x . To be aware of the efficiency of our method and the extent of our results, it is enough to refer to existing work in the semimartingale framework where calculations are tedious and assumptions are numerous. Requiring to treat the problem in an abstract way has clarified many subtleties including the role of the initial conditions and how to deal with the Stieltjes integral near to zero. The different notions of orthogonality introduced for the first time in this type of problem are the keys of this work. These difficulties are particularly well-illustrated in the Markovian economic equilibrium problem. By approaching the problem from the dual point of view to be concentrated on the pricing kernel process only (as in the first part) and exploiting the necessary and sufficient orthogonality condition of our main result yields to the complete resolution of this problem, until now without a satisfactory answer.

Also, this condition of orthogonality undoubtedly plays an important role in Markov framework, because they give us the necessary and sufficient conditions in the form of PDEs. Solving them is still an open question that will be studied in a forthcoming paper.

Finally, since we have no regularity assumptions with respect to the time, our results can be applied in the discrete frame, and in different setting, as preference learning in which the goal is to learn a predictive preference model from observed preference information, see [FH11, FSS06]. As well, reasoning with preferences has been recognized as a particularly promising research direction for artificial intelligence, see [NJ04, QXL]. Other learning problems can also be studied from the viewpoint of an expected utility maximizing as learning a probabilistic models, see for example [FS16, CS03] for more details.

References

- [Art99] W.B. Arthur. Complexity and the economy. *science*, 284(5411):107–109, 1999. (Cited on page 2.)
- [Bic90] A. Bick. On viable diffusion price processes of the market portfolio. *The Journal of Finance*, 45(2):673–689, 1990. (Cited on page 17.)
- [BRT09] F. Berrier, L. Rogers, and M. Tehranchi. A characterization of forward utility functions. *Preprint*, 2009. (Cited on page 21.)
- [CE16] C.P. Chambers and F. Echenique. *Revealed preference theory*, volume 56. Cambridge University Press, 2016. (Cited on pages 2 and 3.)
- [CKO01] U. Chajewska, D. Koller, and D. Ormoneit. Learning an agent’s utility function by observing behavior. In *ICML*, pages 35–42, 2001. (Not cited.)
- [CS03] Friedman C. and S. Sandow. Learning probabilistic models: An expected utility maximization approach. *Journal of Machine Learning Research*, 4:257–291, 2003. (Cited on page 26.)
- [DR97] P.H. Dybvig and L.C.G. Rogers. Recovery of preferences from observed wealth in a single realization. *The Review of Financial Studies*, 10(1):151–174, 1997. (Cited on page 2.)
- [EK81] N. El Karoui. Les aspects probabilistes du contrôle stochastique. In *Ninth Saint Flour Probability Summer School—1979 (Saint Flour, 1979)*, volume 876 of *Lecture Notes in Math.*, pages 73–238. Springer, Berlin, 1981. (Not cited.)
- [EKHM18a] N. El Karoui, C. Hillairet, and M. Mrad. Consistent utility of investment and consumption: a forward/backward spde viewpoint. *Stochastics*, 90(6):927–954, 2018. (Cited on page 12.)
- [EKHM18b] N. El Karoui, C. Hillairet, and M. Mrad. Construction of an aggregate consistent utility, without Pareto optimality. Application to Long-Term yield curve Modeling. 2018. To Appear in *ESAIM: Mathematical Modelling and Numerical Analysis*. (Cited on page 15.)

-
- [EKHR95] N. El Karoui, G. Helyette, , and J.C. Rochet. Changes of numeraire, changes of probability measure and option pricing. *Journal of Applied probability*, 32(2):443–458, 1995. (Not cited.)
 - [EKM13] N. El Karoui and M. Mrad. An Exact connection between two Solvable SDEs and a Non Linear Utility Stochastic PDEs. *SIAM Journal on Financial Mathematics*, 4(1):697–736, 2013. (Cited on page 12.)
 - [EKM17] N. El Karoui and M. Mrad. Mixture of consistent stochastic utilities, and a priori randomness. *preprint.*, 2017. (Cited on page 16.)
 - [FH11] Jo. Fürnkranz and E. Hüllermeier. *Preference learning*. Springer, 2011. (Cited on page 26.)
 - [FS16] C. Friedman and S. Sandow. *Utility-based learning from data*. CRC Press, 2016. (Cited on page 26.)
 - [FSS06] J. Fürnkranz, T. Scheffer, and M. Spiliopoulou. *Machine Learning: ECML 2006: 17th European Conference on Machine Learning, Berlin, Germany, September 18-22, 2006, Proceedings*, volume 4212. Springer Science & Business Media, 2006. (Cited on page 26.)
 - [Gan07] A. Gandhi. The stochastic response dynamic: A new approach to learning and computing equilibrium in continuous games. *University of Chicago*, 2007. (Not cited.)
 - [GR13] J. Gomez-Ramirez. Don’t blame the economists. it is an inverse problem! *European Journal of Futures Research*, 1(1):1, 2013. (Cited on page 2.)
 - [GRS13] J. Gomez-Ramirez and R. Sanz. On the limitations of standard statistical modeling in biological systems: a full bayesian approach for biology. *Progress in biophysics and molecular biology*, 113(1):80–91, 2013. (Not cited.)
 - [HH07] V. Henderson and D. Hobson. Horizon-unbiased utility functions. *Stochastic Process. Appl.*, 117(11):1621–1641, 2007. (Not cited.)
 - [Hib11] B. Hibbard. Model-based utility functions. *CoRR*, abs/1111.3934, 2011. (Not cited.)
 - [HL93] H. He and H. Leland. On equilibrium asset price processes. *The Review of Financial Studies*, 6(3):593–617, 1993. (Cited on pages 3, 15, 17, 18 and 21.)
 - [Kun97] H. Kunita. *Stochastic flows and stochastic differential equations*, volume 24 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, 1997. (Not cited.)
 - [LT11] S. Lasaulce and H. Tembine. *Game theory and learning for wireless networks: fundamentals and applications*. Academic Press, 2011. (Not cited.)
-

-
- [Mra09] M. Mrad. *Utilités Progressives Dynamiques*. Thesis. PASTEL, 2009. (Not cited.)
- [MZ10] Marek Musiela and Thaleia Zariphopoulou. Portfolio choice under space-time monotone performance criteria. *SIAM Journal on Financial Mathematics*, 1(1):326–365, 2010. (Cited on page 21.)
- [NJ04] T. D. Nielsen and F. V. Jensen. Learning a decision maker’s utility function from (possibly) inconsistent behavior. *Artificial Intelligence*, 160(1):53–78, 2004. (Cited on page 26.)
- [NT17] S. Nadtochiy and M. Tehranchi. Optimal investment for all time horizons and martin boundary of space-time diffusions. 27(2):438–470, 2017. (Cited on page 17.)
- [QXL] Y. Qi, M. Xu, and J. Lafferty. Learning high-dimensional concave utility functions for discrete choice models. (Cited on page 26.)
- [Sam38] PA. Samuelson. A note on the pure theory of consumer’s behavior. *Economica*, 5(17):61–71, 1938. (Cited on page 2.)
- [Sam48] PA. Samuelson. Consumption theory in terms of revealed preference. *Economica*, 15(60):243–253, 1948. (Cited on page 2.)
- [Sta04] A. Stahl. Approximation of utility functions by learning similarity measures. In *Logic versus Approximation*, pages 150–172. Springer, 2004. (Not cited.)
- [Wan93] S. Wang. The local recoverability of risk aversion and intertemporal substitution. *Journal of Economic Theory*, 59(2):333–363, 1993. (Cited on page 17.)
- [Wid63] D. V. Widder. The role of the Appell transformation in the theory of heat conduction. *Trans. Amer. Math. Soc.*, 109:121–134, 1963. (Not cited.)
- [Wid75] D. V. Widder. *The heat equation*. Academic Press [Harcourt Brace Jovanovich Publishers], New York, 1975. Pure and Applied Mathematics, Vol. 67. (Not cited.)
- [WTKD04] William E. W., G. Tesauro, J. O. Kephart, and R. Das. Utility functions in autonomous systems. In *1st International Conference on Autonomic Computing (ICAC 2004), 17-19 May 2004, New York, NY, USA*, pages 70–77, 2004. (Not cited.)