Generic separating sets for three-dimensional elasticity tensors

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GENERIC SEPARATING SETS FOR 3D ELASTICITY TENSORS

R. DESMORAT, N. AUFFRAY, B. DESMORAT, B. KOLEV, AND M. OLIVE

Abstract. We define what is a *generic separating set* of invariant functions (a.k.a. a *weak functional basis*) for tensors. We produce then two generic separating sets of polynomial invariants for 3D elasticity tensors, one made of 19 polynomials and one made of 21 polynomials (but easier to compute) and a generic separating set of 18 rational invariants. As a byproduct, a new integrity basis for the fourth-order harmonic tensor is provided.

1. Introduction

Assuming that one could measure the elasticity tensors of two materials, it is a natural question to ask, *if one can decide by finitely many calculations, whether the two materials have identical elastic properties* (are identical as elastic materials), in other words if the two elasticity tensors are related by a rotation. More precisely, two elasticity tensors $\mathbf{E}_1$ and $\mathbf{E}_2$ belonging to the vector space $\mathbb{E}\mathrm{la}$, of fourth-order tensors having major and left/right minor indicial symmetries

$$E_{ijkl} = E_{ijlk} = E_{klij},$$

define the *same elastic material*, if and only if, there exists a rotation $g \in \mathrm{SO}(3)$ such that

$$(\mathbf{E}_2)_{ijkl} = g_{ip}g_{jq}g_{kr}g_{ls}(\mathbf{E}_1)_{pqrs}\,,$$

a relation that we shall denote by

$$\mathbf{E}_2 = g \star \mathbf{E}_1,$$

and we say then that the two tensors *are in the same orbit*. When such a rotation does not exist, the two tensors describe *different* elastic materials.

To give different names to different elastic materials, we need to construct a set of functions on the space of elasticity tensors which :

(1) are constant on each orbit ;
(2) take different values on different orbits.

Such as a set of functions is known in the mathematical community as a *separating set*, and as a *functional basis* in the field of theoretical mechanics [48]. A separating set is minimal if no proper subset of it is a separating set. At the present time there is no known algorithm for constructing a minimal separating set. If the minimality aspect is left aside, an approach

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for obtaining a *separating* set of functions is to consider the algebra of invariant polynomials over the space of elasticity tensors. In the context of *real* elasticity tensors and for the group of rotations, the algebra of invariant polynomials is *finitely generated* [23, 41] and *separates the orbits* [1, Appendix C]. The generating set of such an algebra is classically known as an integrity basis. Nevertheless, calculating explicitly an integrity basis for the invariant algebra is an extremely difficult task. Furthermore, its important to note that if generating basis is a separating basis, the converse is generally false. As a consequence the cardinal of a generating basis is, in general, larger than the one of a separating basis.

The determination of an integrity basis for the elasticity tensor has a long history, which can be traced back, at least, to the work of Betten [6, 7], who obtained some partial results. The question was formulated in rigourous mathematical terms by Boehler et al. in [11], where the link with *invariants of binary forms (i.e. of homogeneous complex polynomials in two variables)* was established for the first time. However, due to the complexity of the required computations the authors did not provide a final answer to the problem. With the help of a Computer Algebra System (CAS), a minimal set of 297 generators for the invariant algebra of the 3D elasticity tensor was finally obtained in 2017, by some of the present authors, in [29], which definitively solved this old problem.

Whether this minimal integrity basis can be reduced to obtain a separating set of lower cardinality is nevertheless still an open problem. The difficulty is that there is no known general procedure to produce explicit general separating sets whereas there are constructive algorithms to obtain integrity bases [17, 27].

There is a huge literature on integrity and functional bases for an n-uplet of second-order symmetric tensors or more generally for a family of second-order symmetric tensors and vectors (including, thus, skew-symmetric second-order tensors) [49, 39, 37, 36, 45, 46]. Usually, these functional bases are polynomial [43, 44, 36]. For higher-order tensors, results are usually sparse or incomplete [11, 38, 28]. Up to the authors best knowledge, nothing is known concerning the elasticity tensor but the 297 invariants of a minimal integrity basis [29, 30].

Since most materials have no symmetry in practice (they are *triclinic*), their membership to higher-symmetry classes is just a convenient approximation of the reality. Therefore, the notion of separating set/functional basis can be weakened again, in order to reduce its cardinal. To be more specific, the notion of *weak separating set* — also known as a *weak functional basis* — has been formulated in [11], in the sense that they separate only *generic* tensors (defined rigorously in Section 3, using Zariski topology). In [11], Boehler, Kirillov and Onat produced a weak separating set of 39 polynomial invariants for $\mathbf{E} \in \mathbb{E}\mathrm{la}$.

In the present paper, by formulating slightly different genericity conditions, we produce a weak separating set of *21 polynomial invariants* for the elasticity tensor. This result, formulated as Theorem 4.2, is our main theorem. Moreover, translating results on rational invariants of the binary form of degree 8 by Maeda in [26], we can shorten this number to 19 (Corollary

4.6), but the corresponding polynomial invariants are more complicated. We can also deduce a set of 18 *rational invariants* which separate generic elasticity tensors (Corollary 4.5).

The paper is organized as follows. In section 2, we recall basic notions on *integrity basis* and produce a new minimal integrity basis for $\mathbb{H}^4(\mathbb{R}^3)$, the space of fourth-order harmonic tensors. In section 3, we introduce various definitions of *separating sets* and formulate rigorously the concept of *genericity*. Formulations of the main result, some corollaries are provided in Section 4. The mathematical material needed to understand the link between *invariant theory of binary forms* and *invariant theory of harmonic tensors* is recalled in Appendix C. A set of 18 *rational invariants* which separate generic fourth-order harmonic tensors is then provided in Appendix D by translating *Maeda invariants* [26] into invariants of the fourth-order harmonic tensor.

Notations. $\mathrm{O}(3)$ is the orthogonal group, that is the group of all isometries of $\mathbb{R}^3$ i.e. $g \in \mathrm{O}(3)$ if $\det g = \pm 1$ and $g^{-1} = g^t$, where the superscript ^{t} denotes the transposition. The group of rotations is $\mathrm{SO}(3)$ is the special orthogonal group, i.e. the subgroup of $\mathrm{O}(3)$ of elements satisfying $\det g = 1$.

We denote by $\mathbb{T}^n(\mathbb{R}^3)$, the space of nth-order tensors on $\mathbb{R}^3$ and by $\mathbb{S}^n(\mathbb{R}^3)$, the subspace of totally symmetric tensors of order n. A traceless tensor $\mathbf{H} \in \mathbb{S}^n(\mathbb{R}^3)$ is called an *harmonic tensor* and the space of nth-order harmonic tensors is noted $\mathbb{H}^n(\mathbb{R}^3)$. The notation $\mathbf{q}$ will stand for the Euclidean metric tensor, and since an orthonormal basis is considered $\mathbf{q} = (\delta_{ij})$. All the tensorial components will be expressed with respect to an orthonormal basis, and hence no distinction will be made between covariant and contravariant components.

The total symmetrization of a tensor $\mathbf{T} \in \mathbb{T}^n(\mathbb{R}^3)$ is the tensor $\mathbf{T}^s \in \mathbb{S}^n(\mathbb{R}^3)$, defined by

$$(T^s)_{i_1\dots i_n} = \frac{1}{n!}\sum_{\sigma\in\mathfrak{S}_n} T_{i_{\sigma(1)}\dots i_{\sigma(n)}},$$

where $\mathfrak{S}_n$ is the permutation group over n elements.

Example 1.1. The total symmetrization $\mathbf{S} = \mathbf{T}^s$ of a third order tensor $\mathbf{T} \in \mathbb{T}^3(\mathbb{R}^3)$ has for components

$$S_{ijk} = \frac{1}{6}\left(S_{ijk} + S_{ikj} + S_{jik} + S_{jki} + S_{kij} + S_{kji}\right). \tag{1}$$

Example 1.2. The total symmetrization of an elasticity tensor $\mathbf{E} \in \mathbb{E}\mathrm{la}$ writes

$$\mathbf{S} = \mathbf{E}^s, \qquad S_{ijkl} = \frac{1}{3}\left(E_{ijkl} + E_{ikjl} + E_{iljk}\right). \tag{2}$$

Compared to elasticity tensors, $\mathbf{S}$ has the additional index symmetry

$$S_{ijkl} = S_{ikjl}.$$

The *symmetric tensor product* between two totally symmetric tensors $\mathbf{S}^k \in \mathbb{S}^{n_k}(\mathbb{R}^3)$ is defined as

$$\mathbf{S}^1 \odot \mathbf{S}^2 := (\mathbf{S}^1 \otimes \mathbf{S}^2)^s \in \mathbb{S}^{n_1+n_2}(\mathbb{R}^3). \tag{3}$$

Example 1.3. The symmetric tensor product $\boldsymbol{v}\odot\boldsymbol{w} = (\boldsymbol{v}\otimes\boldsymbol{w})^s$ of two vectors $\boldsymbol{v}, \boldsymbol{w}$ writes

$$\boldsymbol{v}\odot\boldsymbol{w} = \frac{1}{2}\left(\boldsymbol{v}\otimes\boldsymbol{w} + \boldsymbol{w}\otimes\boldsymbol{v}\right), \qquad (\boldsymbol{v}\odot\boldsymbol{w})_{ij} = \frac{1}{2}\left(v_i w_j + v_j w_i\right).$$

Example 1.4. The symmetric tensor product $\mathbf{a}\odot\mathbf{b} = (\mathbf{a}\otimes\mathbf{b})^s$ of two symmetric second order tensors $\mathbf{a}, \mathbf{b} \in \mathbb{S}^2(\mathbb{R}^3)$ has for components

$$(\mathbf{a}\odot\mathbf{b}) = \frac{1}{6}\left(a_{ij}b_{kl} + a_{ik}b_{jl} + a_{il}b_{jk} + b_{ij}a_{kl} + b_{ik}a_{jl} + b_{il}a_{jk}\right).$$

The *r-contraction* between two tensors $\mathbf{T}^k \in \mathbb{T}^{n_k}(\mathbb{R}^3)$ is defined as

$$(\mathbf{T}^1 \overset{(r)}{\cdot} \mathbf{T}^2)_{i_1\dots i_{n_1-r}j_{r+1}\dots j_{n_2}} := T^1_{i_1\dots i_{n_1-r}k_1\dots k_r}\, T^2_{k_1\dots k_r j_{r+1}\dots j_{n_2}}.$$

In particular, we get

$$\begin{aligned} &(\mathbf{a}\cdot\mathbf{b})_{ij} = a_{ik}b_{kj}, && \mathbf{a}:\mathbf{b} = a_{ij}b_{ij}, \\ &(\mathbf{H}:\mathbf{a})_{ij} = H_{ijkl}a_{kl}, && (\mathbf{H}:\mathbf{K})_{ijkl} = H_{ijpq}K_{pqkl} \\ &(\mathbf{H}\,\vdots\,\mathbf{K})_{ij} = H_{ipqr}K_{pqrj}. \end{aligned}$$

where $\mathbf{a}, \mathbf{b}$ are two second-order tensors and $\mathbf{H}, \mathbf{K}$, two fourth-order tensors. The usual abbreviations $\mathbf{a}^{n+1} = \mathbf{a}^n\cdot\mathbf{a}$, $\mathbf{ab} = \mathbf{a}\cdot\mathbf{b}$ and $\mathbf{H}^{n+1} = \mathbf{H}^n:\mathbf{H}$ shall also be used.

The *symmetric r-contraction* between two totally symmetric tensors $\mathbf{S}^k \in \mathbb{S}^{n_k}(\mathbb{R}^3)$ is defined as

$$\mathbf{S}^1 \underset{s}{\overset{(r)}{\cdot}} \mathbf{S}^2 := (\mathbf{S}^1 \overset{(r)}{\cdot} \mathbf{S}^2)^s \in \mathbb{S}^{n_1+n_2-2r}(\mathbb{R}^3), \tag{4}$$

Example 1.5. The symmetric contraction $\mathbf{a} \underset{s}{\overset{(1)}{\cdot}} \mathbf{b}$ of two symmetric second-order tensors $\mathbf{a}, \mathbf{b} \in \mathbb{S}^2(\mathbb{R}^3)$ is nothing else than $\mathbf{a} \underset{s}{\overset{(1)}{\cdot}} \mathbf{b} = \frac{1}{2}\left(\mathbf{ab} + \mathbf{ba}\right)$.

The *generalized cross product* between two totally symmetric tensors $\mathbf{S}^k \in \mathbb{S}^{n_k}(\mathbb{R}^3)$, which has been introduced in [31], is defined as

$$\mathbf{S}^1 \times \mathbf{S}^2 := (\mathbf{S}^2 \cdot \boldsymbol{\varepsilon} \cdot \mathbf{S}^1)^s \in \mathbb{S}^{n_1+n_2-1}(\mathbb{R}^3), \tag{5}$$

where $\boldsymbol{\varepsilon}$ is the third-order Levi-Civita tensor

Example 1.6. The generalized cross product $\mathbf{a}\times\mathbf{b}$ of two symmetric second order tensors $\mathbf{a}, \mathbf{b} \in \mathbb{S}^2(\mathbb{R}^3)$ has for components

$$(\mathbf{a}\times\mathbf{b})_{ijk} = \frac{1}{6}\big(b_{ip}\varepsilon_{pjq}a_{qk} + b_{ip}\varepsilon_{pkq}a_{qj} + b_{jp}\varepsilon_{piq}a_{qk} + b_{jp}\varepsilon_{pkq}a_{qi} + b_{kp}\varepsilon_{piq}a_{qj} + b_{kp}\varepsilon_{pjq}a_{qi}\big).$$

The *leading harmonic part* $\mathbf{S}' \in \mathbb{H}^n(\mathbb{R}^3)$ of a totally symmetric tensor $\mathbf{S} \in \mathbb{S}^n(\mathbb{R}^3)$ means the harmonic part of highest order of $\mathbf{S}$ in its harmonic decomposition (see [30, Proposition 2.8], where it was noted $\mathbf{S}_0$ rather than $\mathbf{S}'$).

Example 1.7. The leading harmonic part of a symmetric second order tensor $\mathbf{a} \in \mathbb{S}^2(\mathbb{R}^3)$ is noting else than its deviatoric part $\mathbf{a}' = \mathbf{a} - \frac{1}{3}(\operatorname{tr}\mathbf{a})\,\mathbf{q}$.

Example 1.8. The leading harmonic part $\mathbf{H} = (\mathbf{E}^s)'$ of an elasticity tensor $\mathbf{E} \in \mathbb{E}\mathrm{la}$ is harmonic fourth-order tensor

$$\mathbf{H} = \mathbf{S} - \frac{6}{7}\mathbf{q} \odot (\operatorname{tr}\mathbf{S})' - \frac{1}{5}(\operatorname{tr}\operatorname{tr}\mathbf{S})\,\mathbf{q} \odot \mathbf{q}, \tag{6}$$

where $\mathbf{S} = \mathbf{E}^s$ is totally symmetric part (2) of $\mathbf{E}$, $(\operatorname{tr}\mathbf{S})'_{ij} = S_{kkij} - \frac{1}{3}S_{kkll}\delta_{ij}$ and $\operatorname{tr}\operatorname{tr}\mathbf{S} = S_{kkll} = \frac{1}{3}(E_{kkij} + 2E_{kikj})$.

The *harmonic product* between two harmonic tensors $\mathbf{H}^k \in \mathbb{H}^{n_k}(\mathbb{R}^3)$, which has been introduced in [30], is defined as harmonic tensor

$$\mathbf{H}^1 * \mathbf{H}^2 := (\mathbf{H}^1 \odot \mathbf{H}^2)' \in \mathbb{H}^{n_1+n_2}(\mathbb{R}^3). \tag{7}$$

Example 1.9. The harmonic product of two harmonic (deviatoric) second order tensors $\mathbf{a}', \mathbf{b}' \in \mathbb{H}^2(\mathbb{R}^3)$ writes

$$\mathbf{a}' * \mathbf{b}' = \mathbf{a}' \odot \mathbf{b}' - \frac{2}{7}\,\mathbf{q} \odot (\mathbf{a}'\mathbf{b}' + \mathbf{b}'\mathbf{a}') + \frac{2}{35}\operatorname{tr}(\mathbf{a}'\mathbf{b}')\,\mathbf{q} \odot \mathbf{q},$$

with $\mathbf{a}' * \mathbf{b}' \in \mathbb{H}^4(\mathbb{R}^3)$.

2. Integrity basis

Let us first recall some definitions and fundamental aspects concerning integrity bases of real polynomial invariant algebras (such as the invariant algebra $\mathbb{R}[\mathbb{E}\mathrm{la}]^{\mathrm{SO}(3)}$ of elasticity tensors).

We consider a *linear representation* $\mathbb{V}$ of the 3-dimensional orthogonal group $\mathrm{O}(3)$. This means that we have a mapping

$$(g, \boldsymbol{v}) \mapsto g \star \boldsymbol{v}, \qquad \mathrm{O}(3) \times \mathbb{V} \to \mathbb{V},$$

which is linear in $\boldsymbol{v}$ and such that

$$(g_1 g_2) \star \boldsymbol{v} = g_1 \star (g_2 \star \boldsymbol{v}).$$

Remark 2.1. Note that the representations of $\mathrm{O}(3)$ and $\mathrm{SO}(3)$ on *even-order tensors* are the same because, then,

$$(-\mathrm{I}) \star \mathbf{T} = \mathbf{T}, \qquad \forall\, \mathbf{T} \in \mathbb{T}^{2n}(\mathbb{R}^3),$$

where I is the identity in $\mathrm{O}(3)$.

A polynomial function p defined on $\mathbb{V}$ (*i.e* which can be written as a polynomial in components of $\boldsymbol{v} \in \mathbb{V}$ in any basis) is *invariant* if

$$p(g \star \boldsymbol{v}) = p(\boldsymbol{v}), \qquad \forall g \in \mathrm{O}(3),\ \forall \boldsymbol{v} \in \mathbb{V}.$$

The set of $\mathrm{O}(3)$-*invariant polynomial functions* is a subalgebra of the polynomial algebra $\mathbb{R}[\mathbb{V}]$ of real polynomial functions on $\mathbb{V}$, which will be denoted by $\mathbb{R}[\mathbb{V}]^{\mathrm{O}(3)}$.

Definition 2.2 (Integrity basis)**.** A finite set of $\mathrm{O}(3)$-invariant polynomials $\{P_1, \dots, P_k\}$ over $\mathbb{V}$ is a *generating set* (also called an *integrity basis*) of the invariant algebra $\mathbb{R}[\mathbb{V}]^{\mathrm{O}(3)}$ if any $\mathrm{O}(3)$-invariant polynomial J over $\mathbb{V}$ is a polynomial function in $P_1, \dots, P_k$, *i.e* if J can be written as

$$J(\boldsymbol{v}) = p(P_1(\boldsymbol{v}), \dots, P_k(\boldsymbol{v})), \qquad \boldsymbol{v} \in \mathbb{V},$$

where p is a polynomial function in k variables. An integrity basis is minimal if no proper subset of it is an integrity basis.

Example 2.3 ($\mathbb{V} = \mathbb{R}^3 \oplus \cdots \oplus \mathbb{R}^3$). For an n-uplet of vectors $(\boldsymbol{v}_1, \dots, \boldsymbol{v}_n)$, Weyl [47] proved that a minimal integrity basis of the diagonal representation of $\mathrm{O}(3)$

$$g \star (\boldsymbol{v}_1, \dots, \boldsymbol{v}_n) := (g\boldsymbol{v}_1, \dots, g\boldsymbol{v}_n)$$

is given by the family

$$\boldsymbol{v}_i \cdot \boldsymbol{v}_j, \qquad i, j = 1, \dots, n.$$

Example 2.4 ($\mathbb{V} = \mathbb{S}^2(\mathbb{R}^3)$). Another classical example is given by the standard $\mathrm{O}(3)$-representation on $\mathbb{S}^2(\mathbb{R}^3)$, the space of symmetric second-order tensors on $\mathbb{R}^3$. A minimal integrity basis is given by

$$\operatorname{tr}\mathbf{a}, \qquad \operatorname{tr}\mathbf{a}^2, \qquad \operatorname{tr}\mathbf{a}^3,$$

where $\mathbf{a} \in \mathbb{S}^2(\mathbb{R}^3)$.

A minimal integrity basis is not unique but its cardinality as well as the degrees of its members are independent of the basis [18]. For instance, an alternative minimal integrity basis of $\mathbb{R}[\mathbb{S}^2(\mathbb{R}^3)]^{\mathrm{O}(3)}$ is given by the three elementary functions

$$\sigma_1 := \lambda_1 + \lambda_2 + \lambda_3, \qquad \sigma_2 := \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3, \qquad \sigma_3 := \lambda_1\lambda_2\lambda_3,$$

where λ_k are the eigenvalues of the second-order symmetric tensor $\mathbf{a}$. These two minimal integrity bases are related by invertible polynomial relations, more precisely

$$\sigma_1 = \operatorname{tr}\mathbf{a}, \qquad \sigma_2 = \frac{1}{2}\left((\operatorname{tr}\mathbf{a})^2 - \operatorname{tr}\mathbf{a}^2\right), \qquad \sigma_3 = \frac{1}{6}\left((\operatorname{tr}\mathbf{a})^3 - 3\operatorname{tr}\mathbf{a}\,\operatorname{tr}\mathbf{a}^2 + 2\operatorname{tr}\mathbf{a}^3\right),$$

and conversely

$$\operatorname{tr}\mathbf{a} = \sigma_1, \qquad \operatorname{tr}\mathbf{a}^2 = \sigma_1^2 - 2\sigma_2, \qquad \operatorname{tr}\mathbf{a}^3 = \sigma_1^3 - 3\sigma_1\sigma_2 + 3\sigma_3.$$

For a couple $(\mathbf{a}, \mathbf{b})$ of second-order symmetric tensors, that is for

$$\mathbb{V} = \mathbb{S}^2(\mathbb{R}^3) \oplus \mathbb{S}^2(\mathbb{R}^3),$$

a minimal integrity basis for the diagonal $\mathrm{O}(3)$-representation is known since at least 1958 [39], and can be found in many references, for instance [37, 8, 10, 50]. More precisely, the following result holds.

Proposition 2.5. *The following collection of ten polynomial invariants*

$$\operatorname{tr}\mathbf{a}, \quad \operatorname{tr}\mathbf{a}^2, \quad \operatorname{tr}\mathbf{a}^3, \quad \operatorname{tr}\mathbf{b}, \quad \operatorname{tr}\mathbf{b}^2, \quad \operatorname{tr}\mathbf{b}^3, \quad \operatorname{tr}\mathbf{ab}, \quad \operatorname{tr}\mathbf{a}^2\mathbf{b}, \quad \operatorname{tr}\mathbf{ab}^2, \quad \operatorname{tr}\mathbf{a}^2\mathbf{b}^2,$$

is a minimal integrity basis for $\mathbb{R}[\mathbb{S}^2(\mathbb{R}^3) \oplus \mathbb{S}^2(\mathbb{R}^3)]^{\mathrm{O}(3)}$.

For higher-order tensors, the determination of an integrity basis is much more complicated and one way to compute such a basis requires first to decompose the tensor space $\mathbb{V}$ into *irreducible representations* called also an *harmonic decomposition* of $\mathbb{V}$ (see [4, 16, 5, 3, 2, 29] for more details). In this decomposition, the irreducible factors are isomorphic to the spaces $\mathbb{H}^n(\mathbb{R}^3)$, of nth-order *harmonic tensors*. Such a decomposition is, in general, not unique. For the elasticity tensor, $\mathbb{V} = \mathbb{E}\mathrm{la}$, we can use, for instance, the following explicit decomposition:

$$\mathbf{E} = (\lambda, \mu, \mathbf{d}', \mathbf{v}', \mathbf{H}), \tag{8}$$

with

$$\lambda := \operatorname{tr} \mathbf{d}, \qquad \mu := \operatorname{tr} \mathbf{v}, \qquad \mathbf{d}' := \mathbf{d} - (\lambda/3)\mathbf{q}, \qquad \mathbf{v}' := \mathbf{v} - (\mu/3)\mathbf{q}, \tag{9}$$

where $\mathbf{d} := \operatorname{tr}_{12} \mathbf{E}$ (*i.e.* $d_{ij} = E_{kkij}$) is the *dilatation tensor*, $\mathbf{v} := \operatorname{tr}_{13} \mathbf{E}$ (*i.e.* $v_{ij} = E_{kikj}$) is the *Voigt tensor* and, in accordance with (6),

$$\mathbf{H} := (\mathbf{E}^s)' = \mathbf{E}^s - \mathbf{q} \odot \mathbf{a}' - \frac{7}{30}(\operatorname{tr} \mathbf{a})\, \mathbf{q} \odot \mathbf{q}, \quad \text{where} \quad \mathbf{a} := \frac{2}{7}(\mathbf{d} + 2\mathbf{v}), \tag{10}$$

Note that in any decomposition of the elasticity tensor, the fourth-order component $\mathbf{H}$ is uniquely defined, which is not the case of the other components.

A minimal integrity basis of the invariant algebra $\mathbb{R}[\mathbb{H}^4(\mathbb{R}^3)]^{\mathrm{O}(3)}$ of harmonic fourth-order tensors was exhibited for the first time by Boehler, Onat and Kirillov [11] and republished later by Smith and Bao [38]. In both cases, the derivation is based on original mathematical results obtained earlier by Shioda [35] and von Gall [42] on binary forms (see Appendix C). The corresponding minimal integrity basis, provided in [11], uses the following second-order covariants, *i.e.* second-order tensor valued functions $\mathbf{d}(\mathbf{H})$, depending of $\mathbf{H}$ in such a way that

$$g \star \mathbf{d}(\mathbf{H}) = \mathbf{d}(g \star \mathbf{H}), \tag{11}$$

for all $\mathbf{H} \in \mathbb{H}^4(\mathbb{R}^3)$ and $g \in \mathrm{O}(3)$ (see [31] for more details).

Theorem 2.6 (Boehler–Kirillov–Onat)**.** *Let* $\mathbf{H} \in \mathbb{H}^4(\mathbb{R}^3)$ *and set:*

$$\begin{array}{lll} \mathbf{d}_2 = \operatorname{tr}_{13} \mathbf{H}^2, & \mathbf{d}_3 = \operatorname{tr}_{13} \mathbf{H}^3, & \mathbf{d}_4 = \mathbf{d}_2^2, \\ \mathbf{d}_5 = \mathbf{d}_2(\mathbf{H} : \mathbf{d}_2), & \mathbf{d}_6 = \mathbf{d}_2^3, & \mathbf{d}_7 = \mathbf{d}_2^2(\mathbf{H} : \mathbf{d}_2) \\ \mathbf{d}_8 = \mathbf{d}_2^2(\mathbf{H}^2 : \mathbf{d}_2), & \mathbf{d}_9 = \mathbf{d}_2^2(\mathbf{H} : \mathbf{d}_2^2), & \mathbf{d}_{10} = \mathbf{d}_2^2(\mathbf{H}^2 : \mathbf{d}_2^2). \end{array} \tag{12}$$

A minimal integrity basis for $\mathbf{H}$ *is given by the nine following invariants:*

$$J_k = \operatorname{tr} \mathbf{d}_k, \qquad k = 2, \dots, 10. \tag{13}$$

Recall that, even if a minimal integrity basis is not unique, its cardinality and the degree of its elements are the same for all bases [18]. A remarkable observation is that there exists a minimal integrity basis of $\mathbb{H}^4(\mathbb{R}^3)$, involving only the two second-order covariants $\mathbf{d}_2$, $\mathbf{d}_3$ introduced in (12).

Theorem 2.7. *The following nine polynomial invariants*

$$\begin{array}{lll} I_2 = \operatorname{tr} \mathbf{d}_2, & I_3 = \operatorname{tr} \mathbf{d}_3, & I_4 = \operatorname{tr} \mathbf{d}_2^2, \\ I_5 = \operatorname{tr}(\mathbf{d}_2\mathbf{d}_3), & I_6 = \operatorname{tr} \mathbf{d}_2^3, & I_7 = \operatorname{tr}(\mathbf{d}_2^2\mathbf{d}_3), \\ I_8 = \operatorname{tr}(\mathbf{d}_2\mathbf{d}_3^2), & I_9 = \operatorname{tr} \mathbf{d}_3^3, & I_{10} = \operatorname{tr}(\mathbf{d}_2^2\mathbf{d}_3^2). \end{array} \tag{14}$$

form a minimal integrity basis of $\mathbb{R}[\mathbb{H}^4]^{\mathrm{O}(3)}$.

The proof follows from the fact that there are algebraic relations between the two sets of invariants provided below. Since $\{J_k\}$ is an integrity basis elements of $\{I_k\}$ are polynomial functions in $\{J_k\}$: $I_2 = J_2$, $I_3 = J_3$, $I_4 = J_4$,

$I_6 = J_6$, $I_5 = \frac{1}{6}(3J_5 + 2J_2J_3)$, $I_7 = \frac{1}{6}(3J_7 + 2J_4J_3)$ and

$$
\begin{aligned}
I_8 =& \frac{1}{1620}\big(1080J_8 - 1230J_6J_2 + 495J_5J_3 - 216J_4^2 + 1197J_4J_2^2 + 140J_3^2J_2 - 237J_2^4\big),\\
I_9 =& \frac{1}{19440}\big(5184J_9 - 6480J_7J_2 + 9456J_6J_3 + 20255J_2^2 - 7974J_4J_3J_2 + 2500J_3^3 + 1596J_3J_2^3\big),\\
I_{10} =& \frac{1}{1630}\big(1080J_{10} - 675J_8J_2 + 495J_7J_3 + 24J_6J_4 - 117J_6J_2^2 - 171J_4^2J_2\\
&+ 190J_4J_3^2 + 228J_4J_2^3 - 45J_2^5\big),
\end{aligned}
$$

The converse is also observed: $J_5 = \frac{1}{3}\big(6I_5 - 2I_2I_3\big)$, $J_7 = \frac{1}{3}\big(6I_7 - 2I_4I_3\big)$ and

$$
\begin{aligned}
J_8 =& \frac{1}{2160}\big(3240I_8 - 1980I_5I_3 + 2460I_6I_2 + 380I_3^2I_2 + 432I_4^2 - 2394I_4I_2^2 + 474I_2^4\big),\\
J_9 =& \frac{1}{10368}\big(38880I_9 + 25920I_7I_2 - 8100I_5I_2^2 - 5000I_3^3 - 18912I_3I_6 + 7308I_3I_4I_2 - 492I_3I_2^3\big),\\
J_{10} =& \frac{1}{17280}\big(25920I_{10} + 16200I_8I_2 - 15840I_7I_3 - 9900I_5I_3I_2 + 2240I_3^2I_4\\
&+ 1900I_3^2I_2^2 - 384I_6I_4 + 14172I_6I_2^2 + 4896I_4^2I_2 - 15618I_4I_2^3 + 3090I_2^5\big).
\end{aligned}
$$

hence proving that $\{I_k\}$ also constitutes an integrity basis.

3. Separating sets

The weaker concept of *separating set*, often called a *functional basis* in the mechanical community [48, 9] (see [19, 25, 17] for alternative definitions in the mathematical community), is formulated in invariant theory as follows.

Definition 3.1 (Separating set)**.** A finite set of O(3)-invariant functions $\{s_1, \dots, s_n\}$ over $\mathbb{V}$ is a *separating set* if

$$s_i(\boldsymbol{v}_1) = s_i(\boldsymbol{v}_2), \quad i = 1, \dots, n \implies \exists g \in \mathrm{O}(3),\ \boldsymbol{v}_1 = g \star \boldsymbol{v}_2.$$

for all $\boldsymbol{v}_1, \boldsymbol{v}_2 \in \mathbb{V}$. A separating set is minimal if no proper subset of it is a separating set.

In other words for elasticity tensors $\mathbf{E}, \overline{\mathbf{E}} \in \mathbb{E}\mathrm{la}$ (which are of even order), the n equalities $s_i(\mathbf{E}) = s_i(\overline{\mathbf{E}})$ between their separating invariants imply that the elasticity tensor $\overline{\mathbf{E}}$ is obtained by rotating $\mathbf{E}$ (*i.e.* $\overline{\mathbf{E}} = g \star \mathbf{E}$, $g \in \mathrm{SO}(3)$, hence $\overline{\mathbf{E}}$ and $\mathbf{E}$ are in the same orbit).

Note that definition 3.1 is very general and the functions $s_1, \dots, s_n$ are not required to be polynomial in $\boldsymbol{v}$ (resp. in $\mathbf{E}$).

Remark 3.2. A remarkable fact is that an integrity basis of $\mathbb{R}[\mathbb{V}]^{\mathrm{O}(3)}$, the algebra of real O(3)-invariant polynomials over $\mathbb{V}$, is also a *separating set* [1, Appendix C]. However, the cardinal of an integrity basis can be very big (for instance, it is of 297 for the 3D elasticity tensor [29]). But, even if no general result exists, the cardinal n of a polynomial separating set can be smaller than the cardinal of a minimal integrity basis (see for instance [50]).

(a) Genericity – Weak separating set. A weaker concept was suggested in [11], but requires first to define what is meant by *generic tensors* (also called *tensors in general position*). This can be done rigorously by introducing the *Zariski topology* on the real vector space $\mathbb{V}$ ($\mathbb{V} = \mathbb{E}\mathrm{la}$ in next applications), which is defined by specifying its *closed sets* rather than its *open sets* (see [22] for more details). A closed set in the Zariski topology is defined as

$$Z := \{\boldsymbol{v} \in \mathbb{V};\ f(\boldsymbol{v}) = 0, \forall f \in S\}$$

where S is any set of polynomials in $\boldsymbol{v}$.

Remark 3.3. A remarkable fact concerning this topology is that a non-empty closed set is either the whole space or has Lebesgue measure zero [15, 34].

A Zariski open set is defined as the complementary set Z^c of a closed Zariski set. Note that a non-empty Zariski open set in $\mathbb{R}^N$ is moreover *open and dense* in the canonical topology of $\mathbb{R}^n$, as a normed vector space.

Example 3.4. On the vector space $\mathbb{V} = \mathbb{R}^3 \oplus \mathbb{R}^3 \oplus \mathbb{R}^3$, the following set

$$Z := \{(\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3) \in \mathbb{V};\ \det(\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3) = 0\}$$

is a Zariski closed set (as $\det(\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3) = 0$ is a polynomial equation in the components v_1^i, v_2^i, v_3^i of $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$) and

$$Z^c = \{(\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3) \in \mathbb{V};\ \det(\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3) \neq 0\}$$

is a Zariski open set.

Example 3.5. On the vector space $\mathbb{E}\mathrm{la}$, the set of non triclinic materials (*i.e.* of either monoclinic, orthotropic, tetragonal, trigonal, transversely isotropic, cubic or isotropic materials [20]) is defined by polynomial equations in $\mathbf{E}$ [31]. It is a Zariski closed set. The set of triclinic materials is a non-empty Zariski open set.

Definition 3.6 (Genericity)**.** Given a proper closed Zariski set Z of some (finite dimensional) vector space $\mathbb{V}$, a vector $\boldsymbol{v}$ is called *generic* (or as *in general position* by algebraic geometers) if it belongs to the non-empty Zariski open set Z^c.

Coming back to our definition of generic tensors, this means that, informally speaking, the probability of a randomly chosen tensor being generic is 1 and that we omit, in the results, tensors in the Zariski closed set Z. Note that the notion of generic element is not absolute. It depends on some given property which defines a Zariski open subset Z^c and there is a lot of freedom in the choice of *such a class* of generic tensors.

Definition 3.7 (Weak separating set)**.** Given some non-empty Zariski open set $Z^c \subset \mathbb{V}$, a finite set of $\mathrm{O}(3)$-invariant functions $\{s_1, \dots, s_m\}$ over $\mathbb{V}$ is called a *weak separating set* (or a *weak functional basis*) if

$$s_i(\boldsymbol{v}_1) = s_i(\boldsymbol{v}_2), \quad i = 1, \dots, m \implies \exists g \in \mathrm{O}(3),\ \boldsymbol{v}_1 = g \star \boldsymbol{v}_2.$$

for all $\boldsymbol{v}_1, \boldsymbol{v}_2 \in Z^c$.

The notion of *minimal cardinality* for weak separating (functional) bases can also be formulated in a *given class of functions*. We shall say that a weak separating basis is minimal if their is no other weak separating basis with smaller cardinal in the same class of functions. If some results exist for the class of polynomial functions in complex algebraic geometry [19], where some bounds on the cardinal of a minimal weak separating basis are provided, it is not totaly clear how they can be translated into the realm of real algebraic geometry.

(b) Separating sets of rational functions. Besides (weak) separating bases of polynomial functions, there are also results on separating bases of rational functions [24, 21] (which are necessarily *weak* since tensors for which the denominators vanish are forbidden). For instance, Maeda [26] provided a separating set of 6 rational invariants for binary octavics (complex polynomials homogeneous of degree 8 in two variables), which are closely related to harmonic tensors of order 4 (vector space $\mathbb{H}^4(\mathbb{R}^3)$, see Appendix C). Using this result, we provide in Appendix D a separating set of 6 rational invariants for $\mathbb{H}^4(\mathbb{R}^3)$. This set is minimal because one cannot produce a set of separating invariants of cardinality lower than the *transcendence degree*, which is the maximal number of algebraic independent elements in the fractional field of the invariant algebra [14, Page 26]. For the elasticity tensor, this minimal number is

$$\dim \mathbb{E}\mathrm{la} - \dim \mathrm{O}(3) = 18.$$

On this matter, there is a paper by Ostrosablin [33] who suggests a system of 18 separating rational invariants, but no rigourous proof of this result seems to be available in the literature.

(c) Local separability. Finally, there is a third (weaker) notion of separability which should not be confused with the preceding ones: *local separability* (formulated in Appendix A), which has been addressed, for instance, in [13] for the elasticity tensor.

These several notions of separability differ by the size of the subset U of $\mathbb{V}$ ($\mathbb{V}$ taken next as the vector space $\mathbb{E}\mathrm{la}$), on which the separating property is defined. The strongest one is the first one (separating set) because the separating property is global and defined over the whole vector space $\mathbb{V}$. In particular, the minimal integrity basis — of 297 invariants — for elasticity tensors produced in [29] is a global, albeit non minimal, separating set over the full vector space $\mathbb{E}\mathrm{la}$. A Zariski's open sets Z^c being very large (open and dense in the canonical topology of $\mathbb{V}$, the second notion (weak separating set) separates most orbits (except a few ones which constitute a set of zero Lebesgue measure over $\mathbb{V}$. The last one (local separating set) is the weaker, it separates only tensors in a given neighbourhood U of a given point $\boldsymbol{v}_0 \in \mathbb{V}$ (resp. of a given elasticity tensor $\mathbf{E}_0 \in \mathbb{E}\mathrm{la}$).

4. Weak separating sets for elasticity tensors

As already discussed, the harmonic decomposition $\mathbf{E} = (\lambda, \mu, \mathbf{d}', \mathbf{v}', \mathbf{H})$ of an elasticity tensor is given by (9)-(10), with λ, μ two invariants, with $\mathbf{d}(\mathbf{E}) = \operatorname{tr}_{12} \mathbf{E}$ and $\mathbf{v}(\mathbf{E}) = \operatorname{tr}_{13} \mathbf{E}$ two second-order covariants and $\mathbf{H}(\mathbf{E})$ a fourth-order (harmonic) covariant of $\mathbf{E}$ (satisfying (11)).

In [11], Boehler, Kirillov and Onat introduced a set of *generic elasticity tensors* and provided, for this set, a weak separating set of 39 polynomial invariants. Their generic tensors are defined as those for which the second-order covariants $\mathbf{d}_2 = \operatorname{tr}_{13} \mathbf{H}^2$ and $\mathbf{d}_3 = \operatorname{tr}_{13} \mathbf{H}^3$ of the considered elasticity tensor $\mathbf{E}$ do not share a common principal axis. This is equivalent to say that the symmetry class of the pair of second-order tensors $(\mathbf{d}_2, \mathbf{d}_3)$ is triclinic (its symmetry group is reduced to the identity). This condition defines a Zariski open set Z^c. Some polynomial equations defining the complementary set Z were detailed in [11] using tensor's components. An intrinsic and covariant formulation of these polynomial conditions can be formulated as follows (see [31, Theorem 8.5])

$$(\mathbf{d}_2\boldsymbol{v}_5) \times \boldsymbol{v}_5 \neq 0 \quad \text{or} \quad (\mathbf{d}_3\boldsymbol{v}_5) \times \boldsymbol{v}_5 \neq 0,$$

where $\boldsymbol{v}_5 := \boldsymbol{\varepsilon} :(\mathbf{d}_2\mathbf{d}_3 - \mathbf{d}_3\mathbf{d}_2)$ is a first-order covariant of $\mathbf{H}$ (and therefore of $\mathbf{E}$). In the present work, we shall consider a smaller Zariski open set by restricting to elasticity tensors for which $\mathbf{d}_2$ *is furthermore orthotropic* (three distinct eigenvalues). This is equivalent to add the polynomial condition ${\mathbf{d}_2}^2 \times \mathbf{d}_2 \neq 0$ (see [31, Lemma 8.1]).

Remark 4.1. Note that, if the pair $(\mathbf{d}_2, \mathbf{d}_3)$ is triclinic, then $\mathbf{H}$ (and hence $\mathbf{E}$) is triclinic, since a tensor cannot be less symmetric than its covariants. However, the converse does not hold: it is not true that for any triclinic elasticity tensor $\mathbf{E}$, the pair of second-order covariants $(\mathbf{d}_2, \mathbf{d}_3)$ is triclinic, the later condition is stronger.

We will now formulate our main theorem. Recall that $\mathbf{a}' = \mathbf{a} - \frac{1}{3}(\operatorname{tr}\mathbf{a})\mathbf{q}$ stands for deviatoric part of a second-order tensor, $(\mathbf{ab})^s := \frac{1}{2}(\mathbf{ab} + \mathbf{ba})$ is the symmetrized matrix product, and $[\mathbf{a}, \mathbf{b}] := \mathbf{ab} - \mathbf{ba}$ is the commutator of two second-order symmetric tensors $\mathbf{a}$, $\mathbf{b}$.

Theorem 4.2. *Let* $\mathbf{E} = (\lambda, \mu, \mathbf{d}', \mathbf{v}', \mathbf{H})$ *be an elasticity tensor,* $\mathbf{d}_2 = \operatorname{tr}_{13}\mathbf{H}^2$ *and* $\mathbf{d}_3 = \operatorname{tr}_{13}\mathbf{H}^3$*. Then, the following 21 polynomial invariants,* $\lambda = \operatorname{tr}\mathbf{d}$*,* $\mu = \operatorname{tr}\mathbf{v}$*,*

$$I_2 := \operatorname{tr}\mathbf{d}_2, \quad I_3 := \operatorname{tr}\mathbf{d}_3, \quad I_4 := \operatorname{tr}\mathbf{d}_2^2, \quad I_5 := \operatorname{tr}(\mathbf{d}_2\mathbf{d}_3), \quad I_6 := \operatorname{tr}\mathbf{d}_2^3,$$
$$I_7 := \operatorname{tr}(\mathbf{d}_2^2\mathbf{d}_3), \quad I_8 := \operatorname{tr}(\mathbf{d}_2\mathbf{d}_3^2), \quad I_9 := \operatorname{tr}\mathbf{d}_3^3, \quad I_{10} := \operatorname{tr}(\mathbf{d}_2^2\mathbf{d}_3^2),$$
$$D_3 := \mathbf{d}' : \mathbf{d}_2, \quad D_4 := \mathbf{d}' : \mathbf{d}_3, \quad D_5 := \mathbf{d}' : \mathbf{d}_2^2, \quad D_6 := \mathbf{d}' :(\mathbf{d}_2\mathbf{d}_3)^s, \quad D_{11} := \mathbf{d}' :[\mathbf{d}_2, \mathbf{d}_3]^2,$$
$$V_3 := \mathbf{v}' : \mathbf{d}_2, \quad V_4 := \mathbf{v}' : \mathbf{d}_3, \quad V_5 := \mathbf{v}' : \mathbf{d}_2^2, \quad V_6 := \mathbf{v}' :(\mathbf{d}_2\mathbf{d}_3)^s, \quad V_{11} := \mathbf{v}' :[\mathbf{d}_2, \mathbf{d}_3]^2,$$

separate generic tensors $\mathbf{E}$*, satisfying the following conditions:*
(1) the pair $(\mathbf{d}_2, \mathbf{d}_3)$ *is triclinic, and (2)* $\mathbf{d}_2$ *is orthotropic.*

Remark 4.3. Note that if condition (1) is satisfied, then, either $\mathbf{d}_2$ or $\mathbf{d}_3$ is orthotropic (by Lemma 4.4). Thus, one could omit condition (2) (as in [11]) and formulate a new separating result on this larger Zariski open set. However, the price to pay is to add the two invariants $\mathbf{d}' : \mathbf{d}_3^2$ and $\mathbf{v}' : \mathbf{d}_3^2$ to the list in Theorem 4.2, increasing its cardinal from 21 to 23 (but still below the 39 invariants of [11]). Indeed, in that case, $\mathbf{d}_3$ can play the role of $\mathbf{d}_2$ in the proof of Theorem 4.2,

Theorem 4.2 makes use of the following lemma (whose proof is postponed to Appendix B).

Lemma 4.4. *Let* $(\mathbf{a}, \mathbf{b})$ *be a triclinic pair of symmetric second-order tensors. Then at least one of them is orthotropic, say* $\mathbf{a}$*, and in that case*

$$\mathcal{B} = \left(\mathbf{q}, \mathbf{a}, \mathbf{b}, \mathbf{a}^2, (\mathbf{ab})^s, [\mathbf{a}, \mathbf{b}]^2\right)$$

is a basis of $\mathbb{S}^2(\mathbb{R}^3)$*, the space of symmetric second-order tensors.*

Proof of Theorem 4.2. Let $\mathbf{E} = (\lambda, \mu, \mathbf{d}', \mathbf{v}', \mathbf{H})$ be an elasticity tensor satisfying the conditions (1) and (2) of Theorem 4.2. Then, by Lemma 4.4,

$$\mathcal{B} = \left(\mathbf{q}, \mathbf{d}_2, \mathbf{d}_3, \mathbf{d}_2{}^2, (\mathbf{d}_2\mathbf{d}_3)^s, [\mathbf{d}_2, \mathbf{d}_3]^2\right)$$

is a basis of $\mathbb{S}^2(\mathbb{R}^3)$. Thus, if we set

$$\boldsymbol{\epsilon}_1 := \mathbf{d}_2, \quad \boldsymbol{\epsilon}_2 := \mathbf{d}_3, \quad \boldsymbol{\epsilon}_3 := \mathbf{d}_2{}^2, \quad \boldsymbol{\epsilon}_4 := (\mathbf{d}_2\mathbf{d}_3)^s, \quad \boldsymbol{\epsilon}_5 := [\mathbf{d}_2, \mathbf{d}_3]^2,$$

and define $\boldsymbol{\epsilon}'$ as the deviatoric part of $\boldsymbol{\epsilon}$, then, $\mathcal{B}' = (\boldsymbol{\epsilon}'_\alpha)$ is a basis of the 5-dimensional vector space $\mathbb{H}^2(\mathbb{R}^3)$, *i.e.* of the space of deviatoric second-order tensors. In particular, the second-order harmonic components $(\mathbf{d}', \mathbf{v}')$ of $\mathbf{E}$ can be expressed in this basis as

$$\mathbf{d}' = \sum_{\alpha=1}^{5} d'_\alpha \boldsymbol{\epsilon}'_\alpha, \qquad \mathbf{v}' = \sum_{\alpha=1}^{5} v'_\alpha \boldsymbol{\epsilon}'_\alpha.$$

We will now show that the components d'_α and v'_α are rational expressions of the polynomial invariants I_k, D_k and V_k introduced in Theorem 4.2. To do so, we shall introduce the *Gram matrix* $G = (G_{\alpha\beta})$, where

$$G_{\alpha\beta} = \boldsymbol{\epsilon}'_\alpha : \boldsymbol{\epsilon}'_\beta$$

are the components of the canonical scalar product on $\mathbb{H}^2(\mathbb{R}^3)$ in this basis. Note that G is positive definite and that its components are polynomial invariants of $\mathbf{H}$. They can thus be expressed as polynomial functions of the invariants $I_2, \dots, I_{10}$, which form an integrity basis of $\mathbb{R}[\mathbb{H}^4]^{\mathrm{O}(3)}$. Now, we have

$$\mathbf{d}' : \boldsymbol{\epsilon}'_\beta = \sum_{\alpha=1}^{5} d'_\alpha G_{\alpha\beta}, \qquad \mathbf{v}' : \boldsymbol{\epsilon}'_\beta = \sum_{\alpha=1}^{5} v'_\alpha G_{\alpha\beta},$$

and since $\mathbf{d}' : \boldsymbol{\epsilon}' = \mathbf{d}' : \boldsymbol{\epsilon}$, and $\mathbf{v}' : \boldsymbol{\epsilon}' = \mathbf{v}' : \boldsymbol{\epsilon}$, we get

$$(D_3\ D_4\ D_5\ D_6\ D_{11}) = (d'_1\ d'_2\ d'_3\ d'_4\ d'_5)G,$$

and

$$(V_3\ V_4\ V_5\ V_6\ V_{11}) = (v'_1\ v'_2\ v'_3\ v'_4\ v'_5)G.$$

Inverting these linear systems, we deduce that d'_α and v'_α are rational expressions of I_k, D_k and V_k, where the common denominator $\det G$ depends only on the I_k. Consider now two generic elasticity tensors

$$\mathbf{E} = (\lambda, \mu, \mathbf{d}', \mathbf{v}', \mathbf{H}), \quad \text{and} \quad \overline{\mathbf{E}} = (\overline{\lambda}, \overline{\mu}, \overline{\mathbf{d}}', \overline{\mathbf{v}}', \overline{\mathbf{H}})$$

for which the 21 invariants defined in Theorem 4.2 are the same. Then, by Theorem 2.7 and Remark 3.2, there exists $g \in \mathrm{O}(3)$ such that

$$\overline{\mathbf{H}} = g \star \mathbf{H}.$$

We get thus

$$\overline{\mathbf{d}}_2 = g \star \mathbf{d}_2, \qquad \overline{\mathbf{d}}_3 = g \star \mathbf{d}_3.$$

Hence the two bases of $\mathbb{S}^2(\mathbb{R}^3)$, $(\boldsymbol{\epsilon}'_\alpha(\mathbf{H}))$ and $(\boldsymbol{\epsilon}'_\alpha(\overline{\mathbf{H}}))$ are related by g

$$\boldsymbol{\epsilon}'_\alpha(\overline{\mathbf{H}}) = g \star \boldsymbol{\epsilon}'_\alpha(\mathbf{H}),$$

and the corresponding Gram matrices are equal, $\overline{G} = G$. Moreover, the components of $\mathbf{d}'$, $\mathbf{v}'$ in $(\boldsymbol{\epsilon}'_\alpha(\mathbf{H}))$ and the components of $\overline{\mathbf{d}}'$, $\overline{\mathbf{v}}'$ in $(\boldsymbol{\epsilon}'_\alpha(\overline{\mathbf{H}}))$

are the same (since the invariants D_k and V_k have the same value on both tensors). Therefore, we have

$$\overline{\mathbf{d}}' = g \star \mathbf{d}', \qquad \overline{\mathbf{v}}' = g \star \mathbf{v}'.$$

Finally, since $\overline{\lambda} = \lambda$ and $\overline{\mu} = \mu$, we get

$$\overline{\mathbf{E}} = (\overline{\mathbf{H}}, \overline{\mathbf{d}}', \overline{\mathbf{v}}', \overline{\lambda}, \overline{\mu}) = (g \star \mathbf{H}, g \star \mathbf{d}', g \star \mathbf{v}', \lambda, \mu) = g \star \mathbf{E},$$

This ends the proof. □

Note that in the proof of Theorem 4.2, the nine invariants I_k are first used to separate the two fourth-order harmonic tensors $\mathbf{H}$ and $\overline{\mathbf{H}}$ (the fourth-order hamonic components of the two elasticity tensors $\mathbf{E}$ and $\overline{\mathbf{E}}$). Thus these nine invariants can be substituted by any other separating set for the vector space $\mathbb{H}^4(\mathbb{R}^3)$ of harmonic fourth-order tensors without changing the final result. In Appendix D, we provide a set of 6 separating rational invariants for $\mathbb{H}^4(\mathbb{R}^3)$

$$i_2, \quad i_3, \quad i_4, \quad k_4, \quad k_8, \quad k_9,$$

defined in Theorem D.3, obtained by translating the 6 generators of the field of rational invariants of the binary octavic calculated by Maeda in [26]. We get therefore the following first corollary.

Corollary 4.5. *The following 18 rational invariants*

$$\begin{gathered} \lambda, \quad \mu, \quad i_2, \quad i_3, \quad i_4, \quad k_4, \quad k_8, \quad k_9, \\ D_3, \quad D_4, \quad D_5, \quad D_6, \quad D_{11}, \quad V_3, \quad V_4, \quad V_5, \quad V_6, \quad V_{11} \end{gathered}$$

separate generic tensors $\mathbf{E} = (\lambda, \mu, \mathbf{d}', \mathbf{v}', \mathbf{H})$*, satisfying the following conditions: (1) the pair* $(\mathbf{d}_2, \mathbf{d}_3)$ *is triclinic, and (2)* $\mathbf{d}_2$ *is orthotropic.*

In Theorem D.3, it can be observed that the denominator of each rational invariant

$$i_2, \quad i_3, \quad i_4, \quad k_4, \quad k_8, \quad k_9,$$

is a power of the polynomial invariant of degree 12

$$M_{12} := \|\mathbf{d}_2{}^2 \times \mathbf{d}_2\|^2.$$

where the generalized cross product $\times$ was defined in (5). Besides, it was shown in [31, Lemma 8.1] that $\mathbf{d}_2{}^2 \times \mathbf{d}_2 \neq 0$ if and only if $\mathbf{d}_2$ is orthotropic. We have thus the following second corollary.

Corollary 4.6. *The following 19 polynomial invariants*

$$\begin{gathered} \lambda, \qquad \mu, \qquad M_{12} \qquad K_{14} := M_{12}\, i_2, \qquad K_{27} := M_{12}{}^2\, i_3, \\ K_{40i} := M_{12}{}^3\, i_4, \qquad K_{40k} := M_{12}{}^3\, k_4, \qquad K_{80} := M_{12}{}^6\, k_8, \qquad K_{93} := M_{12}{}^7\, k_9, \\ D_3, \quad D_4, \quad D_5, \quad D_6, \quad D_{11}, \quad V_3, \quad V_4, \quad V_5, \quad V_6, \quad V_{11}, \end{gathered}$$

separate generic tensors $\mathbf{E} = (\lambda, \mu, \mathbf{d}', \mathbf{v}', \mathbf{H})$*, satisfying the following conditions: (1) the pair* $(\mathbf{d}_2, \mathbf{d}_3)$ *is triclinic, and (2)* $\mathbf{d}_2$ *is orthotropic.*

5. Conclusion

After having obtained in Section 2 an integrity basis $\{I_k\}$ for fourth-order harmonic tensors $\mathbf{H}$ by means of two of its second-order covariants only ($\mathbf{d}_2(\mathbf{H})$ and $\mathbf{d}_3(\mathbf{H})$), we have proposed in Theorem 4.2 a weak separating set of 21 polynomial invariants

$$\{s_i(\mathbf{E})\} = \{\lambda, \mu, I_k, D_l, V_l\}, \quad k = 2, \cdots, 10, \quad l = 3, 4, 5, 6, 11,$$

for generic triclinic elasticity tensors $\mathbf{E}$. Such a set is also called a weak functional basis in the mechanical community. It is such that the 21 equalities $s_i(\mathbf{E}) = s_i(\overline{\mathbf{E}})$ of separating invariants of generic $\mathbf{E}, \overline{\mathbf{E}} \in \mathbb{E}\mathrm{la}$ imply that elasticity tensor $\overline{\mathbf{E}}$ is obtained by rotation of elasticity tensor $\mathbf{E}$. There is no need to assume that $\overline{\mathbf{E}}$ is in a *neighbourhood* of $\mathbf{E}$ (contrary to the case of the locally separating set given in Theorem A.3).

We also provide in Corollary 4.5 a minimal separating basis of 18 rational invariants for generic elasticity tensors (leading to a minimal weak separating basis of 19 polynomial invariants in Corollary 4.6). This result is important from a theoretical point of view (as $18 = \dim \mathbb{E}\mathrm{la} - \dim \mathrm{O}(3)$ is the transcendence degree so that this set is minimal). We point out that the notion of genericity is not absolute, it depends on some given property which defines a closed subset (by polynomial equations for the Zariski topology used in present work) and that in all cases the probability of a randomly chosen elasticity tensor being generic is 1. Using the genericity condition defined in [11], we improve these authors' result of 39 separating polynomial invariants by the set of 23 polynomial separating invariants $\{\lambda, \mu, I_k, D_l, V_l, \mathbf{d}' : \mathbf{d}_3^2, \mathbf{v}' : \mathbf{d}_3^2\}$ (Remark 4.3).

Appendix A. Local separability

Local separability can be formulated as follows.

Definition A.1 (Locally separating set)**.** A finite set of $\mathrm{O}(3)$-invariant functions $\{s_1, \dots, s_p\}$ over $\mathbb{V}$ is *locally separating in the neighbourhood* $U \subset \mathbb{V}$ *of* $\boldsymbol{v}_0$ (for the usual topology of $\mathbb{V}$) if and only if

$$s_i(\boldsymbol{v}_1) = s_i(\boldsymbol{v}_2), \quad i = 1, \dots, p \implies \exists g \in \mathrm{O}(3),\, \boldsymbol{v}_1 = g \star \boldsymbol{v}_2.$$

for all $\boldsymbol{v}_1, \boldsymbol{v}_2 \in U$.

Such a set can be considered as a "local chart" (*i.e* local coordinates) around the orbit of $\boldsymbol{v}_0$ (resp. $\mathbf{E}_0$) in the orbit space $\mathbb{V}/\mathrm{O}(3)$ (resp. $\mathbb{E}\mathrm{la}/\mathrm{O}(3)$), which is not a smooth manifold anyway. A locally separating set of 18 invariants (but not polynomial) for elasticity tensors which have 6 distinct Kelvin moduli [12] has been produced in [13].

Remark A.2. Since an integrity basis $J = (J_1, \dots, J_{297})$ is known for the elasticity tensor [29], one can find a locally separating set of 18 invariants (*i.e.* the minimal number) around each tensor $\mathbf{E}_0$ for which the Jacobian matrix

$$dJ = \left(\frac{\partial J_p}{\partial E_{ijkl}} \right)$$

has maximal rank 18. Indeed, one can extract from dJ, a submatrix

$$(dJ_{p_1}, \dots, dJ_{p_{18}})$$

of rank 18, construct a local *cross-section* as in [32, Page 161] and show that $J_{p_1}, \dots, J_{p_{18}}$ are locally separating around $\mathbf{E}_0$.

We have then the following result (rank $(dJ_{p_1}, \dots, dJ_{p_{18}})$ equal to 18 checked on a randomly chosen elasticity tensor $\mathbf{E}_0$):

Theorem A.3. *Let $\mathbf{E} = (\lambda, \mu, \mathbf{d}', \mathbf{v}', \mathbf{H})$ be an elasticity tensor. Then, the following 18 polynomial invariants,*

$$\lambda = \operatorname{tr}\mathbf{d}, \quad \mu = \operatorname{tr}\mathbf{v}, \quad \operatorname{tr}\mathbf{d}^2, \quad \operatorname{tr}\mathbf{v}^2, \quad \operatorname{tr}\mathbf{d}^3, \quad \operatorname{tr}\mathbf{v}^3,$$
$$I_2 = \operatorname{tr}\mathbf{d}_2, \quad I_3 = \operatorname{tr}\mathbf{d}_3, \quad I_4 = \operatorname{tr}{\mathbf{d}_2}^2, \quad \mathbf{d} : \mathbf{d}_2, \quad \mathbf{v} : \mathbf{d}_2, \quad \mathbf{d}^2 : \mathbf{d}_2,$$
$$\mathbf{v}^2 : \mathbf{d}_2 \quad \mathbf{d} : \mathbf{H} : \mathbf{d}, \quad \mathbf{v} : \mathbf{H} : \mathbf{v}, \quad \mathbf{d} : \mathbf{H} : \mathbf{v}, \quad \operatorname{tr}(\mathbf{d}\mathbf{d}_2\mathbf{v}), \quad \mathbf{d} : (\mathbf{H}^2)^s : \mathbf{v},$$

separate locally generic elasticity tensors.

Appendix B. Proof of Lemma 4.4

Proof. Note first that $\mathbf{a}$ and $\mathbf{b}$ cannot be both transversely isotropic (*i.e.* having both only two different eigenvalues), otherwise the pair $(\mathbf{a}, \mathbf{b})$ would have necessarily a common eigenvector and would be not be triclinic. Suppose thus that $\mathbf{a}$ is orthotropic. Without loss of generality, we can assume that $\mathbf{a} = \operatorname{diag}(\lambda_1, \lambda_2, \lambda_3)$ is diagonal with $\lambda_i \neq \lambda_j$ for $i \neq j$. But then, $(\mathbf{q}, \mathbf{a}, \mathbf{a}^2)$ is a basis of the space of diagonal matrices, noted Diag, and therefore $\mathcal{B}$ contains $\mathbf{e}_{11}, \mathbf{e}_{22}, \mathbf{e}_{33}$ where

$$\mathbf{e}_{ij} = \begin{cases} \boldsymbol{e}_i \otimes \boldsymbol{e}_i, & \text{if } i = j; \\ \boldsymbol{e}_i \otimes \boldsymbol{e}_j + \boldsymbol{e}_j \otimes \boldsymbol{e}_i, & \text{if } i \neq j. \end{cases}$$

We will now show that $\mathcal{B}$ contains also $\mathbf{e}_{12}, \mathbf{e}_{13}, \mathbf{e}_{23}$. Let's write

$$\mathbf{b} = x\,\mathbf{e}_{23} + y\,\mathbf{e}_{13} + z\,\mathbf{e}_{12}, \quad \text{mod Diag}.$$

where modulo Diag means that the equality holds up to a diagonal matrix that we don't need to precise. We cannot have $(x, y) = (0, 0)$, nor $(x, z) = (0, 0)$, nor $(y, z) = (0, 0)$, otherwise $\mathbf{a}$ and $\mathbf{b}$ would share a common eigenvector and would not be triclinic. We have then

$$2(\mathbf{ab})^s = (\lambda_2 + \lambda_3)x\,\mathbf{e}_{23} + (\lambda_1 + \lambda_3)y\,\mathbf{e}_{13} + (\lambda_1 + \lambda_2)z\,\mathbf{e}_{12}, \quad \text{mod Diag}.$$

and

$$\begin{aligned}[\mathbf{a}, \mathbf{b}]^2 = ((\lambda_1 - \lambda_2)\lambda_3 + \lambda_1\lambda_2 - \lambda_1^2)yz\,\mathbf{e}_{23} + ((\lambda_2 - \lambda_1)\lambda_3 - \lambda_2^2 + \lambda_1\lambda_2)xz\,\mathbf{e}_{13} \\ + (-\lambda_3^2 + (\lambda_2 + \lambda_1)\lambda_3 - \lambda_1\lambda_2)xy\,\mathbf{e}_{12}, \quad \text{mod Diag}.\end{aligned}$$

The question is then reduced to check whether $\mathbf{b}$, $(\mathbf{ab})^s$ and $[\mathbf{a}, \mathbf{b}]^2$ are linearly independent modulo Diag. To do so, we calculate the determinant of the matrix

$$M = \begin{pmatrix} b_{23} & 2(\mathbf{ab})^s{}_{23} & [\mathbf{a}, \mathbf{b}]^2{}_{23} \\ b_{13} & 2(\mathbf{ab})^s{}_{13} & [\mathbf{a}, \mathbf{b}]^2{}_{13} \\ b_{12} & 2(\mathbf{ab})^s{}_{12} & [\mathbf{a}, \mathbf{b}]^2{}_{12} \end{pmatrix}$$

and find

$$\det M = (\lambda_2 - \lambda_1)(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)\left(x^2y^2 + y^2z^2 + z^2x^2\right),$$

which does not vanish since $\mathbf{a}$ is orthotropic. This ends the proof. $\square$

Appendix C. Rational invariants

In this appendix, we detail the link between polynomial and rational invariants of $\mathbb{H}^n(\mathbb{C}^3)$ and the space of binary forms S_{2n}. Recall that a binary form $\mathbf{f}$ of degree k is a homogeneous complex polynomial in two variables u, v of degree k:

$$\mathbf{f}(\boldsymbol{\xi}) = a_0 u^k + a_1 u^{k-1} v + \cdots + a_{k-1} u v^{k-1} + a_k v^k,$$

where $\boldsymbol{\xi} = (u, v)$ and $a_i \in \mathbb{C}$. The set of all binary forms of degree k, noted S_k, is a complex vector space of dimension $k+1$. The special linear group

$$\mathrm{SL}(2,\mathbb{C}) := \left\{ \gamma := \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad ad - bc = 1 \right\}$$

acts naturally on $\mathbb{C}^2$ and induces a left action on S_k, given by

$$(\gamma \star \mathbf{f})(\boldsymbol{\xi}) := \mathbf{f}(\gamma^{-1}\boldsymbol{\xi}),$$

where $\gamma \in \mathrm{SL}(2,\mathbb{C})$.

Binary forms of degree $2n$ are closely related to harmonic tensors of degree n (we refer to [29, 31] for more details) in the following way. Every totally symmetric tensor $\mathbf{S}$ of order n defines an homogeneous polynomial of degree n

$$\mathrm{p}(\boldsymbol{x}) = \mathbf{S}(\boldsymbol{x}, \dots, \boldsymbol{x})$$

which can be seen to be an isomorphism. In this correspondence, harmonic tensors (with vanishing traces) correspond to harmonic polynomials (with vanishing Laplacian). Now, there is an equivariant isomorphism between the space $\mathcal{H}_n(\mathbb{C}^3)$ of complex harmonic polynomials of degree n and binary forms of degree $2n$. This isomorphism is induced by the *Cartan map*

$$\phi : \mathbb{C}^2 \to \mathbb{C}^3, \qquad (u, v) \mapsto \left(\frac{u^2 + v^2}{2}, \frac{u^2 - v^2}{2i}, iuv \right), \tag{15}$$

and is given by

$$\phi^* : \mathcal{H}_n(\mathbb{C}^3) \to \mathrm{S}_{2n}, \qquad \mathrm{h} \mapsto \mathrm{h} \circ \phi.$$

This isomorphism is moreover $\mathrm{SL}(2,\mathbb{C})$-equivariant. Indeed, the adjoint representation Ad of $\mathrm{SL}(2,\mathbb{C})$ on its Lie algebra $\mathfrak{sl}(2,\mathbb{C})$ (which is isomorphic to $\mathbb{C}^3$), preserves the quadratic form $\det m$, where $m \in \mathfrak{sl}(2,\mathbb{C})$, and induces a group morphism from $\mathrm{SL}(2,\mathbb{C})$ to

$$\mathrm{SO}(3,\mathbb{C}) := \left\{ P \in \mathrm{M}_3(\mathbb{C});\ P^t P = \mathrm{I},\ \det P = 1 \right\}.$$

The isomorphism ϕ^* between $\mathcal{H}_n(\mathbb{C}^3)$ and S_{2n} is thus equivariant in the following sense:

$$\phi^*(\mathrm{Ad}_\gamma \star \mathrm{h}) = \gamma \star \phi^*(\mathrm{h}), \qquad \mathrm{h} \in \mathcal{H}_n(\mathbb{C}^3),\ \gamma \in \mathrm{SL}(2,\mathbb{C}),$$

and the invariant algebras $\mathbb{C}[\mathcal{H}_n(\mathbb{C}^3)]^{\mathrm{SO}(3,\mathbb{C})}$ and $\mathbb{C}[\mathrm{S}_{2n}]^{\mathrm{SL}(2,\mathbb{C})}$ are isomorphic.

Definition C.1. The *transvectant* of index r of two binary forms $\mathbf{f} \in \mathrm{S}_p$ and $\mathbf{g} \in \mathrm{S}_q$ is defined as

$$\{\mathbf{f}, \mathbf{g}\}_r = \frac{(p-r)!(q-r)!}{p!q!} \sum_{i=0}^{r} (-1)^i \binom{r}{i} \frac{\partial^r \mathbf{f}}{\partial u^{r-i} \partial v^i} \frac{\partial^r \mathbf{g}}{\partial u^i \partial v^{r-i}}, \tag{16}$$

which is a binary form of degree $p+q-2r$ (which vanishes if $r > \min(p,q)$).

The invariant algebra of S_n is generated by iterated *transvectants* [32]. The tensorial operations between *totally symmetric tensors*, introduced in the notations section, allow to translate these transvectants into tensorial operations. Each of them has a polynomial counterpart (see [31]), which we detail below. In what follows, totally symmetric tensors $\mathbf{S}^1, \mathbf{S}^2$, of respective order n_1, n_2, correspond to the polynomials $\mathrm{p}_1, \mathrm{p}_2$, of respective degree n_1, n_2.

- The *symmetric tensor product* (3) $\mathbf{S}^1 \odot \mathbf{S}^2$ corresponds to the standard product of polynomials
$$\mathrm{p} = \mathrm{p}_1\,\mathrm{p}_2.$$
- The *symmetric r-contraction* (4) $\mathbf{S}^1 \overset{(r)}{\underset{s}{\cdot}} \mathbf{S}^2$ corresponds to the polynomial
$$\mathrm{p} = \frac{(n_1-r)!}{n_1!}\frac{(n_2-r)!}{n_2!}\sum_{k_1+k_2+k_3=r}\frac{r!}{k_1!k_2!k_3!}\frac{\partial^r \mathrm{p}_1}{\partial x^{k_1}\partial y^{k_2}\partial z^{k_3}}\frac{\partial^r \mathrm{p}_2}{\partial x^{k_1}\partial y^{k_2}\partial z^{k_3}}.$$
- The *generalized cross product* (5) $\mathbf{S}^1 \times \mathbf{S}^2$ corresponds to the polynomial
$$\mathrm{p} = \frac{1}{n_1 n_2}\det(\boldsymbol{x}, \nabla \mathrm{p}_1, \nabla \mathrm{p}_2),$$
where $\nabla \mathrm{p}$ is the gradient of p.
- The *harmonic product* (7) $\mathbf{H}^1 * \mathbf{H}^2$ corresponds to the polynomial
$$\mathrm{p} = (\mathrm{p}_1\,\mathrm{p}_2)'.$$

Using these operations and the Cartan map (15), we can translate the transvectants as binary operations between tensors. In the following proposition we have made no difference between an harmonic tensor $\mathbf{H}$ and its polynomial counterpart (which is an abuse of notation). Moreover, the trace of a symmetric tensor of order n is defined as the contraction between any two indices.

Proposition C.2. *Let $\mathbf{F} \in \mathbb{H}^p(\mathbb{C}^3)$ and $\mathbf{G} \in \mathbb{H}^q(\mathbb{C}^3)$ be two harmonic tensors and set $\mathbf{f} := \phi^*\mathbf{F}$ and $\mathbf{g} := \phi^*\mathbf{G}$. Then we have*

$$\{\mathbf{f}, \mathbf{g}\}_{2r} = 2^{-r}\phi^*(\mathbf{F} \overset{(r)}{\underset{s}{\cdot}} \mathbf{G})' \tag{17}$$

and

$$\{\mathbf{f}, \mathbf{g}\}_{2r+1} = \kappa(p,q,r)\phi^*(\mathrm{tr}^r(\mathbf{F} \times \mathbf{G}))' \tag{18}$$

where

$$\kappa(p,q,r) = \frac{1}{2^{2r+1}}\frac{(p+q-1)!(p-r-1)!(q-r-1)!}{(p+q-1-2r)!(p-1)!(q-1)!}.$$

Besides polynomial invariants, one can also define rational invariants for a given representation $\mathbb{V}$ of a group G. These are defined as rational functions on $\mathbb{V}$, which are invariant under the action of G. These functions form a field, the field of rational invariants and is noted $K(\mathbb{V})^G$. An important result is the following theorem which is a corollary of a more general result due to Popov and Vinberg [40, Theorem 3.3] (see also [14, Page 16]).

Theorem C.3. *Let $\mathbb{V}$ be a linear representation of G, where G is either* $\mathrm{SL}(2,\mathbb{C})$*,* $\mathrm{SO}(3,\mathbb{C})$ *or* $\mathrm{SO}(3,\mathbb{R})$ *and the base field K is either $\mathbb{R}$ or $\mathbb{C}$. Then the field of rational invariants $K(\mathbb{V})^G$ is the field of fractions of the invariant algebra $K[\mathbb{V}]^G$. In other words, any rational invariant k can be written as P/Q where P and Q* belong to *$K[\mathbb{V}]^G$.*

A finite system of rational invariants $\mathcal{S} = \{k_1, \dots, k_N\}$ generates the field $K(\mathbb{V})^G$ if any rational invariant $k \in K(\mathbb{V})^G$ can be written as a rational expression in $k_1, \dots, k_N$.

Remark C.4. A remarkable fact is that a finite system $\mathcal{S}$ of rational invariants generates the field $K(\mathbb{V})^G$ if and only if $\mathcal{S}$ is a weak separating set (see [40, Lemma 2.1].

Note that Theorem C.3 allows to translate any generating set of $K(\mathrm{S}_8)^{\mathrm{SL}(2,\mathbb{C})}$ into a generating set of $K(\mathbb{H}^4)^{\mathrm{SO}(3,\mathbb{R})}$.

Appendix D. Maeda Invariants

A minimal generating set of 9 generators for the invariant algebra of S_8 is known since at least 1880 (see [42, 35]). In 1990 [26, Theorem B], Maeda produced a system of 6 rational invariants which generate the invariant field $\mathbb{C}(\mathrm{S}_8)^{\mathrm{SL}(2,\mathbb{C})}$.

Theorem D.1 (Maeada, 1990). *The invariant field of binary octavics over $\mathbb{C}$ is generated by the following six algebraic independent rational functions*

$$I_2^M := \{\boldsymbol{\theta},\boldsymbol{\theta}\}_2/M, \qquad I_3^M := \{\boldsymbol{\theta}^3,\mathbf{t}\}_6/M^2, \qquad I_4^M := \{\boldsymbol{\theta}^4,\{\mathbf{t},\mathbf{t}\}_2\}_8/M^3$$
$$J_2^M := \{\{\boldsymbol{\theta},\mathbf{f}\}_1,\{\mathbf{t},\mathbf{t}\}_2\}_8\,\{\boldsymbol{\theta}^6,\mathbf{j}\}_{12}/M^6,$$
$$J_3^M := \left(36\{\boldsymbol{\theta}^2\mathbf{f},\mathbf{j}\}_{12}/M^2 - 28\{\{\boldsymbol{\theta}^2,\mathbf{f}\}_3,\mathbf{t}\}_6/5M\right)\,\{\boldsymbol{\theta}^6,\mathbf{j}\}_{12}/M^5,$$
$$J_4^M := 2\{\mathbf{f}\boldsymbol{\theta}^3,\mathbf{t}\{\mathbf{t},\mathbf{t}\}_2\}_{14}/M^3 + 20\{\{\mathbf{f},\boldsymbol{\theta}^3\}_1,\mathbf{j}\}_{12}/7M^3 - 70\{\{\mathbf{f},\boldsymbol{\theta}^3\}_4,\mathbf{t}\}_6/99M^2,$$

where $\mathbf{f} \in \mathrm{S}_8$ is a binary form and

$$\mathbf{Q} := \{\mathbf{f},\mathbf{f}\}_6, \quad \mathbf{t} := \{\{\mathbf{Q},\mathbf{Q}\}_2,\mathbf{Q}\}_1, \quad \boldsymbol{\theta} := \{\mathbf{f},\mathbf{t}\}_6,$$
$$M := \{\mathbf{t},\mathbf{t}\}_6, \quad \mathbf{j} := \{\{\mathbf{t},\mathbf{t}\}_2,\mathbf{t}\}_1.$$

Remark D.2. We found a few minor numerical errors in [26] and did the following corrections, which were used in Theorem D.1.

- In [26, Lemma 2.10(3)], we should read
$$\{\mathbf{t},\{\mathbf{t},\mathbf{t}\}_2\}_1 = -\mathbf{j} = \Delta^3\lambda^3/108;$$
- In [26, Lemma 2.12], we should read
$$\lambda^6\nabla = -108\{\boldsymbol{\theta}^6,\mathbf{j}\}_{12}\Delta^3/\lambda^3;$$
- In [26, Lemma 2.13], we should read

$$\lambda J_2/\nabla = 72\{\{\boldsymbol{\theta},\mathbf{f}\}_1,\{\mathbf{t},\mathbf{t}\}_2\}_8/\Delta\lambda^2,$$
$$J_3/\nabla = 108\{\boldsymbol{\theta}^2\mathbf{f},\mathbf{j}\}_{12}/\lambda^5 - 28\{\{\boldsymbol{\theta}^2,\mathbf{f}\}_3,\mathbf{t}\}_6/5\lambda^3,$$
$$J_4 = 54\{\boldsymbol{\theta}^3\mathbf{f},\mathbf{t}\{\mathbf{t},\mathbf{t}\}_2\}_{14}/\lambda^6 + 540\{\{\mathbf{f},\boldsymbol{\theta}^3\}_1,\mathbf{j}\}_{12}/7\lambda^6 - 70\{\{\mathbf{f},\boldsymbol{\theta}^3\}_4,\mathbf{t}\}_6/11\lambda^4.$$

Let $\mathbf{H} \in \mathbb{H}^4$ and $\mathbf{f} = \phi^*\mathbf{H}$, the corresponding binary form of degree 8, where ϕ^* has been defined in Appendix C. Using transvectants' translations obtained in Proposition C.2, we can recast Maeda's invariants of $\mathbf{f}$ as rational invariants of $\mathbf{H}$. We get first

$$\begin{aligned}
\phi^{-*}\mathbf{Q} &= \phi^{-*}\{\mathbf{f},\mathbf{f}\}_6 = \frac{1}{8}\mathbf{d}_2',\\
\phi^{-*}\mathbf{t} &= \phi^{-*}\{\{\mathbf{Q},\mathbf{Q}\}_2,\mathbf{Q}\}_1 = \frac{1}{2^{11}}\mathbf{d}_2^2\times\mathbf{d}_2 = \frac{1}{2^{11}}\mathbf{T}_6,\\
M &= \phi^{-*}\{\mathbf{t},\mathbf{t}\}_6 = \frac{1}{2^{25}}\|\mathbf{T}_6\|^2 = \frac{1}{2^{25}}M_{12},\\
\phi^{-*}\boldsymbol{\theta} &= \phi^{-*}\{\mathbf{f},\mathbf{t}\}_6 = \frac{1}{2^{14}}\boldsymbol{w}_7 = \frac{1}{2^{14}}\mathbf{H}\,\vdots\,\mathbf{T}_6,\\
\phi^{-*}\mathbf{j} &= \phi^{-*}\{\{\mathbf{t},\mathbf{t}\}_2,\mathbf{t}\}_1 = \frac{1}{2^{35}}\left((\mathbf{T}_6 \overset{(1)}{\underset{s}{\cdot}} \mathbf{T}_6)'\times\mathbf{T}_6\right)' = \frac{1}{2^{35}}\mathbf{J}_{18},
\end{aligned}$$

where ϕ^{-*} stands for the inverse of ϕ^* and where we have used the following observations.

(1) If $\mathbf{H}\in\mathbb{H}^n(\mathbb{R}^3)$ and $\mathbf{q}$ is the Euclidean tensor, then,

$$(\odot^k\mathbf{q})\times\mathbf{H} = 0, \qquad \forall k\geq 1,$$

where $\odot^k\mathbf{q}$ is the symmetric tensor product of k copies of $\mathbf{q}$.

(2) If $\mathbf{H}\in\mathbb{H}^n(\mathbb{R}^3)$ and $\boldsymbol{w}\in\mathbb{H}^1(\mathbb{R}^3)$, then, $\boldsymbol{w}\times\mathbf{H}$ is harmonic.

(3) If $\mathbf{a}\in\mathbb{S}^2(\mathbb{R}^3)$, then $\mathbf{a}^2\times\mathbf{a}$ is harmonic (see [31, Remark 8.2]) and

$$\mathbf{a}^2\times\mathbf{a} = \mathbf{a}'^2\times\mathbf{a}'.$$

(4) If $\mathbf{T}^1,\mathbf{T}^2\in\mathbb{T}^n(\mathbb{R}^3)$, then, $\mathbf{T}^1\overset{(n)}{\cdot}\mathbf{T}^2 = \langle\mathbf{T}^1,\mathbf{T}^2\rangle$ is their scalar product and

$$\langle\mathbf{T}_1,\mathbf{T}_2^s\rangle = \langle\mathbf{T}_1^s,\mathbf{T}_2\rangle, \qquad \langle\mathbf{T}_1,(\mathbf{T}_2^s)'\rangle = \langle(\mathbf{T}_1^s)',\mathbf{T}_2\rangle.$$

We get then

$$\phi^{-*}\{\boldsymbol{\theta}^2,\mathbf{f}\}_3 = \frac{5}{6}\operatorname{tr}[(\boldsymbol{w}_7*\boldsymbol{w}_7)\times\mathbf{H}] = -\frac{1}{4}(\mathbf{H}\cdot\boldsymbol{w}_7)\times\boldsymbol{w}_7,$$

which is an harmonic third-order tensor, by (2) and the fact that $\mathbf{H}\cdot\boldsymbol{w}_7$ is itself harmonic. We have finally the following result, where we have introduced the notation $*^k\boldsymbol{w}_7$ for the harmonic product $\boldsymbol{w}_7*\ldots*\boldsymbol{w}_7 = (\boldsymbol{w}_7\otimes\ldots\otimes\boldsymbol{w}_7)'$ of k copies of $\boldsymbol{w}_7$. We point out, moreover, that the first-order covariant $\boldsymbol{w}_7$, the third-order covariant $\mathbf{T}_6$ as well as the sixth-order covariant $\mathbf{J}_{18}$ are all harmonic.

Theorem D.3. *The invariant field of* $\mathbb{H}^4(\mathbb{R}^3)$ *is generated by the following six algebraic independent rational functions*

$$
\begin{aligned}
i_2 &= \frac{\|\boldsymbol{w}_7\|^2}{M_{12}}, \\
i_3 &= \frac{\langle *^3\, \boldsymbol{w}_7, \mathbf{T}_6\rangle}{M_{12}^2}, \\
i_4 &= \frac{\langle *^4\, \boldsymbol{w}_7, \mathbf{T}_6 \cdot \mathbf{T}_6\rangle}{M_{12}^3} \\
k_4 &= \frac{1}{5M_{12}^3}\langle \mathbf{H} * (*^3\, \boldsymbol{w}_7), \mathbf{T}_6 * (\mathbf{T}_6 \overset{(1)}{\underset{s}{\cdot}} \mathbf{T}_6)'\rangle + \frac{1}{7M_{12}^3}\langle \mathbf{H} \times (*^3\, \boldsymbol{w}_7), \mathbf{J}_{18}\rangle \\
&\quad - \frac{7}{99M_{12}^2}\langle \mathbf{H} : (*^3\, \boldsymbol{w}_7), \mathbf{T}_6\rangle. \\
k_8 &= \frac{\langle \boldsymbol{w}_7 \times \mathbf{H}, \mathbf{T}_6 \cdot \mathbf{T}_6\rangle\, \langle *^6\, \boldsymbol{w}_7, \mathbf{J}_{18}\rangle}{M_{12}^6}, \\
k_9 &= \frac{\langle *^6\, \boldsymbol{w}_7, \mathbf{J}_{18}\rangle}{M_{12}^5}\left(\frac{36}{M_{12}^2}\langle (*^2\, \boldsymbol{w}_7) * \mathbf{H}, \mathbf{J}_{18}\rangle + \frac{28}{5M_{12}}\langle (\mathbf{H} \cdot \boldsymbol{w}_7) \times \boldsymbol{w}_7, \mathbf{T}_6\rangle\right),
\end{aligned}
$$

where $\mathbf{H} \in \mathbb{H}^4(\mathbb{R}^3)$ *is the harmonic tensor, and*

$$
\begin{aligned}
\mathbf{T}_6 &:= {\mathbf{d}_2}^2 \times \mathbf{d}_2, & M_{12} &:= \|{\mathbf{d}_2}^2 \times \mathbf{d}_2\|^2, \\
\boldsymbol{w}_7 &:= \mathbf{H} \,\vdots\, \mathbf{T}_6, & \mathbf{J}_{18} &:= (\mathbf{T}_6 \overset{(1)}{\underset{s}{\cdot}} \mathbf{T}_6)' \times \mathbf{T}_6.
\end{aligned}
$$

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