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Computing the k-coverage of a wireless network

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***Abstract*—Coverage is one of the main quality of service of a wireless network. k-coverage, that is to be covered simultaneously by k network nodes, is synonym of reliability and numerous applications such as multiple site MIMO features, or handovers. We introduce here a new algorithm for computing the k-coverage of a wireless network. Our method is based on the observation that k-coverage can be interpreted as k layers of 1-coverage, or simply coverage. We use simplicial homology to compute the network's topology and a reduction algorithm to indentify the layers of 1-coverage. We provide figures and simulation results to illustrate our algorithm.**

I. Introduction

Wireless networks encompass cellular networks, WiFi access points, sensor networks, and so on. With the increasing usage of high data rates mobile devices such as smartphones and tablets, and the development of the Internet of Things (IoT), they have become indispensable in our everyday lives. A common key quality of service of this type of networks is the coverage. The coverage of a wireless network is the set of points that are in the sensing range of at least one network node. The greater the covered area, the more mobile devices can have access to it. For cellular networks, coverage can even be a governmental obligation. The absence of coverage holes inside the covered area is needed to offer a continuous access to services.

However, network nodes are often not regularly deployed on lattice or according to the hexagonal model in practice, see [1] for cellular networks in France for example. And, deciding whether a set of network nodes does cover a given area is not that easy for arbitrary deployments. Simplicial homology can help us do that, considering the network nodes GPS positions and their coverage ranges, it is possible to build a purely combinatorial object, namely an abstract simplicial complex, of which it is possible to compute the topology. Basically an abstract simplicial complex is the generalization of the concept of graph, it is made of k-simplices where 0-simplices are vertices, 1-simplices are edges, 2-simplices are triangles, 3-simplices are tetrahedron and so on. In particular, geometrical simplicial complexes such as the Čech complex and the Vietoris-Rips complex, represent exactly and approximatively respectively, the topology of the union of the coverage disks as stated in [2]. Then algebraic topology, [3], is a mathematical tool that can compute the number of connected components, of coverage holes, and of 3D voids, namely the Betti numbers of the simplicial complex representing the network, as explained in [4]. Since, the computational time to obtain the Betti numbers can explode with the size of the simplicial complex, many works focus on faster ways to compute them, for instance in a decentralized way [5], using persistent homology [6], thanks to chain complexes reduction [7] , or with witness complexes reduction [8]. In our work we use simplicial complex reduction to reduce a simplicial complex to the minimum number of points needed to provide 1-coverage. Precisely, we use the reduction algorithm presented in [9], that can also be found for coverage hole detection in [10] and for energy efficiency in cellular networs in [11].

Coverage can thus be computed mathematically thanks to algebraic topology. However k-coverage computing is not that simple. Indeed, a point is said to be k-covered when it is in the covered area of at least k network nodes. Consequently, an area is k-covered whenever every point in it is k-covered. The expected k-coverage in wireless networks has been studied in [12] in order to propose a node scheduling scheme that conserves energy while retaining network coverage. In [13], the authors use k-order Voronoï diagrams to compute the density of network nodes required to achieve k-coverage.

In this article, we propose a method and an algorithm for computing the k-coverage of a given wireless network. Theoretically it is easy to compute the probability for a point to be k-covered for a wireless network generated by a Poisson point process. However it is more difficult to apprehend the k-coverage of a whole area. Moreover, probabilistic results can not be applied to every wireless network. That is why we need simplicial homology representation to compute the topology of a given network as a whole. We then exhibit that the k-coverage can be seen as k layers of 1-coverage and give an algorithm that compute the k-coverage of a wireless network. For operating purposes, the k layers of network nodes that ensure each 1-coverage are returned by our algorithm.

First in Section II, we define the k-coverage and discuss its application for wireless networks such as IoT sensor networks and cellular networks. We present a probabilistic network model and give some theoretical results in Section III, and introduce few needed mathetical tools in Section IV. Then in Section V, we give our algorithm for computing k-coverage, and simulation results in Section VI. Finally we conclude in Section VII.

II. k-coverage

Cellular networks, Wireless Local Area Networks (WLANs), and sensor networks take part in the family of wireless networks. In these networks, coverage define the utility of the network. In cellular networks or WLANs, users can access the service only if they are covered by a network node. In sensor networks, sensor can communicate only if

they are in the sensing range of each other. In a wireless network, a point is said to be covered if it is in the sensing range of a network node, that is to say if it is in the coverage of this node. An area is then covered, when every point of it is covered. By extension, an area is k-covered when every point of the area is in the coverage of at least k network nodes. A wireless network providing k-coverage for an area with a large k is then a densely deployed network.

In the literature, k-coverage is more often used for sensor networks such as Low-Power Wide Area Networks (LPWANs), and Internet of Things (IoT) [12], [13]. Indeed the benefits of k-coverage include better reliability, better accuracy in sensor measurements, greater throughput by using multiple channels, etc. And these uses concern primarily sensor nodes. Moreover, sensor nodes are small, live on battery, and are cheap to buy and replace, so they can be deployed in large quantities, thus providing easily k-coverage with a great k.

However, k-coverage can also be of interest for cellular networks, where it becomes synonym of multi-site transmitter. The first application is the handover. Actually, a handover is performed when a user changes cells during a communication. It is called a soft-handover when the user is connected simultaneously to multiple cells for the transition between cells, or for interference mitigation in dense area in 3G networks. Therefore, a user in a k-covered area would have the possibility to have handovers with k different cells, which means that telecommunication operators could make trafic off-loading decisions by directing users to less-busy cells, or offer better radio channels thanks to antenna diversity. In 4G and later networks, k-coverage means also that MIMO transmissions from multiple base stations to a user can be performed. That is the basis of the Coordinated Multi-Point Joint Processing (CoMP JP) scheme that allows great capacity gains [14]. In CoMP JP, two or more base stations can cooperate to serve simultaneously a user leading to a throughput multiplied by 2 or more.

III. Probabilistic analysis

We represent the wireless network nodes by a Poisson point process:

Definition 1. *Let $\mathrm{d}\mu = \lambda \mathrm{d}x$ be the Lebesgue measure on $E \subseteq \mathbb{R}^2$, N is a spatial Poisson point process of intensity $\lambda > 0$ on E if:*

- *$N(A)$ the number of points that fall in $A \subset E$ follows a Poisson law*

$$\mathbf{P}[N(A) = k] = e^{-\mu(A)} \frac{\mu(A)^k}{k!}$$

- *If $A, B \subset E$ such that $A \cap B = \emptyset$, then $N(A)$ and $N(B)$ are independant.*

We can note that $\mu(A) = \lambda S(A)$ where $S(A)$ is the area of A. Moreover, conditionnally to $N(A) = n$ for $A \subset E$, then the points of N are independantly and uniformly distributed on A.

We suppose that every network node has the same sensing range r. Thus its coverage area is a disk of radius $r > 0$. Then for $x \in E$, the probability for x to be k-covered is given by:

$$\mathbf{P}[x \text{ } k\text{-covered}] = \mathbf{P}[\exists y_1, \ldots, y_k \in N | x \in \bigcap_{i=1}^{k} B(y_i, r)],$$

where $B(y, r)$ is the ball of center $y \in E$ and radius $r > 0$. We immediatly have that:

$$\begin{aligned}
\mathbf{P}[x \text{ } k\text{-covered}] &= \mathbf{P}[\exists y_1, \ldots, y_k \in N | x \in \bigcap_{i=1}^{k} B(y_i, r)] \\
&= \mathbf{P}[\exists y_1, \ldots, y_k \in N | \forall i, y_i \in B(x, r)] \\
&= \mathbf{P}[N(B(x, r)) \geq k] \\
&= 1 - \mathbf{P}[N(B(x, r)) < k] \\
&= 1 - \sum_{i=0}^{k-1} e^{-\lambda \pi r^2} \frac{(\lambda \pi r^2)^i}{i!}.
\end{aligned}$$

We can see in Fig. 1 the probability for a point to be k-covered for $k = 1, \ldots, 6$, depending on $\lambda \pi r^2$. As r is fixed, only λ varies. Logically, as λ grows, the probability to be k-covered tends to 1, and the greater k is, the smaller the probability to be k-covered is.

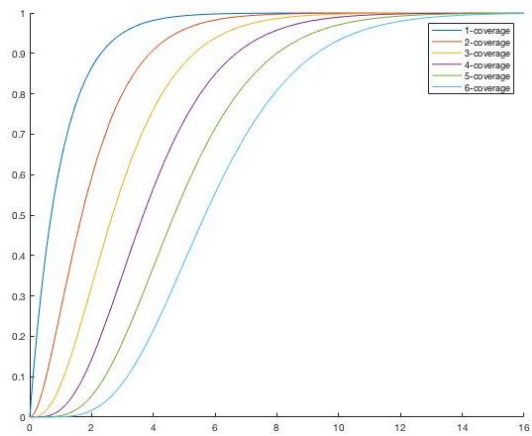

Fig. 1. Probability for a point to be k-covered depending on $\lambda \pi r^2$

The probability that a point $x \in E$ is exactly k-covered and not $(k+1)$-covered is then:

$$\begin{aligned}
\mathbf{P}[x \text{ exactly } k\text{-covered}] &= \mathbf{P}[N(B(x, r)) = k] \\
&= e^{-\lambda \pi r^2} \frac{(\lambda \pi r^2)^k}{k!}.
\end{aligned}$$

We can derive the mean k for which a point $x \in E$ is

k-covered and not $(k+1)$-covered:

$$\begin{aligned}\mathbf{E}[k] &= \sum_{k=1}^{\infty} k\mathbf{P}[x \text{ exactly } k\text{-covered}] \\ &= \sum_{k=1}^{\infty} e^{-\lambda\pi r^2}\frac{(\lambda\pi r^2)^k}{(k-1)!} \\ &= e^{-\lambda\pi r^2}\sum_{k=0}^{\infty}\frac{(\lambda\pi r^2)^k}{k!}(\lambda\pi r^2) \\ &= \lambda\pi r^2.\end{aligned}$$

Therefore, the mean k for which a point $x \in E$ is k-covered and not $(k-1)$-covered is directly proportional to λ.

However, considering the probability for a point to be k-covered is not sufficient to benefit from the advantages of k-coverage. Indeed, in wireless networks, reception devices such as phones are mobile. Therefore, one needs a whole area to be k-covered to offer k-coverage applications such as joint processing or handover. The probability of k-coverage of one point is an upper-bound of the probability of a whole area to be completely k-covered without any hole. Indeed it is easier to ensure that a point is covered, than a whole area without any hole. So probabilistic results can not be used for engineering purposes. That is why we need to consider another approach : to study and compute the coverage of the network as a whole, that is mathematically to study the topology of the network.

IV. Simplicial homology and algebraic topology

Considering a set of points representing network nodes, the first idea to apprehend the topology of the network would be to look at the neighbors graph: if the distance between two points is less than a given parameter then an edge is drawn between them. However this representation is too limited to transpose the network's topology. First, only 2-by-2 relationships are represented in the graph, there is no way to grasp interactions between three or more nodes. Moreover, there is no concept of coverage in a graph. That is why we are interested in more complex objects.

Indeed, graphs can be generalized to more generic combinatorial objects known as simplicial complexes. While graphs model binary relations, simplicial complexes can represent higher order relations. A simplicial complex is thus a combinatorial object made up of vertices, edges, triangles, tetrahedra, and their n-dimensional counterparts. Given a set of vertices X and an integer k, a k-simplex is an unordered subset of $k+1$ vertices $\{x_0, \ldots, x_k\}$ where $x_i \in X, \forall i \in \{0, \ldots, k\}$ and $x_i \neq x_j$ for all $i \neq j$. Thus, a 0-simplex is a vertex, a 1-simplex an edge, a 2-simplex a triangle, a 3-simplex a tetrahedron, etc. See Fig. 2 for instance.

Any subset of vertices included in the set of the $k+1$ vertices of a k-simplex is a face of this k-simplex. A k-face is then a face that is a k-simplex. The inverse notion of face is coface. An abstract simplicial complex is a set of simplices such that all faces of these simplices are also in the set of simplices.

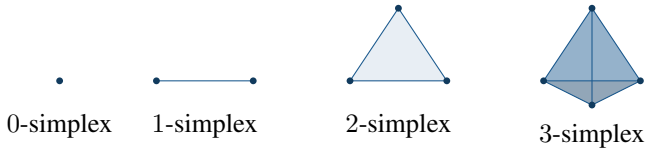

Fig. 2. Examples of k-simplices.

In this article, we are intersted in representing the topology of a wireless network, we introduce the two following abstract simplicial complexes:

Definition 2 (Čech complex)**.** *Let ω be a finite set of points in $\mathbb{R}^2$, and r a real positive number. The Čech complex of parameter r of ω, $\mathcal{C}_r(\omega)$, is the abstract simplicial complex whose k-simplices correspond to the unordered $(k+1)$-tuples of vertices in ω such that the intersection of the $k+1$ balls centered on them is non empty.*

Definition 3 (Vietoris-Rips complex)**.** *Let ω be a finite set of points in $\mathbb{R}^2$, and ϵ a real positive number. The Vietoris-Rips complex of parameter ϵ of ω, $\mathcal{R}_\epsilon(\omega)$, is the abstract simplicial complex whose k-simplices correspond to the unordered $(k+1)$-tuples of vertices in ω which are pairwise within distance less than ϵ of each other.*

The Čech complex provides the representation of the exact topology of the network (see the Nerve lemma in [2]) but can be tricky to compute due to the check of whether three disks intersect or not. One can see easily that the Vietoris-Rips complex $\mathcal{R}_{2r}(\omega)$ is an approximation of the Čech complex $\mathcal{C}_r(\omega)$ that is way easier to compute since it is a clique complex based only on the neighbors graph information. This approximation is quite good: in the case of a random uncorrelated deployment with network nodes deployed according to a Poisson point process the error is less than 0.06% in the computation of the covered area [15]. An example of a Čech complex representing a wireless network can be seen in Fig. 3. We can see 4 coverage holes in the network that are highlighted in the simplicial complex representation.

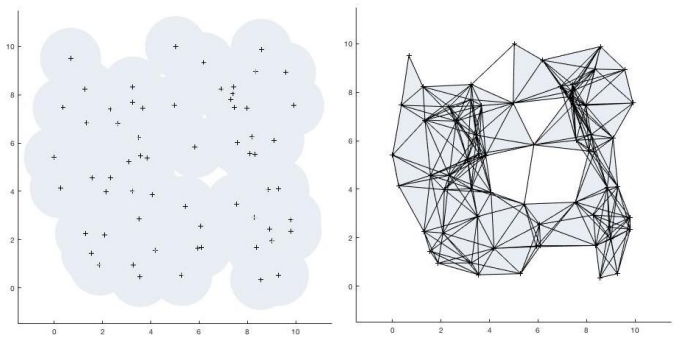

Fig. 3. A Čech complex representing a wireless network

Given an abstract simplicial complex, one can define an orientation on the simplices by defining an order on the

vertices, where a change in the orientation, that is a swap between two vertices, corresponds to a change in the sign. Then let us define the vector spaces of the k-simplices of a simplicial complex, and the associated boundary maps:

Definition 4. *Let S be an abstract simplicial complex.*

For any integer k, $\mathscr{C}_k(S)$ is the vector space spanned by the set of oriented k-simplices of S.

Definition 5. *Let S be an abstract simplicial complex and $\mathscr{C}_k(S)$ the vector space of its k-simplices for any k integer.*

The boundary map ∂_k is defined as the linear transformation $\partial_k : \mathscr{C}_k(S) \to \mathscr{C}_{k-1}(S)$ which acts on the basis elements $[x_0, \ldots, x_k]$ of $\mathscr{C}_k(S)$ via:

$$\partial_k[x_0, \ldots, x_k] = \sum_{i=0}^{k} (-1)^i [x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_k].$$

For example, for a 2-simplex we have:

$$\partial_2([x_0, x_1, x_2]) = [x_1, x_2] - [x_0, x_2] + [x_0, x_1]$$

As its name indicates, the boundary map applied to a linear combination of simplices gives its boundary. The boundary of a boundary is the null application. Therefore the following theorem can be easily demonstrated (see [3] for instance):

Theorem 1. *For any k integer, $\partial_k \circ \partial_{k+1} = 0$.*

Let S be an abstract simplicial complex. Then we can denote the k-th boundary group of S as $B_k(S) = \operatorname{im} \partial_{k+1}$, and the k-th cycle group of S as $Z_k(S) = \ker \partial_k$. We have $B_k(S) \subset Z_k(S)$. We are now able to define the k-th homology group and its dimension:

Definition 6. *The k-th homology group of an abstract simplicial complex S is the quotient vector space:*

$$H_k(S) = \frac{Z_k(S)}{B_k(S)}.$$

The k-th Betti number of the abstract simplicial complex S is:

$$\beta_k(S) = \dim H_k(S).$$

According to its definition, the k-th Betti number counts the number of cycles of k-simplices that are not boundaries of $(k+1)$-simplices, that are the k-th dimensional holes. In small dimensions, they have a geometrical interpretation:

- β_0 is the number of connected components,
- β_1 is the number of coverage holes,
- β_2 is the number of 3D-voids.

For any $k \geq d$ where d is the dimension, we have $\beta_k = 0$.

For further reading on algebraic topology, see [3].

V. Algorithm

Thanks to simplicial homology, we have a representation for a wireless network that allows the computation of the network's topology, that is its coverage, or 1-coverage. Computing the k-coverage is another problem. In order to do that, we choose to view the k-coverage as k layers of coverage:

Lemma 1. *An area is k-covered, for k integer, if there exists k sets of network nodes without any common nodes such that each set provides 1-coverage on the area.*

Proof. If there exists k sets of network nodes that provide 1-coverage, then let x be any point in the area, x is covered by each layer. Thus, there exists k nodes, one per layer, that cover x. And the area is k-covered.

Reciprocally, it is not possible to find k sets of nodes such that each provide 1-coverage. We can suppose that there exists $k-1$ sets of nodes that provide exactly 1-coverage, and a last set with the remaining nodes that do not provide 1-coverage. That is there exists at least one coverage hole in the coverage provided by the k-th set. Then let x be a point in this coverage hole, then x is in the coverage range of exactly one node in each of the first $k-1$ sets, since these sets of node provide exactly 1-coverage. The point x is inside a coverage hole of the remaining nodes, then there is no other node which coverage range covers x. And x is not k-covered. □

Therefore, to compute the k-coverage of a wireless network, we intend to count the number of 1-coverage layers. To slice the network in layers, we use the simplicial complex reduction algorithm that we presented in [9]. This reduction algorithm takes as input a simplicial complex, then removes points and their cofaces (that is the simplices they are part of) until it is no more possible without creating neither a coverage hole nor a disconnectivity in the network. At the end, we obtain a simplicial complex that provides the same coverage as the initial complex with a minimal set of points. Then the set of network nodes associated to these points provide at least 1-coverage on the whole covered area, but not 2-coverage on the whole area or more points could be removed and the simplicial complex could be further reduced. However, locally 2-coverage is provided by the reduced complex, since we consider coverage disks and disks can not tile the plane, there will always exist intersection of disks. It is important to note that the reduction algorithm needs the definition of a boundary (via a list of points) to delimit the area to be covered, here it is the boundary of the area where one need to compute the k-coverage. We can see an example of the reduction algorithm on a Vietoris-Rips simplicial complex in Fig. 4.

Our algorithm for computing k-coverage then takes as input the positions of the network nodes, compute the simplicial complex to represent their topology. Then, the reduction algorithm is applied, its result constitutes of the first layer of 1-coverage. This first layer is then discarded, the simplicial complex is built on the remaining points and the we re-apply the reduction algorithm on it. We continue while the number of

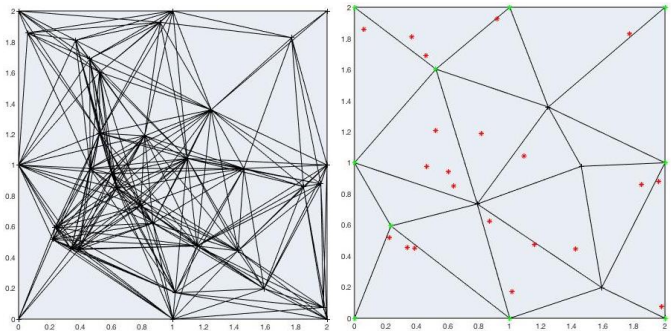

Fig. 4. A complex reduced by the reduction algorithm presented in [9]

connected components stays at 1, and the number of coverage holes stays at 0. At the end, our algorithm provides the k index of k-coverage of the wireless network, and also supplies the k sets of points/network nodes that are the k layers of coverage. The pseudo-code of the algorithm is given in Alg. 1.

Algorithm 1 k-coverage computing algorithm.

Require: set V of n vertices, coverage radius r.
  $S := \mathcal{R}_{2r}(V)$ or $\mathcal{C}_r(V)$
  Computation of $\beta_0(S)$ and $\beta_1(S)$
  $k := 0$
  while $\beta_0(S) = 1$ and $\beta_1(S) = 0$ **do**
    $k = k + 1$
    Apply reduction algorithm to S
    Save reduced complex as k-th layer
    Save list of discarded vertices as V'
    $S := \mathcal{R}_{2r}(V')$ or $\mathcal{C}_r(V')$
    Computation of $\beta_0(S)$ and $\beta_1(S)$
  end while return k and the k layers of coverage

Our algorithm provides a lower-bound for k-coverage, that is that k-coverage is guaranteed in every point of the area. More specifically, when the algorithm returns the value k for a wireless networks on the area A, that means that:

- $\forall x \in A$, x is k-covered,
- $\exists x \in A$, x is not $(k+1)$-covered,
- There may exist some $x \in A$ that are l-covered with $l > k$ (at the intersection of coverage disks).

VI. Simulation results

In this section, we give some figures illustrating the functioning of our algorithm and present some simulation results on the k-coverage of a wireless network simulated by a Poisson point process.

We can see an example of the execution of our k-coverage computation algorithm for the wireless network represented by a Vietoris-Rips complex of Fig. 5. This network was simulated with $N = 40$ points randomly placed in a square of size 10, plus a boundary of fixed points on the square to delimit the area of the square in which we want to compute the k-coverage. The coverage radius is set to 2.5. In order to obtain nicer figures, the process used to draw the points positions is of hard-core type, that is it is forbidden for 2 points to be too close to each other.

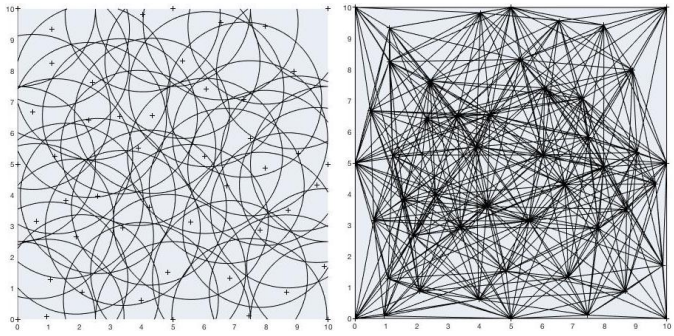

Fig. 5. A wireless network and its Vietoris-Rips representation

We can see in Fig. 6 that our algorithm exhibits 3 layers of coverage, that means that the wireless network provides 3-coverage. The last layer presents a coverage hole in the bottom right corner, so 4-coverage is not available in that part, and thus on the square. In each subfigure, points in red are the remaining points that are not yet part of a layer. On the left of each subfigure is the wireless network representation with the coverage disks, and on the right is the Vietoris-Rips representation.

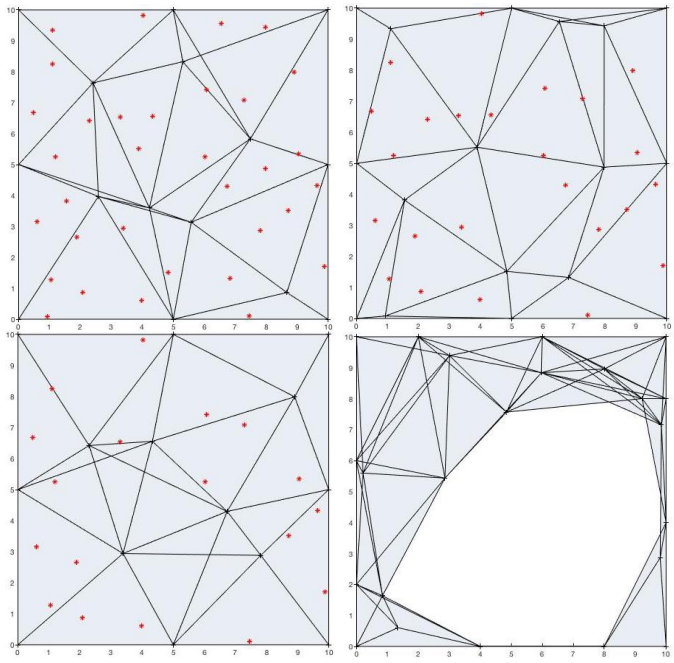

Fig. 6. The 3 layers of coverage and the last incomplete layer.

We also provide an example of our k-coverage algorithm running of the Čech complex of Fig. 7. The configuration set-up is the same as before, except the number of points is initially set to $N = 50$.

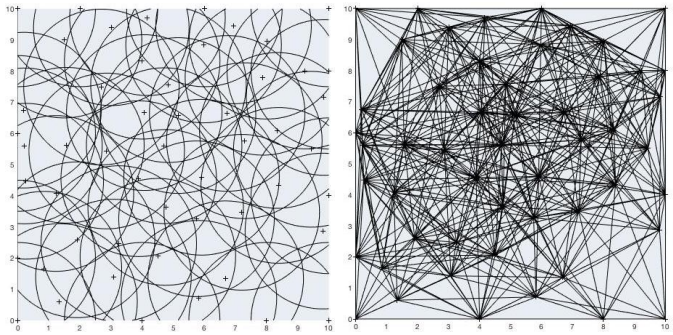

Fig. 7. A wireless network and its Čech representation

This network has more points and provides 4 layers of coverage that is 4-coverage as we can see in Fig. 8.

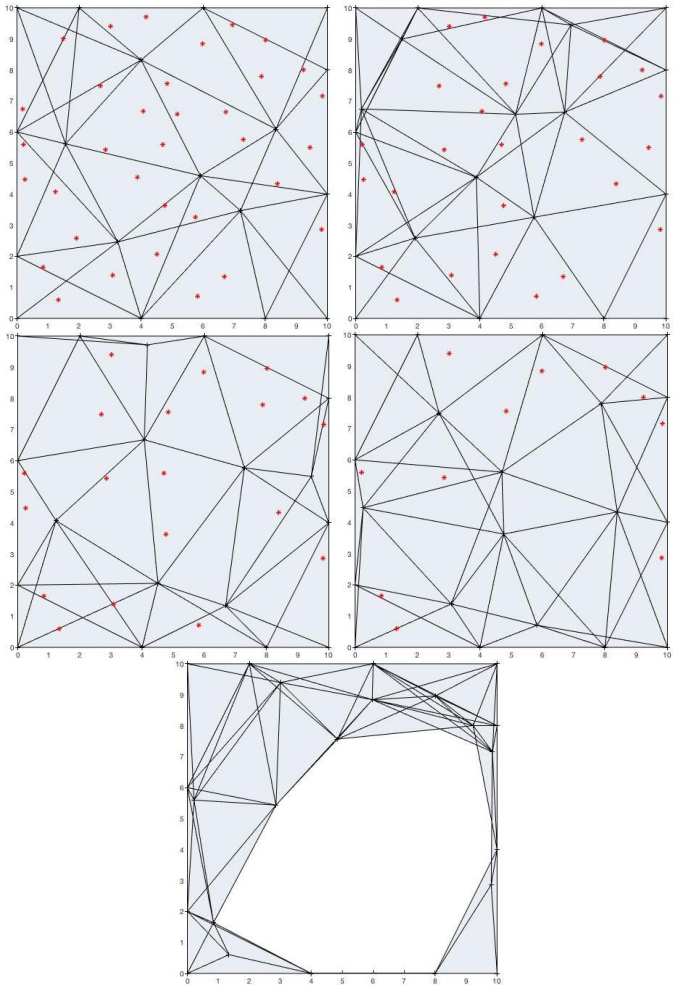

Fig. 8. The 4 layers of coverage and the last incomplete layer.

Finally, we provide some simulation results on the k-coverage of a wireless network generated by a Poisson point process. We look at the value of k, where k is the maximum index such that the network provides k-coverage, depending on the intensity of the process, that is the mean number of points by surface unit, and whether the topology is computed via a Vietoris-Rips or a Čech complex.

We consider a set of points generated with a Poisson point process of intensity λ on a square with side of size 10. We add points on the boundary of the square to delimit the area to be covered. The coverage radius is set to 2.5, one can note that $\pi r^2 \approx 20$. We compute $\bar{k}$ the mean value of k such that the complex provides k-coverage and not $(k+1)$-coverage, on average on 1000 simulations.

TABLE I
MEAN $\bar{k}$ FOR A VIETORIS-RIPS AND FOR FOR A ČECH COMPLEX

λ	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40
$\lambda\pi r^2$	1	2	3	4	5	6	7	8
$\bar{k}$	0.004	0.081	0.268	0.590	0.967	1.396	1.820	

λ	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40
$\lambda\pi r^2$	1	2	3	4	5	6	7	8
$\bar{k}$	0.002	0.079	0.237	0.540	0.859	1.297	1.695	

We can see in Table I the result of the simulations. Moreover, these results are plotted in the graph of Fig. 9. We can compare these results to the theoretic ones of the mean k for which a point x is k-covered: $\mathbf{E}[k] = \lambda\pi r^2$. We can see that, as expected, the simulated values are below the theoretic ones. This is because, theoretically we are only capable of computing the probability of a point to be k-covered, but not the probability that a whole area is k-covered without any coverage holes. The second one, that we can approach by simulation, is smaller than the first one. Furthermore, our algorithm guarantees k-coverage, but locally points may be l-covered with $l > k$.

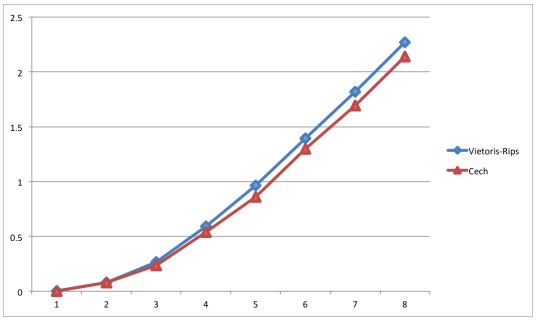

Fig. 9. Mean $\bar{k}$ computed by our algorithm.

VII. CONCLUSION

In this article, we propose a method and an algorithm for computing the k-coverage of a wireless network. The k-coverage is the fact for a point to be covered by k network nodes, this definition can be extended to a whole area: an area is k-covered if every point in it is k-covered. Theoretically it is easy to compute the probability for a point to be k-covered for a wireless network generated by a Poisson point process. However it is more difficult to apprehend the k-coverage for a whole area, and we need simplicial homology representation to compute the topology of the network as a whole. We then

exhibit that the k-coverage can be seen as k layers of 1-coverage and give an algorithm that compute the k-coverage of a wireless network. We provide some figures and simulation results to illustrate our algorithm.

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