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HIGH-DIMENSIONAL INSTRUMENTAL VARIABLES REGRESSION AND
CONFIDENCE SETS - V2/2012

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Background facts: This was a revision of the arXiv preprint [arXiv:1105.2454v1] which was online here. A link is given in footnote 2 of [arXiv:1105.2454v5]. (3.5) is a different STIV estimator from the one in [arXiv:1105.2454v1] where, instead of one conic constraint, there are as many conic constraints as moments (instruments) allowing to use more directly moderate deviations for self-normalized sums. The idea first appeared in formula (6.5) in [arXiv:1105.2454v1] when some instruments can be endogenous. This was called the STIV until the 2014 version. Eventually, the original STIV was put back in the v2 on arXiv because we did not agree on the usefulness of the additional complication. For reference and to avoid confusion with the STIV estimator, this estimator should be called C-STIV.

Keywords: Instrumental variables, sparsity, STIV estimator, endogeneity, high-dimensional regression, conic programming, heteroscedasticity, confidence regions, non-Gaussian errors, variable selection, unknown variance, sign consistency.

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Abstract. We propose an instrumental variables method for inference in structural linear models with endogenous regressors in the high-dimensional setting where the number of possible regressors $K$ can be much larger than $n$. Our new procedure, called $STIV$ (Self Tuning Instrumental Variables) estimator, is realized as a solution of a conic optimization program. We allow for partial identification and very weak distributional assumptions including heteroscedasticity. We do not need prior knowledge of the variances of the errors. A key ingredient is sparsity, i.e., the vector of coefficients has many zeros. The main result of the paper is nested confidence sets, sometimes non-asymptotic, around the identified region under various level of sparsity. In the presence of very many instruments, a number exponential in the sample size, too many non-zero coefficients, or when there is an endogenous regressor which is only instrumented by instruments which are too weak, our confidence sets have infinite volume. We show that a variation on the $STIV$ estimator is a new robust method to estimate low dimensional models without sparsity and possibly many weak instruments. We obtain rates of estimation and show that, under appropriate assumptions, a thresholded $STIV$ estimator correctly selects the non-zero coefficients with probability close to 1. In a non sparse setting, we obtain a sparse oracle inequality that shows how well the regression coefficients are estimated when the model can be well approximated by a sparse model. In our IV regression setting, the standard tools from the literature on sparsity, such as the restricted eigenvalue assumption are inapplicable. Therefore, for our analysis we develop a new approach based on data-driven sensitivity characteristics. We also obtain confidence sets, when the endogenous regressors in the high-dimensional structural equation have a sparse reduced form and we use a two-stage procedure, akin to two-stage least squares. We finally study the properties of a two-stage procedure to deal with the detection of non-valid (endogenous) instruments.

1. Introduction

In this article we consider a structural model of the form

$$y_i = x_i^T \beta + u_i, \quad i = 1, \ldots, n,$$

where $x_i$ are vectors of explanatory variables of dimension $K \times 1$, $u_i$ is a zero-mean random error possibly correlated with some or all regressors. We consider the problem of inference on the structural parameter $\beta$ from $n$ independent, not necessarily identically distributed, realizations $(y_i, x_i^T, z_i^T)$, $i = 1, \ldots, n$. This allows for heteroscedasticity. We denote by $x_{ki}$, $k = 1, \ldots, K$, the components of $x_i$. The regressors $x_{ki}$ are called endogenous if they are correlated with $u_i$ and they are called exogenous otherwise. It is well known that endogeneity occurs when a regressor correlated both with $y_i$ and regressors in the model is unobserved; in the errors-in-variables model when the measurement error
is independent of the underlying variable; when a regressor is determined simultaneously with the response variable \( y_i \). A random vector \( z_i \) of dimension \( L \times 1 \) is called a vector of instrumental variables (or instruments) if it satisfies

\[
(1.2) \quad \forall i = 1, \ldots, n, \quad \mathbb{E}[z_i y_i] = 0,
\]

where \( \mathbb{E}[\cdot] \) denotes the expectation. Having access to instrumental variables makes it possible to identify the vector \( \beta \) when \( \mathbb{E}[z_i x_i^T] \) has full column rank. The case where \( K \geq L \) is typically a case where identification fails (see Example 6 below).

Though the method of this paper can be applied in standard set ups, we are mainly interested in the more challenging high-dimensional setting where \( K \) can be much larger than \( n \) and one of the following two assumptions is satisfied:

(i) only few coefficients \( \beta_k \) are non-zero (\( \beta \) is \textit{sparse}),

(ii) \( \beta \) can be well approximated by a sparse vector (\( \beta \) is \textit{approximately sparse}).

The first assumption means that \( K \) is not the actual number of regressors but the number of potential regressors. All of the regressors do not necessarily appear in the true model underlying \( (1.1) \). The second assumption is likely to be satisfied in a wage equation because in some data sets there are many variables that have a nonzero effect on wage but their marginal effect is too small to matter. The large \( K \) relative to \( n \) problem allows to deal with various important situations encountered in empirical applications. Here are some examples of such applications. In examples 3, 4 and 5 the most likely assumption is the approximate sparsity assumption (iii).

\textbf{Example 1. Economic theory is not explicit enough about which variables belong to the true model.} Sala-i-Martin (1997) and Belloni and Chernozhukov (2001b) give examples from development economics where model selection is important. In macroeconomics, development economics or international finance, it is common to consider cross-country regressions. Because \( n \) is of the order of a few dozens, it is important to allow for \( K \) to be much larger than \( n \).

\textbf{Example 2. Rich heterogeneity.} When there is a rich heterogeneity one usually wants to control for many variables and possibly interactions, or to carry out a stratified analysis where models are estimated in small population sub-groups (e.g., groups defined by the value of an exogenous discrete variable). In both cases \( K \) can be large relative to \( n \).

\begin{itemize}
  \item \( ^1 \)This is the case where \( \beta \) is actually structural.
  \item \( ^2 \)This is also very attractive because it is a feasible, non combinatoric, method, unlike a variable selection method like BIC.
\end{itemize}
Example 3. Many endogenous regressors due to a type of non-parametric specification. This occurs when one considers a structural equation of the form

\[ y_i = f(x_{\text{end},i}) + u_i \]

\[ = \sum_{k=1}^{K} \alpha_k f_k(x_{\text{end},i}) + u_i \]

\[ = x_i^T \beta + u_i \]

(1.4)

and the stronger notion of exogeneity \( E[u_i|z_i] = 0 \) (zero conditional mean assumption), \( x_{\text{end},i} \) is a low dimensional vector of endogenous regressor, \( x_i = (f_1(x_{\text{end},i}), \ldots, f_K(x_{\text{end},i}))^T \), \( \beta = (\alpha_1, \ldots, \alpha_K)^T \) and \( (f_k)_{k=1}^{K} \) are functions from a dictionary. Exogenous regressors could also be included in the right hand side of (1.3) similar to a partial linear model (see next example). Belloni and Chernozhukov (2011b) gives the example of a wage equation with many transformation of education to properly account for nonlinearities. Another typical application is the estimation of Engel curves where it is important to include nonlinearities in the total budget (see, e.g., Blundell, Chen and Kristensen (2007)). When one estimates Engel curves using aggregate data, \( n \) is again usually of the order of a few dozens. It is well known that education in a wage equation and total budget in Engel curves are endogenous variables.

Example 4. Many exogenous regressors due to a type of semi-parametric specification. Consider a partially linear model of the form

\[ y_i = x_{\text{end},i}^T \beta_{\text{end}} + f(x_{\text{exo},i}) + u_i \]

\[ \mathbb{E}[u_i|x_{\text{exo},i}] = 0. \]

\( x_{\text{exo},i} \) is a low dimensional vector of exogenous regressors. If \( f \) can be properly decomposed as a linear combination of functions from a large enough dictionary \( (f_k)_{k=1}^{K_c} \), with coefficients \( \alpha_k \), we obtain

\[ y_i = x_{\text{end},i}^T \beta_{\text{end}} + \sum_{k=1}^{K_c} \alpha_k f_k(x_{\text{exo},i}) + u_i \]

\[ = x_i^T \beta + u_i \]

(1.7)

\footnote{We use the word dictionary, common in the machine learning community, because high-dimensional methods allow to consider very many series terms and to mix bases. This is not exactly the nonparametric IV setup because in a truly nonparametric model there would be an extra approximation error term in (1.4), of order at most \( n^{-1/2} \) when \( K \) is very large and we make minimal smoothness assumptions. High dimensional methods allow in several cases to obtain an adaptive estimation method (that does not need to know the smoothness of the unknown function) but we will not discuss this aspect in this article.}
where $\beta = (\beta_{\text{end}}, \alpha_1, \ldots, \alpha_{Kc})^T$ and $x_i = (x_{\text{end},i}^T, f_1(x_{\text{exo},i}), \ldots, f_{Kc}(x_{\text{exo},i}))^T$. In that case, one is usually interested in the marginal effects $\beta_{\text{end}}$ of $x_{\text{end},i}$ holding fixed $x_{\text{exo},i}$, rather than the whole vector $\beta$.

**Example 5. Many control variables to justify the use of an instrument.** Suppose that we are interested in the parameter $\beta$ in

$$y_i = x_i^T \beta + v_i,$$

where some of the variables in $x_i$ are endogenous, but that we have at our disposal a variable $z_i$ that we want to use as an instrument but which does not satisfy $E[z_i v_i] = 0$. Suppose that we also have observations of vectors of controls $w_i$ such that $E[v_i|w_i, z_i] = E[v_i|w_i]$ (conditional mean independence). Then we can rewrite (1.8) as

$$y_i = \overline{x}_i^T \overline{\beta} + f(w_i) + u_i$$

where $f(w_i) = E[v_i|w_i]$ and $u_i = v_i - E[v_i|w_i, z_i]$ is such that $E[z_i u_i] = 0$. It yields

$$y_i = \overline{x}_i^T \overline{\beta} + \sum_{k=1}^{Kc} \alpha_k f_k(w_i) + u_i$$

when $f$ can be decomposed on $(f_k)_{k=1}^{Kc}$. This model can be rewritten in the form (1.11) with $\beta = (\overline{\beta}^T, \alpha_1, \ldots, \alpha_{Kc})^T$ and $x_i = (\overline{x}_i^T, f_1(w_i), \ldots, f_{Kc}(w_i))^T$. Again one is usually interested in the sub-vector $\overline{\beta}$.

**Example 6. The instruments can have a direct effect on the outcome.** Kolesár, Chetty, Friedman, et al. (2011) considers the case where one wants to allow the instruments to have a direct effect on the outcome, i.e., to be potentially on the right hand side of (1.1). We are thus in a setting where $K > L$ which implies that $\beta$ is not identified from the moment conditions (1.2). One typically needs exclusion restrictions (i.e., that some exogenous variables are instruments and do not appear on the right hand side of (1.1)) which corresponds to some coefficients being equal to zero in the specification where all instruments can potentially appear on the right hand side of (1.1). Exclusion restrictions is therefore a reason for sparsity. This specification is more flexible as we assume that there exists some exclusion restrictions without telling which one in advance. In a setup where there

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4By the law of iterated conditional expectations.
are many (weak) instruments\(^5\), allowing the instruments to potentially have a direct effect, implies that \(K\) is large.

Statistical inference under the sparsity scenario when the dimension is larger than the sample size is now an active and challenging field. The most studied techniques are the Lasso, the Dantzig selector (see, e.g., Candès and Tao (2007), Bickel, Ritov and Tsybakov (2009); more references can be found in the recent book by Bühlmann and van de Geer (2011), as well as in the lecture notes by Koltchinskii (2011), Belloni and Chernozhukov (2011b)), and aggregation methods (see Dalalyan and Tsybakov (2008), Rigollet and Tsybakov (2011) and the papers cited therein). A central concern in this literature is to propose methods that are computationally feasible. The Lasso for example is a convex relaxation of \(l^0\) penalized least squares methods like BIC. The last are \(NP\)-hard and it is impossible in practice to consider the case where \(K\) is larger than a few dozens. The Dantzig selector is solution of a simple linear program. In recent years, these techniques became a reference in several areas, such as biostatistics and imaging. Some important extensions to model from econometrics have been obtained by Belloni and Chernozhukov (2011a) who study the \(\ell_1\)-penalized quantile regression and give an application to cross-country growth analysis and by Belloni, Chernozhukov and Hansen (2010) who use the Lasso to estimate the optimal instruments with an application to the impact of eminent domain on economic outcomes. Städler, Bühlmann and van de Geer (2010) studies the estimation of mixtures of high-dimensional regressions, this is important in econometrics to handle group heterogeneity. Caner (2009) studies a Lasso-type GMM estimator. Rosenbaum and Tsybakov (2010) deal with the high-dimensional errors-in-variables problem and discuss an application to hedge fund portfolio replication. Belloni and Chernozhukov (2011b) also presents several applications of high-dimensional methods to economics. The high-dimensional setting in a structural model with endogenous regressors that we are considering here has not yet been analyzed. Note that the direct implementation of the Lasso or Dantzig selector fails in the presence of a single endogenous regressor simply because the zero coefficients in the structural equation (1.1) do not correspond to the zero coefficients in a linear projection type model. We also obtain confidence sets in a high-dimensional framework.

The main message of this paper is that, in model (1.1) containing endogenous regressors, under point identification, the high-dimensional vector of coefficients or a subset of important coefficients can be estimated together with proper confidence sets using instrumental variables. In partially identified settings, we obtain confidence sets around the identified region under a sequence of sparsity scenarios.

\(^5\)As exemplified in Angrist and Krueger (1991), under the stronger notion of exogeneity based on the zero conditional mean assumption, considering interactions or functionals of instruments can lead to a large number of instruments.
varying the upper bound $s$ on the number of non-zero coefficients of $\beta$ (that we call \textit{sparsity certificate}). Thus there is no restriction on the size of $K$. When $K$ is too large, one loses identification. Our confidence sets can have infinite volume when $s$ and/or $K$ is too large. There is no restriction either on the strength of the instruments. If a very large number of instruments is used (exponential in the sample size), or when all the instruments are weak, the method yields infinite volume confidence sets. The price to pay for including an irrelevant instrument in a preliminary set of $L$ instruments is just a factor of $\sqrt{\log(L + 1)/\sqrt{\log(L)}}$ in the size of the confidence sets. The \textit{STIV} estimator is thus a method that is robust to weak instruments and can handle very many (\textit{i.e.} exponential in the sample size) instruments. Also it is not required in principle to know in advance which regressor is endogenous and which is not, one needs a set of valid (exogenous) instruments. This is achieved by the \textit{STIV} estimator (Self Tuning Instrumental Variables estimator) that we introduce below. Based on it, we can also perform variable selection. Under various assumptions on the data generating process we obtain non-asymptotic or asymptotic results. We also provide meaningful bounds when either (i) or (ii) above holds and $\log(L)$ is small compared to $n$. We believe that a non-asymptotic framework is the most natural setting to consider in high dimensions. We also restrict mostly our attention to a one stage method. This is even more justified in \textit{IV} estimation of structural equations with endogenous regressors and weak instruments where inference usually relies on non-standard asymptotics (see, \textit{e.g.}, Stock, Wright and Yogo (2002) and Andrews and Stock (2007) for a review and the references cited therein). Nelson and Startz (1990a, b) contains a simulation study where, due to weak instruments, the finite sample distribution of the two-stage least squares estimator can be non normal and even bimodal.

The \textit{STIV} estimator is an extension of the Dantzig selector of Candès and Tao (2007). The results of this paper extend those on the Dantzig selector (see Candès and Tao (2007), Bickel, Ritov and Tsybakov (2009) and further references in Bühlmann and van de Geer (2011)) in several ways: By allowing for endogenous regressors when instruments are available, by working under weaker sensitivity assumptions than the restricted eigenvalue assumption of Bickel, Ritov and Tsybakov (2009), which in turns yields tighter bounds, by imposing weak distributional assumptions, by introducing a procedure independent of the noise level and by providing confidence sets.

The \textit{STIV} estimator is also very much inspired by the Square-root Lasso of Belloni, Chernozhukov and Wang (2010) which proposes a pivotal method independent of the variance of the errors.

\footnote{The terminology very many is borrowed from Belloni, Chen, Chernozhukov et al. (2010).}
in the Gaussian linear model with fixed regressors. The Square-root Lasso is a very important contribution to the literature on the Lasso where almost all articles require to know the variance of the errors which is related to the degree of penalization required. The most common practice is to adjust the degree of penalization is to use cross validation or BIC. The STIV estimator adds extra linear constraints coming from the restrictions $\text{(1.2)}$ to the Square-root Lasso which allows one to deal with endogenous regressors. The implementation of the STIV estimator also correspond to solving a simple conic optimization program. Confidence sets require as well solving at most $2K^2$ linear programs for lower bounding the sensitivities that we introduce below. So our method is easy and fast to implement in practice. Our confidence sets rely on moderate deviations for self-normalized sums and either on the sparsity certificates or on perfect model selection. Perfect model selection requires a separation from zero of the non-zero coefficients, while the sparsity certificate approach does not. We are indebted to Belloni, Chernozhukov and Wang (2010) who are the first to use moderate deviations for self-normalized sums results for high-dimensional regression (see also Belloni, Chen, Chernozhukov and Hansen (2011)). We will see that under some distributional assumptions these can lead to confidence sets that have finite sample coverage properties.

2. Basic Definitions and Notation

We set $Y = (y_1, \ldots, y_n)^T$, $U = (u_1, \ldots, u_n)^T$, and we denote by $X$ and $Z$ the matrices of dimension $n \times K$ and $n \times L$ respectively with rows $x_i^T$ and $z_i^T$, $i = 1, \ldots, n$.

The sample mean is denoted by $E_n[\cdot]$. We use the notation

$$E_n[X^a_k U^b] \triangleq \frac{1}{n} \sum_{i=1}^{n} x_{ki}^a u_i^b,$$

$$E_n[Z^a_l U^b] \triangleq \frac{1}{n} \sum_{i=1}^{n} z_{li}^a u_i^b,$$

where $x_{ki}$ is the $k$th component of vector $x_i$, and $z_{li}$ is the $l$th component of $z_i$ for some $k \in \{1, \ldots, K\}$, $l \in \{1, \ldots, L\}$, $a \geq 0$, $b \geq 0$. Similarly, we define the sample mean for vectors; for example, $E_n[UX]$ is a row vector with components $E_n[UX_k]$. We also define the corresponding population means:

$$E[X^a_k U^b] \triangleq \frac{1}{n} \sum_{i=1}^{n} E[x_{ki}^a u_i^b],$$

$$E[Z^a_l U^b] \triangleq \frac{1}{n} \sum_{i=1}^{n} E[z_{li}^a u_i^b].$$

\footnote{We believe that this is unavoidable for model selection but presumably strong for some econometrics applications. However this result is a useful addition to the current state of the art in the theory of high-dimensional regression even with exogenous regressors.}
and set, for $l = 1, \ldots, L$,

$$z_{t^*} \triangleq \max_i |z_{ti}|, \quad (x \cdot z)_l = \left\{ \max_{k=1,\ldots,K} \mathbb{E}_n \left[ \left( \frac{X_k}{\mathbb{E}_n[X_k^2]^{1/2}} Z_l \right)^2 \right] \right\}^{1/2}.$$

We denote by $D_X$ the diagonal $K \times K$ matrices with diagonal entries $\mathbb{E}_n[X_k^2]^{-1/2}$ for $k = 1, \ldots, K$. We use the notation $D_Z^{(I)}$ for the $L \times L$ matrix of entries $(x \cdot z)_l^{-1}$ for $l \in I$ and $z_{l^*}^{-1}$ otherwise. The set $I$ is a subset of $\{1, \ldots, L\}$. It always contains the index of the instrument which is identically equal to 1. The corresponding coefficient $\beta_k$ is the usual constant in (1.1). The matrices $D_X$ and $D_Z^{(I)}$ are normalization matrices. The user specifies the set $I$ based on the observed distribution. For fast implementation of our algorithm, $z_{l^*}^{-1}$ is the preferred normalization. For heavy tail distributions, though, one may want to use the second normalization. The more heavy tail instruments we have, the more computationally intensive our method will be. Indeed, we will add a conic constraint for each heavy tail instrument. For a vector $\beta \in \mathbb{R}^K$, let $J(\beta) = \{ k \in \{1, \ldots, K\} : \beta_k \neq 0 \}$ be its support, i.e., the set of indices corresponding to its non-zero components $\beta_k$. We denote by $|J|$ the cardinality of a set $J \subseteq \{1, \ldots, K\}$ and by $J^c$ its complement: $J^c = \{1, \ldots, K\} \setminus J$. The subset of indices $\{1, \ldots, K\}$ corresponding to variables in (1.1) that are known in advance to be exogenous and serve as their own instruments is denoted by $J_{\text{exo}}$. There might be more true exogenous variables in (1.1), they are excluded from this list of indices if they have not been included as instruments. The $\ell_p$ norm of a vector $\Delta$ is denoted by $|\Delta|_p$, $1 \leq p \leq \infty$. For $\Delta = (\Delta_1, \ldots, \Delta_K)^T \in \mathbb{R}^K$ and a set of indices $J \subseteq \{1, \ldots, K\}$, we consider $\Delta_J \triangleq (\Delta_1 \mathbb{I}_{\{1 \in J\}}, \ldots, \Delta_K \mathbb{I}_{\{K \in J\}})^T$, where $\mathbb{I}_{\{\cdot\}}$ is the indicator function.

For a vector $\beta \in \mathbb{R}^K$, we set $\text{sign}(\beta) \triangleq (\text{sign}(\beta_1), \ldots, \text{sign}(\beta_K))$ where

$$\text{sign}(t) \triangleq \begin{cases} 1 & \text{if } t > 0 \\ 0 & \text{if } t = 0 \\ -1 & \text{if } t < 0 \\ \end{cases}.$$

For $a \in \mathbb{R}$, we set $a_+ \triangleq \max(0, a)$, $a_+^{-1} \triangleq (a_+)^{-1}$, and $a/0 \triangleq \infty$ for $a > 0$. We adopt the convention $0/0 \triangleq 0$ and $1/\infty \triangleq 0$.

We denote by

$$\text{Ident} = \{ \beta : \mathbb{E}[z_i(y_i - x_i^T \beta)] = 0 \}$$

the identified region. It is an affine space which is reduced to a point when the model (1.1)-(1.2) is point identified. It is possible to impose some restrictions like a known sign or a prior upper bound on the size of the coefficients. However, for simplicity, we will just consider the case when we know
an a priori upper bound $s$ on the sparsity of $\beta$, i.e., we know that $|J(\beta)| \leq s$ for some integer $s$. We call this a sparsity certificate. We introduce

$$\mathcal{B}_s = \text{Ident} \cap \{\beta : |J(\beta)| \leq s\}.$$ 

Note that $\mathcal{B}_K = \text{Ident}$. Thus, considering confidence sets around $\mathcal{B}_s$ allows to deal with both the case when we rely on sparsity certificates and the case when we do not.

3. The STIV Estimator

The sample counterpart of the moment conditions (1.2) can be written in the form

$$\frac{1}{n} Z^T (Y - X\beta) = 0.$$ 

This is a system of $L$ equations with $K$ unknown parameters. If $L > K$, it is overdetermined. In general rank$(Z^T X) \leq \min(K, L, n)$, thus when $L \leq K$ or when $n < K$ the matrix does not have full column rank. Furthermore, replacing the population equations (1.2) by (3.1) induces a huge error when $L, K$ or both are larger that $n$. So, looking for the exact solution of (3.1) in the high-dimensional setting makes no sense. However, we can stabilize the problem by restricting our attention to a suitable “small” candidate set of vectors $\beta$, for example, to those satisfying the constraint

$$\frac{1}{n} Z^T (Y - X\beta) \leq \tau,$$ 

where $\tau > 0$ is chosen such that (3.2) holds for $\beta$ in $\mathcal{B}_s$ with high probability. We can then refine the search of the estimator in this “small” random set of vectors $\beta$ by minimizing an appropriate criterion. It is possible to consider different small sets in (3.2), however the use of the sup-norm makes the inference robust in the presence of weak instruments. This will be clarified later.

In what follows, we use this idea with suitable modifications. First, notice that it makes sense to normalize the matrix $Z$. This is quite intuitive because, otherwise, the larger the instrumental variable, the more influential it is on the estimation of the vector of coefficients. For technical reasons, we choose normalization where we multiply $Z$ by $D_Z^{(I)}$. The constraint (3.2) is modified as follows:

$$\frac{1}{n} D_Z^{(I)} Z^T (Y - X\beta) \leq \tau.$$ 

Along with the constraint of the form (3.3), we include more constraints to account for the unknown (average in the case of heteroscedasticity) level $\sigma$ of the “effective noise” $z_i u_i$. Specifically,
we say that a pair \((\beta, \sigma) \in \mathbb{R}^K \times \mathbb{R}^+\) satisfies the IV-constraint if it belongs to the set

\[
\hat{I}^{(I)} \triangleq \left\{ (\beta, \sigma) : \beta \in \mathbb{R}^K, \sigma > 0, \frac{1}{n} D_Z^{(I)} Z^T (Y - X\beta) \right\}_\infty \leq \sigma r, (D_Z^{(I)})_l^2 \hat{Q}_l(\beta) \leq \sigma^2, \forall l \in I \}
\]

for some \(r > 0\) (specified below), and

\[
\hat{Q}_l(\beta) \triangleq \frac{1}{n} \sum_{i=1}^n z_{li}^2 (y_i - x_i^T \beta)^2.
\]

Note that the instrument \(z_{li} = 1\) for all \(i = 1, \ldots, n\) belongs to \(I\) and the corresponding value of \(\hat{Q}_l(\beta)\) is

\[
\hat{Q}_l(\beta) = \frac{1}{n} \sum_{i=1}^n (y_i - x_i^T \beta)^2.
\]

**Definition 3.1.** We call the STIV estimator any solution \(\hat{\beta}(c,I), \hat{\sigma}(c,I)\) of the following minimization problem:

\[
\min_{(\beta, \sigma) \in \hat{I}^{(I)}} \left( |D_X^{-1} \beta|_1 + c \sigma \right),
\]

where \(0 < c < 1\).

In this formulation the instruments show up both in the IV-constraint and in the penalization through \(\sigma\). As discussed above, the IV-constraint includes the constraints coming from the moment conditions (1.2) accounting for instrument exogeneity. We use \(\hat{\beta}(c,I)\) as an estimator of \(\beta\) in \(B_s\).

Finding the STIV estimator is a conic program; it can be efficiently solved, see Section 10.1. Note that the STIV estimator is not necessarily unique. Minimizing the \(\ell_1\) criterion \(|D_X^{-1} \beta|_1\) is a convex relaxation of minimizing the \(\ell_0\) norm, i.e., the number of non-zero coordinates of \(\beta\). This usually ensures that the resulting solution is sparse. The term \(c \sigma\) is included in the criterion to prevent from choosing \(\sigma\) arbitrarily large; indeed, the IV-constraint does not prevent from this. The matrix \(D_X^{-1}\) arises from re-scaling of \(X\). It is natural to normalize the regressors by their size in a procedure that can do variable selection. This way changing units does not change which coefficients are found to be zeros or non-zeros. For the particular case where \(Z = X\) and \(D_Z^{(I)} = D_X\) is the diagonal matrix with entries \((E_n[X_k^2]^{-1/2})_k^{K}_{k=1}\), the STIV estimator provides an extension of the Dantzig selector to the setting with unknown variance of the noise.

In this particular case, the STIV estimator can also be related to the Square-root Lasso of Belloni, Chernozhukov and Wang (2010), which solves the problem of unknown variance in high-dimensional regression with deterministic regressors and i.i.d. errors. The definition of STIV estimator...
contains the additional constraint (3.3), which is not present in the conic program for the Square-root Lasso. This is due to the fact that we have to handle the endogeneity.

4. Summary of the Main Results

The main result of this paper are nested confidence sets around the regions \((B_s)_{s \leq K}\). We allow for set-identification, arbitrarily weak instruments, for a situation where \(K > L\) (for example when all the instruments can have a direct effect on the outcome or for various combinations of discrete regressors and instruments). The dimension \(K\) could be as large as we want, even larger than an exponential in the sample size, in that case the model (1.1) together with (1.2) becomes set-identified. Also, in finite samples and when \(L\) and/or \(K\) are much larger than \(n\), we can obtain finite volume confidence sets for low values of \(s\) and infinite volume confidence sets for larger values of \(s\), even in point-identified settings. These results are presented in Section 6. We first present various assumptions that one can make on the data generating process. These sometimes limit the number of instruments. We give the corresponding way to adjust the constant \(r\) in the definition of the set \(\hat{I}^{(I)}\) appearing in the constrained optimization for the STIV estimator.

In all but the first scenario on the data generating process, our confidence sets are of the form: For every \(\beta\) in \(B_s\), with probability at least \(1 - \alpha\), for any \(c\) in \((0, 1)\) and any set of indices \(I\) containing the index of the instrument which is unity, any solution \((\hat{\beta}^{(c,I)}, \hat{\sigma}^{(c,I)})\) of the minimization problem (3.5) satisfies

\[
\left| \left( DX^{-1}(\hat{\beta}^{(c,I)} - \beta) \right)_{J_0} \right|_p \leq \frac{2\hat{\sigma}^{(c,I)} r}{\kappa^{(c,I)}_{p,J_0}(s)} \left( 1 - \frac{r}{\kappa^{(c,I)}_1(s)} \right)^{-1} \quad \forall p \in [1, \infty], \forall J_0 \subset \{1, \ldots, K\},
\]

where the set \(J_0\) is specified by the econometrician, depending on which subset of the regressor he is interested in, and for all \(k = 1, \ldots, K\),

\[
|\hat{\beta}_k^{(c,I)} - \beta_k| \leq \frac{2\hat{\sigma}^{(c,I)} r}{E_n[X_k^2]^{1/2} \kappa^{(c,I)*}_k(s)} \left( 1 - \frac{r}{\kappa^{(c,I)}_1(s)} \right)^{-1}.
\]

Under the first scenario on the data generating process the \(1 - \alpha\) probability event depends on \(I\) so that the statement is not uniform in \(I\) but for fixed \(I\). The constants \(\kappa^{(c,I)}_{p,J_0}(s), \kappa^{(c,I)}_1(s)\) or \(\kappa^{(c,I)*}_k(s)\) are easy to calculate, data-dependent, lower bounds on the sensitivities that we introduce in Section 5. The sensitivities provide a generalization of the restricted eigenvalues to non-symmetric and non-square matrices (see Section 10.1 in the appendix for a comparison). The most difficult constant to calculate is \(\kappa^{(c,I)}_1(s)\), it requires to solve \(2K^2\) linear programs. The confidence sets could be infinite when \(\kappa^{(c,I)}_1(s) \leq r\). We will see that this occurs either when we have very many instruments or when
there exists a regressor for which all instruments are too weak instruments or \( s \) is too large. We will also discuss how the constants \( \kappa_k^{(c,I)}(s) \) and \( \kappa_1^{(c,I)}(s) \) compare to the usual concentration parameter defined for low-dimensional structural equations with homoscedastic Gaussian errors. Recall that we relax both heteroscedasticity and Gaussian errors. When we are only interested in a subset of the coefficients, our confidence sets are smaller than those that one would obtain by projection of joint confidence sets. They account for the a priori upper bound on the sparsity \( s \). The confidence sets do not require that the non-zero coefficients are large enough but require to make a stand and give a priori upper bound \( s \). As we will see in Section 7 the STIV estimator usually estimates too many non-zero coefficients when the underlying vector is sparse so that \( |J(\hat{\beta}^{(c,I)})| \) can be considered as a conservative upper bound on the true sparsity. For increasing values of \( s \) one obtain nested confidence sets. These become infinite when \( s \) approaches \( K \) and \( K \) is larger than \( n \). If one is interested in a few coefficients, it is possible to choose the value of \( c \) in \((0, 1)\) that yields the smaller confidence sets. This is easily obtained by taking grid values for \( c \). Similarly, for all but the first scenario on the data generating process, it is possible to try several sets \( I \) and choose according to the size of the confidence sets around the vector of the coefficients of interest.

In Section 6 we present a sparse oracle inequality in the case (ii) above where the underlying model is not sparse. It shows that the STIV estimator estimates the coefficients as well as a method that would know in advance which is the best sparse approximation in terms of bias/variance trade-off.

In Section 8 we present rates of convergence and model selection results. Indeed, the right hand-side in the above two upper bounds are random and these inequalities do not tell, for example, how well is the vector of coefficient estimated depending on \( L \) and \( n \). We also obtain that, if the absolute values of the coefficients \( |\beta_k| \) are large enough on the support of \( \beta \), then with probability close to one \( J(\beta) \subseteq J(\hat{\beta}^{(c,I)}) \). This is used to obtain a second type of confidence sets where we plug-in an estimator of \( J(\beta) \) in the definition of the sensitivities to obtain a lower bound, and thus, a proper data-driven upper bound on the estimation error. Recall that the first type of confidence sets uses sparsity certificates and does not rely on such a separation from zero of the non-zero coefficients assumption. The fact that, with probability close to one \( J(\beta) \subseteq J(\hat{\beta}^{(c,I)}) \) is confirmed in the simulation study where the STIV estimator usually selects too many regressors. This result allows to recover exactly, with probability close to 1, the true support of the vector of coefficients as well as the sign of the coefficients by a thresholding rule.

Section 8 discusses some special cases and extensions. Because the STIV estimator is a one stage method that is robust to weak-instruments and allows for very many instruments, we present
a variation on the STIV estimator to estimate low-dimensional \((K < n)\) and non sparse structural models. This is a new method to construct confidence sets that is robust to weak instruments, heteroscedasticity and non-Gaussian errors, and allows for very-many-instruments. We also present the refined properties of our estimator when none of the regressors and instruments have heavy tail distribution. This simplifies its calculation and gives finer results, especially in the very many or weak-instruments case. Finally, we present the properties of a two-stage STIV procedure with estimated sparse linear projection type instruments, akin to two-stage least squares. Here the two stages are high-dimensional regressions and the second stage has endogenous regressors. This is related to the literature on selection of instruments and optimal instruments. Here we do not touch on optimality because it is an open question to define optimality with a high-dimensional structural equation with endogenous regressors. This method only works if the endogenous regressors have a sparse reduced form. In the case of an approximately sparse reduced form (case (ii) above) we would obtain rates of convergence but our method do not yield confidence sets.

In Section \(9\) we consider the following model

\[
\begin{align*}
y_i &= x_i^T \beta + u_i, \\
\mathbb{E}[z_i u_i] &= 0, \\
\mathbb{E}[\tau_i u_i] &= \theta.
\end{align*}
\]

Here, \(z_{il}\) for \(i = 1, \ldots, n\) and \(l = 1, \ldots, L\), is a set of instruments that are known in advance to be valid (exogenous) and \(\tau_i\) for \(i = 1, \ldots, n\) and \(l = 1, \ldots, L\), is a second set of instruments. One wants to decide which instrument from this second list is valid and which is invalid (endogenous). In this setup, we again allow for the dimensions of \(x_i, z_i\) an \(\tau_i\) to be much larger than \(n\).

Section \(10\) discusses way to implement our algorithm and to calculate the constants that drive the size of the confidence sets. This is based either on conic or linear programs. These are readily available in many classical softwares. In this section we also present a simulation study. All the proofs are given in the appendix.

5. Sensitivity Characteristics

In the usual linear regression in low dimension, when \(Z = X\) and the Gram matrix \(X^T X / n\) is positive definite, the sensitivity is given by the minimal eigenvalue of this matrix. In high-dimensional regression, the theory of the Lasso and the Dantzig selector comes up with a more sophisticated sensitivity analysis; there the Gram matrix cannot be positive definite and the eigenvalue conditions
are imposed on its sufficiently small submatrices. This is typically expressed via the restricted isometry property of Candès and Tao (2007) or the more general restricted eigenvalue condition of Bickel, Ritov and Tsybakov (2009). In our structural model with endogenous regressors, these sensitivity characteristics cannot be used, since instead of a symmetric Gram matrix we have a rectangular matrix $Z^T X/n$ involving the instruments. More precisely, we will deal with its normalized version
\[
\Psi_n \triangleq \frac{1}{n} D(Z^T Z) X D X.
\]
In general, $\Psi_n(I)$ is not a square matrix. For $L = K$, it is a square matrix but, in the presence of at least one endogenous regressor, $\Psi_n(I)$ is not symmetric.

We now introduce some scalar sensitivity characteristics related to the action of the matrix $\Psi_n(I)$ on vectors in the cone $C_J^{(c)} \triangleq \{ \Delta \in \mathbb{R}^K : |\Delta_J|_1 \leq \frac{1+c}{1-c} |\Delta|_1 \}$, where $0 < c < 1$ is the constant in the definition of the STIV estimator, $J$ is any subset of $\{1, \ldots, K\}$. When the cardinality of $J$ is small, the vectors $\Delta$ in the cone $C_J^{(c)}$ have a substantial part of their mass concentrated on a set of small cardinality. We call $C_J^{(c)}$ the cone of dominant coordinates. The set $J$ that will be used later is the set $J(\beta)$ which is small if $\beta$ is sparse. The use of similar cones to define sensitivity characteristics is standard in the literature on the Lasso and the Dantzig selector (see, Bickel, Ritov and Tsybakov (2009)); the particular choice of the constant $\frac{1+c}{1-c}$ will become clear from the proofs. It follows from the definition of $C_J^{(c)}$ that
\[
|\Delta|_1 \leq \frac{2}{1-c} |\Delta_J|_1 \leq \frac{2}{1-c} |J|^{1-1/p} |\Delta_J|_p, \quad \forall \Delta \in C_J^{(c)} , \quad 1 \leq p \leq \infty.
\]
For $p \in [1, \infty]$, we define the $\ell_p$ sensitivity as the following random variable:
\[
\kappa_{p,J}^{(c,I)} \triangleq \inf_{\Delta \in C_J^{(c)} : |\Delta|_p=1} \frac{|\Psi_n(I) \Delta|_\infty}{|\Delta|_p}.
\]
Similar, but different, quantities named cone invertibility factors have been introduced in Ye and Zhang (2010).

Given a subset $J_0 \subset \{1, \ldots, K\}$ and $p \in [1, \infty]$, we define the $l_p$-$J_0$-block sensitivity as
\[
\kappa_{p,J_0,J}^{(c,I)} \triangleq \inf_{\Delta \in C_J^{(c)} : |\Delta|_{J_0}=1} \frac{|\Psi_n(I) \Delta|_\infty}{|\Delta|_{J_0}}.
\]
By convention, we set $\kappa_{p,J}^{(c,I)}(\beta) = \infty$. We use the notation $\kappa_{k,J}^{(c,I)}$ for coordinate-wise sensitivities, i.e., for block sensitivities when $J_0 = \{k\}$ is a singleton.

\[
\kappa_{k,J}^{(c,I)}(\beta) \triangleq \inf_{\Delta \in C^{(c,I)}_J: \Delta_k = 1} \left| \frac{\Psi^{(I)}_n(\Delta)}{\infty} \right|.
\]

Note that here we restrict the minimization to vectors $\Delta$ with positive $k$th coordinate, $\Delta_k = 1$, since replacing $\Delta$ by $-\Delta$ yields the same value of $|\Psi^{(I)}_n(\Delta)|_{\infty}$.

The heuristic behind the sensitivity characteristics is the following. As we will see in the Appendix, for a fixed value of $\beta$ in $B_s$, we adjust $r$ in the definition of $\hat{T}^{(I)}$ such that $\left( \hat{\beta}, \max_{l \in I} (D^{(I)}_{\beta})^{1/2} \sqrt{\hat{Q}^{(I)}_l(\beta)} \right)$ belongs to $\hat{T}^{(I)}$ on an event $E_\alpha$ of probability $1 - \alpha$. This yields that for a proper $\tau$, on $E_\alpha$,

\[
|\Psi^{(I)}_n(\Delta)|_{\infty} \leq \tau
\]

where $\Delta = D^{-1}_X (\hat{\beta}^{(c,I)} - \beta)$. When $\tau$ can be calculated directly from the data, this yields a confidence region for $\beta$ in $B_s$. Moreover, because we minimize the objective function (3.5), we will see that, on the same event $E_\alpha$, $\Delta$ is constrained to belong to the subset $C^{(c,I)}_{J(\beta)}$ of $\mathbb{R}^K$. The sensitivities allow to deduce from (5.4) and the cone condition, confidence regions for various losses. Suppose, for example, that one is interested in a confidence region for the subvector of $\beta$ in $B_s$, which corresponds to the coefficients of indices in $J_0 \subset \{1, \ldots, K\}$, and considers the $l_p$-loss, then one gets

\[
|\Delta_{J_0}|_p \leq \frac{\tau}{\kappa_{p,J_0,J}^{(c,I)}(\beta)}.
\]

Indeed, this is trivial when $|\Delta_{J_0}|_p = 0$ and, when it is not, it immediately follows from

\[
\frac{|\Psi^{(I)}_n(\Delta)|_{\infty}}{|\Delta_{J_0}|_p} \geq \inf_{\tilde{\Delta} \neq 0, \tilde{\Delta} \in C^{(c,I)}_{J(\beta)}} \left| \frac{\Psi^{(I)}_n(\tilde{\Delta})}{\infty} \right|_{\tilde{\Delta}_{J_0}|_p}.
\]

Remark that, working with the coordinate-wise sensitivities $\kappa_{k,J}^{(c,I)}(\beta)$ is much better than projecting the region [5.3] onto the axes because it also takes into account the cone condition and, thus, the sparsity of the underlying $\beta$. We see that the sensitivities are quantities that are intrinsic to the estimation properties of the STIV estimator. We show in Section 11.1 that the assumption that the sensitivities $\kappa_{p,J}^{(c,I)}(\beta)$ are positive is weaker and more flexible than the restricted eigenvalue (RE) assumption of Bickel, Ritov and Tsybakov (2009). Unlike the RE assumption, it is applicable to non-square non-symmetric matrices. Another nice feature of the sensitivities that we introduce is that lower bounds on these

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8 They coincide for all values of $p \in [1, \infty]$.

9 They are also intrinsic to the Dantzig selector and the Lasso.
sensitivities can be efficiently calculated (see below), this opens the path to confidence statements in high-dimensional regression.

The coordinate-wise sensitivities are measures of the strength of the instruments. The coordinate-wise sensitivity for index $k$ could be interpreted as restricted maximal partial empirical correlation between the instruments and the regressor $(x_{ki})_{i=1}^n$. Indeed, (5.3) can be written as

$$K_{k,J}^{(c,I)_k^*} = \inf_{\lambda \in \mathbb{R}^{K-1}}: \sum_{l \in J^c} |E_n[|x_i^2|^{1/2}]/\lambda| \leq \frac{1}{n} \left(1+\sum_{l \in J^c} |E_n[|x_i^2|^{1/2}]/\lambda| \right) \max_{l=1,...,L} \left( D_{Z}^{(I)} \right)_{ll} \left| \frac{1}{n} \sum_{i=1}^n z_{li} \left( x_{ki} - x_{k \mid}^T \lambda \right) \right|. $$

It is easy to check that (5.5) yields that when $L < |J|$, $K_{k,J}^{(c,I)_k^*} = 0$. For confidence sets or rates of estimation statements, the set $J$ will correspond to $J(\beta)$ for $\beta$ in $B_\delta$ so that the restriction in the infimum corresponds to vectors having most of their mass on the support of the underlying vector of coefficients $\beta$ from the set $B_\delta$ (which includes the constant). When an exogenous variable serves as its own instrument, the coordinate-wise sensitivity for that regressor should be large. This is because there exists, among the list of instruments, $(x_{ki})_{i=1}^n$ themselves. Otherwise, the coordinate-wise sensitivity of a regressor is small if all instruments have small restricted partial empirical correlation with that regressor. Because of the maximum, one good instrument is enough to have a large coordinate-wise sensitivity. It is small only if all instruments are weak.

**Example 7.** For comparison, consider a structural equation with only one endogenous regressor $(x_{kend})_{i=1}^n$. Assume that $(z_{li})_{i=1}^n$ for $l = 1, \ldots, L$ are fixed and write the reduced form equation

$$(5.6) \quad x_{kend} = \sum_{l \in J^c} \tilde{z}_l \zeta_l + \sum_{l \in J} \tilde{z}_l \zeta_l + v_i, \quad i = 1, \ldots, n,$$

where $(\zeta_l)_{l=1}^{L}$ are unknown coefficients, $J$ is the set of indices of the exogenous regressors that have non-zero coefficients in (1.1) (this is possible when $|J| \leq n$), $\tilde{z}_l = z_{li}$ for $l \in J$ and $i = 1, \ldots, n$, while $(\tilde{z}_l)_{l=1}^{n}$ are residuals from the regression of the original instruments $(z_{li})_{i=1}^n$ for $l \in J^c$ on $(z_{li})_{i=1}^n$ for $l \in J$. Assume finally that $(u_i, v_i)$ are i.i.d. and have a mean zero bivariate normal distribution. Denote by $\sigma^2_u$ the variance of $v_i$. It is easy to check that $\lambda$ in $\mathbb{R}^{K-1}$ such that $x_{kend}^T \zeta_l = \sum_{l \in J} \tilde{z}_l \zeta_l$ satisfies the constraint in (5.5) which yields using (5.6)

$$(5.7) \quad K_{k,end,J}^{(c,I)_k^*} \geq \max \left( \max_{l \in J} \left( D_{Z}^{(I)} \right)_{ll} \left| \frac{1}{n} \sum_{i=1}^n z_{li} v_i \right|, \max_{l \in J^c} \left( D_{Z}^{(I)} \right)_{ll} \left| \frac{1}{n} \sum_{i=1}^n z_{li} v_i + \frac{1}{n} e_l^T \tilde{Z}_j^T \tilde{Z}_j \zeta_l \cdot J^c \right| \right)$$

where $(e_l)_{l=1}^L$ is the canonical basis of $\mathbb{R}^L$. Note that $\left| \frac{1}{n} \sum_{i=1}^n z_{li} v_i \right|$ is small for large $n$ because the instruments are fixed and $v_i$ is mean zero. This shares similarity with the concentration parameter
(see, e.g, Andrews and Stock (2007)), defined as
\[
\mu^2 = \frac{\zeta_T^T \tilde{Z}_j \zeta_j}{\sigma^2_v},
\]
which is a classical measure of the strength of the set of instruments for low dimensional structural equations, under homoscedasticity. The quantity in (5.7) isolates the strength of each instrument. Thus we see that, unlike the concentration parameter which considers the set of all instruments, the coordinate-wise sensitivities depend on the strength of the best instrument. Due to our rescaling of the instruments, which allows to relax a lot the usual distributional assumptions and to handle heteroscedasticity, the coordinate-wise sensitivities and concentration parameter are not comparable in terms of the dependence on \( \sigma^2_v \).

A first reason to come up with lower bounds on the sensitivities is that they depend on \( J(\beta) \) and thus on the unknown \( \beta \). A second reason is to obtain an easy to calculate lower bound. For every sensitivity, we propose an easy to calculate lower bound. The following proposition will be useful.

**Proposition 5.1.**  
(i) Let \( J, \hat{J} \) be two subsets of \( \{1, \ldots, K\} \) such that \( J \subseteq \hat{J} \). Then, for all \( J_0 \subset \{1, \ldots, K\} \), all \( p \in [1, \infty) \), all \( c \) in \((0, 1)\) and all set of indices \( I \) containing the index of unity, \( \kappa_{p,J_0,J}^{(c,I)} \geq \kappa_{p,J_0,\hat{J}}^{(c,I)} \).

(ii) For all \( J_0 \subset \{1, \ldots, K\} \) all \( p \in [1, \infty) \), all \( c \) in \((0, 1)\) and all set of indices \( I \) containing the index of unity, \( \kappa_{p,J_0,J}^{(c,I)} \geq \kappa_{p,J}^{(c,I)} \).

(iii) For all \( p \in [1, \infty) \), all \( c \) in \((0, 1)\) and all set of indices \( I \) containing the index of unity,
\[
\left( \frac{2|J|}{1-c} \right)^{-1/p} \kappa_{\infty,J}^{(c,I)} \leq \kappa_{p,J}^{(c,I)} \leq \left( \frac{2}{1-c} \right)^{1-1/p} |J|^{1-1/p} \kappa_{1,J}^{(c,I)} ,
\]
and for all \( J_0 \subset \{1, \ldots, K\} \), all \( p \in [1, \infty) \), all \( c \) in \((0, 1)\) and all set of indices \( I \) containing the index of unity,
\[
\left| J_0 \right|^{-1/p} \kappa_{\infty,J_0,J}^{(c,I)} \leq \kappa_{p,J_0,J}^{(c,I)} \leq \left| J_0 \right|^{1-1/p} \kappa_{1,J_0,J}^{(c,I)} .
\]

(iv) For all \( J_0 \subset \{1, \ldots, K\} \), \( \kappa_{\infty,J_0,J}^{(c,I)} = \min_{k \in J_0} \kappa_{k,J}^{(c,I)} \).

Results for the \( l_p \)-sensitivities are easily deduced from (i) and (iv) by taking \( J_0 = \{1, \ldots, K\} \). (ii) and (5.9) allow to minorize the bloc sensitivities when \( J_0 \) can be large and a direct calculation would be too difficult from a numerical point of view.

We can control \( \kappa_{p,J_0,J(\beta)}^{(c,I)} \) without knowing \( J(\beta) \) by means of *sparsity certificate*. Assume that we have an upper bound \( s \) on the sparsity of \( \beta \), i.e., we know that \( |J(\beta)| \leq s \) for some integer \( s \) and that we use it as well to define \( B_s \). This does not require that non-zero coefficients are large enough
but just that a maximum of \( s \) are non-zero. In view of (5.1), if \( |J| \leq s \), then for any \( \Delta \) in the cone \( C_j^{(s)} \) we have \( |\Delta|_1 \leq \frac{2s}{c} |\Delta|_\infty \). Thus, for all \( J \) such that \( |J| \leq s \), we can bound the coordinate-wise sensitivities as follows:

\[
\kappa_{k,J}^{(c,I)*} \geq \inf_{\Delta_k=1,|\Delta|_1 \leq a|\Delta|_\infty} \left| \Psi_n^{(I)} \Delta \right|_\infty
\geq \min_{j=1,\ldots,K} \left\{ \min_{\Delta_k=1,|\Delta|_1 \leq a|\Delta|_j} \left| \Psi_n^{(I)} \Delta \right|_\infty \right\} \triangleq \kappa_k^{(c,I)*}(s),
\]

where \( a = \frac{2s}{c} \). For given \( s \), this bound is data-driven since the minimum in curly brackets can be computed by solving \( 2K \) linear programs (see Section 10.1). Then, using (iv), we can deduce a lower bound on \( \kappa_{\infty,J_0,J}^{(c,I)} \)

\[
\kappa_{\infty,J_0,J}^{(c,I)} \geq \min_{k \in J_0} \kappa_k^{(c,I)*}(s).
\]

Using (5.8) and (5.11) we get computable lower bounds for all \( \kappa_{p,J}^{(c,I)} \), \( p \in [1,\infty] \), which depend only on \( s \) and on the data. In particular, for \( |J| \leq s \),

\[
\kappa_{1,J}^{(c,I)} \geq \frac{1-c}{2s} \min_{k=1,\ldots,K} \kappa_k^{(c,I)*}(s) \triangleq \kappa_1^{(c,I)}(s).
\]

This can thus be obtained by solving \( 2K^2 \) linear programs. Analogously to (5.10), the sparsity certificate approach yields a bound for block sensitivities

\[
\kappa_{1,J_0,J}^{(c,I)} \geq \inf_{|\Delta_{J_0}|_1=1,|\Delta|_1 \leq a|\Delta|_\infty} \left| \Psi_n^{(I)} \Delta \right|_\infty
\geq \min_{j=1,\ldots,K} \left\{ \min_{|\Delta_{J_0}|_1=1,|\Delta|_1 \leq a|\Delta|_j} \left| \Psi_n^{(I)} \Delta \right|_\infty \right\} \triangleq \kappa_{1,J_0,J}^{(c,I)}(s).
\]

In Section 10.1 we show that the expression in curly brackets in (5.13) can be computed by solving \( 2|J_0| \) linear programs. Thus, the values \( \kappa_{1,J_0}(s) \) can be readily obtained for sets \( J_0 \) of small cardinality. Otherwise we do not advertise this lower bound but simply to use the following lower bound for \( p = 1 \).

For \( |J| \leq s \), (ii), (5.8) and (5.11) yield

\[
\kappa_{p,J_0,J}^{(c,I)} \geq \max \left( |J_0|^{-1/p} \min_{k \in J_0} \kappa_k^{(c,I)*}(s), \left( \frac{1-c}{2s} \min_{k=1,\ldots,K} \kappa_k^{(c,I)*}(s) \right) \right).
\]

This bound can be calculated by solving \( 2K^2 \) linear programs even for large sets \( J_0 \).

For the sake of completeness, we present an alternative to the sparsity certificate approach. It corresponds to computing \( \kappa_{1,J}^{(c,I)} \) and \( \kappa_{k,J}^{(c,I)*} \) directly. This is numerically feasible only for \( J \) of small cardinality. Indeed, we show in Section 10.1 that obtaining the coordinate-wise sensitivities corresponds to solving \( 2|J| \) linear programs. Using (5.8) and (5.11), we obtain computable lower
bounds for all $\kappa_{p,J}^{(c,I)}$, $p \in [1, \infty]$. The lower bounds are valid for any given index set $J$. However, we will need to compute the characteristics for the inaccessible set $J = J(\beta)$, where $\beta$ is the unknown parameter from $\mathcal{B}_s$ on which we are making inference. To circumvent this problem, we can plug in an estimator $\tilde{J}$ of $J(\beta)$. The confidence bounds remain valid whenever $J(\beta) \subseteq \tilde{J}$, since then $\kappa_{p,J(\beta)}^{(c,I)} \geq \kappa_{p,\tilde{J}}^{(c,I)}$, by Proposition 5.1 (i). Theoretical guarantees for the inclusion $J(\beta) \subseteq J(\tilde{J})$ to hold with probability close to 1 require a separation from zero: $|\beta_k|$ is not too small on the support of $\beta$ (see Theorem 7.4 (iv)). On the other hand, one typically observes in simulations that the relevant set $J(\beta)$ is either estimated exactly or overestimated by its empirical counterpart $\tilde{J} = J(\tilde{\beta}(c,I))$, so that the required inclusion is satisfied for such a simple choice of $\tilde{J}$.

Belloni and Chernozhukov (2010) study the property of post model selection least squares estimation in the case of prediction loss, they obtain theoretical bounds on the error made when we do not have $J(\beta) \subset J(\tilde{\beta}(c,I))$. We do not touch upon the very difficult question of obtaining confidence sets with coverage at most $1 - \alpha$ that allow to deal with the possibility that $J(\beta) \subset J(\tilde{\beta}(c,I))$ is not satisfied. We state that this type of confidence sets has coverage $1 - \gamma$ where $\gamma > \alpha$. They are approximate $1 - \alpha$ confidence sets. In our simulation study we obtain that the confidence regions obtained via the sparsity certificate using for $s = |J(\tilde{\beta}(c,I))|$ are almost identical to the second ones which also require the separation from zero. Because the first method does not require such an assumption and is feasible (non combinatoric) even when $|J(\beta)|$ is large, we strongly advertise the first approach based on the sparsity certificate and to use a conservative sparsity certificate $s$. We also suggest to draw nested confidence sets varying the degree of sparsity that we assume.

In the next proposition, we present a simple lower bound on $\kappa_{p,J}^{(c,I)}$ for general $L \times K$ rectangular matrices $\Psi_n^{(I)}$. Its proof, as well as other lower bounds on $\kappa_{p,J}^{(c,I)}$, can be found in Section 11. It is important to note that adding rows to matrix $\Psi_n^{(I)}$ (i.e., adding instruments) increases the sup-norm $|\Psi_n^{(I)} \Delta|_\infty$, and thus potentially increases the sensitivities $\kappa_{p,J(\beta)}^{(c,I)}$ and their computable lower bounds. This has a positive effect since the inverse of the computable lower bounds of the sensitivities drive the width of the confidence set for $\beta$ in $\mathcal{B}_s$, see Theorem 5.5. Thus, adding instruments potentially improves the confidence set, which is quite intuitive. On the other hand, the price for adding instruments in terms of the rate of convergence is only logarithmic in the number of instruments, as we will see in the next section.
Proposition 5.2. Fix $J \subseteq \{1, \ldots, K\}$. Assume that there exist $\eta_1 > 0$ and $0 < \eta_2 < 1$ such that
\[
\forall k \in J, \exists l(k) : \begin{cases}
|\langle \Psi_n(I) \rangle_{l(k)k} | \geq \frac{\eta_1}{1-c}, \\
\frac{\max_{k' \neq k} |\langle \Psi_n(I) \rangle_{l(k)k'} |}{|\langle \Psi_n(I) \rangle_{l(k)k} |} \leq \frac{(1-\eta_2)(1-c)}{2|J|}.
\end{cases}
\]
Then
\[
k_p(c,I,J)^{(c,I)} \geq (2|J|)^{-1/p}(1-c)^{-1+1/p} \eta_1 \eta_2.
\]

Assumption (5.15) is similar in spirit to the coherence condition introduced by Donoho, Elad and Temlyakov (2006) for symmetric matrices, but it is more general because it deals with rectangular matrices. Since the regressors and instruments are random, the values $\eta_1$ and $\eta_2$ can, in general, be random. Remarkably, for estimation of the coefficients of the endogenous variables, it suffices to have a “good” row of the matrix $\Psi_n(I)$. This means that it is enough to have, among all instruments, one good instrument. The way the instruments are ordered is not important. Good instruments correspond to the rows $l(k)$, for which the value $|\langle \Psi_n(I) \rangle_{l(k)k} |$ measuring the relevance of the instrument for the $k$th variable is high. On the other hand, the value $\max_{k' \neq k} |\langle \Psi_n(I) \rangle_{l(k)k'} |$ accounting for the relation between the instrument and the other variables should be small. An instrument which is well “correlated” with two variables of the model is not satisfactory for this assumption.

6. Confidence Sets and Sparse Oracle Inequality

6.1. Distributional Assumptions and Control of The Stochastic Part. For every $\beta$ in $B_s$, $u_i = y_i - x_i^T \beta$ is such that, for every $l = 1, \ldots, L$, $z_{li} u_i$ are independent and, for every $i = 1, \ldots, n$, $E[z_{li} u_i] = 0$. The size of the constant $r$ in the definition of $\tilde{F}(l)$ is directly related to the coverage probability of the confidence sets. It should be adjusted so that with probability $1 - \alpha$ (the coverage probability),
\[
\max_{l=1, \ldots, L} \left( \frac{1}{n} \sum_{i=1}^n z_{li} u_i \right) \leq r.
\]
This is a sup-norm of so called self-normalized sums. We propose different possible choices of $r$ based on different distributional assumptions (and sometimes an upper bound on the number of instruments).

Scenario 1. Suppose that the errors $u_i$ are identically distributed, independent from the $z_i$’s, and of distribution known up to the variance, then the quantiles of
\[
\max_{l \in I^c} \left( \frac{\max_{l \in I^c} (D_{Z_l}^{(1)})_{l,l} \left( \frac{1}{n} \sum_{i=1}^n z_{li} u_i \right)}{\sqrt{\frac{1}{n} \sum_{i=1}^n u_i^2}}, \max_{l \in I} \left( \frac{1}{n} \sum_{i=1}^n z_{li} u_i \right) \right),
\]
conditional on $z_i$ for $i = 1, \ldots, n$ can be obtained numerically, for example using a Monte-Carlo method, and $r$ is adjusted accordingly. This is an approach proposed by Belloni and Chernozhukov (2010) for the linear regression model without endogeneity. This approach is worth mentioning because it has the advantage to yield a smaller $r$, and thus smaller confidence sets as we will see later. Indeed, it does not rely on a crude union bound but account for the correlation between the regressors.

**Scenario 2.** Under the assumption

**Assumption 6.1.** For every $i = 1, \ldots, n$ and $l = 1, \ldots, L$, $z_{li}u_i$ are symmetric and neither of $z_{li}u_i$ is almost surely equal to 0.

we choose

\begin{equation}
(6.1) \quad r = \sqrt{\frac{2\log(L/2\alpha)}{n}}.
\end{equation}

Symmetry is a very likely assumption if (1.1) is a first difference between two time periods in a linear panel data model.

**Scenario 3.** Under the assumption

**Assumption 6.2.** For every $l = 1, \ldots, L$, $z_{li}u_i$ are i.i.d. and symmetric, neither of $z_{li}u_i$ is almost surely equal to 0 and $L$ is such that

\begin{equation}
L < \frac{9\alpha}{4e^3\Phi(-\sqrt{n})},
\end{equation}

we choose

\begin{equation}
(6.2) \quad r = -\frac{1}{\sqrt{n}}\Phi^{-1}\left(\frac{9\alpha}{4Le^3}\right).
\end{equation}

This is a slightly tighter constant that the one given for Scenario 2 but Assumption (6.2) does not allow for heteroscedastic errors. The upper bound on the number of instruments is of the order of an exponential in $n$.

**Scenario 4.** We denote by $d_{n,\delta} = \min_{l=1,\ldots,L} d_{n,\delta,l}$ where

\begin{equation}
d_{n,\delta,l} \triangleq \frac{\left(\sum_{i=1}^{n} \mathbb{E}[z_{li}^2u_i^2]\right)^{1/2}}{\left(\sum_{i=1}^{n} \mathbb{E}[|z_{li}u_i|^{2+\delta}]\right)^{1/(2+\delta)}}
\end{equation}

when the numerator and denominator are well defined. Under the assumption

**Assumption 6.3.** There exists $\delta$ positive such that, for all $i = 1, \ldots, n$, $l = 1, \ldots, L$, $\mathbb{E}[|z_{li}u_i|^{2+\delta}] < \infty$, neither of $z_{li}u_i$ is almost surely equal to 0 and $L$ is such that

\begin{equation}
L \leq \frac{\alpha}{2\Phi(-d_{n,\delta}) \left(1 + A_0 \left(1 + d_{n,\delta}^{-1}\right)^{2+\delta}\right)}.
\end{equation}
where $A_0 > 0$ is the absolute constant of Theorem 11.6 in Section 11.2.

we choose

\begin{equation}
(6.3) \quad r = -\frac{1}{\sqrt{n}} \Phi^{-1} \left( \frac{\alpha}{2L} \right).
\end{equation}

Note that if, for any fixed $l$ in \{1, \ldots, L\}, the variables $z_i u_i$ are i.i.d., then for $l = 1, \ldots, L$, \( d_{n,\delta,l} = n^{-\frac{1}{4+2\gamma}} \sqrt{\frac{E[(z_i^2 u_i^2)]^{1/2}}{(L+3)^{1/(2+\gamma)}}} \) which tends to infinity with $n$. With this choice of $r$ the confidence sets will only be asymptotically valid\(^{10}\) in the asymptotic where $L$ and $n$ increase in such a way that \( \Phi^{-1} \left( \frac{\alpha}{2L} \right) d_{n,\delta}^{-1} \to 0 \). This scenario allows for heteroscedasticity and non-symmetric errors. This is very desirable in an IV setup.

**Scenario 5.** We denote by $\gamma_4 \triangleq \max_{l=1,\ldots,L} \gamma_{4l}$ where $\gamma_{4l} = \mathbb{E}[(z_i u_i)^4]/(\mathbb{E}[(z_i u_i)^2])^2$. Under the assumption

\begin{equation}
(6.4) \quad L < \frac{\alpha}{2e+1} \exp \left( \frac{n}{c_4} \right).
\end{equation}

we choose

\begin{equation}
(6.5) \quad r = \sqrt{\frac{2\log(L(2e+1)/\alpha)}{n - c_4 \log(L(2e+1)/\alpha)}}.
\end{equation}

It is reasonable to assume that \( n - c_4 \log(L(2e+1)/\alpha) \geq n/2 \) as soon as $n$ is relatively large relative compared to $\log(L/\alpha)$. In that case, we can take

\begin{equation}
(6.6) \quad r = 2 \sqrt{\frac{\log(L(2e+1)/\alpha)}{n}}
\end{equation}

and have confidence sets with valid finite sample properties. It is also possible to proceed in two stages. We first start by choosing $r$ as in (6.5) with a very rough upper bound on $\gamma_4$, for example using (6.6). As we will see, it yields a point estimate and a confidence region for $\beta$, which in turns yields a point estimate and a confidence set for $\gamma_4$. In a second stage, it is possible to plug-in either the point estimate for $\gamma_4$ or the upper bond from the confidence set for $\gamma_4$. This two-stage approach yields approximately valid confidence sets. A similar approach has been used in the simulations in

\(^{10}\)We only consider asymptotically valid confidence sets because the moderate deviations result for the self-normalized sums that we use depends on the constant $A_0$ which is universal but not explicit.
Bertail, Gauthérat and Harari-Kermadec (2005) to obtain finite sample confidence sets for inference based on empirical \( \varphi^* \)-discrepancies and quasi-empirical likelihood methods.\(^{11}\)

Scenarios 2-5 rely on moderate deviations for self-normalized sums that are recalled in Section 11.2 and are respectively from Efron (1969), Pinelis (1994), Jing, Shao and Wang (2003) and Bertail, Gauthérat and Harari-Kermadec (2009).\(^{12}\) Each moderate deviations result relies on a different set of assumptions and our confidence sets with finite sample validity will rely on a restricted class of distributions for the data generating process. This is related to the Bahadur and Savage (1956) impossibility result, see also Romano and Wolf (2000).\(^{13}\)

6.2. Confidence Sets.

**Theorem 6.5.** For every \( \beta \) in \( \mathcal{B}_s \), under one of the scenarios 2-5 of Section 6.1 together with its respective choice of \( r \), with probability at least \( 1 - \alpha \) (approximately at least \( 1 - \alpha \) for scenario 4 and the two-stage procedure with scenario 5), for any \( c \) in \( (0, 1) \) and any set of indices \( I \) containing the index of the instrument which is unity, for any solution \( (\hat{\beta}^{(c,I)}, \hat{\sigma}^{(c,I)}) \) of the minimization problem (3.5) we have

\[
\left| \left( D_X^{-1}(\hat{\beta}^{(c,I)} - \beta) \right)_{J_0} \right| \leq \frac{2\hat{\sigma}^{(c,I)}_r}{\kappa_{p,J_0,J(\beta)}^{(c,I)}} \left( 1 - \frac{r}{\kappa_{1,J(\beta)}^{(c,I)}} \right)^{-1} \quad \forall \, p \in [1, \infty], \, \forall J_0 \subset \{1, \ldots, K\},
\]

\[
\left| \hat{\beta}^{(c,I)}_k - \beta_k \right| \leq \frac{2\hat{\sigma}^{(c,I)}_r}{\mathbb{E}_n[X_k^2]^{1/2}} \kappa_{k,J(\beta)}^{(c,I)} \left( 1 - \frac{r}{\kappa_{1,J(\beta)}^{(c,I)}} \right)^{-1} \quad \forall k = 1, \ldots, K,
\]

and

\[
\hat{\sigma}^{(c,I)} \leq \max_{l \in I} \left( D_Z^{(I)} \right)_l \sqrt{Q_l(\beta)} \left( 1 + \frac{r}{c\kappa_{1,J(\beta),J(\beta)}^{(c,I)}} \right) \left( 1 - \frac{r}{c\kappa_{1,J(\beta),J(\beta)}^{(c,I)}} \right)^{-1}.
\]

\(^{11}\)Self-normalized sums also appear naturally in empirical likelihood contexts.

\(^{12}\)Bertail, Gauthérat and Harari-Kermadec (2009) provides an upper bound of tail probabilities of self-normalized sums with an explicit constant unlike Jing, Shao and Wang (2003). The result of Jing, Shao and Wang (2003) is mostly useful to study large deviations, that is why under Scenario 4 we only consider asymptotically valid confidence sets.

\(^{13}\)As explained in Romano and Wolf (2000), the Bahadur and Savage result implies that we need to make some restriction on the set of distributions \( \mathcal{P} \) on the line in order to construct conservative confidence intervals for the mean of \( \mathcal{P} \) in \( \mathcal{P} \) that are bounded. A conservative interval of coverage level \( 1 - \alpha \) is such that the parameter is contained in the interval with probability at least \( 1 - \alpha \) for all \( \mathcal{P} \) in \( \mathcal{P} \) and sample size \( n \).
Consider now the case of Example 6 where the instruments can have a direct effect on the outcome. In that case \(L < K\) and identification fails without further assumptions. This is simply because \(\dim \ker \left( \mathbb{E}[z_ix_J^T] \right) \geq K - L > 0\). If we knew the number \(s\) of non-zero coefficients, the underlying vector \(\beta\) is the solution of one system
\[
(6.10) \quad \begin{cases} 
\beta_{J^c} = 0 \\
\mathbb{E}[z_i(y_i - x_J^T\beta_J)] = 0
\end{cases}
\]
for each subset \(J\) of \(\{1, \ldots, K\}\) of size \(s\) (including the index of the constant regressor). For every set \(J\) such that \(\mathbb{E}[z_ix_J^T]\) has full column rank, \((6.10)\) has at most one solution. When \(s < L\), \(\mathbb{E}[z_ix_J^T]\beta_J = \mathbb{E}[z_iy_i]\) does not necessarily have a solution because the system is overdetermined. Note as well that, because \(s\) is the number of non-zero coefficients, in addition to \((6.10)\), we know that for every \(k\) in \(\{1, \ldots, K\}\), \(\beta_k \neq 0\). This again restricts the set of solutions. Note that there are also cases where the set of solutions of \((6.10)\) is an affine space. For example, if \(s > L\), \(\mathbb{E}[z_ix_J^T]\) does not have full column rank. Geometrically \(\mathcal{B}_s\) is the union of the intersections of \(\mathcal{Ident}\) with the spaces \(\{\forall k \in J, \beta_k = 0\}\) if for every \(k\) in \(\{1, \ldots, K\}\), \(\beta_k \neq 0\). This is a union of affine spaces. We have observed that, adding the prior information that there are at most \(s < L\) non-zero coefficients, is very likely to yield a set \(\mathcal{B}_s\) which is a union of points. Recall that, in the context of Example 6, \(s < L\) means that there are in reality instruments that do not have a direct effect on the outcome. There are thus unknown exclusion restrictions which have an identification power. Again, because of the cone constraints, the sensitivities \(\kappa_{p,j_0,J(\beta^*)}\) and \(\kappa_{k,J(\beta^*)}\) on the right hand-side of \((6.7)\) and \((6.8)\) can be different from 0 even in the high-dimensional framework where \(n\) is small relative \(K\).

Consider now the term \(\left(1 - r/\kappa_{1,J(\beta)}\right)^{-1}\) in the upper bounds of Theorem 6.5. Observe that the confidence sets of level at least \(1 - \alpha\) for \(\beta\) in \(\mathcal{B}_s\) can have infinite volume on the random event
Indeed, the upper bounds (6.7) and (6.8) are infinite. This occurs either when \( r \) is large or when \( \kappa^{(c,I)}_{1,J(\beta)} \) is too small. Recall that \( r \) is of the order of \( \sqrt{\log(L)/n} \) in all scenarios. Because increasing \( L \) increases the sensitivities, once \( \kappa^{(c,I)}_{1,J(\beta)} \) is sufficiently bounded away from zero, the first condition corresponds to a situation with very many instruments relative to the sample size. This occurs when \( L \) is as large as an exponential in \( n \). To interpret the second condition, note that Proposition 5.2 (ii) yields that for any \( k \) in \( J(\beta) \),

\[
\kappa^{(c,I)}_{1,J(\beta)} \leq \kappa^{(c,I)*}_{k,J(\beta)}.
\]

This can occur on the event where a regressor \((x_{ki})_{i=1}^{n}\) has a restricted maximal partial empirical correlation with the instruments less than \( r \). In the setting of Example 7, this becomes increasingly likely when the parameter \( \zeta_{J(\beta)} \) in the data generating approaches zero (see (5.7)). Confidence sets of infinite volume with positive probability is a desired feature of a procedure that is robust to weak instruments (see Dufour (1997) building on Gleser and Hwang (1987)). We will show in Section 8.1 that for low dimensional models (\( K \) is small and there is no sparsity) a slight modification of the \( STIV \) estimator is a new procedure that is robust to weak instruments (see Andrews and Stock (2007) for a review of existing methods). Note also that the term \((1 - r/\kappa^{(c,I)}_{1,J(\beta)})^{-1}\) appears because we have a procedure that relaxes distributional assumptions and, in the homoscedastic case, does not require to know the variance of the errors or an upper bound. When the regressors are deterministic and the errors are Gaussian, an upper bound on the variance is sufficient to adjust the penalization and obtain a consistent Lasso or Dantzig estimator. In the linear regression with exogenous regressors, \( E[y_{i}^{2}] \) provides such an upper bound on the variance. This no longer works in the presence of endogenous regressors.

When \( \tau_{1} \triangleq 1 - \frac{r}{\kappa^{(c,I)}_{1,J(\beta)}} \) is close to 1 and the sensitivities \( \kappa^{(c,I)}_{p,J(\beta)} \) and \( \kappa^{(c,I)*}_{k,J(\beta)} \) are bounded away from zero, the upper bounds in (6.8) is of the order \( O(r) = O(\sqrt{\log(L)/n}) \). Thus, we have an extra \( \sqrt{\log(L)} \) factor as compared to the usual root-\( n \) rate. It is a modest price for using a large number \( L \) of instruments. Under the premise of Proposition 5.2 for \( \tau_{1} \approx 1 \) it is sufficient to have \( |J(\beta)| \leq Cr^{-1} = O(\sqrt{n/\log(L)}) \) where \( C > 0 \) is a proper constant. This is quite a reasonable condition on the sparsity \( |J(\beta)| \).

The only unknown ingredient of the inequalities (6.7) and (6.8) is the set \( J(\beta) \) that determines the sensitivities. To turn these inequalities into valid confidence bounds, it suffices to provide data-driven lower estimates on the sensitivities. As discussed in Section 5, there are two ways to do it. The
first one is based on the sparsity certificate, i.e., assuming some known upper bound \( s \) on \( |J(\beta)| \); then we get bounds depending only on \( s \) and on the data.

**Corollary 6.6.** For every \( \beta \) in \( B_s \), under scenarios 2-5 of Section 6.1 together with its respective assumption and choice of \( r \), with probability at least \( 1 - \alpha \) (approximately at least \( 1 - \alpha \) for scenario 4 and the two-stage procedure with scenario 5), for any \( c \) in \( (0, 1) \) and any set of indices \( I \) containing the index of the instrument which is unity, for any solution \( (\hat{\beta}(c,I), \hat{\sigma}(c,I)) \) of the minimization problem (3.5) we have

\[
\left\| \mathbf{D}^{-1}(\hat{\beta}(c,I) - \beta) \right\|_{J_0} \leq \frac{2\hat{\sigma}(c,I)_r}{\kappa(p,J_0)(s)} \left( 1 - \frac{r}{\kappa(c,I)(s)} \right)^{-1} \quad \forall \ p \in [1, \infty], \ \forall J_0 \subset \{1, \ldots, K\},
\]

where \( \kappa(p,J_0)(s) \) is any lower bound on \( \kappa(p,J_0,J(\beta)) \) based on the sparsity certificates that is convenient to calculate (see, e.g., (5.13) and (5.14))

\[
|\hat{\beta}(c,I)_k - \beta_k| \leq \frac{2\hat{\sigma}(c,I)_r}{\mathbb{E}[X_k^2]^{1/2} \kappa(k)(s)} \left( 1 - \frac{r}{\kappa(c,I)(s)} \right)^{-1} \quad \forall k = 1, \ldots, K.
\]

Under scenario 1 of Section 6.1 with the corresponding choice of \( r \), for any set of indices \( I \) containing the index of the instrument which is unity, for every \( \beta \) in \( B_s \), with probability at least \( 1 - \alpha \), for any \( c \) in \( (0, 1) \), any solution \( (\hat{\beta}(c,I), \hat{\sigma}(c,I)) \) of the minimization problem (3.5) satisfies (6.11) and (6.12).

In simulations the STIV estimator has always more non-zeros than the truth so it is reasonable to take \( s = |J(\hat{\beta}(c,I))| \) or some larger value. We also advertise the possibility of drawing nested confidence sets for increasing values of \( s \).

The second way is to plug in, instead of \( J(\beta) \), some data-driven upper estimate \( \hat{J} \), i.e., a set satisfying \( J(\beta) \subseteq \hat{J} \) on the intersection of the event of probability at least \( 1 - \alpha \) (or approximately \( 1 - \alpha \) for scenario 4 and the two-stage procedure with scenario 5) of Theorem 6.5 and an event of probability close to 1. Theorem 7.4 provides examples of such estimators \( \hat{J} \). In particular, assertion (iv) of Theorem 7.4 guarantees that, under some assumptions, the estimator \( \hat{J} = J(\hat{\beta}(c,I)) \) has the required property. The statement of the corresponding result is postponed to Section 7.

We do not touch upon optimality of the confidence regions in this article. Because of the popularity of two-stage least squares, we present the properties of a high dimensional two-stage method akin to two-stage least squares in Section 8.3. The STIV estimator depends on the tuning parameter \( c \). A smaller \( c \) implies a smaller cone \( C_{J(\beta)} \) and thus a larger sensitivity. On the other hand, because we penalize less \( \sigma \) in (3.5), \( \hat{\sigma}(c,I) \) is larger. Because the dependence is nonlinear and the sensitivities
are random and depend on the distribution of the data generating process, there does not exist a universally good value for $c$. It is important to note that our results hold on the $1 - \alpha$ probability event, for any value of $c$ and any set of indices $I$ containing the index of the instrument which is unity (for scenarios 2-5 only for the set $I$). It is therefore possible to consider values of $c$ or $I$ which are random and depend on the data. If one is interested in a specific coefficient, it is possible to pick the values of $c$ and $I$ that yields the smaller confidence interval for that specific regressor. Because the procedure is fast to implement, varying $c$ on a grid on $(0, 1)$ and picking the value that yields the smaller confidence interval is an easy thing to do. Varying $I$ is too complicated but, if one hesitates on a few specifications of $I$, it is possible to build confidence sets for these various possibilities and choose the set $I$ that yields the smaller confidence sets.

6.3. Sparse Oracle Inequality. We now consider the approximately sparse setting. The sparsity assumption is quite natural in empirical economics since usually only a moderate number of covariates is included in the model. However, one might also be interested in the case when $\beta$ is only approximately sparse. This means that most of the coefficients $\beta$ are not exactly zero but too small to matter, whereas the remaining ones are relatively large. This setting received some attention in the statistical literature. For example, the performance of Dantzig selector and $MU$-selector under such assumptions is studied by Candès and Tao (2007) and Rosenbaum and Tsybakov (2010) respectively. We will derive a similar result for the $STIV$ estimator.

Consider the enlarged cone
\[
\tilde{C}_J^{(c)} = \{ \Delta \in \mathbb{R}^K : |\Delta_J|_1 \leq \frac{2 + c}{1 - c} |\Delta_J|_1 \}
\]
and define, for $p \in [1, \infty]$ and $J_0 \subset \{1, \ldots, K\}$
\[
\tilde{\kappa}^{(c,I)}_{K,p,J_0,J} \triangleq \inf_{\Delta \in \mathbb{R}^K : |\Delta_{J_0}|_p = 1, \Delta \in \tilde{C}_J^{(c)}} \left| \frac{\Psi(I)}{n} \Delta \right|_\infty
\]
and $\tilde{\kappa}_{1,J}^{(c,I)}$ corresponds to $\tilde{\kappa}^{(c,I)}_{K,p,J_0,J}$ with $p = 1$ and $J_0 = \{1, \ldots, K\}$.

The following theorem is an analog of the above results for the approximately sparse case.

**Theorem 6.7.** For every $\beta$ in $\mathcal{B}_K$, under scenarios 2-5 of Section 6.1 together with its respective assumption and choice of $r$, with probability at least $1 - \alpha$ (approximately at least $1 - \alpha$ for scenario 4 and the two-stage procedure with scenario 5), for any $c$ in $(0, 1)$ and any set of indices $I$ containing

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The same is true for the Square-root Lasso.
the index of the instrument which is unity, for any solution \((\tilde{\beta}(c,I), \tilde{\sigma}(c,I))\) of the minimization problem \((3.5)\) we have, for every \(J_0 \subset \{1, \ldots, K\}\),
\[
(6.13) \quad \left| \left( \mathbf{D}_X^{-1} (\tilde{\beta}(c,I) - \beta) \right) \right|_p \leq \min_{J \subset \{1, \ldots, K\}} \left\{ \max_{\sigma(c,I)} \left( \frac{2\tilde{\sigma}(c,I)}{\kappa_{p,J_0,J}} \left( 1 - \frac{r}{\hat{\kappa}(c,I)} \right)^{-1} + \frac{6 \left| (\mathbf{D}_X^{-1} \beta)_{J_0} \right|_1}{1 - c} \right) \right\}.
\]

Under scenario 1 of Section 6.1 with the corresponding choice of \(r\), for any set of indices \(I\) containing the index of the instrument which is unity, for every \(\beta\) in \(\mathcal{B}_s\), with probability at least \(1 - \alpha\), for any \(c\) in \((0,1)\), any solution \((\tilde{\beta}(c,I), \tilde{\sigma}(c,I))\) of the minimization problem \((3.5)\) satisfies \((6.13)\) for every \(J_0 \subset \{1, \ldots, K\}\).

We can interpret Theorem 6.7 as the fact that the STIV estimator automatically realizes a “bias/variance” trade-off related to a non-linear approximation. Inequality \((6.13)\) means that this estimator performs as well as if the optimal subset \(J\) were known.

### 7. Rates of Convergence and Selection of the Variables

Let us consider rates of convergence of our estimator. We need to replace the random right hand-side of \((6.7), (6.8)\) and \((6.9)\) by deterministic upper bounds. In this section we consider that \(c\) in \((0,1)\) and the set \(I\) are fixed.

**Assumption 7.1.** For every \(\beta \in \mathcal{B}_s\) and \(\gamma_1 \in (0,1)\), there exists a constant \(\sigma_\ast > 0\) such that
\[
\mathbb{P} \left( \max_{l \in I} (\mathbf{D}_Z^{(l)})_l^2 \mathbb{E}_n [Z_l^2 (Y - X^T \beta)^2] \leq \sigma_\ast^2 \right) \geq 1 - \gamma_1.
\]

The second assumption concerns the population counterparts of the sensitivities.

**Assumption 7.2.** For every \(\beta \in \mathcal{B}_s\) and \(\gamma_2 \in (0,1)\), there exists constants \(c_p^{(c,I)} > 0\), \(c_{1,J_0}^{(c,I)} > 0\) for \(J_0 = J(\beta)\) and \(J_0 = \{k\}\) for \(k = 1, \ldots, K\), such that, with probability at least \(1 - \gamma_2\),
\[
(7.1) \quad \kappa_{p,J(\beta)}^{(c,I)} \geq c_p^{(c,I)} |J(\beta)|^{-1/p},
\]
\[
(7.2) \quad \kappa_{1,J_0,J(\beta)}^{(c,I)} \geq c_{1,J_0}^{(c,I)}.
\]

If \(J_0 = \{k\}\) is a singleton we write for brevity \(c_{1,J_0}^{(c,I)} = c_k^{(c,I)}\).

The dependence on \(|J(\beta)|\) of the right hand-side of \((7.1)\) is motivated by Proposition 5.2. In \((7.2)\), we do not indicate the dependence of the bounds on \(|J(\beta)|\) explicitly because it can be different for different sets \(J_0\). For general \(J_0\), combining Proposition 5.1 (ii) and Proposition 5.2 suggests that
the value $c_{1,J_0}$ can be bounded from below by a quantity of the order $|J(\beta)|^{-1}$. Note, however, that this is a coarse bound valid for any set $J_0$.

The last assumption defines a population counterpart of $E_n[X_k^2]^{1/2}$.

**Assumption 7.3.** For every $\gamma_3 \in (0, 1)$ and $k \in \{1, \ldots, K\}$, there exist constants $v_k > 0$ such that

$$\mathbb{P}\left( E_n[X_k^2]^{1/2} \geq v_k, \forall k \in \{1, \ldots, K\} \right) \geq 1 - \gamma_3.$$

Assumptions 7.1, 7.2 and 7.3 are very weak. They are required to obtain rates of convergence, i.e., a deterministic bound on the estimation error on a high probability event (see, e.g., (7.4) below).

We set $\gamma = \alpha + \sum_{j=1}^{3} \gamma_j$, and

$$\tau^{(c,I)*} \triangleq \left( 1 + \frac{r}{c_{c_1,J(\beta)^*}} \right) \left( 1 - \frac{r}{c_{c_1,J(\beta)^*}} \right)^{-1} \left( 1 + \frac{r|J(\beta)|}{c_{c_1,J(\beta)^*}} \right)^{-1}.$$

**Theorem 7.4.** For every $\beta$ in $B$, under the assumptions of Theorem 6.5 and Assumption 7.1, the following holds.

(i) Let part 7.2 of Assumption 7.2 be satisfied. Then, with probability at least $1 - \alpha - \gamma_1 - \gamma_2$, for any solution $\hat{\sigma}^{(c,I)}$ of (3.5) we have

$$\hat{\sigma}^{(c,I)} \leq \sigma^{*} \left( 1 + \frac{r}{c_{c_1,J(\beta)^*}} \right) \left( 1 - \frac{r}{c_{c_1,J(\beta)^*}} \right)^{-1}.$$

(ii) Fix $p \in [1, \infty]$. Let Assumption 7.2 be satisfied. Then, with probability at least $1 - \alpha - \gamma_1 - \gamma_2$, for any solution $\hat{\beta}^{(c,I)}$ of (3.5) we have

$$\left| D_X^{-1} \left( \hat{\beta}^{(c,I)} - \beta \right) \right|_p \leq \frac{2\sigma^{*}r|J(\beta)|^{1/p\tau^{(c,I)*}}}{c_{c_1,J(\beta)^*}}.$$

(iii) Let Assumptions 7.2 and 7.3 be satisfied. Then with probability at least $1 - \gamma$, for any solution $\hat{\beta}^{(c,I)}$ of (3.5) we have

$$|\hat{\beta}_k^{(c,I)} - \beta_k| \leq \frac{2\sigma^{*}r\tau^{(c,I)*}}{c_{c_1,J(\beta)^*}}v_k, \quad k = 1, \ldots, K.$$

(iv) Let the assumptions of (iii) hold, and $|\beta_k| > \frac{2\sigma^{*}r\tau^{(c,I)*}}{c_{c_1,J(\beta)^*}}v_k$ for all $k \in J(\beta)$. Then, with probability at least $1 - \gamma$, for any solution $\hat{\beta}^{(c,I)}$ of (3.5) we have

$$J(\beta) \subseteq J(\hat{\beta}^{(c,I)}).$$
For reasonably large sample size \((n \gg \log(L))\), the value \(r\) is small, and \(\tau^{(c,I)*}\) is approaching 1 as \(r \to 0\). Thus, the bounds \((7.3)\) and \((7.4)\) are of the order of magnitude \(O(r|J(\beta)|^{1/p})\) and \(O(r)\) respectively. These are the same rates, in terms of the sparsity \(|J(\beta)|\), the dimension \(L\), and the sample size \(n\), that were proved for the Lasso and Dantzig selector in high-dimensional regression with Gaussian errors and without endogenous variables (here \(L = K\)) in Candès and Tao (2007), Bickel, Ritov and Tsybakov (2009), Lounici (2008) (see also Bühlmann and van de Geer (2011) for references to more recent work).

The lower bound for estimation of a high dimensional linear regression model, without endogeneity, under fixed and Gaussian design, with \(K\) potential regressor and at most \(s\) non-zero coefficients, is known to be \(\sqrt{|J(\beta)| \log(K/|J(\beta)|n)}\) (see Verzelen (2010), Ye and Zhang (2010) and Raskutti, Wainwright and Yu (2011), rates for prediction are also obtained in Rigollet and Tsybakov (2011)).

Note that Theorem 7.4 assumes that we work on some event such that the lower bounds \(c^{(c,I)}\), \(c^{(c,I)*}_{k,J(\beta)}\) and \(c^{(c,I)}_{1,J(\beta)}\) are positive. When for a particular value of \(\beta\) there are 0 on a large probability event then on that event we do not have convergence rates or model selection results. Also the extra condition in (iv) is not satisfied for every \(\beta\) in \(B_s\). Therefore when \(B_s\) is not a point then the conclusion of (iv) only holds for certain values of the parameter in the set \(B_s\).

From (6.8), Theorem 7.4 (iv) and Proposition 5.1 (i), we obtain the following confidence sets of level \(1 - \gamma\) for \(\beta_k\) in \(B_s\).

**Corollary 7.5.** For every \(\beta\) in \(B_s\), under one of the 5 scenarios of Section 6.1 together with its respective assumption and choice of \(r\), under the assumptions of Theorem 7.4 (iv), with probability at least \(1 - \gamma\) (approximately at least \(1 - \gamma\) for scenario 4 and the two-stage procedure with scenario 5) for any solution \((\hat{\beta}, \hat{\sigma})\) of the minimization problem (3.3) we have

\[
\left| \left( D_x^{-1}(\hat{\beta}^{(c,I)} - \beta) \right)_{J_0} \right|_p \leq \frac{2\hat{\sigma}^{(c,I)} r}{\kappa^{(c,I)}_{p,J_0,J(\hat{\beta}^{(c,I)})}} \left( 1 - \frac{r}{\kappa^{(c,I)}_{1,J(\hat{\beta}^{(c,I)})}} \right)^{-1}, \quad \forall \ p \in [1, \infty], \forall J_0 \subset \{1, \ldots, K\},
\]

where \(\kappa^{(c,I)}_{p,J_0,J(\hat{\beta}^{(c,I)})}\) is any lower bound on \(\kappa^{(c,I)}_{p,J_0,J(\hat{\beta}^{(c,I)})}\) that is convenient to calculate (see, e.g., \((5.13)\) and \((5.14)\)), and for all \(k = 1, \ldots, K\),

\[
|\hat{\beta}^{(c,I)}_k - \beta_k| \leq \frac{2\hat{\sigma}^{(c,I)} r}{E_n[X_k^2]^{1/2} \kappa^{(c,I)*}_{k,J(\hat{\beta}^{(c,I)})}} \left( 1 - \frac{r}{\kappa^{(c,I)}_{1,J(\hat{\beta}^{(c,I)})}} \right)^{-1}.
\]
Here we do not control exactly the coverage probability because its probability is now $1 - \gamma$. Having $\gamma_j$ very small for $j = 1, 2, 3$ and thus coverage probability of approximately at least $1 - \alpha$ requires taking a large $\sigma^*$ and lower bounds on the sensitivities. Such a choice is compatible with (iv) from Theorem 7.4 only when $\beta_k$ are large enough for $k \in J(\beta)$.

Theorem 7.4 (iv) provides an upper estimate on the set of non-zero components of $\beta$. We now consider the problem of the exact selection of variables. For this purpose, we use the thresholded STIV estimator whose coordinates are defined by

$$(7.7) \quad \tilde{\beta}^{(c,I)}_{k} (\omega^{(c,I)}_{k}) \triangleq \begin{cases} \tilde{\beta}^{(c,I)}_{k} & \text{if } |\tilde{\beta}^{(c,I)}_{k}| > \omega^{(c,I)}_{k}, \\ 0 & \text{otherwise}, \end{cases}$$

where $\tilde{\beta}^{(c,I)}_{k}$ are the coordinates of the STIV estimator $\tilde{\beta}^{(c,I)}$, and $\omega^{(c,I)}_{k} > 0$, $k = 1, \ldots, K$, are thresholds that will be specified below. We will use the sparsity certificate approach, so that the thresholds will depend on the upper bound $s$ on the number of non-zero components of $\beta$. Because, for practical use, we want to a data driven thresholding rule we need to strengthen Assumption 7.2 as follows.

**Assumption 7.6.** Fix an integer $s \geq 1$. For every $\gamma_2 \in (0, 1)$, for $\beta$ in $B_s$, there exist constants $c_{1,J(\beta)}^{(c,I)} > 0$, $c_{1,J_0}^{(c,I)}(s) > 0$ for $J_0 = \{k\}$, $\forall k$ and $J_0 = \{1, \ldots, K\}$, such that, with probability at least $1 - \gamma_2$,

$$(7.8) \quad \kappa_{1,J(\beta),J(\beta)}^{(c,I)} \geq c_{1,J(\beta)}^{(c,I)} \quad \text{and} \quad \kappa_{1,J_0}^{(c,I)}(s) \geq c_{1,J_0}^{(c,I)}(s).$$

If $J_0 = \{k\}$ is a singleton we write for brevity $c_{1,J_0}^{(c,I)}(s) = c_{k,J_0}^{(c,I)}(s)$ and $c_{1,\{1, \ldots, K\}}^{(c,I)}(s) = c_{1}^{(c,I)}(s)$. Set

$$\tau_{\{1,\{1, \ldots, K\}\}}^{(c,I)}(s) \triangleq \left( 1 + \frac{r}{c_{1,J(\beta)}^{(c,I)}} \right) \left( 1 - \frac{r}{c_{1,J(\beta)}^{(c,I)}} \right)^{-1} \left( 1 - \frac{r}{c_{1}^{(c,I)}(s)} \right)^{-1}.$$

The following theorem shows that, based on thresholding of the STIV estimator, we can reconstruct exactly the set of non-zero coefficients $J(\beta)$ with probability close to 1. Even more, we achieve the sign consistency, i.e., we reconstruct exactly the vector of signs of the coefficients of $\beta$ with probability close to 1.

**Theorem 7.7.** For every $\beta$ in $B_s$, let the assumptions of Theorem 6.5 and Assumptions 7.1, 7.3, 7.6 be satisfied. Assume that $|J(\beta)| \leq s$, and $|\beta_k| > \frac{4\sigma r + r^{(c,I)}(s)}{c_{1}(c,I)^{*}(s)v_k}$ for all $k \in J(\beta)$. Take the thresholds

$$\omega^{(c,I)}_{k}(s) \triangleq \frac{2\tilde{\beta}_{k}^{(c,I)} r}{\kappa_{k}^{(c,I)^{*}}(s) \mathbb{E}_n[X_k^2]^{1/2}} \left( 1 - \frac{r}{\kappa_{k}^{(c,I)}(s)} \right)^{-1},$$
and consider the estimator \( \tilde{\beta}^{(c,I)} \) with coordinates \( \tilde{\beta}^{(c,I)}_k(s) \), \( k = 1, \ldots, K \). Then, with probability at least \( 1 - \gamma \), we have

\[
\text{sign}(\tilde{\beta}^{(c,I)}) = \text{sign}^{*}(\beta).
\]

As a consequence, \( J(\tilde{\beta}^{(c,I)}) = J(\beta) \).

Sign consistency implies perfect model selection but it is also of independent interest. Indeed, the above result gives conditions under which one can make valid statements on the sign of the coefficients. Unfortunately this is only a theoretical result because it is not possible to verify the condition that \( |\beta_k| > \frac{4\sigma_{\tau(c,I)^*(s)}}{\rho(c,I)^*(s)v_k} \) for all \( k \in J(\beta) \).

8. Further Results on the STIV estimator

8.1. Low Dimensional Models and Very-Many-Weak-Instruments. In this section we consider a slight modification of the STIV estimator to deal with the case where \( K \) is small relative to \( n \) and we know that all \( K \) coefficients are nonzero. It is a new procedure that is robust to a situation where each individual instrument is weak. It can handle very-many instruments, i.e. \( L \) can be much larger than \( n \). It does not rely on normal errors and the confidence sets under Scenarios 2 and 4 (asymptotic) allow for heteroscedasticity. Again, we do not rely on non-standard asymptotics.

Because this is a very active and challenging area of research in econometrics we cannot make an exhaustive literature review and refer to Andrews and Stock (2007) and the references therein.

Because \( K \) is small and \( L \) is large (1.1)-(1.2) is usually point identified, but in the presence of weak instruments one is at the verge of identification. Thus for the sake of completeness we do not exclude partial identification from our result.

Definition 8.1. We call the STIV-R estimator any solution \( (\tilde{\beta}(I), \tilde{\sigma}(I)) \) of the following minimization problem:

\[
\min_{(\beta, \sigma) \in \mathcal{I}(I)} \sigma.
\]

In that case there are no cone constraints in the definition of the sensitivities and we simply drop the index \( J(\beta) \) in the notation of the sensitivities. We also drop the exponent \( c \) everywhere because the minimization problem (8.1) no longer involves the constant \( c \) because the model is low dimensional. Unlike the high-dimensional setup of the previous sections, the sensitivities can be directly calculated from the data and we do not have to rely on lower bounds. Obtaining confidence
sets in a non high-dimensional structural equation is much more direct and easy. We obtain the following result.

**Theorem 8.2.** For every $\beta$ in $\text{Ident}$, under one of the 5 scenarios of Section 6.1 together with its respective assumption and choice of $r$, with probability at least $1 - \alpha$ (approximately at least $1 - \alpha$ for scenario 4 and the two-stage procedure with scenario 5) for any solution $(\hat{\beta}(I), \hat{\sigma}(I))$ of the minimization problem (8.1)

\[
(D_x^{-1}(\hat{\beta}(I) - \beta))_{J_0, p} \leq \frac{2\hat{\sigma}(I) r}{\kappa_{p, J_0}^{(I)}} \left( 1 - \frac{r}{\kappa_1^{(I)}} \right)^{-1}, \quad \forall \ p \in [1, \infty], \ \forall J_0 \subset \{1, \ldots, K\},
\]

for all $k = 1, \ldots, K$,

\[
|\hat{\beta}_k^{(I)} - \beta_k| \leq \frac{2\hat{\sigma}(I) r}{\sqrt{n}[X^2_{\hat{k}]}^{1/2} \kappa_k^{(I)*}} \left( 1 - \frac{r}{\kappa_1^{(I)}} \right)^{-1}
\]

and

\[
\hat{\sigma}(I) \leq \max_{l=1, \ldots, L} (D_z^{(I)})_{ll} \sqrt{Q_l(\beta)}.
\]

The same discussion that we had in the case of a high-dimensional structural equation with endogenous regressors holds. The term

\[
\left( 1 - \frac{r}{\kappa_1^{(I)}} \right)^{-1}
\]

in the upper bounds of Theorem 8.2 can yield infinite volume confidence regions. This occurs either when $r$, of the order of $\sqrt{\log(L)/n}$ in all scenarios, is large or when $\kappa_1^{(I)}$ is too small. The first condition occurs when $L$ is as large as an exponential in $n$. To interpret the second condition, again Proposition 5.2 (ii) yields that for any $k$ in $\{1, \ldots, K\}$,

\[
\kappa_1^{(I)} \leq \kappa_k^{(I)*}.
\]

Therefore a necessary condition for $\kappa_1^{(I)}$ to be small is when there exists in model (1.1) a regressor $(x_{ki})_{i=1}^n$ with low coordinate-wise sensitivity. Without the cone constraint the coordinate wise sensitivities are defined through

\[
\kappa_k^{(I)*} = \inf_{\lambda \in \mathbb{R}^{K-1}} \max_{l=1, \ldots, L} (D_z^{(I)})_{ll} \left| \frac{1}{n} \sum_{i=1}^n z_{li} \left( x_{ki} - x_{T(k)\lambda}^T \right) \right|
\]
and can be interpreted as maximal partial empirical correlations with the instruments. This means that there is a regressor for which all instruments are weak in the sense that it is such that

\[ \inf_{\lambda \in \mathbb{R}^{K-1}} \max_{l=1, \ldots, L} \left( D_{Z_l}^{(1)} \right)_{ll} \left| \frac{1}{n} \sum_{i=1}^{n} z_{li} \left( x_{ki} - x_{\{k\} \setminus i}^T \right) \right| \leq r. \]

Again, in the setting of Example 7, this becomes increasingly likely as \( \zeta J_c \) approaches zero (see (5.7)). Rates of convergence could be obtained as well as in Section 7.

### 8.2. Potentially Sharper Results When None of the Instruments Have Heavy Tail.

In the subsequent sections we restrict the subset of indices \( I \) in (3.4) to a singleton (the index of the regressor which is constant and equal to 1). When \( I \) is a singleton, the STIV estimator is straightforward to obtain because (3.4) only involves one conic constraint. This procedure should be used when all regressors and instruments are bounded or have relatively thin tails. In this section we consider the properties of the STIV estimator with a different matrix \( D_X \) in (3.5). We use for the matrix \( D_X \) the diagonal \( K \times K \) matrix with diagonal entries \( x_{k^*}^{-1}, k = 1, \ldots, K \) where \( x_{k^*} \triangleq \max_i |x_{ki}| \).

The following theorem which is in the same spirit as Theorem 6.5.

**Theorem 8.3.** For every \( \beta \) in \( B_s \), under one of the 5 scenarios of Section 6.1 together with its respective assumption and choice of \( r \), with probability at least \( 1 - \alpha \) (approximately at least \( 1 - \alpha \) for scenario 4 and the two-stage procedure with scenario 5), for any \( c \) in \((0, 1)\), for any solution \((\hat{\beta}(c), \hat{\sigma}(c))\) of the minimization problem (3.7) we have for all \( p \in [1, \infty) \) and \( J_0 \subset \{1, \ldots, K\} \),

\[
(8.5) \quad \left( D_X^{-1} (\hat{\beta}(c) - \beta) \right)_{k, j_0} \leq 2\hat{\sigma}(c) r \frac{1}{\kappa_{p, J_0, \beta}} \left( 1 - \frac{r}{\kappa_{1, J_{exo}, \beta}} - \frac{r^2}{\kappa_{1, J_{exo}, \beta}} \right)^{-1},
\]

for all \( k = 1, \ldots, K \),

\[
(8.6) \quad |\hat{\beta}_k(c) - \beta_k| \leq \frac{2\hat{\sigma}(c) r}{x_{k^*} \kappa_{k, J(\beta)}} \left( 1 - \frac{r}{\kappa_{1, J_{exo}, \beta}} - \frac{r^2}{\kappa_{1, J_{exo}, \beta}} \right)^{-1} + \frac{r}{\kappa_{1, J_{exo}, \beta}} \frac{1}{\kappa_{1, J_{exo}, \beta}}
\]

and

\[
(8.7) \quad \hat{\sigma}(c) \leq \sqrt{\hat{Q}(\beta)} \left( 1 + \frac{r}{\kappa_{1, J(\beta), \beta}} \right) \left( 1 - \frac{r}{\kappa_{1, J_{exo}, \beta}} \right)^{-1}.
\]

\[15\text{Not as a set, see the discussion in Section 5 about the comparison with the usual concentration parameter.}\]
Note that the upper bound (6.7) involves the term
\[ \tau_1 = 1 - \frac{r}{\kappa_{1,J}(\beta)} \]
while (8.5) in Theorem 8.3 involves
\[ \tau_2 \triangleq 1 - \frac{r}{\kappa_{1,J(\beta)}} - \frac{r^2}{\kappa_{1,J(\beta)}}. \]

Though the sensitivities in the two cases are not entirely comparable because we use different matrices \( D_X \), we know, from Proposition 5.1 (ii) that restricting a sensitivity to a sub-block increases the sensitivity. When the set \( J_{exo} \) of regressors that have not been identified as exogenous and therefore used as instruments is small, the sensitivity \( \kappa_{1,J(\beta)} \) is large, whereas the small sensitivity of its complement \( \kappa_{1,J(\beta)} \) is compensated by the small value \( r^2 \) in the numerator. In the extreme case where \( J_{exo} = \emptyset \), we have \( \frac{r}{\kappa_{1,J(\beta)}} = 0 \), so that \( \tau_2 \leq 1 - \frac{r^2}{\kappa_{1,J(\beta)}} \). Under the premise of Proposition 5.2 for it to be positive it is sufficient to have \( |J(\beta)| \leq C r^{-2} = O(n/\log(L)) \) where \( C > 0 \) is a proper constant. Recall that in the case of Theorem 5.3 we need that \( |J(\beta)| \leq C r^{-1} = O(\sqrt{n}/\log(L)). \)

The following results gives proper confidence sets.

**Corollary 8.4.** For every \( \beta \) in \( B_s \), under one of the 5 scenarios of Section 6.1 together with its respective assumption and choice of \( r \), with probability at least \( 1 - \alpha \) (approximately at least \( 1 - \alpha \) for scenario 4 and the two-stage procedure with scenario 5), for any \( c \in (0,1) \), for any solution \( (\beta(c), \hat{\sigma}(c)) \) of the minimization problem \((5.3)\) we have

\[ (D_X^{-1}(\beta(c) - \beta))_{J_0} \leq 2\hat{\sigma}(c) r \left( 1 - \frac{r}{\kappa_{1,J(\beta)}} \right)^{-1}, \quad \forall \, p \in [1, \infty], \forall J_0 \subset \{1, \ldots, K\}, \]

where \( \kappa_{p,J_0}(s), \kappa_{1,J_{exo}}(s), \kappa_{1,J_{exo}}(s) \) are any lower bounds on \( \kappa_{p,J_0}(\beta), \kappa_{1,J_{exo}}(\beta) \) and \( \kappa_{1,J_{exo}}(\beta) \) based on the sparsity certificates that is convenient to calculate (see, e.g., (5.13) and (5.14)), and for all \( k = 1, \ldots, K \),

\[ |\beta_k(c) - \beta_k| \leq \frac{2\hat{\sigma}_k(c) r}{\kappa_{1,J(\beta)}} \left( 1 - \frac{r}{\kappa_{1,J(\beta)}} \right)^{-1}. \]

As we have seen in Section 5, equation (5.13) yields a lower bound for the block sensitivities. It is only computationally feasible for small blocks. So this lower bound could only be used at most for one of the two block sensitivities appearing in \( \tau_2 \), the other one (possibly the two) should be lower.
bounded using \((5.14)\). When \(J_{exo}^c\) is small there are cases when the method of this section yields finite volume confidence regions while due to \(\tau_1\) we can only deduce from Theorem \((5.5)\) infinite volume confidence regions.

It is easy to check that Theorem \((7.4)\) and Corollary \((7.5)\) also hold replacing Assumption \((7.3)\) by Assumption \((8.5)\).

For every \(0 < \gamma_3 < 1\), there exist constants \(v_k > 0\) such that

\[
P(x_{k*} \geq v_k, \forall k \in J(\beta)) \geq 1 - \gamma_3.
\]

by assuming Assumption \((7.6)\) allowing for \(J_0\) to also be \(J_{exo}^c\) and \(J_{exo}^c\) and using the following expression for \(\tau^{(c)}_*\).

\[
\tau^{(c)}_* \triangleq \left(1 + \frac{r}{cc_1(\beta)}\right) \left(1 - \frac{r}{cc_1(\beta)}\right)^{-1} \left(1 - \frac{r}{c_{1,exo}(s)} - \frac{r^2}{c_{1,exo}(s)}\right)^{-1}.
\]

Theorem \((7.7)\) also holds using

\[
\omega_k^{(c)}(s) \triangleq \frac{2\tilde{S}(c)r \kappa^{(c)}_k(s) x_{k*}}{\kappa^{(c)}_k(s) x_{k*} \left(1 - \frac{r}{c_{1,exo}(s)} - \frac{r^2}{c_{1,exo}(s)}\right)^{-1}},
\]

with

\[
\tau^{(c)}_*(s) \triangleq \left(1 + \frac{r}{cc_1(\beta)}\right) \left(1 - \frac{r}{cc_1(\beta)}\right)^{-1} \left(1 - \frac{r}{c_{1,exo}(s)} - \frac{r^2}{c_{1,exo}(s)}\right)^{-1}.
\]

Note that due to the new scaling, the constant \(v_k\) can be much larger than in Section \((7)\). This does not change the rates when the regressors are bounded, but it does if the regressors are unbounded. If a regressor has a normal distribution, this implies an increased \(\log(n)\) in the rate. When \(\tau^{(c,I)}_*\) would be infinite under the scaling of Section \((3)\) this is still a mild loss. We present in Section \((10.2)\) the implementation of this method in the difficult setting where the regressors are Gaussian. This method still performs well in this case.

8.3. The \textit{STIV} Estimator with Linear Projection Instruments. The results of the previous sections show that the \textit{STIV} estimator can handle a very large number of instruments, up to an exponential in the sample size. Moreover, adding instruments always improves the sensitivities. In this section, we consider the case where \(L > K\) but we look for a smaller set of instruments, namely, of size \(K\).

This is a classical problem when the structural equation is low-dimensional. The two-stage least square is a leading and very popular example. Under the stronger zero conditional mean assumption,
optimal instruments attain the semi-parametric efficiency bound (see Amemiya (1974), Chamberlain (1987), Newey (1990) and Hahn (2002)). In the homoscedastic case, the optimal instruments correspond to the projection of the endogenous variables on the space of variables measurable with respect to all the instruments. These optimal instruments are expressed in terms of conditional expectations that are not available in practice and should be estimated. Related work includes, for example, Donald and Newey (2001), Chamberlain and Imbens (2004), Bai and Ng (2009), Okui (2010), Carrasco and Florens (2000) and Carrasco (2012). Belloni, Chen, Chernozhukov et al. (2010) is very related to our approach because it proposes to use the Lasso or post-Lasso to estimate the optimal instrument. This allows to approximate the conditional expectations with a number of functionals of a low dimensional vector of instrument that is exponential in the sample size. They consider a second stage which is the heteroscedastic robust IV estimator and obtain confidence sets for the low dimensional structural equation.

In this section we investigate the properties of a high-dimensional version of the two-stage least squares for high-dimensional structural equations with endogenous regressors. In that case, we have $L > K$ and $K$ is potentially much larger than $n$. For simplicity, we will not consider estimating the optimal instruments in the first stage. One would also have to properly define optimality in high-dimension. This is a difficult question that should be addressed in the future. Moreover, we are aiming at robust confidence sets that have a good approximate finite sample validity. For simplicity of exposure we assume that there is only one endogenous regressor $(x_{k_{\text{end}}}^i)_{i=1}^n$ in (8.1). We write the reduced form equation in the form

$$(8.10) \quad x_{k_{\text{end}}} = \sum_{l=1}^{L} z_{li} \zeta_l + v_i, \quad i = 1, \ldots, n,$$

where $\zeta_l$ are unknown coefficients of the linear combination of instruments and

$$(8.11) \quad \mathbb{E}[z_{li} v_i] = 0$$

for $i = 1, \ldots, n$, $l = 1, \ldots, L$. We call $\sum_{l=1}^{L} z_{li} \zeta_l$ the linear projection instrument.

For simplicity of the proofs we use the normalization of Section 8.2. What we present in this section can easily be extended to a situation with more than one endogenous regressor and with the normalization of Section 3.

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16This is extremely hard in the presence of many instruments due to the curse of dimensionality but it can be done when the instruments are functionals of a low dimensional original instrument.
The first stage consists in estimating the unknown coefficients \( \zeta \). If \( L \geq K > n \) and if the reduced form model \( (8.10) \) has some sparsity (or approximate sparsity), it is natural to use a high-dimensional procedure, such as the Lasso, the Dantzig selector, the Square-root Lasso or the post-Lasso, to produce estimators \( \hat{\zeta}_l \) of the coefficients. Because we are after confidence sets for the coefficients of the structural equation, we will assume that \( (8.10) \) is sparse\(^\text{17}\). When the reduced form equation is not sparse or approximately sparse, for example when all the instruments are equally weak but relevant, this method will fail. This is what we observe in the simulations in Section 10.2. In that situation we recommend to use the one stage STIV estimator.

It is easy to check, from the first order condition that the STIV estimator is, up to the normalization, equivalent to the Square-root Lasso when all the regressors are exogenous. We can deduce from Theorem 8.3 that under one of the 5 scenarios, on an event \( E_\alpha \) of probability at least \( 1 - \alpha \) (or approximately at least \( 1 - \alpha \) for scenario 4 and the two-stage procedure with scenario 5), for any \( c_{RF} \) in \( (0,1) \), any solution \( (\hat{\zeta}^{(c_{RF})}, \sigma_1^{(c_{RF})}) \) to the STIV estimator applied to the reduced form equation is such that

\[
\left| \left( \mathbf{D}_x^{-1}(\hat{\zeta}^{(c_{RF})} - \zeta) \right) \right|_1 \leq \frac{2\sigma_{1}^{(c_{RF})}}{\kappa_{1,J}(\zeta)} \left( 1 - \frac{r^2}{\kappa_{1,J}(\zeta)} \right)^{-1}.
\]

This yields, if one is given a maximum sparsity \( s_{RF} \), the computable upper bound

\[
(8.12) \quad \left| \left( \mathbf{D}_x^{-1}(\hat{\zeta} - \zeta) \right) \right|_1 \leq \frac{2\sigma_{RF}}{\kappa_{1,J}(s_{RF})} \left( 1 - \frac{r^2}{\kappa_{1,J}(s_{RF})} \right)^{-1} \triangleq C_1(r,c_{RF},s_{RF}).
\]

The second stage now consists in using the STIV estimator from Section 8.2 with the enlarged IV-constraint

\[
(8.13) \quad \hat{\mathcal{I}}_{2S} \triangleq \left\{ (\beta, \sigma) : \beta \in \mathbb{R}^K, \sigma > 0, \left| \frac{1}{n} \mathbf{D}_z_{2S} \mathbf{Z}_{2S}^T (\mathbf{Y} - \mathbf{X} \beta) \right|_\infty \leq \sigma r, \hat{Q}(\beta) \leq \sigma^2 \right\}
\]

where \( \mathbf{D}_z_{2S} \) is a \( K \times K \) diagonal matrix such that \( (\mathbf{D}_z_{2S})_{k_{\text{end}},k_{\text{end}}} = (\max_{i=1,\ldots,n} |z^T_i \hat{\zeta}| + 2C_1(r,c_{RF},s_{RF}))^{-1} \) and for every \( k \) in \( \{1,\ldots,K\} \setminus \{k_{\text{end}}\} \), \( (\mathbf{D}_z_{2S})_{kk} = x_k^{-1} \), and the matrix \( \mathbf{Z}_{2S} \) is the stacked matrix of the endogenous regressors and the estimated linear projection instrument \( (z^T_i \hat{\lambda})_{p=1} \). We enlarge the IV-constraint to account for the error made in the first stage estimation of the linear projection instrument. In this section, for simplicity, we restrict our attention to the case where \( I \) is a singleton corresponding to the index of the regressor being unity in \( \hat{\mathcal{I}}_{2S} \). For this reason we drop the index

\(^{17}\text{Belloni, Chen, Chernozhukov et al. (2010) also study the setting where the reduced form has an approximately sparse vectors of coefficients. We could obtain rates of convergence in that case or asymptotically valid confidence sets. Because we are aiming at confidence sets that have some finite sample validity we do not study this case in this article.}
Corollary 8.7. For every $\beta$ in $\mathcal{B}_s$, under one of the 5 scenarios of Section 6.1 together with its respective assumption and choice of $r$, with probability at least $1 - \alpha$ (approximately at least $1 - \alpha$ for scenario 4 and the two-stage procedure with scenario 5), for any $c$ in $(0, 1)$, for any solution $(\hat{\beta}^{(c)}, \hat{\sigma}^{(c)})$ of the minimization problem (8.3), replacing $\hat{\beta}$ by $\hat{\beta}_{2S}$, we have for all $p \in [1, \infty]$ and $J_0 \subset \{1, \ldots, K\}$,

\begin{equation}
\left| \left( D^{-1}(\hat{\beta}^{(c)} - \beta) \right)_{J_0} \right| \leq \frac{2\hat{\sigma}^{(c)}_r}{\kappa^{(c)2S}_p,J_0,J(\beta)} \left( 1 - \frac{r}{\kappa^{(c)2S}_{k_{\text{end}}}J(\beta)} - \frac{r^2}{\kappa^{(c)2S}_{1\{1,\ldots,K\}\setminus\{k_{\text{end}}\},J(\beta)}} \right)^{-1}
\end{equation}

for all $k = 1, \ldots, K$,

\begin{equation}
|\hat{\beta}^{(c)}_k - \beta_k| \leq \frac{2\hat{\sigma}^{(c)}_r}{x_{k_{\text{end}}}\kappa^{(c)2S}_{k_{\text{end}},J(\beta)}} \left( 1 - \frac{r}{\kappa^{(c)2S}_{k_{\text{end}}}J(\beta)} - \frac{r^2}{\kappa^{(c)2S}_{1\{1,\ldots,K\}\setminus\{k_{\text{end}}\},J(\beta)}} \right)^{-1}
\end{equation}

and

\begin{equation}
\hat{\sigma}^{(c)} \leq \sqrt{\hat{Q}(\beta)} \left( 1 + \frac{r}{c\kappa^{(c)2S}_{1\{1,\ldots,K\}\setminus\{k_{\text{end}}\},J(\beta)}} \right) \left( 1 - \frac{r}{c\kappa^{(c)2S}_{1\{1,\ldots,K\}\setminus\{k_{\text{end}}\},J(\beta)}} \right)^{-1}.
\end{equation}

In turn, this yields the following bona-fide confidence sets.

Corollary 8.7. For every $\beta$ in $\mathcal{B}_s$, under one of the 5 scenarios of Section 6.1 together with its respective assumption and choice of $r$, with probability at least $1 - \alpha$ (approximately at least $1 - \alpha$ for scenario 4 and the two-stage procedure with scenario 5), for any $c$ in $(0, 1)$, for any solution $(\hat{\beta}^{(c)}, \hat{\sigma}^{(c)})$ of the minimization problem (8.3), replacing $\hat{\beta}$ by $\hat{\beta}_{2S}$, we have

\begin{equation}
\left| \left( D^{-1}(\hat{\beta}^{(c)} - \beta) \right)_{J_0} \right| \leq \frac{2\hat{\sigma}^{(c)}_r}{\kappa^{(c)2S}_p,J_0,J(\beta)} \left( 1 - \frac{r}{\kappa^{(c)2S}_{k_{\text{end}}}J(\beta)} - \frac{r^2}{\kappa^{(c)2S}_{1\{1,\ldots,K\}\setminus\{k_{\text{end}}\},J(\beta)}} \right)^{-1}, \quad \forall p \in [1, \infty], \forall J_0 \subset \{1, \ldots, K\},
\end{equation}

where $\kappa^{(c)2S}_{p,J_0,J(\beta)}$ is any lower bounds on $\kappa^{(c)2S}_{p,J_0,J(\beta)}$ based on the sparsity certificates that is convenient to calculate (see, e.g., (5.13) and (5.14)), and for all $k = 1, \ldots, K$,

\begin{equation}
|\hat{\beta}^{(c)}_k - \beta_k| \leq \frac{2\hat{\sigma}^{(c)}_r}{x_{k_{\text{end}}}\kappa^{(c)2S}_{k_{\text{end}}}J(\beta)} \left( 1 - \frac{r}{\kappa^{(c)2S}_{k_{\text{end}}}J(\beta)} - \frac{r^2}{\kappa^{(c)2S}_{1\{1,\ldots,K\}\setminus\{k_{\text{end}}\},J(\beta)}} \right)^{-1}.
\end{equation}

All lower bounds on the sensitivities appearing in (8.17) and (8.18) are easy to calculate. Indeed, we have minorized $\kappa^{(c)2S}_{1\{1,\ldots,K\}\setminus\{k_{\text{end}}\}}(s)$ by $\kappa^{(c)2S}_{1}(s)$ using Proposition 5.1 (ii). It is also possible
to proceed using the plug-in strategy where we replace $J(\beta)$ by an estimate $\hat{J}$ and obtain confidence sets of level at least $1 - \gamma$ where $\gamma > \alpha$ but $\gamma$ is very close to $\alpha$ under the assumption that the non-zero coefficients are sufficiently separated from zero.

Rates of convergence and model selection results, similar to those of Section 7, could also be obtained.

9. Models with Possibly Non-valid Instruments

In this section, we study the problem of checking for instrument exogeneity when there is overidentification. This is a classic problem in econometrics (see, e.g., Sargan (1958) and Basmann (1960) for the linear IV model, and Hansen (1982) for GMM). See also Andrews (1999) and Liao (2010) where one can find more references. We propose a two-stage method based on the STIV estimator. The main purpose of the suggested method is to construct confidence sets for non-validity indicators, and to detect non-valid (i.e., endogenous) instruments in the high-dimensional framework. We restrict our attention to the case where (1.1) is point identified, i.e. $\text{Ident} = \{\beta^*\}$. The model can be written in the form:

\begin{align}
(9.1) & \quad y_i = x_i^T \beta^* + u_i, \\
(9.2) & \quad \mathbb{E}[z_iu_i] = 0, \\
(9.3) & \quad \mathbb{E}[\bar{z}_i u_i] = \theta^*,
\end{align}

where $x_i$, $z_i$, and $\bar{z}_i$ are vectors of dimensions $K$, $L$ and $L_1$, respectively. For simplicity of the discussion, we assume in this section that (9.1), (9.2) is point identified. For this reason we use the notation $\beta^*$ instead of $\beta$. The instruments are decomposed in two parts, $z_i$ and $\bar{z}_i$, where $\bar{z}_i^T = (\bar{z}_{1i}, \ldots, \bar{z}_{L_1i})$ is a vector of possibly non-valid instruments. A component of the unknown vector $\theta^* \in \mathbb{R}^{L_1}$ is equal to zero when the corresponding instrument is indeed valid. The component $\theta^*_l$ of $\theta^*$ will be called the non-validity indicator of the instrument $\bar{z}_{li}$. Our study covers the models with dimensions $K$, $L$ and $L_1$ that can be much larger than the sample size.

As above, we assume independence and allow for heteroscedasticity. The difference from the previous sections is only in introducing equation (9.3). We observe realizations of independent random vectors $(x_i^T, y_i, z_i^T, \bar{z}_i^T)$, $i = 1, \ldots, n$. The components $\bar{z}_{li}$ satisfy $\mathbb{E}[\bar{z}_{li} u_i] = \theta^*_l$ for all $l = 1, \ldots, L_1$, $i = 1, \ldots, n$. We denote by $\bar{Z}$ the matrix of dimension $n \times L_1$ with rows $\bar{z}_i^T$, $i = 1, \ldots, n$. Set

$$
\bar{z}_* = \max_{l=1,\ldots,L_1} \left( \frac{1}{n} \sum_{i=1}^n \bar{z}_{li}^2 \right)^{1/2}.
$$

In this section, we assume that we have a pilot estimator \( \hat{\beta} \) and a statistic \( \hat{b} \) such that, with probability close to 1,

\[
(9.4) \quad \left| D_{X^{-1}}(\hat{\beta} - \beta^*) \right|_1 \leq \hat{b}.
\]

For example, \( \hat{\beta} \) can be the STIV estimator based only on the vectors of valid instruments \( z_1, \ldots, z_n \).

In this case, an explicit expression for \( \hat{b} \) can be obtained from (6.11), (7.5), (8.2) or (8.9), depending on the situation.

We define the STIV-NV estimator \((\hat{\theta}(c), \hat{\sigma}_1(c))\) as any solution of the problem

\[
(9.5) \quad \min_{(\theta, \sigma_1) \in \hat{I}_1} (|\theta|_1 + c \sigma_1),
\]

where \( 0 < c < 1 \),

\[
\hat{I}_1 \triangleq \left\{ (\theta, \sigma_1) : \theta \in \mathbb{R}^{L_1}, \sigma_1 > 0, \left| \frac{1}{n} \hat{Z}^T (Y - X \hat{\beta}) - \theta \right|_\infty \leq \sigma_1 r_1 + \hat{b}_z, \ F(\theta, \hat{\beta}) \leq \sigma_1 + \hat{b}_z \right\}
\]

for some \( r_1 > 0 \) (to be specified below), where for all \( \theta = (\theta_1, \ldots, \theta_{L_1}) \in \mathbb{R}^{L_1}, \beta \in \mathbb{R}^K \),

\[
F(\theta, \beta) \triangleq \max_{l=1, \ldots, L_1} \sqrt{Q_l(\theta_l, \beta)}
\]

with

\[
\hat{Q}_l(\theta_l, \beta) \triangleq \frac{1}{n} \sum_{i=1}^n (\tilde{z}_{li} (y_i - x_i^T \beta) - \theta_l)^2.
\]

It is not hard to see that the optimization problem (9.5) can be re-written as a conic program.

The following theorem provides a basis for constructing confidence sets for the non-validity indicators.

**Theorem 9.1.** Under one of the 5 scenarios of Section 6.1, replacing \( z_i \) by \( \tilde{z}_{li} \) and \( L \) by \( L_1 \), together with its respective assumption and choice of \( r \) with \( \alpha \) replaced by \( \alpha_1 \), when \( \hat{\beta} \) is an estimator satisfying (9.4) with probability at least \( 1 - \alpha_2 \) for some \( 0 < \alpha_2 < 1 \), then, with probability at least \( 1 - \alpha_1 - \alpha_2 \) (approximately at least \( 1 - \alpha_1 - \alpha_2 \) for scenario 4 and the two-stage procedure with scenario 5), for any \( c \) in \((0,1)\), for any solution \((\hat{\theta}(c), \hat{\sigma}_1(c))\) of the minimization problem (9.5), we have

\[
|\hat{\theta}(c) - \theta^*|_\infty \leq \frac{2 \left[ \hat{\sigma}_1(c) r_1 + (1 + r_1(1 - c)^{-1}) \hat{b}_z \right]}{(1 - 2r_1(1 - c)^{-1}|J(\theta^*)|)_+} \triangleq V(\hat{\sigma}_1, c, \hat{b}, |J(\theta^*)|),
\]

and

\[
|\hat{\theta}(c) - \theta^*|_1 \leq \frac{2 |J(\theta^*)|(\hat{\sigma}_1(c) r_1 + (1 + r_1 \hat{b}_z) + \hat{c}\hat{b}_z)}{(1 - c - 2r_1|J(\theta^*)|)_+}.
\]

\(^{18}\) (6.13) is not explicit.
This theorem should be naturally applied when \( r_1 \) is small, i.e., \( n \gg \log(L_1) \). In addition, we need a small \( \hat{b} \), which is guaranteed by the results of Section 6 under the condition \( n \gg \log(L) \) if the pilot estimator \( \hat{\beta} \) is the STIV estimator. Note also that the bounds (9.6) and (9.7) are meaningful if their denominators are positive, which is roughly equivalent to the following bound on the sparsity of \( \theta^* \): \( |J(\theta^*)| = O(1/r_1) = O(\sqrt{n}/\log(L_1)) \).

Bounds for all the norms \( |\hat{\theta}^{(c)} - \theta^*|_p \), \( \forall 1 < p < \infty \), follow immediately from (9.6) and (9.7) by the standard interpolation argument. We note that, in Theorem 9.1, \( \hat{\beta}^{(c)} \) can be any estimator satisfying (9.4), not necessarily the STIV estimator.

We now consider that \( c \in (0, 1) \) is fixed. To turn (9.6) and (9.7) into valid confidence bounds, we can replace there \( |J(\theta^*)| \) by \( |J(\hat{\theta}^{(c)})| \), as follows from Theorem 9.3 (ii) below. In addition, Theorem 9.3 establishes the rate of convergence of the STIV-NV estimator and justifies the selection of non-valid instruments by thresholding. To state the theorem, we will need an extra assumption that the random variable \( F(\theta^*, \beta^*) \) is bounded in probability by a constant \( \sigma_1 > 0 \):

**Assumption 9.2.** There exist constants \( \sigma_1 > 0 \) and \( 0 < \varepsilon < 1 \) such that, with probability at least \( 1 - \varepsilon \),

\[
(9.8) \quad \max_{l=1, \ldots, L_1} \frac{1}{n} \sum_{i=1}^{n} (\mathcal{Q}_0 u_i - \theta^*_l)^2 \leq \sigma_1^2.
\]

As in (7.7) we define a thresholded estimator

\[
(9.9) \quad \tilde{\theta}^{(c)}_l \triangleq \begin{cases} 
\hat{\theta}^{(c)}_l & \text{if } |\hat{\theta}^{(c)}_l| > \omega^{(c)}, \\
0 & \text{otherwise},
\end{cases}
\]

where \( \omega^{(c)} > 0 \) is some threshold. For \( b_* > 0, s_1 > 0 \), define

\[
(9.10) \quad \sigma_* = \left(1 - \frac{4r_1s_1}{c(1 - c - 2r_1s_1)_+} \right)^{-1} \left[ \sigma_1 + \frac{2b_*\varepsilon s_1 (1 + 2(1 + r_1)s_1/c)}{(1 - c - 2r_1s_1)_+} \right].
\]

**Theorem 9.3.** Let the assumptions of Theorem 9.1 and Assumption 9.2 be satisfied. Then the following holds.

(i) Let \( \hat{\beta} \) be an estimator satisfying

\[
(9.11) \quad \left| D_X^{-1}(\hat{\beta} - \beta^*) \right|_1 \leq b_*
\]

with probability at least \( 1 - \alpha_2 \) for some \( 0 < \alpha_2 < 1 \) and some constant \( b_* \). Assume that \( |J(\theta^*)| \leq s_1 \). Then, with probability at least \( 1 - \alpha_1 - \alpha_2 - \varepsilon \), for any solution \( \hat{\theta}^{(c)} \) of the
minimization problem (9.5) we have

$$|\hat{\theta}^{(c)} - \theta^*|_{\infty} \leq V(\sigma_*, c, b_*, s_1).$$

(ii) Let $$(\hat{\beta}, \hat{\sigma})$$ be the STIV estimator, and let the assumptions of all the items of Theorem 7.4 be satisfied (with $p = 1$ in item (ii)). Assume that $|J(\theta^*)| \leq s_1$, $|J(\beta^*)| \leq s$, and $|\theta^*_l| > V(\sigma_*, c, b_*, s_1)$ for all $l \in J(\theta^*)$, where

$$b_* = \frac{2\sigma_* r s \tau^{(c, I)^*}(s)}{c_1}.$$

Then, with probability at least $1 - \alpha_1 - \varepsilon - \gamma$, for any solution $\hat{\theta}^{(c)}$ of the minimization problem (9.5) we have

$$J(\theta^*) \subseteq J(\hat{\theta}^{(c)}).$$

(iii) Let the assumptions of item (ii) and Assumption 7.6 hold. Assume that $|\theta^*_l| > 2V(\sigma_*, c, b_*, s_1)$ for all $l \in J(\theta^*)$. Let $\hat{\theta}$ be the thresholded estimator defined in (9.9) where $\hat{\theta}^{(c)}$ is any solution of the minimization problem (9.5), and the threshold is defined by $\omega^{(c)} = V(\sigma_1, c, \hat{\sigma}, s_1)$ with

$$\hat{b} = \frac{2\sigma r s \kappa^{(c)}(s)}{\kappa_1(s)} \left(1 - \frac{r}{\kappa_1(s)}\right)^{-1}.$$

Then, with probability at least $1 - \alpha_1 - \varepsilon - \gamma$, we have

$$\lim_{c \to \infty} \text{sign}(\hat{\theta}^{(c)}) = \text{sign}(\theta^*).$$

As a consequence, $J(\hat{\theta}^{(c)}) = J(\theta^*)$.

In practice, the parameter $s$ may not be known and it can be replaced by $|J(\hat{\theta})|$; this is a reasonable upper bound on $|J(\theta^*)|$ as suggested by Theorem 9.3 (ii). It is interesting to analyze the dependence of the rate of convergence in (9.11) on $r, r_1, s$, and $s_1$. As discussed above, a meaningful framework is to consider small $r, r_1$ and the sparsities $s, s_1$ such that $rs, r_1s_1$ are comfortably smaller than 1. In this case, the value $b_*$ given in (9.14) is of the order $O(rs)$ and the rate of convergence in (9.11) is of the order $O(r_1) + O(rs)$. We see that the rate does not depend on the sparsity $s_1$ of $\theta^*$ but it does depend on the sparsity $s$ of $\beta^*$. It is interesting to explore whether this rate is optimal, i.e., whether it can be improved by estimators different from the STIV-NV estimator.
10. Practical Implementation

10.1. Computational Aspects. For simplicity of exposition, in this section we only consider the case where \( I = \{ i_0 \} \) and \( i_0 \) is the index of unity. Extension to several cones is easy. Finding a solution \( (\hat{\beta}(c,\{i_0\}),\hat{\sigma}(c,\{i_0\})) \) of the minimization problem (3.5) reduces to the following conic program: find \( \beta \in \mathbb{R}^K \) and \( t > 0 \) (\( \sigma = t/\sqrt{n} \)), which achieve the minimum

\[
(10.1) \quad \min_{(\beta,t,v,w) \in V} \left( \sum_{k=1}^{K} w_k + c \frac{t}{\sqrt{n}} \right)
\]

where \( V \) is the set of \((\beta,t,v,w)\), with satisfying:

\[
\begin{align*}
  v &= Y - X\beta, \\
  -rt1 &\leq \frac{1}{\sqrt{n}}DzZ^T(Y - X\beta) \leq rt1, \\
  -w &\leq D^{-1}X\beta \leq w, \\
  w &\geq 0, \\
  (t,v) &\in C.
\end{align*}
\]

Here and below \( \mathbf{0} \) and \( \mathbf{1} \) are vectors of zeros and ones respectively, the inequality between vectors is understood in the componentwise sense, and \( C \) is a cone: \( C \triangleq \{(t,v) \in \mathbb{R} \times \mathbb{R}^n : t \geq |v|_2 \} \). Conic programming is a standard tool in optimization and many open source toolboxes are available to implement it (see, e.g., Sturm (1999)).

The expression in curly brackets in the lower bound (5.10) is equal to the value of the following optimization program:

\[
(10.2) \quad \min_{\epsilon = \pm 1} \min_{(w,\Delta,v) \in V_{k,j}} v
\]

where \( V_{k,j} \) is the set of \((w,\Delta,v)\) with \( w \in \mathbb{R}^K, \Delta \in \mathbb{R}^K, v \in \mathbb{R} \) satisfying:

\[
\begin{align*}
  v &\geq 0, \\
  -v1 &\leq \Psi^{(I)}_n \Delta \leq v1, \\
  w &\geq 0, \\
  -w_{I^c} &\leq \Delta_{I^c} \leq w_{I^c} \quad \text{for} \quad I = \{ j, k \}, \\
  w_I &\leq 0, \\
  \Delta_k &\leq 1, \\
  \epsilon \Delta_j &\geq 0, \\
  \sum_{i=1}^{K} w_i + 1 &\leq \epsilon(a + g)\Delta_j
\end{align*}
\]

where \( g \) is the constant such that

\[
{g = \begin{cases} 
0 & \text{if } k = j \\
-1 & \text{otherwise.}
\end{cases}}
\]

Note that, here, \( \epsilon \) is the sign of \( \Delta_j \), and (10.2) is the minimum of two terms, each of which is the value of a linear program. Analogously, the expression in curly brackets in (5.13) can be computed by solving \( 2^{|J_0|} \) linear programs. The reduction is done in the same way as in (10.2) with the only difference that instead of \( \epsilon \) we introduce a vector \( (\epsilon_k)_{k \in J_0} \) of signs of the coordinates \( \Delta_k \) for indices \( k \in J_0 \).
The coordinate-wise sensitivities

\[ k_{k,J}^{(c,\{i_0\})^*} = \inf_{\Delta_k = 1, |\Delta_J| \leq \frac{1}{c} |\Delta_J|_1} \left| \Psi_n^{(I)} \Delta \right|_\infty \]

can be efficiently computed for given \( J \) when the cardinality \( |J| \) is small. Indeed, it is enough to find the minimum of the values of \( 2^{|J|} \) linear programs:

\[
(10.3) \min_{(\epsilon_j)_{j \in J} \in \{-1,1\}^{|J|}} \min_{(w,\Delta,v) \in U_{k,J}} v
\]

where \( U_{k,J} \) is the set of \((w,\Delta,v)\) with \( w \in \mathbb{R}^K \), \( \Delta \in \mathbb{R}^K \), \( v \in \mathbb{R} \) satisfying:

\[
v \geq 0, \quad -v1 \leq \Psi_n^{(I)} \Delta \leq v1, \quad w \geq 0, \quad -w_{Ic} \leq \Delta_{Ic} \leq w_{Ic} \quad \text{for } I = J \cup \{k\},
\]

\[
w_I = 0, \quad \Delta_k = 1, \quad \epsilon_j \Delta_j \geq 0, \quad \text{for } j \in J, \quad \sum_{i=1}^{K} w_i \leq \frac{1 + c}{1 - c} \sum_{j \in J} \epsilon_j \Delta_j + g.
\]

Here \((\epsilon_j)_{j \in J}\) is the vector of signs of the coordinates \( \Delta_j \) with \( j \in J \) and \( g \) is the constant defined by

\[
g = \begin{cases} 
0 & \text{if } k \in J, \\
-1 & \text{otherwise.} 
\end{cases}
\]

10.2. Simulations. In this section, we consider the performance of the STIV estimator of Section 8.2 on simulated data. The model is as follows:

\[
y_i = \sum_{k=1}^{K} x_{ki} \beta_k^* + u_i,
\]

\[
x_{1i} = \sum_{l=1}^{L-K+1} z_{li} \zeta_l + v_i,
\]

\[
x_{l_l} = z_{li} \quad \text{for } l' = l - L + K \quad \text{and} \quad l \in \{L - K + 2, \ldots, L\},
\]

where \((y_i, x_i^T, z_i^T, u_i, v_i)\) are i.i.d., \((u_i, v_i)\) have the joint normal distribution

\[
\mathcal{N} \left( 0, \begin{pmatrix} \sigma_{\text{struct}}^2 & \rho \sigma_{\text{struct}} \sigma_{\text{end}} \\ \rho \sigma_{\text{struct}} \sigma_{\text{end}} & \sigma_{\text{end}}^2 \end{pmatrix} \right),
\]

\(z_i^T\) is a vector of independent standard normal random variables, and \(z_i^T\) is independent of \((u_i, v_i)\). Clearly, in this model \(\mathbb{E}[z_i u_i] = 0\). We take \(n = 49, L = 50, K = 25, \sigma_{\text{struct}} = \sigma_{\text{end}} = \rho = 0.3, \beta^* = (1, 1, 1, 1, 1, 0, \ldots, 0)^T\) and \(\zeta_l = 0.15\) for \(l = 1, \ldots, L - K + 1\). We have 50 instruments and only 49 observations, so we are in a framework of application of high-dimensional techniques. We set \(c = 0.1\) and set \(r\) according to (6.3) with \(\alpha = 0.05\). The three columns on the left of Table II present simulation results for the STIV estimator. It is straightforward to see that only the first five variables
the true support of $\beta^*$) are eligible to be considered as relevant. This set will be denoted by $\hat{J}$. The second and third columns in Table 1 present the true coordinate-wise sensitivities $\kappa_{k,\hat{J}}^*$ as well as their lower bounds $\kappa_{k}^*(s)$ obtained via the sparsity certificate with $s = 5$. These lower bounds are easy to compute, and we see that they yield reasonable approximations from below of the true sensitivities. The estimate $\hat{\sigma}$ is 0.247 which is quite close to $\sigma_{\text{struct}}$. Next, based on (6.8), the fact that $J_{\text{exo}}^c = \{1\}$, and the bounds on the sensitivities in Proposition 5.1 and in (5.10) – (5.13), we have the following formulas for the confidence intervals

\begin{equation}
|\hat{\beta}_k - \beta_k^*| \leq \frac{2\hat{\sigma}}{x_{ks} \kappa_{k,\hat{J}}^*} \left(1 - \frac{r}{\kappa_{k,\hat{J}}^*} - \frac{r^2}{\kappa_{1,\hat{J}}^*}\right)^{-1}, \tag{10.4}
\end{equation}

\begin{equation}
|\hat{\beta}_k - \beta_k^*| \leq \frac{2\hat{\sigma}}{x_{ks} \kappa_{1}(s)} \left(1 - \frac{r}{\kappa_{1}(s)} - \frac{r^2}{\kappa_{1}(s)}\right)^{-1} \text{ with } s = 5. \tag{10.5}
\end{equation}

Here, $\kappa_{k,\hat{J}}^*$ and $\kappa_{1}(s)$ are computed directly via the programs (10.3) and (10.2) respectively. The value $\kappa_{1}(s)$ is then obtained from (5.12), and for $\kappa_{1,\hat{J}}^*$ we use a lower bound analogous to (5.12):

$$
\kappa_{1,\hat{J}}^* \geq \frac{1 - c}{2|J|} \min_{k=1,\ldots,K} \kappa_{k,\hat{J}}^*.
$$

We get $\kappa_{1,\hat{J}}^* = 0.0096$ and $\kappa_{1}(5) = 0.0072$. In particular, we have $r/\kappa_{1,\hat{J}}^* = 4.40 > 1$, so that (10.4) and (10.5) do not provide confidence intervals in this numerical example.

The columns on the right in Table 1 present the results where we use the same data, estimate the linear projection instrument by the Square-root Lasso and then take only $K$ instruments: $z_{il}$, $l = L - K + 2, \ldots, L$, and $\tilde{x}_{i1} = \sum_{l=1}^{L} \hat{\zeta}_{l}z_{il}$, where $\hat{\zeta}_{l}$ are the Square-root Lasso estimators of $\zeta_{l}$, $l = 1, \ldots, L$. The Square-root Lasso with parameter $c_{\sqrt{\text{lasso}}} = 1.1$ recommended in Belloni, Chernozhukov and Wang (2010) yields all coefficients equal to zero when keeping only the first three digits. This is disappointing since we get an instrument equal to zero. It should be noted that estimation in this setting is a hard problem since the dimension $L$ is larger than the sample size, the number of non-zero coefficients $\zeta_{l}$ is large ($L - K + 1 = 26$), and their values are relatively small (equal to 0.15).

To improve the estimation, we adjusted the parameter $c_{\sqrt{\text{lasso}}}$ empirically, based on the value of the estimates. Ultimately, we have chosen $c_{\sqrt{\text{lasso}}} = 0.3$. This choice is not covered by the theory of Belloni, Chernozhukov and Wang (2010) because $c_{\sqrt{\text{lasso}}}$ should be greater than 1. However, it leads to $\sqrt{Q(\hat{\beta})} = 0.309$, which is very close to $\sigma_{\text{end}}$. The corresponding estimates $\hat{\zeta}_{l}$ are given in Table 2. We see that they are not very close to the true $\zeta_{l}$; some of the relevant coefficients are erroneously

\footnote{One should not confuse the constant $c_{\sqrt{\text{lasso}}}$ denoted by $c$ in Belloni, Chernozhukov and Wang (2010) with the constant $c = c_{STIV}$ in the definition of the STIV estimator; $c_{\sqrt{\text{lasso}}}$ is an equivalent of $\sqrt{n}/c_{STIV}$, up to constants.}
Table 1. Results for the STIV estimator without and with estimated instruments, $n = 49$

<table>
<thead>
<tr>
<th></th>
<th>$\hat{\beta}_1$ (1)</th>
<th>$\kappa^*_1$ (1)</th>
<th>$\kappa^*_1(5)$ (1)</th>
<th>$\hat{\beta}_2$ (1)</th>
<th>$\kappa^*_2$ (2)</th>
<th>$\kappa^*_2(5)$ (2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta^*_1$</td>
<td>1.03</td>
<td>0.107</td>
<td>0.103</td>
<td>1.03</td>
<td>0.085</td>
<td>0.068</td>
</tr>
<tr>
<td>$\beta^*_2$</td>
<td>0.98</td>
<td>0.308</td>
<td>0.157</td>
<td>0.98</td>
<td>0.367</td>
<td>0.075</td>
</tr>
<tr>
<td>$\beta^*_3$</td>
<td>0.96</td>
<td>0.129</td>
<td>0.103</td>
<td>0.96</td>
<td>0.126</td>
<td>0.071</td>
</tr>
<tr>
<td>$\beta^*_4$</td>
<td>0.95</td>
<td>0.150</td>
<td>0.109</td>
<td>0.95</td>
<td>0.115</td>
<td>0.057</td>
</tr>
<tr>
<td>$\beta^*_5$</td>
<td>0.90</td>
<td>0.253</td>
<td>0.175</td>
<td>0.90</td>
<td>0.177</td>
<td>0.086</td>
</tr>
</tbody>
</table>

We use dots because the values that do not appear are similar.

(1): With all the 50 instruments,

(2): With 25 instruments including an estimate of the linear projection instrument.

Table 2. Estimates of the coefficients of the linear projection instrument

<table>
<thead>
<tr>
<th>$\hat{\zeta}_1$</th>
<th>$\hat{\zeta}_2$</th>
<th>$\hat{\zeta}_3$</th>
<th>$\hat{\zeta}_4$</th>
<th>$\hat{\zeta}_6$</th>
<th>$\hat{\zeta}_8$</th>
<th>$\hat{\zeta}_9$</th>
<th>$\hat{\zeta}_{10}$</th>
<th>$\hat{\zeta}_{14}$</th>
<th>$\hat{\zeta}_{15}$</th>
<th>$\hat{\zeta}_{16}$</th>
<th>$\hat{\zeta}_{17}$</th>
<th>$\hat{\zeta}_{18}$</th>
<th>$\hat{\zeta}_{20}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.084</td>
<td>0.130</td>
<td>0.190</td>
<td>0.142</td>
<td>0.115</td>
<td>0.083</td>
<td>0.104</td>
<td>0.126</td>
<td>0.176</td>
<td>0.030</td>
<td>0.023</td>
<td>0.157</td>
<td>0.135</td>
<td>0.082</td>
</tr>
<tr>
<td>$\hat{\zeta}_{21}$</td>
<td>$\hat{\zeta}_{23}$</td>
<td>$\hat{\zeta}_{24}$</td>
<td>$\hat{\zeta}_{25}$</td>
<td>$\hat{\zeta}_{26}$</td>
<td>$\hat{\zeta}_{27}$</td>
<td>$\hat{\zeta}_{28}$</td>
<td>$\hat{\zeta}_{29}$</td>
<td>$\hat{\zeta}_{34}$</td>
<td>$\hat{\zeta}_{35}$</td>
<td>$\hat{\zeta}_{36}$</td>
<td>$\hat{\zeta}_{37}$</td>
<td>$\hat{\zeta}_{38}$</td>
<td>$\hat{\zeta}_{39}$</td>
</tr>
<tr>
<td>0.100</td>
<td>0.125</td>
<td>0.038</td>
<td>0.025</td>
<td>0.026</td>
<td>-0.058</td>
<td>0.108</td>
<td>0.005</td>
<td>-0.053</td>
<td>-0.006</td>
<td>-0.009</td>
<td>-0.063</td>
<td>0.033</td>
<td></td>
</tr>
</tbody>
</table>

We only show the non-zero coefficients.

set to 0 and several superfluous variables are included, sometimes with significant coefficients, such as $\hat{\zeta}_{32}$. We get $\kappa^*_1, \hat{J} = 0.0076$ and $\kappa^*_1(5) = 0.0040$. Again, $r/\kappa^*_1, \hat{J} > 1$, so that we cannot use (10.4) and (10.5) to get the confidence intervals. Note that this approach based on the estimated linear projection instrument gives sensitivities, which are lower than with the full set of instruments. This is mainly due to the fact that the estimation of the linear projection instrument is quite imprecise. Indeed, we add an instrument $\hat{x}_{i1}$, which is not so good, and at the same time we drop a large number of other instruments, which may be not so bad. The overall effect on the sensitivities turns out to be negative. Recall that since the sensitivities involve the maximum of the scalar products of the rows of $\tilde{\Psi}_n(I)$ with $\Delta$, the more we have rows (i.e., instruments) the higher is the sensitivity. The same deterioration of the sensitivities occurred in other simulated data sets. In conclusion, the approach based on estimation of the linear projection instrument was not helpful to realize the above confidence intervals.
Table 3. Monte-Carlo study, 1000 replications

<table>
<thead>
<tr>
<th></th>
<th>5th percentile</th>
<th>Median</th>
<th>95th percentile</th>
<th></th>
<th>5th percentile</th>
<th>Median</th>
<th>95th percentile</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta_1$</td>
<td>0.872</td>
<td>0.986</td>
<td>1.093</td>
<td>$\beta_5$</td>
<td>-0.057</td>
<td>0.000</td>
<td>0.055</td>
</tr>
<tr>
<td>$\beta_2$</td>
<td>0.877</td>
<td>0.970</td>
<td>1.048</td>
<td>$\beta_9$</td>
<td>-0.052</td>
<td>0.000</td>
<td>0.059</td>
</tr>
<tr>
<td>$\beta_3$</td>
<td>0.879</td>
<td>0.970</td>
<td>1.049</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\beta_4$</td>
<td>0.886</td>
<td>0.971</td>
<td>1.051</td>
<td>$\beta_{23}$</td>
<td>-0.051</td>
<td>0.000</td>
<td>0.051</td>
</tr>
<tr>
<td>$\beta_5$</td>
<td>0.877</td>
<td>0.968</td>
<td>1.049</td>
<td>$\beta_{24}$</td>
<td>-0.057</td>
<td>0.000</td>
<td>0.051</td>
</tr>
<tr>
<td>$\beta_6$</td>
<td>-0.048</td>
<td>0.000</td>
<td>0.055</td>
<td>$\beta_{25}$</td>
<td>-0.053</td>
<td>0.000</td>
<td>0.049</td>
</tr>
<tr>
<td>$\beta_7$</td>
<td>-0.059</td>
<td>0.000</td>
<td>0.063</td>
<td>$\hat{\sigma}$</td>
<td>0.181</td>
<td>0.233</td>
<td>0.291</td>
</tr>
</tbody>
</table>

intervals in this small sample situation. However, we will see that it achieves the task when the sample size gets large.

Although in this numerical example we were not able to use (10.4) and (10.5) for the confidence intervals, we got evidence that the performance of the STIV estimator is quite satisfactory. Table 3 shows a Monte-Carlo study where we keep the same values of the parameters of the model, of the sample size $n = 49$, and of the parameter $A$ defining the set $\hat{I}$, simulate 1000 data sets, and compute 1000 estimates. The empirical performance of the STIV estimator is extremely good, even for the endogenous variable. The Monte-Carlo estimation of the variability of $\hat{\beta}_1$ is very similar to that of the exogenous variables. With $c = 0.1$ the estimate $\hat{\sigma}$ is larger than $\sigma_{\text{struct}}$ in 95% of the simulations. This suggests that there remains some margin to penalize less for the “variance” in (3.5), i.e., to decrease $c$ and thus to obtain higher sensitivities.

Next, we study the empirical behavior of the non-pivotal STIV estimator. We consider the same model and the same values of all the parameters, and we choose $\sigma_* = 2 \cdot 0.233$ where 0.233 is the median of $\hat{\sigma}$ from Table 3. Indeed $P(\mathbb{E}_n[U^2] \leq \sigma_*^2)$ should be close to 1 (see Assumption 7.1). The results are given in Table 4. The non-pivotal procedure seems to better estimate as zeros the zero coefficients. This is because we minimize the $\ell_1$ norm of the coefficients without an additional $c\sigma$ term. On the other hand, the non-zero coefficients are better estimated using the pivotal estimator. The non-pivotal procedure yields some shrinkage to zero (especially for large $\sigma_*$). Using the pivotal procedure in the first place allows us to have a good initial guess of $\sigma_*$. Let us now increase $n$ to see whether we can obtain interval estimates and take advantage of thresholding for variable selection. We consider the same model as above and the same values of the parameters of the method but we replace $n = 49$ by $n = 8000$. Then we are no longer in a situation where we must use specific high-dimensional techniques. However, it is still a challenging
Table 4. Monte-Carlo study of the non-pivotal estimator, 1000 replications

<table>
<thead>
<tr>
<th>β*</th>
<th>5th percentile</th>
<th>Median</th>
<th>95th percentile</th>
<th>β*</th>
<th>5th percentile</th>
<th>Median</th>
<th>95th percentile</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.714</td>
<td>0.914</td>
<td>1.110</td>
<td>8</td>
<td>-0.003</td>
<td>0.000</td>
<td>0.016</td>
</tr>
<tr>
<td>2</td>
<td>0.803</td>
<td>0.909</td>
<td>1.010</td>
<td>9</td>
<td>0.000</td>
<td>0.000</td>
<td>0.024</td>
</tr>
<tr>
<td>3</td>
<td>0.789</td>
<td>0.904</td>
<td>1.019</td>
<td>10</td>
<td>0.000</td>
<td>0.000</td>
<td>0.018</td>
</tr>
<tr>
<td>4</td>
<td>0.793</td>
<td>0.904</td>
<td>1.023</td>
<td></td>
<td>:</td>
<td>:</td>
<td>:</td>
</tr>
<tr>
<td>5</td>
<td>0.796</td>
<td>0.907</td>
<td>1.017</td>
<td>23</td>
<td>0.000</td>
<td>0.000</td>
<td>0.021</td>
</tr>
<tr>
<td>6</td>
<td>0.000</td>
<td>0.000</td>
<td>0.020</td>
<td>24</td>
<td>0.000</td>
<td>0.000</td>
<td>0.016</td>
</tr>
<tr>
<td>7</td>
<td>0.000</td>
<td>0.000</td>
<td>0.021</td>
<td>25</td>
<td>0.000</td>
<td>0.000</td>
<td>0.005</td>
</tr>
</tbody>
</table>

task to select among 25 candidate variables, one of them being endogenous. Indeed, classical selection procedures like the BIC would require to solve 2^{25} least squares problems. Our methods are much less numerically intensive. They are based on linear and conic programming, and their computational cost is polynomial in the dimension. We study both the setting with all the 50 instruments and the setting where we estimate the linear projection instrument.

Consider first the case where we use all the instruments. Set for brevity

$$\hat{w} \triangleq \left(1 - \frac{r}{\kappa_1^*} - \frac{r^2}{\kappa_1^*} \right)_+^{-1}, \quad w(5) \triangleq \left(1 - \frac{r}{\kappa_1^*(5)} - \frac{r^2}{\kappa_1^*(5)} \right)_+^{-1}.$$

These are the quantities appearing in (10.4) and (10.5). As above, we take $\hat{\mathcal{J}}$ equal to the set of the first five coordinates; $w(5)$ corresponds to the sparsity certificate approach with $s = 5$. Computing the exact coordinate-wise sensitivities we obtain the bound $\hat{w} \leq 1.6277$. The sparsity certificate approach with $s = 5$ yields $w(5) \leq 1.6306$. We obtain $\hat{\sigma} = 0.2970$ and the estimates in Table 5. The values $\hat{\beta}_{l, \hat{\mathcal{J}}}$ and $\hat{\beta}_{u, \hat{\mathcal{J}}}$ are the lower and upper confidence limits respectively obtained from (10.4); $\hat{\beta}_{l,SC}$ and $\hat{\beta}_{u,SC}$ are the lower and upper confidence limits obtained from (10.5) (sparsity certificate approach with $s = 5$). The thresholds $\omega_{k, \hat{\mathcal{J}}}$ and $\omega_k(5)$ are computed from the formulas

$$\omega_{k, \hat{\mathcal{J}}} = \frac{2 \cdot 1.6277 \hat{\sigma} r}{x_{k^*} \kappa_{k, \hat{\mathcal{J}}}} , \quad \omega_k(5) = \frac{2 \cdot 1.6306 \hat{\sigma} r}{x_{k^*} \kappa_{k}^*(5)} .$$

Table 5 shows that in this example thresholding works well: The true support of $\beta^*$ is recovered exactly by selecting the variables, for which the estimated coefficient is larger than the threshold. Note that the threshold for the endogenous variable is very close to the estimate of the first coefficient $\hat{\beta}_1$ since the confidence intervals are wider for the endogenous variable.

We now consider the case where we use only 25 instruments; the 24 exogenous variables serve as their own instruments and the Square-root Lasso estimator of the linear projection instrument is
Table 5. Confidence intervals and selection of variables, n = 8000

<table>
<thead>
<tr>
<th>$\hat{\beta}_{k,SC}$</th>
<th>$\hat{\beta}_{k,J}$</th>
<th>$\hat{\beta}$</th>
<th>$\hat{\beta}_{u,SC}$</th>
<th>$\hat{\beta}_{u,J}$</th>
<th>$\kappa_{k}^{*}$</th>
<th>$\kappa_{k}^{*}(5)$</th>
<th>$\omega_{k,J}$</th>
<th>$\omega_{k,SC}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta_{1}^{*}$</td>
<td>0.131</td>
<td>0.135</td>
<td>1.048</td>
<td>1.960</td>
<td>1.965</td>
<td>0.134</td>
<td>0.134</td>
<td>0.912</td>
</tr>
<tr>
<td>$\beta_{2}^{*}$</td>
<td>0.795</td>
<td>0.804</td>
<td>0.995</td>
<td>1.185</td>
<td>1.195</td>
<td>0.897</td>
<td>0.855</td>
<td>0.191</td>
</tr>
<tr>
<td>$\beta_{3}^{*}$</td>
<td>0.824</td>
<td>0.829</td>
<td>1.004</td>
<td>1.179</td>
<td>1.185</td>
<td>0.796</td>
<td>0.775</td>
<td>0.175</td>
</tr>
<tr>
<td>$\beta_{4}^{*}$</td>
<td>0.817</td>
<td>0.822</td>
<td>0.998</td>
<td>1.173</td>
<td>1.178</td>
<td>0.858</td>
<td>0.833</td>
<td>0.175</td>
</tr>
<tr>
<td>$\beta_{5}^{*}$</td>
<td>0.833</td>
<td>0.834</td>
<td>1.001</td>
<td>1.168</td>
<td>1.168</td>
<td>0.793</td>
<td>0.790</td>
<td>0.167</td>
</tr>
<tr>
<td>$\beta_{6}^{*}$</td>
<td>-0.163</td>
<td>-0.160</td>
<td>0.003</td>
<td>0.166</td>
<td>0.169</td>
<td>0.807</td>
<td>0.791</td>
<td>0.163</td>
</tr>
<tr>
<td>$\beta_{7}^{*}$</td>
<td>-0.173</td>
<td>-0.168</td>
<td>0.002</td>
<td>0.172</td>
<td>0.177</td>
<td>0.846</td>
<td>0.823</td>
<td>0.170</td>
</tr>
<tr>
<td>$\beta_{8}^{*}$</td>
<td>-0.173</td>
<td>-0.170</td>
<td>0.001</td>
<td>0.173</td>
<td>0.175</td>
<td>0.789</td>
<td>0.779</td>
<td>0.172</td>
</tr>
<tr>
<td>$\beta_{9}^{*}$</td>
<td>-0.163</td>
<td>-0.160</td>
<td>0.003</td>
<td>-0.190</td>
<td>-0.188</td>
<td>0.074</td>
<td>0.099</td>
<td>0.191</td>
</tr>
<tr>
<td>$\beta_{10}^{*}$</td>
<td>-0.171</td>
<td>-0.166</td>
<td>0.001</td>
<td>0.168</td>
<td>0.172</td>
<td>0.842</td>
<td>0.821</td>
<td>0.167</td>
</tr>
<tr>
<td>$\beta_{11}^{*}$</td>
<td>-0.172</td>
<td>-0.169</td>
<td>-0.005</td>
<td>0.158</td>
<td>0.162</td>
<td>0.828</td>
<td>0.811</td>
<td>0.163</td>
</tr>
</tbody>
</table>

Table 6. Estimates of the coefficients in the linear projection instrument

<table>
<thead>
<tr>
<th>$\hat{\zeta}_{1}$</th>
<th>$\hat{\zeta}_{2}$</th>
<th>$\hat{\zeta}_{3}$</th>
<th>$\hat{\zeta}_{4}$</th>
<th>$\hat{\zeta}_{5}$</th>
<th>$\hat{\zeta}_{6}$</th>
<th>$\hat{\zeta}_{7}$</th>
<th>$\hat{\zeta}_{8}$</th>
<th>$\hat{\zeta}_{9}$</th>
<th>$\hat{\zeta}_{10}$</th>
<th>$\hat{\zeta}_{11}$</th>
<th>$\hat{\zeta}_{12}$</th>
<th>$\hat{\zeta}_{13}$</th>
<th>$\hat{\zeta}_{14}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.142</td>
<td>0.145</td>
<td>0.134</td>
<td>0.136</td>
<td>0.137</td>
<td>0.135</td>
<td>0.139</td>
<td>0.139</td>
<td>0.134</td>
<td>0.140</td>
<td>0.146</td>
<td>0.140</td>
<td>0.134</td>
<td>0.136</td>
</tr>
<tr>
<td>0.137</td>
<td>0.138</td>
<td>0.141</td>
<td>0.128</td>
<td>0.142</td>
<td>0.137</td>
<td>0.133</td>
<td>0.135</td>
<td>0.135</td>
<td>0.142</td>
<td>0.137</td>
<td>0.137</td>
<td>0.138</td>
<td></td>
</tr>
</tbody>
</table>

We only show the non-zero coefficients (keeping only three digits).

used for the endogenous variable. This time, we apply the Square-root Lasso with the recommended choice $c_{\sqrt{\text{Lasso}}} = 1.1$. We get $\sqrt{Q(\hat{\beta})} = 0.3012$. The estimates of $\hat{\zeta}_{l}$ are given in Table 6. Next, we use (10.4) and (10.5) to obtain the confidence intervals. Computing the exact coordinate-wise sensitivities we get the bound $\hat{w} \leq 1.0941$. The sparsity certificate approach with $s = 5$ yields $w(5) \leq 1.0990$. We also get $\hat{\sigma} = 0.2970$. The thresholds $\omega_{k,J}$ and $\omega_{k}(5)$ are obtained from the formulas

$$\omega_{k,J} = \frac{2 \cdot 1.0941\hat{\sigma}r}{x_{k}\kappa_{k,J}^{*}}, \quad \omega_{k}(5) = \frac{2 \cdot 1.0990\hat{\sigma}r}{x_{k}\kappa_{k}^{*}(5)}.$$  

The results are summarized in Table 7. Note that the confidence intervals and the thresholds are sharper than in the approach including all the instruments. The particularly good news is that the confidence interval for the coefficient of the endogenous variable becomes much tighter.

In conclusion, when the sample size is large, the coordinate-wise sensitivities based on the sparsity certificate work remarkably well for estimation, confidence intervals, and variable selection.
Table 7. Confidence intervals and selection of variables, \( n = 8000 \)

<table>
<thead>
<tr>
<th>( \hat{\beta}_{l,SC} )</th>
<th>( \hat{\beta}_{l,J} )</th>
<th>( \hat{\beta}_{u,SC} )</th>
<th>( \hat{\beta}_{u,J} )</th>
<th>( \kappa^*_{l,J} )</th>
<th>( \kappa^*_{u,J} )</th>
<th>( \omega_{l,J} )</th>
<th>( \omega_{u,SC} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \beta_1^* )</td>
<td>0.901</td>
<td>0.909</td>
<td>1.148</td>
<td>1.194</td>
<td>0.556</td>
<td>0.531</td>
<td>0.139</td>
</tr>
<tr>
<td>( \beta_2^* )</td>
<td>0.872</td>
<td>0.883</td>
<td>1.095</td>
<td>1.118</td>
<td>0.968</td>
<td>0.882</td>
<td>0.111</td>
</tr>
<tr>
<td>( \beta_3^* )</td>
<td>0.896</td>
<td>0.905</td>
<td>1.103</td>
<td>1.112</td>
<td>0.888</td>
<td>0.821</td>
<td>0.099</td>
</tr>
<tr>
<td>( \beta_4^* )</td>
<td>0.885</td>
<td>0.893</td>
<td>1.012</td>
<td>1.110</td>
<td>0.907</td>
<td>0.848</td>
<td>0.105</td>
</tr>
<tr>
<td>( \beta_5^* )</td>
<td>0.899</td>
<td>0.902</td>
<td>1.100</td>
<td>1.103</td>
<td>0.843</td>
<td>0.823</td>
<td>0.099</td>
</tr>
<tr>
<td>( \beta_6^* )</td>
<td>-0.098</td>
<td>-0.092</td>
<td>0.003</td>
<td>0.099</td>
<td>0.104</td>
<td>0.868</td>
<td>0.822</td>
</tr>
<tr>
<td>( \beta_7^* )</td>
<td>-0.103</td>
<td>-0.098</td>
<td>0.002</td>
<td>0.102</td>
<td>0.107</td>
<td>0.907</td>
<td>0.869</td>
</tr>
<tr>
<td>( \beta_8^* )</td>
<td>-0.109</td>
<td>-0.095</td>
<td>0.001</td>
<td>0.098</td>
<td>0.102</td>
<td>0.886</td>
<td>0.853</td>
</tr>
<tr>
<td>( \beta_{23}^* )</td>
<td>-0.115</td>
<td>-0.109</td>
<td>0.003</td>
<td>0.115</td>
<td>0.121</td>
<td>0.862</td>
<td>0.825</td>
</tr>
<tr>
<td>( \beta_{24}^* )</td>
<td>-0.104</td>
<td>-0.099</td>
<td>0.001</td>
<td>0.101</td>
<td>0.106</td>
<td>0.888</td>
<td>0.848</td>
</tr>
<tr>
<td>( \beta_{25}^* )</td>
<td>-0.109</td>
<td>-0.104</td>
<td>-0.005</td>
<td>0.093</td>
<td>0.098</td>
<td>0.870</td>
<td>0.830</td>
</tr>
</tbody>
</table>

We also get a significant improvement from using the two-stage procedure with estimated linear projection instrument.

11. Appendix

11.1. Lower Bounds on \( \kappa_{p,J}^{(c,I)} \) for Square Matrices \( \Psi_n^{(I)} \). In this section we consider the case where all regressors are exogenous and we take \( D_z^{(I)} = D_X \). We therefore drop the exponent \( I \). The following propositions establish lower bounds on \( \kappa_{p,J}^{(c)} \) when \( \Psi_n \) is a square \( K \times K \) matrix. For any \( J \subseteq \{1, \ldots, K\} \) we define the following restricted eigenvalue (RE) constants

\[
\kappa^{(c)}_{RE,J} \triangleq \inf_{\Delta \in \mathbb{R}^{K \setminus \{0\}}: \Delta \in C_J^{(c)}} \frac{|\Delta^T \Psi_n \Delta|}{|\Delta_J|^2}, \quad \kappa^{(c)}_{RE,J} \triangleq \inf_{\Delta \in \mathbb{R}^{K \setminus \{0\}}: \Delta \in C_J^{(c)}} \frac{|\Delta|^2 |\Delta^T \Psi_n \Delta|}{|\Delta_J|^4}.
\]

**Proposition 11.1.** For any \( J \subseteq \{1, \ldots, K\} \) and \( c \) in \((0, 1)\) we have

\[
\kappa_{1,J}^{(c)} \geq \frac{(1 - c)^2}{4|J|} \kappa^{(c)}_{RE,J} \geq \frac{(1 - c)^2}{4|J|} \kappa^{(c)}_{RE,J}.
\]

**Proof.** For such that \( |\Delta_J|^2|\Delta|_1 \leq \frac{1 + c}{1 - c} |\Delta_J|_1 \) we have \( |\Delta|_1 \leq \frac{2}{1 - c} |\Delta_J|_1 \). Thus,

\[
\frac{|\Delta^T \Psi_n \Delta|}{|\Delta_J|^2} \leq \frac{|\Delta|_1 |\Psi_n \Delta|_\infty}{|\Delta_J|^2} \leq \frac{4}{(1 - c)^2} \frac{|\Psi_n \Delta|_\infty}{|\Delta|_1}.
\]

This proves the first inequality of the proposition. The second inequality is obvious. \( \square \)
Proposition 11.2. Let $J \subseteq \{1, \ldots, K\}$ and $c$ in $(0, 1)$ be such that
\begin{equation}
\inf_{\Delta \in \mathbb{R}^{K} \setminus \{0\} : \Delta \in C_{J}^{(c)}} \frac{|XD_{X} \Delta|_{2}}{\sqrt{n}||\Delta||_{2}} \geq \tilde{\kappa}^{(c)}
\end{equation}
for some $\tilde{\kappa} > 0$, and let there exist $0 < \delta < 1$ such that
\begin{equation}
\left|\frac{1}{n} (XD_{X} - ZD_{Z})^{T} XD_{X}\right|_{\infty} \leq \frac{\delta(1 - c)^{2}(\tilde{\kappa}^{(c)})^{2}}{4|J|}.
\end{equation}
Then
\begin{equation}
\kappa^{(c)}_{1,J} \geq \frac{(1 - \delta)(1 - c)^{2}(\tilde{\kappa}^{(c)})^{2}}{4|J|}.
\end{equation}

Proof. We have
\begin{align*}
|\Psi_{n} \Delta|_{\infty} |\Delta|_{1} & \geq |\Delta^{T} \Psi_{n} \Delta| \\
& \geq \left| \Delta^{T} \frac{1}{n} D_{X} X^{T} XD_{X} \Delta \right| - \left| \Delta^{T} \frac{1}{n} (XD_{X} - ZD_{Z})^{T} XD_{X} \Delta \right|
\end{align*}
where
\begin{align*}
\left| \Delta^{T} \frac{1}{n} (XD_{X} - ZD_{Z})^{T} XD_{X} \Delta \right| & \leq \left| \frac{1}{n} (XD_{X} - ZD_{Z})^{T} XD_{X} \right|_{\infty} |\Delta|^{2}_{1} \\
& \leq \frac{\alpha(1 - c)^{2} \tilde{\kappa}^{2}}{4|J|} |\Delta|^{2}_{1}.
\end{align*}
Combining these inequalities and using that $|\Delta|^{2}_{1} \leq \frac{4}{(1-c)^{2}} |J||\Delta|^{2}_{2}$ for all $\Delta \in C_{J}^{(c)}$ (cf. proof of Proposition 11.2) we get the result. \hfill \Box

Relation (11.2) accounts for the closeness between the normalized instruments and the normalized regressors. In the case where there is only one endogenous regressor in the structural equation and all exogenous regressors are used as their own instrument, $XD_{X}$ and $ZD_{Z}$ only differ through one column. They are equal if there is no endogeneity. In that case we are left with the usual condition (11.1). It is the restricted eigenvalue condition of Bickel, Ritov and Tsybakov (2009) for the Gram matrix of $X$-variables, up to the normalization by $D_{X}$.

We now obtain bounds for sensitivities $\kappa^{(c)}_{p,J}$ with $1 < p \leq 2$ and $c$ in $(0, 1)$. For any $s \leq K$, we consider a uniform version of the restricted eigenvalue constant: $\kappa_{RE}(s) \triangleq \min_{|J| \leq s} \kappa^{(c)}_{RE,J}$.

Proposition 11.3. For any $s \leq K/2$ and $1 < p \leq 2$, we have
\begin{equation}
\kappa^{(c)}_{p,J} \geq C(p) s^{-1/p} \kappa_{RE}(2s), \quad \forall J : |J| \leq s,
\end{equation}
where $C(p) = 2^{-1/p - 1/2}(1 - c) \left(1 + \frac{1+c}{1-c} (p - 1)^{-1/p}\right)^{-1}$. 

Proof. For $\Delta \in R^K$ and a set $J \subset \{1, \ldots, K\}$, let $J_1 = J_1(\Delta, J)$ be the subset of indices in $\{1, \ldots, K\}$ corresponding to the $s$ largest in absolute value components of $\Delta$ outside of $J$. Define $J_+ = J \cup J_1$. If $|J| \leq s$ we have $|J_+| \leq 2s$. It is easy to see that the $k$th largest absolute value of elements of $\Delta_{J_c}$ satisfies $|\Delta_{J_c}|(k) \leq |\Delta_{J_c}|1/k$. Thus,

$$|\Delta_{J_+}^\ast|_p^p \leq \frac{|\Delta_{J_c}|^p_1}{(p-1)^{1/p} s^1 - 1/p} \leq \frac{c_0 |\Delta_{J_c}|_1}{(p-1)^{1/p} s^1 - 1/p},$$

where $c_0 = 1 + c$. Therefore, for $\Delta \in C^{(c)}_J$,

$$|\Delta|_p \leq (1 + c_0(p - 1)^{1/p}) |\Delta_{J_+}|_p \leq (1 + c_0(p - 1)^{1/p}) (2s)^{1/p - 1/2} |\Delta_{J_+}|_2.$$

Using (11.3) and the fact that $|\Delta|_1 \leq \frac{2\sqrt{s}}{1-c} |\Delta_{J_c}|_2$ for $\Delta \in C^{(c)}_J$, we get

$$\left| \frac{\Delta^T \Psi_n \Delta}{|\Delta_{J_+}|_2^2} \right| \leq \frac{|\Delta|_1 |\Psi_n \Delta|_\infty}{|\Delta_{J_+}|_2^2} \leq \frac{2\sqrt{s} |\Psi_n \Delta|_\infty}{(1-c) |\Delta_{J_+}|_2} \leq \frac{s^{1/p} |\Psi_n \Delta|_\infty}{C(p)|\Delta|_p}.$$

Since $|J_+| \leq 2s$, this proves the proposition.

The lower bounds in Propositions 11.1 and 11.3 require to control from below $|\Delta^T \Psi_n \Delta|$ (where $\Psi_n$ is a non-symmetric possibly non-positive definite matrix) by a quadratic form with many zero eigenvalues for vectors in a cone of dominant coordinates. This is potentially a strong restriction on the instruments that we can use. In other words, the sensitivity characteristics $\kappa^{(c)}_{p,J}$ can be much larger than the above bounds. The propositions of this section imply that, even in the case of symmetric matrices, these characteristics are more general and potentially lead to better results than the restricted eigenvalues $\kappa^{(c)}_{RE}(\cdot)$ appearing in the usual RE condition of Bickel, Ritov and Tsybakov (2009).

11.2. Moderate Deviations for Self-normalized Sums. Throughout this section $X_1, \ldots, X_n$ are independent random variables such that, for every $i$, $\mathbb{E}[X_i] = 0$. The following result is due to Efron (1969).
Theorem 11.4. If $X_i$ for $i = 1, \ldots, n$ are symmetric, then for every $r$ positive,

$$\Pr\left(\frac{\frac{1}{n} \sum_{i=1}^{n} X_i}{\sqrt{\frac{1}{n} \sum_{i=1}^{n} X_i^2}} \geq r\right) \leq 2 \exp\left(-\frac{nr^2}{2}\right).$$

This upper bound is refined in Pinelis (1994) for i.i.d. random variables.

Theorem 11.5. If $X_i$ for $i = 1, \ldots, n$ are symmetric and identically distributed, then for every $r$ in $[0, 1),$

$$\Pr\left(\frac{\frac{1}{n} \sum_{i=1}^{n} X_i}{\sqrt{\frac{1}{n} \sum_{i=1}^{n} X_i^2}} \geq r\right) \leq \frac{4e^3}{9} \Phi\left(-\sqrt{nr}\right).$$

The following result is from Jing, Shao and Wang (2003).

Theorem 11.6. Assume that $0 < \mathbb{E}[|X_i|^{2+\delta}] < \infty$ for some $0 < \delta \leq 1$ and set

$$B_n^2 = \sum_{i=1}^{n} \mathbb{E}[X_i^2], \quad L_{n, \delta} = \sum_{i=1}^{n} \mathbb{E}\left[|X_i|^{2+\delta}\right], \quad d_{n, \delta} = B_n/L_{n, \delta}^{1/(2+\delta)}.

Then

$$\forall 0 \leq r \leq \frac{d_{n, \delta}}{\sqrt{n}}, \quad \Pr\left(\frac{\frac{1}{n} \sum_{i=1}^{n} X_i}{\sqrt{\frac{1}{n} \sum_{i=1}^{n} X_i^2}} \geq r\right) \leq 2 \Phi\left(-\sqrt{nr}\right) \left(1 + A_0 \left(\frac{1 + \sqrt{nr}}{d_{n, \delta}}\right)^{2+\delta}\right)$$

where $A_0 > 0$ is an absolute constant.

Despite of its great interest to understand the large deviations behavior of self-normalized sums, the bound has limited practical use because $A_0$ is not an explicit constant.

The following result is a corollary of Theorem 1 in Bertail, Gauthérat and Harari-Kermadec (2009).

Theorem 11.7. Assume that $X_i$ for $i = 1, \ldots, n$ are identically distributed and $0 < \mathbb{E}[|X_i|^4] < \infty$. Then

$$\forall r \geq 0, \quad \Pr\left(\frac{\frac{1}{n} \sum_{i=1}^{n} X_i}{\sqrt{\frac{1}{n} \sum_{i=1}^{n} X_i^2}} \geq r\right) \leq (2e + 1) \exp\left(-\frac{nr^2}{2 + \gamma_4 r^2}\right)$$

where $\gamma_4 = \frac{\mathbb{E}[X_i^4]}{\mathbb{E}[X_i^2]^2}$, while

$$\forall r \geq \sqrt{n}, \quad \Pr\left(\frac{\frac{1}{n} \sum_{i=1}^{n} X_i}{\sqrt{\frac{1}{n} \sum_{i=1}^{n} X_i^2}} \geq r\right) = 0.$$
Proof Bertail, Gauthérat and Harari-Kermadec (2009) obtain the upper bound for \( r \geq \sqrt{n} \) and that for \( 0 \leq r < \sqrt{n} \)

\[
P\left( \left| \frac{1}{n} \sum_{i=1}^{n} X_i \right| \geq r \right) \leq \inf_{a > 1} \left\{ 2e \exp \left( - \frac{nr^2}{2(1 + a)} \right) + \exp \left( - \frac{n}{2\gamma^2} \left( 1 - \frac{1}{a} \right)^2 \right) \right\}.
\]

Because

\[
\frac{1}{1 + a} = \frac{1}{a + \frac{1}{1 + a}} \geq \frac{1}{a} \left( 1 - \frac{1}{a} \right)
\]

we obtain

\[
- \frac{r^2}{1 + a} \leq - \frac{r^2}{a} \left( 1 - \frac{1}{a} \right).
\]

This yields (11.4) by choosing \( a \) to equate the two exponential terms.

\[\square\]

### 11.3. Proofs. Proof of Proposition 5.1

Parts (ii), (iii) and (iv) of the proposition are straightforward. The upper bound in (5.8) follows immediately from (5.1). Next, obviously, \(|\Delta|_p \leq |\Delta|_1^{1/p} |\Delta|_\infty^{1-1/p}\) and we get that, for \( \Delta \neq 0 \),

\[
\frac{|\Psi_n(I) \Delta|_\infty}{|\Delta|_p} \geq \frac{|\Psi_n(I) \Delta|_\infty}{|\Delta|_\infty} \left( \frac{|\Delta|_\infty}{|\Delta|_1} \right)^{1/p}.
\]

Furthermore, (5.1) implies \(|\Delta|_1 \leq \frac{2}{\eta_1} \max\{\eta(I)\}|\Delta|_\infty\) for \( \Delta \in C_I \). Combining this with the above inequality we obtain the lower bound in (5.8). The sequence of inequalities in (5.9) follow from the fact that

\[
|J_0| - 1 + 1/p |\Delta J_0|_1 \leq |\Delta J_0|_p \leq |J_0| 1/p |\Delta J_0|_1.
\]

\[\square\]

### Proof of Proposition 5.2

For all \( 1 \leq k \leq K \) and \( 1 \leq l \leq L \),

\[
\left| \left( \psi_n(I) \Delta \right)_l - \left( \psi_n(I) \right)_{lk} \Delta_k \right| \leq |\Delta|_1 \max_{k' \neq k} \left| \psi_n(I)_{lk'} \right|,
\]

which yields

\[
\left| \left( \psi_n(I) \right)_{lk} \right| |\Delta_k| \leq |\Delta|_1 \max_{k' \neq k} \left| \psi_n(I)_{lk'} \right| + \left| \left( \psi_n(I) \Delta \right)_l \right|.
\]

The two inequalities of the assumption yield

\[
\left| \left( \psi_n(I) \right)_{l(k)k} \right| |\Delta_k| \leq |\Delta|_1 \frac{(1 - \eta_2)(1 - c)}{2 |J|} \left| \left( \psi_n(I) \right)_{l(k)k} \right| + \frac{1 - c}{\eta_1} \left| \left( \psi_n(I) \Delta \right)_l \right| \left| \left( \psi_n(I) \right)_{l(k)k} \right|.
\]

This inequality, together with the fact that \( \left| \left( \psi_n(I) \Delta \right)_l \right| \leq \left| \psi_n(I) \Delta \right|_\infty \), we obtain

\[
|\Delta_j| \leq |\Delta|_1 \frac{(1 - \eta_2)(1 - c)}{2 |J|} + \frac{1 - c}{\eta_1} \left| \psi_n(I) \Delta \right|_\infty
\]

(11.5)
Summing the inequalities over $j$ in $J$, yields
\[ |\Delta J| \leq \frac{(1 - \eta_2)(1 - c)}{2} |\Delta|_1 + \frac{|J|(1 - c)}{\eta_1} \left| \frac{1}{\sqrt{n}} \Psi_n^{(I)} \Delta \right|_\infty. \]

This and the first inequality in (5.1) imply that
\[ \frac{1 - c}{2} |\Delta|_1 \leq \frac{(1 - \eta_2)(1 - c)}{2} |\Delta|_1 + \frac{|J|(1 - c)}{\eta_1} \left| \frac{1}{\sqrt{n}} \Psi_n^{(I)} \Delta \right|_\infty \]
which yields
\[ \frac{\eta_2(1 - c)}{2} |\Delta|_1 \leq \frac{|J|(1 - c)}{\eta_1} \left| \frac{1}{\sqrt{n}} \Psi_n^{(I)} \Delta \right|_\infty \]
and
\[ \frac{\eta_1 \eta_2}{2|J|} |\Delta|_1 \leq \left| \frac{1}{\sqrt{n}} \Psi_n^{(I)} \Delta \right|_\infty. \]

We conclude, using the definition of the $l_1$-sensitivity, that
\[ \kappa_{1,J} = \frac{\eta_1 \eta_2}{2|J|}. \]

Next, plugging (11.6) into (11.5), we deduce
\[ |\Delta|_j \leq \left( \frac{1 - \eta_2}{\eta_1} + \frac{1}{\eta_1} \right) (1 - c) \left| \frac{1}{\sqrt{n}} \Psi_n^{(I)} \Delta \right|_\infty \]
\[ \leq \frac{1 - c}{\eta_1 \eta_2} \left| \frac{1}{\sqrt{n}} \Psi_n^{(I)} \Delta \right|_\infty, \]
which implies
\[ \kappa_{\infty,J} \geq \frac{\eta_1 \eta_2}{1 - c}. \]

This and the lower bound in (5.8) yield the result. \(\square\)

**Proof of Theorem 6.5** Fix $\beta$ in $B_s$, denote by $u_i = y_i - x_i^T \beta$ and define the event
\[ G = \left\{ \forall l = 1,\ldots,L, \left\{ D_z^{(I)} \right\}_{ll} \frac{1}{n} \sum_{i=1}^{n} z_i u_i \leq \max_{l \in I} \left( D_z^{(I)} \right)_{ll} \sqrt{\hat{Q}_l(\beta)} r \right\}. \]

Note that $\hat{Q}_l(\beta) = \mathbb{E}_u[Z_l^2 U^2]$. We have
\[ G^c = \left( \bigcup_{l \in I^c} \left\{ \frac{1}{n} \left| \sum_{i=1}^{n} z_i u_i \right| \geq r \right\} \right) \cup \left( \bigcup_{l \in I} \left\{ \frac{1}{n} \left| \sum_{i=1}^{n} z_i u_i \right| \geq r \right\} \right). \]

Under scenario 1, $\mathbb{P}(G^c) = \alpha$. For the other scenarios, we use
\[ G^c \subset \bigcup_{l=1,\ldots,L} \left\{ \left| \frac{\sum_{i=1}^{n} z_i u_i}{\sqrt{\sum_{i=1}^{n} (z_i u_i)^2}} \right| \geq \sqrt{nr} \right\}. \]

(11.8)
The union bound yields
\[
\mathbb{P}(G^c) \leq \sum_{i=1}^{L} \mathbb{P}\left( \left| \sum_{i=1}^{n} \frac{z_{ii}u_{i}}{\sqrt{\sum_{i=1}^{n}(z_{ii}u_{i})^2}} \right| \geq \sqrt{n}r \right).
\]

We conclude, using the moderate deviations result from Section 11.2, that the event \(G^c\) holds with probability at least \(1 - \alpha\) (approximately at least \(1 - \alpha\) for scenario 4 and the two-stage procedure with scenario 5) under the respective choices of \(r\). Because the event containing \(G^c\) on the right hand side of (11.17) does not depend on \(I\), we obtain statements which are uniform in \(I\). Because \(G^c\) does not depend on \(c\) we obtain statements which are uniform in \(c\) in \((0, 1)\) under all 5 scenarios. Set \(\Delta \triangleq D^{-1}(\hat{\beta}^{(c,I)} - \beta)\). On the event \(G\) we have:

\[
(11.9) \quad \left| \Psi^{(I)}_{n}(\Delta) \right|_{\infty} \leq \left| \frac{1}{n} D^{(I)}_{Z} Z^T (Y - X \hat{\beta}^{(c,I)}) \right|_{\infty} + \left| \frac{1}{n} D^{(I)}_{Z} Z^T Q_{\hat{\beta}^{(c,I)}} \right|_{\infty}
\]

\[
(11.10) \quad \leq r\hat{\sigma}^{(c,I)} + \frac{1}{n} D^{(I)}_{Z} Z^T U \right|_{\infty}
\]

\[
\leq r \left( \hat{\sigma}^{(c,I)} + \max_{l \in I} (D(\hat{\beta}^{(c,I)})) \right).
\]

The inequality (11.10) holds because \((\hat{\beta}^{(c,I)}, \hat{\sigma}^{(c,I)})\) belongs to the set \(\tilde{T}^{(I)}\) by definition. Notice that, on the event \(G\), the pair \((\beta, \max_{l=1,...,L}(D(\hat{\beta}^{(c,I)})))\) belongs to the set \(\tilde{T}^{(I)}\). On the other hand, \((\hat{\beta}^{(c,I)}, \hat{\sigma}^{(c,I)})\) minimizes the criterion \(|D^{-1}(\beta)|_{1} + c\sigma\) on the same set \(\tilde{T}^{(I)}\). Thus, on the event \(G\),

\[
(11.11) \quad \left| D^{-1}(\beta) \right|_{1} + c\sigma^{(c,I)} \leq \left| D^{-1}(\beta) \right|_{1} + c \max_{l \in I} (D(\hat{\beta}^{(c,I)})) \right).
\]

This implies, again on the event \(G\),

\[
(11.12) \quad \left| \Delta_{J(\beta)^{c}} \right|_{1} = \sum_{k \in J(\beta)^{c}} \left| \mathbb{E}_{n} [X^2_{k}^{1/2} \beta_{k}^{(c,I)}] \right|
\]

\[
\leq \sum_{k \in J(\beta)} \left( \left| \mathbb{E}_{n} [X^2_{k}^{1/2} \beta_{k}] \right| - \left| \mathbb{E}_{n} [X^2_{k}^{1/2} \beta_{k}^{(c,I)}] \right| \right) + c \left( \max_{l \in I} (D(\hat{\beta}^{(c,I)})) \right).
\]

For every \(l\) in \(I, \gamma \mapsto (D(\hat{\beta}^{(c,I)})) \) is almost surely differentiable. Thus almost surely its subgradient is the gradient and thus the singleton

\[
\partial \left( (D(\hat{\beta}^{(c,I)})) \right) (\beta) = \nabla \left( (D(\hat{\beta}^{(c,I)})) \right) (\beta) = \frac{\mathbb{E}_{n} [UXTZ]}{\sqrt{\mathbb{E}_{n} [(UZ)^2]}}.
\]

Because the function is also convex, almost surely,

\[
(D(\hat{\beta}^{(c,I)})) - (D(\hat{\beta}^{(c,I)})) \leq \nabla \left( (D(\hat{\beta}^{(c,I)})) \right) (\beta)^{T} (\beta - \hat{\beta}^{(c,I)}).
\]
\[ \leq (D_Z^{(1)})_t \left( \frac{\mathbb{E}[U^TXZ]}{\sqrt{\mathbb{E}[UZ]}} \right)^T D_X D_X^{-1}(\beta - \hat{\beta}(c,I)) \]

\[ \leq -(D_Z^{(1)})_t \frac{\mathbb{E}[U^TXZ]}{\sqrt{\mathbb{E}[UZ]}}^T D_X \Delta. \]

By the Cauchy-Schwartz inequality,

\[ \left| -(D_Z^{(1)})_t \frac{\mathbb{E}[U^TXZ]}{\sqrt{\mathbb{E}[UZ]}}^T D_X \right| \leq 1. \]

We will now use the same notation for \( G \) and \( G \) intersected with the probability 1 event where each mapping is differentiable and work on the later event. By the Dubovitsky-Milutin theorem (see, e.g., Alekseev, Tikhomirov and Fomin (1987), Chapter 2)

\[ \left( \max_{l \in I} (D_Z^{(1)})_t \sqrt{\hat{Q}_l(\beta)} - \max_{l \in I} (D_Z^{(1)})_t \sqrt{\hat{Q}_l(\hat{\beta}(c,I))} \right) \leq |\Delta|_1. \]

Thus (11.12) implies

(11.13)\[ |\Delta_{J(\beta)}|_1 \leq \frac{1 + c}{1 - c} |\Delta_{J(\beta)}|_1. \]

Thus, \( \Delta \in C^{(c,I)}_{J(\beta)} \) on the event \( G \). Using (11.9) and arguing as in (11.12) we find

(11.14)\[ |\Psi_n^{(I)} \Delta|_\infty \leq r \left( 2\sigma^{(c,I)} + \max_{l \in I} (D_Z^{(1)})_t \sqrt{\hat{Q}_l(\beta)} - \sigma^{(c,I)} \right) \]

\[ \leq r \left( 2\sigma^{(c,I)} + \max_{l \in I} (D_Z^{(1)})_t \sqrt{\hat{Q}_l(\beta)} - \max_{l \in I} (D_Z^{(1)})_t \sqrt{\hat{Q}_l(\hat{\beta}(c,I))} \right) \]

\[ \leq r \left( 2\sigma^{(c,I)} + |\Delta|_1 \right). \]

This again uses the convexity for every \( l \) in \( I \), of \( \gamma \mapsto (D_Z^{(1)})_t \sqrt{\hat{Q}_l(\gamma)} \). Using the definition of the sensitivities we get that, on the event \( G \),

\[ |\Psi_n^{(I)} \Delta|_\infty \leq r \left( 2\sigma^{(c,I)} + \frac{|\Psi_n^{(I)} \Delta|_\infty}{\kappa_{1,J(\beta)}} \right), \]

which implies

(11.15)\[ |\Psi_n^{(I)} \Delta|_\infty \leq 2\sigma^{(c,I)} r \left( 1 - \frac{r}{\kappa_{1,J(\beta)}} \right)^{-1}. \]

This inequality and the definition of the sensitivities yield (6.7) and (6.8).
To prove (6.9), it suffices to note that, by (11.11) and by the definition of \( \kappa_{(c,I)}^{(1)} \),
\[
c\sigma^{(c,I)} \leq |\Delta_{J(\beta)}|_1 + c \max_{l \in L} (D^{(1)}_Z) u \sqrt{\tilde{Q}_l(\beta)}
\]
\[
\leq \frac{|\Psi_n(\beta)|_\infty}{\kappa_{c,(J),J(\beta)}} + c \max_{l \in L} (D^{(1)}_Z) u \sqrt{\tilde{Q}_l(\beta)},
\]
and to combine this inequality with (11.9).

**Proof of Theorem 6.7.** Take \( \beta \) in \( B_s \). Fix an arbitrary subset \( J \) of \( \{1, \ldots, K\} \). Acting as in (11.12) with \( J \) instead of \( J(\beta) \), we get:
\[
\sum_{k \in J^c} \left| \mathbb{E}_n[X_k^2]^{1/2} \beta_k^{(c,I)} \right| + \sum_{k \in J} \left| \mathbb{E}_n[X_k^2]^{1/2} \beta_k \right| \leq \sum_{k \in J} \left( \left| \mathbb{E}_n[X_k^2]^{1/2} \beta_k \right| - \left| \mathbb{E}_n[X_k^2]^{1/2} \beta_k^{(c,I)} \right| \right) + 2 \sum_{k \in J^c} \left| \mathbb{E}_n[X_k^2]^{1/2} \beta_k \right|
\]
\[
+ c \left( \sqrt{\hat{Q}(\beta)} - \sqrt{\hat{Q}(\beta^{(c,I)})} \right)
\]
\[
\leq |\Delta_J|_1 + 2 \left| (D^{-1}_X \beta)^{J^c} \right|_1 + c|\Delta|_1.
\]

This yields
\[
|\Delta_{J^c}|_1 \leq |\Delta_J|_1 + 2 \left| (D^{-1}_X \beta)^{J^c} \right|_1 + c|\Delta|_1.
\]

Assume now that we are on the event \( \mathcal{G} \). Consider the two possible cases. First, if \( 2 \left| (D^{-1}_X \beta)^{J^c} \right|_1 \leq |\Delta_J|_1 \), then \( \Delta \in \tilde{C}_{J}^{(c,I)} \). From this, using the definition of the sensitivity \( \tilde{\kappa}_{p,J_0,J}^{(c,I)} \), we get that \( |\Delta_{J_0}|_p \) is bounded from above by the first term of the maximum in (6.13). Second, if \( 2 \left| (D^{-1}_X \beta)^{J^c} \right|_1 > |\Delta_J|_1 \), then for any \( p \in [1, \infty) \) we have a simple bound
\[
|\Delta_{J_0}|_p \leq |\Delta|_1 = |\Delta_{J^c}|_1 + |\Delta_J|_1 \leq \frac{6}{1 - c} \left| (D^{-1}_X \beta)^{J^c} \right|_1.
\]

In conclusion, \( |\Delta|_p \) is smaller than the maximum of the two bounds.

**Proof of Theorem 7.4.** Part (i) of the theorem is a consequence of (6.9) and Assumptions 7.1 and 7.2. Parts (ii) and (iii) follow immediately from (6.7), (6.8), and Assumptions 7.1 and 7.2. Part (iv) is straightforward in view of (7.4).

**Proof of Theorem 7.7.** Fix \( \beta \) in \( B_s \). Let \( \mathcal{G}_j \) be the events of probabilities at least \( 1 - \gamma_j \) respectively appearing in Assumptions 7.1 7.3 7.6. Assume that all these events hold, as well as the event \( \mathcal{G} \). Then, using Theorem 7.4 (i),
\[
\omega_k^{(c)}(s) \leq \frac{2\sigma_r}{c_k^{(c)}(s) v_k} \left( 1 + \frac{r}{c_{c1,J(\beta)}} \right) \left( 1 - \frac{r}{c_{c1,J(\beta)}} \right)^{-1} \left( 1 - \frac{r}{c_1} \right) \Delta \omega_k^{(c)}.
\]
By assumption, $|\beta_k| > 2\omega_k^{(c)*}$ for $k \in J(\beta)$. Note that the following two cases can occur. First, if $k \in J(\beta)^c$ (so that $\beta_k = 0$) then, using (6.8) and Assumptions 7.1 and 7.6, we obtain $|\beta_k^{(c,I)}| = \omega_k^{(c,I)}$, which implies $\beta_k^{(c,I)} = 0$. Second, if $k \in J(\beta)$, then using again (6.8) we get $||\beta_k| - |\beta_k^{(c,I)}|| \leq |\beta_k - \beta_k^{(c,I)}| \leq \omega_k^{(c,I)} \leq \omega_k^{(c)*}$. Since $|\beta_k| > 2\omega_k^{(c)*}$ for $k \in J(\beta)$, we obtain that $|\beta_k^{(c,I)}| = \omega_k^{(c,I)}$, so that $\beta_k^{(c,I)} = \beta_k^{(c,I)}$ and the signs of $\beta_k$ and $\beta_k^{(c,I)}$ coincide. This yields the result.

**Proof of Theorem 8.3** Fix $\beta$ in $B_\alpha$ and define the event

$$
\mathcal{G} = \left\{ \left| \frac{1}{n} D_z Z^T U \right|_\infty \leq \sqrt{\tilde{Q}(\beta)r} \right\}.
$$

Note that $\tilde{Q}(\beta) = \mathbb{E}_n[U^2]$. Under scenario 1, the event $\mathcal{G}$ holds with probability at least $1 - \alpha$. For the other scenarios, the union bound yields

$$
P(\mathcal{G}^c) \leq \sum_{l=1}^{L} P \left( \left| \frac{1}{n} \sum_{i=1}^{n} z_i u_i \right| \leq r \right)
\leq \sum_{l=1}^{L} P \left( \left| \frac{1}{n} \sum_{i=1}^{n} z_i u_i \right| \leq \sqrt{nr} \right).
$$

We conclude, using the moderate deviations result from Section 11.2 that the event $\mathcal{G}$ holds with probability at least $1 - \alpha$ (approximately at least $1 - \alpha$ for scenario 4 and the two-stage procedure with scenario 5).

Take $c$ in $(0,1)$, set $\Delta \triangleq D_x^{-1}(\tilde{\beta}^{(c)} - \beta)$. On the event $\mathcal{G}$ we have:

$$
|\Psi_n \Delta|_\infty \leq \left| \frac{1}{n} D_z Z^T (Y - X \tilde{\beta}^{(c)}) \right|_\infty + \left| \frac{1}{n} D_z Z^T (Y - X \beta) \right|_\infty
\leq r \tilde{\sigma}^{(c)} + \frac{1}{n} D_z Z^T U
\leq r \left( \tilde{\sigma}^{(c)} + \sqrt{\tilde{Q}(\beta)} \right).
$$

Notice that, on the event $\mathcal{G}$, the pair $\left( \beta, \sqrt{\tilde{Q}(\beta)} \right)$ belongs to the set $\tilde{\mathcal{I}}$. On the other hand, $(\tilde{\beta}^{(c)}, \tilde{\sigma}^{(c)})$ minimizes the criterion $|D_x^{-1}\beta|_1 + c\sigma$ on the same set $\tilde{\mathcal{I}}$. Thus, on the event $\mathcal{G}$,

$$
|D_x^{-1}\tilde{\beta}^{(c)}|_1 + c\tilde{\sigma}^{(c)} \leq |D_x^{-1}\beta|_1 + c\sqrt{\tilde{Q}(\beta)}.
$$

This implies, again on the event $\mathcal{G}$,

$$
|\Delta_{J(\beta)^c}|_1 = \sum_{k \in J(\beta)^c} |x_k \tilde{\beta}^{(c)}_k|.
$$
Note that (11.20) can be re-written as a cone condition:

\[
\Delta_{J(\beta)} \leq \frac{\sum_{k \in J(\beta)} (|x_{k, \beta_k}| - |x_{k, \tilde{\beta}_k(c)|}) + c \left( \sqrt{ \hat{Q}(\beta) - \sqrt{\hat{Q}(\hat{\beta}(c))} \right)}
\]

\[
\leq |\Delta_{J(\beta)}|_1 + c \left( \sqrt{ \hat{Q}(\beta) - \sqrt{\hat{Q}(\hat{\beta}(c))} \right)
\]

\[
\leq |\Delta_{J(\beta)}|_1 + c \left( \frac{\mathbb{E}_n[U X^T] D X \Delta}{\sqrt{\mathbb{E}_n[U^2]}} \right) \quad \text{(by convexity of } \beta \mapsto \sqrt{\hat{Q}(\beta)})
\]

\[
\leq |\Delta_{J(\beta)}|_1 + c \left( \frac{\mathbb{E}_n[U X^T] D X \Delta}{\sqrt{\mathbb{E}_n[U^2]}} \right) \quad \text{by the Cauchy-Schwarz inequality).
\]

Note that (11.20) can be re-written as a cone condition:

\[
|\Delta_{J(\beta)}|_1 \leq \frac{1 + c}{1 - c} |\Delta_{J(\beta)}|_1.
\]

Thus, \( \Delta \in C^{(c)}_{J(\beta)} \) on the event \( \mathcal{G} \). Using (11.9) and arguing as in (11.2) we find

\[
|\Psi_n \Delta|_\infty \leq r \left( 2\bar{\sigma}(c) + \sqrt{\hat{Q}(\beta) - \bar{\sigma}(c)} \right)
\]

\[
\leq r \left( 2\bar{\sigma}(c) + \sqrt{\hat{Q}(\beta) - \sqrt{\hat{Q}(\hat{\beta}(c))}} \right) \quad \text{(since } \sqrt{\hat{Q}(\hat{\beta}(c))} \leq \bar{\sigma}(c))
\]

\[
\leq r \left( 2\bar{\sigma}(c) + \frac{\mathbb{E}_n[U X^T] D X \Delta}{\sqrt{\mathbb{E}_n[U^2]}} \right)
\]

\[
\leq r \left( 2\bar{\sigma}(c) + \max_{j \in J_{\text{exo}}} \left| \frac{\mathbb{E}_n[U X_j]}{\sqrt{\mathbb{E}_n[X_j^2 U^2]}} \right| |\Delta_{J_{\text{exo}}}|_1 + \max_{j \in J_{\text{exo}}} \left| \frac{\mathbb{E}_n[U X_j]}{\sqrt{\mathbb{E}_n[X_j^2 U^2]}} \right| |\Delta_{J_{\text{exo}}}|_1 \right)
\]

\[
\leq r \left( 2\bar{\sigma}(c) + |\Delta_{J_{\text{exo}}}|_1 + \max_{j \in J_{\text{exo}}} \left| \frac{\mathbb{E}_n[U X_j]}{\sqrt{\mathbb{E}_n[X_j^2 U^2]}} \right| |\Delta_{J_{\text{exo}}}|_1 \right) \quad \text{(by the Cauchy-Schwarz inequality)}
\]

Since the exogenous variables serve as their own instruments, we obtain that, on the event \( \mathcal{G} \),

\[
\max_{j \in J_{\text{exo}}} \left| \frac{\mathbb{E}_n[U X_j]}{\sqrt{\mathbb{E}_n[X_j^2 U^2]}} \right| \leq r.
\]

Combining this with (11.22) and using the definition of the block sensitivity \( \kappa^{(c)}_{J_0, J(\beta)} \) with \( J_0 = J_{\text{exo}} \), \( J_0 = J_{\text{exo}} \), we get that, on the event \( \mathcal{G} \),

\[
|\Psi_n \Delta|_\infty \leq r \left( 2\bar{\sigma}(c) + \sqrt{\hat{Q}(\beta) - \bar{\sigma}(c)} \right)
\]
\[ r \left( 2\bar{\sigma}(c) + \frac{\left| \Psi_n \Delta \right|_{\infty}}{\kappa_1^{(c)}_{J_{exo}, J(\beta)}} + r \frac{\left| \Psi_n \Delta \right|_{\infty}}{\kappa_1^{(c)}_{J_{exo}, J(\beta)}} \right), \]

which implies

\[ (11.24) \quad \left| \Psi_n \Delta \right|_{\infty} \leq 2\bar{\sigma}(c) r \left( 1 - \frac{r}{\kappa_1^{(c)}_{J_{exo}, J(\beta)}} - \frac{r^2}{\kappa_1^{(c)}_{J_{exo}, J(\beta)}} \right)^{-1}. \]

This inequality and the definition of the sensitivities yield (8.5) and (8.6).

To prove (8.7), it suffices to note that, by (11.19) and by the definition of \( \kappa_1^{(c)}_{J(\beta), J(\beta)} \),

\[
c\widehat{\sigma}^{(c)} \leq |\Delta_{J(\beta)}|_1 + c\sqrt{Q(\beta)} \leq \frac{\left| \Psi_n \Delta \right|_{\infty}}{\kappa_1^{(c)}_{J_{exo}, J(\beta)}} + c\sqrt{Q(\beta)},
\]

and to combine this inequality with (11.18). \( \square \)

**Proof of Theorem 8.6** Take \( \beta \) in \( B_\delta \). Note that

\[
|z_i^T \widehat{\zeta}^{(cRF)}| \geq |z_i^T \zeta| - \left| (D_Z)^{-1} (\widehat{\zeta}^{(cRF)} - \zeta) \right|_1 \geq |z_i^T \zeta^{(cRF)}| - C_1(r, c_{RF}, s_{RF})
\]

thus

\[
\max_{i=1,\ldots,n} |z_i^T \zeta^{(cRF)}| + C_1(r, c_{RF}, s_{RF}) r \geq \max_{i=1,\ldots,n} |z_i^T \zeta|.
\]

As well, we have on \( E_\alpha \),

\[
\frac{1}{n} |\zeta^T Z^T U| \leq \left| (D_Z)^{-1} (\widehat{\zeta}^{(cRF)} - \zeta) \right|_1 \sqrt{Q(\beta)} r + \frac{1}{n} |\zeta^T Z^T U| \leq C_1(r, c_{RF}, s_{RF}) \sqrt{Q(\beta)} r + \frac{1}{n} |\zeta^T Z^T U|,
\]

thus

\[
\frac{1}{n} |\zeta^{(cRF)T} Z^T U| \leq \frac{C_1(r, c_{RF}, s_{RF}) \sqrt{Q(\beta)} r}{\max_{i=1,\ldots,n} |z_i^T \zeta^{(cRF)}|} + \frac{1}{n} |\zeta^T Z^T U| \leq \frac{C_1(r, c_{RF}, s_{RF}) \sqrt{Q(\beta)} r}{\max_{i=1,\ldots,n} |z_i^T \zeta|} + \sqrt{Q(\beta)} r \leq \sqrt{Q(\beta)} r \left( 1 + \frac{C_1(r, c_{RF}, s_{RF})}{\max_{i=1,\ldots,n} |z_i^T \zeta^{(cRF)}| + C_1(r, c_{RF}, s_{RF})} \right).
\]
This yields
\[
\frac{1}{n} |\hat{\beta}^{(CF)}_T Z^T U|_{\infty} \leq \sqrt{Q(\beta)} r.
\]

The rest of the proof is the same as for Theorem 8.3.

**Proof of Theorem 9.1.** Throughout the proof, we assume that we are on the event of probability at least $1 - \alpha_2$ where (9.4) holds. It follows easily from (9.4) that

(11.25) \[
\left| \frac{1}{n} Z^T (\hat{\beta} - \beta^*) \right|_{\infty} \leq \tilde{\nu}_z.
\]

Next, an argument similar to (11.17) and Theorem 11.6 yield that, with probability at least $1 - \alpha_1$,

(11.26) \[
\left| \frac{1}{n} Z^T U - \theta^* \right|_{\infty} \leq r_1 \max_{l=1,\ldots,L} \left( \frac{1}{n} \sum_{i=1}^n (z_i u_i - \theta^*)^2 \right) = r_1 F(\theta^*, \beta^*).
\]

In what follows, we assume that we are on the event of probability at least $1 - \alpha_1 - \alpha_2$ where both (11.25) and (11.26) are satisfied.

We will use the properties of $F(\theta, \beta)$ stated in the next lemma that we prove in Section 11.4

**Lemma 11.8.** We have

(11.27) \[
F(\theta^*, \hat{\beta}) - F(\hat{\theta}, \hat{\beta}) \leq |\hat{\theta} - \theta^*|_1,
\]

(11.28) \[
|F(\theta^*, \hat{\beta}) - F(\theta^*, \beta^*)| \leq \tilde{\nu}_z \left| D_x^{-1} (\hat{\beta} - \beta^*) \right|_1 \leq \tilde{\nu}_z.
\]

We proceed now to the proof of Theorem 9.1. First, we show that the pair $(\theta, \sigma_1) = (\theta^*, F(\theta^*, \beta^*))$ belongs to the set $\hat{L}_1$. Indeed, from (11.25) and (11.26) we get

\[
\left| \frac{1}{n} Z^T (Y - X \hat{\beta}) - \theta^* \right|_{\infty} \leq \left| \frac{1}{n} Z^T U - \theta^* \right|_{\infty} + \left| \frac{1}{n} Z^T (\hat{\beta} - \beta^*) \right|_{\infty} \leq r_1 F(\theta^*, \beta^*) + \tilde{\nu}_z.
\]

Thus, the pair $(\theta, \sigma_1) = (\theta^*, F(\theta^*, \beta^*))$ satisfies the first constraint in the definition of $\hat{L}_1$. It satisfies the second constraint as well, since $F(\theta^*, \hat{\beta}) \leq F(\theta^*, \beta^*) + \tilde{\nu}_z$ by (11.28).

Take any $c$ in $(0, 1)$, as $(\theta^*, F(\theta^*, \beta^*)) \in \hat{L}_1$ and $(\hat{\theta}^{(c)}, \hat{\sigma}^{(c)}_1)$ minimizes $|\theta|_1 + c\sigma_1$ over $\hat{L}_1$, we have

(11.29) \[
|\hat{\theta}^{(c)}|_1 + c\hat{\sigma}^{(c)}_1 \leq |\theta^*|_1 + cF(\theta^*, \beta^*),
\]

which implies

(11.30) \[
|\Sigma_{J(\theta^*)}|_1 \leq |\Sigma_{J(\theta^*)}|_1 + c(F(\theta^*, \beta^*) - \hat{\sigma}^{(c)}_1),
\]
where $\Delta = \hat{\theta}^{(c)} - \theta^*$. Using the fact that $F(\hat{\theta}^{(c)}, \hat{\beta}) \leq \hat{\sigma}_1^{(c)} + \hat{b}\pi_*$, (11.27), and (11.28) we obtain

\[
F(\theta^*, \beta^*) - \hat{\sigma}_1^{(c)} \leq F(\theta^*, \beta^*) - F(\hat{\theta}^{(c)}, \hat{\beta}) + 2\hat{b}\pi_*.
\]

This inequality and (11.30) yield

\[
|\Delta_{J(\theta^*)}|_1 \leq |\Delta_{J(\theta^*)}|_1 + 2c|\hat{\theta}^{(c)} - \theta^*|_1 + 2\hat{b}\pi_*,
\]

or equivalently,

\[
|\Delta_{J(\theta^*)}|_1 \leq \frac{1 + c}{1 - c} |\Delta_{J(\theta^*)}|_1 + \frac{2c}{1 - c} \hat{b}\pi_*.
\]

Next, using (11.25), (11.26) and the second constraint in the definition of $(\hat{\theta}^{(c)}, \hat{\sigma}_1^{(c)})$, we find

\[
|\hat{\theta}^{(c)} - \theta^*|_\infty \leq \left| \frac{1}{n} Z^T (Y - X\hat{\beta}) - \hat{\theta}^{(c)} \right|_\infty + \left| \frac{1}{n} Z^T U - \theta^* \right|_\infty + \left| \frac{1}{n} Z^T X(\hat{\beta} - \beta^*) \right|_\infty \leq r_1(\hat{\sigma}_1^{(c)} + F(\theta^*, \beta^*)) + 2\hat{b}\pi_*.
\]

This and (11.31) yield

\[
|\hat{\theta}^{(c)} - \theta^*|_\infty \leq r_1(2\hat{\sigma}_1^{(c)} + |\hat{\theta}^{(c)} - \theta^*|_1) + 2(1 + r_1)\hat{b}\pi_*.
\]

On the other hand, (11.32) implies

\[
|\hat{\theta}^{(c)} - \theta^*|_1 \leq \frac{2}{1 - c} |\Delta_{J(\theta^*)}|_1 + \frac{2c}{1 - c} \hat{b}\pi_* \leq \frac{2|J(\theta^*)|}{1 - c} |\hat{\theta}^{(c)} - \theta^*|_\infty + \frac{2c}{1 - c} \hat{b}\pi_*.
\]

Inequalities (9.6) and (9.7) follow from solving (11.33) and (11.34) with respect to $|\hat{\theta}^{(c)} - \theta^*|_\infty$ and $|\hat{\theta}^{(c)} - \theta^*|_1$ respectively.

\[\square\]

**Proof of Theorem 9.3.** We first prove part (i). We will assume that we are on the event of probability at least $1 - \alpha_1 - \alpha_2 - \varepsilon$ where (11.26), (9.8), and (9.10) are simultaneously satisfied. From (11.29) and the fact that (9.8) can be written as $F(\theta^*, \beta^*) \leq \sigma_1$, we obtain

\[
\hat{\sigma}_1^{(c)} \leq |\hat{\theta}^{(c)} - \theta^*|_1 / c + \sigma_1.
\]
Note also that the argument in the proof of Theorem 9.1 and the results of that theorem remain obviously valid with \( \hat{b} \) replaced by \( b^* \). Thus, we can use (9.7) with \( \hat{b} \) replaced by \( b^* \), and combining it with (11.35) we obtain

\[
\hat{\sigma}_1^{(c)} \leq \sigma^*.
\]

This and (9.6) yield (9.11).

We now prove part (ii) of the theorem. In the rest of the proof, we assume that we are on the event \( G' \) of probability at least \( 1 - \alpha_1 - \varepsilon - \gamma \) where (11.26) and the events \( G, G_j \) defined in the proofs of Theorems 6.5, 7.4 are simultaneously satisfied. Then item (ii) of Theorem 7.4 with \( p = 1 \) implies (9.10) with \( b^* \) defined in (9.12). This and (9.11) easily give part (ii) of the theorem.

To prove part (iii), note that, by Theorem 7.4 (i) and Assumption 7.6,

\[
\hat{b} = 2\hat{\sigma}_r \left( 1 - \frac{r}{\kappa_1(s)} \right)^{-1} \leq b^*
\]

for \( b^* \) defined in (9.12). This and (11.36) imply that the threshold \( \omega \) satisfies \( \omega \leq V(\hat{\sigma}_1^{(c)}, c, \hat{b}, J(\hat{\theta})) \leq V(\sigma^*, c, b^*, s_1) \) on the event \( G' \). On the other hand, (9.6) guarantees that \( |\hat{\theta}_l^{(c)} - \theta_l^*| \leq \omega^{(c)} \) and, by assumption, \( |\theta_l^*| > 2\omega^{(c)} \) for all \( l \in J(\theta^*) \). In addition, by (6.7) and (9.6) for all \( l \in J(\theta^*) \) we have \( |\theta_l^*| < \omega^{(c)} \), which implies \( \hat{\theta}_l^{(c)} = 0 \). We finish the proof in the same way as the proof of Theorem 7.4.

\[\square\]

11.4. Proof of Lemma 11.8. Set \( f_l(\theta_l) \triangleq \sqrt{Q_l(\theta_l, \beta)} \) and \( f(\theta) \triangleq \max_{l=1,\ldots,L} f_l(\theta_l) \equiv F(\theta, \hat{\beta}) \).

The mappings \( \theta \mapsto f_l(\theta_l) \) are convex, so that by the Dubovitsky-Milutin, the subdifferential of their maximum \( f \) is contained in the convex hull of the union of the subdifferentials of the \( f_l \):

\[
\partial f \subseteq \text{Conv} \left( \bigcup_{l=1}^{L_1} \partial f_l \right).
\]

Since, obviously, \( \partial f_l(\theta_l) \subseteq [-1, 1] \), we find that \( \partial f(\theta) \subseteq \{ w \in \mathbb{R}^{L_1} : |w|_{\infty} \leq 1 \} \) for all \( \theta \in \mathbb{R}^{L_1} \). Using this property and the convexity of \( f \), we get

\[
f(\theta^*) - f(\hat{\theta}) \leq \langle w, \theta^* - \hat{\theta} \rangle \leq |\hat{\theta} - \theta^*|_1, \quad \forall \ w \in \partial f(\theta^*),
\]

where \( \langle \cdot, \cdot \rangle \) denotes the standard inner product in \( \mathbb{R}^{L_1} \). This yields (11.27). The proof of (11.28) is based on similar arguments. Instead of \( f_l \), we now introduce the functions \( g_l \) defined by \( g_l(\beta) \triangleq \sqrt{Q_l(\theta_l^*, \beta)} \), and set \( g(\beta) \triangleq \max_{l=1,\ldots,L} g_l(\beta) \equiv F(\theta^*, \beta) \). Next, notice that the subdifferential of \( g_l \)
satisfies $\partial g_l(\beta) \subseteq \{ w \in \mathbb{R}^K : |w_k| \leq a_{lk}, \ k = 1, \ldots, K \}$ for all $\beta \in \mathbb{R}^K, \ l = 1, \ldots, L_1$, where

$$a_{lk} = \frac{\sqrt{\frac{1}{n} \sum_{i=1}^{n} \bar{z}_{li} x_{ki} (\bar{z}_{li} (y_i - x_i^T \beta) - \theta_i^*)}}{\sqrt{\frac{1}{n} \sum_{i=1}^{n} \bar{z}_{li} (y_i - x_i^T \beta) - \theta_i^*)^2}}.$$

Consequently, by the Cauchy-Schwarz inequality, $D_X \partial g_l(\beta) \subseteq \{ w \in \mathbb{R}^K : |w|_\infty \leq \tau_* \}$ for all $\beta \in \mathbb{R}^K, \ l = 1, \ldots, L_1$. This and (11.38) with $g, g_l$ instead of $f, f_l$ imply $D_X \partial g(\beta) \subseteq \{ w \in \mathbb{R}^K : |w|_\infty \leq \tau_* \}$ for all $\beta \in \mathbb{R}^K$. Using this property and the convexity of $g$, we get

$$g(\beta) - g(\beta') \leq \langle w, (\beta - \beta') \rangle \leq |D_X w|_\infty |D_X^{-1}(\beta - \beta')|_1 \leq \tau_* |D_X^{-1}(\beta - \beta')|_1, \ \forall \ w \in \partial g(\beta),$$

for any $\beta, \beta' \in \mathbb{R}^K$. This proves (11.28). □

**Proof of Theorem 8.2.** The only difference with the proof of Theorem 6.5 is that, because we do not have the $l_1$-norm in the objective function (8.1), we drop the discussion leading to (11.13). □

**References**


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