An Illustrated Guide of the Modern Approaches of Hamilton-Jacobi Equations and Control Problems with Discontinuities

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Information: At the end of the book, in addition to the usual index, two appendices gather the main notations and assumptions which are used throughout this book. Also included is a list of the different notions of solutions as a quick guide.

Foreword

In general, in Mathematics, a book is written on a topic when the theory starts being well-established, when the key results have reached their definitive forms (or are very close to that point) and there is a global understanding of the phenomenas which is validated by the community. When we started this project of book, we were not at all in this idyllic situation: on one hand, the connections between Hamilton-Jacobi Equations with co-dimension 1 discontinuities and equations set on networks have just been noticed and there was still controversies about the most satisfying approach for such problems, and in particular on the notions of solutions to be used. On the other hand, for more general discontinuities, we had just finished revisiting "stratified problems" in the whole space and the theory needed to be further developed. In that direction, the course of P.L. Lions at the College de France when he presented the "tanker problem" was illuminating on the possibilities of using stratified formulations for such problems with boundary conditions.

These two points were the main motivations to start writing this book: on one hand, our first aim was to revisit the recent progress made in the study of Hamilton-Jacobi Equations with co-dimension 1 discontinuities, and in particular those coming from the existing theories for equations set on networks (a subject which we are going to consider a little bit even if we did not want to enter too deeply into it). Our aim was both to clarify and simplify the existing results and their proofs and it seemed to us that there was a lot of room to do it!

On the other hand, we wished to develop and pushed as far as possible the ideas we had for stratified problems, and in particular with the aim of treating problems with boundary conditions. Maybe the imperfection or even the lack of the existing theories in these different directions explain why it took us five years to reach this last version.

In the case of co-dimension 1 discontinuities, revisiting the recent progress means that we are not merely copy-pasting with few modifications the existing articles. Instead we have tried to emphasize the main common ideas, either technical or more fundamental ones. This is why, while thinking about all the common points in several works, we have decided to dedicate an entire part (Part I) to the "basic results", which are common —bricks, used very often under perhaps slightly different forms, to prove the main results. This has the advantage to lighten the presentation of the main

results and their proofs, but that creates a rather technical –and perhaps difficult to read– part, although it is not uninteresting to see some classical ideas revisited in (sometimes) unusual ways.

In the second and third parts, we describe and compare the different notions of solutions for codimension-1 type discontinuities: we begin with the classical Ishii's notion of viscosity solutions (Part II) and then, in Part III, we consider the different approaches used for networks (flux-limited and junction solutions). We have tried to analyze all these different approaches in full details, trying to give to the reader the most precise comparison of their advantages and disadvantages in terms of the generality of assumptions and results. Ending up with the idea that they are complementary and, being based on rather different technical ideas, they provide all together a very interesting toolbox which, we hope, will be useful in the future progress of the domain. Even if they are different, they share a lot of common points which partly justifies our first part on common tools.

In this third part dedicated to what we call the "Network approach" for problems with a co-dimension 1 discontinuity, we do not just describe the state-of-the-art but we also provide new results and applications, based on a combinations of the presented tools, in order to illustrate them. In particular, we give some applications to KPP (Kolmogorov-Petrovsky-Piskunov) type problems and we also give a simple idea of the results one can get in the framework which is the HJ analogue of 1-d scalar conservation laws with a discontinuous flux, for the reader who just wants to have a quick overview of them. In order to conclude this short overview of the codimension-1 discontinuities case, we point out that a very intriguing question, which enlights the difference and complementarity of the different notions of solutions, is the convergence of the vanishing viscosity approximation: one has a complete answer in this codimension-1 framework by combining the arguments from these various notions, but it remains open in more general situations, like on chessboard-type configurations for example.

A large part of this book is dedicated to stratified problems where discontinuities of any co-dimensions are present: this opens a very large range of applications, the newest ones being for problems with boundary conditions where both the boundary may be non-smooth and the data may present discontinuities. Some a priori very singular problems can be addressed and even treated, the most fascinating ones being in the boundary conditions case. Here, to our point of view, the main message is the identification of what we believe as being the "good framework" for studying discontinuities in Hamilton-Jacobi Equations, namely the assumptions of "tangential

continuity", "normal controllability" and the suitable notion of solution. The reader who is familiar with either exit time, state-constraint control problems or boundary conditions for Hamilton-Jacobi Equations will recognize some common features. With these assumptions, it is surprising to see how some applications can be treated without major additional difficulty compared to the continuous case.

For treating applications in a simpler way we address, in particular, the following question: under which conditions can one prove that Ishii viscosity solutions are stratified solutions? The motivation is clear: classical viscosity solutions have nice stability properties and passages to the limit, even in rather complicated situations, are rather simple. On the other hand, we have a general comparison result for stratified solutions. Hence in the cases where these two concepts of solutions coincide, we benefit of all the advantages. Of course, as it is clear from the study of codimension 1 discontinuities where we have a very precise and complete picture, this strategy does not lead to the most general results. But still, it provides interesting results in a rather cheap way and in cases where our understanding of discontinuities is not so satisfactory (any discontinuity of codimension bigger than 1, in fact...).

This book is called "illustrative" since we have tried to provide applications to concrete problems but also we have pointed out several key open problems. We hope that the reader of this manuscript will enjoy reading it and that it will be useful for him/her. Of course, we will be very happy to hear that he will be able to solve some of the open problems we mention.

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Contents

Introduction

Chapter 1

General Introduction

Viscosity solutions: a simple, efficient and universal notion of solutions.

In 1983, the introduction of the notion of viscosity solutions by Crandall and Lions [68] solved the main questions concerning first-order Hamilton-Jacobi Equations (HJE in short), at least those set in the whole space $\mathbb{R}^N$, for both stationary and evolution equations: this framework provided the right notion of solutions for which uniqueness and stability hold, allowing to prove (for example) the convergence of the vanishing viscosity method. In this founding article the definition was very inspired by the works of Kružkov [127, 128, 130, 129] and, in fact, viscosity solutions appeared as the L^∞-analogue of the L^1-entropy solutions for scalar conservation laws. This initial, rather complicated Kružkov-type definition, was quickly replaced by the present definition, given in the article of Crandall, Evans and Lions [66], emphasizing the key role of the Maximum Principle and of the degenerate ellipticity, thus preparing the future extension to second-order equations.

The immediate success of the notion of viscosity solutions came from both its simplicity but also universality: only one definition for all equations, no matter whether the Hamiltonian was convex or not. A single theory was providing a very good framework to treat all the difficulties connected to the well-posedness (existence, uniqueness, stability...etc.) but it was also fitting perfectly with the applications to deterministic control problems, differential games, front propagations, image analysis etc.

Of course, a second key breakthrough was made with the first proofs of comparison results for second-order elliptic and parabolic, possibly degenerate, fully nonlinear partial differential equations (pde in short) by Jensen [124] and Ishii [120]. They allow the extension of the notion of viscosity solutions to its natural framework and

open the way to more applications. This extension definitively clarifies the connections between viscosity solutions and the Maximum Principle since, for second-order equations, the Maximum Principle is a standard tool and viscosity solutions (for degenerate equations) are those for which the Maximum Principle holds when testing with smooth test-functions.

The article of Ishii and Lions [117] was the first one in which the comparison result for second-order equations was presented in the definitive form; we recommend this article which contains a lot of results and ideas, in particular for using the ellipticity in order to obtain more general comparison results or Lipschitz regularity of solutions.

We refer to the User's guide of Crandall, Ishii and Lions [67] for a rather complete introduction of the theory (See also Bardi and Capuzzo-Dolcetta[13] and Barles [31] for first-order equations, Fleming and Soner [87] for second-order equations together with applications to deterministic and stochastic control, Bardi, Crandall, Evans, Soner and Souganidis [12] ot the CIME course [2] for a more modern presentation of the theory with new applications).

Discontinuities: a weakness of viscosity solutions?

Despite all these positive points, the notion of viscosity solutions had a little weakness: it only applies with the maximal efficiency when solutions are continuous and, this is even more important, when the Hamiltonians in the equations are continuous. This fact is a consequence of the keystone of the theory, namely the comparison result, which is mainly proved by the "doubling of variables" technic, relying more or less on continuity both of the solutions and the Hamiltonians.

Yet, a definition of discontinuous solutions has appeared very early (in 1985) in Ishii [118] and a first attempt to use it in applications to control problems was proposed in Barles and Perthame [22]. The main contribution of [22] is the "half-relaxed limit method", a stability result for which only a L^∞-bound on the solutions is needed. But this method, based on the Ishii's notion of discontinuous viscosity solutions for discontinuous Hamiltonians, uses discontinuous solutions more as an intermediate tool than as an interesting object by itself.

Discontinuities: the end of universality?

However, in the late 80's, two other types of works considered discontinuous solutions and Hamiltonians, breaking the universality feature of viscosity solutions. The first one was the study of measurable dependence in time in time-dependent equation

(cf. Barron and Jensen [43], Lions and Perthame [133], see also the case of second-order equations in Nunziante [140, 141], Bourgoing [51, 52] with Neumann boundary conditions, and Camilli and Siconolfi [58]): in these works, the pointwise definition of viscosity solutions has to be modified to take into account the measurable dependence in time. It is worth pointing out that there was still no difference between convex and non-convex Hamiltonians.

On the contrary, Barron and Jensen [42] in 1990 considered semi-continuous solutions of control problems (See also [30] for a slightly simpler presentation of the ideas of [42] and Frankowska [92], Frankowska and Plaskacz [94], Frankowska and Mazzola [93] for different approaches): they introduced a particular notion of viscosity solution which differs according to whether the control problem consists in minimizing some cost or maximizing some profit; thus treating differently convex and concave Hamiltonians. This new definition had the important advantage to provide a uniqueness result for lower semi-continuous solutions in the case of convex Hamiltonians, a very natural result when thinking in terms of optimal control.

In the period 1990-2010, several attempts were made to go further in the understanding of Hamilton-Jacobi Equations with discontinuities. A pioneering work is the one of Dupuis [72] whose aim was to construct and study a numerical method for a calculus of variation problem with discontinuous integrand, motivated by a Large Deviations problem. Then, control problems with a discontinuous running cost were addressed by Garavello and Soravia [98, 97] and Soravia [157] who highlight some non-uniqueness feature for the Bellman Equations in optimal control, but identify the maximal and minimal solutions. To the best of our knowledge, all the uniqueness results use either a special structure of the discontinuities or different notions solutions, which are introduced to try to tackle the main difficulties as in [70, 71, 100, 101, 107] or an hyperbolic approach as in [8, 62]. For the boundary conditions, Blanc [46, 47] extended the [22] and [42] approaches to treat problems with discontinuities in the boundary data for Dirichlet problems. Finally, even the case of measurability in the state variable was considered for Eikonal type equations by Camilli and Siconolfi [57].

Before going further, we point out that we do not mention here the L^p-viscosity solutions nor viscosity solutions for stochastic pdes, two very interesting subjects but too far from the scope of this book.

Towards more general discontinuities

In this period, the most general contribution for first-order Hamilton-Jacobi-Bellman Equations was the work of Bressan and Hong [53] who considered the case of control problems in *stratified domains*. In their framework, the Hamiltonians can have

discontinuities on submanifolds of $\mathbb{R}^N$ of any codimensions which form a Whitney stratification and the viscosity solutions inequalities are disymmetric between sub and supersolutions (we come back on this important point later on). In this rather general setting, they are able to provide comparison results by combining pde and control methods. Of course, we are very far from the context of an universal definition but it seems difficult to have more general discontinuities. Before going further, we refer the reader to Whitney [162, 161] for the notion of *Whitney stratified space*.

Networks and discontinuities: same battle?

In the years 2010's, a lot of efforts have been spent to understand Hamilton-Jacobi Equations on networks and, maybe surprisingly, this had a key impact on the study of discontinuities in these equations. An easy way to understand why is to look at an HJ-equation set on the real line $\mathbb{R}$, with only one discontinuity at $x = 0$. Following this introduction, it seems natural to jump on to Ishii's definition and to address the problem as an equation set on $\mathbb{R}$. But another point of view consists in seeing $\mathbb{R}$ as a network with two branches $\mathbb{R}^-$ and $\mathbb{R}^+$. This way, $x = 0$ becomes the intersection of the two branches and it is conceivable that the test-functions could be quite different in each branch, leading to a different notion of solution. Moreover, a "junction condition" is needed at 0 which might come from the two Hamiltonians involved (one for each branch) but also a specific inequality at 0 coming from the model and the transmission condition we have in mind. Therefore, at first glance, these "classical approach" and "network approach" seem rather different.

Surprisingly (with today's point of view), these two approaches were investigated by different people and (almost) completely independently until Briani, Imbert and the authors of this book made the simple remark which is described in the last above paragraph. But, in some sense, this "mutual ignorance" was a good point since different complementary questions were investigated and we are going to described these questions now.

For the "classical approach", in the case of the simplest codimension 1 discontinuity in $\mathbb{R}$ or $\mathbb{R}^N$ and for deterministic control problems, *i.e.* with convex Hamiltonians, these questions were

(*i*) Is Ishii's definition of viscosity solutions providing a unique solution which is the value-function of an associated control problem?

(*ii*) If not, can we identify the minimal and maximal solutions in terms of value functions of *ad hoc* control problems?

(*iii*) In non-uniqueness cases, is it possible to recover uniqueness by imposing some additional condition on the discontinuity?

(*iv*) Can the limit of the vanishing viscosity method be identified? Is it the maximal or minimal solution? Or can it change depending on the problem?

These questions were investigated by Rao [144, 145], Rao and Zidani [146], Rao, Siconolfi and Zidani [143] by optimal control method, and Barles, Briani and Chasseigne [16, 17] by more pde methods. In [16, 17], there are some complete answers to questions (*i*) and (*ii*), almost complete for (*iii*) and really incomplete for (*iv*).

For the "network approach", in the case of two (or several) $1-$dimensional (or multi-dimensional) branches, the questions were different and the convexity of the Hamiltonians appears as being less crucial

(*v*) What is the correct definition of solution at the junction? What are the different possible junction conditions and their meanings in the applications?

(*vi*) Does a comparison result for such network problems hold?

(*vii*) Does the Kirchhoff condition (involving derivatives of the solution in all branches) differ from tangential conditions (which just involve tangential derivatives)?

(*viii*) What are the suitable assumptions on the Hamiltonians to get comparison?

(*ix*) Can we identify the limit of the vanishing viscosity method?

Questions (*v*)-(*vi*) were investigated under different assumptions in Schieborn [152], Camilli and Marchi [55], Achdou, Camilli, Cutrì and Tchou [3], Schieborn and Camilli [153], Imbert, Monneau and Zidani [115], Imbert and Monneau [113] for 1-dimensional branches and Achdou, Oudet and Tchou [4, 6], Imbert and Monneau [114] for all dimensions; while Graber, Hermosilla and Zidani [103] consider the case of discontinuous solutions. The most general comparison result (with some restrictions anyway) is the one of Lions and Souganidis [137, 138] which is valid with very few, natural assumptions on the Hamiltonians, and not only in the case of Kirchhoff conditions but also for general junction conditions. It allows to answer in full generality to question (*ix*) which is also investigated in Camilli, Marchi and Schieborn [56].

In fact, Lions and Souganidis use a notion of solution which we call in this book "junction viscosity solution" and which is rather close to the classical notion of viscosity solutions; the only difference which is imposed by the network framework is the space of test-functions but this is a common feature of all the notions of solution in

this context. Because of this similarity, the half-relaxed limits' method extends without any difficulty and, taking into account the very general ideas of their comparison result, almost all the above questions seem to be solved by this notion of solution.

It remains anyway two questions: on one hand, despite of its generality, the comparison result of Lions and Souganidis requires in higher dimensions some unnatural hypotheses and, on the other hand, this result is originally proved in [137, 138] for Kirchhoff type junction conditions which is not the most natural conditions for control problems but which appear when studying the convergence of the vanishing viscosity method. Hence a very concrete question is: is it possible in the case of convex or concave Hamiltonians to give formulas of representation for such problems with Kirchhoff type junction conditions? To answer this question, it seems clear that one has to investigate the connections between Kirchhoff type junction conditions and "flux-limited conditions" in the terminology of Imbert and Monneau [113, 114] which are the natural junction conditions for control problems.

The extensive study of "flux-limited conditions" by Imbert and Monneau [113, 114] uses the notion of "flux-limited solutions": contrarily to the notion of "junction viscosity solution", this notion is less general and requires quasi-convex Hamiltonians on each branch of the network. It has also the defect to lead to a rather complicated (and limited) stability result. But it perfectly fits with control problems and the comparison result is proved under very natural and general assumptions.

In this book, we completely describe these two notions of solutions and theirs properties but we also show the connections between general Kirchhoff conditions and flux-limited conditions in the quasi-convex case, allowing the complete identification of the vanishing viscosity limit.

How this book was written; key points

In this book, our aim is to consider various problems with different types of discontinuities and to describe the different approaches to treat them. Thinking about all the common points that can be found in the works mentioned above, we have decided to dedicate an entire part to the "basic results", which are common bricks, used very often under perhaps slightly different forms. This organisation has the advantage to lighten the presentation of the main results and their proofs, but this clearly creates a rather technical—and perhaps difficult to read—first part. Anyway we think that collecting some classical ideas, revisited in sometimes unusual ways presents sufficient advantages to accept this flaw.

Then, the first problems we address are "simple" co-dimension 1 discontinuities (a

discontinuity along an hyperplane or an hypersurface, for example) in the whole space $\mathbb{R}^N$. For these problems, we provide in Part II and III a full description of the "classical approach" and the connections with the "network approach" with different comparison proofs (the Lions-Souganidis one and the Barles-Briani-Chasseigne-Imbert one). We also analyze their advantages and disadvantages.

Main results of these parts are

(i) Identification of the minimal viscosity supersolution and maximal viscosity subsolution with explicit controls formulas. Furthermore, we provide an easy-to-check condition on the Hamiltonians ensuring that these minimal supersolution and maximal subsolution are equal, i.e. that there is a unique viscosity solution. This condition turns out to be useful in different applications.

(ii) For the different notions of solutions in the "network approach", we provide comparison and stability results, and a complete analysis of the connections between these different types of solutions (classical Ishii viscosity solutions, flux-limited solutions and junction viscosity solutions).

(iii) Several versions of the convergence of the vanishing viscosity method, for convex and non-convex Hamiltonians, each of them relying on a particular notion of solution; the most complete form uses all the results of (i) and (ii) above, in particular the links between the different notions of solutions in the "network approach".

The reader who wants to have a quick idea of all these results can take a look at Section 18. This section gives a flavor of them in a simple framework, the Hamilton-Jacobi analogue of 1-d scalar conservation laws with a discontinuous flux.

In these parts, we make a point to emphasize the following fundamental issues which play a key role in all the other parts. They also seem to be the key assumptions to be used in problems with discontinuities in order to have a continuous solution and a comparison result between sub and supersolutions

(NC) Normal controlability (or coercivity): in the framework of control problems, this property means that one should be able either to reach the interface (here the codimension 1 manifold where we have the discontinuity) or, on the contrary, to leave it in any directions, in order to take advantage of a more favorable situation in terms of cost. Such assumption ensures that this potentially favorable situation is "seen". This is translated into a coercivity-type assumption in the normal coordinates on the associated Hamiltonian.

(TC) Tangential continuity : with respect to the coordinates of the interface, the Hamiltonians have to satisfy standard comparison under $\mathbb{R}^N$-type assumptions.

We insist on the fact that these assumptions will be used for ANY type of results: comparison, but also stability and connections with control. These are really key assumptions and we will find them everywhere throughout the books, expressed in different ways.

The fourth part is devoted to the case of "stratified problems" in the whole space $\mathbb{R}^N$, i.e. to the case where discontinuities of any codimensions can appear, provided they form a Whitney stratification. In this part, we describe the extension of the work by Bressan and Hong [53] obtained in [35], with some extensions and applications.

We point out anyway two differences with [35]: first, we introduce a notion of *weak stratified subsolutions*[(1)]. While the *strong stratified subsolutions* used in [35] are Ishii subsolutions which satisfy additional subsolution inequalities on each manifold of the Whitney stratification, the weak stratified subsolutions are not assumed to satisfy the usual "global" Ishii subsolution inequality on the manifolds of codimension bigger than 1; hence they are not a priori Ishii subsolutions.

Despite being rather natural from the control point of view, this notion has the defect to allow "artificial values" on the discontinuities of the equation since no connection between these values on the different part of the Whitney stratification is imposed by the subsolutions inequalities. This is the second key difference with [35] where the "global" Ishii subsolution inequality and **(NC)** imply the "regularity of subsolutions", i.e. the fact that on a discontinuity, the values of a subsolution is the lim sup of its values outside this discontinuity. Hence strong stratified subsolutions are necessarely "regular" while it may not be the case for the weak ones.

This is a key point since, as it is remarked in [35], this property is playing a very important role for all the results. To summarize the content of Part IV, we can just say that all the results of [35] hold for regular weak stratified subsolutions and even that regular weak stratified subsolutions are strong stratified subsolutions under suitable assumptions, which are, to our point of view, the natural hypotheses to be used in this framework.

The main result is, of course, the comparison result between either regular weak or strong stratified subsolutions and supersolutions; it uses in a key way **(NC)**-**(TC)** but also standard reductions presented in Part I. Then we present different stability results where we improve the one given in [35] by taking into account changes in the structure of discontinuities: indeed we handle cases where some discontinuities may

[(1)] The situation for supersolutions is simpler since stratified supersolutions are just Ishii supersolutions.

either disappear or appear. It is worth remarking that the notion of weak stratified subsolutions has the advantage to simplify the proofs of these stability results.

Finally, we provide conditions under which classical viscosity (sub)solutions are stratified (sub)solutions. Under these conditions, classical viscosity solutions and stratified solutions are the same, which allows to treat in a rather precise way some applications. This applies in particular to KPP-type problems, even in rather complicated domains. Indeed, we can take advantage at the same time of the good properties of viscosity solutions in terms of stability, and the uniqueness of stratified solutions.

In Part V, we extend these ideas to consider "stratified problems" set in a domain with state-constraint boundary conditions. In fact, the stratified formulation allows to treat various boundary conditions (Dirichlet, Neumann, sliding boundary conditions,...) in the same framework. This, without assuming the boundary of the domain to be smooth, and with the possibility of mixing boundary conditions in some rather exotic way, as in the Tanker problem which is presented at the beginning of this part.

Roughly speaking, all the results of Part IV can be extended to this more general framework since, essentially, the boundary and the discontinuities in the boundary conditions just create new parts of the stratification and new associated Hamiltonians. Only the "one-sided feature" coming from the absence of exterior controlability at the boundary generates some technical difficulties. For instance, the regularity of subsolutions which comes automatically from **(NC)** in $\mathbb{R}^N$ is not so simple here. We show in this part how to reformulate classical boundary conditions and conclude with the non-standard example of the Tanker problem.

Different approaches for control problems in stratified frameworks, more in the spirit of Bressan & Hong have been developed by Hermosilla, Wolenski and Zidani [111] for Mayer and Minimum Time problems, Hermosilla and Zidani [112] for classical state-constraint problems, Hermosilla, Vinter and Zidani [110] for (very general) state-constraint problems (including a network part).

A final remark.

We conclude this introduction by a remark on "how to read this book?" vs "how not to read this book?".

As we already mentioned it above, we have decided to start by an entire part (Part I) gathering basics results which are identified as the key bricks appearing in any type of problems involving Hamilton-Jacobi Bellman Equations and deterministic control problems. This part is unavoidably a "little bit technical" and admittedly hard to read without a serious motivation... Which we hope can be found in the next parts!

We have tried to draft all the proofs by emphasizing the role of the related key bricks but in order to be readable without knowing the details of these bricks: in that way, one can avoid reading the different independent sections of Part I before being completely convinced that it is necessary.

Part II is certainly the most unavoidable one since it describes all the challenges and potential solutions at hand in a rather simple context of a co-dimension 1 discontinuity. Yet the difficulty of this part is to extract a clear global vision and we try to provide our point of view in Section 19.

Stratified problems require a non-neglectable investment but we have tried to point out the main ideas to keep in mind and to start from the easiest case and before going towards the most sophisticated ones. We hope that the general treatment of singular boundary conditions in non-smooth domains will be a sufficient motivation for enduring all the difficulties! But also the applications of Chapter 23.

Chapter 2

Basic Continuous Framework and Classical Assumptions Revisited

Viscosity solutions' theory relies on two types of key results: comparison results and stability results. If the "half-relaxed limits" method provides stability in a very general discontinuous framework where both solutions and Hamiltonians may be discontinuous (see Section 3.1), the situation is completely different for comparison. If most of the classical arguments for comparison can handle discontinuous sub and supersolutions, none of them can really handle discontinuous Hamiltonians, even in the simplest cases of discontinuities.

In this section, we first describe one of the most classical result in the continuous framework: our aim is to comment the assumptions and the methods of proof in order to introduce and partially justify the general approach we develop afterwards.

We only sketch the approach and results in this chapter since they are classical. We refer the reader to well-known references on this subject for more details: Lions [134], Bardi and Capuzzo-Dolcetta [13], Fleming and Soner [87], the CIME courses [12, 2] and Barles [31].

2.1 The value function and the associated pde

We consider a finite horizon control problem in $\mathbb{R}^N$ on the time interval $[0, T_f]$ for some $T_f > 0$, where, for $x \in \mathbb{R}^N$ and $t \in [0, T_f]$, the state of the system is described by the solution $X(\cdot)$ of the ordinary differential equation

$$\dot{X}(s) = b(X(s), t-s, \alpha(s)) \ , \ X(0) = x \in \mathbb{R}^N \ .$$

Here, $\alpha(\cdot) \in \mathcal{A} := L^\infty(0, T_f; A)$ is the control which takes values in the compact metric space A and b is a continuous function of all its variables. More precise assumptions are introduced later on.

For a finite horizon problem, the value function is classically defined by

$$U(x,t) = \inf_{\alpha(\cdot)\in\mathcal{A}} \left\{ \int_0^t l(X(s), t-s, \alpha(s)) \exp\left(- \int_0^s c(X(\tau), t-\tau, \alpha(\tau)) d\tau \right) \, \mathrm{d}s \right.$$
$$\left. + u_0(X(t)) \exp\left(- \int_0^t c(X(\tau), t-\tau, \alpha(\tau)) d\tau \right) \right\} ,$$

where l is the running cost, c is the discount factor and u_0 is the final cost. All these functions are assumed to be continuous on $\mathbb{R}^N \times [0, T_f] \times A$ (for l and c) and on $\mathbb{R}^N$ (for u_0) respectively.

The most classical framework use the following assumptions which will be refered below as $(\mathbf{H}^{\text{class.}}_{\mathbf{BA-CP}})$ for *Basic Assumptions on the Control Problem – Classical case*:

(*i*) The function $u_0 : \mathbb{R}^N \to \mathbb{R}$ is a bounded, uniformly continuous function.

(*ii*) The functions b, c, l are bounded, uniformly continuous on $\mathbb{R}^N \times [0, T_f] \times A$.

(*iii*) There exists a constant $C_1 > 0$ such that, for any $x, y \in \mathbb{R}^N$, $t \in [0, T_f]$, $\alpha \in A$, we have

$$|b(x,t,\alpha) - b(y,t,\alpha)| \ \leq C_1 |x - y| \ .$$

One of the most classical results connecting the value function with the associated Hamilton-Jacobi-Bellman Equation is the

Theorem 2.1.1 *If Assumption* $(\mathbf{H}^{\text{class.}}_{\mathbf{BA-CP}})$ *holds, the value function* U *is continuous on* $\mathbb{R}^N \times [0, T_f]$ *and is the unique viscosity solution of*

$$\begin{cases} u_t + H(x, t, u, D_x u) = 0 \quad \text{in } \mathbb{R}^N \times (0, T_f) \ , & (2.1) \\ u(x, 0) = u_0(x) \quad \text{in } \mathbb{R}^N \ . & (2.2) \end{cases}$$

where

$$H(x,t,r,p) := \sup_{\alpha\in A} \left\{ -b(x,t,\alpha) \cdot p + c(x,t,\alpha) r - l(x,t,\alpha) \right\} \ .$$

In Theorem 2.1.1, we have used the notation u_t for the time derivative of the function $(x,t) \mapsto u(x,t)$ and $D_x u$ for its derivatives with respect to the space variable x. These notations will be used throughout this book.

Sketch of Proof — Of course, there exists a lot of variants of this result with different assumptions on b, c, l and u_0 but, with technical variants, the proofs use mainly the same steps.

(a) The first one consists in proving that U is continuous and satisfies a Dynamic Programming Principle (DPP in short), *i.e.* that for any $0 < h < t$,

$$U(x,t) = \inf_{\alpha(\cdot)\in\mathcal{A}} \left\{ \int_0^h l(X(s), t-s, \alpha(s)) \exp\left(- \int_0^s c(X(\tau), t-\tau, \alpha(\tau)) d\tau \right) \mathrm{d}s \right.$$
$$\left. + U(X(h), t-h) \exp\left(- \int_0^h c(X(\tau), t-\tau, \alpha(\tau)) d\tau \right) \right\}.$$

This is obtained by using the very definition of U and taking suitable controls.

(b) If U is smooth, using the DPP on $[0,h]$ and performing expansions of the different terms with respect to the variable h, we deduce that U is a classical solution of (2.1)-(2.2). If U is not smooth, this has to be done with test-functions and we obtain that U is a viscosity solution of the problem.

(c) Finally one proves a comparison result for (2.1)-(2.2), which shows that U is the unique viscosity solution of (2.1)-(2.2).

Q.E.D.

We point out that, in this sketch of proof, the continuity (or uniform continuity) of U is not as crucial as it seems to be. Of course continuity can be obtained directly by working on the definition of U in this framework. But one may also show that U is a discontinuous viscosity solution (see Section 3.1) and deduce continuity from the comparison result. We insist on the fact that in this classical framework, people are mainly interested in cases where U is continuous and therefore in assumptions ensuring this continuity.

Concerning Assumption $(\mathbf{H}^{\text{class.}}_{\mathbf{BA-CP}})$, it is clear that (iii) together with (ii) ensure that for any choice of control $\alpha(\cdot)$ there is a well-defined trajectory, by the Cauchy-Lipschitz Theorem. Moreover, this trajectory $X(\cdot)$ exists for all times, thanks to the boundedness of b. On the other hand, the boundedness of l, c allows to show that $U(x,t)$ is well-defined, bounded in $\mathbb{R}^N \times [0,T_f]$ and even uniformly continuous there. Therefore we get all the necessary information at the control level.

But Assumption $(\mathbf{H}^{\text{class.}}_{\mathbf{BA-CP}})$ plays also a key role at the pde level, in view of the comparison result: indeed, it implies that the Hamiltonian H satisfies the following property: for any $R \geq 1$

There exists $M > 0$, C_1 *and a modulus of continuity* $m : [0,+\infty) \to [0,+\infty)$ *such*

that, for any $x, y \in \mathbb{R}^N$, $t, s \in [0, T_f]$, $-R \le r_1 \le r_2 \le R \in \mathbb{R}$ *and* $p, q \in \mathbb{R}^N$

$$\begin{cases} |H(x,t,r_1,p) - H(y,s,r_1,p)| \le (C_1|x-y| + m(|t-s|))\,|p| + m\left((|x-y| + |t-s|)R\right) , \\ H(x,t,r_2,p) - H(x,t,r_1,p) \ge -M(r_2 - r_1) , \\ |H(x,t,r_1,p) - H(x,t,r_1,q)| \le M|p-q| . \end{cases}$$

Of course, these properties are satisfied with $M = \max(||b||_\infty, ||c||_\infty, ||l||_\infty)$ and m is the modulus of uniform continuity of b, c, l.

2.2 Important remarks on the comparison proof

We want to insist on several points here, and highlight several remarks that are important to understand the methods and strategies we develop throughout this book.

On proper Hamiltonians – in the process of performing comparison between a subsolution u and a supersolution v (See Section 3.1), the initial step is to reduce the proof to the case when $r \mapsto H(x,t,r,p)$ is increasing (or even non-decreasing) for any x, t, p. Such Hamiltonians are often called "proper".

This can be done through the classical change of unknown functions

$$u(x,t) \to \tilde{u}(x,t) := u(x,t)\exp(-Kt) ,$$

and the same for $v \to \tilde{v}$, for some $K \ge M$. The Hamiltonian H is changed into

$$\tilde{H}(x,t,r,p) := \sup_{\alpha \in A} \left\{-b(x,t,\alpha)\exp(-Kt)\cdot p + [c(x,t,\alpha) + K]r - l(x,t,\alpha)\exp(-Kt)\right\} .$$

This allows to reduce to the case where $c(x,t,\alpha) \ge 0$ for any x, t, α, or even ≥ 1.

- ⋆ *We will always assume in this book that, one way or the other, we can reduce to the case when* $c \ge 0$.

On the x **and** t**-dependence** – the second point we want to emphasize is the t-dependence of b. It is well-know that, in the comparison proof, the term

$$\mathcal{Q} := (C_1|x-y| + m(|t-s|))\,|p|$$

is playing a key role. In order to handle the difference in the behavior of b in x and t, one has to perform a proof with a "doubling of variable" technique which is different in x and t. Namely we have to consider the function

$$(x,t,y,s) \mapsto \tilde{u}(x,t) - \tilde{v}(y,s) - \frac{|x-y|^2}{\varepsilon^2} - \frac{|t-s|^2}{\beta^2} - \eta(|x|^2+|y|^2)\ ,$$

where $0 < \beta \ll \varepsilon \ll 1$ and $0 < \eta \ll 1$. We recall that the η-term ensures that this function achieves its maximum while the ε, β-terms ensure (x,t) is close to (y,s). Therefore the maximum of this function is close to $\sup_{\mathbb{R}^N}(\tilde{u}-\tilde{v})$.

The idea behind this different doubling in x and t is the following: the proof requires a quantity similar to $\mathcal{Q}$ above to be small. Now, since $|p|$ behaves like $o(1)\varepsilon^{-1}$, while $|x-y|$ is like $o(1)\varepsilon$ and $|t-s|$ like $o(1)\beta$, the product $C_1|x-y||p|$ is indeed small. But in order to ensure that the product $m(|t-s|)|p|$ is also small, we need to choose β small enough compared to ε.

⋆ *In this book, we want to handle cases when b, c, l can be discontinuous on submanifolds of $\mathbb{R}^N \times [0,T_f]$. From a technical point of view, one quickly realizes that the x and t variables often play a similar role in this framework. For this reason, our assumptions on the behavior of b, c, l or H w.r.t. x and t will essentially be the same. In particular, we will assume that b is also Lipschitz continuous in t. This unnatural hypothesis simplifies the proofs but we indicate in Section 24.1 how it can be removed at the expense of more technicalities.*

On localization arguments – last but not least, this classical comparison proof does not use a real "localization" procedure. Of course, the role of the $-\eta(|x|^2+|y|^2)$-term is to ensure that the function associated to the "doubling of variable" achieves its maximum. However, the way to play with the parameters, letting first η tend to 0 and then sending β and ε to zero afterwards implies that these maximum points do not remain a priori bounded.

⋆ *In all the arguments in the book, we will use in a more central way the Lipschitz continuity of H in p in order to have a more local comparison proof. We systematically develop this point of view in Section 3.2.*

2.3 Basic assumptions

The previous remarks lead us to replace $(\mathbf{H}^{\text{class.}}_{\mathbf{BA-CP}})$ by the following basic (yet less classical) set of assumptions on the control problem:

($\mathbf{H_{BA-CP}}$) *Basic Assumptions on the Control Problem*:

(*i*) The function $u_0 : \mathbb{R}^N \to \mathbb{R}$ is a bounded, continuous function.

(*ii*) The functions b, c, l are bounded, continuous functions on $\mathbb{R}^N \times [0, T_f] \times A$ and the sets $(b, c, l)(x, t, A)$ are convex compact subsets of $\mathbb{R}^{N+2}$ for any $x \in \mathbb{R}^N$, $t \in [0, T_f]$ [(1)].

(*iii*) For any ball $B \subset \mathbb{R}^N$, there exists a constant $C_1(B) > 0$ such that, for any $x, y \in \mathbb{R}^N$, $t \in [0, T_f]$, $\alpha \in A$, we have

$$|b(x, t, \alpha) - b(y, s, \alpha)| \ \leq C_1(B) \left(|x - y| + |t - s|\right) \ .$$

We will explain in Section 24.1 how to handle a more general dependence in time when the framework allows it. In terms of equations and Hamiltonians, and although the following assumption is not completely equivalent to ($\mathbf{H_{BA-CP}}$), we will use the

($\mathbf{H_{BA-HJ}}$) *Basic Assumptions on the Hamilton-Jacobi equation*:

There exists a constant $C_2 > 0$ and, for any ball $B \subset \mathbb{R}^N \times [0, T_f]$, for any $R > 0$, there exists constants $C_1(B, R) > 0, \gamma(R) \in \mathbb{R}$ and a modulus of continuity $m(B, R) : [0, +\infty) \to [0, +\infty)$ such that, for any $x, y \in B$, $t, s \in [0, T_f]$, $-R \leq r_1 \leq r_2 \leq R$ and $p, q \in \mathbb{R}^N$

$$|H(x, t, r_1, p) - H(y, s, r_1, p)| \leq C_1(B, R)[|x-y|+|t-s|]|p|+m(B, R)(|x-y|+|t-s|) \ ,$$

$$|H(x, t, r_1, p) - H(x, t, r_1, q)| \leq C_2|p - q| \ ,$$

$$H(x, t, r_2, p) - H(x, t, r_1, p) \geq \gamma(R)(r_2 - r_1) \ .$$

In the next part "Tools", we introduce the key ingredients which allow to pass from the above standard framework to the discontinuous one; they are concerned with

a. **Hamilton-Jacobi Equations:** we recall the notion of viscosity solutions and we revisit the comparison proof in order to have an easier generalization to the discontinuous case. We immediately point out that the regularization of sub and supersolutions by sup or inf-convolutions will play a more important role in the discontinuous setting than in the continuous one.

[(1)] The last part of this assumption which is not a loss of generality will be used for the connections with the approach by differential inclusions.

b. **Control problems:** the discontinuous framework leads to introduce **Differential inclusions** in order to define properly the dynamic, discount and cost when b, c, l are discontinuous. We provide classical and less classical results on the DPP in this setting.

c. **Stratifications:** we describe the notion of Whitney stratification which is the notion used in Bressan and Hong [53] for the structure of the discontinuities of H or the (b, c, l).

Using these tools requires to make some basic assumptions for each of them, which are introduced progressively in this next part. Apart from $(\mathbf{H_{BA-HJ}})$ and $(\mathbf{H_{BA-CP}})$ that we introduced above, we will use $(\mathbf{H_{BCL}})$ and $(\mathbf{H_{ST}})$ respectively for the Differential Inclusion and the Stratification. We have also compiled the various assumptions in this book in an appendix for the reader's convenience.

Part I

A Toolbox for Discontinuous Hamilton-Jacobi Equations and Control Problems

Chapter 3

Pde Tools

3.1 Discontinuous viscosity solutions for equations with discontinuities, "half-relaxed limits" method

In this section, we recall the classical definition of discontinuous viscosity solutions introduced by Ishii[118] for equations which present discontinuities. We have chosen to present it in the first-order framework since, in this book, we are mainly interested in Hamilton-Jacobi Equations but it extends without major changes to the case of fully nonlinear elliptic and parabolic pdes. We refer to the Users' guide of Crandall, Ishii and Lions [67], the books of Bardi and Capuzzo-Dolcetta [13] and Fleming and Soner [87] and the CIME courses [12, 2] for more detailed presentations of the notion of viscosity solutions in this more general setting.

We (unavoidably) complement this definition by the description of the discontinuous stability result, often called "Half-Relaxed Limits Method", being clearly needed when dealing with discontinuities. We recall that it allows passage to the limit in fully nonlinear elliptic and parabolic pdes with just an L^∞–bound on the solutions. The "Half-Relaxed Limits Method" was introduced by Perthame and the first author in [22] and developed in a series of works [23, 24]. One of its first striking consequences was the "Perron's method" of Ishii [156], proving the existence of viscosity solutions for a very large class of first- and second-order equations (see also the above references for a complete presentation).

The definition of viscosity solutions uses the upper semicontinuous (u.s.c.) envelope and lower semicontinuous (l.s.c.) envelope of both the (sub and super) solutions and of the Hamiltonians and we introduce the following notations: if $f : A \subset \mathbb{R}^p \to \mathbb{R}$ is a

locally bounded function (possibly discontinuous), we denote by f^* its u.s.c. envelope

$$f^*(X) = \limsup_{\tilde{X}\to X} \; f(\tilde{X}) \quad \text{for } X \in A\,,$$

and by f_* its l.s.c. envelope

$$f_*(X) = \liminf_{\tilde{X}\to X} \; f(\tilde{X}) \quad \text{for } X \in A\,.$$

Throughout this section, we use $X \in \mathbb{R}^N$ as the generic variable to cover both the stationary and evolution cases where respectively, $X = x \in \mathbb{R}^n$ or $X = (x,t) \in \mathbb{R}^n \times \mathbb{R}$.

3.1.1 Discontinuous viscosity solutions

We consider a generic Hamiltonian $\mathbb{G} : \overline{\mathcal{O}} \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ where $\mathcal{O}$ is an open subset of $\mathbb{R}^N$ and $\overline{\mathcal{O}}$ denotes its closure. We just assume that $\mathbb{G}$ is a locally bounded function which is defined pointwise.

The definition of viscosity sub and supersolution is the following

Definition 3.1.1 (Discontinuous Viscosity Solutions) *A locally bounded function $u : \overline{\mathcal{O}} \to \mathbb{R}$ is a viscosity subsolution of the equation*

$$\mathbb{G}(X, u, Du) = 0 \qquad \text{on } \overline{\mathcal{O}} \tag{3.1}$$

if, for any $\varphi \in C^1(\overline{\mathcal{O}})$, at a maximum point $X_0 \in \overline{\mathcal{O}}$ of $u^* - \varphi$, one has

$$\mathbb{G}_*(X_0, u^*(X_0), D\varphi(X_0)) \le 0\,.$$

A locally bounded function $v : \overline{\mathcal{O}} \to \mathbb{R}$ is a viscosity supersolution of Equation (3.1) if, for any $\varphi \in C^1(\overline{\mathcal{O}})$, at a minimum point $X_0 \in \overline{\mathcal{O}}$ of $v_* - \varphi$, one has

$$\mathbb{G}^*(X_0, v_*(X_0), D\varphi(X_0)) \ge 0\,.$$

A (discontinuous) solution is a function which is both viscosity sub and supersolution of the equation.

Several classical remarks on this definition:

(i) In general, the notion of subsolution is given for u.s.c. functions while the notion of super-solution is given for l.s.c. functions: this may appear natural when looking at the above definition where just u^* and v_* play a role and actually we can reformulate

the above definition for general functions as: u is a subsolution if and only if the u.s.c. function u^* is a subsolution and v is a supersolution if and only if the l.s.c. function v_* is a supersolution. The interest of this more general definition comes from the applications, for example to control problems, where we face functions which are a priori neither u.s.c. nor l.s.c. and still we wish to prove that they are sub and supersolution of some equations. Therefore such a formulation is needed. But when we will have to give a result which holds for subsolutions (or supersolutions), we will assume the subsolution to be u.s.c. (or the supersolution to be l.s.c.) in order to lighten the notations in the statement.

(*ii*) If the space of "test-functions" φ which is here $C^1(\overline{\mathcal{O}})$ is changed into $C^2(\overline{\mathcal{O}})$, $C^k(\overline{\mathcal{O}})$ for any $k > 1$ or $C^\infty(\overline{\mathcal{O}})$, we obtain an equivalent definition. Then, for a classical stationary equation (say in $\mathbb{R}^n$) like

$$H(x, u, Du) = 0 \quad \text{in } \mathbb{R}^n \,,$$

the variable X is just x, $N = n$ and Du stand for the usual gradient of u in $\mathbb{R}^n$. But this framework also contains the case of evolution equations

$$u_t + H(x, t, u, D_x u) = 0 \quad \text{in } \mathbb{R}^n \times (0, T_f) \,,$$

where $X = (x, t) \in \mathbb{R}^n \times (0, T_f)$, $N = n + 1$ and $Du = (D_x u, u_t)$ where u_t denotes the time-derivative of u and $D_x u$ is the derivative with respect to the space variables x, and the Hamiltonian reads

$$G(X, r, P) = p_t + H(x, t, r, p_x) \,,$$

for any $(x, t) \in \mathbb{R}^n \times (0, T_f)$, $r \in \mathbb{R}$ and $P = (p_x, p_t)$.

(*iii*) This definition is a little bit strange since the equation is set on a closed subset, a very unusual situation. There are two reasons for introducing it this way: the first one is to unify equation and boundary condition in the same formulation as we will see below. With such a general formulation, we avoid to have a different results for each type of boundary conditions. The second one, which provides also a justification of the "boundary conditions in the viscosity sense" is the convergence result we present in the next section.

To be more specific, let us consider the problem

$$\begin{cases} F(x, u, Du) = 0 & \text{in } \mathcal{O} \subset \mathbb{R}^n, \\ L(x, u, Du) = 0 & \text{on } \partial\mathcal{O}, \end{cases}$$

where F, L are given continuous functions. If we introduce the function G defined by

$$\mathbb{G}(x, r, p) = \begin{cases} F(x, r, p) & \text{if } x \in \mathcal{O}, \\ L(x, r, p) & \text{if } x \in \partial\mathcal{O}. \end{cases}$$

we can just rewrite the above problem as

$$\mathbb{G}(x,u,Du)=0 \quad \text{on } \overline{\mathcal{O}}\,,$$

where the first important remark is that $\mathbb{G}$ is a priori a discontinuous Hamiltonian. Hence, even if we assume F and L to be continuous, we face a typical example which we want to treat in this book!

The interpretation of this new problem can be done by setting the equation in $\overline{\mathcal{O}}$ instead of $\mathcal{O}$. Applying blindly the definition, we see that u is a subsolution if $\mathbb{G}_*(x,u^*,Du^*)\le 0$ on $\overline{\mathcal{O}}$, i.e. if

$$\begin{cases} F(x,u^*,Du^*)\le 0 & \text{in } \mathcal{O}\,,\\ \min(F(x,u^*,Du),L(x,u^*,Du^*))\le 0 & \text{on } \partial\mathcal{O}\,, \end{cases}$$

while v is a supersolution if $\mathbb{G}^*(x,v_*,Dv_*)\ge 0$ on $\overline{\mathcal{O}}$, *i.e.* if

$$\begin{cases} F(x,v_*,Dv_*)\ge 0 & \text{in } \mathcal{O}\,,\\ \max(F(x,v_*,Dv_*),L(x,v_*,Dv_*))\ge 0 & \text{on } \partial\mathcal{O}\,. \end{cases}$$

Indeed, we have just to compute $\mathbb{G}_*$ and $\mathbb{G}^*$ on $\overline{\mathcal{O}}$ and this is where the "min" and the "max" come from on $\partial\mathcal{O}$.

Of course, these properties have to be justified and this can be done by the discontinuous stability result of the next section which can be applied for example to the most classical way to solve the above problem, namely the vanishing viscosity method

$$\begin{cases} -\varepsilon\Delta u_\varepsilon + F(x,u_\varepsilon,Du_\varepsilon)=0 & \text{in } \mathcal{O}\,,\\ L(x,u_\varepsilon,Du_\varepsilon)=0 & \text{on } \partial\mathcal{O}\,. \end{cases}$$

Indeed, by adding a $-\varepsilon\Delta$ term, we regularize the equation in the sense that one can expect to have more regular solutions for this approximate problem—typically in $C^2(\mathcal{O})\cap C^1(\overline{\mathcal{O}})$.

To complete this section, we turn to a key example: the case of a two half-spaces problem, which presents a discontinuity along an hyperplane. We use the following framework: in $\mathbb{R}^N$, we set $\Omega_1=\{x_N>0\}$, $\Omega_2=\{x_N<0\}$ and $\mathcal{H}=\{x_N=0\}$. We assume that we are given three continuous Hamiltonians, H_1 on $\overline{\Omega}_1$, H_2 on $\overline{\Omega}_2$ and H_0 on $\mathcal{H}$. Here, $X=(x,t)$ and let us introduce

$$\mathbb{G}(X,u,p):=\begin{cases} p_t+H_1(x,t,u,p_x) & \text{if } x\in\Omega_1\,,\\ p_t+H_2(x,t,u,p_x) & \text{if } x\in\Omega_2\,,\\ p_t+H_0(x,t,u,p_x) & \text{if } x\in\mathcal{H}\,. \end{cases}$$

Then solving $\mathbb{G}(X,u,Du)=0$ for $X=(x,t)\in\mathbb{R}^{N+1}$ means to solve the equations $u_t+H_i(x,t,u,Du)=0$ in each Ω_i $(i=1,2)$ with the "natural" conditions on $\mathcal{H}$ given by the Ishii's conditions for the sub and super-solutions, namely

$$\begin{cases} \min(u_t+H_1(x,t,u^*,Du^*),u_t+H_2(x,t,u^*,Du^*),u_t+H_0(x,t,u^*,Du^*))\le 0 & \text{on } \mathcal{H}\,,\\ \max(u_t+H_1(x,t,v_*,Dv_*),u_t+H_2(x,t,v_*,Dv_*),u_t+H_0(x,t,v_*,Dv_*))\ge 0 & \text{on } \mathcal{H}\,. \end{cases}$$

Remark 3.1.2 *We have decided to present the definition of viscosity solution on a closed space $\overline{\mathcal{O}}$ for the reasons we explained above. But we can define as well equations set in open subset of $\mathbb{R}^N$ (typically $\mathcal{O}$) or open subsets of $\overline{\mathcal{O}}$ (typically $\overline{\mathcal{O}}\cap B(X,r)$ for some $X\in\overline{\mathcal{O}}$ and $r>0$). The definition is readily the same, considering local maximum points of $u^*-\varphi$ or minimum points of $v_*-\varphi$ which are in $\mathcal{O}$ or $\overline{\mathcal{O}}\cap B(X,r)$.*

We end this section with a classical "trick" that is used in many stability results like the half-relaxed limit method, which is detailed in the next section.

Lemma 3.1.3 *When testing the sub or supersolution condition for an equation of the type $\mathbb{G}(X,u,Du)=0$, if $u-\varphi$ reaches a local extremum at X_0, we can always assume that X_0 is a strict maximum or minimum point, without changing $D\varphi(X_0)$.*

We point out that a immediate consequence of this lemma is that we have an equivalent definition of viscosity sub and supersolutions by considering only strict local maximum/minimum points.

Proof — In the case of a maximum point, we just need to replace φ by $\psi(X):=\varphi(X)-c|X-X_0|^2$ where $c>0$: it is clear that $u-\psi$ has a strict maximum at X_0 and moreover since $D\varphi(X_0)=D\psi(X_0)$, the subsolution condition still takes the form

$$\mathbb{G}(X_0,u(X_0),D\varphi(X_0))\le 0\,.$$

Of course, the same argument applies for the supersolution condition by adding this time $c|X-X_0|^2$ to φ.

Q.E.D.

Notice that the same trick works for second-order equations, but in order to keep the second-order derivatives unchanged we have to use $\varphi(X)\pm c|X-X_0|^4$.

3.1.2 The half-relaxed limits method

In order to state it we use the following notations: if $A\subset\mathbb{R}^p$ and if $(f_\varepsilon)_\varepsilon$ is a sequence of uniformly locally bounded real-valued functions defined on A, the half-relaxed

limits of $(f_\varepsilon)_\varepsilon$ are defined, for any $X \in A$, by

$$\limsup{}^* f_\varepsilon(X) = \limsup_{\substack{Y\to X\\ \varepsilon\to 0}} f_\varepsilon(Y) \quad \text{and} \quad \liminf{}_* f_\varepsilon(X) = \liminf_{\substack{Y\to X\\ \varepsilon\to 0}} f_\varepsilon(Y)\,.$$

Theorem 3.1.4 *Assume that, for $\varepsilon > 0$, u_ε is a viscosity subsolution [resp. a supersolution] of the equation*

$$\mathbb{G}_\varepsilon(X, u_\varepsilon, Du_\varepsilon) = 0 \qquad \text{on } \overline{\mathcal{O}}\,,$$

where $(\mathbb{G}_\varepsilon)_\varepsilon$ is a sequence of uniformly locally bounded functions in $\overline{\mathcal{O}} \times \mathbb{R} \times \mathbb{R}^N$. If the functions u_ε are uniformly locally bounded on $\overline{\mathcal{O}}$, then $\overline{u} = \limsup^* u_\varepsilon$ [resp. $\underline{u} = \liminf_* u_\varepsilon$] is a subsolution [resp. a supersolution] of the equation

$$\underline{\mathbb{G}}(X, u, Du) = 0 \qquad \text{on } \overline{\mathcal{O}}\,,$$

where $\underline{\mathbb{G}} = \liminf_* \mathbb{G}_\varepsilon$. [resp. of the equation

$$\overline{\mathbb{G}}(X, u, Du) = 0 \qquad \text{on } \overline{\mathcal{O}}\,,$$

where $\overline{\mathbb{G}} = \limsup^* \mathbb{G}_\varepsilon$].

In order to compare them, we recall that the first stability result for viscosity solutions is given in the introductory article of Crandall and Lions [68]: it takes the form

Theorem 3.1.5 *Assume that, for $\varepsilon > 0$, $u_\varepsilon \in C(\mathcal{O})$ is a viscosity subsolution [resp. a supersolution] of the equation*

$$\mathbb{G}_\varepsilon(X, u_\varepsilon, Du_\varepsilon) = 0 \qquad \text{in } \mathcal{O}\,,$$

where $(\mathbb{G}_\varepsilon)_\varepsilon$ is a sequence of continuous functions in $\mathcal{O} \times \mathbb{R} \times \mathbb{R}^N$. If $u_\varepsilon \to u$ in $C(\mathcal{O})$ and if $\mathbb{G}_\varepsilon \to \mathbb{G}$ in $C(\mathcal{O} \times \mathbb{R} \times \mathbb{R}^N)$, then u is a subsolution [resp. a supersolution] of the equation

$$\mathbb{G}(X, u, Du) = 0 \qquad \text{in } \mathcal{O}\,.$$

We recall that the convergence in the space of continuous functions ($C(\mathcal{O})$ or $C(\mathcal{O} \times \mathbb{R} \times \mathbb{R}^N)$) is the local uniform convergence.

Theorem 3.1.5 is, in fact, a particular case of Theorem 3.1.4. Indeed, as the proof will show, the result of Theorem 3.1.4 remains valid if we replace $\overline{\mathcal{O}}$ by $\mathcal{O}$ and if u_ε and $\mathbb{G}_\varepsilon$ converge uniformly then $u = \overline{u} = \underline{u}$ and $\mathbb{G} = \overline{\mathbb{G}} = \underline{\mathbb{G}}$.

Hence Theorem 3.1.4 is more general when applied to either sub or supersolutions: its main interest is to allow the passage to the limit in the notion of sub and supersolutions with very weak assumptions on the solutions but also on the equations: only uniform local L^∞–bounds. In particular, phenomenas like *boundary layers* can be handled with such a result. This is a striking difference with Theorem 3.1.5 which, in practical uses, requires some compactness of the u_ε's in the space of continuous functions (typically some gradient bounds) in order to have a converging subsequence.

The counterpart is that we do not have a limit anymore, but two half-limits $\overline{u}$ and $\underline{u}$ which have to be connected in order to obtain a real convergence result. In fact, the complete **Half-Relaxed Limit Method** is performed as follows

1. Get a locally (or globally) uniform L^∞–bound for the $(u_\varepsilon)_\varepsilon$.

2. Apply the above discontinuous stability result.

3. The inequality $\underline{u} \leq \overline{u}$ on $\overline{\mathcal{O}}$ holds by definition.

4. To obtain the converse inequality, use a **Strong Comparison Result**, **(SCR)** in short, i.e a comparison result which is valid for *discontinuous* sub and supersolutions, which yields

$$\overline{u} \leq \underline{u} \quad \text{in} \quad \mathcal{O} \,(\text{or on} \quad \overline{\mathcal{O}}\,)\,.$$

5. From the **(SCR)**, we deduce that $\overline{u} = \underline{u}$ in $\mathcal{O}$ (or on $\overline{\mathcal{O}}$). Setting $u := \overline{u} = \underline{u}$, it follows that u is continuous (because $\overline{u}$ is u.s.c. and $\underline{u}$ is l.s.c.) and it is easy to show that, u is *the unique solution* of the limit equation, by using again the **(SCR)**.

6. Finally, we also get the convergence of u_ε to u in $C(\mathcal{O})$ (or in $C(\overline{\mathcal{O}})$) (see Lemma 3.1.7 below).

It is clear that, in this method, **(SCR)** play a central role and one of the main challenge in this book is to show how to obtain them in various contexts.

Now we give the **Proof of Theorem 3.1.4**. We do it only for the subsolution case, the supersolution one being analogous.

We first remark that $\limsup^* u_\varepsilon = \limsup^* u_\varepsilon^*$ and therefore changing u_ε in u_ε^*, we can assume without loss of generality that u_ε is u.s.c.. Recall also that by Lemma 3.1.3, we are always reduced to consider strict extremum points in viscosity inequalities testing. The proof is based on the

Lemma 3.1.6 *Let $(w_\varepsilon)_\varepsilon$ be a sequence of uniformly bounded u.s.c. functions on $\overline{\mathcal{O}}$ and $\overline{w} = \limsup^* w_\varepsilon$. If $X \in \overline{\mathcal{O}}$ is a strict local maximum point of $\overline{w}$ on $\overline{\mathcal{O}}$, there exists a subsequence $(w_{\varepsilon'})_{\varepsilon'}$ of $(w_\varepsilon)_\varepsilon$ and a sequence $(X_{\varepsilon'})_{\varepsilon'}$ of points in $\overline{\mathcal{O}}$ such that, for all ε', $X_{\varepsilon'}$ is a local maximum point of $w_{\varepsilon'}$ in $\overline{\mathcal{O}}$, the sequence $(X_{\varepsilon'})_{\varepsilon'}$ converges to X and $w_{\varepsilon'}(X_{\varepsilon'}) \to \overline{w}(X)$.*

We first prove Theorem 3.1.4 by using the lemma. Let $\varphi \in C^1(\overline{\mathcal{O}})$ and let $X \in \overline{\mathcal{O}}$ be a strict local maximum point de $\overline{u} - \varphi$. We apply Lemma 3.1.6 to $w_\varepsilon = u_\varepsilon - \varphi$ and $\overline{w} = \overline{u} - \varphi = \limsup^* (u_\varepsilon - \varphi)$. There exists a subsequence $(u_{\varepsilon'})_{\varepsilon'}$ and a sequence $(X_{\varepsilon'})_{\varepsilon'}$ such that, for all ε', $X_{\varepsilon'}$ is a local maximum point of $u_{\varepsilon'} - \varphi$ on $\overline{\mathcal{O}}$. But $u_{\varepsilon'}$ is a subsolution of the $\mathbb{G}_{\varepsilon'}$-equation, therefore

$$\mathbb{G}_{\varepsilon'}(X_{\varepsilon'}, u_{\varepsilon'}(X_{\varepsilon'}), D\varphi(X_{\varepsilon'})) \leq 0 \, .$$

Since $X_{\varepsilon'} \to X$ and since φ is smooth $D\varphi(X_{\varepsilon'}) \to D\varphi(X)$; but we have also $u_{\varepsilon'}(X_{\varepsilon'}) \to \overline{u}(X)$, therefore by definition of $\underline{\mathbb{G}}$

$$\underline{\mathbb{G}}(X, \overline{u}(X), D\varphi(X)) \leq \liminf \mathbb{G}_{\varepsilon'}(X_{\varepsilon'}, u_{\varepsilon'}(X_{\varepsilon'}), D\varphi(X_{\varepsilon'})) \, .$$

This immediately yields

$$\underline{\mathbb{G}}(X, \overline{u}(X), D\varphi(X)) \leq 0 \, ,$$

and the proof is complete.

Proof of Lemma 3.1.6 — Since X is a strict local maximum point of $\overline{w}$ on $\overline{\mathcal{O}}$, there exists $r > 0$ such that

$$\forall Y \in \overline{\mathcal{O}} \cap \overline{B}(X, r) \, , \quad \overline{w}(Y) \leq \overline{w}(X) \, ,$$

the inequality being strict for $Y \neq X$. But $\overline{\mathcal{O}} \cap \overline{B}(X, r)$ is compact and w_ε is u.s.c., therefore, for all $\varepsilon > 0$, there exists a maximum point X_ε of w_ε on $\overline{\mathcal{O}} \cap \overline{B}(X, r)$. In other words

$$\forall Y \in \overline{\mathcal{O}} \cap \overline{B}(X, r) \, , \quad w_\varepsilon(Y) \leq w_\varepsilon(X_\varepsilon) \, . \tag{3.2}$$

Now we take the lim sup as $Y \to X$ and $\varepsilon \to 0$: we obtain

$$\overline{w}(X) \leq \limsup_{\varepsilon \to 0} w_\varepsilon(X_\varepsilon) \, .$$

Next we consider the right-hand side of this inequality: extracting a subsequence denoted by ε', we have $\limsup_\varepsilon w_\varepsilon(X_\varepsilon) = \lim_{\varepsilon'} w_{\varepsilon'}(X_{\varepsilon'})$ and since $\overline{\mathcal{O}} \cap \overline{B}(X, r)$ is compact, we may also assume that $X_{\varepsilon'} \to \bar{X} \in \overline{\mathcal{O}} \cap \overline{B}(X, r)$. But using again the definition of the $\limsup^*$ at $\bar{X}$, we get

$$\overline{w}(X) \leq \limsup_{\varepsilon \to 0} w_\varepsilon(X_\varepsilon) = \lim_{\varepsilon' \to 0} w_{\varepsilon'}(X_{\varepsilon'}) \leq \overline{w}(\bar{X}) \, .$$

Since X is a strict maximum point of $\overline{w}$ in $\overline{\mathcal{O}} \cap \overline{B}(X,r)$ and that $\bar{X} \in \overline{\mathcal{O}} \cap \overline{B}(X,r)$, this inequality implies that $\bar{X} = X$ and that $w_{\varepsilon'}(X_{\varepsilon'}) \to \overline{w}(X)$, so that the proof is complete.

Q.E.D.

Controlling the liminf and limsup also implies local uniform convergence:

Lemma 3.1.7 *If $\mathcal{K}$ is a compact subset of $\overline{\mathcal{O}}$ and if $\overline{u} = \underline{u}$ on $\mathcal{K}$ then u_ε converges uniformly to the function $u := \overline{u} = \underline{u}$ on $\mathcal{K}$.*

Proof of Lemma 3.1.7 : Since $\overline{u} = \underline{u}$ on $\mathcal{K}$ and since $\overline{u}$ is u.s.c. and $\underline{u}$ is l.s.c. on $\overline{\mathcal{O}}$, u is continuous on $\mathcal{K}$. We first consider

$$M_\varepsilon = \sup_{\mathcal{K}} \left(u_\varepsilon^* - u\right) .$$

The function u_ε^* being u.s.c. and u being continuous, this supremum is in fact a maximum and is achived at a point X_ε. The sequence $(u_\varepsilon)_\varepsilon$ being locally uniformly bounded, the sequence $(M_\varepsilon)_\varepsilon$ is also bounded and, $\mathcal{K}$ being compact, we can extract subsequences such that $M_{\varepsilon'} \to \limsup_\varepsilon M_\varepsilon$ and $X_{\varepsilon'} \to \bar{X} \in \mathcal{K}$. But by the definition of the $\limsup^*$, $\limsup u_{\varepsilon'}^*(X_{\varepsilon'}) \leq \overline{u}(\bar{X})$ while we have also $u(X_{\varepsilon'}) \to u(\bar{X})$ by the continuity of u. We conclude that

$$\limsup_{\varepsilon \to 0} M_\varepsilon = \lim_{\varepsilon' \to 0} M_{\varepsilon'} = \lim_{\varepsilon' \to 0} \left(u_{\varepsilon'}^*(X_{\varepsilon'}) - u(X_{\varepsilon'})\right) \leq \overline{u}(\bar{X}) - u(\bar{X}) = 0 .$$

This part of the proof gives half of the uniform convergence, the other part being obtained analogously by considering $\tilde{M}_\varepsilon = \sup_{\mathcal{K}} \left(u - (u_\varepsilon)_*\right)$.

3.2 Strong comparison results: how to cook them?

In the previous section, we have seen that **(SCR)** are key tools which are needed to use the "Half-Relaxed Limit Method". We have used the terminology "strong" because such comparison results have to hold for discontinuous sub and supersolutions, which are only u.s.c. and l.s.c. respectively. From a technical point of view, it is easier to compare at least continuous sub and supersolutions and of course, some comparison results may even fail in the discontinuous framework. However, in this book we mainly prove **(SCR)** therefore the expression "comparison result" always refers to a *strong* one.

In general, a comparison result is a *global* inequality (i.e. on the whole domain) between sub and supersolutions. However, in the case of Hamilton-Jacobi Equations

with discontinuities it is far easier, if not necessary, to argue locally. This is why in this section we explain how to reduce the proof of *global* comparison results to the proof of *local* comparison results. We do not pretend this section to cover all cases but we tried to make it as general as we could.

3.2.1 Stationary equations

In this section we are in the situation where $X = x$ is the space variable in $\mathbb{R}^N$ or a subset of it, and no time variable is involved here. We consider a general equation

$$\mathbb{G}(x, u, Du) = 0 \quad \text{on } \mathcal{F}\,, \tag{3.3}$$

where $\mathcal{F}$ is a closed subset of $\mathbb{R}^N$ and $\mathbb{G}$ is a continuous or discontinuous function on $\mathcal{F} \times \mathbb{R} \times \mathbb{R}^N$.

We introduce the following notations: USC-Sub($\mathcal{F}$) is a subset of u.s.c. subsolutions of (3.3) while LSC-Sup($\mathcal{F}$) is a subset of l.s.c. supersolutions of (3.3). We prefer to remain a little bit vague on these subsets but the reader may have in mind that they are generally defined by some growth conditions at infinity if $\mathcal{F}$ is an unbounded subset of $\mathbb{R}^N$. In these definitions, we may replace below $\mathcal{F}$ by a subset (open or closed) of $\mathcal{F}$ and we use below the following notations

$$\mathcal{F}^{x,r} := B(x,r) \cap \mathcal{F} \quad \text{and} \quad \partial\mathcal{F}^{x,r} := \partial B(x,r) \cap \mathcal{F}\,.$$

Finally we denote by USC-Sub($\mathcal{F}^{x,r}$) [*resp.* LSC-Sup($\mathcal{F}^{x,r}$)] the set of u.s.c. [*resp.* l.s.c.] functions on $\overline{\mathcal{F}^{x,r}}$ which are subsolutions [*resp.* supersolutions] of $\mathbb{G} = 0$ in $\mathcal{F}^{x,r}$. Notice that, for these sub and supersolutions, no viscosity inequality is imposed on $\partial B(x,r)$.

By "global" and "local" comparison results we mean the following

(GCR)$^{\mathcal{F}}$ *Global Comparison Result in* $\mathcal{F}$:
For any $u \in$ USC-Sub($\mathcal{F}$), *for any* $v \in$ LSC-Sup($\mathcal{F}$), *we have* $u \le v$ *on* $\mathcal{F}$.

(LCR)$^{\mathcal{F}}$ *Local Comparison Result in* $\mathcal{F}$:
For any $x \in \mathcal{F}$, *there exists* $r > 0$ *such that, if* $u \in$ USC-Sub($\mathcal{F}^{x,r}$), $v \in$ LSC-Sup($\mathcal{F}^{x,r}$) *then*

$$\max_{\overline{\mathcal{F}^{x,r}}} (u - v)^+ \le \max_{\partial\mathcal{F}^{x,r}} (u - v)^+.$$

Writing the$(\mathbf{LCR})^{\mathcal{F}}$ with the $(...)^+$ inequality is quite standard, see for instance [102]. The meaning of this formulation is the following: either $u \leq v$ in $\overline{\mathcal{F}^{x,r}}$ and we are done locally speaking, or the maximum of $u - v$ is positive, but controlled by the values at the boundary.

In the rest of this section, we skip the reference to $\mathcal{F}$ in **(LCR)** and **(GCR)** since there is no ambiguity here. It is clear that proving **(LCR)** seems much easier because of the compactness of $\overline{\mathcal{F}^{x,r}}$ and the fact that r may depend on x: indeed, on one hand, the behavior at infinity of u and v does not play a role anymore, and, on the other hand, we only use local properties of $\mathbb{G}$; in particular, if $\mathbb{G}$ has discontinuities which form a stratification, we can use this localization to restrict to a flat stratification (cf Section 3.3).

Now we formulate two assumptions which allow to reduce **(GCR)** to **(LCR)**.

(**LOC1**): *If $\mathcal{F}$ is unbounded, for any $u \in$ USC-Sub$(\mathcal{F})$, for any $v \in$ LSC-Sup$(\mathcal{F})$, there exists a sequence $(u_\alpha)_{\alpha>0}$ of u.s.c. subsolutions of (3.3) such that $u_\alpha(x) - v(x) \to -\infty$ when $|x| \to +\infty$, $x \in \mathcal{F}$. Moreover, for any $x \in \mathcal{F}$, $u_\alpha(x) \to u(x)$ when $\alpha \to 0$.*

In the above assumption, we do not write that $u_\alpha \in$ USC-Sub$(\mathcal{F})$ because this is not the case in general: typically, USC-Sub$(\mathcal{F})$ may be the set of *bounded* subsolutions of (3.3) while u_α is not expected to be bounded.

In order to introduce the second localization hypothesis, let us define

$$\lfloor f \rfloor^x_{\partial \mathcal{F}^{x,r}} := f(x) - \max_{y \in \partial \mathcal{F}^{x,r}} f(y)\,,$$

which in some sense measures the variation of f between x and the boundary. Notice that since $\max(f+g) \leq \max(f) + \max(g)$, this operator enjoys the following property

$$\lfloor f \rfloor^x_{\partial \mathcal{F}^{x,r}} + \lfloor g \rfloor^x_{\partial \mathcal{F}^{x,r}} \leq \lfloor f + g \rfloor^x_{\partial \mathcal{F}^{x,r}}\,. \tag{3.4}$$

(**LOC2**): *For any $x \in \mathcal{F}$, $r > 0$, if $u \in$ USC-Sub$(\mathcal{F}^{x,r})$, there exists a sequence $(u^\delta)_{\delta>0}$ of functions in* USC-Sub$(\mathcal{F}^{x,r})$ *such that $\lfloor u^\delta - u \rfloor^x_{\partial \mathcal{F}^{x,r}} \geq \eta(\delta) > 0$ for any δ. Moreover, for any $y \in \mathcal{F}^{x,r}$, $u^\delta(y) \to u(y)$ when $\delta \to 0$.*

The role of (**LOC1**) and (**LOC2**) will be clear in the proof of the property "(**LCR**) implies (**GCR**)" below

- (**LOC1**) leads to a standard localization procedure: instead of having to prove the comparison in $\mathcal{F}$ which can be unbounded, it allows to do it only on a compact subset of $\mathcal{F}$. This has several advantages: first, we can consider maximum

points for the u.s.c. function $u_\alpha - v$ in such a compact subset, while this is not, in general, the case for $u - v$ in $\mathcal{F}$ since u, v can also be unbounded. But we can also have more general assumptions on $\mathbb{G}$ (cf. ($\mathbf{H_{BA-CP}}$) and ($\mathbf{H_{BA-HJ}}$) in Section 2.3).

- (**LOC2**) is a technical assumption which allows to make sure that, in **(LCR)**, the max is not attained at the boundary by replacing u with another subsolution which has a greater variation between x and the boundary. This is a key point in the proof "**(LCR)** implies **(GCR)**".

Proposition 3.2.1 *Under Assumptions* (**LOC1**) *and* (**LOC2**)*, then* (**LCR**) *implies* (**GCR**).

Proof — Given $u \in$ USC-Sub($\mathcal{F}$) and $v \in$ LSC-Sup($\mathcal{F}$), we have to prove that $u \leq v$ on $\mathcal{F}$.

Instead of comparing u and v, we are going to compare u_α and v for u_α given by (**LOC1**) and then to let α tend to 0. Arguing in that way and droping the α for simplifying the notations means that we can assume without loss of generality that $u(x) - v(x) \to -\infty$ when $|x| \to +\infty$, $x \in \mathcal{F}$ and therefore we can consider $M := \max_{\mathcal{F}}(u - v)$ and we argue by contradiction, assuming that $M > 0$.

Since $\mathcal{F}$ is closed, $u - v$ is u.s.c. and tends to $-\infty$ at infinity, this function achieves its maximum at some point $x \in \mathcal{F}$. Considering $r > 0$ for which **(LCR)** holds, this means that $\lfloor u - v \rfloor^x_{\partial \mathcal{F}^{x,r}} \geq 0$.

Now we apply (**LOC2**). Since $u^\delta \in$ USC-Sub($\mathcal{F}^{x,r}$) and **(LCR)** holds, we get the following alternative

(i) either $u^\delta \leq v$ in $\overline{\mathcal{F}^{x,r}}$, but this cannot be the case for δ small enough since $u^\delta(x) - v(x) \to u(x) - v(x) > 0$;

(ii) or $\max_{\overline{\mathcal{F}^{x,r}}}(u^\delta - v) > 0$ and

$$\max_{\overline{\mathcal{F}^{x,r}}}(u^\delta - v) \leq \max_{\partial \mathcal{F}^{x,r}}(u^\delta - v).$$

In particular, this implies that $\lfloor u^\delta - v \rfloor^x_{\partial \mathcal{F}^{x,r}} \leq 0$. But using (3.4), we deduce that

$$\lfloor u - v \rfloor^x_{\partial \mathcal{F}^{x,r}} \leq \lfloor u^\delta - v \rfloor^x_{\partial \mathcal{F}^{x,r}} - \lfloor u^\delta - u \rfloor^x_{\partial \mathcal{F}^{x,r}} \leq -\eta(\delta) < 0\,,$$

which yields a contradiction. The conclusion is that M cannot be positive, hence $u_\alpha \leq v$ in $\mathcal{F}$ for any α and we get the **(GCR)** by sending $\alpha \to 0$.

Q.E.D.

Now an important key question is: how can we check (**LOC1**) and (**LOC2**)? We provide some typical examples.

The Lipschitz case — We assume that there exists a constant $c > 0$ such that the function $\mathbb{G}$ satisfies, for all $x \in F$, $z_1 \leq z_2$ and $p, q \in \mathbb{R}^N$

$$\mathbb{G}(x, z_1, p) - \mathbb{G}(x, z_2, p) \geq c^{-1}(z_1 - z_2) \ ,$$

$$|\mathbb{G}(x, z_1, p) - \mathbb{G}(x, z_1, q) \leq c|p - q| \ .$$

In the case when USC-Sub, LSC-Sup are sets of bounded sub or supersolutions then (**LOC1**) is satisfied with $u_\alpha(x) = u(x) - \alpha[(|x|^2 + 1)^{1/2} + c^2]$, indeed

$$\begin{aligned}\mathbb{G}(x, u_\alpha(x), Du_\alpha(x)) &\leq \mathbb{G}(x, u(x), Du(x)) - c^{-1}(\alpha[(|x|^2 + 1)^{1/2} + c^2)) + c\alpha\frac{|x|}{(|x|^2 + 1)^{1/2}} \ , \\ &\leq -c^{-1}(\alpha c^2)) + c\alpha = 0.\end{aligned}$$

Concerning (**LOC2**), for any $r > 0$ we can use

$$u^\delta(y) = u(y) - \delta(|y - x|^2 + k)$$

for some well-chosen constant k. Indeed

$$\begin{aligned}\mathbb{G}(y, u^\delta(y), Du^\delta(y)) &\leq \mathbb{G}(y, u(y), Du(y)) - c^{-1}\delta(|y - x|^2 + k) + 2c\delta|y - x| \ , \\ &\leq -\frac{\delta}{c}(|y - x|^2 + k - 2c^2|y - x|),\end{aligned}$$

and with the choice $k = c^4$ we get a subsolution since $X^2 - 2c^2X + c^4$ has no real roots. On the other hand, if $y \in \partial\mathcal{F}^{x,r}$

$$u^\delta(x) - u(x) = -\delta k \geq -\delta(|y - x|^2 + k) + \delta r^2 = u^\delta(y) - u(y) + \delta r^2 \ ,$$

so that $\lfloor u^\delta - u \rfloor^x_{\partial\mathcal{F}^{x,r}} \geq \eta(\delta) = \delta r^2$.

The convex case — Here we assume that $\mathbb{G}(x, z, p)$ is convex in z and p; the advantage is to avoid the restriction due to the Lipschitz continuity of $\mathbb{G}$ in p. For the localization (**LOC1**), we do not propose any explicit building of u_α since it strongly depends on (typically) the growth at infinity of the solutions we want to handle. But a classical construction is described by the following assumption which emphasizes not only the role of the growth of solutions (via ψ_1) but also of the convexity of $\mathbb{G}$ (via the way the u_α are built)

(Subsol1): For any $u \in$ USC-Sub($\mathcal{F}$), $v \in$ LSC-Sup($\mathcal{F}$), there exists an u.s.c. subsolution $\psi_1 : F \to \mathbb{R}$ such that for any $0 < \alpha < 1$, $u_\alpha(x) := (1-\alpha)u(x) + \alpha\psi_1(x)$ satisfies (**LOC1**).

Concerning (**LOC2**), we may also use a similar construction relying on the convexity

(Subsol2): For any $x \in \mathcal{F}$, there exists $r > 0$ and $\psi_2 \in$ USC-Sub($\mathcal{F}^{x,r}$) such that for any $0 < \delta < 1$, $u_\delta(y) = (1-\delta)u(y) + \delta\psi_2(y)$ satisfies (**LOC2**).

A typical candidate is $\psi_2^K(x) = -(K|y-x|^2 + k)$ for $k > 0$ large enough depending on K, but we need an additional argument.

We first replace the subsolution u by $\tilde{u}(y) = \max(u(y), \psi_2^1(y))$; since the maximum of two subsolutions is a subsolution, $\tilde{u}$ is a subsolution and, if k is large enough $\tilde{u}(x) = u(x)$. Therefore we can use $\tilde{u}$ instead of u in the proof of Proposition 3.2.1.

With this construction, $\tilde{u}$ satisfies **(Subsol2)** with ψ^K for K large enough. If $\psi_2^K(x) = -(K|y-x|^2 + \tilde{k})$, for any $y \in B(x,r)$ we have

$$\tilde{u}_\delta(y) - \tilde{u}(y) = \delta(\psi_2^K(y) - \tilde{u}(y)) \le \delta(\psi_2^K(y) - \psi_2^1(y)) = -\delta((K-1)r^2 + \tilde{k} - k)\ ,$$

while $\tilde{u}_\delta(x) - \tilde{u}(x) = \delta(\psi_2^K(x) - \tilde{u}(x)) = -\delta(\tilde{k} + \tilde{u}(x))$. Hence, provided k is chosen large enough (with respect to K), we get

$$\lfloor \tilde{u}_\delta - \tilde{u}(y) \rfloor^x_{\partial\mathcal{F}^{x,r}} \le -\delta((K-1)r^2 - k - \tilde{u}(x)) = -\eta(\delta) < 0\ .$$

3.2.2 The evolution case

There are some key differences in the evolution case due to the fact that the time-variable is playing a particular role since we are mainly solving a Cauchy problem, hence we have to reformulate the results with the "parabolic boundary". Using here the variable $X = (x,t)$, we first write the equation as

$$\mathbb{G}(x,t,u,(D_x u, u_t)) = 0 \quad \text{on } \mathcal{F} \times (0,T_f]\ , \tag{3.5}$$

where $\mathcal{F}$ is a closed subset of $\mathbb{R}^N$ and $\mathbb{G}$ is a continuous or discontinuous function on $\mathcal{F} \times [0,T_f] \times \mathbb{R} \times \mathbb{R}^{N+1}$.

This equation has to be complemented by an initial data at time $t = 0$ which can be of an usual form, namely

$$u(x,0) = u_0(x) \quad \text{on } \mathcal{F}\ , \tag{3.6}$$

where u_0 is a given function defined on $\mathcal{F}$, or this initial value of u can be obtained by solving an equation of the type

$$\mathbb{G}_{init}(x, 0, u(x,0), D_x u(x,0)) = 0 \quad \text{on } \mathcal{F}\,, \tag{3.7}$$

where $\mathbb{G}_{init}$ is a continuous or discontinuous function on $\mathcal{F} \times [0, T_f] \times \mathbb{R} \times \mathbb{R}^N$.

A strong comparison result for either (3.5)-(3.6) or (3.5)-(3.7) which is denoted below by **(GCR)**-evol can be defined in an analogous way as **(GCR)**: subsolutions (in a certain class of functions) are below supersolutions (in the same class of functions), USC-Sub($\mathcal{F}$) and LSC-Sup($\mathcal{F}$) being just replaced by USC-Sub($\mathcal{F} \times [0, T_f]$) and LSC-Sup($\mathcal{F} \times [0, T_f]$); we just point out that the initial data is included in the equation in this abstract formulation: for example, a subsolution u satisfies either

$$u(x,0) \le (u_0)^*(x) \quad \text{on } \mathcal{F}\,,$$

in the case of (3.6) or the function $x \mapsto u(x,0)$ satisfies

$$\mathbb{G}_{init}(x, 0, u(x,0), D_x u(x,0)) \le 0 \quad \text{on } \mathcal{F}\,,$$

in the viscosity sense, in the case of (3.7).

As it is even more clear in the case of (3.7), a comparison result in the evolution case consists in two steps

(i) proving that for any $u \in$ USC-Sub($\mathcal{F} \times [0, T_f]$) and $v \in$ LSC-Sup($\mathcal{F} \times [0, T_f]$),

$$u(x,0) \le v(x,0) \quad \text{on } \mathcal{F}, \tag{3.8}$$

(ii) showing that this inequality remains true for $t > 0$, i.e.

$$u(x,t) \le v(x,t) \quad \text{on } \mathcal{F} \times [0, T_f]\,.$$

Of course, in the case of (3.6), (3.8) is obvious if u_0 is a continuous function; but, in the case of (3.7), the proof of such inequality is nothing but a stationary **(GCR)** in $\mathcal{F} \times \{0\}$.

Therefore the main additional difficult consists in showing that Property (ii) holds true and we are going to explain now the analogue of the approach of the previous section *assuming that we have* (3.8).

To redefine **(LCR)**, we have to introduce, for $x \in \mathcal{F}$, $t \in (0, T_f]$, $r > 0$ and $0 < h < t$, the sets

$$Q^{x,t}_{r,h}[\mathcal{F}] := (B(x,r) \cap \mathcal{F}) \times (t-h, t]\,.$$

As in the stationary case, we introduce the set USC-Sub$(Q^{x,t}_{r,h}[\mathcal{F}])$, LSC-Sup$(Q^{x,t}_{r,h}[\mathcal{F}])$ of respectively u.s.c. subsolutions and l.s.c. supersolution of $G(x,t,u,(D_xu,u_t))=0$ in $Q^{x,t}_{r,h}[\mathcal{F}]$. This means that the viscosity inequalities holds in $Q^{x,t}_{r,h}[\mathcal{F}]$ and not necessarily on its closure, but these sub and supersolutions are u.s.c. or l.s.c. on $\overline{Q^{x,t}_{r,h}[\mathcal{F}]}$.

On the other hand, including $(B(x,r)\cap\mathcal{F})\times\{t\}$ in the set where the subsolution or supersolution inequalities hold is important in order to have the suitable comparison up to time t and we also refer to Proposition 3.2.4 for the connection between sub and supersolutions in $(B(x,r)\cap\mathcal{F})\times(t-h,t)$ and on $(B(x,r)\cap\mathcal{F})\times(t-h,t]$.

With this definition we have

(LCR)-*evol* : *For any $(x,t)\in\mathcal{F}\times(0,T_f]$, there exists $r>0$, $0<\bar h<t$ such that, for any $0<h<\bar h$, if $u\in$ USC-Sub$(Q^{x,t}_{r,\bar h}[\mathcal{F}])$, $v\in$ LSC-Sup$(Q^{x,t}_{r,\bar h}[\mathcal{F}])$,*

$$\max_{\overline{Q^{x,t}_{r,h}[\mathcal{F}]}}(u-v)^+\le\max_{\partial_pQ^{x,t}_{r,h}[\mathcal{F}]}(u-v)^+\,,$$

where $\partial_pQ^{x,t}_{r,h}[\mathcal{F}]$ stands for the parabolic boundary of $Q^{x,t}_{r,h}[\mathcal{F}]$, composed of a "lateral" part and an "initial" part as follows

$$\begin{aligned}\partial_pQ^{x,t}_{r,h}[\mathcal{F}]&=\Big\{(\partial B(x,r)\cap\mathcal{F})\times[t-h,t]\Big\}\bigcup\Big\{(\overline{B(x,r)}\cap\mathcal{F})\times\{t-h\}\Big\}\\&=:\partial_{\rm lat}Q\cup\partial_{\rm ini}Q\,.\end{aligned}$$

We point out that, in the sequel, we are going to play with the parameter h to obtain the comparison result. This explains the formulation of **(LCR)** where the local comparison result has to hold in $Q^{x,t}_{r,h}$ for any $0<h\le\bar h$.

The corresponding evolution versions of (**LOC1**) and (**LOC2**) are given by

(**LOC1**)-*evol*: *If $\mathcal{F}$ is unbounded, for any $u\in$ USC-Sub$(\mathcal{F}\times[0,T_f])$, for any $v\in$ LSC-Sup$(\mathcal{F}\times[0,T_f])$, there exists a sequence $(u_\alpha)_{\alpha>0}$ of u.s.c. subsolutions of (3.3) such that $u_\alpha(x,t)-v(x,t)\to-\infty$ when $|x|\to+\infty$, $x\in\mathcal{F}$, uniformly for $t\in[0,T_f]$. Moreover, for any $x\in\mathcal{F}$, $u_\alpha(x,t)\to u(x,t)$ when $\alpha\to0$.*

(**LOC2**)-*evol*: *For any $x\in\mathcal{F}$, if $u\in$ USC-Sub$(Q^{x,t}_{r,\bar h}[\mathcal{F}])$ for some $0<\bar h<t$, there exists $0<h\le\bar h$ and a sequence $(u^\delta)_{\delta>0}$ of functions in* USC-Sub$(Q^{x,t}_{r,h}[\mathcal{F}])$ *such that $\lfloor u^\delta-u\rfloor^{(x,t)}_{\partial_{\rm lat}Q}\ge\tilde\eta(\delta)>0$ with $\tilde\eta(\delta)\to0$ as $\delta\to0$. Moreover, for any $y\in\mathcal{F}$, $u^\delta(y)\to u(y)$ when $\delta\to0$.*

Notice that (**LOC2**)-*evol* is only concerned with a property at the lateral boundary. As we see in the proof, the initial boundary is easily left out by a minimality argument.

With these assumptions, we have the

Proposition 3.2.2 *Under Assumptions* (**LOC1**)*-evol and* (**LOC2**)*-evol, then* **(LCR)***-evol implies* **(GCR)***-evol.*

Proof — There is no main change in the proof except the following point: using (**LOC1**)*-evol*, we may assume that the maximum of $u - v$ is achieved at some point (x, t). Here we choose t as the minimal time such that we have a maximum of $u - v$. And we assume that this maximum is positive.

Notice first that $t > 0$ because $u \leq v$ on $\mathcal{F} \times \{0\}$ and, if r and $h \leq \bar{h}$ are given by **(LCR)***-evol*, notice also that by the minimality property of t,

$$\max_{(\overline{B(x,r)}\cap\mathcal{F})\times\{t-h\}} (u - v) < \max_{\overline{Q^{x,t}_{r,h}[\mathcal{F}]}} (u - v) = u(x, t) - v(x, t)\,.$$

In other words, the maximum of $u - v$ is not attained on the initial boundary, $\partial_{\rm ini}Q$. On the other hand, on the lateral boundary we obviously get

$$\lfloor u - v \rfloor^{(x,t)}_{\partial_{\rm lat}Q} = (u - v)(x) - \max_{\partial\mathcal{F}^{x,r}\times[t-h,t]} (u - v) \geq 0\,.$$

Then we apply (**LOC2**)*-evol*. Using the property of $\tilde{\eta}(\delta)$ we can choose δ small enough in order that again, the max of $u^\delta - v$ is not attained at time $t - h$. Of course, if $\max(u^\delta - v) \leq 0$ we reach a contradiction for δ small enough so using the **(LCR)***-evol* yields

$$\begin{aligned} u^\delta(x, t) - v(x, t) &\leq \max_{\overline{Q^{x,t}_{r,h}[\mathcal{F}]}} (u^\delta - v) \\ &\leq \max_{\partial_p Q^{x,t}_{r,h}[\mathcal{F}]} (u^\delta - v) = \max_{\partial\mathcal{F}^{x,r}\times[t-h,t]} (u^\delta - v)\,. \end{aligned}$$

In other words, $\lfloor u^\delta - v \rfloor^{(x,t)}_{\partial_{\rm lat}Q} \leq 0$ and the rest of the proof follows the same arguments as in the stationary case

$$\lfloor u - v \rfloor^{(x,t)}_{\partial_{\rm lat}Q} \leq \lfloor u^\delta - v \rfloor^{(x,t)}_{\partial_{\rm lat}Q} - \lfloor u^\delta - u \rfloor^{(x,t)}_{\partial_{\rm lat}Q} \leq -\tilde{\eta}(\delta) < 0\,,$$

which leads to a contradiction.

Q.E.D.

In the evolution case, where the equation (or part of the equation) contains some u_t-term, building the u_α and u^δ turns out to be easy. For example, in the Lipschitz case

$$u_\alpha(x, t) = u(x, t) - \alpha[(|x|^2 + 1)^{1/2} + Kt]\,,$$

for $K > 0$ large enough. And for u^δ,

$$u^\delta(y,s) = u(y,s) - \delta[(|y-x|^2+1)^{1/2} - 1 + K(s-t)] \ ,$$

where K has to be chosen large enough to have a subsolution and h small enough to have the right property on the parabolic boundary. This is because of this property on the parabolic boundary that (**LOC2**)-*evol* has this formulation for h.

In the convex case, the arguments which are presented in the stationary part can be extended.

Remark 3.2.3 *Since we are going to use these localization properties throughout the book in order to treat discontinuities, let us make two important comments here.*

(*i*) *As the proofs show it (both in the stationary and evolution case), in order to have* **(GCR)**, *we do not need* **(LCR)** *to hold on the whole set $\mathcal{F}$: indeed, if we already know that $u \le v$ on some subset $\mathcal{A}$ of $\mathcal{F}$, then* **(LCR)** *is required only in $\mathcal{F} \setminus \mathcal{A}$ to have* **(GCR)**.

(*ii*) *Both in the Lipschitz and convex case we can check* (**LOC1**), (**LOC2**) *(and their evolution variations) in standard ways. It should be noticed that, in both cases, the localization procedure is independant of the possible discontinuities in x. Which is why it will be systematically applied to get the various* **(GCR)** *throughout this book.*

3.2.3 Viscosity inequalities at $t = T_f$ in the evolution case

We conclude this section by examining the viscosity sub and supersolutions inequalities at $t = T_f$ and their consequences on the properties of sub and supersolutions. To do so, we have to be a little bit more precise on the assumptions on the function $\mathbb{G}$ appearing in (3.5). We introduce the following hypothesis

($\mathbf{H_{BA}}$–p_t) *For any $(x,t,r,p_x,p_t) \in \mathcal{F} \times (0,T_f] \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}$, the function $p_t \mapsto \mathbb{G}\big(x,t,r,(p_x,p_t)\big)$ is increasing and $\mathbb{G}\big(x,t,r,(p_x,p_t)\big) \to +\infty$ as $p_t \to +\infty$, uniformly for bounded x,t,r,p_x.*

This assumption is obviously satisfied in the standard case, *i.e.* for equations like

$$u_t + H(x,t,u,D_x u) = 0 \quad \text{in } \mathbb{R}^N \times (0,T_f] \ ,$$

provided H is continuous (or only locally bounded) since in this case $\mathbb{G}(x,t,r,(p_x,p_t)) = p_t + H(x,t,r,p_x)$.

Proposition 3.2.4 *Under Assumption* ($\mathbf{H_{BA}}$–p_t), *we have*

(*i*) *If* $u : \mathcal{F} \times (0,T_f) \to \mathbb{R}$ [*resp.* $v : \mathcal{F} \times (0,T_f) \to \mathbb{R}$] *is an u.s.c. viscosity subsolution* [*resp. lsc supersolution*] *of*

$$\mathbb{G}(x,t,w,(D_x w, w_t)) = 0 \quad \text{on } \mathcal{F} \times (0,T_f)\ ,$$

then, for any $0 < T' < T_f$, u [*resp.* v] *is an u.s.c. viscosity subsolution* [*resp. lsc supersolution*] *of*

$$\mathbb{G}(x,t,w,(D_x w, w_t)) = 0 \quad \text{on } \mathcal{F} \times (0,T']\ .$$

(*ii*) *Under the same conditions on* u *and* v *and if*

$$u(x,T_f) = \limsup_{\substack{(y,s)\to(x,T_f)\\ s<T_f}} u(y,s) \quad [\textit{resp.} \quad v(x,T_f) = \liminf_{\substack{(y,s)\to(x,T_f)\\ s<T_f}} v(y,s)]\ , \tag{3.9}$$

then u *and* v *are respectively sub and supersolution of* (3.5).

(*iii*) *If* $u : \mathcal{F} \times (0,T_f] \to \mathbb{R}$ *is an u.s.c. viscosity subsolution of* (3.5), *then, for any* $x \in \mathcal{F}$, (3.9) *holds for* u.

(*iv*) *If* $\mathbb{G}$ *satisfies* $\mathbb{G}(x,t,r,(p_x,p_t)) \to -\infty$ *as* $p_t \to -\infty$, *uniformly for bounded* x,t,r,p_x *and if* $v : \mathcal{F} \times (0,T_f] \to \mathbb{R}$ *is a l.s.c. viscosity supersolution of* (3.5), *then* (3.9) *holds for* v.

This result clearly shows the particularities of the viscosity inequalities at the terminal time $t = T_f$ or $t = T'$: sub and supersolutions in $\mathcal{F} \times (0,T_f)$ are automatically sub and supersolutions on $\mathcal{F} \times (0,T']$ for any $0 < T' < T_f$ and even for $T' = T_f$ provided that they are extended in the right way up to time T_f, according to (3.9). And conversely sub and supersolutions on $\mathcal{F} \times (0,T_f]$ satisfy (3.9) provided that G has some suitable properties which clearly hold for the standard H-equation above. Here there is a difference between sub and supersolutions due to the disymmetry of Assumption ($\mathbf{H_{BA}}$–p_t). We will come back later on this point with the control interpretation.

Proof — We only prove the first and second part of the result in the subsolution case, the proof for the supersolution being analogous. Let φ be a smooth function (say, in $\mathcal{F} \times [0,T_f]$) and let (x,T') be a strict local maximum point of $u - \varphi$ in $\mathcal{F} \times [0,T']$. We introduce the function

$$(y,s) \mapsto u(y,s) - \varphi(y,s) - \frac{[(s-T')^+]^2}{\varepsilon}\ .$$

An easy application of Lemma 6.4.1 implies that this function has a local maximum point at $(x_\varepsilon, t_\varepsilon)$ and we have

$$(x_\varepsilon, t_\varepsilon) \to (x, T') \quad \text{and} \quad u(x_\varepsilon, t_\varepsilon) \to u(x, T') \text{ as } \varepsilon \to 0 \, ,$$

because of both the strict maximum point property and the ε-penalisation. Moreover, for ε small enough, the penalization implies that $t_\varepsilon < T_f$.

Since u is a subsolution of the $\mathbb{G}$-equation in $\mathcal{F} \times (0, T_f)$ and as we noticed, $(x_\varepsilon, t_\varepsilon)$ is a local maximum point in $\mathcal{F} \times (0, T_f)$, we have

$$\mathbb{G}_*\Big(x_\varepsilon, t_\varepsilon, u(x_\varepsilon, t_\varepsilon), (D_x\varphi(x_\varepsilon, t_\varepsilon), \varphi_t(x_\varepsilon, t_\varepsilon) + 2\varepsilon^{-1}(s - T_f)^+)\Big) \leq 0 \, .$$

But, by $(\mathbf{H_{BA}}\text{-}p_t)$, $\mathbb{G}(y, s, r, (p_x, p_t))$ and therefore $\mathbb{G}_*(y, s, r, (p_x, p_t))$ is increasing in the p_t-variable and we have

$$\mathbb{G}_*(x_\varepsilon, t_\varepsilon, u(x_\varepsilon, t_\varepsilon), (D_x\varphi(x_\varepsilon, t_\varepsilon), D_t\varphi(x_\varepsilon, t_\varepsilon))) \leq 0 \, .$$

The conclusion follows from the lower semicontinuity of $\mathbb{G}_*$ by letting ε tend to 0.

For the proof of (ii), we argue in an analogous way: if (x, T_f) is a strict local maximum point of $u - \varphi$ in $\mathcal{F} \times [0, T_f]$, we introduce the function

$$(y, s) \mapsto u(y, s) - \varphi(y, s) - \frac{\varepsilon}{(T_f - s)} \, .$$

By Lemma 6.4.1, this function has a local maximum point at $(x_\varepsilon, t_\varepsilon)$ and we have

$$(x_\varepsilon, t_\varepsilon) \to (x, T_f) \quad \text{and} \quad u(x_\varepsilon, t_\varepsilon) \to u(x, T_f) \text{ as } \varepsilon \to 0 \, .$$

It is worth pointing out that, in this case, the proof of such properties uses not only the strict maximum point property and the fact that the ε-penalisation is vanishing, but also strongly Property (3.9) for u which provides Assumption-(iii) of Lemma 6.4.1.

We are led to

$$\mathbb{G}_*\Big(x_\varepsilon, t_\varepsilon, u(x_\varepsilon, t_\varepsilon), (D_x\varphi(x_\varepsilon, t_\varepsilon), \varphi_t(x_\varepsilon, t_\varepsilon) + \frac{\varepsilon}{(T_f - s)^2})\Big) \leq 0 \, ,$$

and we conclude by similar arguments as in the proof of (i).

Finally we prove (iii) since the supersolution one, (iv), follows again from similar arguments with the additional assumption on $\mathbb{G}$.

We pick some $(x, T_f) \in \mathcal{F} \times \{T_f\}$ and we aim at proving (3.9). We argue by contradiction: if this is not the case then $u(x, T_f) > \limsup u(y, s)$ as $(y, s) \to (x, T_f)$,

with $s < T_f$. This implies that for any $\varepsilon > 0$ small enough and any $C > 0$, the function

$$(y,s) \mapsto u(y,s) - \frac{|y-x|^2}{\varepsilon^2} - C(s-T_f)$$

can only have a maximum point for $s = T_f$, say at $y = x_\varepsilon$ close to x. The viscosity subsolution inequality reads

$$\mathbb{G}_*\Big(x_\varepsilon, T_f, u(x_\varepsilon, T_f), (\frac{2(x_\varepsilon - x)}{\varepsilon^2}, C)\Big) \leq 0\ .$$

But if we fix ε (small enough), all the arguments in $\mathbb{G}_*$ remains bouded, except C. So, choosing C large enough, we have a contradiction because of $(\mathbf{H_{BA-}}p_t)$.

Q.E.D.

Remark 3.2.5 *We point out that, even if Proposition 3.2.4 only provides the result for sub or supersolutions inequalities in sets of the form $\mathcal{F} \times (0,T_f)$, a similar result can be obtained, under suitable assumptions, for sub and supersolution properties at any point (x,T_f) of $\mathcal{M}$ where $\mathcal{M}$ is the restriction to $\mathbb{R}^N \times (0,T_f]$ to a submanifold of $\mathbb{R}^N \times \mathbb{R}$. Indeed, it is clear from the proof that only Assumption $(\mathbf{H_{BA-}}p_t)$ is really needed to have such properties.*

3.2.4 The simplest examples of (SCR): continuous Hamilton-Jacobi equations in the whole space $\mathbb{R}^N$

As a simple example, we consider the standard continuous Hamilton-Jacobi Equation

$$u_t + H(x,t,u,D_x u) = 0 \quad \text{in } \mathbb{R}^N \times (0,T_f)\,, \tag{3.10}$$

where $H : \mathbb{R}^N \times [0,T_f] \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a continuous function, u_t denotes the time-derivative of u and $D_x u$ is the derivative with respect to the space variables x. Of course, this equation has to be complemented by an initial data

$$u(x,0) = u_0(x) \quad \text{in } \mathbb{R}^N\ . \tag{3.11}$$

We provide comparison results in the two cases we already consider above, namely the Lipschitz case and the convex case, the later one allowing more general Hamiltonians coming from unbounded control problems.

Our result is the following

Theorem 3.2.6 (Comparison for the Lipschitz case) *Let* USC-Sub$(\mathbb{R}^N \times [0,T_f])$ *be the set of bounded u.s.c. subsolution* u *of* (3.10) *such that* $u(x,0) \le u_0(x)$ *in* $\mathbb{R}^N$ *and* LSC-Sup$(\mathbb{R}^N \times [0,T_f])$ *is the set of bounded l.s.c. supersolutions* v *of* (3.10) *such that* $u_0(x) \le v(x,0)$ *in* $\mathbb{R}^N$. *Under Assumption* $(\mathbf{H_{BA-HJ}})$, *there exists a (GCR-evol) for sub and supersolutions of* (3.10)-(3.11) *in* USC-Sub$(\mathbb{R}^N \times [0,T_f])$ *and* LSC-Sup$(\mathbb{R}^N \times [0,T_f])$ *respectively.*

Proof — We just sketch it since it is the standard comparison proof that we recast in a little unsual way.

By the argument of the previous section, it suffices to prove **(LCR)**-*evol*. Therefore, we argue in $\overline{Q^{\bar{x},\bar{t}}_{r,h}}$ for some $\bar{x} \in \mathbb{R}^N$, $0 < \bar{t} < T_f$, $r,h > 0$ and we assume that $\max_{\overline{Q^{\bar{x},\bar{t}}_{r,h}}}(u-v) > 0$ where $u \in$ USC-Sub$(Q^{\bar{x},\bar{t}}_{r,h})$, $v \in$ LSC-Sup$(Q^{\bar{x},\bar{t}}_{r,h})$.

It is worth pointing out that, in $\overline{Q^{\bar{x},\bar{t}}_{r,h}}$, taking into account the fact that u and v are bounded, we have fixed constants and modulus in $(\mathbf{H_{BA-HJ}})$ (that we denote below by C_1, γ and m). Moreover, we can assume w.l.o.g. that $\gamma > 0$ through the classical change $u(x,t) \to \exp(Kt)u(x,t)$, $v(x,t) \to \exp(Kt)v(x,t)$ for some large enough constant K.

We argue by contradiction, assuming that

$$\max_{\overline{Q^{\bar{x},\bar{t}}_{r,h}}}(u-v) > \max_{\partial_p Q^{\bar{x},\bar{t}}_{r,h}}(u-v)\,,$$

and we introduce the classical doubling of variables

$$(x,t,y,s) \mapsto u(x,t) - v(y,s) - \frac{|x-y|^2}{\varepsilon^2} - \frac{|t-s|^2}{\varepsilon^2}\,.$$

Using Lemma 6.4.1, this u.s.c. function has a maximum point at $(x_\varepsilon,t_\varepsilon,y_\varepsilon,s_\varepsilon)$ with $(x_\varepsilon,t_\varepsilon),(y_\varepsilon,s_\varepsilon) \in Q^{\bar{x},\bar{t}}_{r,h}$ and

$$u(x_\varepsilon,t_\varepsilon) - v(y_\varepsilon,s_\varepsilon) \to \max_{\overline{Q^{\bar{x},\bar{t}}_{r,h}}}(u-v) \quad \text{and} \quad \frac{|x_\varepsilon-y_\varepsilon|^2}{\varepsilon^2} + \frac{|t_\varepsilon-s_\varepsilon|^2}{\varepsilon^2} \to 0\,.$$

It remains to write the viscosity inequalities which reads

$$a_\varepsilon + H(x_\varepsilon,t_\varepsilon,u(x_\varepsilon,t_\varepsilon),p_\varepsilon) \le 0 \quad \text{and} \quad a_\varepsilon + H(y_\varepsilon,s_\varepsilon,v(y_\varepsilon,s_\varepsilon),p_\varepsilon) \ge 0\,,$$

with

$$a_\varepsilon = \frac{2(t_\varepsilon - s_\varepsilon)}{\varepsilon^2} \quad \text{and} \quad p_\varepsilon = \frac{2(x_\varepsilon - y_\varepsilon)}{\varepsilon^2}\,.$$

Subtracting the two inequalities, we obtain

$$H(x_\varepsilon, t_\varepsilon, u(x_\varepsilon, t_\varepsilon), p_\varepsilon) - H(y_\varepsilon, s_\varepsilon, v(y_\varepsilon, s_\varepsilon), p_\varepsilon) \leq 0 \ ,$$

that we can write as

$$[H(x_\varepsilon, t_\varepsilon, u(x_\varepsilon, t_\varepsilon), p_\varepsilon) - H(x_\varepsilon, t_\varepsilon, v(x_\varepsilon, t_\varepsilon), p_\varepsilon)] \leq [H(x_\varepsilon, t_\varepsilon, v(x_\varepsilon, t_\varepsilon), p_\varepsilon) - H(y_\varepsilon, s_\varepsilon, v(y_\varepsilon, s_\varepsilon), p_\varepsilon)] \ .$$

It remains to apply $(\mathbf{H_{BA-HJ}})$

$$\gamma(u(x_\varepsilon, t_\varepsilon), p_\varepsilon) - v(x_\varepsilon, t_\varepsilon)) - C_1(|x_\varepsilon - y_\varepsilon| + |t_\varepsilon - s_\varepsilon|)|p_\varepsilon| - m(|x_\varepsilon - y_\varepsilon| + |t_\varepsilon - s_\varepsilon|) \leq 0 \ .$$

But, as $\varepsilon \to 0$, $m(|x_\varepsilon - y_\varepsilon| + |t_\varepsilon - s_\varepsilon|) \to 0$ since $|x_\varepsilon - y_\varepsilon| + |t_\varepsilon - s_\varepsilon| = o(\varepsilon)$ and

$$(|x_\varepsilon - y_\varepsilon| + |t_\varepsilon - s_\varepsilon|)|p_\varepsilon| = \frac{2|x_\varepsilon - y_\varepsilon|^2}{\varepsilon^2} + \frac{2|t_\varepsilon - s_\varepsilon||x_\varepsilon - y_\varepsilon|}{\varepsilon^2} \to 0 \ .$$

Therefore we have a contradiction for ε small enough since

$$\gamma(u(x_\varepsilon, t_\varepsilon), p_\varepsilon) - v(x_\varepsilon, t_\varepsilon)) \to \gamma \max_{\overline{Q^{\bar{x},\bar{t}}_{r,h}}}(u - v) > 0 \ .$$

And the proof is complete.

Q.E.D.

It is worth pointing out the simplifying effect of the localization argument in this proof: the core of the proof becomes far simpler since we do have to handle several penalization terms at the same time (the ones for the doubling of variables and the localization ones).

We have formulated and proved Theorem 3.2.6 in a classical way and in a way which is consistent with the previous sections but in this Lipschitz framework, we may have the stronger result based on *a finite speed of propagation type phenomena* which we present here since it follows from very similar arguments

Theorem 3.2.7 (Finite speed of propagation) *Let* USC-Sub$(\mathbb{R}^N \times [0, T_f])$ *be the set of locally bounded u.s.c. subsolution* u *of* (3.10) *and* LSC-Sup$(\mathbb{R}^N \times [0, T_f])$ *is the set of locally bounded bounded l.s.c. supersolutions* v *of* (3.10)*. Assume that* $(\mathbf{H_{BA-HJ}})$ *holds with* $\gamma(R)$ *independent of* R*; if* $u \in$ USC-Sub$(\mathbb{R}^N \times [0, T_f])$ *and* $v \in$ LSC-Sup$(\mathbb{R}^N \times [0, T_f])$ *satisfy* $u(x, 0) \leq v(x, 0)$ *for* $|x| \leq R$ *for some* $R > 0$*, then*

$$u(x, t) \leq v(x, t) \quad \textit{for } |x| \leq R - C_2 t \ ,$$

where C_2 *is given by* $(\mathbf{H_{BA-HJ}})$*.*

Proof — Let $\chi : (-\infty, R) \to \mathbb{R}$ be a smooth function such that $\chi(s) \equiv 0$ if $s \leq 0$, χ is increasing on $\mathbb{R}$ and $\chi(s) \to +\infty$ when $s \to R^-$. We set

$$\psi(x,t) := \exp(-|\gamma|t)\chi(|x| + C_2 t) \ .$$

This function is well-defined in $\mathcal{C} := \{(x,t) : \ |x| + C_2 t \leq R\}$.

We claim that, for $0 < \alpha \ll 1$, the function $u_\alpha(x,t) := u(x,t) - \alpha\psi(x,t)$ in a subsolution of (3.10) in $\mathcal{C}$ and satisfies $u_\alpha(x,t) \to -\infty$ if $(x,t) \to \partial\mathcal{C} \cap \{t > 0\}$ and $u_\alpha(x,0) \leq u(x,0)$ for $|x| \leq R$.

The second part of the claim is obvious by the properties of ψ. To prove the first one, we first compute formally

$$(u_\alpha)_t + H(x,t,u_\alpha, D_x u_\alpha) \leq u_t + H(x,t,u,D_x u) - \alpha(\psi_t - |\gamma|\psi - C_2|D_x\psi|) \ .$$

But an easy -again formal- computation shows that $\psi_t - |\gamma|\psi - C_2|D_x\psi| \geq 0$ in $\mathcal{C}$ and since the justification of these formal computations is straightforward by regularizing $|x|$ in order that ψ becomes C^1, the claim is proved.

The rest of the proof consists in comparing u_α and v in $\mathcal{C}$, which follows from the same arguments as in the proof of Theorem 3.2.6.

Q.E.D.

Now we turn to the *convex case* where we may have some more general behavior for H and in particular no Lipschitz continuity in p. To simplify the exposure, we do not formulate the assumption in full generality but in the most readable way (at least, we hope so!)

($\mathbf{H_{BA-Conv}}$) $H(x,t,r,p)$ is a locally Lipschitz function which is convex in (r,p). Moreover, for any ball $B \subset \mathbb{R}^N \times [0,T_f]$, for any $R > 0$, there exists constants $L = L(B,R), K = K(B,R) > 0$ and a function $G = G(B,R) : \mathbb{R}^N \to [1, +\infty[$ such that, for any $x, y \in B$, $t, s \in [0,T_f]$, $-R \leq u \leq v \leq R$ and $p \in \mathbb{R}^N$

$$D_p H(x,t,r,p) \cdot p - H(x,t,u,p) \geq G(p) - L \ ,$$

$$|D_x H(x,t,r,p)|, |D_t H(x,t,r,p)| \leq KG(p)(1 + |p|) \ ,$$

$$D_r H(x,t,r,p) \geq 0 \ .$$

On the hand, we assume the existence of a subsolution

($\mathbf{H_{Sub-HJ}}$) There exists an C^1-function $\psi : \mathbb{R}^N \times [0,T_f] \to \mathbb{R}$ which is a subsolution of (3.10) and which satisfies $\psi(x,t) \to -\infty$ as $|x| \ \to +\infty$, uniformly for $t \in [0,T_f]$

and $\psi(x,0) \le u_0(x)$ in $\mathbb{R}^N$.

The result is

Theorem 3.2.8 (Comparison for the Convex case) *Assume* $(\mathbf{H_{BA-HJ-U}})$ *and* $(\mathbf{H_{Sub-HJ}})$. *Let* USC-Sub$(\mathbb{R}^N \times [0,T_f])$ [*resp.* LSC-Sup$(\mathbb{R}^N \times [0,T_f])$] *be the set of bounded u.s.c. subsolution* u *of* (3.10) [*resp. the set of bounded l.s.c. supersolutions* v *of* (3.10)] *such that*

$$\limsup_{|x|\to+\infty} \frac{u(x,t)}{\psi(x,t)} \ge 0 \quad \left[\textit{resp. } \liminf_{|x|\to+\infty} \frac{v(x,t)}{\psi(x,t)} \le 0\right] \quad \textit{uniformly for } t \in [0,T_f]\ .$$

Then there exists a **(GCR)**-*evol for sub and supersolutions of* (3.10)-(3.11) *in* USC-Sub$(\mathbb{R}^N \times [0,T_f])$ *and* LSC-Sup$(\mathbb{R}^N \times [0,T_f])$ *respectively.*

Proof — The first step consists as above in replacing u by $u_\alpha := (1-\alpha)u + \alpha\psi$ for $0 < \alpha \ll 1$. The convexity of $H(x,t,r,p)$ in (r,p) implies that u_α is still a subsolution of (3.10) and $u_\alpha(x,0) \le u_0(x)$ in $\mathbb{R}^N$. Moreover, by the definition of USC-Sub$(\mathbb{R}^N \times [0,T_f])$ and LSC-Sup$(\mathbb{R}^N \times [0,T_f])$,

$$\lim(u_\alpha(x,t) - v(x,t)) = -\infty \quad \text{as } |x| \ \to +\infty, \text{ uniformly for } t \in [0,T_f].$$

Therefore the subsolution ψ plays its localization role.

For **(LCR)**-*evol*, we argue exactly in the same way as in the proof of Theorem 3.2.6 in $Q^{\bar{x},\bar{t}}_{r,h}$ (and therefore with fixed contants L, K and a fixed function G) but with the following preliminary reductions: changing u, v in $u(x,t) + Lt$ and $v(x,t) + Lt$, we may assume that $L = 0$. Finally we perform Kruzkov's change of variable

$$\tilde{u}(x,t) := -\exp(-u(x,t)) \quad , \quad \tilde{v}(x,t) := -\exp(-v(x,t))\ .$$

The function $\tilde{u}, \tilde{v}$ are respectively sub and supersolution of

$$w_t + \tilde{H}(x,t,w,Dw) = 0 \quad \text{in } Q^{\bar{x},\bar{t}}_{r,h}\ ,$$

with $\tilde{H}(x,t,r,p) = -rH(x,t,-\log(-r),-p/r)$.

Computing $D_r\tilde{H}(x,t,r,p)$, we find $(D_pH\cdot p - H)(x,t,-\log(-r),-p/r)) \ge G(-p/r)$, while $D_x\tilde{H}(x,t,r,p)$, $D_t\tilde{H}(x,t,r,p)$ are estimated by $|r||D_xH(x,t,-\log(-r),-p/r)|$, $|r||D_tH(x,t,-\log(-r),-p/r)|$, i.e. by $|r|KG(-p/r)(1+|p/r|)$.

Following the proof of Theorem 3.2.6, we have to examine an inequality like

$$\tilde{H}(x_\varepsilon,t_\varepsilon,\tilde{u}(x_\varepsilon,t_\varepsilon),p_\varepsilon) - \tilde{H}(y_\varepsilon,s_\varepsilon,\tilde{v}(y_\varepsilon,s_\varepsilon),p_\varepsilon) \le 0\ .$$

To do so, we argue as if $\tilde{H}$ was C^1 (the justification is easy by a standard approximation argument) and we introduce the function

$$f(\mu) := \tilde{H}(\mu x_\varepsilon + (1-\mu)y_\varepsilon, \mu t_\varepsilon + (1-\mu)s_\varepsilon, \mu\tilde{u}(x_\varepsilon,t_\varepsilon) + +(1-\mu)\tilde{v}(y_\varepsilon,s_\varepsilon), p_\varepsilon)\ ,$$

which is defined on $[0,1]$. The above inequality reads $f(1)-f(0)\le 0$ while

$$f'(\mu) = D_x\tilde{H}.(x_\varepsilon - y_\varepsilon) + D_t\tilde{H}.(t_\varepsilon - s_\varepsilon) + D_r\tilde{H}.(\tilde{u}(x_\varepsilon,t_\varepsilon) - \tilde{v}(y_\varepsilon,s_\varepsilon))\ ,$$

where all the $\tilde{H}$ derivatives are computed at the point

$$(\mu x_\varepsilon + (1-\mu)y_\varepsilon, \mu t_\varepsilon + (1-\mu)s_\varepsilon, \mu\tilde{u}(x_\varepsilon,t_\varepsilon) + (1-\mu)\tilde{v}(y_\varepsilon,s_\varepsilon), p_\varepsilon)\ .$$

If we denote by $r_\varepsilon = \mu\tilde{u}(x_\varepsilon,t_\varepsilon) + (1-\mu)\tilde{v}(y_\varepsilon,s_\varepsilon)$, we have, by the above estimates,

$$\begin{aligned} f'(\mu) \ge & -|r_\varepsilon| K G(-p_\varepsilon/r_\varepsilon)(1+|p_\varepsilon/r_\varepsilon|)(|x_\varepsilon - y_\varepsilon| + |t_\varepsilon - s_\varepsilon|) \\ & + G(-p_\varepsilon/r_\varepsilon).(\tilde{u}(x_\varepsilon,t_\varepsilon) - \tilde{v}(y_\varepsilon,s_\varepsilon)) \\ \ge & G(-p_\varepsilon/r_\varepsilon)\Big[-K(|r_\varepsilon| + |p_\varepsilon|)(|x_\varepsilon - y_\varepsilon| + |t_\varepsilon - s_\varepsilon|) + (\tilde{u}(x_\varepsilon,t_\varepsilon) - \tilde{v}(y_\varepsilon,s_\varepsilon))\Big]\ . \end{aligned}$$

But if $M := \max_{\overline{Q^{\bar{x},\bar{t}}_{r,h}}}(\tilde{u}-\tilde{v}) > 0$, the arguments of the proof of Theorem 3.2.6 show that the bracket is larger than $M/2$ if ε is small enough. Therefore $f'(\mu) \ge M/2 > 0$, a contradiction with $f(1)-f(0)\le 0$.

Q.E.D.

We conclude this part by an application of Theorem 3.2.6 and 3.2.8 to the equation

$$u_t + a(x,t)|D_x u|^q - b(x,t)\cdot D_x u = f(x,t) \quad \text{in } \mathbb{R}^N \times (0,T_f)\,,$$

where a, b, f are at least continuous function in $\mathbb{R}^N \times [0,T_f]$ and $q \ge 1$.

Of course, Theorem 3.2.6 applies if $q=1$ and a, b are locally Lipschitz continuous functions and f is a uniformly continuous function on $\mathbb{R}^N \times [0,T_f]$.

Theorem 3.2.8 is concerned with the case $q>1$ and $a(x,t)\ge 0$ in $\mathbb{R}^N \times [0,T_f]$ in order to have a convex Hamiltonian.

Next the computation gives

$$D_p H(x,t,r,p)\cdot p - H(x,t,u,p) = a(x,t)(q-1)|p|^q - b(x,t)\cdot p + f(x,t)\ .$$

and in order to verify ($\mathbf{H_{BA-HJ-U}}$), we have to reinforce the convexity assumption by assuming $a(x,t) > 0$ in $\mathbb{R}^N \times [0,T_f]$. If B is a ball in $\mathbb{R}^N \times [0,T_f]$, we set $m(B) = \min_B a(x,t)$ and we have, using Young's inequality

$$D_p H(x,t,r,p)\cdot p - H(x,t,u,p) = m(B)(q-1)|p|^q + 1 - L(B)\ .$$

Here the "+1" is just a cosmetic term to be able to set $G(p) := m(B)(q-1)|p|^q+1 \geq 1$ and $L(B)$ is a constant depending on the L^∞-norm of b and f on B.

Finally, a, b, f being locally Lipschitz continuous, it is clear enough that the estimates on $|D_x H(x,t,r,p)|, |D_t H(x,t,r,p)|$ hold. It is worth pointing out that the behavior at infinity of a, b, f does not play any role since we have the arguments of the comparison proof are local. But, of course, we do not pretend that this strategy of proof is optimal...

The checking of $(\mathbf{H_{Sub-HJ}})$ is more "example-dependent" and we are not going to try to find "good frameworks". If $b = 0$ and if there exists $\eta >$ such that

$$\eta \leq a(x,t) \leq \eta^{-1} \quad \text{in } \mathbb{R}^N \times (0, T_f)\,,$$

the Oleinik-Lax Formula suggests subsolutions of the form

$$\psi(x,t) = -\alpha(t+1)(|x|^{q'}+1) - \beta\,,$$

where q' is the conjugate exponent of q, i.e. $\dfrac{1}{q} + \dfrac{1}{q'} = 1$ and α, β are large enough constants. Indeed

$$\psi_t + a(x,t)|D_x\psi|^q - f(x,t) \leq -\alpha(|x|^{q'}+1) + \eta^{-1}[q'\alpha(t+1)]^q|x|^{q'} - f(x,t)\,.$$

If there exists $c > 0$ such that

$$f(x,t) \geq -c(|x|^{q'}+1) \quad \text{in } \mathbb{R}^N \times (0, T_f)\,,$$

then, for large α, namely $\alpha > \eta^{-1}[q'\alpha]^q + c$, one has a subsolution BUT only on a short time interval $[0, \tau]$. Therefore one has a comparison result if, in addition, the initial data satisfies for some $c' > 0$

$$u_0(x) \geq -c'(|x|^{q'}+1) \quad \text{in } \mathbb{R}^N\,,$$

in which case, we should also have $\alpha > c'$.

In good cases, the comparison result on $[0, \tau]$ can be iterated on $[\tau, 2\tau]$, $[2\tau, 3\tau]$,. . . ,etc to get a full result on $[0, T_f]$.

3.3 Whitney stratifications: a good framework for Hamilton-Jacobi equations

In this section, we introduce the notion of Whitney stratification (based on the Whitney conditions found in [162, 161]). This yields a well-adapted structure to deal with

the general discontinuities we are considering in this book, especially in Part IV. We first do it in the case of a flat stratification; the non-flat case is reduced to the flat one by suitable local charts.

Before we begin, notice that for the moment we consider stratifications in $\mathbb{R}^N$ but

(i) since the definition of a (flat or not flat) stratification is purely local, a stratification of an open subset $\mathcal{O} \subset \mathbb{R}^N$ can be defined exactly in the same way.

(ii) When considering time-dependent problems, we have to consider stratifications in $\mathbb{R}^{N+1}$—or more precisely of $\mathbb{R}^N \times (0, T_f)$—, adding one dimension for time and using the remark of Point (i). This allows to treat the case of time-depending stratifications, see Chapter 20.

(iii) Stratifications can also be considered in a closed set, typically the closure of a domain $\Omega \subset \mathbb{R}^N$. In this case, as we will see in Part V, both the interior of the set and the boundary—typically Ω and $\partial\Omega$—can be stratified. Of course, this last point can also be combined with (i) and (ii) and this is what we will do in Part V, looking at stratifications of $\overline{\Omega} \times (0, T_f)$.

3.3.1 Admissible flat stratifications

We consider here the stratification introduced in Bressan and Hong [53]:

$$\mathbb{R}^N = \mathbf{M}^0 \cup \mathbf{M}^1 \cup \cdots \cup \mathbf{M}^N \ ,$$

where the $\mathbf{M}^k$ ($k = 0..N$) are disjoint submanifolds of $\mathbb{R}^N$. Each $\mathbf{M}^k$ can be decomposed as the union of various connected components,

$$\mathbf{M}^k = \bigcup_{i \in \mathcal{I}_k} \mathbf{M}^k_i \ .$$

To start with, we examine the case when the different embedded submanifolds of $\mathbb{R}^N$ are locally affine subspace of $\mathbb{R}^N$.

Notations:

- for $k = 0..N$, $V^{(k)}$ is the set of all k-dimensional affine subspaces of $\mathbb{R}^N$;
- For $x \in \mathbb{R}^N$, $V^{(k)}(x) \subset V^{(k)}$ is the subset of affine subspaces containing x. In other words, $V \in V^{(k)}(x)$ if $V = x + V_k$ where V_k is a k-dimensional vector subspace of $\mathbb{R}^N$.

Definition 3.3.1 *We say that* $\mathbb{M} = (\mathbf{M}^k)_{k=0..N}$ *is an Admissible Flat Stratification —* (AFS) *in short— if the following set of hypotheses* $(\mathbf{H_{ST}})_{flat}$ *is satisfied*

(*i*) *If* $x \in \mathbf{M}^k$ *for some* $k = 0..N$*, there exists* $r = r_x > 0$ *such that*

- (*a*) $B(x,r) \cap \mathbf{M}^k = B(x,r) \cap (x + V_k)$ *for some* $(x + V_k) \in V^{(k)}(x)$ *;*
- (*b*) *For any* $l < k$, $B(x,r) \cap \mathbf{M}^l = \emptyset$ *;*
- (*c*) *For any* $l > k$, $B(x,r) \cap \mathbf{M}^l$ *is either empty or has at most a finite number of connected components ;*
- (*d*) *For any* $l > k$, $B(x,r) \cap \mathbf{M}^l_j \neq \emptyset$ *if and only if* $x \in \partial \mathbf{M}^l_j$.

(*ii*) *If* $\mathbf{M}^k_i \cap \overline{\mathbf{M}^l_j} \neq \emptyset$ *for some* $l > k$ *then* $\mathbf{M}^k_i \subset \overline{\mathbf{M}^l_j}$.

(*iii*) *The following inclusion holds:* $\overline{\mathbf{M}^k} \subset \mathbf{M}^0 \cup \mathbf{M}^1 \cup \cdots \cup \mathbf{M}^k$.

We first notice that Condition $(\mathbf{H_{ST}})_{flat}$-(i) implies that the set $\mathbf{M}^0$, if not void, consists of isolated points. Indeed, in the case $k = 0$, $(x + V_k) = \{x\}$ so that $\mathbf{M}^0 \cap B(x,r) = \{x\}$.

Let us also mention that assumption $(\mathbf{H_{ST}})_{flat}$-$(i)(d)$ is just a consequence of $(i)(c)$ provided we choose the radius $r_x > 0$ small enough. Indeed, since, by $(i)(c)$, we have only a finite number of connected components $\mathbf{M}^l_j$ for $l > k$ such that $B(x,r) \cap \mathbf{M}^l_j \neq \emptyset$, we can exclude all those such that $\mathrm{dist}(x, \mathbf{M}^l_j) > 0$ by choosing a smaller radius r.

Let us also mention another direct consequence of $(\mathbf{H_{ST}})_{flat}$

Lemma 3.3.2 *Let* $\mathbb{M} = (\mathbf{M}^k)_{k=0..N}$ *be an* (AFS) *of* $\mathbb{R}^N$*. Then, for any* $k = 0..N$ *and* $i \in \mathcal{I}_k$*, there exists an open set* $\mathcal{O} = \mathcal{O}(i,k) \subset \mathbb{R}^N$ *and* $V^{(k)}_i \in V^{(k)}$ *such that*

$$\mathbf{M}^k_i = \mathcal{O} \cap V^{(k)}_i \, .$$

In other words, if $x \in \mathbf{M}^k_i$*, then* $\mathbf{M}^k_i = \mathcal{O} \cap (x + V_k)$ *for some* k*-dimensional vector space* V_k *independent of* x.

Proof — Let $k \in \{0, .., N\}$, $i \in \mathcal{I}_k$, and fix $x \in \mathbf{M}^k_i$. By $(\mathbf{H_{ST}})_{flat}$-$(i)(a)$, for any $z \in \mathbf{M}^k_i$, there exists $V^{(k)}_{i(z)} \in V^{(k)}$ such that $z \in V^{(k)}_{i(z)}$. Now, consider the function

$$\chi : \quad \mathbf{M}^k_i \to \{0,1\}$$
$$z \mapsto \begin{cases} 1 \text{ if } V^{(k)}_{i(z)} = V^{(k)}_{i(x)} \, , \\ 0 \text{ otherwise.} \end{cases}$$

This function is obviously locally constant: indeed, by $(\mathbf{H_{ST}})_{flat}$-$(i)(a)$, if $z \in \mathbf{M}_i^k$ then $B(z,r_z) \cap \mathbf{M}_i^k = B(z,r_z) \cap V_{i(z)}^{(k)}$ and therefore if $z' \in B(z,r_z) \cap \mathbf{M}_i^k$, necessarily $V_{i(z')}^{(k)} = V_{i(z)}^{(k)}$.

Therefore, since $\mathbf{M}_i^k$ is connected, it follows that χ is in fact constant, so that $i(z) = i(x) = i$ for all $z \in \mathbf{M}_i^k$. In other words, $(\mathbf{H_{ST}})_{flat}$-$(i)(a)$ can be written for all $z \in \mathbf{M}_i^k$ by means of only one affine subspace

$$B(z,r_z) \cap \mathbf{M}_i^k = B(z,r_z) \cap V_i^{(k)} \, .$$

We then set $\mathcal{O} := \cup_{z \in \mathbf{M}_i^k} B(z,r_z)$ which is an open set in $\mathbb{R}^N$. We deduce from the previous set equality that $\mathcal{O} \cap \mathbf{M}_i^k = \mathcal{O} \cap V_i^{(k)}$.

Q.E.D.

Before providing comments on the difference between the assumptions $(\mathbf{H_{ST}})_{flat}$ and the ones used in Bressan & Hong [53], we consider the simplest relevant example of a flat stratification.

Example 3.1 — We consider in $\mathbb{R}^2$ a chessboard-type configuration, see Figure 3.1. In this case, we have the following decomposition:

$$\mathbf{M}^0 = \mathbb{Z} \times \mathbb{Z} \, ,$$

$$\mathbf{M}^1 = \Big\{ (\mathbb{Z} \times \mathbb{R}) \cup (\mathbb{R} \times \mathbb{Z}) \Big\} \setminus \mathbb{Z}^2 \, ,$$

and $\mathbf{M}^2 = \mathbb{R}^2 \setminus (\mathbf{M}^0 \cup \mathbf{M}^1)$. In this simple case, the checking of the $(\mathbf{H_{ST}})_{flat}$-assumptions is straightforward.

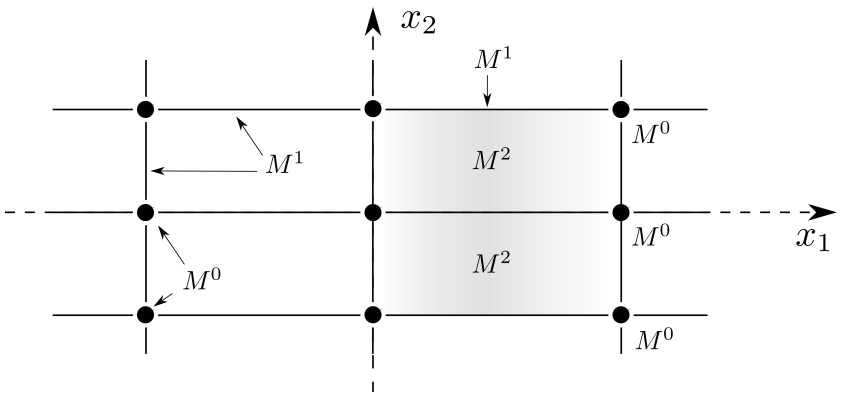


Figure 3.1: The chessboard-type configuration

We point out that, even if the formulation is slightly different, Assumptions $(\mathbf{H_{ST}})_{flat}$ are equivalent (for the flat case) to the assumptions of Bressan & Hong [53]. Indeed, we both assume that we have a partition of $\mathbb{R}^N$ with disjoints submanifolds but we define a different way the submanifolds $\mathbf{M}^k$. The key point is that for us $\mathbf{M}^k$ is here a k-dimensional submanifold while, in [53], the $\mathbf{M}^j$ can be of any dimension. In other words, *our* $\mathbf{M}^k$ is the union of all submanifolds of dimension k in the stratification of Bressan & Hong.

With this in mind it is easier to see that our assumptions $(\mathbf{H_{ST}})_{flat}$-(ii)-(iii) are equivalent to the following assumption of Bressan and Hong: if $\mathbf{M}^k \cap \overline{\mathbf{M}^l} \neq \emptyset$ then $\mathbf{M}^k \subset \overline{\mathbf{M}^l}$ for all indices l, k without asking $l > k$ in our case. But according to the last part of $(\mathbf{H_{ST}})_{flat}$-(i), $\mathbf{M}^k \cap \overline{\mathbf{M}^l} = \emptyset$ if $l < k$: indeed for any $x \in \mathbf{M}^k$, there exists $r > 0$ such that $B(x, r) \cap \mathbf{M}^l = \emptyset$. This property clearly implies $(\mathbf{H_{ST}})_{flat}$-(iii).

A typical example – Let us consider a stratification in $\mathbb{R}^3$ induced by the upper half-plane $\{x_3 > 0, x_2 = 0\}$ and the x_2-axis (see figure 3.2).

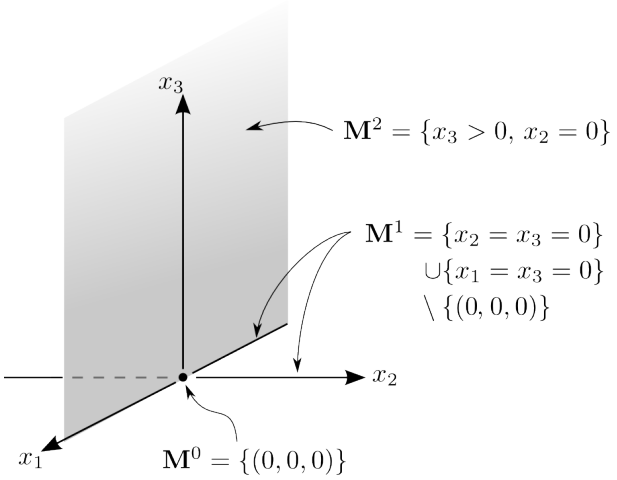


Figure 3.2: Example of a 3-D stratification

• *The "good" stratification* consists in setting first $\mathbf{M}^2 = \{x_3 > 0, x_2 = 0\}$. By $(\mathbf{H_{ST}})_{flat}$-(iii), the boundary of $\mathbf{M}^2$ which is the x_1-axis is included in $\mathbf{M}^1 \cup \mathbf{M}^0$ and we also have x_2-axis in the stratification. In this case, $\mathbf{M}^1 \cup \mathbf{M}^0$ is the cross formed by the x_1 and x_2-axis but in order for $\mathbf{M}^1$ to be a manifold, $(0, 0, 0)$ has to be excluded and we have to set here $\mathbf{M}^0 = \{(0, 0, 0)\}$. Thus, $\mathbf{M}^1$ consists of four connected components which are induced by the x_1- and x_2-axis (but excluding the origin, which is in $\mathbf{M}^0$). Notice that in this situation, the x_3-axis has no particular

status, it is included in $\mathbf{M}^2$.

• *A wrong approach* would be the following alternative decomposition:

$$\mathbf{M}^2 = \{x_3 > 0, x_2 = 0\},\ \mathbf{M}^1 = \{x_1 = x_3 = 0\} \cup \{x_2 = x_3 = 0\},\ \mathbf{M}^3 = \mathbb{R}^3 - \mathbf{M}^2 - \mathbf{M}^1\,.$$

Because $(0,0,0) \in \mathbf{M}^1 \cap \overline{\mathbf{M}^2}$ but clearly $\mathbf{M}^1$ is not included in $\overline{\mathbf{M}^2}$, so $(\mathbf{H_{ST}})_{flat}$-(ii) forbids this decomposition of $\mathbb{R}^3$.

As a consequence of the definition we have following result which will be useful in a tangential regularization procedure (see Figure 2 below)

Lemma 3.3.3 *Let $\mathbb{M} = (\mathbf{M}^k)_{k=0..N}$ be an* (AFS) *of $\mathbb{R}^N$. Let $x \in \mathbf{M}^k$ and $r > 0$, V_k be as in $(\mathbf{H_{ST}})_{flat}$-(i). If $y \in B(x,r) \cap \mathbf{M}^l_j$ for some $l > k$ and $j \in \mathcal{I}_l$ then $x \in \overline{\mathbf{M}^l_j}$ and*

$$B(x,r) \cap (y + V_k) \subset B(x,r) \cap \mathbf{M}^l_j\,.$$

Proof — Notice that by $(\mathbf{H_{ST}})_{flat}$-$(i)(d)$ we already know that $x \in \overline{\mathbf{M}^l_j}$, but moreover $(\mathbf{H_{ST}})_{flat}$-(ii) implies that $\mathbf{M}^k_i \subset \overline{\mathbf{M}^l_j}$. Using open sets $\mathcal{O}_k$ and $\mathcal{O}_l$ defined in Lemma 3.3.2 we get

$$\mathcal{O}_k \cap (x + V_k) \subset \overline{\mathcal{O}_l} \cap (y + V_l) = \overline{\mathcal{O}_l} \cap (x + V_l)\,,$$

the last equality being justified by the fact that $x \in \partial \mathbf{M}^l_j$.

This implies that V_k is a subspace of V_l, so that clearly for any $y \in \mathbf{M}^l_j$, $y + V_k \subset y + V_l$. The result directly follows after intersecting with $B(x,r)$.

Q.E.D.

Remark 3.3.4 *In this flat situation, the tangent space of $\mathbf{M}^k$ at x is $T_x := x + V_k$ while the tangent space of $\mathbf{M}^l$ at y is $T_y := y + V_l$, where $l > k$. The previous lemma implies that if $(y_n)_n$ is a sequence converging to x, then the limit tangent plane of the T_{y_n} is $x + V_l$ and it contains T_x, which is exactly the Whitney condition—see* [162, 161].

3.3.2 General regular stratifications

Definition 3.3.5 *We say that $\mathbb{M} = (\mathbf{M}^k)_{k=0..N}$ is a general regular stratification of $\mathbb{R}^N$* — (RS) *in short— if it satisfies the two following assumptions denoted by $(\mathbf{H_{ST}})_{reg}$*

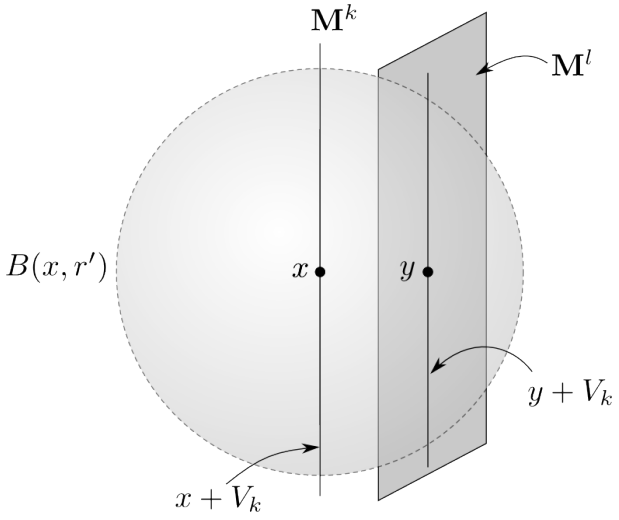


Figure 3.3: local situation

(*i*) *the following decomposition holds:* $\mathbb{R}^N = \mathbf{M}^0 \cup \mathbf{M}^1 \cup \cdots \cup \mathbf{M}^N$*;*

(*ii*) *for any* $x \in \mathbb{R}^N$*, there exists* $r = r(x) > 0$ *and a* $C^{1,1}$*-change of coordinates* $\Psi^x : B(x,r) \to \mathbb{R}^N$ *such that* $\Psi^x(x) = x$ *and* $\{\Psi^x(\mathbb{M} \cap B(x,r))\}_{k=0..N}$ *is the restriction to* $\Psi^x(B(x,r))$ *of an* (AFS) *in* $\mathbb{R}^N$*.*

Remark 3.3.6 *If we need to be more specific, we also say that* $(\mathbb{M}, \Psi)$ *is a stratification of* $\mathbb{R}^N$*, keeping the reference* Ψ *for the collection of changes of variables* $(\Psi^x)_x$*. This will be usefull in Section 22 when we consider sequences of stratifications.*

Tangent spaces – The definition of regular stratifications (flat or not) allows to define, for each $x \in \mathbf{M}^k$, the tangent space to $\mathbf{M}^k$ at x, denoted by $T_x\mathbf{M}^k$. To be more precise, if $x \in \mathbf{M}^k$ and $r > 0$, V_k are as in $(\mathbf{H_{ST}})_{flat}$-(i), then

$$T_x\mathbf{M}^k = (D\Psi^x(x))^{-1}(V_k)\,,$$

which can be identified to $\mathbb{R}^k$. Moreover, we can decompose $\mathbb{R}^N = V_k \oplus V_k^\perp$, where $V_k^\perp$ is the orthogonal space to V_k. For any $p \in \mathbb{R}^N$, we have $p = p_\top + p_\perp$ with $p_\top \in V_k$ and $p_\perp \in V_k^\perp$. In the special case $x \in \mathbf{M}^0$, we have $V_0 = \{0\}$, $p = p_\perp$ and $T_x\mathbf{M}^0 = \{0\}$.

The notion of stratification is introduced above as a pure geometrical tool and it remains to connect it with the singularities of Hamilton-Jacobi Equations. Our aim is to define below a "natural framework" allowing to treat Hamilton-Jacobi Equations

(or control problems) with discontinuities, which will involve two types of information: some conditions on the kinds of singularities we can handle and some assumptions on the Hamiltonians in a neighborhood of those singularities.

We provide here a first step in this direction by considering the simple example of an equation set in the whole space $\mathbb{R}^N$

$$H(x, u, Du) = 0 \quad \text{in } \mathbb{R}^N \ ,$$

where the Hamiltonian H has some discontinuities (in the x-variable) located on some set $\Gamma \subset \mathbb{R}^N$. The first question is: what kind of sets Γ can be handled?

The approach we systematically use consists in assuming that Γ provides a stratification $\mathbb{M} = (\mathbf{M}^k)_{k=0..N}$ of $\mathbb{R}^N$. This means that $\mathbf{M}^N$ is the open subset of $\mathbb{R}^N$ where H is continuous while $\mathbf{M}^k$ contains the discontinuities of dimension $0 \leq k \leq (N-1)$. Of course, some of the $\mathbf{M}^k$ can be empty.

What should be done next is to clarify the structure of the Hamiltonian H in a neighborhood of each point $x \in \mathbf{M}^k$ and for each $k \leq (N-1)$. This is where the previous analysis on stratifications allows to reduce locally the problem to the following situation: if $x \in \mathbf{M}^k$, there is a ball $B(x, r)$ for some $r > 0$, and a C^1-diffeomorphism Ψ such that

$$B(x, r) \cap \Psi(\mathbf{M}^k) = B(x, r) \cap \bigcup_{j=0}^{k} \left(x + V_j\right) .$$

In other words, through a suitable C^1 change of coordinates, we are in a *flat* situation where x is only possibly "touched" by j-dimensional vector spaces for $j \geq k$.

3.3.3 The limits of the stratified approach

The notion of regular stratification seems to provide a very general framework in which one could think that many (if not all) situations can be treated. As we have seen, several quite special geometric structures can be handled, corresponding to a great variety of discontinuities in the equations we consider.

However, it should be noticed that there are still very simple situations that the stratified framework cannot handle. Let us focus here on curves in $\mathbb{R}^2$ in order to better understand the problems that may occur.

Cusps are the typical examples of geometric structures which cannot be included in regular stratifications: consider the curve

$$\Gamma := \left\{ y = \sqrt{|x|} : x \in \mathbb{R} \right\} \subset \mathbb{R}^2 .$$

The natural (and only) stratification of Γ would be to set

$$\mathbf{M}^0 = \{(0,0)\}\,, \text{ and } \mathbf{M}^1 = \{y = \sqrt{-x} : x < 0\} \cup \{y = \sqrt{x} : x > 0\}\,.$$

However, condition (ii) of the regular stratification definition cannot hold. More precisely, at the singular point $z = (0,0)$, there is no $C^{1,1}$ change of variables Ψ^z which can transform the cusp into a flat stratification since such a change of variables could not be Lipschitz regular.

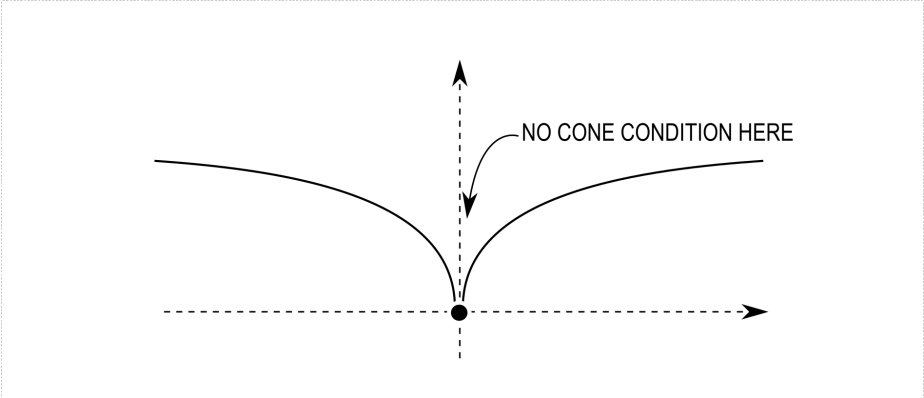


Figure 3.4: A cusp

Nevertheless, a piecewise C^1 curve $\Gamma \subset \mathbb{R}^2$ satisfying a double-sided cone condition at junction points can always be considered as a regular stratification, after choosing $\mathbf{M}^0$ as the set of singular points. Indeed, if $x \in \mathbf{M}^0$, the $C^{1,1}$ diffeormorphism Ψ^x just has to "flatten the angle" in order to get a flat stratification (see fig 3.5), which is of course possible.

In order to give a general result that stratifications must satisfy, we need to introduce some objects.

Extended tangent spaces — Let $x \in \mathbf{M}_i^k$ and Ψ^x, $B(x,r)$ as in the definition of (RS). If $x \in \partial\mathbf{M}_j^l$, then, combining Lemma 3.3.3 and Remark 3.3.4, there exists a l-dimensional vector space $V_{l,j}$ such that

$$\Psi^x(\mathbf{M}_j^l \cap B(x,r)) \subset (x + V_{l,j})\,,$$

and we can extend the tangent space to $\mathbf{M}_j^l$ up to x by setting

$$\overline{T}_x\mathbf{M}_j^l := D(\Psi^x(x))^{-1}(V_{l,j})\,.$$

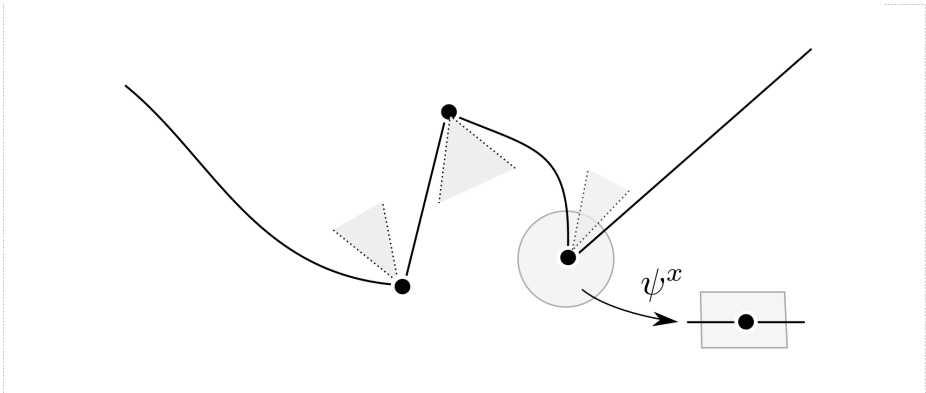


Figure 3.5: A piecewise smooth example

Inward pointing cones — Let $\mathbb{M}$ be an (AFS) and fix $x \in \mathbf{M}_i^k$ for some $k \in \{0..N-1\}$, $i \in \mathcal{I}_k$. We assume that $x \in \partial \mathbf{M}_j^l$ for some $l > k$. We first introduce the notion of *inward directions to* $\mathbf{M}_j^l$ *at* x: a direction $v \in \mathbb{R}^N \setminus \{0\}$ is said to point inward to $\mathbf{M}_j^l$ at x if $x + hv \in \mathbf{M}_j^l$ for $h > 0$ small enough. Since $\mathbf{M}_j^l = \mathcal{O}_{l,j} \cap (x + V_{l,j})$ is flat, all these inward directions v belongs to $V_{l,j}$. Then we define the *inward pointing cone* $C^+_{\text{flat}}(l,j)(x)$ as the set containing all these inward directions to $\mathbf{M}_j^l$ at x. This vector set is strictly positively homogeneous by definition and it does not contain the tangential directions in $\partial \mathbf{M}_j^l$ nor 0.

More generally, in the case of a (RS) the definition of the inward pointing cone is given by

$$C_x^+ \mathbf{M}_j^l := (D\Psi^x(x))^{-1}\Big(C^+_{\text{flat}}(l,j)(x)\Big) \subset \overline{T}_x \mathbf{M}_j^l \,.$$

Here also, the vectors in $C_x^+ \mathbf{M}_l^j$ are pointing strictly inwards $\mathbf{M}_j^l$, excluding the directions tangent to $\partial \mathbf{M}_j^l$ at x and 0. Notice finally that since $(D\Psi^x(x))^{-1}$ is linear, $C_x^+ \mathbf{M}_j^l$ is also strictly positively homogeneous.

An intrinsic characterization of the inward pointing cone can be given. To do so, for a given $x \in \partial \mathbf{M}_j^l$, we consider the C^1-curves $\gamma : \mathbb{R} \to \mathbb{R}^N$ such that $\gamma(0) = x$ and $\gamma(s) \in \mathbf{M}_j^l$ if $s \in (0, s_0)$ for some $s_0 > 0$. We will say that $\gamma \in \Lambda_j^l(x)$ if there exists $\eta > 0$ such that

$$\text{dist}(\gamma(s), \partial \mathbf{M}_j^l) \geq \eta s \quad \text{for all } s \in (0, s_0) \,. \tag{3.12}$$

Then the following characterization holds:

Lemma 3.3.7 *Given $x \in \mathbf{M}_i^k \cap \partial \mathbf{M}_j^l$, we have $C_x^+ \mathbf{M}_j^l = \big\{\dot\gamma(0) : \gamma \in \Lambda_j^l(x)\big\}$.*

Proof — We first prove the result in the case of an (AFS).

Direct inclusion – For the inclusion $C^+_{\rm flat}(l,j)(x) \subset C^+_x\mathbf{M}^l_j$, we have to show that if $v \in C^+_{\rm flat}(l,j)(x)$ there exists $\eta > 0$ such that $\gamma(s) := x + sv$ satisfies (3.12) for $s \in (0, s_0)$, s_0 being small enough. We argue by contradiction: if (3.12) does not hold, there exists a sequence of positive numbers $s_\varepsilon \to 0$ such that

$$0 < \mathrm{dist}(x + s_\varepsilon v, \partial\mathbf{M}^l_j) \le \varepsilon s_\varepsilon$$

(of course the distance is positive because $x + s_\varepsilon v$ is in $\mathbf{M}^l_j$, not on its boundary).

By $(\mathbf{H_{ST}})_{flat}$-(i)-(c) and (iii), we can extract a subsequence of $(s_\varepsilon)_\varepsilon$ (still denoted in the same way to simplify the exposure) such that the distance is achieved for y_ε in *the same* $\mathbf{M}^n_m$ for some $n < l$ and $m \in \mathcal{I}_n$. Hence, if $\mathbf{M}^n_m = (x + V_{n,m}) \cap \mathcal{O}_{n,m}$,

$$\Big|(x + s_\varepsilon v) - (x + w_\varepsilon)\Big| \le \varepsilon s_\varepsilon \quad \text{for some } w_\varepsilon \in V_{n,m}\ .$$

We deduce from this property that

$$\left|v - \frac{w_\varepsilon}{s_\varepsilon}\right| \le \varepsilon\ ,$$

and since $w_\varepsilon/s_\varepsilon \in V_{n,m}$ for any $\varepsilon > 0$, by letting ε tend to 0 we deduce that $v \in V_{n,m}$. It follows that $x + s_\varepsilon v \in (x + V_{n,m})$ and thus, for $\varepsilon > 0$ small enough, $x + s_\varepsilon v \in (x + V_{n,m}) \cap \mathcal{O}_{n,m} \subset \partial\mathbf{M}^l_j$ which contradicts $\mathrm{dist}(x + s_\varepsilon v, \partial\mathbf{M}^l_j) > 0$. Hence (3.12) is proved.

Converse inclusion – In order to prove that $C^+_x\mathbf{M}^l_j \subset C^+_{\rm flat}(l,j)(x)$, we take any $\gamma \in \Lambda^l_j(x)$ and we have to show that $\dot\gamma(0) \in C^+_{\rm flat}(l,j)(x)$. Notice first that $\gamma(s) \in x + V_{l,j}$ for any $s \in (0, s_0)$ and therefore $\dot\gamma(0) \in V_{l,j}$. On the other hand, by the differentiability of γ at 0,

$$\gamma(s) = x + \dot\gamma(0)s + o(s)\ ,$$

and $x + \dot\gamma(0)s \in x + V_{l,j}$. Now, by (3.12) we see that for $s > 0$ small enough,

$$\mathrm{dist}(x + \dot\gamma(0)s, \partial\mathbf{M}^l_j) \ge \mathrm{dist}(\gamma(s), \partial\mathbf{M}^l_j) + o(s) \ge (\eta + o(1))s > 0\ ,$$

which implies that $x + \dot\gamma(0)s \in \mathbf{M}^l_j$ for any $s > 0$ small enough. Hence $\dot\gamma(0) \in C^+_{\rm flat}(l,j)(x)$ and we are done.

The (RS) **case** – Here we use in an essential way the Lipschitz continuity of Ψ^x and its C^1-property.

If $v \in C^+_x\mathbf{M}^l_j$, we claim that the curve $\gamma(s) := (\Psi^x)^{-1}(x + sv)$ belongs to $\Lambda^l_j(x)$: indeed, $\Psi^x(\gamma(s)) = x + sv$ with $v \in C^+_{\rm flat}(l,j)(x)$ and the first part of the proof implies

that it satisfies (3.12) for the (local) flat stratification. Using the Lipschitz continuity of Ψ^x, we deduce that γ also satisfies (3.12), for some other parameters $\tilde{\eta}, \tilde{s}_0 > 0$.

Conversely, if $\gamma \in \Lambda_j^l(x)$, then the curve $\Gamma(\cdot) := \Psi^x(\gamma(\cdot))$ is also in the set $\Lambda_j^l(x)$ (but for the flat stratification) and therefore

$$\dot{\Gamma}(0) = (D\Psi^x)(x)\big(\dot{\gamma}(0)\big) \in C^+_{\rm flat}(l,j)(x)\ .$$

By definition of $C_x^+\mathbf{M}_j^l$, it follows that $\dot{\gamma}(0) \in C_x^+\mathbf{M}_j^l$, and the proof is complete.
Q.E.D.

The main result of this section is the

Proposition 3.3.8 *Let $\mathbb{M}$ be a stratification of $\mathbb{R}^N$, $0 \le k < N$ and $x \in \mathbf{M}_i^k$ for some $i \in \mathcal{I}_k$. Assume that $x \in \partial\mathbf{M}_j^l \cap \partial\mathbf{M}_{j'}^{l'}$ for some $k < l, l' \le N$. If $(l,j) \neq (l,j')$, then*

(i) the following inclusion holds: $\mathbf{M}_i^k \subset \partial\mathbf{M}_j^l \cap \partial\mathbf{M}_{j'}^{l'}$;

(ii) for any $x \in \mathbf{M}_i^k$, $C_x^+\mathbf{M}_j^l \cap C_x^+\mathbf{M}_{j'}^{l'}(x) = \emptyset$.

Though this proposition is simple in its form, it rules out several cusp-like configurations involving various dimensions (see below examples after the proof). In particular, in the case of the piecewise smooth curve in dimension $N = 2$, we recover that the tangents from both sides of a singular point cannot be equal in the limit at such point. Notice that of course, they can possibly make a π-angle but in that case, the inward pointing directions are opposite.

Proof — Concerning (i), the result follows directly from $(\mathbf{H_{ST}})_{flat}$-(ii): since

$$x \in \overline{\mathbf{M}_j^l} \cap \overline{\mathbf{M}_{j'}^{l'}}\,,$$

we get that $\mathbf{M}_i^k$ is included in both $\overline{\mathbf{M}_j^l} \cap \overline{\mathbf{M}_{j'}^{l'}}$. But since $l > k$, $\mathbf{M}_i^k$ does not intersect with $\mathbf{M}_j^l$ nor with $\mathbf{M}_{j'}^{l'}$, so that (i) holds.

We now turn to (ii) and consider first the case of an (AFS). Since $(l,j) \neq (l',j')$ then $\mathbf{M}_j^l \cap \mathbf{M}_{j'}^{l'} = \emptyset$ which clearly implies that the inward pointing cones are disjoint. Indeed, as we noticed before, if $e \in C^+_{\rm flat}(l,j) \cap C^+_{\rm flat}(l',j')$ then for h small enough, we get that $x + he \in \mathbf{M}_j^l \cap \mathbf{M}_{j'}^{l'}$ which is a contradiction.

In the (RS) case, the conclusion follows from the fact that since $D(\Psi^x(x))^{-1}$ is invertible it cannot map two different directions on the same one. More precisely, assume that

$$e \in C_x^+\mathbf{M}_j^l \cap C_x^+\mathbf{M}_{j'}^{l'} \neq \emptyset\,.$$

Then there exist two vectors $w \in C^+_{\text{flat}}(i,j)$, $w' \in C^+_{\text{flat}}(l',j')$ such that

$$e = D(\Psi^x(x))^{-1}(w) = D(\Psi^x(x))^{-1}(w')\,.$$

But since $w \neq w'$ because they belong to $C^+_{\text{flat}}(l,j)$ and $C^+_{\text{flat}}(l',j')$ respectively, we get a contradiction with the bijectivity of $D(\Psi^x(x))^{-1}$.

Q.E.D.

Remark 3.3.9 *The fact that the cones $C_x^+\mathbf{M}_j^l$ do not intersect implies that various cone conditions hold at x, separating the manifolds touching at this point (which are in finite number, see $(\mathbf{H_{ST}})_{flat}$-$(i)(c)$). However, building explicitly such cones is quite difficult in all its generality and we wil not try to state it here. But notice that there is a lot of freedom in choosing the directions of such cones: if $C_x^+\mathbf{M}_j^l \cap C_x^+\mathbf{M}_{j'}^{l'} = \emptyset$, any direction e at positive distance from both cones allows to build a separating cone.*

Typical situations — Of course very complex situations can occur involving different dimensions but let us see two simple situations to understand the meaning of Proposition 3.3.8-(ii).

Fig. 3.6 — On the left the situation is allowed since at the point $\{x\} = \mathbf{M}^0$, $C_x^+\mathbf{M}_1^2 = \mathbf{M}_1^2$, $C_x^+\mathbf{M}_2^2 = \mathbf{M}_2^2$ and therefore $C_x^+\mathbf{M}_1^2 \cap C_x^+\mathbf{M}_2^2 = \emptyset$. Notice however that the boundaries intersect, which corresponds to the direction of $\mathbf{M}_1^1$.

On the right, it is clear that the problem does not come from $C_x^+\mathbf{M}_{1/2/3}^2$ which do not intersect (although $C_x^+\mathbf{M}_3^2 = \emptyset$), but from the $\mathbf{M}^1$ manifolds since $C_x^+\mathbf{M}_1^1 \cap C_x^+\mathbf{M}_2^1 = \mathbf{M}_1^1 \neq \emptyset$. This cusp-type situation is of course not allowed.

Fig. 3.7 — On the left the situation is allowed since the semi-line $\mathbf{M}^1$ makes a non-zero contact angle with the plane $\mathbf{M}^2$. However, using for instance the caracterization in Lemma 3.3.7 we see that $C_x^+\mathbf{M}^1 = \mathbb{R}_*^+ e$, while $C_x^+\mathbf{M}^2 = \mathbf{M}^2$. Hence $C_x^+\mathbf{M}^1 \cap C_x^+\mathbf{M}^2 = \mathbb{R}_*^+ e \neq \emptyset$, another cusp-type situation that is not allowed.

Stratifications in domains — The question is whether we can extend or not the notion of stratification in $\mathbb{R}^N$ to the case of open sets Ω.

Figure 3.6: Examples in 2-D

Figure 3.7: Examples in 3-D

At this point of the book, we do not enter into details on this because we devote a complete part of the book (Part V) to the case of state-constraint problems. Let us just mention that when we consider a domain Ω, its boundary $\partial\Omega$ has to be understood as a specific part of the stratification. And if the boundary is not regular, we use the stratified approach to decompose it in various manifolds of different dimensions.

The conditions on the inward pointing cones that we proved above imply that Ω has to satisfy a double-sided cone condition in order to deal with it in the stratified approach. This cone condition (at least the interior one) is also used in Section 26.4 in order to get a suitable boundary regularity for subsolutions.

3.4 Partial regularity, partial regularization

In this section, motivated by Sections 3.2 and 3.3, we present some key ingredients in the proof of local comparison results for HJ equations with discontinuities. The assumptions we are going to use are those which are needed everywhere in this book to prove any kind of results and therefore we define at the end of the section a "good local framework for HJ Equations with discontinuities".

Local comparison results lead to consider HJ-Equations in a ball, namely

$$\mathbb{G}(X, u, Du) = 0 \quad \text{in } B_\infty(\bar{X}, r)\ , \tag{3.13}$$

where $\bar{X} \in \mathbb{R}^N$ and $r > 0$ are fixed. We recall that the notation X can refer to either $X = x$ or $X = (x, t)$. Because of the previous section, it is natural to assume that the discontinuities in this equation have a general *flat stratification-type structure*: the variable $X \in \mathbb{R}^N$ can be decomposed as $(Y, Z) \in \mathbb{R}^k \times \mathbb{R}^{N-k}$ and $\mathbb{G}$ is continuous w.r.t. u, p and Y but not with respect to Z. In particular we have in mind that locally around $\bar{X}$, Hamiltonian $\mathbb{G}$ has a discontinuity on $\Gamma_0 = \{(Y, Z);\ Z = 0\}$ which can be identified with $\mathbb{R}^k$.

The properties of discontinuous sub and supersolutions on Γ are playing a key role in the proof of such local comparison results and the aim of the next section is to introduce the notion of "regular discontinuous function".

3.4.1 Regular discontinuous functions

The following definition provides several notions of regularity for discontinuous functions.

Definition 3.4.1 *Let $A \subset \mathbb{R}^k$, $f : A \to \mathbb{R}$ an u.s.c. [resp. lsc] function and $\omega \subset A$.*

(*i*) *The function f is said to be ω-regular at $x \in \partial\omega \cap A$ if*

$$f(x) = \limsup_{\substack{y \to x \\ y \in \omega}} f(y) \qquad \big[resp.\ f(x) = \liminf_{\substack{y \to x \\ y \in \omega}} f(y)\big].$$

(*ii*) *If $\mathcal{E} \subset \partial\omega \cap A$, f is said to be ω-regular on $\mathcal{E}$ if it is ω-regular at any point of $\mathcal{E}$.*

(*iii*) *If A is a closed subset of $\mathbb{R}^k$ and $\mathcal{E} \subset A$ is such that $\overline{A \setminus \mathcal{E}} = A$, f is said to be regular on $\mathcal{E}$ if, at any point $x \in \mathcal{E}$ and for any connected component ω of $A \setminus \mathcal{E}$ such that $x \in \overline{\omega}$, f is ω-regular at x.*

The regularity of u.s.c. subsolution or l.s.c. supersolutions is used in several type of situations: the most classical one is when we consider a stationary HJ-equation set in a domain Ω of $\mathbb{R}^N$; a natural choice is $A = \overline{\Omega}$, $\omega = \Omega$, $\mathcal{E} = \partial\Omega$. In the study of the Dirichlet problem (cf. for example [22, 23, 24]), such regularity of the sub and/or supersolution is needed to have a comparison result up to the boundary. The point is to avoid "artificial values" of these sub or supersolution on $\partial\Omega$. For the case of evolution

equations, one may also choose $A = \overline{\Omega} \times (0, T_f)$, $\omega = \Omega \times (0, T_f)$, $\mathcal{E} = \partial\Omega \times (0, T_f)$. In the same context, some result can be formulated using the ω-regularity of the sub or supersolution at some point of $\partial\Omega \times (0, T_f)$ (cf. Section 3.5).

In this book, an other important example is the case when $A = \mathbb{R}^N \times (0, T_f)$ and $\mathcal{E} = \mathbb{M} \times (0, T_f)$ where $\mathbb{M} \subset \mathbb{R}^N$ is a k-dimensional manifold. But here there are two different cases: if $k < N-1$, $\omega = A \setminus \mathcal{E}$ is connected and there is no difference between (ii) and (iii). But if $\mathcal{E}$ is an hyperplane, then $A \setminus \mathcal{E}$ has two connected components ω_1, ω_2 and, roughly speaking, the regularity property has to hold in both side of $\mathcal{E}$, i.e. both for ω_1 and ω_2. This is actually the case which will be studied in Part II and III.

3.4.2 Regularity of subsolutions

The aim of this section is to study subsolutions of (3.13) and to prove that, under suitable assumptions, they satisfy some "regularity properties".

We immediately point out that, for reasons which will clear later on in this book, we are not going to use only subsolutions in the Ishii sense and therefore, we are not going to use only the lower semi-continuous enveloppe of some Hamiltonian as in the Ishii definition. To simplify matter, we assume here that the function $\mathbb{G}$ contains all the necessary information for subsolutions. In other words, by subsolution of (3.13), we mean an u.s.c. function u which satisfies

At any maximum point $X \in B_\infty(\bar{X}, r)$ of $u - \phi$, where ϕ is a smooth test-function, we have

$$\mathbb{G}(X, u(X), D\phi(X)) \leq 0 \, .$$

In the sequel, we decompose Du as $(D_Y u, D_Z u)$ (the same convention is used for the test-functions ϕ) and the corresponding variable in $\mathbb{G}$ will be $p = (p_Y, p_Z)$.

In order to state our main result on the "regularity of subsolutions", we introduce the assumption

($\mathbf{NC}_w$) *Weak Normal Controllability*:

(i) If $N - k > 1$, there exists $e \in \mathbb{R}^{N-k}$ such that, for any $R > 0$, we have

$$\mathbb{G}(X, u, (p_Y, Ce)) \to +\infty \quad \text{when } C \to +\infty \, ,$$

uniformly for $X = (Y, Z) \in B_\infty(\bar{X}, r)$, $|u| \leq R$, $|p_Y| \leq R$.

(ii) If $N - k = 1$, this property holds for $e = +1$.

(iii) If $N - k = 1$, this property holds for $e = -1$.

Our result is the following.

Proposition 3.4.2 *We have*

(a) *Assume that* $(\mathbf{NC}_w)$ *holds. If* u *be a bounded, u.s.c. subsolution of* (3.13) *and if* $\Gamma_c := B_\infty(\bar{X}, r) \cap \{(Y, Z)\ ;\ Z = c\} \neq \emptyset$, *then* u *is regular on* Γ_c. *In particular,* u *is regular on* $\Gamma = \Gamma_0$.

(b) *If* u *be a bounded, u.s.c. subsolution of* (3.13), *if* $N - k = 1$ *and if* $(\mathbf{NC}_w)$*-(ii) holds, then* u *is regular on* Γ_0 *with respect to* $B_\infty(\bar{X}, r) \cap \{Z > 0\}$. *In the same way, if* $N - k = 1$ *and if* $(\mathbf{NC}_w)$*-(iii) holds, then* u *is regular on* Γ_0 *with respect to* $B_\infty(\bar{X}, r) \cap \{Z < 0\}$.

(c) *If* u *is a subsolution of* $\mathbb{G} = 0$ *on* $B_\infty(\bar{X}, r) \cap \{Z \geq 0\}$ *and if either* $(\mathbf{NC}_w)$*-(ii) or* $(\mathbf{NC}_w)$*-(iii) holds then* u *is is regular on* Γ_0 *with respect to* $B_\infty(\bar{X}, r) \cap \{Z > 0\}$.

This proposition means that in $B_\infty(\bar{X}, r)$, subsolutions cannot have "singular values" on affine subspaces of the form $\{(Y, Z);\ Z = c\}$. By singular values we mean here values which are not given by limits coming from outside of those affine subspaces. The three above results can be interpreted in the following way: (a) is the general "good case" of a subsolution which is regular on Γ_0, a set of discontinuity for $\mathbb{G}$, when we use the entire assumption $(\mathbf{NC}_w)$. Result (b) is the case when Γ_0 is an affine hyperplan but only one part of assumption $(\mathbf{NC}_w)$. Result (c) deals with boundary regularity; such regularity property is useful in order to use the results of Section 3.5.

Proof — We start by (a). We recall that, thanks to Definition 3.4.1, in the case when $k < N - 1$, we have to show that, for any $X = (Y, Z) \in \Gamma_c$

$$u(X) = \limsup\{u(Y', Z')\ ;(Y', Z') \to X,\ Z' \neq Z\}\ . \tag{3.14}$$

since $B_\infty(\bar{X}, r) \setminus \Gamma_c$ is connected and $Z' \neq Z$ is equivalent to $(Y', Z') \notin \Gamma_c$. Moreover, if $N - k = 1$, we also have to show

$$\begin{aligned} u(X) &= \limsup\{u(Y', Z');(Y', Z') \to X,\ Z' > Z\} \\ &= \limsup\{u(Y', Z');(Y', Z') \to X,\ Z' < Z\}, \end{aligned} \tag{3.15}$$

since in this case, $B_\infty(\bar{X}, r) \setminus \Gamma_c$ has two connected components. In order to prove (3.14) we argue by contradiction assuming that

$$u(X) > \limsup\{u(Y', Z')\ ;(Y', Z') \to X,\ Z' \neq Z\}\ .$$

Therefore there exists some $\delta > 0$ small enough such that $u(Y', Z') < u(X) - \delta$ if $|(Y', Z') - X| < \delta$, with $Z' \neq Z$. Next, for $\varepsilon > 0$, we consider the function

$$Y' \mapsto u(Y', Z) - \frac{|Y - Y'|^2}{\varepsilon} .$$

If ε is small enough, this function has a local maximum point at Y_ε which satisfies $|Y_\varepsilon - Y| < \delta$ and $u(Y_\varepsilon, Z) \geq u(X)$. But because of the above property, there exists a neighborhood $\mathcal{V}$ of (Y_ε, Z) such that, if $(Y', Z') \in \mathcal{V}$ and $Z' \neq Z$, $u(Y', Z') < u(Y_\varepsilon, Z) - \delta$.

This implies that (Y_ε, Z) is also a local maximum point of the function

$$(Y', Z') \mapsto u(Y', Z') - \frac{|Y - Y'|^2}{\varepsilon} - Ce \cdot (Z' - Z) .$$

for any positive constant C and the vector e of $\mathbb{R}^{N-k}$ given by $\mathbf{(NC}_w\mathbf{)}$. But, by the subsolution property, we have

$$\mathbb{G}\left((Y_\varepsilon, Z), u(Y_\varepsilon, Z), \left(\frac{2(Y_\varepsilon - Y)}{\varepsilon}, Ce\right)\right) \leq 0 .$$

But, using $\mathbf{(NC}_w\mathbf{)}$ with $R = \max(||u||_\infty, 2\delta\varepsilon^{-1})$, we reach a contradiction for C large enough.

For the case $N - k = 1$, we repeat the same argument by choosing either $e = +1$ or $e = -1$.

Indeed, if we assume by contradiction that $u(X) > \limsup\{u(Y', Z') \, ; (Y', Z') \to X, \ Z' > Z\}$, we argue as above but looking at a local maximum point of the function

$$(Y', Z') \mapsto u(Y', Z') - \frac{|Y - Y'|^2}{\varepsilon} + C(Z' - Z) ,$$

therefore with the choice $e = -1$. We first look at a maximum point of this function in compact set of the form

$$\{(Y', Z'); |Y' - Y| + |Z' - Z| \leq \delta, \ Z' \leq Z\} .$$

Notice that, in this set, the term $C \cdot (Z' - Z)$ is negative (therefore it has the right sign) and this function has a local maximum point which depends on ε and C, but, in order to simplify the notations, we denote it by $(\bar{Y}, \bar{Z})$. We have $u(\bar{Y}, \bar{Z}) \geq u(X)$ by the maximum point property and we have $(\bar{Y}, \bar{Z}) \to (Y_\varepsilon, 0)$ when $C \to +\infty$, where $(Y_\varepsilon, 0)$ is a maximum point of the function

$$Y' \mapsto u(Y', 0) - \frac{|Y - Y'|^2}{\varepsilon} .$$

Using that $u(X) > \limsup\{u(Y',Z')\ ;(Y',Z') \to X,\ Z' > Z\}$, we clearly have the same property at $(Y_\varepsilon, 0)$ and therefore, for C large enough, at $(\bar{Y},\bar{Z})$ which is also a maximum point of the above function for all (Y',Z') such that $|Y'-Y|+|Z'-Z| \leq \delta$ if δ is chosen small enough. And we reach a contradiction as in the first part of the proof using $\mathbf{(NC}_w\mathbf{)}$.

Hence u is regular with respect to the the $\{Z' > Z\}$ side but an analogous proof shows the same property for the other side.

Finally the proofs of (b) and (c) rely on analogous arguments, therefore we skip them. We just point out that, for (c), the fact that $B_\infty(\bar{X},r) \cap \{Z < 0\}$ is not part of the domain allows to do the proof as in the first case of (a).

Q.E.D.

Remark 3.4.3

(*i*) *We have stated and proved Proposition 3.4.2 under Assumption* $\mathbf{(NC}_w\mathbf{)}$ *but, in the sequel, we will mainly use Assumption* $\mathbf{(NC)}$ *which will be introduced in the next section. Clearly* $\mathbf{(NC)}$ *implies* $\mathbf{(NC}_w\mathbf{)}$.

On an other hand, we point out that, in control problems, provided that the Hamiltonian $\mathbb{G}$ *is defined in a suitable way,* $\mathbf{(NC}_w\mathbf{)}$ *is equivalent to the existence of a non-tangential dynamic in the case* $N-k>1$ *while, in the case when* $N-k=1$*, it is equivalent to the existence of two dynamics pointing strictly inward each of the two half-spaces defined by the hyperplan* Γ_0.

(*ii*) *Notice that a similar result still holds for l.s.c. subsolutions à la Barron-Jensen, where we consider minimum points of* $u-\phi$*. Of course in this case, the regularity property has to be expressed with a liminf instead of a limsup but the modifications are straightforward. We refer to Section 23.4 where the Barron-Jensen approach is detailed and we use this liminf regularity property.*

3.4.3 Regularization of subsolutions

The aim of this section is to construct, for a given subsolution, a suitable approximation by Lipschitz continuous subsolutions which are even C^1 in Y in the convex case.

To do so, we use for $\mathbb{G}$ the following assumptions: for any $R>0$, there exist some constants $C_i^R > 0$ for $i = 1\dots 4$, a modulus of continuity $m^R : [0,+\infty[\to [0,+\infty[$ and either a constant $\lambda^R > 0$ or $\mu^R > 0$ such that

(TC) *Tangential Continuity*: for any $X_1 = (Y_1, Z), X_2 = (Y_2, Z) \in B_\infty(\bar{X}, r)$, $|u| \leq R$ and $p \in \mathbb{R}^N$, then

$$|\mathbb{G}(X_1, u, p) - \mathbb{G}(X_2, u, p)| \leq C_1^R |Y_1 - Y_2|.|p| + m^R\big(|Y_1 - Y_2|\big) .$$

(NC) *Normal Controllability*: for any $X = (Y, Z) \in B_\infty(\bar{X}, r)$, $|u| \leq R$, $p = (p_Y, p_Z) \in \mathbb{R}^N$, then

$$\mathbb{G}(X, u, p) \geq C_2^R |p_Z| - C_3^R |p_Y| - C_4^R .$$

Notice that **(NC)** and **(TC)** have counterparts in terms of control elements i.e. dynamic and cost, see **(NC-BCL)**, **(TC-BCL)**, p. 139. For the last assumption, if $p_Y \in \mathbb{R}^k$, we set $p_Y = (p_{Y_1}, \cdots, p_{Y_k})$

(Mon) *Monotonicity*: for any $R > 0$, there exists $\lambda_R, \mu_R \in \mathbb{R}$, such that one of the two following properties holds

(**Mon**-u): $\lambda_R > 0$ and for any $X \in B_\infty(\bar{X}, r)$, $p = (p_Y, p_Z) \in \mathbb{R}^N$, any $R \leq u_1 \leq u_2 \leq R$,

$$\mathbb{G}(X, u_2, p) - \mathbb{G}(X, u_1, p) \geq \lambda^R (u_2 - u_1) ; \tag{3.16}$$

(**Mon**-p): (3.16) holds with $\lambda_R = 0$, we have $\mu_R > 0$ and

$$\mathbb{G}(X, u_1, q) - \mathbb{G}(X, u_1, p) \geq \mu^R (q_{Y_1} - p_{Y_1}) , \tag{3.17}$$

for any $q = (q_Y, p_Z)$ with $p_{Y_1} \leq q_{Y_1}$ and $p_{Y_i} = q_{Y_i}$ for $i = 2, ..., p$.

Before providing results using these assumptions, we give an example showing the type of properties hidden behind these general assumptions.

Example – We consider an equation in $\mathbb{R}^{N+1}$ written as

$$\mu u_t + H((x_1, x_2), t, u, (D_{x_1} u, D_{x_2} u)) = 0 \quad \text{in } \mathbb{R}^k \times \mathbb{R}^{N-k} \times (0, +\infty) ,$$

Here the constant μ satisfies $0 \leq \mu \leq 1$ and in order to simplify we can assume that H is a continuous function. To be in the above framework, we write $X = (t, x_1, x_2) \in (0, +\infty) \times \mathbb{R}^k \times \mathbb{R}^{N-k}$ and we set $Y = (t, x_1) \in \mathbb{R}^{k+1}$, $Z = x_2 \in R^{N-k}$ and

$$\mathbb{G}(X, u, P) = \mu p_t + H((x_1, x_2), t, u, (p_{x_1}, p_{x_2})) ,$$

where $P = (p_t, (p_{x_1}, p_{x_2}))$.

In order to formulate **(TC)**, **(NC)** and **(Mon)** in a simple way, we assume that $(x_1, t, u) \mapsto H((x_1, x_2), t, u, (p_{x_1}, p_{x_2}))$ is locally Lipschitz continuous for any x_2, p_{x_1}, p_{x_2}. Then these assumptions can be formulated in the following way

- For **(TC)**, recalling that we always argue *locally*, one has to assume that, for any $R > 0$, there exists a constant $C_1^R > 0$ such that, for any $(t, x_1, x_2) \in [0, +\infty) \times \mathbb{R}^k \times \mathbb{R}^{N-k}$ with $t + |x_1| + |x_2| \leq R$, $|u| \leq R$ and $(p_{x_1}, p_{x_2}) \in \mathbb{R}^k \times \mathbb{R}^{N-k}$, we have

$$|D_{x_1} H((x_1, x_2), t, u, (p_{x_1}, p_{x_2}))|, |D_t H((x_1, x_2), t, u, (p_{x_1}, p_{x_2}))| \leq C_1^R(|(p_{x_1}, p_{x_2})| + 1) \ .$$

Here we are in the simple case when $m^R(\tau) = C_1^R \tau$ for any $\tau \geq 0$. One can easily check that these assumptions imply the right property for $\mathbb{G}$ with $Y = (t, x_1)$.

- Next since $p_{Y_1} = p_t$, **(Mon)** reduces to either $\mu > 0$ or $D_u H((x_1, x_2), t, u, (p_{x_1}, p_{x_2})) \geq \lambda_R > 0$ for the same set of $(t, x_1, x_2), u, (p_{x_1}, p_{x_2})$ as for **(TC)**. Hence, either we are in a real time evolution context ($\mu > 0$), or $\mu = 0$ and the standard assumption "H strictly increasing in u" has to hold.

- Finally **(NC)** holds if H satisfies the following coercivity assumption in p_{x_2}

$$H((x_1, x_2), t, u, (p_{x_1}, p_{x_2})) \geq C_2^R |p_{x_2}| - C_3^R |p_{x_1}| - C_4^R \, ,$$

again for the same set of $(t, x_1, x_2), u, (p_{x_1}, p_{x_2})$ as for **(TC)**. Notice that in order to check **(NC)** for $\mathbb{G}$, the constant C_3^R may have to be changed in order to incorporate the μp_t-term if $\mu \neq 0$.

Our result concerning the approximation by Lipschitz subsolutions is the

Proposition 3.4.4 *Let u be a bounded subsolution of (3.13) and assume that* **(TC)**, **(NC)** *and* **(Mon)** *hold. Then there exists a sequence of Lipschitz continuous functions $(u^\varepsilon)_\varepsilon$ defined in $B_\infty(\bar{X}, r - a(\varepsilon))$ where $a(\varepsilon) \to 0$ as $\varepsilon \to 0$ such that*

(i) each u^ε is a subsolutions of (3.13) in $B_\infty(\bar{X}, r - a(\varepsilon))$,

(ii) each u^ε are semi-convex in the Y-variable

(iii) $\limsup^* u^\varepsilon = u$ *as* $\varepsilon \to 0$.

Remark 3.4.5 *Equations of the form*

$$\max(u_t + G_1(x, D_x u); G_2(x, u, D_x u)) = 0 \ ,$$

do not satisfy **(Mon)** *even if G_2 satisfies* (**Mon**-u) *and the Hamiltonian $p_t + G_1(x, p_x)$ satisfies* (**Mon**-p). *To overcome this difficulty, we have to use a change of variable of*

the form $v = \exp(Kt) \cdot u$ *in order that both Hamiltonians satisfy* (**Mon**-u)*, which is a natural change (cf. Section 5.5). Of course, suitable assumptions on* G_1 *and* G_2 *are needed in order to have* **(TC)** *and* **(NC)**.

Proof — First we can drop the R in all the constants appearing in the assumptions by remarking that, u being bounded, we can use the constants with $R = ||u||_\infty$.

In the case, when **(Mon)** holds with $\lambda > 0$ we set for $X = (Y, Z)$

$$u^\varepsilon(X) := \max_{Y' \in \mathbb{R}^k} \left\{ u(Y', Z) - \frac{(|Y - Y'|^2 + \varepsilon^4)^{\alpha/2}}{\varepsilon^\alpha} \right\},$$

for some (small) $\alpha > 0$ to be chosen later on, while, in the other case we set

$$u^\varepsilon(X) := \max_{Y' \in \mathbb{R}^k} \left\{ u(Y', Z) - \exp(KY_1) \frac{|Y - Y'|^2}{\varepsilon^2} \right\},$$

for some constant K to be chosen later on.

In both cases, the maximum is achieved for some Y' such that $|Y - Y'| \leq O(\varepsilon)$, hence with a point $(Y', Z) \in B_\infty(\bar{X}, r)$ for $a(\varepsilon) > O(\varepsilon)$, and therefore u^ε is well-defined in $B_\infty(\bar{X}, r - a(\varepsilon))$. By standard properties of the sup-convolution, the u^ε's are continuous in Y but, for the time being, not necessarily in Z, despite of Proposition 3.4.2.

To prove that u^ε is a subsolution in $B_\infty(\bar{X}, r - a(\varepsilon))$, we consider a smooth test-function ϕ and we assume that $X \in B_\infty(\bar{X}, r - a(\varepsilon))$ is a maximum point of $u^\varepsilon - \phi$. We first consider the "$\lambda > 0$" case : if

$$u^\varepsilon(X) = u(Y', Z) - \frac{(|Y - Y'|^2 + \varepsilon^4)^{\alpha/2}}{\varepsilon^\alpha},$$

then (Y', Z) is a maximum point of $(\tilde{Y}, \tilde{Z}) \mapsto u(\tilde{Y}, \tilde{Z}) - \dfrac{\left(|Y - \tilde{Y}|^2 + \varepsilon^4\right)^{\alpha/2}}{\varepsilon^\alpha} - \phi(Y, \tilde{Z})$, and therefore, by the subsolution property for u

$$\mathbb{G}((Y', Z), u(Y', Z), (p_Y, D_Z\phi(Y, Z))) \leq 0 \, ;$$

where

$$p_Y := \alpha(Y' - Y) \frac{(|Y - Y'|^2 + \varepsilon^4)^{\alpha/2 - 1}}{\varepsilon^\alpha} \, .$$

On the other hand the maximum point property in Y, implies that $p_Y = D_Y\phi(Y, Z)$.

To obtain the right inequality, we have to replace (Y', Z) by $X = (Y, Z)$ in this inequality and $u(Y', Z)$ by $u^\varepsilon(X)$. To do so, we have to use **(TC)**; in order to do it, we need to have a precise estimate on the term $|Y - Y'||(p_Y, D_Z\phi(Y, Z))|$. The explicit form of p_Y gives it for $|Y - Y'||p_Y|$ but this is not the case for $|Y - Y'|.|D_Z\phi(Y, Z)|$ since we have not such a precise information on $D_Z\phi(Y, Z)$. Instead we have to use **(NC)** which implies

$$C_2|D_Z\phi(Y, Z)| - C_3|p_Y| - C_4 \leq 0 \ .$$

(remember that we have dropped the dependence in R for all the constants). On the other hand, we have combining **(TC)** and **(Mon)**

$$\mathbb{G}(X, u^\varepsilon(X), (D_Y\phi(Y, Z), D_Z\phi(Y, Z))) \leq \mathbb{G}((Y', Z), u(Y', Z), (p_Y, D_Z\phi(Y, Z)))+$$

$$C_1|Y - Y'||D\phi(X)| + m(|Y - Y'|) - \lambda\frac{\left(|Y - \tilde{Y}|^2 + \varepsilon^4\right)^{\alpha/2}}{\varepsilon^\alpha} \ .$$

It remains to estimate the right-hand side of this inequality: we have seen above that $|Y - Y'| = O(\varepsilon)$ and **(NC)** implies that

$$|D\phi(X)| \leq \bar{K}(|p_Y| + 1) \ ,$$

for some large constant $\bar{K}$ depending only on C_2, C_3, C_4. Finally

$$|Y - Y'||p_Y| = \alpha|Y - Y'|^2\frac{(|Y - Y'|^2 + \varepsilon^4)^{\alpha/2-1}}{\varepsilon^\alpha} \leq \alpha\frac{\left(|Y - \tilde{Y}|^2 + \varepsilon^4\right)^{\alpha/2}}{\varepsilon^\alpha} \ .$$

By taking $\alpha < \bar{K}$, we finally conclude that

$$\mathbb{G}(X, u^\varepsilon(X), (D_Y\phi(Y, Z), D_Z\phi(Y, Z)) \leq O(\varepsilon) + m(O(\varepsilon)) \ ,$$

and changing u^ε in $u^\varepsilon - \lambda^{-1}(O(\varepsilon) + m(O(\varepsilon)))$, we have the desired property.

In the μ-case, the equality $p_Y = D_Y\phi(Y, Z)$ is replaced by

$$D_Y\phi(Y, Z) = -K\exp(KY_1)\frac{|Y - Y'|^2}{\varepsilon^2}e_1 + \exp(Kt)\frac{(Y' - Y)}{\varepsilon^2} \ ,$$

where e_1 is the vector $(1, 0, \cdots, 0)$ in $\mathbb{R}^k$. The viscosity subsolution inequality for u at (Y', Z) reads

$$\mathbb{G}((Y', Z), u(Y', Z), (\tilde{p}_Y, D_Z\phi(Y, Z)) \leq 0 \ ,$$

where $\tilde{p}_Y = \exp(Kt)\dfrac{(Y' - Y)}{\varepsilon^2}$.

We first use **(NC)**, which implies

$$|D\phi(X)| \leq \bar{K}(|\tilde{p}_Y| + 1) = \bar{K}(\exp(Kt)\frac{|Y' - Y|}{\varepsilon^2} + 1) \ .$$

Then we combine **(TC)** and **(Mon)** to obtain

$$\mathbb{G}(X, u^\varepsilon(X), (D_Y\phi(Y,Z), D_Z\phi(Y,Z)) \leq \mathbb{G}((Y',Z), u(Y',Z), (\tilde{p}_Y, D_Z\phi(Y,Z)) +$$

$$C_1|Y - Y'||D\phi(X)| + m(|Y - Y'|) - \mu K \exp(KY_1)\frac{|Y - Y'|^2}{\varepsilon^2} \ .$$

We conclude easily as in the first case choosing K such that $\mu K > C_1\bar{K}$.

Properties (ii) and (iii) are classical properties which are easy to obtain and we drop the proof.

We conclude this proof by sketching the proof of the Lipschitz continuity of u^ε in Z. To do so, we write $\bar{X} = (\bar{Y}, \bar{Z})$ and for any fixed Y such that $|Y - \bar{Y}| \ < r - a(\varepsilon)$, we consider the function $Z \mapsto u^\varepsilon(Y,Z)$. By using **(NC)** and the Lipschitz continuity of u^ε in the Y-variable, it is easy to prove that this function is a subsolution of

$$C_2|D_Z w| \leq C_3 K_\varepsilon + C_4 \ ,$$

where $K_\varepsilon = ||D_Y u^\varepsilon||_\infty$ and the estimates of $D_Z u^\varepsilon$ follows.

Q.E.D.

The convex case – The above regularization result can be improved when some convexity property of the Hamiltonian holds. More precisely, let us introduce the following assumption

($\mathbf{H_{Conv}}$) : *For any $X \in B_\infty(\bar{X}, r)$, the function $(u,p) \mapsto \mathbb{G}(X,u,p)$ is convex.*

We begin with a result concerning convex combinations of subsolutions. While the result is interesting in itself even in the case of continuous Hamiltonians, we actually need it to make a suitable regularization of subsolutions. By a convex combination of subsolutions u_i for $i = 1, .., n$, we mean of course a finite sum

$$W := \sum_{i=1}^n \mu_i u_i \ , \quad \text{where for all } i, \ \mu_i \geq 0 \quad \text{and} \quad \sum_{i=1}^n \mu_i = 1 \ .$$

Lemma 3.4.6 *Assume that* $(X,r,p) \mapsto \mathbb{G}(X,r,p)$ *is l.s.c. and satisfies* $(\mathbf{H_{Conv}})$. *Then any convex combination of Lipschitz continuous subsolutions of* $\mathbb{G}=0$ *is a subsolution of* $\mathbb{G}=0$.

Proof — We just sketch the proof since most of the arguments are rather standard. We have only to prove the result for a convex combination of two subsolutions $W := \lambda w_1 + (1-\lambda)w_2$, the general case involving n subsolutions for $n>2$ deriving immediately by iteration of the result. Of course, we can assume w.l.o.g. that $0<\lambda<1$.

Let ϕ be a smooth test-function and $\tilde{X} \in B_\infty(\bar{X},r)$ a local strict maximum point of $W-\phi$ in $\overline{B(\tilde{X},\tilde{r})} \subset B_\infty(\bar{X},r)$. We use a tripling of variables by considering in $\overline{B(\tilde{X},\tilde{r})}^3$ the function

$$\Psi(X_1,X_2,X) := \lambda w_1(X_1)+(1-\lambda)w_2(X_2)-\phi(X)-\lambda\frac{|X_1-X|^2}{\varepsilon}-(1-\lambda)\frac{|X_2-X|^2}{\varepsilon}\,.$$

Denoting by $(X_1^\varepsilon, X_2^\varepsilon, X^\varepsilon)$ a maximum point of this function and applying Lemma 6.4.1 in Section 6.4, we have $(X_1^\varepsilon, X_2^\varepsilon, X^\varepsilon) \to (\tilde{X},\tilde{X},\tilde{X})$ when $\varepsilon \to 0$, therefore $X_1^\varepsilon, X_2^\varepsilon, X^\varepsilon \in B(\tilde{X},\tilde{r})$ for ε small enough. Hence we get the viscosity inequalities

$$\mathbb{G}(X_1^\varepsilon, w_1(X_1^\varepsilon), P_1^\varepsilon) \le 0 \quad , \quad \mathbb{G}(X_2^\varepsilon, w_2(X_2^\varepsilon), P_2^\varepsilon) \le 0\,,$$

and the property $D\phi(X) = \lambda P_1^\varepsilon + (1-\lambda)P_2^\varepsilon$, where, for $i=1,2$,

$$P_i^\varepsilon = \frac{2(X_i^\varepsilon - X^\varepsilon)}{\varepsilon}\,.$$

Using the Lipschitz continuity of w_1, w_2, the P_i^ε are uniformly bounded with respect to ε and extracting if necessary subsequences, we can assume that they converge respectively to P_i when $\varepsilon \to 0$.

Letting ε tend to 0, using in a crucial way the lower semi-continuity of $\mathbb{G}$, we are lead to the same situation as above:

$$\mathbb{G}(\tilde{X}, w_1(\tilde{X}), P_1) \le 0 \quad , \quad \mathbb{G}(\tilde{X}, w_2(\tilde{X}), P_2) \le 0\,.$$

Because of the continuity of $D\phi$, $D\phi(\tilde{X}) = \lambda P_1 + (1-\lambda)P_2$. So, making the convex combinaison of the above inequalities, after using $(\mathbf{H_{Conv}})$ we finally get

$$\mathbb{G}(\tilde{X}, W(\tilde{X}), D\phi(\tilde{X})) \le 0\,,$$

which proves that W is a viscosity subsolution of $\mathbb{G}=0$.

Q.E.D.

We can now state the regularization result

Proposition 3.4.7 *Under the assumptions of Proposition 3.4.4, if $\mathbb{G}$ is l.s.c. and if $(\mathbf{H_{Conv}})$ holds then the sequence $(u^\varepsilon)_\varepsilon$ of Lipschitz continuous subsolutions of (3.13) can be built in such a way that they are C^1 (and even C^∞) in the Y variable.*

Proof — By Proposition 3.4.4, we can assume without loss of generality that u is Lipschitz continuous. In order to obtain further regularity, we are going to use a standard convolution with a sequence of mollifying kernels but only in the Y-variable.

Let us introduce a sequence $(\rho_\varepsilon)_\varepsilon$ of positive, C^∞-functions on $\mathbb{R}^k$, ρ_ε having a compact support in $B_\infty(0,\varepsilon)$ and with $\int_{\mathbb{R}^k} \rho_\varepsilon(e)de = 1$. Then we set, for $X = (Y,Z) \in B_\infty(\bar{X}, r-\varepsilon)$

$$u^\varepsilon(X) := \int_{|e|_\infty<\varepsilon} u(Y-e,Z)\rho_\varepsilon(e)de\ .$$

By standard arguments, it is clear that u^ε is smooth in Y.

We first want to prove that the u^ε are approximate subsolutions of (3.13), i.e. there exists some $\eta(\varepsilon) \to 0$ as $\varepsilon \to 0$ such that

$$\mathbb{G}(X, u^\varepsilon, Du^\varepsilon) \le \eta(\varepsilon) \quad \text{in } B_\infty(\bar{X}, r-\varepsilon)\ . \tag{3.18}$$

To do so we follow the strategy of [38][Lemma A.3], approximating the integral by a Riemann sum. We are lead to consider a function u_n^ε defined by

$$u_n^\varepsilon(X) := \sum_{i=1}^n \mu_i\, u(Y-e_i,Z)\,,$$

for some $|e_i| < \varepsilon$ and for coefficients $\mu_i \ge 0$ such that $\sum \mu_i = \int \rho_\varepsilon = 1$.

Using **(TC)** and the Lipschitz continuity of u, it is clear that there exists $\eta(\varepsilon)$, satisfying the above mentioned properties and independent of i, such that the functions $X = (Y,Z) \mapsto u(Y-e_i,Z)$ are all subsolutions of $\mathbb{G} - \eta(\varepsilon) = 0$.

Applying Lemma 3.4.6, u_n^ε is also a subsolution of $\mathbb{G} - \eta(\varepsilon) = 0$ and since u_n^ε converges uniformly to u^ε when $n \to +\infty$, a standard stability result (cf. Theorem 3.1.4) implies that u^ε a subsolution of $\mathbb{G} - \eta(\varepsilon) = 0$ as well.

Finally, in order to drop the $\eta(\varepsilon)$-term in (3.18), we can either replace u_ε by $u^\varepsilon - \lambda^{-1}\eta(\varepsilon)$ in the (**Mon**-u)-case, or $u^\varepsilon - \mu^{-1}\eta(\varepsilon)Y_1$ in the (**Mon**-p) case, and we get indeed a subsolution of $\mathbb{G} = 0$.

Q.E.D.

Remark 3.4.8 *Let us make three complementary comments.*

(i) It is clear from the proof of Proposition 3.4.7 that the convexity of $\mathbb{G}(X,r,P)$ in r is not necessary to obtain such a result, the continuity in r being enough, as we explain now. Notice first that, by the Lipschitz continuity of u,

$$|u(Y-e_i,Z)-u(Y,Z)|, |u(Y,Z)-u^\varepsilon(Y,Z)|, |u(Y,Z)-u_n^\varepsilon(Y,Z)| \le K\varepsilon\ ,$$

K being the Lipschitz constant. Then, we are reduced to a version of Lemma 3.4.6 with no r-dependence by using an approximate Hamiltonian of the form $\tilde G(X,P) := \mathbb{G}(X,u^\varepsilon(X),P)-\tilde\eta(\varepsilon)$, depending only on X and P. Indeed, taking into account the $K\varepsilon$ error term into $\tilde\eta(\varepsilon)$, the functions $u(Y-e_i,Z)$, u_n^ε and u^ε become all subsolutions of $\tilde G=0$, which satisfies $(\mathbf{H_{Conv}})$. The rest of the proof is then the same as above.

(ii) The next remark concerns "tangential regularizations" in the case of a "tangential viscosity inequalities". In several situations, and in particular in stratified problems, the subsolution u of (3.13) satisfies also a subsolution inequality of the form

$$\mathbb{G}^\Gamma(Y,u(Y,0),D_Yu(Y,0)) \le 0 \quad \text{on } \Gamma\ ,$$

where the precise meaning of this subsolution inequality is obtained by looking at maximum points of $u(Y,0)-\phi(Y)$ on Γ, not in all $\mathbb{R}^N$. As the proofs of Proposition 3.4.4 and 3.4.7 show, if $\mathbb{G}^\Gamma$ satisfies **(TC)**, *then the u^ε given by the regularization processes of these results are also semi-convex or C^1 subsolutions of $\mathbb{G}^\Gamma\le 0$; indeed the main difficulty in the proofs of these results comes from the Z-variable which does not appear here. A remark which plays a crucial in the case of stratified problems.*

(iii) The result still applies to quasi-convex Hamiltonians. Indeed, for instance using (i) above for simplicity, the convexity of $\mathbb{G}$ is used to prove essentially that if $G(X,u,P_i)\le 0$ for $i=1,2$, then $\mathbb{G}(X,u,sP_1+(1-s)P_2)\le 0$. But of course this is also true in the case of quasi-convexity since

$$\mathbb{G}(X,u,sP_1+(1-s)P_2) \le \max\{\mathbb{G}(X,u,P_1),\mathbb{G}(X,u,P_2)\} \le 0\,.$$

However, in the context of evolution equations this means that we need a "full" quasi-convexity assumption: under the form $H(x,t,u,(D_xu,u_t))=0$, the quasi-convexity is required to hold with respect to both (D_xu,u_t). Suprisingly, this assumption leaves out "natural" evolution equations under the form $u_t+F(x,t,u,D_xu)=0$ where F is quasi-convex in D_xu. Indeed, the full Hamiltonian $H=u_t+F$ is not quasi-convex with respect to both variables in general.

3.4.4 What about regularization for supersolutions?

The previous section shows how to regularize subsolutions and we address here the question: is it possible to do it for supersolutions, changing (of course) the sup-convolution into an inf-convolution?

Looking at the proof of Theorem 3.4.4, the answer is not completely obvious: on one hand, the arguments for an inf-convolution may appear as being analogous but, on the other hand, we use in a key way Assumption **(NC)** which allows to control the derivatives in Z of the sup-convolution (or the test-function), *an argument which is, of course, valid only for subsolutions.*

Actually, regularizing a supersolution v of (3.13)—a notion which is defined exactly in the same way as for subsolutions—requires additional assumptions on either v or $\mathbb{G}$. For $\mathbb{G}$, we introduce the following stronger version of **(TC)**

(TC-s) *Strong Tangential Continuity*: *for any $R > 0$, there exists $C_1^R > 0$ and a modulus of continuity $m^R : [0, +\infty[\to [0, +\infty[$ such that for any $X_1 = (Y_1, Z), X_2 = (Y_2, Z) \in B_\infty(\bar{X}, r)$, $|u| \leq R$, $p = (p_Y, p_Z) \in \mathbb{R}^N$, then*

$$|\mathbb{G}(X_1, u, p) - \mathbb{G}(X_2, u, p)| \leq C_1^R |Y_1 - Y_2|.|p_Y| + m^R\big(|Y_1 - Y_2|\big) \, .$$

We point out that, compared to **(TC)**, the "$|p|$" is replaced by "$|p_Y|$". This assumption is typically satisfied by equations of the form

$$\mathbb{G}(X, u, p) = \mathbb{G}_1(X, u, p_Y) + \mathbb{G}_2(z, u, p) \, ,$$

since, for $\mathbb{G}_1$, **(TC-s)** reduces to **(TC)** and $\mathbb{G}_2$ readily satisfies **(TC-s)**.

Another possibility is to assume that $v(X) = v(Y, Z)$ is Lipschitz continuous in Z in $B_\infty(\bar{X}, r)$, uniformly in Y, i.e. there exists a constant $K > 0$ such that, for any $X_1 = (Y, Z_1), X_2 = (Y, Z_2) \in B_\infty(\bar{X}, r)$

$$|v(X_1) - v(X_2)| \leq K|Z_1 - Z_2| \, . \tag{3.19}$$

The result for the supersolutions is the

Proposition 3.4.9 *Let v be a bounded supersolution of (3.13) and assume that*
(a) either **(TC-s)** *and* **(Mon)** *hold*
(b) or **(TC)**, **(Mon)** *and (3.19) hold.*
Then there exists a sequence $(v^\varepsilon)_\varepsilon$ defined in $B_\infty(\bar{X}, r - a(\varepsilon))$ where $a(\varepsilon) \to 0$ as $\varepsilon \to 0$ such that

(i) each v^ε is a supersolution of (3.13) in $B_\infty(\bar{X}, r - a(\varepsilon))$,

(ii) each v^ε is semi-concave in the Y-variable

(iii) $\liminf_* v^\varepsilon = v$ as $\varepsilon \to 0$.

Two remarks on this proposition: first, the proof is readily the same as for subsolutions, the only difference is that we do not need to control the Z-derivative in case (a) because of the form of **(TC-s)** while it is clearly bounded in case (b) because of (3.19). The second remark is that, a priori, the v^ε are not continuous in Z in case (a). But of course, they are Lipschitz continuous in Y and Z in case (b).

3.5 Equations on the boundary

We conclude this chapter with several results concerning the properties of viscosity sub and supersolutions of an HJ-equation at the boundary of the domain where the equation is set. Those results will be mainly applied in Part III but we formulate both in a quite general way here, considering a general HJ-equation of the form

$$u_t + H(x,t,u,Du) = 0 \quad \text{in } Q\ , \tag{3.20}$$

where $Q := \Omega \times (0,T_f)$, H is a continuous function and Ω is a C^1-domain of $\mathbb{R}^N$. We also set $\partial_\ell Q := \partial\Omega \times (0,T_f)$ and $\overline{Q}^\ell = \overline{\Omega} \times (0,T_f)$.

The first result is used below in the proof of Proposition 17.2.1: in terms of control, it means that viscosity subsolution inequalities hold up the boundary for all dynamics which are pointing inward the domain. Here, $d(z) = \mathrm{dist}(z,\partial\Omega)$ denotes the distance to the boundary which is C^1 in a neighborhood of $\partial\Omega$.

Proposition 3.5.1 *Assume that u is an u.s.c., locally bounded function on $\overline{Q}^\ell$ which is a subsolution of (3.20). If there exists $(x,t) \in \partial_\ell Q$ and $r > 0$ such that*

(i) The u.s.c. function u is Q-regular on $\partial_\ell Q \cap [B(x,r) \times (t-r,t+r)]$.

(ii) The distance function d to $\partial\Omega$ is smooth in $\overline{\Omega} \cap B(x,r)$,

(iii) There exists a function $L : \overline{Q}^\ell \cap [B(x,r) \times (t-r,t+r)] \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ such that $L \le H$ on $\overline{Q}^\ell \cap [B(x,r) \times (t-r,t+r)] \times \mathbb{R} \times \mathbb{R}^N$ and

$$\lambda \mapsto L(y,s,u,p + \lambda Dd(y))\ ,$$

is a decreasing function for any $(y,s,u,p) \in \overline{Q}^\ell \cap [B(x,r) \times (t-r,t+r)] \times \mathbb{R} \times \mathbb{R}^N$.

Then u is a subsolution of

$$u_t + L(x,t,u,Du) = 0 \quad \text{on } \partial_\ell Q \cap [B(x,r) \times (t-r,t+r)]\ .$$

Moreover, if we can take $L = H$ the same result is valid for supersolutions.

We point out that this result holds for "regular subsolutions", i.e. which satisfy (i), a regularity which is a consequence of Proposition 3.4.2 if we have suitable normal controlability and tangential continuity type assumptions.

Proof — We consider a test-function ψ which is C^1 on $\overline{Q}^\ell$ and we assume that $(y,s) \in \partial_\ell Q \cap [B(x,r) \times (t-r, t+r)]$ is a strict local maximum point of $u - \psi$ (again we refer to Lemma 3.1.3 to see why we can always assume the maximum point to be strict). To prove the L-inequality, we consider the function

$$(z,\tau) \mapsto u(z,\tau) - \psi(z,\tau) - \frac{\alpha}{d(z)} \, ,$$

where $\alpha > 0$ is a parameter devoted to tend to 0.

We apply Lemma 6.4.1 with

$$\begin{aligned} &w(z,\tau) := u(z,\tau) - \psi(z,\tau) \, , \;\; \chi_\alpha(z,\tau) = \frac{\alpha}{d(z)} \, , \\ &K = F = \overline{Q}^\ell \cap (\overline{B(x,r)} \times [t-r, t+r]) \, . \end{aligned}$$

Assumption (i) of Lemma 6.4.1 is clearly satisfied and since $\liminf_* \chi_\alpha = 0$ in K (even on $\partial\Omega$), Assumption (ii) also holds. We now turn to condition (iii) which requires some explanations.

By the Q-regularity of u on $\partial_\ell Q \cap [B(x,r) \times (t-r, t+r)]$, there exists a sequence (y_k, t_k) converging to (y,t) such that $u(y_k,t_k) \to u(y,t)$ and $y_k \in \Omega$. We may assume without loss of generality that $d(y_k) \geq k^{-1/2}$.

Then, considering the sequence $(y^\alpha, t^\alpha) := (y_{[\alpha^{-1}]}, t_{[\alpha^{-1}]})$ where $[\alpha^{-1}]$ is the integer part of α^{-1}, we have $(y^\alpha, t^\alpha) \to (y,t)$. Moreover, since $d(y^\alpha) \geq [\alpha^{-1}]^{1/2}$, we deduce also that $\chi_\alpha(y^\alpha, t^\alpha) \to 0$ and $w(y^\alpha, t^\alpha) \to w(y,t)$. In other words, this sequence corresponds to the sequence $(z_0^\varepsilon)_\varepsilon$ required in Assumption (iii) of Lemma 6.4.1.

Now, for α small enough, this function has a local maximum at $(\bar{z}, \bar{\tau}) \in K$, depending on α but we drop this dependence for the sake of simplicity of notations. The strict maximum property at (y,s) implies its uniqueness, hence Lemma 6.4.1 ensures that up to extraction, as $\alpha \to 0$ we get

$$(\bar{z}, \bar{\tau}) \to (y,s) \, , \;\; u(\bar{z}, \bar{\tau}) \to u(y,s) \, .$$

Writing the viscosity subsolution inequality for u, we have

$$\psi_t(\bar{z}, \bar{\tau}) + H(\bar{z}, \bar{\tau}, u(\bar{z}, \bar{\tau}), D\psi(\bar{z}, \bar{\tau}) - \frac{\alpha}{[d(\bar{z})]^2} Dd(\bar{z})) \leq 0 \, ,$$

which implies that the same inequality holds for L since $L \leq H$. Finally we use the monotonicity property of L in the $Dd(y)$-direction which yields

$$\psi_t(\bar{z},\bar{\tau}) + L(\bar{z},\bar{\tau},u(\bar{z},\bar{\tau}),D\psi(\bar{z},\bar{\tau})) \leq 0\ .$$

The conclusion follows by letting α tends to 0, using the continuity of L.

For the supersolution property, we argue in an analogous way, looking at a minimum point and introducing a "$+\dfrac{\alpha}{d(z)}$" term instead of the "$-\dfrac{\alpha}{d(z)}$"-one.

Q.E.D.

Then we turn to the classical notions of sub and superdifferentials: we describe their properties on the boundary $\partial_\ell Q$ since those on Q are well-known and, as we already mentioned it above, some of these properties play a crucial role in Part III. Here we add the term "relatively to $\overline{Q}^\ell$" since, in the sequel, we are going to consider at least two domains with a common boundary. Therefore, on $\partial_\ell Q$ we can consider both sub and super-differentials relatively either to $\overline{Q}^\ell$ or to its complementary.

We first give the general definition for any point in $\overline{Q}^\ell$.

Definition 3.5.2 *Sub and superdifferentials relatively to* $\overline{Q}^\ell$.

(*i*) *The superdifferential relatively to* $\overline{Q}^\ell$ *of an u.s.c. function* $u : \overline{Q}^\ell \to \mathbb{R}$ *at a point* $(\bar{x},\bar{t}) \in \overline{Q}^\ell$ *is the, possibly empty, closed convex set* $D^+_{\overline{Q}^\ell}u(\bar{x},\bar{t}) \subset \mathbb{R}^{N+1}$, *defined by:* $(p_x,p_t) \in D^+_{\overline{Q}^\ell}u(\bar{x},\bar{t})$ *if and only if, for any* $(x,t) \in \overline{Q}^\ell$,

$$u(x,t) \leq u(\bar{x},\bar{t}) + p_x \cdot (x-\bar{x}) + p_t(t-\bar{t}) + o(|t-\bar{t}| + |x-\bar{x}|)\ ,$$

(*ii*) *The subdifferential relatively to* $\overline{Q}^\ell$ *of an l.s.c. function* $v : \overline{Q}^\ell \to \mathbb{R}$ *at a point* $(\bar{x},\bar{t}) \in \overline{Q}^\ell$ *is the, possibly empty, closed convex set* $D^-_{\overline{Q}^\ell}v(\bar{x},\bar{t}) \subset \mathbb{R}^{N+1}$, *defined by:* $(p_x,p_t) \in D^-_{\overline{Q}^\ell}v(\bar{x},\bar{t})$ *if and only if, for any* $(x,t) \in \overline{Q}^\ell$,

$$v(x,t) \geq v(\bar{x},\bar{t}) + p_x \cdot (x-\bar{x}) + p_t(t-\bar{t}) + o(|t-\bar{t}| + |x-\bar{x}|)\ .$$

Of course, the terminology "relatively to $\overline{Q}^\ell$" only makes sense for points $(\bar{x},\bar{t}) \in \partial_\ell Q$ and if u (or v) is defined not only on $\overline{Q}^\ell \times (0,T_f)$ but on a larger domain, typically $\mathbb{R}^N \times (0,T_f)$. Moreover, for points in Q, Definition 3.5.2 is the classical definition.

The first lemma is classical and we leave its proof to the reader.

Lemma 3.5.3 (Sub/superdifferentials on $\overline{Q}^\ell$ and test-functions)

(*i*) *Let $u : \overline{Q}^\ell \to \mathbb{R}$ be an u.s.c. function and $(\bar{x},\bar{t}) \in \overline{Q}^\ell$. An element (p_x,p_t) is in $D^+_{\overline{Q}^\ell}u(\bar{x},\bar{t})$ if and only if there exists a C^1-function φ such that $(\bar{x},\bar{t})$ is a strict local maximum point of $u-\varphi$ on $\overline{Q}^\ell$ and $D_x\varphi(\bar{x},\bar{t}) = p_x$, $\varphi_t(\bar{x},\bar{t}) = p_t$.*

(*ii*) *Let $v : \overline{Q}^\ell \to \mathbb{R}$ be an l.s.c. function and $(\bar{x},\bar{t}) \in \overline{Q}^\ell$. An element (p_x,p_t) is in $D^-_{\overline{Q}^\ell}v(\bar{x},\bar{t})$ if and only if there exists a C^1-function φ such that $(\bar{x},\bar{t})$ is a strict local minimum point of $u-\varphi$ on $\overline{Q}^\ell$ and $D_x\varphi(\bar{x},\bar{t}) = p_x$, $\varphi_t(\bar{x},\bar{t}) = p_t$.*

We have formulated Lemma 3.5.3 with "strict" local maximum or minimum point but, obviously, this is a fortiori true with just local maximum or minimum.

Now we turn to the structure of the sub and superdifferentials on the boundary and the connections with Equation (3.20). With the notations of Proposition 3.5.1, we have

Proposition 3.5.4 (Structure of the sub and superdifferentials on $\partial_\ell Q$ and inequalities up to the boundary)

(*i*) *Assume that $u : \overline{Q}^\ell \to \mathbb{R}$ is an u.s.c., locally bounded subsolution of (3.20) which is Q-regular at the point $(x,t) \in \partial_\ell Q$. If $(p_x,p_t) \in D^+_{\overline{Q}^\ell}u(x,t)$, then the set $I = \{\lambda \in \mathbb{R} : (p_x + \lambda Dd(x), p_t) \in D^+_{\overline{Q}^\ell}u(x,t)\}$ is an interval, either $I = \mathbb{R}$ or $I = [\underline{\lambda},+\infty)$ for some $\underline{\lambda} \le 0$ and in this latter case,*

$$p_t + H(x,t,u(x,t),p_x + \underline{\lambda}Dd(x)) \le 0\ .$$

(*ii*) *Assume that $v : \overline{Q}^\ell \to \mathbb{R}$ is a l.s.c., locally bounded supersolution of (3.20) which is Q-regular at the point $(x,t) \in \partial_\ell Q$. If $(p_x,p_t) \in D^-_{\overline{Q}^\ell}v(x,t)$, then the set $J = \{\lambda \in \mathbb{R} : (p_x + \lambda Dd(x), p_t) \in D^-_{\overline{Q}^\ell}v(x,t)\}$ is an interval, either $J = \mathbb{R}$ or $J = (-\infty,\overline{\lambda}]$ for some $\overline{\lambda} \ge 0$ and in this latter case,*

$$p_t + H(x,t,v(x,t),p_x + \overline{\lambda}Dd(x)) \ge 0\ .$$

We recall that, for $x \in \partial\Omega$, $Dd(x)$ is the unit normal vector to $\partial\Omega$ at x pointing inward Ω. Therefore Proposition 3.5.4 gives informations on the structure of the sub and superdifferentials on the boundary in the normal direction.

Proof — We provide a complete proof only in the subsolutions case, the case of supersolutions follows from similar arguments.

(a) Since $D^+_{\overline{Q}^\ell}u(x,t)$ is a non-empty closed convex subset of $\mathbb{R}^{N+1}$, it is clear that I is also a non-empty closed convex subset of $\mathbb{R}$, hence an interval. Moreover, we claim that since $(p_x,p_t)\in D^+_{\overline{Q}^\ell}u(x,t)$, then also $(p_x+\lambda Dd(x),p_t)\in D^+_{\overline{Q}^\ell}u(x,t)$ for any $\lambda\geq 0$. Indeed, if $y\in\overline{\Omega}$, by the regularity of $d(\cdot)$,

$$0\leq d(y)=d(x)+Dd(x)\cdot(y-x)+o(|y-x|)=Dd(x)\cdot(y-x)+o(|y-x|)\,.$$

So, for any $\lambda\geq 0$, $\lambda Dd(x)\cdot(y-x)\geq o(|y-x|)$ and the claim follows. Hence I is either $\mathbb{R}$ or of the form $[\underline{\lambda},+\infty)$ for some $\underline{\lambda}\in\mathbb{R}$, and necessarily $\underline{\lambda}\leq 0$ because $\lambda=0\in I$.

It remains to prove the viscosity inequality when $\underline{\lambda}>-\infty$.

(b) Since $(p_x+\underline{\lambda}Dd(x),p_t)\in D^+_{\overline{Q}^\ell}u(x,t)$, by Lemma 3.5.3, there exists a C^1-function φ such that (x,t) is a strict local maximum point of $u-\varphi$ on $\overline{Q}^\ell$ and $D_x\varphi(x,t)=p_x+\underline{\lambda}Dd(x)$, $\varphi_t(x,t)=p_t$. Then, for $0<\varepsilon\ll 1$, we consider the function

$$\psi_\varepsilon(y,s)=u(y,s)-\varphi(y,s)+\varepsilon d(y)\,.$$

Since (x,t) is a strict local maximum point of $u-\varphi$ on $\overline{Q}^\ell$, for ε small enough, there exists a maximum point $(y_\varepsilon,s_\varepsilon)$ of ψ_ε near (x,t) and we have $(y_\varepsilon,s_\varepsilon)\to(x,t)$, $u(y_\varepsilon,s_\varepsilon)\to u(x,t)$ as $\varepsilon\to 0$.

We claim that $(y_\varepsilon,s_\varepsilon)\in Q$, at least for ε small enough. Indeed, if $(y_\varepsilon,s_\varepsilon)\in\partial_\ell Q$, then necessarily $(y_\varepsilon,s_\varepsilon)=(x,t)$; otherwise, by the strict maximum point property, we would have

$$\psi_\varepsilon(y_\varepsilon,s_\varepsilon)=(u-\varphi)(y_\varepsilon,s_\varepsilon)<(u-\varphi)(x,t)=\psi_\varepsilon(x,t)$$

which would contradict the maximality of $(y_\varepsilon,s_\varepsilon)$ for ψ_ε. But $(y_\varepsilon,s_\varepsilon)=(x,t)$ is not possible since it would imply that $(p_x+(\underline{\lambda}-\varepsilon)Dd(x),p_t)\in D^+_{\overline{Q}^\ell}u(x,t)$, a contradiction to the minimality of $\underline{\lambda}$.

(c) Therefore $(y_\varepsilon,s_\varepsilon)\in Q$ and the viscosity subsolution inequality holds, namely

$$\varphi_t(y_\varepsilon,s_\varepsilon)+H(y_\varepsilon,s_\varepsilon,u(y_\varepsilon,s_\varepsilon),D_x\varphi+(\underline{\lambda}-\varepsilon)Dd(y_\varepsilon))\leq 0\,.$$

The result follows by letting $\varepsilon\to 0$, using the continuity of H and the fact that φ is C^1.

Q.E.D.

Remark 3.5.5 (Sub and superdifferentials on $\partial_\ell Q$ and regularity)
In Proposition 3.5.4, we assume the sub and supersolutions to be Q-regular at the point $(x,t) \in \partial_\ell Q$: this is, of course, to obtain the viscosity inequalities for $\underline{\lambda}$ and $\overline{\lambda}$. We point out anyway that
– even if these regularity properties hold, $\underline{\lambda}$ and $\overline{\lambda}$ can be infinite. Take $Q = (0,+\infty)\times(0,T_f)$ and consider the functions $u(x,t) = -x^{1/2}$ or $v(x,t) = x^{1/2}$.
– If u is NOT Q-regular at the point $(x,t) \in \partial_\ell Q$ and if $D^+_{\overline{Q}^\ell} u(x,t)$ is non-empty then $I = \mathbb{R}$ and, in the same way, if v is NOT Q-regular at the point $(x,t) \in \partial_\ell Q$ and if $D^-_{\overline{Q}^\ell} v(x,t)$ is non-empty then $J = \mathbb{R}$.

We conclude this section by a "two-domain" result. More precisely we consider a domain $\tilde{\Omega} \subset \mathbb{R}^N$ which can be written as

$$\tilde{\Omega} = \Omega_1 \cup \Omega_2 \cup \mathcal{H} \ ,$$

where Ω_1, Ω_2 are two disjoints domains of $\mathbb{R}^N$ and $\mathcal{H} = \partial\Omega_1 \cap \partial\Omega_2$ is a smooth $(N-1)$-manifold. We use the notations $Q_i = \Omega_i \times (0,T_f)$, $\overline{\Omega_i}^\ell = \overline{\Omega}_i \times (0,T_f)$ and we notice that $\mathcal{H}\times(0,T_f) \subset \partial_\ell \Omega_i$ for $i = 1,2$. Finally we denote by $d(\cdot)$ the distance function to $\mathcal{H}$ and by $n(x)$ the unit normal vector to $\mathcal{H}$ pointing inward to Ω_1.

Given $\Lambda = (\lambda_1, \lambda_2) \in \mathbb{R}$, we define the continuous function $\chi^\Lambda : \tilde{\Omega} \to \mathbb{R}$ by

$$\chi^\Lambda(x) = \begin{cases} \lambda_1 d(x) & \text{if } x \in \overline{\Omega}_1, \\ \lambda_2 d(x) & \text{if } x \in \overline{\Omega}_2, \end{cases}$$

Lemma 3.5.6 (Sub and superdifferentials on $\mathcal{H}\times(0,T_f)$ and test-functions)

(i) Let $u : \tilde{\Omega}\times(0,T_f) \to \mathbb{R}$ be an u.s.c. function and $(\bar{x},\bar{t})$ a point of $\mathcal{H}\times(0,T_f)$. We assume that there exists $(p_x,p_t) \in \mathbb{R}^{N+1}$ and $\Lambda = (\lambda_1,\lambda_2) \in \mathbb{R}^2$ such that

$$\begin{cases} (p_x + \lambda_1 n(x), p_t) \in D^+_{\overline{\Omega_1}^l} u(\bar{x},\bar{t}) \, , \\ (p_x - \lambda_2 n(x), p_t) \in D^+_{\overline{\Omega_2}^l} u(\bar{x},\bar{t}) \, . \end{cases}$$

Then there exists a C^1-function φ such that $(\bar{x},\bar{t})$ is a strict local maximum point of $u - \chi^\Lambda - \varphi$ on $\tilde{\Omega}\times(0,T_f)$ and $D_x\varphi(\bar{x},\bar{t}) = p_x$, $\varphi_t(\bar{x},\bar{t}) = p_t$.

(ii) Let $v : \tilde{\Omega}\times(0,T_f) \to \mathbb{R}$ be a l.s.c. function and $(\bar{x},\bar{t})$ a point of $\mathcal{H}\times(0,T_f)$. We assume that there exists $(p_x,p_t) \in \mathbb{R}^{N+1}$ and $\Lambda = (\lambda_1,\lambda_2) \in \mathbb{R}^2$ such that

$$\begin{cases} (p_x + \lambda_1 n(x), p_t) \in D^-_{\overline{\Omega_1}^l} v(\bar{x},\bar{t}) \\ (p_x - \lambda_2 n(x), p_t) \in D^-_{\overline{\Omega_2}^l} v(\bar{x},\bar{t}) \, . \end{cases}$$

Then there exists a C^1-function φ such that $(\bar{x}, \bar{t})$ is a strict local maximum point of $u - \chi^\Lambda - \varphi$ on $\tilde{\Omega} \times (0, T_f)$ and $D_x\varphi(\bar{x}, \bar{t}) = p_x$, $\varphi_t(\bar{x}, \bar{t}) = p_t$.

We refer the reader to Part III where we introduce test-functions which are piecewise C^1 like $\chi^\Lambda + \varphi$ above. Lemma 3.5.6 will be useful in this context.

Proof — The proof is short and we provide it only in the subdifferential case, the proof for the superdifferential being analogous. We just notice that (p_x, p_t) is in the super-differential of the u.s.c. function $u - \chi^\Lambda$ at $(\bar{x}, \bar{t})$. The existence of φ is therefore a consequence of the classical results on subdifferentials.

Q.E.D.

Chapter 4

Control Tools

Of course, the key ingredients used in this chapter are not new, we just try to revisit them in a more modern way: we refer the reader to the founding article of Filippov [86] and to Aubin and Cellina [10], Aubin and Frankowska [11], Clarke [60], Clarke, Ledyaev, Stern and Wolenski [61] for the classical approach of deterministic control problems by non-smooth analysis methods.

4.1 Introduction: how to define deterministic control problems with discontinuities? The two half-spaces problem

As in the basic example of a two half-space discontinuity that was introduced in Section 3.1, we consider a partition of $\mathbb{R}^N$ into

$$\mathcal{H} = \{x_N = 0\}\,,\ \Omega_1 = \{x_N > 0\}\,,\ \Omega_2 = \{x_N < 0\}\,,$$

and we assume that we have three different control problems in each of these subsets given by $(b_{\mathcal{H}}, c_{\mathcal{H}}, l_{\mathcal{H}}), (b_1, c_1, l_1), (b_2, c_2, l_2)$. For the sake of simplicity, we can assume that they are all defined on $\mathbb{R}^N \times [0,T] \times A_i$ for $i = \mathcal{H}, 1, 2$ and even that they all satisfy $(\mathbf{H_{BA-CP}})$.

For such problems, the first question consists in defining properly the dynamic since, when the trajectory reaches $\mathcal{H}$, we have a discontinuity in b and the controller may have access to dynamics b_1 and b_2, but also to the specific dynamics $b_{\mathcal{H}}$. But how? And of course, a similar question holds for the cost and discount factor.

The natural tool consists in using the theory of *differential inclusions* that we first

introduce on the simple example of Chapter 2. The idea consists in looking at the set valued map

$$\mathbf{BCL}(x,t) := \{(b(x,t,\alpha), c(x,t,\alpha), l(x,t,\alpha)) \ : \ \alpha \in A\} \ ,$$

and to solve the differential inclusion

$$(\dot{X}(s), \dot{D}(s), \dot{L}(s)) \in \mathbf{BCL}(X(s), t-s) \ , \ (X,D,L)(0) = (x,0,0) \ ,$$

which only required that the set valued map **BCL** is upper-semicontinuous, with values in compact, convex sets (which is almost satisfied here, at least, adding the assumptions that the $\mathbf{BCL}(x,t)$ are convex or solving with their convex hull). Then

$$\tilde{U}(x,t) = \inf_{(X,D,L)} \left(\int_0^t \dot{L}(s) \exp(D(s))) \, \mathrm{d}s + u_0(X(t)) \exp(D(t)) \right) ,$$

The advantage of this approach is to allow to define the dynamic, discount and cost without any regularity in b, c, l.

The next step is the half-space discontinuity for which we are going to define **BCL** in the same way for $x \in \Omega_1$ and $x \in \Omega_2$ by just setting

$$\begin{cases} (b(x,t,\alpha), c(x,t,\alpha), l(x,t,\alpha)) = (b_1(x,t,\alpha_1), c_1(x,t,\alpha_1), l_1(x,t,\alpha_1)) & \text{if } x \in \Omega_1 \\ (b(x,t,\alpha), c(x,t,\alpha), l(x,t,\alpha)) = (b_2(x,t,\alpha_2), c_2(x,t,\alpha_2), l_2(x,t,\alpha_2)) & \text{if } x \in \Omega_2 \end{cases}$$

where $\alpha \in A = A_{\mathcal{H}} \times A_1 \times A_2$, the "extended control space".

For $x \in \mathcal{H}$ and $t \in [0,T]$, we just follow the theory of differential inclusions: by the upper semi-continuity of **BCL**, we necessarely have in $\mathbf{BCL}(x,t)$ all the $(b_i(x,t,\alpha_i), c_i(x,t,\alpha_i), l_i(x,t,\alpha_i))$ for $i = \mathcal{H}, 1, 2$ but we have also to take the convex hull of all these elements, namely all the convex combinations of them. In particular, for the dynamic, we have (a priori) all the $b = \mu_1 b_1 + \mu_2 b_2 + \mu_3 b_{\mathcal{H}}$ such that $\mu_1 + \mu_2 + \mu_3 = 1$, $\mu_i \geq 0$ but we will show that such b play a role only if the trajectory stays on $\mathcal{H}$ and therefore if we have $b \cdot e_N = 0$. A more precise statement will be given in Section 8.

4.2 Statement of the deterministic control problem and dynamic programming principle

Based on the ideas that we sketched in last section, we consider a general approach of finite horizon control problems with differential inclusions. We use an *extended*

trajectory (X, T, D, L) in which we also embed the running time variable T, pointing out that, in the basic example we introduced in the previous section, we just have $T(s) = t - s$.

This framework may seem complicated but we made this choice because it allows us to consider all the applications we have in mind: on one hand, time and space will play analogous roles when we face time-dependent discontinuities, or for treating some unbounded control type features; on the other hand, discount factors will be necessarily involved when dealing with boundary condition — see Part V.

In this section, we present general and classical results which do not require any particular assumption concerning neither the structure of the discontinuities, nor on the control sets.

In the following, we denote by $\mathcal{P}(E)$ the set of all subsets of E.

4.2.1 Dynamics, discount and costs

The first hypothesis we make is

$(\mathbf{H_{BCL}})_{fund}$: The set-valued map $\mathbf{BCL} : \mathbb{R}^N \times [0, T_f] \to \mathcal{P}(\mathbb{R}^{N+3})$ satisfies

(*i*) the map $(x, t) \mapsto \mathbf{BCL}(x, t)$ has compact, convex images and is upper semi-continuous;

(*ii*) there exists $M > 0$, such that, for any $x \in \mathbb{R}^N$ and $t > 0$,

$$\mathbf{BCL}(x, t) \subset \left\{(b, c, l) \in \mathbb{R}^{N+1} \times \mathbb{R} \times \mathbb{R} : |b| \leq M; |c| \leq M; |l| \leq M\right\},$$

where $|\cdot|$ stands for the usual euclidian norm in any euclidean space $\mathbb{R}^p$ (which reduces to the absolute value in $\mathbb{R}$, for the c and l variables). If $(b, c, l) \in \mathbf{BCL}(x, t)$, b corresponds to the dynamic (both in space and time), c to the discount factor and l to the running cost. Assumption $(\mathbf{H_{BCL}})_{fund}$-$(ii)$ means that dynamics, discount factors and running costs are uniformly bounded. In the following, we sometimes have to consider separately dynamics, discount factors and running costs. To do so, we set

$$\mathbf{B}(x, t) = \left\{b \in \mathbb{R}^{N+1};\ \text{there exists } c, l \in \mathbb{R} \text{ such that } (b, c, l) \in \mathbf{BCL}(x, t)\right\},$$

and analogously for $\mathbf{C}(x, t), \mathbf{L}(x, t) \subset \mathbb{R}$. Finally, we decompose any $b \in \mathbf{B}(x, t)$ as (b^x, b^t), where b^x and b^t are respectively the space and time dynamics.

We recall the definition of upper semi-continuity we use here: a set-valued map $x \mapsto F(x)$ is upper-semi continuous at x_0 if for any open set $\mathcal{O} \supset F(x_0)$, there

exists an open set ω containing x_0 such that $F(\omega) \subset \mathcal{O}$. Expressed in other terms, $F(x) \supset \limsup_{y \to x} F(y)$.

4.2.2 The control problem

We look for trajectories $(X, T, D, L)(\cdot)$ of the differential inclusion

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}(X, T, D, L)(s) \in \mathbf{BCL}\big(X(s), T(s)\big) & \text{for a.e. } s \in [0, +\infty)\,, \\ (X, T, D, L)(0) = (x, t, 0, 0)\,. \end{cases} \tag{4.1}$$

The key existence result is the

Theorem 4.2.1 *Assume that* $(\mathbf{H_{BCL}})_{fund}$ *holds. Then*

(i) for any $(x, t) \in \mathbb{R}^N \times [0, T_f)$ *there exists a Lipschitz function* $(X, T, D, L) : [0, T_f] \to \mathbb{R}^N \times \mathbb{R}^3$ *which is a solution of the differential inclusion* (4.1)*.*

(ii) for each solution (X, T, D, L) *of* (4.1) *there exist measurable functions* $(b, c, l)(\cdot)$ *such that for a.e.* $s \in (t, T_f)$*,*

$$(\dot X, \dot T, \dot D, \dot L)(s) = (b, c, l)(s) \in \mathbf{BCL}(X(s), T(s))\,.$$

Throughout this chapter, we mostly write

$$\begin{aligned} (\dot X(s), \dot T(s)) &= b\big(X(s), T(s)\big) \\ \dot D(s) &= c\big(X(s), T(s)\big) \\ \dot L(s) &= l\big(X(s), T(s)\big) \end{aligned}$$

in order to remember that b, c and l correspond to a specific choice in $\mathbf{BCL}(X(s), T(s))$. Later on, we will also introduce a control $\alpha(\cdot)$ to represent the (b, c, l) as

$$(b, c, l)(X(s), T(s), \alpha(s))\,.$$

In order to simplify the notations, we just use the notation X, T, D, L when there is no ambiguity but we may also use the notations $X^{x,t}, T^{x,t}, D^{x,t}, L^{x,t}$ when the dependence in x, t plays an important role.

Let us introduce a point of vocabulary here: by a state-constraint control problem in a set $\mathcal{W}$, we mean that the controller can only use trajectories which remain in $\mathcal{W}$: $(X, T, D, L)(s) \in \mathcal{W}$ for any $s \in [0, +\infty)$. In general, such constraint only concerns

the state variable X, which is required to satisfy $X(s) \in \overline{\Omega}$ for some domain Ω: we study these state-constraint problems in Part V.

However, throughout this book we have chosen a framework with a dynamic on T in order to describe finite horizon control problems in $\mathbb{R}^N \times [0, T_f]$ (or $\overline{\Omega} \times [0, T_f]$). Hence, the T-variable is also constrained to satisfy $T(s) \in [0, T_f]$. This property is at the origin of some of the hypotheses below. In this setting, the usual terminal cost is changed into a running cost, which also requires some assumptions in order to have a bounded value function.

Before describing the value function, we are going to make the following structure assumptions on the **BCL**-set valued map

$(\mathbf{H_{BCL}})_{struct}$: There exists $\underline{c}, K > 0$ such that

(*i*) For all $x \in \mathbb{R}^N$, $t \in [0, T_f]$ and $b = (b^x, b^t) \in \mathbf{B}(x, t)$, $-1 \leq b^t \leq 0$. Moreover, there exists $b = (b^x, b^t) \in \mathbf{B}(x, t)$ such that $b^t = -1$.

(*ii*) For all $x \in \mathbb{R}^N$, $t \in [0, T_f]$, if $((b^x, b^t), c, l) \in \mathbf{BCL}(x, t)$, then $-Kb^t + c \geq 0$.

(*iii*) For any $x \in \mathbb{R}^N$, there exists an element in $\mathbf{BCL}(x, 0)$ of the form $((0, 0), c, l)$ with $c \geq \underline{c}$.

(*iv*) For all $x \in \mathbb{R}^N$, $t \in [0, T_f]$, if $(b, c, l) \in \mathbf{BCL}(x, t)$ then $\max(-b^t, c, l) \geq \underline{c}$.

By introducing this general framework, our aim is to gather different type of control problems and treat them within the same setting. In classical finite horizon problems $b^t = -1$, which indicates a time direction associated to the u_t-term, and in this case $T(s) = t - s$. Here we choose the more general assumption $-1 \leq b^t \leq 0$ in order to respect this monotonicity in time, but allowing also $b^t = 0$ which can corresponds either to a control problem with a stopping time or an unbounded control problem or a combination of the two. In particular, we point out that a classical final cost is treated as associated to a stopping time control problem. Of course, by the convexity assumption all the interval $b^t \in [-1, 0]$ has to be potentially considered.

Assumption (*iii*) and a part of (*iv*) concern the final cost (u_0 in the example of the previous section) which is in general the initial data for the Hamilton-Jacobi-Bellman Equation. As we pointed out above, the value function we define below is associated to a state-constraint problem in $\mathbb{R}^N \times [0, T_f]$, and therefore it is necessary that strategies with $b^t = 0$ for any point $(x, 0) \in \mathbb{R}^N \times \{0\}$ exist.

Assumption (*iii*) means that we can stop the trajectory at any point $(x, 0)$, as for the case of a classical initial data, the assumption on c being necessary, in general,

to keep the integral of the running cost bounded. However, strategies with $b^t = 0$, $b^x \neq 0$ are also allowed provided that they satisfy (iv) at time $t = 0$ in order, again, that the associated cost remains bounded: indeed, either the trajectory is associated to a positive discount factor $c \geq \underline{c}$ which ensures the boundedness of the integral of the running cost or it has a positive cost $l \geq \underline{c}$ in order to avoid the long use of this strategy.

Such situations may also happen for $t > 0$, either to model a possible stopping time (obstacle type problem) or an exit cost (see in Part V, Dirichlet boundary condition), which is why (iv) is written for all $t \in [0, T_f]$.

On the other hand, the consequence of (ii) is that the change of unknown function $u \to \exp(-Kt)u$ allows to reduce to the easier case of a positive discount factor. Such assumption is necessary in this framework since the formulation below leads to a stationary type equation, because we treat time as a space variable.

Finally, notice that the fact that b^t can be 0 (or close to it) includes the unbounded control case. In particular if $b^t = 0$, the trajectory can stay at a constant time $\bar{t}$ for, say, $s \in [s_1, s_2]$ while if $b^x \neq 0$, the trajectory can be seen as an instantaneous jump from the point $X(s_1)$ to the point $X(s_2)$ since time does not vary on this interval.

In all the rest of the book, $(\mathbf{H_{BCL}})$ means that both $(\mathbf{H_{BCL}})_{fund}$ and $(\mathbf{H_{BCL}})_{struct}$ are fulfilled.

Before introducing the value-function, we state a result allowing to reduce always the case $c \geq 0$ for any $(b, c, l) \in \mathbf{BCL}(x, t)$ and for any $(x, t) \in \mathbb{R}^N \times [0, T_f]$

Lemma 4.2.2 *Assume that* $(\mathbf{H_{BCL}})$ *hold and let* (X, T, D, L) *be a solution of* (4.1) *associated to* $(b, c, l)(\cdot)$ *such that the following integral is well-defined*

$$J(X, T, D, L) = \int_0^{+\infty} l\big(X(s), T(s)\big) \exp(-D(s))\, \mathrm{d}t \ .$$

Then, considering the constant K *given by* $(\mathbf{H_{BCL}})_{struct}$*, we have*

$$\exp(-Kt) J(X, T, D, L) = J(\tilde{X}, \tilde{T}, \tilde{D}, \tilde{L}) \ ,$$

where $(\tilde{X}, \tilde{T}, \tilde{D}, \tilde{L})$ *is the solution of* (4.1) *associated to* $(b, c - Kb_t, l \exp(-KT(s)))(\cdot)$*. In particular* $\tilde{X} = X$*,* $\tilde{T} = T$*,* $\tilde{D} = D + K(T - t)$ *and of course we still have* $(\tilde{X}, \tilde{T}, \tilde{D}, \tilde{L})(0) = (x, t, 0, 0)$*.*

The use of this lemma will be clear in the next sections but it is obvious from $(\mathbf{H_{BCL}})_{struct}$-$(ii)$ that the replacement of c by $c - Kb_t \geq 0$ allows as we wish to reduce $c \in \mathbb{R}$ to the case when $c \geq 0$.

4.2.3 The value function

Now we introduce the value function which is defined on $\mathbb{R}^N \times [0, T_f]$ by

$$U(x,t) = \inf_{\mathcal{T}(x,t)} \Big\{ \int_0^{+\infty} l\big(X(s), T(s)\big) \exp(-D(s))\, \mathrm{d}s \Big\}, \tag{4.2}$$

where $\mathcal{T}(x,t)$ stands for all the Lipschitz trajectories (X,T,D,L) of the differential inclusion which start at $(x,t) \in \mathbb{R}^N \times [0,T_f]$ and such that $(X(s),T(s)) \in \mathbb{R}^N \times [0,T_f]$ for all $s > 0$.

As we explained above, Assumption $(iii) - (iv)$ imply formally the existence of trajectories (X,T,D,L) satisfying the constraint $(X,T) \in \mathbb{R}^N \times [0,T_f]$ with a bounded cost $J(X,T,D,L)$. Hence we expect both that $\mathcal{T}(x,t) \neq \emptyset$ for all $(x,t) \in \mathbb{R}^N \times [0,T_f]$ and that U is bounded. A rigourous proof of this claim is contained in the

Lemma 4.2.3 *Assume that* $(\mathbf{H_{BCL}})$ *holds. Then*

(i) *The value-function U is locally bounded on $\mathbb{R}^N \times [0,T_f]$.*

(ii) *For any trajectory (X,T,D,L) of the differential inclusion such that*

$$J(X,T,D,L) := \int_0^{+\infty} l\big(X(s),T(s)\big) \exp(-D(s)) ds < \infty\,,$$

then $D(s) \to +\infty$ as $s \to +\infty$.

Proof — We first use Lemma 4.2.2 to reduce the proof in the case when c is positive.

(a) In order to prove (i) we first show that $\mathcal{T}(x,t) \neq \emptyset$. Let us solve differential inclusion (4.1), replacing **BCL** by

$$\mathbf{BCL_\flat}(x,t) := \mathbf{BCL}(x,t) \cap \{(b,c,l) \in \mathbb{R}^{N+3};\ b^t = -1\}\,.$$

The reader can easily check that this new set-valued map satisfies all the required assumptions $(\mathbf{H_{BCL}})_{fund}$ and $(\mathbf{H_{BCL}})_{struct}$. Moreover, for any trajectory associated with $\mathbf{BCL_\flat}$ starting at $(x,t,0,0)$, it is clear that $T(t) = 0$ since $T(s) = t - s$. Notice that for $s \in [0,t]$, this trajectory may be seen as a trajectory associated to the original **BCL** since $\mathbf{BCL_\flat} \subset \mathbf{BCL}$.

Then, for any $s \geq t$ we redefine the trajectory by solving

$$(\dot{X}, \dot{T}, \dot{D}, \dot{L})(s) = ((0,0), c, l)$$

where $((0,0),c,l)$ is given by Assumption $(\mathbf{H_{BCL}})_{struct}$-$(iii)$ for the original **BCL**, at $(x,0)=(X(t),T(t))$. This defines a new trajectory for all $s\in[0,+\infty)$ associated to **BCL** and obviously, $(X(s),T(s))\in\mathbb{R}^N\times[0,T_f]$ so that the constructed trajectory (X,T,D,L) belongs to $\mathcal{T}(x,t)$.

(b) Next we show that for any trajectory, $J(X,T,D,L)$ is bounded from below. To do so, we use $(\mathbf{H_{BCL}})_{struct}$-$(iv)$ and introduce the sets

$$E_1:=\{s:-b^t\geq\underline{c}\}\ ,\ E_2:=\{s\notin E_1:c\geq\underline{c}\}\ ,\ E_3=[0,+\infty)\setminus(E_1\cup E_2)\ .$$

By $(\mathbf{H_{BCL}})_{struct}$-$(iv)$, we have $[0,+\infty)=E_1\cup E_2\cup E_3$ and E_1,E_2,E_3 are disjoint by construction. We now evaluate the integral on each of these three sets.

Concerning the E_1-contribution, we notice that, using that $\dot T(s)=b^t$,

$$|E_1|\,\underline{c}\leq\int_{E_1}-b^t(X(s),T(s))\,\mathrm{d}s\leq T(0)=t\,.$$

Since l is bounded, $0\leq\exp(-D(s))\leq 1$ and $|E_1|\leq t/\underline{c}$, the contribution on E_1 in J is finite.

The one on E_3 is positive so it is bounded from below. Finally on E_2, since $\dot D(s)=c(s)\geq\underline{c}$, it follows that

$$\begin{aligned}\int_{E_2}l\big(X(s),T(s)\big)\exp(-D(s))\,\mathrm{d}s&\geq -M\int_{E_2}\exp(-D(s))ds\\&\geq -M\int_{E_2}\frac{\dot D(s)}{\underline{c}}\exp(-D(s))\,\mathrm{d}s\\&\geq -M\int_{[0,+\infty)}\frac{\dot D(s)}{\underline{c}}\exp(-D(s))\,\mathrm{d}s\geq-\frac{M}{\underline{c}}\ ,\end{aligned}$$

which completes the proof of (i).

(c) In order to prove (ii), we examine carefully the sets E_1,E_2,E_3 defined above. We recall first that $|E_1|\leq t/\underline{c}<+\infty$, so that necessarily, either E_2 or E_3 has infinite Lebesgue measure. Now, on E_2, $\dot D(s)=c(s)\geq\underline{c}$ so that

$$\underline{c}\cdot|E_2\cap[0,S]|\leq\int_{E_2\cap[0,S]}\dot D(s)\,\mathrm{d}s\leq D(S)\,.$$

We deduce that if the increasing function $s\mapsto D(s)$ does not tend to $+\infty$ when $s\to+\infty$, then $|E_2|\leq\sup_s D(s)/\underline{c}<\infty$, so that $|E_3|=+\infty$.

By the monotonicity of D, if $D(s)$ does not tend to $+\infty$ when $s \to +\infty$, there exists $\gamma > 0$ such that $\exp(-D(s)) \geq \gamma$ on $[0, +\infty)$ but on E_3, since $l(s) \geq \underline{c}$ we see that

$$\int_{E_3} l\big(X(s), T(s)\big) \exp(-D(s))\, \mathrm{d}s \geq \int_{E_3} \underline{c} \cdot \gamma \, \mathrm{d}s = \underline{c} \cdot \gamma \cdot |E_3| = +\infty \,,$$

and we reach a contradiction because integral $J(X, T, D, L)$ is bounded.

Q.E.D.

4.3 Ishii solutions for the Bellmann equation

In this section we prove that the value function is a (discontinuous) viscosity solution of the Bellmann equation associated with the control problem, namely

$$\mathbb{F}(x, t, u, Du) = 0 \quad \text{in} \quad \mathbb{R}^N \times [0, T_f] \,, \tag{4.3}$$

where, for any $x \in \mathbb{R}^N$, $t \in [0, T_f]$, $r \in \mathbb{R}$ and $p = (p_x, p_t) \in \mathbb{R}^{N+1}$

$$\mathbb{F}(x, t, r, p) := \sup_{(b,c,l) \in \mathbf{BCL}(x,t)} \Big\{ -b \cdot p + cr - l \Big\} \,. \tag{4.4}$$

Writing the Bellmann equation under the form (4.3) is a little bit formal: if a more or less classical definition of viscosity sub and supersolutions can be used in $\mathbb{R}^N \times]0, T_f]$ following Definition 3.1.1, the case of $t = 0$ requires a particular treatment.

Indeed, it is well-known that the supersolution inequality for such Bellman Equation is related to the optimality of one or several trajectories while the subsolution one reflects the fact that any trajectory for any possible control is sub-optimal. At a point $(x, 0)$, the standard $\mathbb{F} \geq 0$ supersolution inequality does not seem to cause any problem, even if the optimal trajectory has to stay on $\mathbb{R}^N \times \{0\}$. On the contrary, there is a problem with the standard subsolution inequality since we cannot use any solution (X, T, D, L) of the **BCL**-differential inclusion, but only those for which $b^t = 0$. This is why the constraint to remain in $\mathbb{R}^N \times [0, T_f]$ obliges us to change the definition of subsolution for $t = 0$.

This leads to introduce the "initial Hamiltonian"

$$\mathbb{F}_{init}(x, r, p_x) := \sup_{((b^x,0),c,l) \in \mathbf{BCL}(x,0)} \Big\{ -b^x \cdot p_x + cr - l \Big\} \,. \tag{4.5}$$

Before going further, we describe the properties of $\mathbb{F}$ and $\mathbb{F}_{init}$ in the following result.

Lemma 4.3.1 *The Hamiltonians $(x,t,r,p) \mapsto \mathbb{F}(x,t,r,p)$ and $(x,t,r,p) \mapsto \mathbb{F}_{init}(x,r,p)$ are u.s.c. with respect to all the variables, and convex and Lipschitz as a function of r and p.*

Proof — We only provide the proof for $\mathbb{F}$, the one for $\mathbb{F}_{init}$ being analogous.

For the upper semi-continuity, let us take a sequence $(x_n,t_n,r_n,p_n) \to (x,t,r,p) \in \mathbb{R}^N \times [0,T_f] \times \mathbb{R} \times \mathbb{R}^{N+1}$. Since, for any n, $\mathbf{BCL}(x_n,t_n)$ is compact, there exists $(b_n,c_n,l_n) \in \mathbf{BCL}(x_n,t_n)$ such that

$$\mathbb{F}(x_n,t_n,r_n,p_n) = -b_n \cdot p_n + c_n r_n - l_n \,.$$

Since $\mathbf{BCL}(\cdot,\cdot)$ is u.s.c. as a set-valued map, it follows that, for any $\delta > 0$, if n is large enough,

$$(b_n,c_n,l_n) \in \mathbf{BCL}(x_n,t_n) \subset \mathbf{BCL}(x,t) + \delta B_{2N+3} \,,$$

where B_{2N+3} is the unit ball in $\mathbb{R}^{2N+3}$. For such n, (b_n,c_n,l_n) can be decomposed as the sum $(\tilde{b}_n,\tilde{c}_n,\tilde{l}_n)+\delta e_n$ for some $(\tilde{b}_n,\tilde{c}_n,\tilde{l}_n) \in \mathbf{BCL}(x,t)$ and some $e_n \in B_{2N+3}$. Now, since (x_n,t_n,r_n,p_n) is bounded,

$$\begin{aligned}
\mathbb{F}(x,t,r,p) &\geq -\tilde{b}_n \cdot p + \tilde{c}_n r - \tilde{l}_n \\
&\geq -b_n \cdot p_n + c_n r_n - l_n - o_\delta(1) \\
&\geq \mathbb{F}(x_n,t_n,r_n,p_n) - o_\delta(1) \,.
\end{aligned}$$

Passing to the limsup on n and sending $\delta \to 0$ yields the upper semi-continuity property.

The Lipschitz continuity is just a consequence of the boundedness of the b and c components in $\mathbf{BCL}(x,t)$ for any x and t: if $F(x,t,r,p) = -b \cdot p + cr - l$, then since $\mathbb{F}(x,t,r',q) \geq -b \cdot q + cr' - l$, we have

$$\mathbb{F}(x,t,r,p) - \mathbb{F}(x,t,r',q) \leq |c||r-r'| + |b||p-q| \leq M\big(|r-r'| + |p-q|\big) \,,$$

and of course the converse inequality is also true.

Finally, the convexity of $\mathbb{F}$ with respect to (r,p) just comes from the fact that $\mathbb{F}$ is the supremum of affine functions in (r,p).

Q.E.D.

4.3.1 Discontinuous viscosity solutions

Let us first give the definition based on the notion of discontinuous (or Ishii) viscosity solution exposed in Chapter 3, but modified in a suitable way to take into account the particularity of $t = 0$.

Definition 4.3.2 *A locally bounded function u is a subsolution of (4.3) if its u.s.c. enveloppe satisfies*

$$\mathbb{F}_*(x,t,u^*,Du^*) \leq 0 \quad \text{on } \mathbb{R}^N \times]0,T_f] \,, \tag{4.6}$$

and, for $t = 0$

$$\min(\mathbb{F}_*(x,0,u^*,Du^*), (\mathbb{F}_{init})_*(x,u^*(x,0),D_x u^*(x,0))) \leq 0 \quad \text{in } \mathbb{R}^N \,. \tag{4.7}$$

A locally bounded function v is a supersolution (4.3) if its l.s.c. enveloppe satisfies

$$\mathbb{F}(x,t,v_*,Dv_*) \geq 0 \quad \text{on } \mathbb{R}^N \times [0,T_f] \,. \tag{4.8}$$

A locally bounded function is a viscosity solution of (4.3) if it is both a subsolution and a supersolution of (4.3).

For the supersolution property, the simple formulation comes from the fact that $\mathbb{F}$ is u.s.c. in $\mathbb{R}^N \times [0,T_f] \times \mathbb{R} \times \mathbb{R}^N$. For the subsolution, the inequality is the expected one on $\mathbb{R}^N \times]0,T_f]$ but is modified for $t = 0$. In fact, we show below that the value function satisfies

$$(\mathbb{F}_{init})_*(x,U^*(x,0),D_x U^*(x,0))) \leq 0 \quad \text{in } \mathbb{R}^N \,,$$

and Section 5.1 (see Proposition 5.1.1) will confirm that the $\mathbb{F}_*$-contribution in (4.7) is not necessary, the initial data condition being totally equivalent to $(\mathbb{F}_{init})_* \leq 0$.

4.3.2 The dynamic programming principle

The first step towards establishing the sub/supersolution properties of U is to prove the classical

Theorem 4.3.3 (Dynamic Programming Principle)
Under hypothesis $(\mathbf{H_{BCL}})$*, the value function U satisfies*

$$U(x,t) = \inf_{\mathcal{T}(x,t)} \Big\{ \int_0^\theta l\big(X(s),T(s)\big) \exp(-D(s))\, \mathrm{d}s + U\big(X(\theta),T(\theta)\big) \exp(-D(\theta)) \Big\} \,,$$

for any $(x,t) \in \mathbb{R}^N \times (0,T_f]$, $\theta > 0$.

Proof — Let us denote by $J_\theta(X,T,D,L)$ the integral over $(0,\theta)$ inside the inf and by $\hat{U}(x,t)$ the complete right-hand side, while $U(x,t) = \inf_{\mathcal{T}(x,t)} J(X,T,D,L)$ and $J(\cdot)$ stands for the integral over $(0,+\infty)$.

(a) Let us prove that $U \le \hat{U}$. We first take any trajectory $(X,T,D,L) \in \mathcal{T}(x,t)$. Then, noting $(x_\theta,t_\theta) := (X(\theta),T(\theta))$, we select an ε-optimal trajectory $(X^\varepsilon,T^\varepsilon,D^\varepsilon,L^\varepsilon) \in \mathcal{T}(x_\theta,t_\theta)$, in the sense that

$$U(x_\theta,t_\theta) \le J(X^\varepsilon,T^\varepsilon,D^\varepsilon,L^\varepsilon) + \varepsilon\,.$$

We then construct a new trajectory in $\mathcal{T}(x,t)$ by setting

$$(\hat{X},\hat{T},\hat{D},\hat{L})(s) := \begin{cases} (X,T,D,L)(s) & \text{if } 0 \le s \le \theta\,, \\ (X^\varepsilon,T^\varepsilon,D^\varepsilon + D(\theta), L^\varepsilon + L(\theta))(s-\theta) & \text{if } s > \theta\,. \end{cases}$$

Using the definition of $U(x,t)$ we get

$$\begin{aligned} U(x,t) &\le J(\hat{X},\hat{T},\hat{D},\hat{L}) \\ &\le J_\theta(X,T,D,L) + \int_\theta^{+\infty} l(X^\varepsilon(s-\theta),T^\varepsilon(s-\theta))\exp(-D^\varepsilon(s-\theta) - D(\theta))\,\mathrm{d}s \\ &\le J_\theta(X,T,D,L) + \exp(-D(\theta))\int_0^{+\infty} l(X^\varepsilon(s),T^\varepsilon(s))\exp(-D^e(s))\,\mathrm{d}s \\ &\le J_\theta(X,T,D,L) + \exp(-D(\theta))(U(X(\theta),T(\theta)) + \varepsilon) \end{aligned}$$

Notice that the trajectory $(X,T,D,L) \in \mathcal{T}(x,t)$ is artbitrary and does not depend on ε. Hence, taking the infimum over $\mathcal{T}(x,t)$ and sending ε to zero, we conclude that indeed $U \le \hat{U}$.

(b) The converse inequality follows from similar manipulations: let us take an ε-optimal trajectory $(X^\varepsilon,T^\varepsilon,D^\varepsilon,L^\varepsilon) \in \mathcal{T}(x,t)$ for estimating $U(x,t)$. After separating the integral in two parts and changing variable $s \mapsto s-\theta$ in the second part we get

$$\begin{aligned} U(x,t) + \varepsilon \ge\ & J_\theta(X^\varepsilon,T^\varepsilon,D^\varepsilon,L^\varepsilon) \\ & + \int_0^{+\infty} l(X^\varepsilon(s+\theta),T^\varepsilon(s+\theta))\exp(-D^\varepsilon(s+\theta))\,\mathrm{d}s\,. \end{aligned} \tag{4.9}$$

The trajectory $(X,T,D,L)(s) := (X^\varepsilon,T^\varepsilon,D^\varepsilon,L^\varepsilon)(s+\theta) - (0,0,D^\varepsilon(\theta),L^\varepsilon(\theta))$ belongs to $\mathcal{T}(X^\varepsilon(\theta),T^\varepsilon(\theta))$, and (4.9) can be written as

$$\begin{aligned} U(x,t) + \varepsilon &\ge J_\theta(X^\varepsilon,T^\varepsilon,D^\varepsilon,L^\varepsilon) \\ &\qquad + \exp(-D^\varepsilon(\theta))\int_0^{+\infty} l(X(s),T(s))\exp(-D(s))\,\mathrm{d}s\,; \\ &\ge J_\theta(X^\varepsilon,T^\varepsilon,D^\varepsilon,L^\varepsilon) + \exp(-D^\varepsilon(\theta))J(X,T,D,L)\,. \end{aligned}$$

Now, using (X, T, D, L) as an admissible trajectory starting at $(X^\varepsilon(\theta), T^\varepsilon(\theta))$ we use the estimate

$$U(X^\varepsilon(\theta), T^\varepsilon(\theta)) \leq J(X, T, D, L)$$

to get the inequality

$$U(x,t) + \varepsilon \geq J_\theta(X^\varepsilon, T^\varepsilon, D^\varepsilon, L^\varepsilon) + \exp(-D^\varepsilon(\theta))U(X^\varepsilon(\theta), T^\varepsilon(\theta))\,.$$

Finally, $\hat{U}(x,t)$ being the infimum of all trajectories in $\mathcal{T}(x,t)$, the right-hand side is greater than or equal to $\hat{U}(x,t)$ and the conclusion follows.

Q.E.D.

4.3.3 The value function is an Ishii solution

Following the definition recalled in Section 4.3, we first prove the Following Definition 4.3.2, we first prove the

Theorem 4.3.4 (Supersolution Property) *Under Assumptions* ($\mathbf{H_{BCL}}$)*, the value function U is a viscosity supersolution of the Bellmann equation* (4.3)*.*

Proof — We keep here the notation $J_\theta(X, T, D, L)$ introduced in the proof of Proposition 4.3.3 for the integral over $(0, \theta)$ in the dynamic programming principle.

Let $(x,t) \in \mathbb{R}^N \times [0, T_f]$ be a local minimum point of $U_* - \phi$ where $\phi \in C^1(\mathbb{R}^N \times [0, T_f])$. We can assume without loss of generality that $U_*(x,t) = \phi(x,t)$. In particular, $U \geq U_* \geq \phi$ in a neighborhood of (x,t). Moreover, by definition of the lower semi-continuous envelope, there exists a sequence $(x_n, t_n) \to (x,t)$ such that $U(x_n, t_n) \to U_*(x,t)$.

We apply the dynamic programming principle for U at (x_n, t_n):

$$U(x_n, t_n) = \inf_{\mathcal{T}(x_n,t_n)} \Big(J_\theta(X_n, T_n, D_n, L_n) + U(X_n(\theta), T_n(\theta)) \exp(-D_n(\theta)) \Big)\,.$$

On one hand, for the left-hand side, using the definition of the sequence (x_n, t_n), the fact that $U_*(x,t) = \phi(x,t)$ and the continuity of ϕ, there exists a sequence $(\varepsilon_n)_n$ of non-negative real numbers converging to 0 such that $U(x_n, t_n) \leq \phi(x_n, t_n) + \varepsilon_n$.

On the other hand, since $|b| \leq M$ is bounded, if θ is small enough the trajectory $(X_n(s), T_n(s))$ remains close enough to (x,t) and we can use the inequalities $U \geq U_* \geq \phi$, the last one coming from the local minimum point property. This yields

$$\phi(x_n, t_n) + \varepsilon_n \geq \inf_{\mathcal{T}(x_n,t_n)} \Big(J_\theta(X_n, T_n, D_n, L_n) + \phi(X_n(\theta), T_n(\theta)) \exp(-D_n(\theta)) \Big)\,. \quad (4.10)$$

For simplicity of notations, we set $Z_s := (X_n(s), T_n(s))$. Since ϕ is C^1, the following expansion holds

$$\begin{aligned}\phi(Z_\theta)\exp(-D(\theta)) - \phi(Z_0) &= \int_0^\theta \frac{d}{ds}\Big(\phi(Z_s)\exp(-D_n(s))\Big)\,\mathrm{d}s \\ &= \int_0^\theta \Big(b(Z_s)\cdot D\phi(Z_s) - c(Z_s)\phi(Z_s)\Big)\exp(-D_n(s))\,\mathrm{d}s\,.\end{aligned} \tag{4.11}$$

Combining with (4.10) yields

$$\begin{aligned}0 &\geq \inf_{\mathcal{T}(x_n,t_n)} \int_0^\theta \Big\{b(Z_s)\cdot D\phi(Z_s) - c(Z_s)\phi(Z_s) + l(Z_s)\Big\}\exp(-D_n(s))\,\mathrm{d}s - \varepsilon_n\,, \\ &\geq \int_0^\theta -\mathbb{F}(X_n(s), T_n(s), \phi(X_n(s), T_n(s)), D\phi(X_n(s), T_n(s)))\exp(-D_n(s))\,\mathrm{d}s - \varepsilon_n\,.\end{aligned}$$

Since θ is arbitrary, we can choose a sequence θ_n in order that $\varepsilon_n\theta_n^{-1} \to 0$. We remark that $(X_n(s), T_n(s), \phi(X_n(s), T_n(s)), D\phi(X_n(s), T_n(s)) \to (x, t, \phi(x,t), D\phi(x,t))$. Therefore, if $\delta > 0$ is fixed and small, provided n is large enough we have

$$\mathbb{F}(X_n(s), T_n(s), \phi(X_n(s), T_n(s)), D\phi(X_n(s), T_n(s)) \leq \mathbb{F}(x, t, \phi(x,t), D\phi(x,t)) + \delta\,.$$

In addition, $\exp(-D_n(s)) = 1 + O(\theta_n)$ so, using all these informations in the above inequality, we deduce that

$$0 \geq \theta_n\,(-\mathbb{F}(x, t, \phi(x,t), D\phi(x,t)) - \delta(\,1 + O(\theta_n)) - \varepsilon_n\,.$$

Dividing by θ_n and letting n tend to infinity, we obtain $\mathbb{F}(x, t, \phi(x,t), D\phi(x,t)) + \delta \geq 0$ and this inequality being true for any $\delta > 0$, the result is proved.

Q.E.D.

Now we turn to the subsolutions properties.

Theorem 4.3.5 (Subsolution Properties) *Under Assumptions* ($\mathbf{H_{BCL}}$)*, the value function U is a viscosity subsolution of*

$$\mathbb{F}_*(x, t, U, DU) \leq 0 \quad \text{on } \mathbb{R}^N \times]0, T_f]\,, \tag{4.12}$$

and for $t = 0$, it satisfies

$$(\mathbb{F}_{init})_*(x, U(x,0), p_x U(x,0)) \leq 0 \quad \text{in } \mathbb{R}^N, \tag{4.13}$$

hence it is a subsolution of (4.3)*.*

Proof — The proof is more involved than for the supersolution condition, first because we need to consider $\mathbb{F}_*$ which a priori differs from $\mathbb{F}$, but also because we face here the potential discontinuities of b, c, l with respect to x, t.

We first prove (4.12). We consider a maximum point $(x,t) \in \mathbb{R}^N \times]0, T_f]$ of $U^* - \phi$ where ϕ is a C^1 test-function and, as above, we assume that $U^*(x,t) = \phi(x,t)$. By definition of U^*, there exists a sequence $(x_n, t_n) \to (x,t)$ such that $U(x_n,t_n) \to U^*(x,t)$ and, by the continuity of ϕ, we also have $\phi(x_n,t_n) \le U(x_n,t_n) + \varepsilon_n$ for some sequence $(\varepsilon_n)_n$ of non-negative real numbers converging to 0.

Applying the dynamic programming principle, we have

$$U(x_n,t_n) = \inf_{\mathcal{T}(x_n,t_n)} \Big(J_\theta(X,T,D,L)) + U(X(\theta),T(\theta)) \exp(-D(\theta)) \Big) .$$

If $\theta > 0$ is small enough, the maximum point property implies

$$U(X(\theta),T(\theta)) \le U^*(X(\theta),T(\theta)) \le \phi(X(\theta),T(\theta))$$

and therefore we obtain

$$\phi(x_n,t_n) - \varepsilon_n \le \inf_{\mathcal{T}(x_n,t_n)} \Big(J_\theta(X,T,D,L)) + \phi(X(\theta),T(\theta)) \exp(-D(\theta)) \Big) .$$

Using expansion (4.11)—here also with the notation $Z_s = (X(s),T(s))$—leads to

$$\int_0^\theta \Big(- b(Z_s) D\phi(Z_s) + c(Z_s)\phi(Z_s) - l(Z_s) \Big) \exp(-D(s))\, \mathrm{d}s \le \varepsilon_n \,, \tag{4.14}$$

for any trajectory $(X,T,D,L)_n \in \mathcal{T}(x_n,t_n)$.

In order to conclude, we have to show that, for any n, we can choose a trajectory $(X,T,D,L)_n \in \mathcal{T}(x_n,t_n)$ such that the integral is close to $\mathbb{F}_*(x,t,\phi(x,t),D\phi(x,t))$.

To do so, we are going to solve a suitable differential inclusion for a set-valued map that we build in the following way. We consider the auxiliary function $h_\phi(b,c,l) := -b \cdot D\phi(x,t) + c\phi(x,t) - l$ and for $\delta > 0$, we define a restricted set-valued map for (y,s) in a neighborhood of (x,t) as follows

$$\mathbf{BCL}^\delta_{\mathrm{loc}}(y,s) := \mathbf{BCL}(y,s) \cap \Big\{ h_\phi(b,c,l) \ge \mathbb{F}_*(x,t,\phi(x,t),D\phi(x,t)) - \delta \Big\} .$$

We claim that $\mathbf{BCL}^\delta_{\mathrm{loc}}$ is not empty and satisfies $(\mathbf{H_{BCL}})_{fund}$, at least for (y,s) close enough to (x,t).

Indeed, if, on the contrary, $\mathbf{BCL}^\delta_{\mathrm{loc}}(y_n,s_n)$ is empty for some sequence $(y_n,s_n) \to (x,t)$, this means that, for any $(b,c,l) \in \mathbf{BCL}(y_n,s_n)$, we have

$$h_\phi(b,c,l) < \mathbb{F}_*(x,t,\phi(x,t),D\phi(x,t)) - \delta \,,$$

which implies that

$$\mathbb{F}(y_n, s_n, \phi(x,t), D\phi(x,t)) = \sup_{(b,c,l)\in \mathbf{BCL}(y_n,s_n)} h_\phi(b,c,l) \le \mathbb{F}_*(x,t,\phi(x,t),D\phi(x,t)) - \delta\,.$$

But using the lower semi-continuity of $\mathbb{F}_*$ we are led to a contradiction since

$$\begin{aligned}\mathbb{F}_*(x,t,\phi(x,t),D\phi(x,t)) &\le \liminf_{n\to\infty} \mathbb{F}(y_n,s_n,\phi(x,t),D\phi(x,t))\\ &\le \mathbb{F}_*(x,t,\phi(x,t),D\phi(x,t)) - \delta\,.\end{aligned}$$

Concerning the images $\mathbf{BCL}^\delta_{\rm loc}(y,s)$, they are clearly convex and compact from the properties of **BCL** and the fact that the set $\{h_\phi \ge \alpha\}$ is closed and convex. Moreover, the u.s.c. property derives from the fact that **BCL** is u.s.c. while $\{h_\phi \ge \delta\}$ is a fixed set.

Hence we can solve the differential inclusion associated to $\mathbf{BCL}^\delta_{\rm loc} \subset \mathbf{BCL}$ with initial data (x_n, t_n) on a small time interval $(0,\theta)$. For this specific trajectory, up to taking θ smaller and n larger, using that ϕ is C^1, we get for $s \in [0,\theta]$

$$\begin{aligned}-b(Z_s)\cdot D\phi(Z_s) + c(Z_s)\phi(Z_s) - l(Z_s) &= h_\phi(b(Z_s),c(Z_s),l(Z_s)) + O(\theta)\\ &\ge \mathbb{F}_*(x,t,\phi(x,t),D\phi(x,t)) - \delta + O(\theta)\,.\end{aligned}$$

Plugging this into (4.14), using also that $\exp(-D(s)) = 1 + O(\theta)$, we get

$$\theta\big(\mathbb{F}_*(x,t,\phi(x,t),D\phi(x,t)) - \delta + O(\theta)\big)(1+O(\theta)) \le \varepsilon_n\,.$$

To conclude, we send $n \to \infty$ and then we divide by θ and we send it to 0. We end up with the inequality $\mathbb{F}_*(x,t,\phi(x,t),D\phi(x,t)) \le \delta$ for any $\delta > 0$ and therefore $\mathbb{F}_*(x,t,\phi(x,t),D\phi(x,t)) \le 0$.

Now we turn to (4.13). This case is treated by the same technique as above provided that we use the

Lemma 4.3.6 *Under Assumptions* ($\mathbf{H_{BCL}}$)*, we have, for any* $x \in \mathbb{R}^N$

$$U^*(x,0) = \limsup_{y\to x} U(y,0)\,.$$

In other word, the u.s.c. enveloppe of U *at points* $(x,0)$ *can be computed by using only* U *on* $\mathbb{R}^N \times \{0\}$.

If Lemma 4.3.6 holds, the same above proof readily applied since we can choose $t_n = 0$ and therefore, in the definition of $\mathbf{BCL}^\delta_{\rm loc}$, we can consider only the b such

that $b^t = 0$ and replace $\mathbb{F}_*$ by $(\mathbb{F}_{init})_*$. Indeed any relevant trajectory starting from $(x_n, 0)$ necessarely satisfies $b^t(Z_s) = 0$.

Proof of Lemma 4.3.6 — By definition of U^*, there exists a sequence $(x_\varepsilon, t_\varepsilon) \to (x, 0)$ such that $U(x_\varepsilon, t_\varepsilon) \to U^*(x, 0)$. Then we apply the dynamic programming principle

$$U(x_\varepsilon, t_\varepsilon) = \inf_{\mathcal{T}(x_\varepsilon, t_\varepsilon)} \Big(J_\theta(X_\varepsilon, T_\varepsilon, D_\varepsilon, L_\varepsilon) + U(X_\varepsilon(\theta), T_\varepsilon(\theta)) \exp(-D_\varepsilon(\theta)) \Big) .$$

We consider a trajectory $(X_\varepsilon, T_\varepsilon, D_\varepsilon, L_\varepsilon)$ which is solution of the differential inclusion associated with $\mathbf{BCL}_\flat$ defined in the proof of Lemma 4.2.3, i.e. with $b^t = -1$, and we use it in the dynamic programming principle with $\theta = t_\varepsilon$. Since $T_\varepsilon(\theta) = T_\varepsilon(t_\varepsilon) = 0$, we obtain

$$U(x_\varepsilon, t_\varepsilon) \le J_{t_\varepsilon}(X_\varepsilon, T_\varepsilon, D_\varepsilon, L_\varepsilon) + U(X_\varepsilon(t_\varepsilon), 0) \exp(-D_\varepsilon(t_\varepsilon)) .$$

But $J_{t_\varepsilon}(X_\varepsilon, T_\varepsilon, D_\varepsilon, L_\varepsilon) = O(t_\varepsilon)$ and $\exp(-D_\varepsilon(t_\varepsilon)) = 1 + O(t_\varepsilon)$, therefore:

$$U(x_\varepsilon, t_\varepsilon) \le U(X_\varepsilon(t_\varepsilon), 0) + O(t_\varepsilon) ,$$

and $U^*(x, 0) \le \limsup U(x_\varepsilon, t_\varepsilon) \le \limsup U(X_\varepsilon(t_\varepsilon), 0) \le U^*(x, 0)$. Proving the claim.

Q.E.D.

Q.E.D.

As we shall see later on in this book, Ishii solutions are not unique in general in the presence of discontinuities. Nevertheless, we prove below that U is the minimal one, see Corollary 4.4.3, and we will explain later on several ways in which we can recover some uniqueness.

4.4 Supersolutions of the Bellmann equation

4.4.1 Super-dynamic programming principle

We prove here that supersolutions always satisfy a super-dynamic programming principle. Again, we remark that this result is independent of the possible discontinuities for the dynamic, discount factor and cost. But to prove it, we need an additional ingredient in which we assume that we have already used Lemma 4.2.2 to reduce to the case when $c \ge 0$.

Lemma 4.4.1 *Assume* $(\mathbf{H_{BCL}})_{struct}$ *holds and let* $\chi(t) = -K(t+1)$ *for* $K > 0$ *large enough. Then, for any* $(x, t) \in \mathbb{R}^N \times [0, T_f]$ *and any* $(b, c, l) \in \mathbf{BCL}(x, t)$,

$$-b \cdot D\chi(t) + c\chi(t) - l \le -\underline{c} < 0 .$$

Proof — This is just obtained by direct computation: $-b \cdot D\chi(t) = Kb^t \leq 0$ while $c\chi(t) - l \leq -Kc - l$. By taking $K \geq (\underline{c} + l)/c$, we get the result.

Q.E.D.

Lemma 4.4.1, which is valid both for $t > 0$ and $t = 0$, provides a very classical property: the underlying HJB equation has a strict subsolution, which is a key point in comparison results. Of course, in this time-dependent case, one could say that such property is obvious. But we are not completely in a standard time-dependent case since we recall that $b^t = 0$ is allowed potentially for any $t \geq 0$.

Our next result is the

Lemma 4.4.2 *Under Assumptions* $(\mathbf{H_{BCL}})$*, if v is a bounded l.s.c. supersolution of* (4.8) *in $\mathbb{R}^N \times (0, T_f]$, then, for any $(\bar{x}, \bar{t}) \in \mathbb{R}^N \times (0, T_f]$ and any $\sigma > 0$,*

$$v(\bar{x}, \bar{t}) \geq \inf_{\mathcal{T}(\bar{x},\bar{t})} \left\{ \int_0^\sigma l\big(X(s), T(s)\big) \exp(-D(s))\, \mathrm{d}s + v\big(X(\sigma), T(\sigma)\big) \exp(-D(\sigma)) \right\} \tag{4.15}$$

Proof — To begin with, because of Lemma 4.2.2 we can assume that $c \geq 0$ for any $(b, c, l) \in \mathbf{BCL}(x, t)$ and for any (x, t). Fixing $(\bar{x}, \bar{t})$ and $\sigma > 0$, we argue through a three-step proof involving a regularization procedure and comparison result in the compact domain

$$\mathcal{K}_{(\bar{x},\bar{t})} := \overline{B(\bar{x}, M\sigma)} \times [0, \bar{t}]\,,$$

where M is given by $(\mathbf{H_{BCL}})_{fund}$.

Step 1: regularization — We consider a sequence of regularized Hamiltonians using the penalization function

$$\psi(b, c, l, x, t) = \inf_{(y,s)\in\mathbb{R}^N\times[0,T_f]} \Big(\operatorname{dist}\big((b, c, l), \mathbf{BCL}(y, s)\big) + |y - x| + |t - s| \Big)\,,$$

where $\operatorname{dist}(\cdot, \mathbf{BCL}(y, s))$ denotes the distance to the set $\mathbf{BCL}(y, s)$. We notice that ψ is Lipschitz continuous and that $\psi(b, c, l, x, t) = 0$ if $(b, c, l) \in \mathbf{BCL}(x, t)$. Then we set

$$\mathbb{F}_\delta(x, t, r, p) := \sup_{(b_\delta,c_\delta,l_\delta)\in\mathbf{BCL}_\delta(x,t)} \big\{ -b_\delta \cdot p + c_\delta r - l_\delta \big\}\,,$$

where $\mathbf{BCL}_\delta(x, t)$ is the set of all $(b_\delta, c_\delta, l_\delta) \in \mathbb{R}^{N+1} \times \mathbb{R} \times \mathbb{R}$ such that $|b^x_\delta| \leq M$, $-1 \leq b^t_\delta \leq 0$, $0 \leq c_\delta \leq M$ and

$$l_\delta = l + \delta^{-1}\psi\Big(b_\delta, c_\delta, l, x, t\Big) \quad \text{for some } |l| \leq M\,.$$

This sequence of Hamiltonians enjoys the following straightforward properties:

(*i*) for any $\delta > 0$, $\mathbb{F}_\delta \geq \mathbb{F}$ and therefore v is a l.s.c. supersolution of $\mathbb{F}_\delta \geq 0$ on $B(\bar{x}, M\sigma) \times (0, t]$;

(*ii*) the Hamiltonians $\mathbb{F}_\delta$ are (globally) Lipschitz continuous w.r.t. all variables;

(*iii*) $\mathbb{F}_\delta \downarrow \mathbb{F}$ as $\delta \to 0$, all the other variables being fixed.

On the other hand, v being l.s.c. on $\mathcal{K}_{(\bar{x},\bar{t})}$, there exists an increasing sequence $(v_\delta)_\delta$ of Lipschitz continuous functions such that $v_\delta \leq v$ and $\sup_\delta v_\delta = v$ on $\mathcal{K}_{(\bar{x},\bar{t})}$.

For $(x,t) \in \mathcal{K}_{(\bar{x},\bar{t})}$, we now introduce the function

$$u_\delta(x,t) := \inf \Big\{ \int_0^{\sigma\wedge\theta} l_\delta\big(X_\delta(s), T_\delta(s)\big) \exp(-D_\delta(s))\, \mathrm{d}s \\ + v_\delta\big(X_\delta(\sigma\wedge\theta), T_\delta(\sigma\wedge\theta)\big) \exp(-D_\delta(\sigma\wedge\theta)) \Big\} ,$$

where $(X_\delta, T_\delta, D_\delta, L_\delta)$ is a solution of the differential inclusion

$$\begin{cases} (\dot{X}_\delta, \dot{T}_\delta, \dot{D}_\delta, \dot{L}_\delta)(s) \in \mathbf{BCL}_\delta(X_\delta(s), T_\delta(s)) , \\ (X_\delta, T_\delta, D_\delta, L_\delta)(0) = (x, t, 0, 0) , \end{cases}$$

the infimum being taken over all trajectories X_δ which stay in $\overline{B(\bar{x}, M\sigma)}$ till time $\sigma\wedge\theta$ and any stopping time θ such that either $X_\delta(\theta)$ on $\partial B(\bar{x}, M\sigma)$ or $T_\delta(\theta) = 0$.

By classical arguments, u_δ is continuous since all the data involved are continuous, $u_\delta \leq v_\delta$ on $(\partial B(\bar{x}, M\sigma) \times [0,\bar{t}]) \cup (B(\bar{x}, M\sigma) \times \{0\})$ (for the same reason) and u_δ satisfies

$$\mathbb{F}_\delta(x, t, u, Du) = 0 \quad \text{in } B(\bar{x}, M\sigma) \times (0, \bar{t}] .$$

Notice that this equation and the one for v_δ hold up to time $\bar{t}$, as a consequence of the fact that $b^t \leq 0$ for all $b \in \mathbf{B}(x,t)$ and all (x,t).

Step 2: comparison for the approximated problem — In order to show that $u_\delta \leq v$ in $\mathcal{K}_{(\bar{x},\bar{t})}$ we argue by contradiction assuming that $\max_{\mathcal{K}_{(\bar{x},\bar{t})}}(u_\delta - v) > 0$.

We consider the function χ given by Lemma 4.4.1: using the definition of l_δ, it is easy to show that

$$\mathbb{F}_\delta(x, t, \chi, D\chi) \leq -\underline{c} < 0 \quad \text{in } B(\bar{x}, M\sigma) \times (0, \bar{t}] ,$$

and, by convexity, for any $0 < \mu < 1$, $u_{\delta,\mu} = \mu u_\delta + (1-\mu)\chi$ is a subsolution of

$$\mathbb{F}_\delta(x, t, u_{\delta,\mu}, Du_{\delta,\mu}) \leq -(1-\mu)\underline{c} < 0 \quad \text{in } B(\bar{x}, M\sigma) \times (0, \bar{t}] .$$

Moreover, if $\mu < 1$ is close enough to 1, we still have $\max_{\mathcal{K}_{(\bar{x},\bar{t})}}(u_{\delta,\mu} - v) > 0$ and we can choose K large enough in order to have $u_{\delta,\mu} \le v_\delta$ on $(\partial B(\bar{x}, M\sigma) \times [0,\bar{t}]) \cup (B(\bar{x}, M\sigma) \times \{0\}$.

If $(\tilde{x},\tilde{t}) \in \mathcal{K}_{(\bar{x},\bar{t})}$ is a maximum point of $u_{\delta,\mu} - v$, we remark that $(\tilde{x},\tilde{t})$ cannot be on $(\partial B(\bar{x}, M\sigma) \times [0,\bar{t}]) \cup (B(\bar{x}, M\sigma) \times \{0\})$ since on these parts of the boundary $u_{\delta,\mu} \le v$.

Now we perform the standard proof using the doubling of variables with the test-function

$$u_{\delta,\mu}(x,t) - v(y,s) - \frac{|x-y|^2}{\epsilon^2} - \frac{|t-s|^2}{\epsilon^2} - (x-\tilde{x})^2 - (t-\tilde{t})^2 \, .$$

By standard arguments, see Lemma 6.4.1, this function has a maximum point $(x_\varepsilon, t_\varepsilon, y_\varepsilon, s_\varepsilon)$ which converges to $(\tilde{x},\tilde{t},\tilde{x},\tilde{t})$ since $(\tilde{x},\tilde{t})$ is a strict global maximum point of $(y,s) \mapsto u_{\delta,\mu}(y,s) - v(y,s) - (y-\tilde{x})^2 - (s-\bar{t})^2$ in $\mathcal{K}_{(\bar{x},\bar{t})}$.

We use now the $\mathbb{F}_\delta$-supersolution inequality for v, the strict subsolution inequality for $u_{\delta,\mu}$ and the regularity of $\mathbb{F}_\delta$ together with the fact that $c \ge 0$ for all $(b,c,l) \in \mathbf{BCL}(y,s)$ [or $\mathbf{BCL}_\delta(y,s)$] and any $(y,s) \in \mathcal{K}_{(\bar{x},\bar{t})}$. We are led to the inequality

$$o(1) \le -(1-\mu)\exp(-K\bar{t})\eta < 0 \, ,$$

which yields a contradiction. Sending $\mu \to 1$, we get that $u_\delta \le v$ in $\mathcal{K}_{(\bar{x},\bar{t})}$.

Step 3: passing to the limit — To conclude the proof, we use the inequality $u_\delta(\bar{x},\bar{t}) \le v(\bar{x},\bar{t})$ and we first remark that, in the definition of $u_\delta(\bar{x},\bar{t})$, necessarily $\sigma \wedge \theta = \sigma$ since the trajectory X_δ cannot exit $B(\bar{x}, M\sigma)$ before time σ. Then, in order to let δ tend to 0 in this inequality, we pick a δ-optimal trajectory $(X_\delta, T_\delta, D_\delta, L_\delta)$.

By the uniform bounds on $(\dot{X}_\delta, \dot{T}_\delta, \dot{D}_\delta, \dot{L}_\delta)$, Ascoli-Arzela's Theorem implies that up to the extraction of a subsequence, we may assume that $(X_\delta, T_\delta, D_\delta, L_\delta)$ converges locally uniformly on $[0,+\infty)$ to some (X,T,D,L). We may also assume that their derivatives converge in L^∞ weak-* topology (in particular $\dot{L}_\delta = l_\delta$).

Using the δ-optimal trajectory for approching u_δ leads to

$$\begin{aligned} v(\bar{x},\bar{t}) \ge \int_0^\sigma & l_\delta\big(X_\delta(s), T_\delta(s)\big)\exp(-D_\delta(s))\,\mathrm{d}s \\ & + v_\delta\big(X_\delta(\sigma), T_\delta(\sigma)\big)\exp(-D_\delta(\sigma)) - \delta \, , \end{aligned} \tag{4.16}$$

an inequality that we use in two ways.

First, by multiplying by δ and using that v and v_δ are bounded. Writing $Z_s = (X_\delta(s), T_\delta(s))$ for simplicity, we obtain

$$\int_0^\sigma \psi\Big(b_\delta(Z_s), c_\delta(Z_s), l_\delta(z_s), X_\delta(s), T_\delta(s)\Big)\exp(-D_\delta(s))ds = O(\delta) \, .$$

By classical results on weak convergence, since the functions $(b_\delta, c_\delta, l_\delta)$ converge weakly to (b, c, l), there exists $\mu_s \in L^\infty\big(0, t; \mathbb{P}(B(0,M)\times[-M,M]^2)\big)$ where $\mathbb{P}(B(0,M)\times[-M,M]^2)$ is the set of probability measures on $B(0,M)\times[-M,M]^2$ such that, taking into account the uniform convergence of X_δ, T_δ and D_δ, we have

$$\int_0^\sigma \int_{B(0,M)\times[-M,M]^2} \psi\Big(b, c, l, X(s), T(s)\Big) \exp(-D(s))\, \mathrm{d}\mu_s(b,c,l)\, \mathrm{d}s =$$

$$\lim_{\delta\to 0} \int_0^\sigma \psi\Big(b_\delta(s), c_\delta(s), l_\delta(s), X_\delta(s), T_\delta(s)\Big) \exp(-D_\delta(s)) ds = 0\ .$$

We remark that $\psi \geq 0$ and $\psi(b,c,l,x,t) = 0$ if and only if $(b,c,l) \in \mathbf{BCL}(x,t)$, therefore (X,T,D,L) is a solution of the **BCL**-differential inclusion.

Second, we come back to (4.16) after recalling that ψ is nonnegative, which implies that $l_\delta\big(X_\delta(s), T_\delta(s)\big) \geq l\big(X_\delta(s), T_\delta(s)\big)$ and therefore

$$\int_0^\sigma l\big(X_\delta(s), T_\delta(s)\big) \exp(-D_\delta(s))\, \mathrm{d}s + v_\delta\big(X_\delta(\sigma), T_\delta(\sigma)\big) \exp(-D_\delta(\sigma)) - \delta \leq v(x,t)\ .$$

We pass to the limit in this inequality using the lower-semicontinuity of v, together with the uniform convergence of $X_\delta, T_\delta, D_\delta$ and the dominated convergence theorem for the l-term. In particular,

$$\liminf_{\delta\to 0} \Big(v_\delta\big(X_\delta(\sigma), T_\delta(\sigma)\big)\Big) \geq v\big(X(\sigma), T(\sigma)\big)\ ,$$

which yields

$$\int_0^\sigma l(X(s), T(s)) \exp(-D(s))\, \mathrm{d}s + v\big(X(\sigma), T(\sigma)\big) \exp(-D(\sigma)) \leq v(\bar{x}, \bar{t})\ .$$

Finally, recalling that (X,T,D,L) is a solution of the **BCL**-differential inclusion, taking the infimum in the left-hand side over all solutions of this differential inclusion gives the desired inequality.

Q.E.D.

4.4.2 The value function is the minimal supersolution

An easy consequence of Lemma 4.4.2 is the

Corollary 4.4.3 *Under Assumptions* ($\mathbf{H_{BCL}}$)*, the value function U is the minimal Ishii supersolution of* (4.8)*.*

Proof — Let v be any bounded l.s.c. supersolution in the Ishii sense of $\mathbb{F} = 0$. Using (4.15) we see that for any $(x,t) \in \mathbb{R}^N \times (0,T_f]$ and $\sigma > 0$,

$$v(x,t) \geq \inf_{\mathcal{T}(x,t)} \left\{ \int_0^\sigma l\big(X(s),T(s)\big) \exp(-D(s))\, \mathrm{d}s + v\big(X(\sigma),T(\sigma)\big) \exp(-D(\sigma)) \right\}.$$

Sending $\sigma \to +\infty$, we see that in particular for any trajectory (X,T,D,L) the integral $J(X,T,D,L)$ in Lemma 4.2.3-(ii) is bounded by $2\|v\|_\infty$.

Therefore, $D(\sigma) \to 0$ as $\sigma \to +\infty$ and passing to the limit in the dynamic programming principle yields

$$v(x,t) \geq \inf_{\mathcal{T}(x,t)} \int_0^{+\infty} l\big(X(s),T(s)\big) \exp(-D(s))\, \mathrm{d}s = U(x,t)\,.$$

The conclusion is that $v \geq U$, which proves the minimality of the value function.

Q.E.D.

We end this chapter by some comment: as we saw, the situation is not totally symmetric between general Ishii supersolutions and subsolutions. For supersolutions, properties derive directly from the Bellman equation while the treatment of general subsolutions requires more advanced tools and some structure assumption on the discontinuities. This is done in Chapter 5.

Chapter 5

Mixed Tools

5.1 Initial condition for sub and supersolutions of the Bellman Equation

In this section, we consider a little bit more precisely the conditions satisfied by sub and supersolutions of the Bellman Equation at time $t = 0$ according to Definition 4.3.2.

In the classical cases where one has a standard initial data u_0, these conditions read $\min(\mathbb{F}_*, u - u_0) \leq 0$ for the subsolution and $\max(\mathbb{F}, v - u_0) \geq 0$ for the supersolution, and it is known that they just reduce to either $u \leq u_0$ in $\mathbb{R}^N$ if u is a subsolution or $v \geq u_0$ in $\mathbb{R}^N$ if v is a supersolution.

Here we have an analogous result but which is more complicated, involving the initial Hamiltonian $\mathbb{F}_{init}$ defined in Section 4.3.

The result is the following.

Proposition 5.1.1 *Under Assumptions* ($\mathbf{H_{BCL}}$)*, if $u : \mathbb{R}^N \times [0, T_f] \to \mathbb{R}$ is an u.s.c. viscosity subsolution of the Bellman Equation $\mathbb{F} = 0$, then $u(x, 0)$ is a subsolution in $\mathbb{R}^N$ of*

$$(\mathbb{F}_{init})_*\big(x, u(x,0), D_x u(x,0)\big) \leq 0 \quad \text{in } \mathbb{R}^N .$$

Similarly, if $v : \mathbb{R}^N \times [0, T_f] \to \mathbb{R}$ is a l.s.c. supersolution of the Bellman Equation, then $v(x, 0)$ is a supersolution of

$$\mathbb{F}_{init}(x, v(x,0), D_x v(x,0)) \geq 0 \quad \text{in } \mathbb{R}^N .$$

Proof — We provide the full proof in the supersolution case and we will add additional

comments in the subsolution one. Let $\phi : \mathbb{R}^N \to \mathbb{R}$ be a smooth function and let x be a local strict minimum point of the function $y \mapsto v(y,0) - \phi(y)$. In order to use the supersolution property of v, we consider for $0 < \varepsilon \ll 1$ the function $(y,t) \mapsto v(y,t) - \phi(y) + \varepsilon^{-1}t$.

By an easy application of Lemma 6.4.1 in a compact neighborhood of $(x,0)$, this function has a local minimum point at $(x_\varepsilon, t_\varepsilon)$ and we have at the same time $(x_\varepsilon, t_\varepsilon) \to (x,0)$ and $v(x_\varepsilon, t_\varepsilon) \to v(x,0)$ as $\varepsilon \to 0$. The viscosity supersolution inequality reads

$$\sup_{(b,c,l)\in \mathbf{BCL}(x_\varepsilon,t_\varepsilon)} \left\{\varepsilon^{-1}b^t - b^x \cdot D_x\phi(x_\varepsilon) + cv(x_\varepsilon,t_\varepsilon) - l\right\} \geq 0\,.$$

We denote by $(b_\varepsilon, c_\varepsilon, l_\varepsilon)$ the (b,c,l) for which the supremum is achieved and which exists since $BCL(x_\varepsilon, t_\varepsilon)$ is compact. By Assumptions $(\mathbf{H_{BCL}})$, we may assume that up to extraction, $(b_\varepsilon, c_\varepsilon, l_\varepsilon) \to (\bar b, \bar c, \bar l) \in \mathbf{BCL}(x,0)$. Moreover, since $b^t_\varepsilon \leq 0$ and the other terms are bounded, the above inequality implies that $\varepsilon^{-1}b^t_\varepsilon$ is also bounded independently of ε. In other words, $b^t_\varepsilon = O(\varepsilon)$ and $\bar b = (\bar b^x, 0)$.

Dropping the negative $\varepsilon^{-1}b^t_\varepsilon$-term in the supersolution inequality, we obtain

$$-b^x_\varepsilon \cdot D_x\phi(x_\varepsilon) + c_\varepsilon v(x_\varepsilon,t_\varepsilon) - l_\varepsilon \geq 0\,,$$

and letting $\varepsilon \to 0$, we end up with $-\bar b^x \cdot D_x\phi(x) + \bar c v(x,0) - \bar l \geq 0$. since $(\bar b, \bar c, \bar l) \in \mathbf{BCL}(x,0)$, we deduce that

$$\sup_{((b^x,0),c,l)\in \mathbf{BCL}(x,0)} \left\{ - b^x \cdot D_x\phi(x) + cv(x,0) - l\right\} \geq 0\,,$$

in other words: $\mathbb{F}_{init}\big(x, v(x,0), D_xv(x,0)\big) \geq 0$ holds in the viscosity sense.

In the subsolution case, the proof is analogous but we consider local strict maximum point of the function $y \mapsto u(y,0) - \phi(y)$. Introducing the function $(y,t) \mapsto u(y,t) - \phi(y) - \varepsilon^{-1}t$ for $0 < \varepsilon \ll 1$, we have a sequence of local maximas $(x_\varepsilon, t_\varepsilon)$ such that $(x_\varepsilon, t_\varepsilon) \to (x,0)$ and $u(x_\varepsilon, t_\varepsilon) \to u(x,0)$ as $\varepsilon \to 0$.

If $t_\varepsilon > 0$, the subsolution inequality reads

$$\mathbb{F}_*(x_\varepsilon, t_\varepsilon, u(x_\varepsilon,t_\varepsilon), (D_x\phi(x_\varepsilon), \varepsilon^{-1})) \leq 0\,.$$

This time, we cannot bound $\varepsilon^{-1}b^t$ as we did for the supersolution case, but because of $(\mathbf{H_{BCL}})_{struct}$-$(i)$, in all $\mathbf{BCL}(x,t)$ for $t \geq 0$, there exists an element with $b^t = -1$. Since the other terms are bounded, this implies that the $\mathbb{F}_*$-term in the above inequality is larger than $\varepsilon^{-1} + O(1)$ and therefore, for ε small enough, the $\mathbb{F}_*$-inequality above cannot hold.

Hence, necessarily $t_\varepsilon = 0$ and the strict maximum point property for $u - \phi$ implies that $x_\varepsilon = x$. But for the same reason as above, for $\varepsilon > 0$ small enough the viscosity inequality

$$\mathbb{F}_*(x, 0, u(x,0), (D_x\phi(x), \varepsilon^{-1})) \leq 0$$

cannot hold unless it corresponds to a $(b, c, l) \in \mathbf{BCL}(x, 0)$ such that $b^t = 0$. Which leads finally to

$$(\mathbb{F}_{init})_*(x, u(x,0), D_x\phi(x)) \leq 0\,,$$

the inequality we wanted to prove.

Q.E.D.

The above result means that, in order to compute the initial data, one has to solve an equation. A fact which is already known in the case of unbounded control.

In the case of classical problems, a typical situation is when for $t > 0$, the elements of $\mathbf{BCL}(x, t)$ are of the form $((b^x, -1), c, l)$ while for $t = 0$ we consider a l.s.c. cost u_0 in $\mathbb{R}^N$. In order to satisfy the upper semi-continuity of $\mathbf{BCL}$ at $t = 0$, we need a priori to consider both elements of the form $((b^x, -1), c, l)$ and $((0, 0), 1, u_0(x))$. But in that situation, the result above leads back to the standard initial data conditions

$$u(x,0) \leq (u_0)^*(x) \quad \text{and} \quad v(x,0) \geq u_0(x) \quad \text{in } \mathbb{R}^N\,,$$

due to the fact that $\mathbb{F}_{init}(x, u, p_x) = u - u_0(x)$ and $(\mathbb{F}_{init})_*(x, u, p_x) = u - (u_0)^*(x)$.

5.2 A second relevant example involving unbounded control

We want to consider here a problem that we first write as

$$\max(u_t + H(x, t, u, D_x u), |D_x u| - 1) = 0 \quad \text{in } \mathbb{R}^N \times (0, T_f), \tag{5.1}$$

with an "initial data" g is a bounded, continuous function in $\mathbb{R}^N$ (we are going to make more precise what we mean by initial data). Here the Hamiltonian H is still given by

$$H(x, t, r, p) := \sup_{\alpha \in A} \{-b(x, t, \alpha) \cdot p + c(x, t, \alpha)r - l(x, t, \alpha)\}\,,$$

but the functions b, c, l may be discontinuous. Our first aim is to connect this problem with the above framework and deduce the key assumptions which have to be imposed on b, c, l in order to have our assumptions being satisfied.

First we have to give the sets **BCL** and to do so, we set, for $x \in \mathbb{R}^N$, $t \in (0, T_f]$

$$\mathbf{BCL}_1(x,t) := \{((b(x,t,\alpha),-1), c(x,t,\alpha), l(x,t,\alpha)) \ : \ \alpha \in A\} \ ,$$

and

$$\mathbf{BCL}_2(x,t) := \{((\beta,0),0,1) \ : \ \beta \in \overline{B(0,1)}\} \ .$$

Then we introduce

$$\mathbf{BCL}(x,t) = \overline{\text{co}}\big(\mathbf{BCL}_1(x,t) \cup \mathbf{BCL}_2(x,t)\big) \ ,$$

where, if $E \subset \mathbb{R}^k$ for some k, $\overline{\text{co}}(E)$ denotes the closed convex of E; computing $\mathbb{F}(x,t,r,p) = \sup_{(b,c,l)\in\mathbf{BCL}(x,t)} \big\{ -b\cdot p + cr - l\big\}$, we actually find that, for any x,t,r,p_x,p_t

$$\mathbb{F}(x,t,r,(p_x,p_t)) = \max(p_t + H(x,t,u,p_x), |p_x| - 1) \ .$$

For $t = 0$, we have to add the following

$$\mathbf{BCL}_0(x,0) := \{((0,0),1,g(x))\} \ .$$

and $\mathbf{BCL}(x,0) = \overline{\text{co}}\,(\mathbf{BCL}_0(x,0) \cup \mathbf{BCL}_1(x,0) \cup \mathbf{BCL}_2(x,0))$.

We first consider Assumption $(\mathbf{H_{BCL}})_{fund}$ which is satisfied if the three functions $b(x,t,\alpha)$, $c(x,t,\alpha)$, $l(x,t,\alpha)$ are bounded on $\mathbb{R}^N \times [0,T_f] \times A$ and if $\mathbf{BCL}_1(x,t)$ has compact, convex images and is upper semi-continuous. Next we remark that $(\mathbf{H_{BCL}})_{struct}$ obviously holds and we are going to assume in addition that $c(x,t,\alpha) \geq 0$ for all x,t,α (this is not really an additional assumption since we can reduce to this case by the $\exp(-Kt)$- change).

Since all these assumptions hold, this means that all the results of Section 4.2 also holds. Moreover we have for the initial data $\mathbb{F}_{init}(x,u,p_x) := \max\big\{|p_x| - 1, u - g(x)\big\}$ and therefore the computation of the "real" initial data comes from the resolution of the stationary equation

$$\max(|D_x u| - 1, u - g(x)) = 0 \quad \text{in } \mathbb{R}^N. \tag{5.2}$$

Remark 5.2.1 *Of course, this example remains completely standard as long as we are in the continuous case (typically under the assumptions* $(\mathbf{H_{BA-CP}})$*); it will be more interesting when we will treat examples in which we have discontinuities in the dynamics, discount factors and costs or when the term "*$|D_x u| - 1$*" will be replaced by, for instance, "*$|D_x u| - a(x)$*" where* $a(\cdot)$ *is a discontinuous functions satisfying suitable assumptions and in particular* $a(x) \geq \eta > 0$ *in* $\mathbb{R}^N$.

5.3 Dynamic programming principle for subsolutions

In this section, we provide a sub-dynamic programming principle for subsolutions of Bellman Equations, but in a more general form than usual, due to the very general framework we use in Section 4.1 allowing dynamics to have some $b^t = 0$. Roughly speaking, we show that if a **(LCR)** holds in a suitable subdomain $\mathcal{O}$ of $\mathbb{R}^N \times [0, T_f]$ and for a suitable equation, then subsolutions satisfy a sub-dynamic programming principle inside $\mathcal{O}$.

This formulation is needed in order to get sub-dynamic principles away from the various manifolds on which the singularities are located, and to deal with situations where the definition of "subsolution" may be different from the standard one: even if, to simplify matter, we write below the equation in a usual form (cf. (5.3)), the notion of "subsolution" can be either an Ishii subsolution or a stratified subsolution, depending on the context. These specific sub-dynamic programming principles will play a key role in the proofs of most of our global comparison results, via Lemma 5.4.1.

In order to be more specific, we consider $(x_0, t_0) \in \mathbb{R}^N \times (0, T_f]$ and the same equation as in the previous section set in $Q_{r,h}^{x_0,t_0}$ for some $r > 0$ and $0 < h < t_0$, namely

$$\mathbb{F}(x, t, u, Du) = 0 \quad \text{on } Q_{r,h}^{x_0,t_0} \,, \tag{5.3}$$

where $\mathbb{F}$ is defined by (4.4), and we recall that $Du = (D_x u, u_t)$. We point out that we assume that **BCL** and $\mathbb{F}$ are defined in the whole domain $\mathbb{R}^N \times [0, T_f]$.

In the sequel, $\mathcal{M}$ is a closed subset of $\overline{Q_{r,h}^{x_0,t_0}}$ such that $(x_0, t_0) \notin \mathcal{M}$ and $\mathcal{O} = Q_{r,h}^{x_0,t_0} \setminus \mathcal{M} \neq \emptyset$. We denote by $\mathcal{T}_{\mathcal{O}}^h(x_0, t_0)$ the set of trajectories starting from (x_0, t_0), such that $(X(s), T(s)) \in \mathcal{O}$ for all $s \in [0, h]$. For simplicity here, we assume that the size of the cylinder satisfies $Mh < r$. This is not restrictive at all since when we use the following sub-dynamic programming principle, we can always apply it in situations where r is fixed and we can choose a smaller h.

Our result is the

Theorem 5.3.1 *Let $h, r > 0$ be such that $Mh < r$. Let u be a subsolution of (5.3) and let us assume that, for any continuous function ψ such that $\psi \geq u$ on $\overline{Q_{r,h}^{x_0,t_0}}$, a* **(LCR)** *holds in $\mathcal{O}$ for*

$$\max(\mathbb{F}(x, t, u, Du), u - \psi) = 0 \quad \text{in } \mathcal{O} \,. \tag{5.4}$$

If $\mathcal{T}^h_{\mathcal{O}}(x_0,t_0)\neq\emptyset$, *then for any* $\eta\le h$

$$u(x_0,t_0)\le \inf_{X\in\mathcal{T}^h_{\mathcal{O}}(x_0,t_0)}\left\{\int_0^\eta l\big(X(s),T(s)\big)\exp(-D(s))\,\mathrm{d}s+u\big(X(\eta),T(\eta)\big)\exp(-D(\eta))\right\}. \tag{5.5}$$

Proof — In order to prove (5.5), the strategy is the following: we build suitable value functions $v^{\varepsilon,\delta}$, depending on two small parameters ε,δ which are supersolutions of some problems of the type $\max(\mathbb{F}(x,t,v,Dv),v-\psi^\delta)\ge 0$, for some function $\psi^\delta\ge u$ on $\overline{Q^{x_0,t_0}_{r,h}}$. Then, comparing the supersolutions $v^{\varepsilon,\delta}$ with the subsolution u and choosing properly the parameters ε,δ we obtain (5.5) after using the dynamic programming principle satisfied by $v^{\varepsilon,\delta}$.

The main difficulty is that we have a comparison result which is not valid up to $\mathcal{M}$, only in $\mathcal{O}$. Therefore we need to make sure that the supersolution enjoys suitable properties not only on $\partial Q^{x_0,t_0}_{r,h}$ but also on $\mathcal{M}$.

To do so, we introduce a control problem in $\mathbb{R}^N\times[t_0-h,t_0]$ with a large penalization both in a neighborhood of $\partial Q^{x_0,t_0}_{r,h}$ and outside $\overline{Q^{x_0,t_0}_{r,h}}$, but also in a neighborhood of $\mathcal{M}$. Unfortunately, the set valued map **BCL** does not necessarily satisfy assumption $(\mathbf{H_{BCL}})_{struct}$-$(iii)$ at time $t=t_0-h$, which plays the role of the initial time $t=0$ here. We need also to take care of the possibility that b^t vanishes inside $\overline{Q^{x_0,t_0}_{r,h}}$. For these reasons, we need to enlarge not only the "restriction" of **BCL** to $\mathbb{R}^N\times\{t_0-h\}$ in order to satisfy $(\mathbf{H_{BCL}})_{struct}$, but also on the whole domain $\mathbb{R}^N\times[t_0-h,t_0]$.

For doing so, since u is u.s.c., it can be approximated a decreasing sequence $(u^\delta)_\delta$ of bounded continuous functions and we enlarge $\mathbf{BCL}(x,t)$ for $t\in[t_0-h,t_0]$ by adding elements of the form

$$((b^x,b^t),c,l)=((0,0),1,u^\delta(x,t)+\delta)\quad\text{for } 0\le\delta\ll 1\,.$$

On the other hand, we introduce, for $0<\varepsilon\ll 1$, the penalization function

$$\chi_\varepsilon(x,t):=\frac{1}{\varepsilon^4}\Big[\big(2\varepsilon-d((x,t),\mathcal{M})\big)^+ +(2\varepsilon-(r-|x-x_0|))^+ +(2\varepsilon-(t-t_0+h))^+\Big],$$

so that $\chi_\varepsilon(x,t)\ge\varepsilon^{-3}$ if either $d((x,t),\mathcal{M})\le\varepsilon$, $d(x,\partial B(x_0,r))\le\varepsilon$ or $t-(t_0-h)\le\varepsilon$.

We use this penalization in order to modify the original elements in $\mathbf{BCL}(x,t)$, where $l(x,t)$ is replaced by $l(x,t)+\chi_\varepsilon(x,t)$. We denote by $\mathbf{BCL}^{\delta,\varepsilon}$ this new set-valued map where, at the same time, **BCL** is enlarged and modified; the elements of $\mathbf{BCL}^{\delta,\varepsilon}$ are referenced as $(b^{\delta,\varepsilon},c^{\delta,\varepsilon},l^{\delta,\varepsilon})$. We recall that we can assume that for the

original **BCL**, we have $c \geq 0$ and therefore we also have $c^{\delta,\varepsilon} \geq 0$ for all (x,t) and $(b^{\delta,\varepsilon}, c^{\delta,\varepsilon}, l^{\delta,\varepsilon}) \in \mathbf{BCL}^{\delta,\varepsilon}(x,t)$.

In $\mathbb{R}^N \times [t_0 - h, t_0]$, we introduce the value-function $v^{\varepsilon,\delta}$ given by

$$v^{\varepsilon,\delta}(x,t) = \inf_{\mathcal{T}^{\delta,\varepsilon}(x,t)} \left\{ \int_0^{+\infty} l^{\delta,\varepsilon}\big(X^{\delta,\varepsilon}(s), T^{\delta,\varepsilon}(s)\big) \exp(-D^{\delta,\varepsilon}(s)) ds \right\},$$

where $(X^{\delta,\varepsilon}, T^{\delta,\varepsilon}, D^{\delta,\varepsilon}, L^{\delta,\varepsilon})$ are solutions of the differential inclusion associated with $\mathbf{BCL}^{\delta,\varepsilon}$, constrained to stay in $\mathbb{R}^N \times [t_0 - h, t_0]$, $\mathcal{T}^{\delta,\varepsilon}(x,t)$ standing for the set of such trajectories.

Borrowing arguments from Section 4.1 and computing carefully the new Hamiltonian, we see that $v^{\varepsilon,\delta}$ is a l.s.c. supersolution of the HJB-equation

$$\max(\mathbb{F}(x,t,w,Dw), w - (u^\delta + \delta)) = 0 \quad \text{in } \mathbb{R}^N \times (t_0 - h, t_0],$$

because $l(x,t) + \chi_\varepsilon(x,t) \geq l(x,t)$ for any x and t, and we notice that u is a subsolution of this equation since $u \leq u^\delta + \delta$ in $\mathbb{R}^N \times (t_0 - h, t_0]$. We also remark that, due to the enlargement of **BCL**, $v^{\varepsilon,\delta}(x,t) \leq u^\delta(x,t) + \delta$, which is the value obtained by solving the differential inclusion with $(b,c,l) = ((0,0), 1, u^\delta(x,t) + \delta)$. We want to show that $v^{\varepsilon,\delta} \geq u$ in $\overline{\mathcal{O}}$. In order to do so, we have to examine the behavior of $v^{\varepsilon,\delta}$ in a neighborhood of $\partial\mathcal{O}$ first, which is provided by the

Lemma 5.3.2 *For $\varepsilon > 0$ small enough, $v^{\varepsilon,\delta}(x,t) \geq u^\delta(x,t)$ on $\partial\mathcal{O}$.*

We postpone the proof of this result and finish the argument. Since $v^{\varepsilon,\delta} \geq u^\delta \geq u$ on the boundary of $\mathcal{O}$, we have just to look at maximum points of $u - v^{\varepsilon,\delta}$ in $\mathcal{O}$ but, in this set, **(LCR)** holds for (5.4) with $\psi := u^\delta + \delta$. Therefore the comparison is valid and we end up with $v^{\varepsilon,\delta} \geq u$ everywhere in $\overline{\mathcal{O}}$.

Ending the proof and getting the sub-dynamic principle is done in three steps as follows.

Step 1 – at the specific point (x_0, t_0) we have $u(x_0,t_0) \leq v^{\varepsilon,\delta}(x_0,t_0)$, and using the Dynamic programming Principle for $v^{\varepsilon,\delta}$ at (x_0,t_0) gives that for any $\eta > 0$,

$$u(x_0,t_0) \leq \inf_{\mathcal{T}^{\varepsilon,\delta}(x_0,t_0)} \left\{ \int_0^{\eta} l^{\delta,\varepsilon}\big(X(s), T(s)\big) \exp(-D(s)) ds + v^{\varepsilon,\delta}\big(X(\eta), T(\eta)\big) \exp(-D(\eta)) \right\}. \tag{5.6}$$

we want to get the same inequality, but for trajectories in $\mathcal{T}^h_{\mathcal{O}}(x_0,t_0)$. This relies on the following step.

Step 2 – *Claim: if (X,T,D,L) is a given trajectory in $\mathcal{T}^h_{\mathcal{O}}(x_0,t_0)$ and if $\eta<h$, then, for $\varepsilon>0$ small enough, (X,T,D,L) coincides with a trajectory in $\mathcal{T}^{\varepsilon,\delta}(x_0,t_0)$ on $[0,\eta]$.*

The main argument in order to prove this claim is to notice that for ε small enough, such trajectories satisfy $\psi_\varepsilon(X(s),T(s))=0$ on $[0,\eta]$.

Indeed, let us fix $\eta<h$ and take ε small enough such that $t_0-h+2\varepsilon<t_0-\eta$. Then, for any trajectory (X,T,D,L) in $\mathcal{T}^h_{\mathcal{O}}(x_0,t_0)$, $T(s)\in[t_0-\eta,t_0]$ for $s\in[0,\eta]$, so that $T(s)>t_0-h+2\varepsilon$. Similarly, since $Mh<r$ and $|b|\le M$, we get that $d(X(s);\partial B(x_0,r))>2\varepsilon$ for $s\in[0,\eta]$. Of course, by definition of $\mathcal{T}^h_{\mathcal{O}}(x_0,t_0)$, the trajectory does not reach $\mathcal{M}$ hence, if ε is small enough, $d((X(s),T(s));\mathcal{M})>2\varepsilon$ for any $s\in[0,\eta]$. In other words, for each fixed trajectory in $\mathcal{T}^h_{\mathcal{O}}(x_0,t_0)$, if we take ε small enough (depending on the trajectory) we have $\psi_\varepsilon(X(s),T(s))=0$ on $[0,\eta]$.

Therefore, for any trajectory $(X,T,D,L)\in\mathcal{T}^h_{\mathcal{O}}(x_0,t_0)$, $l^{\delta,\varepsilon}(X(s),T(s))=l(X(s),T(s))$ if $\varepsilon>0$ is small enough and $0\le s\le\eta<h$. This means that (X,T,D,L) can be seen as a trajectory associated to the extended $\mathbf{BCL}^{\delta,\varepsilon}$, with initial data $(x_0,t_0,0,0)$. Hence it belongs to $\mathcal{T}^{\delta,\varepsilon}(x,t)$, which proves the claim.

Step 3 – *Passing to the limit in ε and δ.*

We take a specific trajectory $(X,T,D,L)\in\mathcal{T}^h_{\mathcal{O}}(x_0,t_0)$ and take ε small enough so that we can use it in (5.6). As we already noticed, $v^{\varepsilon,\delta}\le(u^\delta+\delta)$ everywhere in $Q^{x_0,t_0}_{r,h}$ due to the enlargement of **BCL**. Passing to the limit as $\varepsilon\to0$ yields

$$u(x_0,t_0)\le\int_0^\eta l\big(X(s),T(s)\big)\exp(-D(s))ds+(u^\delta+\delta)\big(X(\eta),T(\eta)\big)\exp(-D(\eta))\Big\}\,.$$

Then, we can let $\delta\to0$ in this inequality, using that $(u^\delta+\delta)_\delta$ is a decreasing sequence which converges to u and that the trajectory (X,T,D,L) and η are fixed.

Therefore $(u^\delta+\delta)\big(X(\eta),T(\eta)\big)\to u\big(X(\eta),T(\eta)\big)$ and we get

$$u(x_0,t_0)\le\int_0^\eta l\big(X(s),T(s)\big)\exp(-D(s))ds+u\big(X(\eta),T(\eta)\big)\exp(-D(\eta))\,.$$

Taking the infimum over all trajectories in $\mathcal{T}^h_{\mathcal{O}}(x_0,t_0)$ yields the conclusion when $\eta<h$. The result for $\eta=h$ is obtained by letting η tend to h, arguing once more trajectory by trajectory.

Q.E.D.

Proof of Lemma 5.3.2 — We need to consider three portions of $\partial\mathcal{O}$: $t=t_0-h$, $x\in\partial B(x_0,r)$ and $(x,t)\in\mathcal{M}$. We detail the first estimate which is technically

involved, then the last two ones are done with similar arguments. In the following, we use an optimal trajectory for $v^{\delta,\varepsilon}$, denoted by $(X^{\delta,\varepsilon}, T^{\delta,\varepsilon}, D^{\delta,\varepsilon}, L^{\delta,\varepsilon})$.

Part A. Initial estimates – if $t = t_0 - h$, we have to consider

– the running costs $l\big(X^{\delta,\varepsilon}(s), T^{\delta,\varepsilon}(s)\big) + \chi_\varepsilon(X^{\delta,\varepsilon}(s), T^{\delta,\varepsilon}(s))$, with (perhaps) a non-zero dynamic b^x.

– the running costs $u^\delta(X^{\delta,\varepsilon}(s), T^{\delta,\varepsilon}(s)) + \delta$ coming from the enlargement with a zero dynamic;

– and the convex combinations of the two above possibilities, obtained by using a weight $\mu^{\delta,\varepsilon}(s) \in [0,1]$.

We first notice that since $t = t_0 - h$, we have $T^{\delta,\varepsilon}(s) = t_0 - h$ for any $s \geq 0$ since $b^t \leq 0$ and the trajectories have the constraint to stay in $\mathbb{R}^N \times [t_0 - h, t_0]$. In the following, we make various estimates (for ε small enough) involving constants $\kappa_0, \kappa_1, \kappa_2, \kappa_3$ depending on the datas of the problem and $\delta > 0$ but neither on ε nor on $x \in \overline{B(x_0, r)}$.

Next we set

$$E := \big\{ s \in [0, +\infty) : l^{\delta,\varepsilon}\big(X^{\delta,\varepsilon}(s), T^{\delta,\varepsilon}(s)\big) = l^{\delta,\varepsilon}\big(X^{\delta,\varepsilon}(s), t_0 - h\big) \geq \varepsilon^{-3/2} \big\} ,$$

where $l^{\delta,\varepsilon}$ is given by the convex combination

$$\begin{aligned} l^{\delta,\varepsilon}\big(X^{\delta,\varepsilon}(s), t_0 - h\big) = \mu^{\delta,\varepsilon}(s) \Big\{ l\big(X^{\delta,\varepsilon}(s), t_0 - h\big) + \chi_\varepsilon\big(X^{\delta,\varepsilon}(s), t_0 - h\big) \Big\} \\ + \big(1 - \mu^{\delta,\varepsilon}(s)\big)(u^\delta + \delta)\big(X^{\delta,\varepsilon}(s), t_0 - h\big) . \end{aligned}$$

By definition of $l^{\delta,\varepsilon}$ and in particular because of the χ_ε-term, we have, for any $s \geq 0$, if ε is small enough

$$l\big(X^{\delta,\varepsilon}(s), t_0 - h\big) + \chi_\varepsilon\big(X^{\delta,\varepsilon}(s), t_0 - h\big) \geq \kappa_0 \varepsilon^{-3} ,$$

while $(1 - \mu^{\delta,\varepsilon})(u^\delta + \delta)(X^{\delta,\varepsilon}(s), t_0 - h\big)$ is bounded uniformly with respect to ε, s and x. Therefore, on E^c, we necessarily have $\mu^{\delta,\varepsilon}(s) \leq \kappa_1 \varepsilon^{3/2}$ for some $\kappa_1 > 0$.

Estimates on E – As we noticed in the proof of Theorem 5.3.1, $v^{\varepsilon,\delta} \leq u^\delta + \delta$. In

particular,

$$\begin{aligned}(u^\delta+\delta)(x,0) &\geq v^{\varepsilon,\delta}(x,0)\\ &\geq \int_0^{+\infty} l^{\delta,\varepsilon}\big(X^{\delta,\varepsilon}(s),T^{\delta,\varepsilon}(s)\big)\exp(-D^{\delta,\varepsilon}(s))ds\\ &\geq \int_E l^{\delta,\varepsilon}\big(X^{\delta,\varepsilon}(s),T^{\delta,\varepsilon}(s)\big)\exp(-D^{\delta,\varepsilon}(s))ds\\ &+\int_{E^c} l^{\delta,\varepsilon}\big(X^{\delta,\varepsilon}(s),T^{\delta,\varepsilon}(s)\big)\exp(-D^{\delta,\varepsilon}(s))ds\end{aligned}$$

By definition of E, the first integral is estimated by

$$\int_E l^{\delta,\varepsilon}\big(X^{\delta,\varepsilon}(s),T^{\delta,\varepsilon}(s)\big)\exp(-D^{\delta,\varepsilon}(s))ds \geq \int_E \varepsilon^{-3/2}\exp(-D^{\delta,\varepsilon}(s))ds\ ,$$

while, using the boundedness of l and $(u^\delta+\delta)$ there exists $C>0$ such that

$$\int_{E^c} l^{\delta,\varepsilon}\big(X^{\delta,\varepsilon}(s),T^{\delta,\varepsilon}(s)\big)\exp(-D^{\delta,\varepsilon}(s))ds \geq -C\int_{E^c}\exp(-D^{\delta,\varepsilon}(s))ds\ .$$

To get an estimate on the Lebesgue measure of E, we need an upper estimate of $\int_{E^c}\exp(-D^{\delta,\varepsilon}(s))ds$. Notice that on E^c, because of the estimate on $\mu^{\delta,\varepsilon}(s)$ we have

$$\dot D^{\delta,\varepsilon}(s)=c^{\delta,\varepsilon}\big(X^{\delta,\varepsilon}(s),T^{\delta,\varepsilon}(s)\big)=\mu^{\delta,\varepsilon}(s)c\big(X^{\delta,\varepsilon}(s),T^{\delta,\varepsilon}(s)\big)+\big(1-\mu^{\delta,\varepsilon}(s)\big)=1+O(\varepsilon^{3/2})\ ,$$

where the $|O(\varepsilon^{3/2})|\leq M\kappa_1\varepsilon^{3/2}$ is independent of x. Hence, since $\dot D^{\delta,\varepsilon}(s)\geq 0$ for any $s\geq 0$,

$$\begin{aligned}\int_{E^c}\exp(-D^{\delta,\varepsilon}(s))ds &= \int_{E^c}\frac{\dot D^{\delta,\varepsilon}(s)}{(1+O(\varepsilon^{3/2}))}\exp(-D^{\delta,\varepsilon}(s))ds\\ &\leq (1+O(\varepsilon^{3/2}))^{-1}\int_0^{+\infty}\dot D^{\delta,\varepsilon}(s)\exp(-D^{\delta,\varepsilon}(s))ds\\ &\leq (1+O(\varepsilon^{3/2}))^{-1}\ .\end{aligned}$$

Gathering all the above informations, we finally conclude that

$$\int_E \varepsilon^{-3/2}\exp(-D^{\delta,\varepsilon}(s))ds\leq\kappa_2\ ,$$

for some constant κ_2 which is independent of ε and x.

We introduce now a parameter $S > 0$ and denote by $E_S := E \cap [0, S]$. Since $0 \le \dot{D}^{\delta,\varepsilon}(s) \le M$ for any $s \ge 0$, we have

$$\exp(-MS)|E_S| \le \int_{E_S} \exp(-D^{\delta,\varepsilon}(s))ds \le \int_E \exp(-D^{\delta,\varepsilon}(s))ds \le \kappa_2 \varepsilon^{3/2} ,$$

where $|E_S|$ denotes the Lebesgue measure of E_S. We choose $S = S_\varepsilon$ such that $\exp(MS_\varepsilon) = \varepsilon^{-1/6}$ which yields

$$|E_{S_\varepsilon}| \le \kappa_2 \varepsilon^{3/2} \exp(MS_\varepsilon) = \kappa_2 \varepsilon^{4/3} .$$

We remark that S_ε behaves like $\ln(\varepsilon^{-1/6})$, uniformly in x. The reason why we choose S_ε in order to get a power $4/3 > 1$ in $|E_{S_\varepsilon}|$ will become clear in the lateral estimates. For Part A, any power in $(0, 3/2)$ is convenient.

Consequences on $v^{\varepsilon,\delta}$ – We first apply the Dynamic Programming Principle for $v^{\varepsilon,\delta}$ which gives

$$\begin{aligned} v^{\varepsilon,\delta}(x, t_0 - h) = &\int_0^{S_\varepsilon} l^{\delta,\varepsilon}\big(X^{\delta,\varepsilon}(s), t_0 - h\big) \exp(-D^{\delta,\varepsilon}(s))ds \\ &+ v^{\varepsilon,\delta}(X^{\delta,\varepsilon}(S_\varepsilon), t_0 - h) \exp(-D^{\delta,\varepsilon}(S_\varepsilon)) . \end{aligned}$$

Now we have to examine each term carefully. We first come back to the equation of $D^{\delta,\varepsilon}$: we have seen above that $|\dot{D}^{\delta,\varepsilon}(s) - 1| \le M\kappa_1 \varepsilon^{3/2}$ on E^c, while $|E_{S_\varepsilon}| \le \kappa_2 \varepsilon^{4/3}$. We deduce that, for $s \in [0, S_\varepsilon]$

$$|D^{\delta,\varepsilon}(s) - s| \le M(\kappa_1 \varepsilon^{3/2} S_\varepsilon + \kappa_2 \varepsilon^{4/3}) \le \kappa_3 \varepsilon^{4/3} \tag{5.7}$$

for some $\kappa_3 > 0$. In particular, since $S_\varepsilon \to +\infty$ as $\varepsilon \to 0$, $\exp(-D^{\delta,\varepsilon}(S_\varepsilon)) \to 0$ as $\varepsilon \to 0$ and

$$\liminf_{\varepsilon \to 0} \big(v^{\varepsilon,\delta}(X^{\delta,\varepsilon}(S), t_0 - h) \exp(-D^{\delta,\varepsilon}(S_\varepsilon))\big) \ge 0 ,$$

uniformly w.r.t. x since $v^{\varepsilon,\delta}$ is bounded from below.

On an other hand, for the $X^{\delta,\varepsilon}$-equation, we also have, on E^c (in fact only the b^x part is useful here)

$$b^{\delta,\varepsilon}(X^{\delta,\varepsilon}(s), t_0 - h) = \mu^{\delta,\varepsilon}(s) b(X^{\delta,\varepsilon}(s), t_0 - h) + (1 - \mu^{\delta,\varepsilon}(s))(0, 0) = O(\varepsilon^{3/2}) ,$$

more precisely the bound takes the form $M\kappa_2 \varepsilon^{3/2}$. Using the decomposition with E_{S_ε} and its complementary $E^c_{S_\varepsilon} = E^c \cap [0, S_\varepsilon]$ as in (5.7), it follows that

$$\begin{aligned} \int_0^{S_\varepsilon} |b^{\delta,\varepsilon}(\tau)| d\tau &= \int_0^{S_\varepsilon} |b^{\delta,\varepsilon}(\tau)| \mathbb{1}_{\{E_{S_\varepsilon}\}}(s) d\tau + \int_0^{S_\varepsilon} |b^{\delta,\varepsilon}(\tau)| \mathbb{1}_{\{E^c_{S_\varepsilon}\}}(s)\, d\tau \\ &\le M(\kappa_2 \varepsilon^{4/3} + \kappa_1 \varepsilon^{3/2} S_\varepsilon) \le \kappa_3 \varepsilon^{4/3} . \end{aligned}$$

We deduce that if $s \in [0, S_\varepsilon]$, $X^{\delta,\varepsilon}(s) - x = O(\varepsilon^{4/3})$ and since u^δ is continuous,

$$(u^\delta + \delta)(X^{\delta,\varepsilon}(s), t_0 - h) = (u^\delta + \delta)(x, t_0 - h) + o_\varepsilon(1) \geq (u^\delta + \delta/2)(x, t_0 - h)\ .$$

For a similar reason, on $E^c_{S_\varepsilon}$ we can absorb the $o_\varepsilon(1)$-term by a $\delta/2$ for ε small enough

$$l^{\delta,\varepsilon}\big(X^{\delta,\varepsilon}(s), t_0 - h\big) \geq (u^\delta + \delta/2)\big(x, t_0 - h\big)\ .$$

Gathering all these informations, using (5.7) and that $(l + \psi_\varepsilon) \geq 0$ on E_{S_ε} we get

$$\begin{aligned} I_\varepsilon &:= \int_0^{S_\varepsilon} l^{\delta,\varepsilon}\big(X^{\delta,\varepsilon}(s), t_0 - h\big) \exp(-D^{\delta,\varepsilon}(s))ds \\ &\geq \int_{E^c_{S_\varepsilon}} \Big((u^\delta + \delta/2)\big(x, t_0 - h\big)\Big) \exp\big(-s + O(\varepsilon^{4/3})\big)ds\,. \end{aligned}$$

Then, since S_ε behaves like $\ln(\varepsilon^{-1/6})$ and $|E_{S_\varepsilon}| \leq \kappa_2 \varepsilon^{4/3}$, we get

$$\begin{aligned} I_\varepsilon &\geq (u^\delta + \delta/2)\big(x, t_0 - h\big) \int_{E^c_{S_\varepsilon}} \exp(-s)ds + o_\varepsilon(1) \\ &\geq (u^\delta + \delta/2)\big(x, t_0 - h\big) + o_\varepsilon(1)\ . \end{aligned}$$

Hence $v^{\varepsilon,\delta}(x, t_0 - h) \geq (u^\delta + \delta/2)\big(x, t_0 - h\big) + o_\varepsilon(1)$ where the "$o_\varepsilon(1)$" is independent of x and for ε small enough, we have $v^{\varepsilon,\delta}(x, t_0 - h) \geq u^\delta\big(x, t_0 - h\big)$ on $\overline{B(x_0, r)}$.

Part B. Lateral estimates – Essentially, the proof is the same as for the initial estimates: the only difference is that the trajectory may exit the region where χ_ε is large. But, if $d((x,t), \mathcal{M}) \leq \varepsilon$ or if $d(x, \partial B(x_0, r)) \leq \varepsilon$, the running cost satisfies again the estimate $l\big(X^{\delta,\varepsilon}(s), T^{\delta,\varepsilon}(s)\big) + \chi_\varepsilon(X^{\delta,\varepsilon}(s), T^{\delta,\varepsilon}(s)) \geq \kappa_0 \varepsilon^{-3} \geq 0$.

We consider the case when $(x,t) \in \mathcal{M}$, the proof being the same if $(x,t) \in \partial B(x_0, r)$. Since the dynamic b is bounded by M, a trajectory (X, T) starting at (x,t) satisfies $d((X(s), T(s)), \mathcal{M}) \leq Ms$ and therefore, it stays in an ε-neighborhood of $\mathcal{M}$ for $s < \varepsilon/M$.

For an optimal trajectory, we repeat the same proof as in Part A, but on $E \cap [0, \tau_\varepsilon \wedge S_\varepsilon]$, where τ_ε is the first time for which $d((X^{\delta,\varepsilon}(s), T^{\delta,\varepsilon}(s)), \mathcal{M}) = \varepsilon$ and $a \wedge b = \min(a, b)$.

If we set as above

$$E := \big\{s \in [0, \infty) : l^{\delta,\varepsilon}\big(X^{\delta,\varepsilon}(s), T^{\delta,\varepsilon}(s)\big) \geq \varepsilon^{-3/2}\big\}\ ,$$

then the Lebesgue measure of $E \cap [0, \tau_\varepsilon \wedge S_\varepsilon]$ is less than $\kappa_3 \varepsilon^{4/3}$ for some $\kappa_3 > 0$, while on $E^c \cap [0, \tau_\varepsilon \wedge S_\varepsilon]$ we have $\mu^{\delta,\varepsilon}(s) \leq \kappa_4 \varepsilon^{3/2}$ for some $\kappa_4 > 0$. As in Part A, using the decomposition on $E \cap [0, \tau_\varepsilon \wedge S_\varepsilon]$ and its complementary we deduce that

$$\int_0^{\tau_\varepsilon \wedge S_\varepsilon} |b^{\delta,\varepsilon}(s)|\, ds \leq M\big\{\kappa_3 \varepsilon^{4/3} + \kappa_4 \varepsilon^{3/2}(\tau_\varepsilon \wedge S_\varepsilon)\big\}\,,$$

while by definition the distance between (x,t) and $(X^{\delta,\varepsilon}(\tau_\varepsilon), T^{\delta,\varepsilon}(\tau_\varepsilon))$ is larger than ε (if τ_ε is finite, of course).

We claim that for ε small enough, $\tau_\varepsilon \wedge S_\varepsilon = S_\varepsilon$. Indeed, assume on the contrary that for some subsequence $\varepsilon_n \to 0$, $\tau_{\varepsilon_n} < S_{\varepsilon_n}$. From the previous estimate it follows that

$$\varepsilon_n \leq |(X^{\delta,\varepsilon}(\tau_\varepsilon), T^{\delta,\varepsilon}(\tau_\varepsilon)) - (x,t)| \leq M\big\{\kappa_3 \varepsilon_n^{4/3} + \kappa_4 \varepsilon_n^{3/2} \tau_{\varepsilon_n}\big\}\,.$$

The fact that the power in the first term is greater than 1 implies that τ_{ε_n} goes to infinity, at least like $\varepsilon_n^{-1/2}$. But since by construction S_{ε_n} behaves like $\ln(\varepsilon_n^{-1/6})$, we reach a contradiction.

We deduce that necessarily $\tau_\varepsilon > S_\varepsilon$ as $\varepsilon \to 0$, and that on $[0, S_\varepsilon]$, the trajectory remains "trapped" in an ε-neighborhood of $\mathcal{M}$. We end the proof exactly as in Part A, sending $\varepsilon \to 0$.

The proof if $x \in \partial B(x_0, r)$ being the same, in conclusion we have shown that $v^{\delta,\varepsilon} \geq u^\delta$ on $\partial\mathcal{O}$ for ε small enough.

Q.E.D.

In the case when b^t is not allowed to vanish, obtaining the sub-dynamic principle is a bit easier since we do not need to consider an obstacle-type problem like (5.4).

Theorem 5.3.3 *Let $h, r > 0$ be such that $Mh < r$ and assume that, for any $(x,t) \in \overline{Q_{r,h}^{x_0,t_0}}$ and any $(b,c,l) \in \mathbf{BCL}(x,t)$, $b^t = -1$. If u is a subsolution of (5.3), if $\mathcal{T}_{\mathcal{O}}^h(x_0,t_0) \neq \emptyset$ and if a* **(LCR)** *holds in $\mathcal{O}$ for the equation $\mathbb{F} = 0$, then for any $\eta \leq h$*

$$u(x_0,t_0) \leq \inf_{X \in \mathcal{T}_{\mathcal{O}}^h(x_0,t_0)} \Big\{ \int_0^\eta l\big(X(s),T(s)\big) \exp(-D(s))\, \mathrm{d}s + u\big(X(\eta),T(\eta)\big) \exp(-D(\eta)) \Big\}\,. \tag{5.8}$$

Proof — The difference between the two cases comes from the fact that, under the assumption of Theorem 5.3.3, we could have $T(h) > t_0 - h$ in (5.6) (Step 1) for a trajectory starting from (x_0,t_0) since b^t was allowed to be different from -1: this is why the strategy of the proof of this theorem uses $\eta < h$ and, for handling this

situation, we need to have $v^{\varepsilon,\delta}(x,t) \le u^\delta(x,t) + \delta$ in the whole domain to conclude after using the Dynamic Programming Principle for $v^{\varepsilon,\delta}$ (cf. Step 3).

Here on the contrary we are sure that $T(h) = t_0 - h$ for any such trajectory and we are going the Dynamic Programming Principle for $v^{\varepsilon,\delta}$ up to time $t_0 - h$, i.e. with $s = h$.

For this reason, we are going to prove (5.8) for $\eta = h$, the inequality for $\eta < h$ being obtained by applying the result with h replaced by η.

For all these reasons the proof is similar to that of Theorem 5.3.1 but there are substantial simplifications.

(a) We enlarge **BCL** in the same way BUT ONLY at time $t = t_0 - h$. The consequence is that $v^{\varepsilon,\delta}$ is a supersolution for the HJB-equation $\mathbb{F} = 0$ and not of (5.4), since we have no enlargement for $t \in (t_0 - h, t_0)$. Hence we just have to deal with the comparison results for the $\mathbb{F}$-equation, we do not need to assume some obstacle-type comparison property.

(b) The penalization function we use here does not require a specific penalization for the initial time and we just write it as

$$\chi_\varepsilon(x,t) := \frac{1}{\varepsilon^4}\left[\big(2\varepsilon - d((x,t),\mathcal{M})\big)^+ + (2\varepsilon - (r - |x - x_0|))^+\right].$$

The initial inequality $v^{\varepsilon,\delta}(x, t_0 - h) \ge (u^\delta + \delta)(x, t_0 - h)$ for any $x \in B(x_0, r)$ follows from the following argument: since $b^t = -1$ in **BCL**, the only possibility for a constrained trajectory $(X^{\delta,\varepsilon}, T^{\delta,\varepsilon}, D^{\delta,\varepsilon}, L^{\delta,\varepsilon}) \in \mathcal{T}^{\delta,\varepsilon}(x, t_0 - h)$ to remain in $\mathbb{R}^N \times [t_0 - h, t_0]$ is to solve the differential inclusion by using the elements $((0,0), 1, (u^\delta + \delta)(x, t_0 - h))$ of $\mathbf{BCL}^{\delta,\varepsilon}$. This implies directly that $v^{\varepsilon,\delta}(x, t_0 - h) \ge (u^\delta + \delta)(x, t_0 - h)$.

(c) With these simplications, the proof remains the same as in the general case $b^t \in [-1, 0]$: we first get that $v^{\varepsilon,\delta} \ge u$ on $t = t_0 - h$, for $x \in \partial B(x_0, r)$ and for $(x,t) \in \mathcal{M}$. Using that we have a **(LCR)** in $\mathcal{O}$ implies that $v^{\varepsilon,\delta} \ge u$ on $\overline{\mathcal{O}}$. Then we proceed as above using the dynamic programming principle for $v^{\varepsilon,\delta}$. For $\eta \le h$[(1)], taking $\varepsilon > 0$ small enough allows to restrict this dynamic principle to the trajectories in $\mathcal{T}^h_{\mathcal{O}}(x_0, t_0)$, which avoid $\mathcal{M}$. Sending $\varepsilon \to 0$ and $\delta \to 0$ is done "trajectory by trajectory".

Q.E.D.

[(1)] Here we do not have to treat separately the cases when $\eta < h$ and $\eta = h$ since we have dropped the penalization term in a neighborhood of $t = t_0 - h$ and we know that $v^{\varepsilon,\delta}(x, t_0 - h) \ge (u^\delta + \delta)(x, t_0 - h)$.

5.4 Local comparison for discontinuous HJB equations

The aim of this section is to provide an argument which is a keystone in several comparison results we give for HJB equations with discontinuities, and in particular for stratified problems.

To do so, we consider a C^1-manifold $\mathcal{M} \subset \mathbb{R}^N \times (0,T)$ (which will be in the sequel a set of discontinuity for the HJB equation) and for any $(x,t) \in \mathcal{M}$, we denote by $T_{(x,t)}\mathcal{M}$, the tangent space of $\mathcal{M}$ at (x,t). Then we define the *tangential Hamiltonian* associated with $\mathcal{M}$ by setting

$$\mathbb{F}^{\mathcal{M}}(x,t,u,p) := \sup_{(b,c,l)\in \mathbf{BCL}_T(x,t)} \left\{ -b\cdot p + cu - l \right\}, \tag{5.9}$$

where $\mathbf{BCL}_T(x,t) := \left\{(b,c,l) \in \mathbf{BCL}(x,t) : b \in T_{(x,t)}\mathcal{M}\right\}$. This tangential Hamiltonian is defined for any $(x,t) \in \mathcal{M} \times [0,T]$, $u \in \mathbb{R}$ and $p \in T_{(x,t)}\mathcal{M}$. But by a slight abuse of notation, we also write $\mathbb{F}^{\mathcal{M}}(x,t,u,p)$ when $p \in \mathbb{R}^{N+1}$, meaning that only the projection of p onto $T_{(x,t)}\mathcal{M}$ is used for the computation. We also recall that $Du = (D_x u, u_t)$.

Our main argument comes from the

Lemma 5.4.1 (The "Magical Lemma")
We assume that $(\mathbf{H_{BCL}})$ *holds, that* $v : \overline{Q^{x,t}_{r,h}} \to \mathbb{R}$ *is a l.s.c. supersolution of* $\mathbb{F}(x,t,v,Dv) = 0$ *in* $Q^{x,t}_{r,h}$ *where* $(x,t) \in \mathcal{M}$ *and* $0 < t-h < t \le T$, *and that* $u : \overline{Q^{x,t}_{r,h}} \to \mathbb{R}$ *has the following properties*

(i) $u \in C^0(\overline{Q^{x,t}_{r,h}}) \cap C^1(\mathcal{M})$,

(ii) $\mathbb{F}^{\mathcal{M}}(y,s,u,Du) < 0$ *on* $\mathcal{M}$,

(iii) *u satisfies a "strict" subdynamic principle in* $Q^{x,t}_{r,h}[\mathcal{M}^c] = (B(x,r)\times(t-h,t])\backslash\mathcal{M}$, *i.e. there exists* $\eta > 0$, *such that, for any* $(\bar{x},\bar{t}) \in Q^{x,t}_{r,h}[\mathcal{M}^c]$, *for any solution* (X,T,D,L) *of the differential inclusion such that* $X(0) = \bar{x}$, $T(0) = \bar{t}$ *and* $(X(s),T(s)) \in Q^{x,t}_{r,h}[\mathcal{M}^c]$ *for* $0 < s \le \bar{\tau}$, *we have, for any* $0 < \tau \le \bar{\tau}$

$$u(y,s) \le \int_0^\tau (l(X(s),T(s)) - \eta)\exp(-D(s))\,\mathrm{d}s + u(X(\tau),T(\tau))\exp(-D(\tau)). \tag{5.10}$$

If $\max_{\overline{Q^{x,t}_{r,h}}}(u-v) > 0$, *then, for any* $(y,s) \in \overline{Q^{x,t}_{r,h}} \setminus \partial_p Q^{x,t}_{r,h}$,

$$(u-v)(y,s) < m := \max_{\partial_p Q^{x,t}_{r,h}} (u-v)\,.$$

Proof — Using $(\mathbf{H_{BCL}})_{struct}$, we can assume without loss of generality that $c \geq 0$ for all $(b,c,l) \in \mathbf{BCL}(y,s)$ and for all $(y,s) \in \overline{Q^{x,t}_{r,h}}$.

We assume by contradiction that $(u-v)$ reaches its maximum on $\overline{Q^{x,t}_{r,h}}$ at a point $(\bar{x},\bar{t}) \in Q^{x,t}_{r,h}$. If $(\bar{x},\bar{t}) \in Q^{x,t}_{r,h} \setminus \mathcal{M}$, we easily reach a contradiction: by Lemma 4.4.2, v satisfies (4.15) and for sufficiently small τ, all the trajectories (X,T,D,L) are such that $(X(s),T(s)) \in Q^{x,t}_{r,h}[\mathcal{M}^c]$. We consider an optimal trajectory for v at $(\bar{x},\bar{t})$, (X,T,D,L) and we gather the information given by (4.15) and (5.10) for some time τ small enough: substracting these inequalities, we get

$$u(\bar{x},\bar{t}) - v(\bar{x},\bar{t}) \leq -\eta\tau + (u(X(\tau),T(\tau)) - v(X(\tau),T(\tau)))\exp(-D(\tau))\,. \tag{5.11}$$

But $(\bar{x},\bar{t})$ is a maximum point of $u-v$ in $\overline{Q^{x,t}_{r,h}}$ and therefore we have at the same time $u(\bar{x},\bar{t}) - v(\bar{x},\bar{t}) > 0$ and $u(\bar{x},\bar{t}) - v(\bar{x},\bar{t}) \geq u(X(\tau),T(\tau)) - v(X(\tau),T(\tau))$; hence, since $\exp(-D(\tau)) \geq 0$

$$u(\bar{x},\bar{t}) - v(\bar{x},\bar{t}) \leq -\eta\tau + (u(\bar{x},\bar{t}) - v(\bar{x},\bar{t}))\exp(-D(\tau))\,,$$

which is a contradiction since $\exp(-D(\tau)) \leq 1$.

If $(u-v)$ reaches its maximum on $\overline{Q^{x,t}_{r,h}}$ at a point $(\bar{x},\bar{t}) \in Q^{x,t}_{r,h} \cap \mathcal{M}$, we face two cases

A. – In (4.15) for $(\bar{x},\bar{t})$, there exists a trajectory (X,T,D,L) and $\tau > 0$ such that $X(0) = \bar{x}$, $T(0) = \bar{t}$ and

$$v(\bar{x},\bar{t}) \geq \int_0^\tau l\big(X(s),T(s)\big)\exp(-D(s))\,\mathrm{d}s + v\big(X(\tau),T(\tau)\big)\exp(-D(\tau))\,, \tag{5.12}$$

AND $(X(s),T(s)) \in Q^{x,t}_{r,h} \setminus \mathcal{M}$ for $s \in (0,\tau]$. In this case we argue essentially as above: we use as a starting point $(x_\varepsilon,t_\varepsilon) := (X(\varepsilon),T(\varepsilon)) \in Q^{x,t}_{r,h}[\mathcal{M}^c]$ for $0 < \varepsilon \ll 1$ and we use (5.10) for the specific trajectory (X,T,D,L) but on the time interval $[\varepsilon,\tau]$

$$u(x_\varepsilon,t_\varepsilon) \leq \int_\varepsilon^\tau (l(X(s),T(s)) - \eta)\exp(-D(s))\,\mathrm{d}s + u(X(\tau),T(\tau))\exp(-D(\tau))\,.$$

But in this inequality, we can send ε to 0, using the continuity of u and finally get, combining it with the above inequality for v to obtain (5.11) and a contradiction.

B. – If Case A cannot hold, this means that, for any τ and for any trajectory (X, T, D, L) such that (5.12) holds, then there exists a sequence $t_n \searrow 0$ such that $X(t_n) \in \mathcal{M}$ for any $n \in \mathbb{N}$. We first use the dynamic programming inequality for v between $s = 0$ and $s = t_n$, which yields

$$v(\bar{x}, \bar{t}) \geq \int_0^{t_n} l(X(s), T(s)) \exp(-D(s))\, \mathrm{d}s + v(X(t_n), T(t_n)) \exp(-D(t_n))\,.$$

Since $u - v$ reaches a maximum at $(\bar{x}, \bar{t})$ and since this maximum is positive, we can replace v by u in this inequality which leads to

$$\frac{u(\bar{x}, \bar{t}) - u(X(t_n), T(t_n)) \exp(-D(t_n))}{t_n} \geq \frac{1}{t_n} \int_0^{t_n} l(X(s), T(s)) \exp(-D(s))\, \mathrm{d}s\,.$$

Now, since u is C^1-smooth on $\mathcal{M} \times (t - h, t)$, we have (recall that $Du = (D_x u, u_t)$ and that here we use only derivatives which are in the tangent space of $\mathcal{M}$)

$$\begin{aligned} u(X(t_n), T(t_n)) =& u(\bar{x}, \bar{t}) + Du(\bar{x}, \bar{t})(X(t_n) - \bar{x}, T(t_n) - \bar{t}) + o(|X(t_n) - \bar{x}| + |T(t_n) - \bar{t}|) \\ =& u(\bar{x}, \bar{t}) + Du(\bar{x}, \bar{t})(X(t_n) - \bar{x}, T(t_n) - \bar{t}) + o(t_n)\ , \end{aligned}$$

and writing

$$(X(t_n) - \bar{x}, T(t_n) - \bar{t}) = \int_0^{t_n} b(s) ds\ ,\ \exp(-D(t_n)) = \int_0^{t_n} -c(s) \exp(-D(s)) ds$$

we obtain

$$\frac{1}{t_n} \int_0^{t_n} \{-b(s) \cdot Du(\bar{x}, \bar{t}) + c(s) u(\bar{x}, \bar{t}) - l(X(s), T(s))\} \exp(-D(s)) ds \geq 0\ .$$

And since $\exp(-D(s)) = 1 + O(t_n)$, we can write this inequality as

$$-b_n \cdot Du(\bar{x}, \bar{t}) + c_n u(\bar{x}, \bar{t}) - l_n \geq o_n(1)\ ,$$

where

$$b_n = \left(\frac{1}{t_n} \int_0^{t_n} b(s) ds\right)\ ,\ c_n = \left(\frac{1}{t_n} \int_0^{t_n} c(s) ds\right)\ ,\ l_n \left(\frac{1}{t_n} \int_0^{t_n} l(X(s), T(s)) ds\right).$$

But the b_n, c_n, l_n are uniformly bounded and therefore we can assume that $b_n \to \bar{b}, c_n \to \bar{c}, l_n \to \bar{l}$. Using the convexity and upper semi-continuity of **BCL**, we have $(\bar{b}, \bar{c}, \bar{l}) \in \mathbf{BCL}(\bar{x}, \bar{t})$ and by the definition of b_n, we also have $\bar{b} \in T_{(\bar{x}, \bar{t})}\mathcal{M}$. Finally, passing to the limit in the above inequality yields

$$-\bar{b} \cdot Du(\bar{x}, \bar{t}) + \bar{c} u(\bar{x}, \bar{t}) - \bar{l} \geq 0\ .$$

But, thanks to the definition of $\mathbb{F}^{\mathcal{M}}$ and the properties of u, we have the inequalities

$$0 \leq -\bar{b} \cdot Du(\bar{x},\bar{t}) + \bar{c}u(\bar{x},\bar{t}) - \bar{l} \leq \mathbb{F}^{\mathcal{M}}(\bar{x},\bar{t},u(\bar{x},\bar{t}),Du(\bar{x},\bar{t})) < 0 \ ,$$

which is the desired contradiction.

Q.E.D.

Remark 5.4.2 *There are possible variants for this lemma. In particular, in Part II, we use one of them where the sub and supersolution properties for u and v are defined in a slightly different way, namely with taking a more restrictive set of control on $\mathcal{M}$. Of course, in that case, $\mathbb{F}^{\mathcal{M}}$ is replaced by an Hamiltonian which defined in a different way. The proof is still valid if the Dynamic Programming argument of **B.** leads to the right inequality.*

5.5 The "good framework for HJ-equations with discontinuities"

The study of Hamilton-Jacobi with discontinuities or the associated control problems in the convex case leads to various situations, many of which we consider in Parts II, III, IV or V. These situations may appear to be quite different, but still we can identify some common structure on the equations and the discontinuities of the Hamiltonians which seems quite "natural" to get most of the results. Of course, what we are going to describe as the "good framework for HJ-Equations with discontinuities" does not perfectly fit all situations and some adaptations have to be made in each case. But the definition below provides a good idea of the key assumptions which are required to treat those problems.

5.5.1 General definition at the pde level

Definition 5.5.1 *We say that we are in the "good framework for HJ-Equations with discontinuities" for the equation*

$$\mathbb{G}(X,u,Du) = 0 \quad \text{in } \mathcal{O} \subset \mathbb{R}^N \tag{5.13}$$

if (**LOC1**), (**LOC2**) *hold and if there exists a stratification $\mathbb{M} = (\mathbf{M}^k)_{k=0..N}$ of $\mathbb{R}^N$ such that, for any $k = 0,..,N$*

(*i*) *if* $\bar{X} \in \mathbf{M}^k \cap \mathcal{O}$, *there is a ball* $B(\bar{X}, r) \subset \mathcal{O}$ *for some* $r > 0$ *and a* $C^{1,1}$*-diffeomorphism* $\Psi : B(\bar{X}, r) \to \mathbb{R}^N$ *such that* $\Psi(\bar{X}) = \bar{X}$, *the* $\Psi(\mathbf{M}^l \cap B(x, r))$ *for* $l = 0..N$ *form a* $(\mathbf{H_{ST}})_{flat}$ *and*

$$\Psi(B(\bar{X}, r) \cap \mathbf{M}^k) = \left(\bar{X} + \mathbb{R}^k \times \{0_{\mathbb{R}^{N-k}}\}\right) \cap \Psi(B(\bar{X}, r))\,.$$

(*ii*) *Denoting* $\Psi(X) = \bar{X} + (Y, Z)$ *with* $Y \in \mathbb{R}^k$, $Z \in \mathbb{R}^{N-k}$ *and*

$$\tilde{\mathbb{G}}((Y, Z), r, (p_Y, p_Z)) = \mathbb{G}(\Psi^{-1}\left(\bar{X} + (Y, Z)\right), r, [(\Psi^{-1})']^T\left(\bar{X} + (Y, Z)\right)(p_Y, p_Z))\,,$$

where $[(\Psi^{-1})']^T$ *denotes the transpose matrix of* $(\Psi^{-1})'$, *then* **(TC)**, **(NC)**,**(Mon)** *hold for* $\tilde{\mathbb{G}}$ *on* $\Psi(B(\bar{X}, r) \cap \mathbf{M}^k)$.

In this case, we will say that $\mathbb{M}$ *is associated to Equation* (5.13).

As we already mentioned it in Section 3.4, the difficulty when stating such definition is that it is supposed to cover very different situations for which the sense of $\mathbb{G} = 0$ may vary and may also involve several Hamiltonians. In these various situations, we use the following convention

(TC), **(Mon)** *have to be satisfied* - up to some change of variables - *by ANY Hamiltonians which are involved in the sub and supersolutions inequalities while* **(NC)** *has to be satisfied by the Hamiltonians which are involved in the subsolutions inequalities related to local maximum points in* $\mathcal{O}$*—or* $\Psi(B(\bar{X}, r))$*—but not by the Hamiltonians related to local maximum points on the* $\mathbf{M}^k$ *for* $k < N$.

But before coming back to this point, let us explain the key ideas beyond this "good framework for HJ-Equations with discontinuities".

The very first idea is that the discontinuities of $\mathbb{G}$ form a regular stratification. Since we always argue locally (using (**LOC1**), (**LOC2**), for comparison results), we can use Definition 3.3.5—with perhaps a smaller r—to reduce to the case when the discontinuities in $\mathbb{G}$ form an AFS (here $\mathbf{M}^k$ is replaced by $\bar{X} + \mathbb{R}^k \times \{0_{\mathbb{R}^{N-k}}\}$). This is the first important reduction. We immediately point out that, in Definition 5.5.1, the diffeomorphism Ψ is assumed to be $C^{1,1}$ which is needed in general to get **(TC)** but, in coercive cases, i.e. when G is coercive in p, C^1-diffeomorphisms may be enough.

Once this change is done, we are in the framework of Section 3.4 and using a combination **(TC)**, **(NC)**,**(Mon)** allows us to regularize subsolutions in order to be able to apply Lemma 5.4.1. The triptych "*Tangential continuity + normal controllability + some suitable monotonicity*" seems to us the basis of most of our results, and not only the comparison ones.

The two extreme cases have also to be commented: if $k = N$, then there is no normal directions, **(TC)** has to be satisfied by all coordinates, $\mathbb{G}$ is continuous in a neighborhood of $\bar{X}$, no change Ψ is really needed and, through **(TC)**, we just recover the classical assumption for the uniqueness of viscosity solutions for a standard HJ-Equations without discontinuity. If $k = 0$, $\bar{X}$ is an isolated point, we have no "tangent coordinates" and **(TC)** is void but **(NC)** implies that $\mathbb{G}$ is coercive in p in a neighborhood of $\bar{X}$.

5.5.2 The stratified case and the associated "good assumptions" on the BCL

Now let us come back on the sense of the equation $\mathbb{G} = 0$ and the way the above convention has to be applied. Anticipating Part IV on the full stratified case, we have an HJ-equation of the type

$$\mathbb{F}(x, t, U, DU) = 0 \quad \text{in} \quad \mathbb{R}^N \times [0, T_f]\,,$$

where $DU = (D_x U, D_t U)$ and

$$\mathbb{F}(x, t, r, p) := \sup_{(b,c,l)\in \mathbf{BCL}(x,t)} \big\{ -b \cdot p + cr - l \big\}\,.$$

Assuming that $(\mathbf{H_{BCL}})$ holds, what does it mean to be in the "good framework for HJ-Equations with discontinuities" here?

In the case of stratified problems, roughly speaking, the sense of the equation is $F^* \geq 0$ in $\mathbb{R}^N \times (0, T_f]$ for supersolutions and, for subsolutions, $F_* \leq 0$ in $\mathbb{R}^N \times (0, T_f]$ with the additional conditions $\mathbb{F}^k \leq 0$ on $\mathbf{M}^k$ where the "tangential Hamiltonians" $\mathbb{F}^k$ for $k = 0..N$ are defined for $(x, t) \in \mathbf{M}^k$, $r \in \mathbb{R}$ and $p \in T_{(x,t)}\mathbf{M}^k$, by

$$\mathbb{F}^k(x, t, u, p) := \sup_{\substack{(b,c,l)\in \mathbf{BCL}(x,t) \\ b\in T_{(x,t)}\mathbf{M}^k}} \big\{ -b \cdot p + cu - l \big\}\,.$$

For $t = 0$, we have analogous properties but for $\mathbb{F}_{init}$. We refer to Chapter 20 for more precise definitions.

Now we examine the needed assumptions on the **BCL** in $\mathbb{R}^N \times (0, T]$ in order to have **(TC)** and **(NC)**: we are going to do it precisely for **(TC)** and **(NC)** since, for **(Mon)**, this is a more standard consequence of $(\mathbf{H_{BCL}})$ and we come back on that point in Chapter 20, more specifically in Section 20.4. On the other hand, for $t = 0$, such checking is analogous using $\mathbb{F}_{init}$ and the associated Hamiltonians on $\mathbf{M}_0^k$.

Since these assumptions are local and invariant by the Ψ-changes, we can state them in a ball $B((x,t),r)$ centered at $(x,t) \in \mathbf{M}^k$ with a small radius $r > 0$ and we can assume that, in $B((x,t),r)$, $\mathbb{M}$ is an (AFS) with $\mathbf{M}^k = (x,t) + V_k$, where V_k is a k-dimensional vector space in $\mathbb{R}^{N+1}$ and $B((x,t),r)$ intersects only $\mathbf{M}^k, \mathbf{M}^{k+1}, \cdots, \mathbf{M}^{N+1}$. We denote by $V_k^\perp$ the orthogonal space to V_k and by $P^\perp$ the orthogonal projector on $V_k^\perp$. We trust the reader to be able to translate them for the original stratification and **BCL**.

In this framework, **(TC)** & **(NC)** are satisfied if, with the above notations

(TC-BCL) *For any $0 \le k \le N+1$ and for any $(x,t) \in \mathbf{M}^k$, there exists a constant $C_1 > 0$ and a modulus $m : [0,+\infty) \to \mathbb{R}^+$ such that, for any $j \ge k$, if $(y_1,t_1),(y_2,t_2) \in \mathbf{M}^j \cap B((x,t),r)$ with $(y_1,t_1) - (y_2,t_2) \in V_k$, then for any $(b_1,c_1,l_1) \in \mathbf{BCL}(y_1,t_1)$, there exists $(b_2,c_2,l_2) \in \mathbf{BCL}(y_2,t_2)$ such that*

$$|b_1 - b_2| \le C_1(|y_1 - y_2| + |t_1 - t_2|) \quad , \quad |c_1 - c_2| + |l_1 - l_2| \le m\big(|y_1 - y_2| + |t_1 - t_2|\big) \, .$$

(NC-BCL) For any $0 \le k \le N+1$ and for any $(x,t) \in \mathbf{M}^k$, there exists $\delta = \delta(x,t) > 0$, such that, for any $(y,s) \in B((x,t),r)$, one has

$$B(0,\delta) \cap V_k^\perp \subset P^\perp\left(\mathbf{B}(y,s)\right) \, .$$

Of course, the case $k = 0$ is particular since $V_k = \{0\}$: here we impose a complete controllability of the system in a neighborhood of $x \in \mathbf{M}^0$ since the condition reduces to $B(0,\delta) \subset \mathbf{B}(y,t)$ because $V_k^\perp = \mathbb{R}^{N+1}$.

As we will see it throughout this book, the normal controllability assumption plays a key role in all our analysis: first, at the control level, to obtain the viscosity subsolution inequalities for the value function on each $\mathbf{M}^k$, then in the comparison proof to allow the regularization (in a suitable sense) of the subsolutions and, last but not least, for the stability result.

It is rather easy to prove that **(NC-BCL)** implies **(NC)** in this (AFS) framework. We therefore concentrate on **(TC-BCL)** and the following result first gives an important consequence of these assumptions: the continuity of all the Hamiltonians $\{\mathbb{F}^k\}_{k=0..N}$, whose proof uses a combination of **(TC-BCL)** and **(NC-BCL)**. We point out that, on the contrary, it is easy to prove that $\mathbb{F}^{N+1}$ satisfies **(TC)** in $\mathbf{M}^{N+1}$.

With the same notations as above we set, for $(y,s) \in B((x,t),r) \cap \mathbf{M}^k$

$$\mathbf{BCL}^k(y,s) := \{(b,c,l) \in \mathbf{BCL}(y,s);\ b \in T_{(y,s)}\mathbf{M}^k = V_k\} \, ,$$

and $\mathbf{B}^k(y,s)$ is the set of all b such that there exists c,l for which $(b,c,l)\in\mathbf{BCL}^k(y,s)$.

We have the

Lemma 5.5.2 *If* **(TC-BCL)** *and* **(NC-BCL)** *hold, then*

(*i*) $\mathbf{BCL}^k(y,s)\neq\emptyset$ *for any* $(y,s)\in B((x,t),r)\cap\mathbf{M}^k$.

(*ii*) *There exists* $\bar{C}_1>0$ *and a modulus* $\bar{m}$ *such that, if* $(y_1,t_1),(y_2,t_2)\in B((x,t),r)\cap\mathbf{M}^k$ *and if* $(b_1,c_1,l_1)\in\mathbf{BCL}^k(y_1,t_1)$, *there exists* $(b_2,c_2,l_2)\in\mathbf{BCL}^k(y_2,t_2)$ *such that*

$$|b_1-b_2|\leq\bar{C}_1(|y_1-y_2|+|t_1-t_2|)\ ,\ |c_1-c_2|+|l_1-l_2|\leq\bar{m}\big(|y_1-y_2|+|t_1-t_2|\big)\ .$$

In particular, the Hamiltonian $\mathbb{F}^k$ *satisfies* **(TC)** *on* $\mathbb{M}^k$, *i.e. for any* $R>0$, *for any* $(y_1,t_1),(y_2,t_2)\in B((x,t),r)\cap\mathbf{M}^k$, $|r|\leq R$, $p\in V_k$ *(or* $p\in\mathbb{R}^{N+1}$*)*

$$\begin{aligned}|\mathbb{F}^k(y_1,t_1,r,p)-\mathbb{F}^k(y_2,t_2,r,p)|\leq&\bar{C}_1(|y_1-y_2|+|t_1-t_2|)|p|\\&+(R+1)\bar{m}\big(|y_1-y_2|+|t_1-t_2|\big)\ .\end{aligned}$$

(*iii*) *For any* $j\geq k$, *there exists* $\tilde{C}_1>0$ *and a modulus* $\tilde{m}$ *such that, if* $(y_1,t_1),(y_2,t_2)\in\mathbf{M}^j\cap B(x,r)$ *with* $(y_1,t_1)-(y_2,t_2)\in V_k$, *if* $(b_1,c_1,l_1)\in\mathbf{BCL}^j(y_1,t_1)$, *there exists* $(b_2,c_2,l_2)\in\mathbf{BCL}^j(y_2,t_2)$ *such that*

$$|b_1-b_2|\leq\tilde{C}_1(|y_1-y_2|+|t_1-t_2|)\ ,\ |c_1-c_2|+|l_1-l_2|\leq\tilde{m}\big(|y_1-y_2|+|t_1-t_2|\big)\ .$$

In particular, the Hamiltonian $\mathbb{F}^j$ *satisfies* **(TC)** *on* $\mathbb{M}^j$, *i.e. for any* $R>0$, *for any* $(y_1,t_1),(y_2,t_2)\in B((x,t),r)\cap\mathbf{M}^k$, $|r|\leq R$, $p\in V_k$ *(or* $p\in\mathbb{R}^{N+1}$*)*

$$\begin{aligned}|\mathbb{F}^j(y_1,t_1,r,p)-\mathbb{F}^j(y_2,t_2,r,p)|\leq&\tilde{C}_1(|y_1-y_2|+|t_1-t_2|)|p|\\&+(R+1)\tilde{m}\big(|y_1-y_2|+|t_1-t_2|\big)\ .\end{aligned}$$

Proof — The first part of the result is a direct consequence of **(NC-BCL)**: indeed $0\in P^\perp(\mathbf{B}(y,s))$, hence there exists $(b,c,l)\in\mathbf{BCL}(y,s)$ such that $P^\perp(b)=0$, i.e. $b\in V_k=T_{(y,s)}\mathbf{M}^k$.

For the second part of the result, we use **(TC-BCL)**: if $(b_1,c_1,l_1)\in\mathbf{BCL}^k(y_1,t_1)\subset\mathbf{BCL}(y_1,t_1)$, there exists $(b_2,c_2,l_2)\in\mathbf{BCL}(y_2,t_2)$ such that

$$|b_1-b_2|\leq C_1(|y_1-y_2|+|t_1-t_2|)\quad,\quad|c_1-c_2|+|l_1-l_2|\leq m\big(|y_1-y_2|+|t_1-t_2|\big)\ .$$

We have to modify (b_2, c_2, l_2) in order to obtain $(\tilde{b}_2, \tilde{c}_2, \tilde{l}_2) \in \mathbf{BCL}^k(y_2, t_2)$ with the right property. To do so, we notice that, since $P^\perp(b_1) = 0$ then $|P^\perp(b_2)| \leq \eta := C_1(|y_1 - y_2| + |t_1 - t_2|)$.

If $P^\perp(b_2) = 0$ the resulu holds, hence we may assume that $P^\perp(b_2) \neq 0$ and set

$$e = \frac{P^\perp(b_2)}{|P^\perp(b_2)|} .$$

Using **(NC-BCL)**, there exists $(\bar{b}_2, \bar{c}_2, \bar{l}_2) \in \mathbf{BCL}(y_2, t_2)$ such that $P^\perp(\bar{b}_2) = -(\delta/2)e$ and we consider the convex combination

$$(\tilde{b}_2, \tilde{c}_2, \tilde{l}_2) := (1 - \alpha)(b_2, c_2, l_2) + \alpha(\bar{b}_2, \bar{c}_2, \bar{l}_2) .$$

Since

$$P^\perp(\tilde{b}_2) = (1 - \alpha)P^\perp(b_2) + \alpha P^\perp(\bar{b}_2) = (1 - \alpha)\eta e - \frac{\delta}{2}\alpha e ,$$

choosing $\alpha = \eta/(\eta + \delta/2)$ we get $P^\perp(\tilde{b}_2) = 0$. Therefore $(\tilde{b}_2, \tilde{c}_2, \tilde{l}_2) \in \mathbf{BCL}^k(y_2, t_2)$ and the estimates on $|b_1 - \tilde{b}_2|$, $|c_1 - \tilde{c}_2|$, $|l_1 - \tilde{l}_2|$ are an easy consequence of the value of α, because of the definition of η and the properties of b_2, c_2, l_2. Indeed, the difference between $(\tilde{b}_2, \tilde{c}_2, \tilde{l}_2)$ and (b_2, c_2, l_2) behaves like $3M\alpha \leq 3M\delta^{-1}\eta$ and therefore the result holds with

$$\bar{C}_1 := (1 + 3M\delta^{-1})C_1 \quad \text{and} \quad \bar{m}(\tau) = m(\tau) + 3M\delta^{-1}C_1\tau .$$

Finally the **(TC)** inequality for $\mathbb{F}^k$ is a direct consequence of the previous result. The third result follows from analogous arguments as in (ii).

Q.E.D.

5.5.3 The case of Ishii viscosity solutions for an Hamilton-Jacobi Equation with a codimension 1 discontinuity

We conclude this section by some remarks on the model problem which is studied in Part II and III where $\mathcal{O} = \mathbb{R}^N \times (0, T_f)$, $X = (x, t)$ and

$$\mathbb{G}(x, t, r, (p_x, p_t)) := \begin{cases} p_t + H_1(x, t, r, p_x) & \text{if } x_N > 0, \\ p_t + H_2(x, t, r, p_x) & \text{if } x_N < 0. \end{cases}$$

For Part II, we are in the control case and we use the standard Ishii inequalities, namely $G^* \geq 0$ in $\mathcal{O}$ and $G_* \leq 0$ in $\mathcal{O}$. We can use **(TC-BCL)** and **(NC-BCL)**

which are satisfied if ($\mathbf{H_{BA-HJ}}$) holds and if H_1, H_2 satisfies the assumption ($\mathbf{NC}_{\mathcal{H}}$) (see p. 163) on $\mathbf{M}^N = \mathcal{H} := \{x : \ x_N = 0\} \times (0, T_f)$. In fact, **(Mon)** but also (**LOC1**), (**LOC2**) are also satisfied under these assumptions.

Concerning Part III, we point out that essentially the same type of assumptions are needed but since the Hamiltonians H_1, H_2 will only be assumed to be quasi-convex, we have to come back to the **(TC)**, **(NC)** formulations.

Chapter 6

Other Tools

6.1 Semi-convex/semi-concave functions: the main properties

The aim of this section is to describe the properties of semi-convex and semi-concave functions which will be used throughout this book, in particular those connected to their differentiability. Considering Section 3.4.3, it is clear that we are not going to manipulate functions which are semi-convex or semi-concave *w.r.t.* all variables but only in the "tangential variables"; anyway, since this latter case consists only in applying the results of the first one by fixing the normal coordinates, we will only be interested in this section in the case of the functions which are semi-convex/semi-concave *w.r.t.* all variables.

We first recall that, if $\mathcal{O} \subset \mathbb{R}^N$ is a convex domain and $f : \mathcal{O} \to \mathbb{R}$, the function f is semi-convex [*resp.* semi-concave] if there exists a constant $C \geq 0$ such that $x \mapsto f(x) + C|x|^2$ is convex [*resp.* $x \mapsto f(x) - C|x|^2$ is concave].

In the sequel, we consider only the semi-convex case, the semi-concave one being deduced by changing f in $-f$ in the results below. In addition, we point out that all the properties we are going to describe are nothing but properties of convex functions which are translated in a suitable (and easy) way, the term $C|x|^2$ being smooth and therefore causing no problem for the differentiability.

We list all the properties in the following result

Proposition 6.1.1 *If $f : \mathcal{O} \to \mathbb{R}$ is a locally bounded function which is semi-convex for a constant $C \geq 0$, we have*

(i) *f is locally Lipschitz continuous in $\mathcal{O}$ and if $\overline{B(x,2r)} \subset \mathcal{O}$, the Lipschitz constant of f in $\overline{B(x,r)}$ depends only on $||f||_{L^\infty(\overline{B(x,r)})}$.*

(ii) *f is differentiable a.e. in $\mathcal{O}$.*

(iii) *For any $x \in \mathcal{O}$, $D^-_{\mathcal{O}} f(x) \neq \emptyset$ and if $p \in D^-_{\mathcal{O}} f(x)$, we have, for all $y \in \mathcal{O}$,*

$$f(y) \geq f(x) + p \cdot (y-x) - 2C|y-x|^2 \, . \tag{6.1}$$

(iv) *Let $(f_\varepsilon)_\varepsilon$ be a sequence of functions which are semi-convex with the same constant C and which are converging to f locally uniformly in $\mathcal{O}$ and let $(x_\varepsilon)_\varepsilon$ a sequence of points of $\mathcal{O}$ which converges to $x \in \mathcal{O}$. If $p_\varepsilon \in D^-_{\mathcal{O}} f(x_\varepsilon)$ and if $(p_{\varepsilon'})_{\varepsilon'}$ is subsequence of $(p_\varepsilon)_\varepsilon$ which converges to p then $p \in D^-_{\mathcal{O}} f(x)$. In particular, if f_ε is differentiable at x_ε for any ε and if f is differentiable at x, then $Df_\varepsilon(x_\varepsilon) \to Df(x)$.*

(v) *If φ is either a C^1 or a semi-concave function defined on $\mathcal{O}$ and if x is a maximum point of $f - \varphi$, then f is differentiable at x, φ is also differentiable at x in the semi-concave case and $Df(x) = D\varphi(x)$.*

Of course we are not going to give a complete proof of Proposition 6.1.1: as we mentioned it above, most of the results are very classical for convex functions and extend without any difficulty to the case of semi-convex ones. But we provide some comments for each of them.

- (i) and (ii) are famous classical results for convex functions, (ii) being a consequence of (i) through Rademacher's Theorem (even if historically Rademacher's Theorem is more a consequence of (ii)).
- (iii) also reflects a classical property of convex function, in particular Inequality (6.1) with the correcting term $-2C|y-x|^2$.
- (iv) is an easy consequence of Inequality (6.1). We point out that the existence of converging subsequences $(p_{\varepsilon'})_{\varepsilon'}$ is a consequence of (i) since it is easy to show that $|p_\varepsilon|$ is controlled by the Lipschitz constant of f_ε and these Lipschitz constants are uniformly bounded by (i) and the local uniform convergence of the sequence $(f_\varepsilon)_\varepsilon$. An interesting particular case is the choice when $f_\varepsilon \equiv f$ where we have some kind of "continuity of the gradient" since, if we have a sequence $(x_\varepsilon)_\varepsilon$ of points where f is differentiable which converges to $x \in \mathcal{O}$ where f is differentiable, then $Df(x_\varepsilon) \to Df(x)$. This is proved by a standard compactness argument since $Df(x)$ is the only possible limit of subsequences of $(Df(x_\varepsilon))_\varepsilon$.

- Property (v) will play a key role for us since we are going to be any time in this context (we recall here that this will be only a property to be used in the "tangential variables"). This property is a consequence of the following result: if $D^-_{\mathcal{O}}f(x) \neq \emptyset$ AND $D^+_{\mathcal{O}}f(x) \neq \emptyset$ then f is differentiable at x and $D^-_{\mathcal{O}}f(x) = D^+_{\mathcal{O}}f(x) = \{Df(x)\}$. In our context, we know by (iii) that $D^-_{\mathcal{O}}f(x) \neq \emptyset$ and then we have two cases

  – if φ is C^1, the maximum point property implies $D\varphi(x) \in D^+_{\mathcal{O}}f(x)$ which is therefore non-empty and the conclusion follows readily.

  – If φ is semi-concave, then $D^+_{\mathcal{O}}\varphi(x) \neq \emptyset$ by an analogous property of (iii) for semi-concave function and the maximum point property both implies $D^+_{\mathcal{O}}\varphi(x) \subset D^+_{\mathcal{O}}f(x)$ and $D^-_{\mathcal{O}}f(x) \subset D^-_{\mathcal{O}}\varphi(x)$. Hence both f and φ are differentiable at x and $Df(x) = D\varphi(x)$.

Remark 6.1.2 *Property (iv) will mainly be used in the case when f is differentiable at x. Then, for any sequence $(x_\varepsilon)_\varepsilon$ of points of $\mathcal{O}$ which converges to $x \in \mathcal{O}$ and for any choice of $p_\varepsilon \in D^-_{\mathcal{O}}f_\varepsilon(x_\varepsilon)$, the sequence of $(p_\varepsilon)_\varepsilon$ converges to $Df(x)$. Indeed, the sequence $(p_\varepsilon)_\varepsilon$ is bounded, hence it lies in a compact subset of $\mathbb{R}^N$ and $Df(x)$ is the only possible limit for converging subsequences of $(p_\varepsilon)_\varepsilon$.*

6.2 Quasi-convexity: definition and main properties in $\mathbb{R}$

Let $\mathcal{C} \subset \mathbb{R}^N$ be a convex set. A *quasi-convex* function $f : \mathcal{C} \to \mathbb{R}$ is a function such that, for any $a \in \mathbb{R}$, the lower level set $\{x : f(x) \leq a\}$ is convex.

An equivalent definition is: for any $x, y \in \mathcal{C}$ and $\lambda \in (0,1)$,

$$f(\lambda x + (1-\lambda)y) \leq \max\{f(x), f(y)\}\,.$$

Of course, convex functions are quasi-convex but the converse is false since quasi-convex functions can be discontinuous, even if they are bounded: for example, take, in $\mathbb{R}^N$, the indicator function of the complementary of a convex set. Hence, one of the differences between convex and quasi-convex functions is that quasi-convex functions may have various "flat" zones, not only where they achieve their minimum.

In the sequel, we consider the following class of functions

Definition 6.2.1 *We say that $f : \mathbb{R} \to \mathbb{R}$ satisfies $(\mathbf{H_{QC-\mathbb{R}}})$ if f is continuous, coercive and quasi-convex.*

6.2.1 Representation of quasi-convex functions

The first (classical) result we have for such functions is the

Lemma 6.2.2 *If $f : \mathbb{R} \to \mathbb{R}$ satisfies $(\mathbf{H}_{\mathbf{QC}-\mathbb{R}})$ then*

(i) there exists $m^-(f) \leq m^+(f)$ such that the set where f achieves its minimum is exactly the interval $[m^-(f), m^+(f)]$.

(ii) f is nonincreasing on $]-\infty, m^-(f)[$ and nondecreasing on $]m^+(f), +\infty[$.

(iii) $f = \max\{f^\sharp, f_\flat\}$ where $f^\sharp$ is nondecreasing and $f_\flat$ is nonincreasing.

Proof — The proof of (i) is easy: since f is continuous and coercive, it is bounded from below and achieves its minimum. Moreover by quasi-convexity, the set $\{x : f(x) \leq \min_{\mathbb{R}}(f)\}$ is convex, hence this is an interval $[m^-(f), m^+(f)]$.

For (ii), we consider $x, y \in]-\infty, m^-(f)[$ with $x < y$. If $f(x) < f(y)$, then, by the quasi-convexity of f, the convex set $\{t : \ f(t) \leq f(x)\}$ contains x and $m^-(f)$, hence all the interval $[x, m^-(f)]$. A contradiction since $y \in [x, m^-(f)]$. Hence f is nonincreasing on $]-\infty, m^-(f)[$ and an analogous proof shows that f nondecreasing on $]m^+(f), +\infty[$.

For (iii), we consider

$$f^\sharp(x) = \min\{f(t); t \geq x\} \ , \ f_\flat(x) = \min\{f(t); t \leq x\}.$$

Clearly we have

$$f^\sharp(x) = \min_{\mathbb{R}}(f) \text{ if } x \leq m^+(f) \quad , \quad f_\flat(x) = \min_{\mathbb{R}}(f) \text{ if } x \geq m^-(f) \ ,$$

while, by using (ii),

$$f^\sharp(x) = f(x) \text{ if } x > m^+(f) \quad , \quad f_\flat(x) = \min_{\mathbb{R}}(f) \text{ if } x < m^-(f) \ .$$

The conclusion follows by analyzing the different cases $x < m^-(f)$, $m^-(f) \leq x \leq m^+(f)$ and $x > m^+(f)$.

Q.E.D.

6.2.2 On the maximum of two quasi-convex functions

In this section, we describe a result which is crucial in order to give sufficient conditions for the uniqueness of Ishii solutions in problems with codimension 1 discontinuities (see Section 10.2).

Let $f, g : \mathbb{R} \to \mathbb{R}$ satisfy $(\mathbf{H_{QC-\mathbb{R}}})$ and define

$$M(s) := \max\{f(s), g(s)\}\,, \qquad M^{\rm reg}(s) := \max\{f^\sharp(s), g_\flat(s)\}\,.$$

We point out that we use the strange notation $M^{\rm reg}$ to be consistent with Section 10.2. Notice that the definition of $M^{\rm reg}$ is not symmetric on f and g.

Lemma 6.2.3 *We assume that f, g satisfy $(\mathbf{H_{QC-\mathbb{R}}})$. There exists $\nu_1 \le \nu_2$ such that*

$$M^{\rm reg}(s) := \begin{cases} g_\flat(s) > f^\sharp(s) & \text{if } s < \nu_1\,, \\ f^\sharp(s) = g_\flat(s) & \text{if } \nu_1 \le s \le \nu_2\,, \\ f^\sharp(s) > g_\flat(s) & \text{if } s > \nu_2\,. \end{cases}$$

Of course, $\min_{s\in\mathbb{R}} M^{\rm reg}(s)$ is attained on $[\nu_1, \nu_2]$.

Proof — We introduce the function $\varphi(s) := f^\sharp(s) - g_\flat(s)$. Due to the properties of $f^\sharp$ and $b_\flat$, the function φ is nondecreasing. Moreover, due to the coercivity assumption, $\varphi(s) \to -\infty$ as $x \to -\infty$ and $\varphi(s) \to +\infty$ as $x \to +\infty$. Therefore, there exists $\nu_1 \le \nu_2$ such that $\varphi(s) < 0$ if $s < \nu_1$, $\varphi(s) > 0$ if $s > \nu_2$ and $\varphi(s) = 0$ on $[\nu_1, \nu_{,2}]$. The lemma directly follows.

Q.E.D.

Proposition 6.2.4 *Let $f, g : \mathbb{R} \to \mathbb{R}$ satisfy $(\mathbf{H_{QC-\mathbb{R}}})$. If $m^+(f) \le m^-(g)$ then the following property holds*

$$\min_{s\in\mathbb{R}} M(s) = \min_{s\in\mathbb{R}} M^{\rm reg}(s)\,.$$

Proof — Notice first that of course the inequality $\max\{f, g\} \ge \max\{f^\sharp, g_\flat\}$ holds simply because of the definition of $f^\sharp$ and $g_\flat$; therefore the same inequality holds when taking the minimum over s.

In order to get the opposite inequality, we first remark that, by Lemma 6.2.3, the minimum of $M^{\rm reg}$ is attained at some point s_0 which satisfies $s_0 \in [\nu_1, \nu_2]$. Moreover, $f^\sharp(s_0) = g_\flat(s_0)$. There are three cases, some of which may be void.

First case: $s_0 \in [m^+(f), m^-(g)]$. In this case the conclusion easily follows from the fact that $f^\sharp(s_0) = f(s_0) = g_\flat(s_0) = g(s_0)$: we deduce immediately that $\min_{\mathbb{R}}(M^{\rm reg}) = M^{\rm reg}(s_0) = M(s_0) \geq \min_{\mathbb{R}}(M)$.

Second case: $s_0 \leq m^+(f) \leq m^-(g)$. This implies that $f^\sharp(s_0) = \min_{\mathbb{R}}(f) = g_\flat(s_0)$ and $\min_{\mathbb{R}}(M^{\rm reg}) = M^{\rm reg}(s_0) = \min_{\mathbb{R}}(f)$.

Considering the situation at $s = m^+(f)$ we see that

$$\begin{aligned} g(m^+(f)) &= g_\flat(m^+(f)) \quad \text{because } m^+(f) \leq m^-(g) \\ &\leq g_\flat(s_0) \quad \text{because } g_\flat \text{ is nonincreasing} \\ &\leq f^\sharp(s_0) \quad \text{by the definition of } s_0 \\ &\leq f^\sharp(m^+(f)) \quad \text{because } f^\sharp \text{ is flat for } s \leq m^+(f) \\ &\leq f(m^+(f)) = \min_{\mathbb{R}}(f)\,. \end{aligned}$$

We deduce that, at $s = m^+(f)$, $M(m^+(f)) = \min_{\mathbb{R}}(f) = \min_{\mathbb{R}}(M^{\rm reg})$. Hence, we conclude that $\min_{\mathbb{R}}(M^{\rm reg}) \geq \min_{\mathbb{R}}(M)$.

Third case: if $s_0 \geq m^-(g) \geq m^+(f)$, the proof is the same after reversing the roles of $f^\sharp$ and $g_\flat$.

The conclusion is that, in any case, $\min_{\mathbb{R}}(M^{\rm reg}) \geq \min_{\mathbb{R}}(M)$ which implies that those minima are equal.

Q.E.D.

6.2.3 Application to quasi-convex Hamiltonians

As we have seen in the previous sections, throughout this book we deal with Hamiltonians of the form $H(x,t,r,p)$. Those may be either convex, Lipschitz, or have a quasi-convexity property that we describe now.

The quasi-convex case (mainly exposed in Part III) is defined in the following way: if we set $p = (p', p_N)$ with $p' \in \mathbb{R}^{N-1}$ and $p_N \in \mathbb{R}$, we will say that we are in the *quasi-convex case* if

($\mathbf{H_{QC}}$) *for any* (x,t,r,p'), *the function* $h : s \mapsto H(x,t,r,p' + se_N)$ *satisfies* ($\mathbf{H_{QC-\mathbb{R}}}$).

Using the previous sections, we can introduce the new Hamiltonians

$$H^-(x,t,r,p) = h^\sharp(p_N) = \Big[H(x,t,r,p' + p_N e_N)\Big]^\sharp\,,$$

$$H^+(x,t,r,p) = h_\flat(s) = \Big[H(x,t,r,p' + se_N)\Big]_\flat\,.$$

Thanks to the above results, we have $H = \max(H^+, H^-)$. We use extensively this decomposition in Part III and we point out that, if H satisfies ($\mathbf{H_{BA-HJ}}$), then the Hamiltonians H^+, H^- also satisfy ($\mathbf{H_{BA-HJ}}$).

6.3 A strange, Kirchhoff-related lemma

In Part III, the following lemma will be useful in order to connect general Kirchhoff type conditions with flux-limited type conditions on the interface.

Lemma 6.3.1 *Assume that $f,g : \mathbb{R} \to \mathbb{R}$ and $h : \mathbb{R}^2 \to \mathbb{R}$ are continuous functions such that*

(*i*) *f is an increasing function with $f(t) \to +\infty$ as $t \to +\infty$,*

(*ii*) *g is a decreasing function with $g(t) \to +\infty$ as $t \to -\infty$,*

(*iii*) *there exists $\alpha > 0$ such that, for any $t_2 \geq t_1$ and $s_2 \leq s_1$, we have*

$$h(t_2, s_2) - h(t_1, s_1) \leq -\alpha(t_2 - t_1) + \alpha(s_2 - s_1) \, .$$

If $\psi : \mathbb{R}^2 \to \mathbb{R}$ is the function defined by

$$\psi(t, s) := \max(f(t), g(s), h(t, s)) \, ,$$

then ψ is a coercive continuous function in $\mathbb{R}^2$ and there exists $(\bar{t}, \bar{s})$ such that

$$\psi(\bar{t}, \bar{s}) = \min_{t,s} (\psi(t, s)) \tag{6.2}$$

and

$$f(\bar{t}) = g(\bar{s}) = h(\bar{t}, \bar{s}) \, . \tag{6.3}$$

Moreover, if a point $(\tilde{t}, \tilde{s}) \in \mathbb{R}^2$ satisfies (6.3) then $(\tilde{t}, \tilde{s})$ is a minimum point of ψ. Finally,

$$\min_{t,s} \{\max(f(t), g(s), h(t, s))\} = \max_{t,s} \{\min(f(t), g(s), h(t, s))\} \, .$$

In the statement of the above lemma, we point out that the assumption on h implies that $h(t, s)$ is a strictly decreasing function of t and a strictly increasing function of s with $h(t, s) \to +\infty$ if $t \to -\infty$, s remaining bounded or if $s \to +\infty$, t remaining bounded.

Proof — Using the three properties we impose on f, g, h, and in particular, the consequences of the assumption on h we describe above, it is easy to prove that ψ is actually continuous and coercive; therefore such a minimum point $(\bar{t}, \bar{s})$ exists.

We have to show that (6.3) holds and to do so, we may assume without loss of generality that f is strictly increasing and g is strictly decreasing. Otherwise, we may prove the result for $f(t) + \varepsilon t$ and $g(s) - \varepsilon s$ for $\varepsilon > 0$ and pass to the limit $\varepsilon \to 0$ remarking that the associated minimum points remain in a fixed compact subset of $\mathbb{R}^2$.

If $m = \min_{t,s}(\psi(t,s))$, we first notice that $h(\bar{t}, \bar{s}) = m$. Otherwise $h(\bar{t}, \bar{s}) < m$ and it is clear enough by using the monotonicity of f and g that, for $\delta > 0$ small enough, then

$$\psi(\bar{t} - \delta, \bar{s} + \delta) < \psi(\bar{t}, \bar{s}) \ ,$$

a contradiction.

In the same way, if $f(\bar{t}) < m$, using the properties of h, there exists $\delta, \delta' > 0$ small enough such that $h(\bar{t} + \delta, \bar{s} + \delta') < m$, $g(\bar{s} + \delta') < m$ and $\psi(\bar{t} + \delta, \bar{s} + \delta') < \psi(\bar{t}, \bar{s})$, again a contradiction.

A similar proof allowing to conclude that $g(\bar{s}) = m$, (6.3) holds.

Notice that if we have replaced $f(t)$ by $f(t) + \varepsilon t$ and $g(t)$ by $g(s) - \varepsilon s$, we can let ε tend to 0 and keep this property for at least one minimum point.

Now we consider a point $(\tilde{t}, \tilde{s}) \in \mathbb{R}^2$ which satisfies (6.3) and we pick any point $(t, s) \in \mathbb{R}^2$. We examine the different possible cases, taking into account the particular form of ψ and the monotonicity properties of f, g, h, using that, of course, $\psi(\tilde{t}, \tilde{s}) = f(\tilde{t}) = g(\tilde{s}) = h(\tilde{t}, \tilde{s})$

- If $t \geq \tilde{t}$, $\psi(t,s) \geq f(t) \geq f(\tilde{t}) = \psi(\tilde{t}, \tilde{s})$.
- If $s \leq \tilde{s}$, the same conclusion holds by using that g is decreasing.
- If $t \leq \tilde{t}$ and $s \geq \tilde{s}$, then $\psi(t,s) \geq h(t,s) \geq h(\tilde{t}, \tilde{s}) = \psi(\tilde{t}, \tilde{s})$.

And the conclusion follows since we have obtained that ψ reaches its minimum at $(\tilde{t}, \tilde{s})$.

For the last property, we set

$$\chi(t,s) = \min(f(t), g(s), h(t,s)) \ .$$

If, as above, $(\tilde{t}, \tilde{s}) \in \mathbb{R}^2$ is a point which satisfies (6.3), we have $\chi(\tilde{t}, \tilde{s}) = f(\tilde{t}) = g(\tilde{s}) = h(\tilde{t}, \tilde{s})$ and by similar arguments as above

- If $t \leq \tilde{t}$, $\chi(t,s) \leq f(t) \leq f(\tilde{t}) = \chi(\tilde{t},\tilde{s})$.
- If $s \geq \tilde{s}$, the same conclusion holds by using that g is decreasing.
- If $t \geq \tilde{t}$ and $s \leq \tilde{s}$, then $\chi(t,s) \leq h(t,s) \leq h(\tilde{t},\tilde{s}) = \chi(\tilde{t},\tilde{s})$.

And the proof is complete.

Q.E.D.

Remark 6.3.2 *A similar result to the last part of Lemma 6.3.1, but with a simpler proof, is*

$$\min_t\{\max(f(t),g(t))\} = \max_t\{\min(f(t),g(t))\}\ . \tag{6.4}$$

This equality is also useful in Part III.

6.4 A few results for penalized problems

In viscosity solutions' theory, several proofs require penalization arguments, i.e. approximations of maxima or minima by penalizing the function. The most emblematic example is certainly the doubling of variables in comparison proofs but there are several other examples, such as the treatment of some boundary conditions (evolution equations set in $(0,T)$ which hold up to time T or more generally boundary conditions in the case when all dynamics are pointing inward the domain) or the convergence of regularization by inf or sup-convolution...etc.

Instead of refering to these (rather easy) results as "standard results" all along this book, we have decided to provide two general lemmas gathering the key informations, one for penalization in compact sets, the other one (more restrictive) concerns the penalization at infinity.

6.4.1 The compact case

Lemma 6.4.1 *Let $w : K \to \mathbb{R}$ be an u.s.c. function defined on some compact set $K \subset \mathbb{R}^p$ and $F \subset K$ be closed. We denote by $M := \max_{z\in F} w(z)$. For any $\varepsilon > 0$ let $\chi_\varepsilon : K \to \mathbb{R} \cup \{+\infty\}$ satisfying*

(i) the functions $\{\chi_\varepsilon\}$ are uniformly bounded from below and l.s.c.[(1)];

[(1)] in the expected generalized sense in order to take into account the $+\infty$ value at some points if necessary.

(*ii*) $\liminf_*\, \chi_\varepsilon(z) = \begin{cases} 0 & \text{if } z \in F\,, \\ +\infty & \text{if } z \in K \setminus F\,; \end{cases}$

(*iii*) *for any* $z_0 \in F$, *there exists* $(z_0^\varepsilon)_\varepsilon$ *such that* $w(z_0^\varepsilon) - \chi_\varepsilon(z_0^\varepsilon) \to w(z_0)$ *as* $\varepsilon \to 0$.

Then

1. $M_\varepsilon := \max_{z\in K} (w(z) - \chi_\varepsilon(z)) \to M$ *as* $\varepsilon \to 0$.
2. *For any* $\varepsilon > 0$ *let* z_ε *be a maximum point of* $z \mapsto w(z) - \chi_\varepsilon(z)$. *If* $(z_{\varepsilon'})_{\varepsilon'}$ *is a subsequence of* (z_ε) *converging to some* $\bar z$, *then*
$$\bar z \in F\ ,\ w(\bar z) = M\ ,\ w(z_{\varepsilon'}) \to w(\bar z)\ ,\ \chi_{\varepsilon'}(z_{\varepsilon'}) \to 0\ .$$
3. *If* $w = w_1 - w_2$ *where* w_1 *is u.s.c. and* w_2 *is l.s.c., then* $w_1(z_{\varepsilon'}) \to w_1(\bar z)$ *and* $w_2(z_{\varepsilon'}) \to w_2(\bar z)$.
4. *If there is a unique maximum point* $\bar z$ *of* w *on* F *then* $z_\varepsilon \to \bar z$, $w(z_\varepsilon) \to w(\bar z)$ *and* $\chi_\varepsilon(z_\varepsilon) \to 0$.

Proof — Since K is compact and F is a closed subset of K, there exists z_0 such that $w(z_0) = M$. By the definition of z_ε and (*iii*), we have
$$M + o_\varepsilon(1) = w(z_0^\varepsilon) - \chi_\varepsilon(z_0^\varepsilon) \le w(z_\varepsilon) - \chi_\varepsilon(z_\varepsilon) = M_\varepsilon$$
and this inequality immediately gives $\limsup M_\varepsilon \ge M$.

On the other hand, if we extract a converging subsequence $z_{\varepsilon'} \to \bar z \in K$, by letting $\varepsilon' \to 0$ and using the upper semicontinuity of w we obtain
$$M \le \liminf (w(z_{\varepsilon'}) - \chi_{\varepsilon'}(z_{\varepsilon'})) \le \limsup (w(z_{\varepsilon'}) - \chi_{\varepsilon'}(z_{\varepsilon'})) \le w(\bar z) - \liminf_*\, \chi_\varepsilon(\bar z)\ .$$
Using (*i*), we see that necessarily $\bar z \in F$ since $\liminf_*\, \chi_\varepsilon(\bar z)$ cannot be $+\infty$, therefore $\liminf_*\, \chi_\varepsilon(\bar z) = 0$. We deduce from this property and the above inequality that $w(\bar z) \ge M$ but since $\bar z \in F$, we conclude that $w(\bar z) = M$.

Gathering all these informations, the above inequality can be rewritten as
$$M \le \liminf (w(z_{\varepsilon'}) - \chi_{\varepsilon'}(z_{\varepsilon'})) \le \limsup (w(z_{\varepsilon'}) - \chi_{\varepsilon'}(z_{\varepsilon'})) \le M\ ,$$
and therefore $M_{\varepsilon'} = w(z_{\varepsilon'}) - \chi_{\varepsilon'}(z_{\varepsilon'}) \to M$.

Extracting first a subsequence such that $\lim M_{\varepsilon'} = \liminf M_\varepsilon$ and then a converging subsequence out of $(z_{\varepsilon'})_{\varepsilon'}$, the above argument shows that $\liminf M_\varepsilon = M$ and therefore $M_\varepsilon \to M$. This proves 1.

Point 2 is a direct consequence of the above argument: for any converging subsequence $z_{\varepsilon'} \to \bar{z} \in K$, we have $\bar{z} \in F$, $w(\bar{z}) = M$ and since $\limsup w(z_{\varepsilon'}) \leq w(\bar{z}) \leq M$ by the upper semi-continuity of w and $\liminf \chi_{\varepsilon'}(z_{\varepsilon'}) \geq \liminf_* \chi_\varepsilon(\bar{z}) = 0$, the only possibility to have such a convergence to M is $w(z_{\varepsilon'}) \to M = w(\bar{z})$ and $\lim \chi_{\varepsilon'}(z_{\varepsilon'}) = 0$.

For Point 3, the argument is analogous: since $\limsup w_1(z_{\varepsilon'}) \leq w_1(\bar{z})$ and $\liminf w_2(z_{\varepsilon'}) \geq w_2(\bar{z})$, the only possibility to have $w(z_{\varepsilon'}) \to w(\bar{z})$ is to have at the same time $w_1(z_{\varepsilon'}) \to w_1(\bar{z})$ and $w_2(z_{\varepsilon'}) \to w_2(\bar{z})$.

Finally 4. comes from a standard compactness argument.

Q.E.D.

Typical application: the doubling of variables – After the localization procedure described in Section 3.2, we get two functions $u, v : \overline{B(x,r)} \to \mathbb{R}$ for some $x \in \mathbb{R}^N$ and $r > 0$, u being u.s.c. while v is l.s.c. and we are considering $M := \max_{x \in \overline{B(x,r)}}(u(x) - v(x))$, that we approximate by the maximum of the function

$$\psi_\varepsilon(x,y) = u(x) - v(y) - \frac{|x-y|^2}{\varepsilon^2} \, .$$

We apply Lemma 6.4.1 with $K = \overline{B(x,r)} \times \overline{B(x,r)}$, $F = K \cap \{(x,y) : \ x = y\}$,

$$z = (x,y) \, , \ w(x,y) = u(x) - v(y) \, , \ \chi_\varepsilon(x,y) = \frac{|x-y|^2}{\varepsilon^2}$$

and $w_1 = u$, $w_2 = v$. We notice that Assumptions $(i) - (ii) - (iii)$ for χ_ε are obviously satisfied with $z_0^\varepsilon = z_0$ for any ε.

So, if $(x_\varepsilon, y_\varepsilon) \in K$ is a maximum point of ψ_ε in K and if $(x_{\varepsilon'}, y_{\varepsilon'})$ is a converging subsequence of maximum points of $\psi_{\varepsilon'}$, we first have that $(x_{\varepsilon'}, y_{\varepsilon'}) \to (\bar{x}, \bar{x}) \in F$ and

$$u(x_{\varepsilon'}) \to u(\bar{x}) \ , \ v(y_{\varepsilon'}) \to v(\bar{x}) \ , \ \frac{|x_{\varepsilon'} - y_{\varepsilon'}|^2}{(\varepsilon')^2} \to 0 \ ,$$

which is the classical result we use.

Remarks on the assumptions

(a) As a first comment, we point out that, one way or the other, the "compactness" assumption on K in Lemma 6.4.1 is necessary, although it may be replaced by a

stronger assumption on w like coercivity which prevents infinity to play a role, see Subsection 6.4.2 below.

Moreover, this type of lemma does not hold in non-compact situations, in general, even if we replace max by sup. Indeed if we look at the following penalization

$$\psi_\varepsilon(x,y) = \sin(x^2) - \sin(y^2) - \frac{|x-y|^2}{\varepsilon^2} - \varepsilon|x| \ ,$$

but with $K = \mathbb{R}\times\mathbb{R}$ and $F = \{(x,y) : \ x = y\}$, the reader will easily check, using the non-uniform continuity of $\sin(x^2)$, that M_ε exists and $M_\varepsilon \to 2$ as $\varepsilon \to 0$ while $M = \sup_{(x,y)\in F}(\sin(x^2) - \sin(y^2)) = 0$.

(b) Notice that χ_ε can take the value $+\infty$, a case which gives important applications too. For instance if $K = [0,T]$, we can handle terms like $\varepsilon/(T-t)$ in χ_ε, which prevent the maximum to be attained at $t = T$. The lower semicontinuity property for χ_ε holds since

$$\lim_{\substack{t\to T\\ t<T}} \chi_\varepsilon(t) = +\infty \ .$$

Similarly if Ω is a bounded smooth domain, we can use a penalization like $\varepsilon[d(x)]^{-1}$ in $\chi_\varepsilon : \overline{\Omega} \to \mathbb{R}\cup\{+\infty\}$ where $d(\cdot)$ stands for the distance to the boundary of Ω. Such penalizations avoid maximum points at the boundary —See for instance Proposition 3.5.1 where this approached is used.

(c) Finally, let us explain the (admittedly strange) Assumption (iii) for χ_ε. In state-constraint problems where the subsolution inequalities hold only in a domain Ω while the supersolution ones hold on $\overline{\Omega}$, one needs to "push inside Ω" the point x corresponding to the subsolution. In order to prove comparison result for such problems, Soner [154, 155] introduces penalization terms of the form

$$\left|\frac{x-y}{\varepsilon} + n(y)\right|^2$$

where, if $\partial\Omega$ is smooth, n denotes an extension to a neighborhood of $\partial\Omega$ of the unit outward normal to $\partial\Omega$. But such penalization terms do not tend to 0 if we choose as above $x = y$. Moreover, it is known that a cone condition should hold for the subsolution. So, here we require by (iii) that for any $\bar{x} \in \partial\Omega$, there exist $(x_\varepsilon, y_\varepsilon) \to (\bar{x},\bar{y})$ such that

$$u(x_\varepsilon) - v(y_\varepsilon) - \left|\frac{x_\varepsilon - y_\varepsilon}{\varepsilon} + n(y_\varepsilon)\right|^2 \to u(\bar{x}) - v(\bar{x}) \ .$$

This assumption is satisfied by $x_\varepsilon = \bar{x} - \varepsilon n(\bar{x})$, $y_\varepsilon = \bar{x}$ if u is continuous or if the cone condition holds for u.

6.4.2 Penalization at infinity

The following result is connected to our localization procedure.

Proposition 6.4.2 *Let $w : \mathbb{R}^N \to \mathbb{R}$ a bounded u.s.c. function and $(w_\alpha)_{\alpha>0}$ a sequence of u.s.c. functions such that*
(i) $w_\alpha(x) \to -\infty$ as $|x| \to +\infty$,
(ii) $w_\alpha(x) \to w(x)$ when $\alpha \to 0$ for any $x \in \mathbb{R}^N$.
Then, if $M_\alpha := \max_{\mathbb{R}^N}(w_\alpha)$ and $M := \sup_{\mathbb{R}^N}(w)$, we have

$$\liminf M_\alpha \geq M \ .$$

Moreover, if $w_\alpha(x) = w(x) - \alpha\chi(x)$ where $\chi : \mathbb{R}^N \to \mathbb{R}$ is a coercive, locally bounded, l.s.c. function and if x_α is such that $w_\alpha(x_\alpha) = M_\alpha$ then $w(x_\alpha) \to M$ and $\alpha\chi(x_\alpha) \to 0$.

Proof — By definition of the supremum, there exists a sequence $(x_k)_k$ of points in $\mathbb{R}^N$ such that $w(x_k) \to M$ and, for any k,

$$w_\alpha(x_k) \leq M_\alpha \ .$$

Taking the liminf as α tend to 0 and letting k tend to infinity, we obtain the first part of the result.

For the second part, we use the fact that χ is bounded from below and therefore $M_\alpha \leq M - \alpha m$, where $m = \min_{\mathbb{R}^N}(\chi)$. Hence $\limsup M_\alpha \leq M$ and therefore $M_\alpha \to M$. In other words

$$w_\alpha(x_\alpha) = w(x_\alpha) - \alpha\chi(x_\alpha) \to M \ .$$

But $-\alpha\chi(x_\alpha) \leq -\alpha m$ and therefore

$$w(x_\alpha) = M_\alpha + \alpha\chi(x_\alpha) \geq M_\alpha + \alpha m \ .$$

Hence $\liminf w(x_\alpha) \geq M$ but obviously $\limsup w(x_\alpha) \leq M$. This yields $\lim w(x_\alpha) = M$ and, as a consequence, $-\alpha\chi(x_\alpha) = M_\alpha - w(x_\alpha) \to 0$.

Q.E.D.

Part II

Deterministic Control Problems and Hamilton-Jacobi Equations for Codimension One Discontinuities

Chapter 7

Introduction : Ishii Solutions for the Hyperplane Case

In this part, we consider one of the simplest and emblematic case of discontinuity for an equation or a control problem: the case when this discontinuity is an hyperplane, say $\mathcal{H} = \{x_N = 0\}$. In terms of stratification, as introduced in Section 3.3, this is one of the simplest examples of stratification of $\mathbb{R}^N \times (0, T_f)$ for which $\mathbf{M}^{N+1} = (\Omega_1 \cup \Omega_2) \times (0, T_f)$, $\mathbf{M}^N = \mathcal{H} \times (0, T_f)$ and $\mathbf{M}^k = \emptyset$ for any $k = 0..(N-1)$, where

$$\Omega_1 = \{x_N > 0\}\,,\ \Omega_2 = \{x_N < 0\}\,.$$

For simplicity of notations, we also write $\Omega_0 = \mathcal{H}$ and we take the convention to denote by $e_N = (0, \dots, 0, 1)$ the unit vector pointing inside Ω_1, so that e_N is also the outward unit normal to Ω_2, see figure 7.1 below.

Two types of questions can be addressed whether we choose the pde or control point of view and, in this part, both will be very connected since we mainly consider Hamilton-Jacobi-Bellman type equations.

7.1 The pde viewpoint

From the pde viewpoint, the main question concerns the existence and uniqueness of solutions to the problem

$$\begin{cases} u_t + H_1(x,t,u,Du) = 0 & \text{for } x \in \Omega_1 \times (0,T_f)\,, \\ u_t + H_2(x,t,u,Du) = 0 & \text{for } x \in \Omega_2 \times (0,T_f)\,, \\ u(x,0) = u_0(x) & \text{for } x \in \mathbb{R}^N\,, \end{cases} \tag{7.1}$$

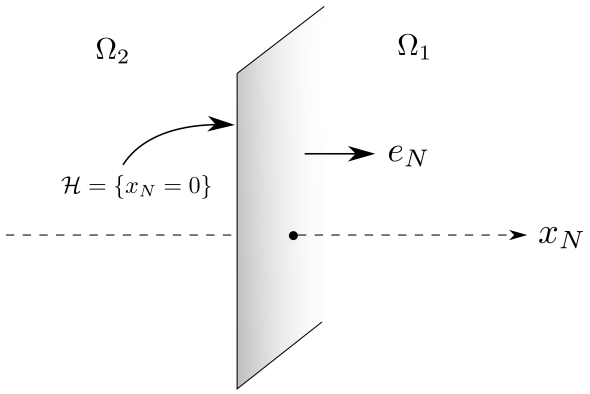


Figure 7.1: Setting of the codimension one case

under some standard assumptions on H_1, H_2 and u_0. It is also very natural to consider a specific control problem or pde on $\mathcal{H}$, which amounts to adding an equation

$$u_t + H_0(x, t, u, D_T u) = 0 \text{ for } x \in \mathcal{H}\,, \tag{7.2}$$

where $D_T u$ stands for the *tangential derivative* of u, *i.e.* the $(N-1)$ first components of the gradient, leaving out the normal derivative. However, for reasons that will be exposed later in Section 11, adding such a condition is not completely tractable in the context of Ishii solutions and is more relevant in the context of flux-limited solutions or junction conditions (see Part III). Therefore, except for Section 11, we restrict ourselves to problem (7.1).

As we explained in Section 3.1, the conditions on $\mathcal{H}$ for those equations have to be understood in the relaxed (Ishii) sense, namely

$$\begin{cases} \max\Big(u_t + H_1(x, t, u, Du), u_t + H_2(x, t, u, Du)\Big) \geq 0\,, \\ \min\Big(u_t + H_1(x, t, u, Du), u_t + H_2(x, t, u, Du)\Big) \leq 0\,, \end{cases} \tag{7.3}$$

meaning that for the supersolution [*resp.* subsolution] condition, at least one of the inequation for H_1 or H_2 has to hold.

7.2 The control viewpoint

From the control viewpoint, we are in the situation where different dynamics, discount factors and costs are defined on Ω_1 and Ω_2. A double question arises: (i) how to define

a global control problem in $\mathbb{R}^N$? (*ii*) once this is done, if each Hamiltonian in (7.1) is associated to the control problem in the corresponding domain, is the "usual" value function still the unique solution of (7.1)?

In this chapter, we combine several tools introduced in Part I in order to address these problems. Notice that the present stratification of $\mathbb{R}^N$ is obviously a typical AFS. So, assuming moreover that each Hamiltonian satisfies **(NC)**, **(TC)** and **(Mon)**, we are in what we called a "good" framework for treating discontinuities in the sense of Definition 5.5.1 (here, no diffeomorphism is needed since the stratification is flat).

7.3 The uniqueness question

As we will see, Ishii's notion of solution is not strong enough to ensure comparison (and uniqueness) in this setting in general: this is already true for Equation (7.1) but the situation is even worse when adding (7.2) on $\mathcal{H}$. Let us give a brief overview of this story here.

The general formulation of control problems described in Chapter 4 provides a "natural" control solution of (7.1), obtained by minimizing a cost over all the possible trajectories. We denoted this solutin by $\mathbf{U}^-$. By Corollary 4.4.3, $\mathbf{U}^-$ is in fact the minimal supersolution (and solution) of (7.1).

But we introduce another value function denoted by $\mathbf{U}^+$ where we minimize over a subset of those trajectories, that are called *regular*. We will show that $\mathbf{U}^+$ is also an Ishii solution of (7.1), and it is even the maximal Ishii (sub)solution of (7.1). In general $\mathbf{U}^- \neq \mathbf{U}^+$ and we provide an explicit example of such a configuration. Finally both $\mathbf{U}^-$ and $\mathbf{U}^+$ can be characterized by means of an additional "tangential" Hamiltonian on $\mathcal{H}$. Later in this part, we will also see that $\mathbf{U}^+$ is the limit of the vanishing viscosity method.

At this point, the reader may think that there is no difference when adding (7.2) to problem (7.1), after modifying in a suitable way the specific control problem on $\mathcal{H}$. It is, of course, the case for $\mathbf{U}^-$ where again the general results of Chapter 4 apply.

But the determination of the maximal Ishii (sub)solution is more tricky: to understand why, we refer the reader to the Dirichlet/exit time problem for deterministic control problem in a domain; it is shown in [23] that, if the minimal solution of the Dirichlet problem is actually given by an analogue of the value function $\mathbf{U}^-$ for such problems, the maximal one is obtained by considering the "worse stopping time" on the boundary (see also [31]). This *differential game* feature arises here in a more complicated way and we give some elements to understand it in Section 11.

In the next four Chapters, we give a complete study of (7.1): we first introduce the control problem, define and characterize $\mathbf{U}^-$. Then we construct and study $\mathbf{U}^+$. Some uniqueness and non-uniqueness results are proved and we discuss the problem of adding (7.2) in the last Chapter 11.

Chapter 8

The Control Problem and the "Natural" Value Function

Assuming that (7.1) is associated to a control problem means that there exists some triplets dynamics-discount-cost $(b_i, c_i, l_i) : \overline{\Omega_i} \times [0, T_f] \times A_i \to \mathbb{R}^{N+3}$ for $i = 1, 2$, such that for any $(x, t, u, p) \in \overline{\Omega_i} \times (0, T_f] \times \mathbb{R} \times \mathbb{R}^N$,

$$H_i(x, t, u, p) = \sup_{\alpha_i \in A_i} \{-b_i(x, t, \alpha_i) \cdot p + c_i(x, t, \alpha_i)u - l_i(x, t, \alpha_i)\} .$$

All these (b_i, c_i, l_i) can be assumed as well to be defined on $\mathbb{R}^N \times [0, T_f] \times A_i$. Moreover, in the following we assume that they satisfy the basic assumptions $(\mathbf{H_{BA-CP}})$ and the normal controllability assumption

$(\mathbf{NC}_{\mathcal{H}})$ — *Normal Controllability.* For any $(x, t) \in \mathcal{H} \times [0, T_f]$, there exists $\delta = \delta(x, t)$ and a neighborhood $\mathcal{V} = \mathcal{V}(x, t)$ such that, for any $(y, s) \in \mathcal{V}$

$$[-\delta, \delta\,] \subset \{b_1(y, s, \alpha_1) \cdot e_N,\ \alpha_1 \in A_1\} \quad \text{if } (y, s) \in \overline{\Omega_1}\ ,$$
$$[-\delta, \delta\,] \subset \{b_2(y, s, \alpha_2) \cdot e_N,\ \alpha_2 \in A_2\} \quad \text{if } (y, s) \in \overline{\Omega_2}\ ,$$

where $e_N = (0, 0 \cdots, 0, 1) \in \mathbb{R}^N$.

It is easy to check that Assumption $(\mathbf{NC}_{\mathcal{H}})$ implies **(NC)** for H_1 and H_2 and we refer below to assumptions $(\mathbf{H_{BA-CP}})$ for (b_i, c_i, l_i), $i = 1, 2$ and $(\mathbf{NC}_{\mathcal{H}})$ as the "standard assumptions in the co-dimension-1 case".

8.1 Finding trajectories by differential inclusions

In order to introduce the set-valued map **BCL**, we first notice that all the equations in (7.1) have the form "$u_t + H(x,t,u,Du)$", which means that $b_i^t(x,s,\alpha_i) = -1$ for all $i = 1,2$ and all $(x,s,\alpha_i) \in \bar{\Omega}_i \times (0,T_f] \times A_i$. Therefore, for $i = 1,2$, $x \in \Omega_i$ and $t \in [0,T_f]$ we set

$$\mathbf{BCL}_i(x,t) := ((b_i,-1),c_i,l_i)(x,t,A_i)$$

and, for $x \in \mathbb{R}^N, t \in (0,T_f]$,

$$\mathbf{BCL}(x,t) := \begin{cases} \mathbf{BCL}_1(x,t) & \text{if } x \in \Omega_1\,, \\ \mathbf{BCL}_2(x,t) & \text{if } x \in \Omega_2\,, \\ \overline{\text{co}}(\mathbf{BCL}_1,\mathbf{BCL}_2)(x,t) & \text{if } x \in \mathcal{H}\,, \end{cases}$$

where $\overline{\text{co}}(E_1,E_2)$ denotes the closure of the convex hull of the sets E_1, E_2. Notice that here, since $\mathbf{BCL}_1$ and $\mathbf{BCL}_2$ have compact images, the convex closure reduces to the union of all possible convex combinations of elements.

For $t = 0$ we need to add more information: since we consider a finite horizon problem, we have to be able to stop the trajectory at time $s = 0$, and we want the initial condition $u(0) = u_0$ to be encoded through the Hamiltonian $H_{init}(x,u,Du) = u - u_0$. So, setting $\text{Init}(x) := \{(0,0),1,u_0(x)\}$, we are led to define

$$\mathbf{BCL}(x,0) := \begin{cases} \overline{\text{co}}(\mathbf{BCL}_1(x,0),\text{Init}(x)) & \text{if } x \in \Omega_1\,, \\ \overline{\text{co}}(\mathbf{BCL}_2(x,0),\text{Init}(x)) & \text{if } x \in \Omega_2\,, \\ \overline{\text{co}}(\mathbf{BCL}_1(x,0),\mathbf{BCL}_2(x,0),\text{Init}(x)) & \text{if } x \in \mathcal{H}\,. \end{cases} \tag{8.1}$$

At this stage, we have defined rigourously **BCL** following the general framework described in Part I – Chapter 4 but, since we are mainly in a case where $b^t = -1$, we are going to drop from now on the b^t-part in **BCL** and, in order to simplify the notations, we just write $b = b^x$. In fact, the only place where b^t plays a role is $t = 0$. Indeed, because of the convex hull, $\mathbf{BCL}(x,0)$ contains all the time dynamics $b^t \in [-1,0]$ However, in our case the initial conditions reduce to

$$u(x,0) \le (u_0)^*(x) \quad \text{and} \quad v(x,0) \ge u_0(x) \quad \text{in } \mathbb{R}^N\,,$$

for a subsolution u and a supersolution v, hence they produce no additional difficulty.

The very first checking in order to solve the control problem is the

Lemma 8.1.1 *The set-valued map* **BCL** *satisfies* ($\mathbf{H_{BCL}}$).

Proof — Concerning $(\mathbf{H_{BCL}})_{fund}$, the proof is quite straightforward by construction: first notice that since all the b_i, l_i, c_i are bounded by some constant $M > 0$, then it is the same for all the elements in **BCL**. Then, by construction $\mathbf{BCL}(x,t)$ is closed, hence compact, and it is convex. It remains to see that $(x,t) \mapsto \mathbf{BCL}(x,t)$ is upper semi-continuous which is clear since each $\mathbf{BCL}_i(x,t)$ is upper semi-continuous and we just make a convex hull of them.

We turn now to $(\mathbf{H_{BCL}})_{struct}$, which follows almost immediatly from (8.1): (i) is obviously satisfied by our choice for b^t which always belongs to $[-1,0]$. Point (ii) clearly holds if $s > 0$. Indeed, if we choose $K = M$ (the constant appearing in $(\mathbf{H_{BCL}})_{fund}$), since $b^t = -1$ for $s > 0$ we get the inequality. Now, if $s = 0$ the inequality comes from the fact that $-Kb^t + c \geq c = 1$. Point (iii) is included in (8.1) and point (iv) follows from the fact that this condition can only happen for $s = 0$ here (otherwise $b^t = -1$), in which case we have $\underline{c} = c = 1 > 0$.

Q.E.D.

Thanks to Theorem 4.2.1 (and recalling that we have dropped the $b^t = -1$ term), we solve the differential inclusion

$$\begin{cases} (\dot{X}, \dot{D}, \dot{L})(s) \in \mathbf{BCL}\big(X(s), t-s\big) & \text{for a.e. } s \in [0, +\infty)\,, \\ (X, D, L)(0) = (x, 0, 0)\,. \end{cases} \tag{8.2}$$

Notice that we have used the fact that $T(s) = t - s$ when the starting point of the (X,T)-trajectory is (x,t). As we saw in Chapter 4, we we mostly write

$$\begin{cases} \dot{X}(s) &= b\big(X(s), t-s\big) \\ \dot{D}(s) &= c\big(X(s), t-s\big) \\ \dot{L}(s) &= l\big(X(s), t-s\big) \end{cases} \tag{8.3}$$

in order to remember that b, c and l correspond to a specific choice in the set $\mathbf{BCL}(X(s), t-s)$, but when needed we will also introduce a control $\alpha(\cdot)$ to represent (b,c,l) as $(b,c,l)(X(s), t-s), \alpha(s))\,.$

Now the aim is to give a more precise description of each trajectory. For the sake of clarity, we denote by $(b_{\mathcal{H}}, c_{\mathcal{H}}, l_{\mathcal{H}})$ the (b,c,l) when $X(s) \in \mathcal{H}$ which are of course obtained through a convex combination of all the (b_i, c_i, l_i), $i = 1, 2$. So, in order to take this into account, we introduce the "extended control space"

$$A := A_1 \times A_2 \times \tilde{\Delta} \quad \text{where} \quad \tilde{\Delta} := \{(\mu_1, \mu_2) \in [0,1]^2 : \mu_1 + \mu_2 = 1\}\,,$$

and $\mathcal{A} := L^\infty(0, T_f; A)$. The extended control takes the form $a = (\alpha_1, \alpha_2, \mu_1, \mu_2)$ and if $x \in \mathcal{H}$,

$$(b_{\mathcal{H}}, c_{\mathcal{H}}, l_{\mathcal{H}}) = \mu_1(b_1, c_1, l_1) + \mu_2(b_2, c_2, l_2)\,,$$

with $\mu_1 + \mu_2 = 1$, where b_1, c_1, l_1 are computed at the point (x, t, α_1) and b_2, c_2, l_2 at the point (x, t, α_2).

Lemma 8.1.2 *For any trajectory (X, D, L) of (8.2) there exists a control $a(\cdot) = (\alpha_1, \alpha_2, \mu_1, \mu_2)(\cdot) \in \mathcal{A}$ such that*

$$\begin{aligned}(\dot X, \dot D, \dot L)(s) = {} & (b_1, c_1, l_1)(X(s), t-s, \alpha_1(s))\mathbb{1}_{\{X(s)\in\Omega_1\}} \\ & + (b_2, c_2, l_2)(X(s), t-s, \alpha_2(s))\mathbb{1}_{\{X(s)\in\Omega_2\}} \\ & + (b_{\mathcal{H}}, c_{\mathcal{H}}, l_{\mathcal{H}})(X(s), t-s, a(s))\mathbb{1}_{\{X(s)\in\mathcal{H}\}}\end{aligned}$$

and $b_{\mathcal{H}}(X(s), t-s, a(s)) \cdot e_N = 0$ for almost any $s \in (t, T_f)$ such that $X(s) \in \mathcal{H}$.

Proof — Given a trajectory, we apply Filippov's Lemma (cf. [11, Theorem 8.2.10]). To do so, we define the map $g : \mathbb{R}^+ \times A \to \mathbb{R}^N$ as follows

$$g(s,a) := \begin{cases} b_1\big(X(s), t-s, \alpha_1\big) & \text{if } X(s) > 0 \\ b_2\big(X(s), t-s, \alpha_2\big) & \text{if } X(s) < 0 \\ b_{\mathcal{H}}\big(X(s), t-s, a\big) & \text{if } X(s) = 0\,, \end{cases}$$

where $a = (\alpha_1, \alpha_2, \mu_1, \mu_2) \in A$.

We claim that g is a Caratheodory map. Indeed, it is first clear that, for fixed s, the function $a \mapsto g(s,a)$ is continuous. Then, in order to check that g is measurable with respect to its first argument we fix $a \in A$, an open set $\mathcal{O} \subset \mathbb{R}^N$ and evaluate

$$g_a^{-1}(\mathcal{O}) = \big\{s > 0 : g(s,a) \cap \mathcal{O} \neq \emptyset\big\}$$

that we split into three components, the first one being

$$g_a^{-1}(\mathcal{O}) \cap \{s > 0 : X(s) < 0\} = \big\{s > 0 : b_1(X(s), t-s, \alpha_1) \in \mathcal{O}\big\} \cap \{s > 0 : X(s) < 0\}\,.$$

Since the function $s \mapsto b_1(X(s), t-s, \alpha_1)$ is continuous, this set is the intersection of open sets, hence it is open and therefore measurable. The same argument works for the other components, namely $\{s > 0 : X(s) < 0\}$ and $\{s > 0 : X(s) = 0\}$ which finishes the claim.

The function $s \mapsto \dot X(s)$ is measurable and, for any s, the differential inclusion implies that

$$\dot X(s) \in g(s, A)\,,$$

therefore, by Filippov's Lemma, there exists a measurable map $a(\cdot) = (\alpha_1, \alpha_2, \mu_1, \mu_2)(\cdot) \in \mathcal{A}$ such that (8.4) is fulfilled. In particular, by the definition of g, we have for a.e. $s \in [0, T_f]$

$$\dot{X}(s) = \begin{cases} b_1\big(X(s), t-s, \alpha_1(s)\big) & \text{if } X(s) > 0 \\ b_2\big(X(s), t-s, \alpha_2(s)\big) & \text{if } X(s) < 0 \\ b_{\mathcal{H}}\big(X(s), t-s, a(s)\big) & \text{if } X(s) = 0. \end{cases} \tag{8.4}$$

The last property is a consequence of Stampacchia's theorem (see for instance [102]): setting $y(s) := X_N(s)$, then $\dot{y}(s) = 0$ almost everywhere on the set $\{y(s) = 0\}$. But $\dot{y}(s) = b_{\mathcal{H}}(X(s), t-s, a(s)) \cdot e_N$ on this set, so the conclusion follows.

Q.E.D.

8.2 The $\mathbf{U}^-$ value-function

Solving (8.2) with **BCL** yields a set $\mathcal{T}(x,t)$ of all admissible trajectories, without specific condition on $\mathcal{H}$ for (7.1) (see Section 4.2.3). Changing slightly the notations of this section to emphasize the role of the control $a(\cdot)$, we first define the value function

$$\mathbf{U}^-(x,t) := \inf_{\mathcal{T}(x,t)} \left\{ \int_0^t l(X(s), t-s, a(s)) \exp(-D(s))\, \mathrm{d}s + u_0(X(t)) \exp(-D(t)) \right\},$$

and the aim is now to prove that $\mathbf{U}^-$ is a viscosity solution of (7.1). To do so, we use the control approach described in Section 4.2: recalling that we use the notation b for b^x, the "global" Hamiltonian is given by

$$\mathbb{F}(x,t,u,(p_x,p_t)) := \sup_{(b,c,l)\in\mathbf{BCL}(x,t)} \big(-(b,-1)\cdot(p_x,p_t) + cu - l \big).$$

Writing p for p_x in order to simplify the notations, we decompose

$$\mathbb{F}(x,t,u,(p_x,p_t)) = p_t + H(x,t,u,p),$$

where $H(x,t,u,p) = H_i(x,t,u,p)$ if $x \in \Omega_i$ for $i = 1,2$. By the upper-semicontinuity of **BCL**, H and $\mathbb{F}$ are upper-semi-continuous and we have the

Lemma 8.2.1 *If $x \in \mathcal{H}$ then, for all $t \in [0,T_f]$, $r \in \mathbb{R}$, $p_x = p \in \mathbb{R}^N$*

$$H(x,t,r,p) = \max\Big(H_1(x,t,r,p), H_2(x,t,u,p)\Big).$$

As a direct consequence, for any $x \in \mathcal{H}$, $t \in [0, T_f]$, $u \in \mathbb{R}$, $p_x = p \in \mathbb{R}^N$, $p_t \in \mathbb{R}$

$$\mathbb{F}(x,t,u,(p_x,p_t)) = \max\left(p_t + H_1(x,t,u,p), p_t + H_2(x,t,u,p)\right) ,$$
$$\mathbb{F}_*(x,t,u,(p_x,p_t)) = \min\left(p_t + H_1(x,t,u,p), p_t + H_2(x,t,u,p)\right) .$$

Proof — If $(b,c,l) \in \mathbf{BCL}(x,t)$, it can be written as a convex combination of some $(b_i,c_i,l_i) \in \mathbf{BCL}_i(x,t)$, $i = 1,2$, and thefore the same is true for $-b\cdot p + cu - l$, namely

$$-b\cdot p + cr - l = \sum_i \mu_i(-b_i\cdot p + c_i r - l_i) ,$$

for some $0 \le \mu_i \le 1$ with $\sum_i \mu_i = 1$. Since $(-b_i\cdot p + c_i r - l_i) \le H_i(x,t,u,p)$, we deduce that $-b\cdot p + cr - l \le \max\left(H_1(x,t,r,p), H_2(x,t,u,p)\right)$ and therefore

$$H(x,t,r,p) \le \max\Big(H_1(x,t,r,p), H_2(x,t,u,p)\Big) .$$

But $H(x,t,r,p) \ge (-b_i\cdot p + c_i r - l_i)$ for any $(b_i,c_i,l_i) \in \mathbf{BCL}_i(x,t)$ so that $H(x,t,r,p) \ge H_i(x,t,r,p)$ for $i = 1,2$. The representation of H as the max follows immediately.

Concerning $\mathbb{F}$, the first equality (as a maximum) is trivial and the representation formula for $\mathbb{F}_*$ derives directly from its definition as the lim inf, knowing that of course H_1 and H_2 are both continuous up to $\mathcal{H}$.

Q.E.D.

Then, by using all the results of Section 4.2, we have the

Proposition 8.2.2 *Assume that the "standard assumptions in the co-dimension-*1 *case" are satisfied. Then the value function* $\mathbf{U}^-$ *is an Ishii viscosity solutions of* (7.1). *Moreover* $\mathbf{U}^-$ *is the minimal supersolution of* (7.1).

We leave the proof of the reader since it immediately follows from Theorem 4.3.4 and Corollary 4.4.3. This result gives a good amount of information on $\mathbf{U}^-$ but not all of them.

To go further, we have to examine more carefully the viscosity inequality on $\mathcal{H}$ which is done in the next section. However, in order to do so we need first to make sure that $(\mathbf{U}^-)^*$ is regular in the sense of Definition 3.4.1. We provide below a direct "control proof" of this fact but for a pde proof, the reader can also check that Proposition 3.4.2 applies here since we assume $(\mathbf{NC}_{\mathcal{H}})$. Notice also that the proof below only uses "outward normal controlability" both from Ω_1 and Ω_2.

Lemma 8.2.3 *Assume that the "standard assumptions in the co-dimension-*1 *case" are satisfied, then*

$$((\mathbf{U}^-)_{|\mathcal{H}\times(0,T_f)})^* = (\mathbf{U}^-)^* \quad \text{on } \mathcal{H}\times(0,T_f)\,,$$

where $(\mathbf{U}^-)_{|\mathcal{H}\times(0,T_f)}$ *denotes the restriction to* $\mathcal{H}\times(0,T_f)$ *of* $\mathbf{U}^-$.

Proof — Let $(x,t)\in\mathcal{H}\times(0,T_f)$. By definition of $(\mathbf{U}^-)^*$, there exists a sequence $(x_n,t_n)\to(x,t)$ such that $\mathbf{U}^-(x_n,t_n)\to(\mathbf{U}^-)^*(x,t)$. The statement of Lemma 8.2.3 means that we can assume that $x_n\in\mathcal{H}$. Indeed, if $x_n\in\Omega_1$, we use the normal controllability assumption $(\mathbf{NC}_{\mathcal{H}})$ at (x,t): there exists $\delta>0$ and a control α_1 such that $b_1(x,t,\alpha_1)\cdot e_N=-\delta<0$. Considering the trajectory with constant control α_1

$$\dot{Y}(s)=b_1(Y(s),t_n-s,\alpha_1)\quad,\quad Y(0)=x_n, \tag{8.5}$$

it is easy to show that τ_n^1, the first exit time of the trajectory Y from Ω_1 tends to 0 as $n\to+\infty$. By the Dynamic Programming Principle, denoting $(\tilde{x}_n,\tilde{t}_n)=(X(\tau_n^1),t-\tau_n^1)$, we have

$$\mathbf{U}^-(x_n,t_n)\le\int_0^{\tau_n^1} l\big(Y(s),t_n-s,\alpha_1\big)\,e^{-D(s)}\,\mathrm{d}s+\mathbf{U}^-(\tilde{x}_n,\tilde{t}_n)\,e^{-D(\tau_n^1)}=\mathbf{U}^-(\tilde{x}_n,\tilde{t}_n)+o_n(1)\,,$$

where $o_n(1)\to0$. Therefore $(\tilde{x}_n,\tilde{t}_n)\to(x,t)$, $\mathbf{U}^-(\tilde{x}_n,\tilde{t}_n)\to(\mathbf{U}^-)^*(x,t)$ and $\tilde{x}_n\in\mathcal{H}$, which is exactly what we wanted to prove. The same results holds if $x_n\in\Omega_2$ using a control such that $b_2(x,t,\alpha_2)\cdot e_N=\delta>0$.

Q.E.D.

8.3 The complemented equations

This section is motivated in particular by Lemma 8.1.2 where the term $(b_{\mathcal{H}},c_{\mathcal{H}},l_{\mathcal{H}})$ plays a key role as a coupling between the control problems in Ω_1 and Ω_2.

Following Section 5.4, we introduce the tangential elements in **BCL** which maintain the trajectories on $\mathcal{H}$: for any $x\in\mathcal{H}$, $t\in[0,T_f]$, we set

$$\mathbf{BCL}_T(x,t):=\big\{(b,c,l)\in\mathbf{BCL}(x,t):b\cdot e_N=0\big\}\,.$$

Similarly we define $\mathbf{B}_T(x,t)$ for the set-valued map of tangential dynamics: any $b\in\mathbf{B}_T(x,t)$ can be expressed as a convex combination

$$b=\mu_1b_1+\mu_2b_2 \tag{8.6}$$

for which $(\mu_1 b_1 + \mu_2 b_2)\cdot e_N = 0$ with $\mu_1+\mu_2=1$, $\mu_1,\mu_2\in[0,1]$. We also introduce tangential Hamiltonian which was already considered

$$H_T(x,t,u,p) := \sup_{\mathbf{BCL}_T(x,t)} \left\{ -b\cdot p + cu - l \right\}. \tag{8.7}$$

Notice that $p_t + H_T(x,t,u,p) = \mathbb{F}^N(x,t,u,(p,p_t))$ on $\mathbf{M}^N = \mathcal{H}\times(0,T_f)$ and, by Lemma 5.5.2 with $k=N$, the Hamiltonian H_T satisfies **(TC)**; in particular, $\mathcal{H}_T$ is continuous in x,t, uniformly with respect to (u,p) in compact sets. Such property can also be obtained by using the representation formula given by Lemma 10.2.1.

Before deriving an H_T-subsolution property, we need first the following preliminary result which allows us to build trajectories which remains on $\mathcal{H}$, at least for some time.

Lemma 8.3.1 *Let $(x,t)\in\mathcal{H}\times(0,T_f)$ and $(b,c,l)\in\mathbf{BCL}_T(x,t)$, obtained as a convex combination $(b,c,l)=\mu_1(b_1,c_1,l_1)+\mu_2(b_2,c_2,l_2)$. If*

$$(b_1(x,t,\alpha_1)\cdot e_N)\ \cdot\ (b_2(x,t,\alpha_1)\cdot e_N) < 0\,,$$

there exists a neighborhood $\mathcal{V}$ of (x,t) in $\mathcal{H}\times(0,T_f)$ and a Lipschitz continuous map $\psi:\mathcal{V}\to\mathbb{R}^N\times\mathbb{R}\times\mathbb{R}$, such that $\psi(x,t)=(b,c,l)$ and $\psi(y,s)=(\tilde b(y,s),\tilde c(y,s),\tilde l(y,s))\in\mathbf{BCL}_T(y,s)$ for any $(y,s)\in\mathcal{V}$.

Proof — Our assumption means that

$$(\mu_1 b_1(x,t,\alpha_1)+\mu_2 b_2(x,t,\alpha_2))\cdot e_N = 0\ .$$

Now, if (y,s) is close enough to (x,t) we set

$$\mu_1^\sharp(y,s) := \frac{b_2(y,s,\alpha_2)\cdot e_N}{(b_2(y,s,\alpha_1)-b_1(y,s,\alpha_1))\cdot e_N}\,,\quad \mu_2^\sharp := 1-\mu_1^\sharp\,.$$

By this choice we have $0\le\mu_1^\sharp,\mu_2^\sharp\le 1$ and $\Big(\mu_1^\sharp(y,s)b_1(y,s,\alpha_1)+\mu_2^\sharp(y,s)b_2(y,s,\alpha_2)\Big)\cdot e_N=0$, which yileds a tangential dynamic which is well-defined as long as $(b_2(y,s,\alpha_1)-b_1(y,s,\alpha_1))\cdot e_N\neq 0$. In particular this is true in a neighborhood of (x,t).

Then the function ψ given by

$$\psi(y,s) := \mu_1^\sharp(y,s)(b_1,c_1,l_1)+\mu_2^\sharp(y,s)(b_2,c_2,l_2)\ ,$$

satisfies all the desired properties: it is Lipschitz continuous since b_1,b_2 are Lipschitz continuous in x,t and since $\mu_1^\sharp(x,t)=\mu_1$, $\mu_2^\sharp(x,t)=\mu_2$, $\psi(x,t)=(b,c,l)$.

Q.E.D.

We now prove that a complementary subsolution inequlity holds on $\mathcal{H}$:

Proposition 8.3.2 *Assume that the "standard assumptions in the co-dimension-*1 *case" are satisfied. Then the value function* $\mathbf{U}^-$ *satisfies the viscosity inequality*

$$(\mathbf{U}^-)^*_t + H_T\Big(x,t,(\mathbf{U}^-)^*, D_T(\mathbf{U}^-)^*\Big) \leq 0 \quad \text{on } \mathcal{H}\times(0,T_f)\,.$$

We point out that in Proposition 8.3.2, the $\mathcal{H}\times(0,T_f)$-viscosity inequality means that we look at maximum points of $(\mathbf{U}^-)^* - \phi$ on $\mathcal{H}\times(0,T_f)$ where ϕ is a smooth test-function on $\mathcal{H}\times(0,T_f)$.

Remark 8.3.3 *In other words,* $\mathbf{U}^-$ *is an Ishii solution satisfying a complemented* H_T*-inequality on* $\mathcal{H}$*. As we will see in Part IV, this can be interpreted as* $\mathbf{U}^-$ *being a stratified solution of the problem. We will actually prove that it is the unique stratified solution.*

Proof — If ϕ is a smooth test-function on $\mathcal{H}\times(0,T_f)$, we have to prove that, if $(x,t)\in\mathcal{H}\times(0,T_f)$ is a maximum point on $\mathcal{H}\times(0,T_f)$ of $(\mathbf{U}^-)^*-\phi$, then (assuming without loss of generality that $(\mathbf{U}^-)^*(x,t)=\phi(x,t)$),

$$\phi_t(x,t) + H_T(x,t,\phi(x,t),D_T\phi(x,t)) \leq 0 \quad \text{on } \mathcal{H}\times(0,T_f)\,.$$

(a) *Using the dymamic programming principle* — By Lemma 8.2.3, we can pick a sequence $(x_n,t_n)\to(x,t)$ such that $\mathbf{U}^-(x_n,t_n)\to(\mathbf{U}^-)^*(x,t)$ with $x_n\in\mathcal{H}$ for all $n\in\mathbb{N}$. By the dynamic programming principle, for any $\tau>0$ and any trajectory (X_n,a_n) in $\mathcal{T}(x_n,t_n)$ we have

$$\mathbf{U}^-(x_n,t_n) \leq \int_0^\tau l\big(X_n(s),t_n-s,a_n(s)\big)\,e^{-D_n(s)}\,\mathrm{d}s + \mathbf{U}^-(X_n(\tau),t_n-\tau)\,e^{-D_n(\tau)}\,. \quad (8.8)$$

Our aim is to show that this inequality implies

$$\phi_t(x,t) - b\cdot D\phi(x,t) + c\phi(x,t) - l \leq 0\,,$$

for any $(b,c,l)\in\mathbf{BCL}_T(x,t)$, which will give the conclusion $H_T\leq 0$. However, replacing $\mathbf{U}^-$ by ϕ above can be done only for trajectories which stay on $\mathcal{H}$, at least for some interval $[0,\tau]$.

(b) *Constructing a trajectory which stays on* $\mathcal{H}$ — We start from the fact that by definition of $\mathbf{BCL}_T(x,t)$, (b,c,l) can be expressed as a convex combination of the (b_i,c_i,l_i) for $i=1,2$, namely

$$(b,c,l) = \mu_1(b_1,c_1,l_1) + \mu_2(b_2,c_2,l_2)$$

with $\mu_1 + \mu_2 = 1$, $\mu_1, \mu_2 \in [0,1]$ and $(\mu_1 b_1 + \mu_2 b_2) \cdot e_N = 0$. We denote by α_i the control which is associated to (b_i, c_i, l_i)

Slightly modifying b_1 and b_2 by using the normal controllability on $\mathcal{H}$, we may assume without loss of generality that $b_1 \cdot e_N \neq 0$ and $b_2 \cdot e_N \neq 0$ while keeping $(\mu_1 b_1 + \mu_2 b_2) \cdot e_N = 0$. Therefore, either $b_1 \cdot e_N < 0 < b_2 \cdot e_N$ or $b_1 \cdot e_N > 0 > b_2 \cdot e_N$ but in both cases Lemma 8.3.1 provides us with a function ψ that we use to solve the ode

$$(\dot{X}_n(s), \dot{D}_n(s), \dot{L}_n(s)) = \psi(X_n(s), t_n - s)\ ,$$

with $(X_n(0), D_n(0), L_n(0)) = (x_n, 0, 0)$.

Because of the properties of ψ, the Cauchy-Lipschitz Theorem implies that there exists a unique solution which, for (x_n, t_n) close enough to (x,t), is defined on a small but fixed (*i.e.* independent of n) interval of time $[0,\tau]$ and $(X_n, D_n, L_n) \in \mathcal{T}(x_n, t_n)$ for any n. Moreover, $X_n \in \mathcal{H}$ on $[0,\tau]$.

(c) *Deriving the tangential inequality* — Since $\mathbf{U}^-(x_n,t_n) = (\mathbf{U}^-)^*(x,t) + o_n(1) = \phi(x,t) + o_n(1)$ while $\mathbf{U}^- \leq \phi$ on $\mathcal{H} \times (0, T_f)$, using X_n in (8.8) we get

$$\phi(x_n,t_n) + o_n(1) \leq \int_0^\tau \dot{L}_n(s)\, e^{-D_n(s)}\, \mathrm{d}s + \phi(X_n(\tau), t_n - \tau)\, e^{-D_n(\tau)}\,. \tag{8.9}$$

We first let n tend to infinity. Due to the Lipschitz property of ψ, up to extraction we see that $(X_n, D_n, L_n) \to (X, D, L)$ in $W^{1,\infty}$ where at least on $[0,\tau]$,

$$(\dot{X}(s), \dot{D}(s), \dot{L}(s)) = \psi(X(s), t - s)\ ,$$

$X(s) \in \mathcal{H}$ for any $s \in [0,\tau]$ and $(X(0), D(0), L(0)) = (x, 0, 0)$. So, passing to the limit in (8.9) yields

$$\phi(x,t) \leq \int_0^\tau \dot{L}(s)\, e^{-D(s)}\, \mathrm{d}s + \phi(X(\tau), t - \tau)\, e^{-D(\tau)}\,.$$

On the other hand, since ϕ is smooth on $\mathcal{H} \times (0, T_f)$, the following expansion holds:

$$\phi(X(\tau), t - \tau))e^{-D(\tau)} = \phi(x,t) + \int_0^\tau \Big(D_x\phi(\xi_s)\dot{X}(s) - \partial_t\phi(\xi_s) - \dot{D}(s)\phi(\xi_s)\Big) e^{-D(s)}\ \mathrm{d}s$$

where ξ_s stands for $(X(s), t - s)$. Combining both integrals, we arrive at

$$0 \leq \int_0^\tau \Big(-\partial_t\phi(\xi_s) + \dot{X}(s) \cdot D\phi(\xi_s) - \dot{D}(s)\phi(\xi_s) + \dot{L}(s)\, e^{-D(s)}\Big) \exp(-D(s))\, \mathrm{d}s\,.$$

Finally, after divinding by τ and sending $\tau \to 0$ the conclusion follows from the fact that ψ is continuous and $\psi(x,t) = (\dot{X}(0), \dot{D}(0), \dot{L}(0)) = (b,c,l)$: we get

$$\phi_t(x,t) - b \cdot D\phi(x,t) + c\phi(x,t) - l \leq 0$$

for any $(b,c,l) \in \mathbf{BCL}_T(x,t)$, which implies that $H_T(x,t,\phi,D\phi) \leq 0$.

Q.E.D.

8.4 A characterization of $\mathbf{U}^-$

The previous section showed that $\mathbf{U}^-$ satisfies an additional subsolution inequality on $\mathcal{H} \times (0,T_f)$. The aim of this section is to prove that this additional inequality is enough to characterize it.

The precise result is the

Theorem 8.4.1 *Assume that the "standard assumptions in the co-dimension-*1 *case" are satisfied. Then* $\mathbf{U}^-$ *is the unique Ishii solution of* (7.1) *such that*

$$u_t + H_T(x,t,u,D_T u) \leq 0 \quad \text{on} \quad \mathcal{H} \times (0,T_f)\,. \tag{8.10}$$

Proof — The proof is obtained by a combination of arguments which will also be used in Part IV for stratified problems.

We recall that we already know (cf. Proposition 8.2.2) that $\mathbf{U}^-$ is the minimal Ishii supersolution of (7.1). Therefore we only need to compare $\mathbf{U}^-$ with subsolutions u such that $u_t + H_T(x,t,u,D_T u) \leq 0$ on $\mathcal{H} \times (0,T_f)$, showing that $\mathbf{U}^- \geq u$ in $\mathbb{R}^N \times [0,T]$.

Though the proof can be reduced to a mere list of several arguments already exposed in Part I, we provide below more explanations and redo most of them in the simpler hyperplane context for the readers's convenience.

Step 1: *Reduction to a local comparison result* **(LCR)** – As already noticed in Part I (see Remarks on page 28), setting $\tilde{u}(x,t) := \exp(Kt)u(x,t)$ for $K > 0$ large enough allows to reduce the proof to the case where $c_i \geq 0$ for any $(b_i,c_i,l_i) \in \mathbf{BCL}_i(x,t)$, $i = 1,2$. As a consequence, we can assume that the H_i $(i = 1,2)$ are nondecreasing in the u-variable, and that H_T enjoys the same property.

Then, rewriting here some arguments already given in Section 3.2 and using that the c_i are positive, we notice that, for $\delta > 0$ small enough, $\psi(x,t) = -\delta(1+|x|^2)^{1/2} -$

$\delta^{-1}(1+t)$ is not only a $\delta/2$-strict subsolution (7.1), but also for the H_T-equation on $\mathcal{H}\times(0,T_f)$ and we can also assume that $\psi \le u$ in $\mathbb{R}^N\times[0,T_f]$. For $\mu\in(0,1)$, setting

$$u_\mu(x,t) := \mu u(x,t) + (1-\mu)\psi(x,t)$$

yields an η-strict subsolution u_μ for some $\eta(\mu,\delta)>0$. By this, we mean that each inequality in (7.1) is η-strict for u_μ but also that $(u_\mu)_t + H_T(x,t,u_\mu,Du_\mu) \le \eta < 0$ on $\mathcal{H}\times(0,T_f)$. This claim is obvious for the initial data, let us prove it for instance for H_1.

Using the convexity property of H_1 in r,p, we get successively

$$\begin{aligned}
&(u_\mu)_t + H_1(x,t,u_\mu,Du_\mu)\\
&= \mu u_t + (1-\mu)\psi_t + H_1(x,t,\mu u + (1-\mu)\psi, \mu Du + (1-\mu)D\psi)\\
&\le \mu u_t + (1-\mu)\psi_t + \mu H_1(x,t,u,Du) + (1-\mu)H_1(x,t,\psi,D\psi)\\
&\le \mu\big\{u_t + H_1(x,t,u,Du)\big\} + (1-\mu)\big\{\psi_t + H_1(x,t,\psi,D\psi)\big\}\\
&\le \mu\big\{u_t + H_1(x,t,u,Du)\big\} - (1-\mu)(\delta/2) \le -(1-\mu)(\delta/2) < 0\,.
\end{aligned}$$

The same is valid for H_2 and H_T for similar reasons. Moreover, by construction $u_\mu - \mathbf{U}^- \to -\infty$ as $|x|\to+\infty$ since $\psi(x,t)\to-\infty$ as $|x|\to+\infty$, so that (**LOC1**) is satisfied for any of those Hamiltonians.

Checking (**LOC2**) is easier: if we are looking for a comparison result around the point (x_0,t_0), it is enough to use

$$u_{\delta'}(x,t) := u(x,t) - \delta'(|x-x_0|^2 + |t-t_0|^2)$$

for $\delta'>0$ small enough. Thus we are in the situation where a **(LCR)** is enough to ensure a **(GCR)**.

In order to prove that **(LCR)** holds, we introduce $Q^{x,t}_{r,h}$, a (small) cylinder around (x,t) where we want to perform the **(LCR)**. Notice that of course, if $x\in\Omega_1$ or Ω_2, then taking r small enough reduces the proof to the standard comparison result since in this case, $Q^{x,t}_{r,h}$ does not intersect with $\mathcal{H}$. Thus, we assume in the following that $x\in\mathcal{H}$. Our aim is to use Lemma 5.4.1 with $\mathcal{M} := (\mathcal{H}\times[0,T_f]) \cap \overline{Q^{x,t}_{r,h}}$ and $\mathbb{F}^{\mathcal{M}}(x,t,r,(p_x,p_t)) := p_t + H_T(x,t,r,p_x)$.

Step 2: *Approximation of the subsolution* – We wish to use an approximation by convolutions (inf-convolution and usual convolution with a smoothing kernel) for the subsolution as in Proposition 3.4.7; to do so, we introduce a slightly larger cylinder $Q^{x,t}_{r',h'}$ where $r'>r$ and $h'>h$ are fixed in order to have some "room" for those convolutions. From Step 1, we know that u_μ is an η-strict subsolution of (7.1) in $Q^{x,t}_{r',h'}$ for some $\eta=\eta(\mu,\delta)$.

Since (**H**$_{\mathbf{Conv}}$), **(NC)**, **(TC)** and (**Mon**-u) are satisfied for all the Hamiltonians, we deduce from Proposition 3.4.7 that there exists a sequence $(u_{\mu,\varepsilon})_\varepsilon$ of $C^0(\overline{Q^{x,t}_{r,h}})\cap C^1(\mathcal{M})$ functions which are all $(\eta/2)$-strict subsolutions of (7.1) in some smaller cylinder $Q(\varepsilon) \subset Q^{x,t}_{r',h'}$, and $Q(\varepsilon) \to Q^{x,t}_{r',h'}$ as $\varepsilon \to 0$ in the sense of the euclidian distance in $\mathbb{R}^{N+1}$. Hence, for ε small enough, we can assume with no restriction that $Q^{x,t}_{r,h} \subset Q(\varepsilon) \subset Q^{x,t}_{r',h'}$ so that $u_{\mu,\varepsilon}$ is an $(\eta/2)$-strict subsolution in $Q^{x,t}_{r,h}$.

This has two consequences:

(a) for any $\varepsilon > 0$ small enough, $(u_{\mu,\varepsilon})_t + H_T(x,t,u_{\mu,\varepsilon}, D_T u_{\mu,\varepsilon}) \leq -\eta/2 < 0$ in $\mathcal{M}$ and in a classical sense since $u_{\mu,\varepsilon}$ is C^1 on $\mathcal{M}$;

(b) since $u_{\mu,\varepsilon}$ is an $(\eta/2)$-strict subsolution in $\mathcal{O} := Q^{x,t}_{r,h} \setminus \mathcal{M}$ (for the Hamiltonians H_1, H_2) and a **(LCR)** holds there, we use the subdynamic programming principle for subsolutions (cf. Theorem 5.3.3) which implies that each $u_{\mu,\varepsilon}$ satisfies an $(\eta/2)$-strict dynamic programming principle in $Q^{x,t}_{r,h}[\mathcal{M}^c]$.

These two properties allow us to make a **(LCR)** in $Q^{x,t}_{r,h}$ in the final step.

Step 3: *Performing the local comparison* – From the previous step we know that for each $\varepsilon > 0$, $u = u_{\mu,\varepsilon}$ satisfies the hypotheses of Lemma 5.4.1. Using $v := \mathbf{U}^-$ as supersolution in this lemma, we deduce that

$$\forall (y,s) \in \overline{Q^{x,t}_{r,h}} \setminus \partial_P Q^{x,t}_{r,h}\,, \quad (u_{\mu,\varepsilon} - \mathbf{U}^-)(y,s) < \max_{\overline{Q^{x,t}_{r,h}}} (u_{\mu,\varepsilon} - \mathbf{U}^-)\,.$$

Using that $u_\mu = \limsup{}^*\, u_{\mu,\varepsilon}$, this yields a local comparison result (with inequality in the large sense) between u_μ and $\mathbf{U}^-$ as $\varepsilon \to 0$. By step 1, we deduce that the **(GCR)** holds: $u_\mu \leq \mathbf{U}^-$ in $\mathbb{R}^N \times [0,T_f]$, and sending finally $\mu \to 1$ gives that $u \leq \mathbf{U}^-$.

The conclusion is that if u is an Ishii solution such that $u_t + H_T(x,t,u,D_T u) \leq 0$ on $\mathcal{H}$, necessarily $u \equiv \mathbf{U}^-$, which ends the proof.

Q.E.D.

Chapter 9

A Less Natural Value-Function, Regular and Singular Dynamics

While studing $\mathbf{U}^-$ we introduced the set $\mathbf{BCL}_T$, containing the dynamics tangent to $\mathcal{H}$ in order to examining the trajectories which remain on $\mathcal{H}$. The new point in this section is to remark that there are two different kinds of dynamics that allow to stay on $\mathcal{H}$, leading to the construction of a second value function.

9.1 Introducing $\mathbf{U}^+$

Let us first begin with regular trajectories:

Definition 9.1.1 *We say that $b \in \mathbf{B}_T(x,t)$ is regular if $b = \mu_1 b_1 + \mu_2 b_2$ while the condition $b_1 \cdot e_N \leq 0 \leq b_2 \cdot e_N$ holds. We denote by*

$$\mathbf{BCL}_T^{\mathrm{reg}}(x,t) := \left\{ (b,c,l) \in \mathbf{BCL}_T(x,t) : b \text{ is regular } \right\}$$

the set containing the regular tangential dynamics, and $\mathcal{T}^{\mathrm{reg}}(x,t)$ the set of controlled trajectories with regular dynamics on $\mathcal{H}$, i.e.

$$\mathcal{T}^{\mathrm{reg}}(x,t) := \Big\{ (X,D,L) \text{ solution of (8.2) such that } \dot{X}(s) \in \mathbf{B}_T^{\mathrm{reg}}(X(s), t-s) \text{ a.e. when } X(s) \in \mathcal{H} \Big\}.$$

In other terms, a regular dynamic corresponds to a "push-push" strategy: the trajectory is maintained on $\mathcal{H}$ because it is pushed on $\mathcal{H}$ from both sides, using only dynamics coming from Ω_1 and Ω_2; we may also have tangent dynamics, i.e. $b_1 \cdot e_N = b_2 \cdot e_N = 0$.

On the contrary, the dynamic is said *singular* if $b_1 \cdot e_N > 0$ and $b_2 \cdot e_N < 0$, which is a "pull-pull" strategy, a quite instable situation where the trajectory remains on $\mathcal{H}$ because each side pulls in the opposite direction. We also recall the notations (8.3) that we use throughout this chapter.

We remark that, by $(\mathbf{NC}_{\mathcal{H}})$, the sets $\mathbf{BCL}_T(x,t)$ and $\mathbf{BCL}^{\rm reg}_T(x,t)$ are non-empty for any $(x,t) \in \mathcal{H}$ (see Lemma 8.3.1). Next, for $(x,t) \in \mathcal{H} \times (0,T_f)$, $r \in \mathbb{R}$ and $p = (p',0) \in \mathbb{R}^N$, we define a second tangential Hamiltonian

$$H^{\rm reg}_T(x,t,r,p) := \sup_{\mathbf{BCL}^{\rm reg}_T(x,t)} \left\{ -b \cdot p + cu - l \right\}, \tag{9.1}$$

and a second value-function can be defined by minimizing only on regular trajectories:

$$\mathbf{U}^+(x,t) := \inf_{\mathcal{T}^{\rm reg}(x,t)} \left\{ \int_0^\infty l(X(s), t-s, a(s)) \exp(-D(s))\, \mathrm{d}s \right\}.$$

Of course it is clear that $\mathbf{U}^- \le \mathbf{U}^+$ in $\mathbb{R}^N \times [0,T_f]$ but we are going to prove more interesting properties on $\mathbf{U}^+$.

The Hamiltonian $H^{\rm reg}_T$ satisfies **(TC)** on $\mathcal{H} \times [0,T_f]$; in particular, $H^{\rm reg}_T$ is continuous with respect to (x,t). Contrarily to $\mathcal{H}_T$, this does not follow directly from Lemma 5.5.2, but a carefull look at the proof will convince the reader that the arguments also apply to $H^{\rm reg}_T$. As it is the case for H_T, an alternative proof consists in using the representation formulas given by Lemma 10.2.1.

Proving the dynamic programming principle for $\mathbf{U}^+$ is done as for $\mathbf{U}^-$ (see Theorem 4.3.3), but using regular trajectories. So, we skip the proof of the

Lemma 9.1.2 *Under hypothesis* $(\mathbf{H_{BCL}})$*, the value function* $\mathbf{U}^+$ *satisfies*

$$\mathbf{U}^+(x,t) = \inf_{\mathcal{T}^{\rm reg}(x,t)} \left\{ \int_0^\theta l\big(X(s), t-s, a(s)\big) \exp(-D(s))\, \mathrm{d}s + \mathbf{U}^+\big(X(\theta), t-\theta\big) \exp(-D(\theta)) \right\},$$

for any $(x,t) \in \mathbb{R}^N \times (0,T_f]$, $\theta > 0$.

The dynamic programming principle naturally leads to a system of pde's satisfied by $\mathbf{U}^+$. But before proving this result, we want to make the following important remark: most of the results we provided in the previous chapter for $\mathbf{U}^-$ were more or less direct consequences of results given in Chapter 4, in particular all the supersolution inequalities using Lemma 8.2.1. However, this is not the case for $\mathbf{U}^+$ which requires specific adaptations.

Proposition 9.1.3 *Assume that the "standard assumptions in the co-dimension-*1 *case" are satisfied. Then the value function* $\mathbf{U}^+$ *is an Ishii solution of* (7.1)*. Moreover* $\mathbf{U}^+$ *satisfies on* $\mathcal{H}\times(0,T_f)$ *the inequality*

$$(\mathbf{U}^+)^*_t + H_T^{\rm reg}(x,t,(\mathbf{U}^+)^*, D_T(\mathbf{U}^+)^*) \le 0 \quad \text{on } \mathcal{H}\times(0,T_f)\,.$$

Proof — Of course, the only difficulties comes from the discontinuity on $\mathcal{H}\times(0,T_f)$, therefore we concentrate on this case.

(a) *Ishii supersolution condition in* $\mathbb{R}^N$ — Since a priori $\mathbf{U}^+$ is not continuous, we have to use semi-continuous envelopes as we did for $\mathbf{U}^-$. In order to prove that $(\mathbf{U}^+)_*$ is a supersolution we assume that $(x,t)\in\mathcal{H}\times(0,T_f)$ is a strict local minimum point of $(\mathbf{U}^+)_*-\phi$ where ϕ is a smooth test-function in $\mathbb{R}^N\times(0,T_f)$, and we can suppose w.l.o.g that $(\mathbf{U}^+)_*(x,t)=\phi(x,t)$.

The first part consists in using the dynamic programming principle and follows the same lines as several proofs we already established so we condense a little bit some of the arguments below. By definition of $(\mathbf{U}^+)_*$, there exists a sequence (x_n,t_n) which converges to (x,t) such that $\mathbf{U}^+(x_n,t_n)\to(\mathbf{U}^+)_*(x,t)$ and by the dynamic programming principle,

$$\mathbf{U}^+(x_n,t_n) = \inf_{\mathcal{T}^{\rm reg}(x_n,t_n)}\Big\{\int_0^\tau \dot L_n(s)\,e^{-D(s)}\,{\rm d}s + \mathbf{U}^+\big(X_n(\tau),t_n-\tau\big)\,e^{-D(\tau)}\Big\}\,,$$

where $\tau\ll 1$ and the n-index is to recall that this trajectory is associated with $X_n(0)=x_n$. We use that (i) $\mathbf{U}^+(x_n,t_n)=(\mathbf{U}^+)_*(x,t)+o_n(1)$ where $o_n(1)\to 0$, (ii) $\mathbf{U}^+\big(X_n(\tau),t_n-\tau\big)\ge(\mathbf{U}^+)_*\big(X_n(\tau),t_n-\tau\big)$ and (iii) the minimum point property, to obtain

$$\phi(x_n,t_n)+o_n(1) \ge \inf_{\mathcal{T}^{\rm reg}(x_n,t_n)}\Big\{\int_0^\tau \dot L_n(s)\,e^{-D(s)}\,{\rm d}s + \phi\big(X_n(\tau),t_n-\tau\big)\,e^{-D(\tau)}\Big\}\,.$$

Next we use the expansion of ϕ along the trajectory of the differential inclusion, writing $\xi_s=(X_n(s),t_n-s)$ for simplicity:

$$\phi(X_n(\tau),t_n-\tau)\,e^{-D(\tau)} = \phi(x_n,t_n)+\int_0^\tau\Big(-\partial_t\phi(\xi_s)+\dot X_n(s)\cdot D\phi(\xi_s)-\dot D_n(s)\phi(\xi_s)\Big)e^{-D(s)}\,{\rm d}s\,.$$

Pluging this expansion into the dynamic programming principle and using that the global Hamiltonian H is the sup over all the (b,c,l), we are led to

$$o_n(1)\le\int_0^\tau\Big(\partial_t\phi(\xi_s)+H(X_n(s),t_n-s,\phi(\xi_s),D\phi(\xi_s)\Big)e^{-D(s)}\,{\rm d}s\,.$$

Using the smoothness of ϕ and the upper semicontinuity of H together with the facts that $|X_n(s)-x|, |(t_n-s)-t| = o_n(1)+O(s)$, $e^{-D(s)} = 1+O(s)$, we can replace $X_n(s)$ by x and $t_n - s$ by t in the integral. Hence, for τ small enough

$$o_n(1) \leq \tau\Big(\partial_t\phi(x,t) + H(x,t,\phi(x,t),D\phi(x,t))\Big) + \tau o_n(1) + o(\tau)\ .$$

It remains to let first $n \to \infty$, then divide by $\tau > 0$ and send $\tau \to 0$, which yields that $\partial_t\phi(x,t) + H(x,t,\phi,D\phi) \geq 0$. Hence $\mathbf{U}^+$ satisfies the Ishii supersolution condition on $\mathcal{H} \times (0,T_f)$.

(b) *The Ishii subsolution condition in* $\mathbb{R}^N$ — We have to consider $(x,t) \in \mathcal{H}\times(0,T_f)$, a local maximum points of $(\mathbf{U}^+)^* - \phi$, ϕ being a smooth function and we assume again that $(\mathbf{U}^+)^*(x,t) = \phi(x,t)$.

By definition of the upper semicontinuous envelope, there exists a sequence $(x_n,t_n) \to (x,t)$ such that $\mathbf{U}^+(x_n,t_n) \to (\mathbf{U}^+)^*(x,t)$ and we first claim that we can assume $x_n \in \mathcal{H}$. To prove this claim, we use exactly the same argument as in the proof of Lemma 8.2.3 for $\mathbf{U}^-$ since it relies only on the normal controllability assumption $(\mathbf{NC}_{\mathcal{H}})$ at (x,t).

Therefore, assuming that $x_n \in \mathcal{H}$, using the maximum point property we insert the test-function ϕ in the dynamic programming principle and get that for any regular control $a(\cdot)$,

$$\phi(x_n,t_n) + o_n(1) \leq \int_0^\tau l\big(X_n(s),t_n-s,a(s)\big)\, e^{-D(s)}\,\mathrm{d}s + \phi(X_n(\tau),t_n-\tau)\, e^{-D(\tau)}\ . \quad (9.2)$$

Then we argue by contradiction: if

$$\min\big\{\phi_t(x,t) + H_1\big(x,t,\phi(x,t),D\phi(x,t)\big), \phi_t(x,t) + H_2\big(x,t,\phi(x,t),D\phi(x,t)\big)\big\} > 0\,,$$

there exists some $(\alpha_1,\alpha_2) \in A_1 \times A_2$, such that, for all $i = 1,2$

$$\phi_t(x,t) - b_i(x,t,\alpha_i)\cdot D\phi(x,t) + c_i(x,t,\alpha_i)\phi(x,t) - l_i(x,t,\alpha_i) > 0\,, \quad (9.3)$$

and the same is true, for n large enough, if we replace (x,t) by (x_n,t_n). Notice that, though the control $a(\cdot)$ in (9.2) is regular, this may not be the case a priori for α_1,α_2. Now we separate the proof in three cases according to the different configurations. For the sake of simplicity of notations, we just note below by b_i the quantity $b_i(x,t,\alpha_i)$.

Case 1 – Either $b_1 \cdot e_N > 0$ or $b_2 \cdot e_N < 0$. In the first case, we use the trajectory (X_n,D_n,L_n) defined by with the constant control α_1. In particular

$$\dot{X}_n(s) = b_1(X_n(s),t_n-s,\alpha_1) \quad , \quad X_n(0) = x_n. \quad (9.4)$$

Then there exists a time $\tau > 0$ such that $X_n(s) \in \Omega_1$ for $s \in (0, \tau]$. Choosing such constant control α_1 in (9.2) and arguing as above, we are led to

$$\phi_t(x,t) - b_1(x,t,\alpha_1) \cdot D\phi(x,t) + c_1(x,t,\alpha_1)\phi(x,t) - l_1(x,t,\alpha_1) \leq 0\,,$$

which yields a contradiction with (9.3). And the proof is the same in the second case, considering the trajectory associated with the constant control α_2 in b_2.

We point out that this case could have been also covered by arguments of Proposition 3.5.1, by extending the equation to the boundary.

Case 2 – if $b_1 \cdot e_N < 0 < b_2 \cdot e_N$, then borrowing arguments of the proof of Lemma 8.3.1, for (y,s) close enough to (x,t), we can set

$$\mu_1^\sharp(y,s) := \frac{b_2(y,s,\alpha_2)\cdot e_N}{(b_2(y,s,\alpha_2) - b_1(y,s,\alpha_1))\cdot e_N}\,, \quad \mu_2^\sharp := 1 - \mu_1^\sharp\,.$$

By this choice we have $0 \leq \mu_1^\sharp, \mu_2^\sharp \leq 1$ and $\Big(\mu_1^\sharp(y,s)b_1(y,s,\alpha_1) + \mu_2^\sharp(y,s)b_2(y,s,\alpha_2)\Big)\cdot e_N = 0$, hence we have a regular dynamic that we use in (9.2).

We solve the ode

$$\dot{X}^\sharp(s) = \mu_1^\sharp(X^\sharp(s), t_n - s)b_1(X^\sharp(s), t_n - s, \alpha_1) + \mu_2^\sharp(X^\sharp(s), t_n - s)b_2(X^\sharp(s), t_n - s, \alpha_2)\,.$$

By our hypotheses on b_1 and b_2, the right-hand side is Lipschitz continuous so that the Cauchy-Lipschitz theorem applies and gives a solution $X^\sharp(\cdot)$ which remains on $\mathcal{H}$, at least until some time $\tau > 0$.

Using $X^\sharp(\cdot)$ in (9.2) together with the associated discount and cost and arguing as above, we are led to

$$\begin{aligned}&\mu_1^\sharp\Big(\phi_t(x,t) - b_1(x,t,\alpha_1)\cdot D\phi(x,t) + c_1(x,t,\alpha_1)\phi(x,t) - l_2(x,t,\alpha_1)\Big)\\ &+\mu_2^\sharp\Big(\phi_t(x,t) - b_2(x,t,\alpha_2)\cdot D\phi(x,t) + c_2(x,t,\alpha_2)\phi(x,t) - l_2(x,t,\alpha_2)\Big) \leq 0\,,\end{aligned}$$

a contradiction.

Case 3 – The last case is when we have either $b_1 \cdot e_N = 0 < b_2 \cdot e_N$ or $b_1 \cdot e_N < 0 = b_2 \cdot e_N$. But using $(\mathbf{NC}_\mathcal{H})$, we can slightly modify b_1 or b_2 by a suitable convex combination in order to be in the framework of Case 1 or Case 2. This completes the proof that the Ishii subsolution condition holds on $\mathcal{H} \times (0, T_f)$.

(c) *The H_T^{reg}-inequality* — We do not give a specific proof here since this property holds for any Ishii subsolution (hence for $\mathbf{U}^+$ too), see Lemma 9.4.1. Alternatively,

this property can also be proved by similar arguments as for the H_T-inequality for $\mathbf{U}^-$, but using of course regular trajectories.

Q.E.D.

9.2 More on regular trajectories

Let us begin by stating the stability of regular trajectories:

Lemma 9.2.1 *Assume that all the (b_i, c_i, l_i) satisfy $(\mathbf{H_{BA-CP}})$. For any $\varepsilon > 0$, let $(X, D, L)^\varepsilon \in \mathcal{T}^{\rm reg}(x,t)$ be a sequence of regular trajectories converging uniformly to (X, D, L) on $[0,t]$. Then $(X, D, L) \in \mathcal{T}^{\rm reg}(x,t)$.*

Though it may seem quite natural, this result is quite difficult to obtain. It is a direct corollary of Proposition 9.5.1 (with constant **BCL** and initial data) which we prove in Subsection 9.5 below. We recall here that since $T(s) = t - s$, we just use trajectories in the form (X, D, L) instead of (X, T, D, L).

Let us focus now on the immediate consequences:

Corollary 9.2.2 *Assume that all the (b_i, c_i, l_i) satisfy $(\mathbf{H_{BA-CP}})$. Then, for any $(x,t) \in \mathbb{R}^N \times (0, T_f)$, there exists a regular trajectory $(X, D, L) \in \mathcal{T}^{\rm reg}(x,t)$ such that*

$$\mathbf{U}^+(x,t) = \int_0^t l\big(X(s), t-s, a(s)\big) e^{-D(s)}\, {\rm d}s + u_0(X(t)) e^{-D(t)}\,, \tag{9.5}$$

therefore there is an optimal trajectory. Moreover, the value-function $\mathbf{U}^+$ satisfies the sub-optimality principle, i.e., for any $(x,t) \in \mathbb{R}^N \times [0, T_f]$ and $0 < \tau < t$, we have

$$(\mathbf{U}^+)^*(x,t) \le \inf_{\mathcal{T}^{\rm reg}(x,t)} \left\{ \int_0^\tau l\big(X(s), t-s, a(s)\big) e^{-D(s)}\, {\rm d}s + (\mathbf{U}^+)^*(X(\tau), t-\tau) e^{-D(\tau)} \right\}\,,$$

and the super-optimality principle, i.e.

$$(\mathbf{U}^+)_*(x,t) \ge \inf_{\mathcal{T}^{\rm reg}(x,t)} \left\{ \int_0^\tau l\big(X(s), t-s, a(s)\big) e^{-D(s)}\, {\rm d}s + (\mathbf{U}^+)_*(X(\tau), t-\tau) e^{-D(\tau)} \right\}\,.$$

Corollary 9.2.2 provides slightly different (and maybe more direct) arguments to prove that $\mathbf{U}^+$ is an Ishii solution of (7.1) but it relies on the extraction of regular trajectories, which is again a rather delicate result to prove.

Proof — We just sketch things here since everything is a straightforward application of Lemma 9.2.1. For the existence of an optimal trajectory, we consider ε-optimal trajectories $(X^\varepsilon, D^\varepsilon, L^\varepsilon)$, *i.e.* trajectories which satisfy

$$\mathbf{U}^+(x,t) \le \int_0^t l\big(X^\varepsilon(s), t-s, a^\varepsilon(s)\big)e^{-D^\varepsilon(s)}\,\mathrm{d}s + u_0(X^\varepsilon(t))e^{-D^\varepsilon(t)} + \varepsilon\,.$$

By applying Ascoli's Theorem on the differential inclusion, we can assume without loss of generality that $(X^\varepsilon, D^\varepsilon, L^\varepsilon) \to (X, D, L)$ in $C([0,t])$ and $\dot{L}^\varepsilon \to \dot{L}$ in L^∞-weak$\star$, so that for some control $a(\cdot)$, we have

$$\int_0^t l\big(X^\varepsilon(s), t-s, a^\varepsilon(s)\big)e^{-D^\varepsilon(s)}\,\mathrm{d}s \to \int_0^t l\big(X(s), t-s, a(s)\big)e^{-D(s)}\,\mathrm{d}s.$$

Then, applying Lemma 9.2.1 shows that (X, D, L) is actually a regular trajectory and (9.5) holds.

The proofs of the sub and super-optimality principle follow from similar arguments considering, for example, a sequence $(x_k, t_k) \to (x,t)$ such that $\mathbf{U}^+(x_k,t_k) \to (\mathbf{U}^+)_*(x,t)$ and passing to the limit in an analogous way.

Q.E.D.

9.3 A magical lemma for $\mathbf{U}^+$

Now we turn a key result in the proof that $\mathbf{U}^+$ is the maximal Ishii solution of (7.1).

Theorem 9.3.1 *Assume that the "standard assumptions in the co-dimension-*1 *case" are satisfied. Let $\phi \in C^1\big(\mathcal{H} \times [0,T_f]\big)$ and suppose that $(x,t) \in \mathcal{H} \times (0,T_f)$ is a local minimum point of $(z,s) \mapsto (\mathbf{U}^+)_*(z,s) - \phi(z,s)$ in $\mathcal{H} \times [0,T_f]$. Then the following alternative holds*

A) *either there exist $\eta > 0$, $i \in \{1,2\}$ and a control $\alpha_i(\cdot)$ such that the associated trajectory (X,D,L) satisfies $X(s) \in \bar{\Omega}_i$ with $\dot{X}(s) = b_i(X(s), t-s, \alpha_i(s))$ for all $s \in]0,\eta]$ and*

$$(\mathbf{U}^+)_*(x,t) \ge \int_0^\eta l_i(X(s), t-s, \alpha_i(s))e^{-D(s)}\,\mathrm{d}s + (\mathbf{U}^+)_*(X(\eta), t-\eta)e^{-D(\eta)}\,; \quad (9.6)$$

B) *or the following viscosity inequality holds*

$$\partial_t\phi(x,t) + H_T^{\mathrm{reg}}\big(x,t,(\mathbf{U}^+)_*(x,t), D_{\mathcal{H}}\phi(x,t)\big) \ge 0. \quad (9.7)$$

Proof — Using the result and the proof of Corollary 9.2.2, for any $0 < \eta < t$, there exists a regular trajectory X and a control a such that

$$(\mathbf{U}^+)_*(x,t) \geq \int_0^\eta l\big(X(s), t-s, a(s)\big)e^{-D(s)}\,\mathrm{d}s + (\mathbf{U}^+)_*(X(\eta), t-\eta)e^{-D(\eta)}\,.$$

Indeed, for any η the infimum in the sub-optimality principle is achieved. Now there are two cases:

(*i*) Either there exists $\eta > 0$ and $i \in \{1,2\}$ such that $X(s) \in \bar{\Omega}_i$ with $\dot{X}(s) = b_i(X(s), t-s, \alpha_i(s))$ for all $s \in]0,\eta]$, from which **A)** follows.

(*ii*) Or this is not the case, which means that there exists a sequence $(\eta_k)_k$ converging to 0 such that $\eta_k > 0$ and $X(\eta_k) \in \mathcal{H}$.

In this second case,

$$(\mathbf{U}^+)_*(x,t) \geq \int_0^{\eta_k} l\big(X(s), t-s, a(s)\big)e^{-D(s)}\,\mathrm{d}s + (\mathbf{U}^+)_*(X(\eta_k), t-\eta_k)e^{-D(\eta_k)}\,,$$

and, assuming w.l.o.g that $\phi(x,t) = (\mathbf{U}^+)_*(x,t)$, the minimum point property on $\mathcal{H}$ yields

$$\phi(x,t) \geq \int_0^{\eta_k} l\big(X(s), t-s, a(s)\big)e^{-D(s)}\,\mathrm{d}s + \phi(X(\eta_k), t-\eta_k)e^{-D(\eta_k)}\,.$$

Using the notation $\xi_s = (X(s), t-s)$, we rewrite this inequality as

$$\int_0^{\eta_k} A[\phi](s)\,\mathrm{d}s \geq 0\,, \quad \text{where}$$
$$A[\phi](s) := \Big(\phi_t(\xi_s) - \dot{X}(s)\cdot D_x\phi(\xi_s) + c\big(\xi_s, a(s)\big)\phi(\xi_s) - l\big(\xi_s, a(s)\big)\Big)e^{-D(s)}\,.$$

In order to prove **B)**, we argue by contradiction, assuming that

$$\partial_t\phi(x,t) + H_T^{\mathrm{reg}}\big(x,t,(\mathbf{U}^+)_*(x,t), D_{\mathcal{H}}\phi(x,t)\big) < 0\,, \tag{9.8}$$

and to get a contradiction we examine the sets $\mathcal{E}_i := \{s \in (0,\eta_k) : X(s) \in \Omega_i\}$ and $\mathcal{E}_{\mathcal{H}} := \{s \in (0,\eta_k) : X(s) \in \mathcal{H}\}$.

(a) The case $\mathcal{E}_{\mathcal{H}}$ is easy: since $\dot{X}(s) = b_{\mathcal{H}}(X(s), t-s, a(s))$ a.e. if $X(s) \in \mathcal{H}$, by definition of H_T^{reg} as the supremum we get directly

$$\int_0^{\eta_k} A[\phi](s)\mathbb{1}_{\{s\in\mathcal{E}_{\mathcal{H}}\}}\,\mathrm{d}s \leq \int_0^{\eta_k} \Big\{\partial_t\phi(\xi_s) + H_T^{\mathrm{reg}}\big(\xi_s, (\mathbf{U}^+)_*(\xi_s), D_{\mathcal{H}}\phi(\xi_s)\big)\Big\}\mathbb{1}_{\{s\in\mathcal{E}_{\mathcal{H}}\}}\,\mathrm{d}s\,,$$

and this integral is stricly negative provided η_k is small enough, thanks to (9.8) and the continuity of H_T^{reg}.

(b) On the other hand, the sets $\mathcal{E}_i$ are open and therefore $\mathcal{E}_i = \cup_k (a_{i,k}, b_{i,k})$ with $a_{i,k}, b_{i,k} \in \mathcal{H}$. On each interval $(a_{i,k}, b_{i,k})$, $\dot{X}(s) = b_i\big(X(s), t-s, \alpha_i(s)\big)$ and introducing the function $d(y) = |y_N|$, we have

$$0 = d(X(b_{i,k})) - d(X(a_{i,k})) = \int_{a_{i,k}}^{b_{i,k}} e_N \cdot b_i\big(X(s), t-s, \alpha_i(s)\big)\,\mathrm{d}s\,. \tag{9.9}$$

By the regularity of (b_i, c_i, l_i) with respect to $X(s)$ we have

$$\int_{a_{i,k}}^{b_{i,k}} (b_i, c_i, l_i)\big(\xi_s, \alpha_i(s)\big)\,\mathrm{d}s = \int_{a_{i,k}}^{b_{i,k}} (b_i, c_i, l_i)\big(x, t, \alpha_i(s)\big)\,\mathrm{d}s + O(\eta_k)(b_{i,k} - a_{i,k})\,.$$

Then, using the convexity of the images of $\mathbf{BCL}_i$, there exists a control $a_{i,k}^\flat$ such that

$$\int_{a_{i,k}}^{b_{i,k}} (b_i, c_i, l_i)\big(\xi_s, a(s)\big)\,\mathrm{d}s = (b_{i,k} - a_{i,k})\,(b_i, c_i, l_i)\big(x, t, \alpha_{i,k}^\flat\big)\,\mathrm{d}s + O(\eta_k)(b_{i,k} - a_{i,k})\,,$$

and (9.9) implies that $b_i\big(x, t, \alpha_{i,k}^\flat\big) \cdot e_N = O(\eta_k)$. In terms of $\mathbf{BCL}$, this means we have a $(b_i^\flat, c_i^\flat, l_i^\flat) \in \mathbf{BCL}_i(x,t)$ such that $b_i^\flat \cdot e_N = O(\eta_k)$.

Using the normal controllabilty and regularity properties of $\mathbf{BCL}_i$, for η_k small enough, there exists a $(b_i^\sharp, c_i^\sharp, l_i^\sharp) \in \mathbf{BCL}_i(x,t)$ which is $O(\eta_k)$-close to $(b_i^\flat, c_i^\flat, l_i^\flat)$ such that $b_i^\sharp \cdot e_N = 0$. This means that there exists a control $\alpha_{i,k}^\sharp \in A_i$ such that still

$$\int_{a_{i,k}}^{b_{i,k}} (b_i, c_i, l_i)\big(\xi_s, a(s)\big)\,\mathrm{d}s = (b_{i,k} - a_{i,k})\,(b_i, c_i, l_i)\big(x, t, a_{i,k}^\sharp\big)\,\mathrm{d}s + O(\eta_k)(b_{i,k} - a_{i,k})$$

holds, and $b_i\big(x, t, a_{i,k}^\sharp\big) \cdot e_N = 0$. In other words, this specific control provides a regular dynamic.

Hence, using the regularity of ϕ, since $a_{i,k}^\sharp$ is regular we get

$$\begin{aligned}\int_{a_{i,k}}^{b_{i,k}} A[\phi](s)\,\mathrm{d}s &= (b_{i,k} - a_{i,k})\Big\{\phi_t(x,t) - b_i(x,t,a_{i,k}^\sharp)\cdot D_x\phi(x,t)\\ &\qquad + c(x,t,a_{i,k}^\sharp)\phi(x,t) - l_i(x,t,a_{i,k}^\flat) + O(\eta_k)\Big\}\,,\\ &\le (b_{i,k} - a_{i,k})\Big\{\partial_t\phi(x,t) + H_T^{\mathrm{reg}}\big(x,t,(\mathbf{U}^+)_*(x,t), D_{\mathcal{H}}\phi(x,t) + O(\eta_k)\Big\} < 0\,.\end{aligned}$$

Therefore, for η_k small enough, on each connected component of $\mathcal{E}_1$, $\mathcal{E}_2$ and on $\mathcal{E}_{\mathcal{H}}$, the integral is strictly negative and we get the desired contradiction.

Q.E.D.

Remark 9.3.2 *Notice that the alternative above with $H_T^{\rm reg}$ only holds for $\mathbf{U}^+$, and not for any arbitrary supersolution (see Theorem 8.4.1 where H_T is used and not $H_T^{\rm reg}$).*

9.4 Maximality of $\mathbf{U}^+$

In order to prove that $\mathbf{U}^+$ is the maximal subsolution, we need the following result on subsolutions

Lemma 9.4.1 *Assume that the "standard assumptions in the co-dimension-1 case" are satisfied. If $u : \mathbb{R}^N \times (0,T_f) \to \mathbb{R}$ is an u.s.c. subsolution of (7.1), then it satisfies*

$$u_t + H_T^{\rm reg}(x,t,u,D_T u) \le 0 \quad \text{on } \mathcal{H} \times (0,T_f)\,. \tag{9.10}$$

Proof — Let ϕ be a C^1–test-function on $\mathcal{H} \times (0,T_f)$. Using the decomposition of $x \in \mathbb{R}^N$ in (x', x_N) with $x' \in \mathbb{R}^{N-1}$, we can assume that ϕ is just a function of x' and t, and we can see ϕ as a function defined in $\mathbb{R}^N \times (0,T_f)$ as well.

If $(\bar{x},\bar{t}) \in \mathcal{H} \times (0,T_f)$ is a strict local maximum point of $u(x,t) - \phi(x',t)$ on $\mathcal{H} \times (0,T_f)$, we have to show that

$$\phi_t(\bar{x}',\bar{t}) + H_T^{\rm reg}(\bar{x},\bar{t},u(\bar{x},\bar{t}),D_T\phi(\bar{x},\bar{t})) \le 0\,,$$

where $D_T\phi(\bar{x},\bar{t})$ is nothing but $D_{x'}\phi(\bar{x}',\bar{t})$ and we also identify it below with the vector $(D_{x'}\phi(\bar{x}',\bar{t}),0)$. So, setting $a = \phi_t(\bar{x}',\bar{t})$ and $p_T = D_T\phi(\bar{x},\bar{t})$, we have to prove that for any $(b,c,l) \in \mathbf{BCL}_T^{\rm reg}(\bar{x},\bar{t})$,

$$\mathcal{I} := a - b\cdot p_T + cu(\bar{x},\bar{t}) - l \le 0\,.$$

By definition of $\mathbf{BCL}_T^{\rm reg}(\bar{x},\bar{t})$, we can write

$$(b,c,l) = \mu_1(b_1,c_1,l_1) + \mu_2(b_2,c_2,l_2)\,,$$

with $b_1\cdot e_N \le 0 \le b_2\cdot e_N$ and $\mu_1+\mu_2 = 1$. Using the normal controllability and an easy approximation argument, we can assume without loss of generality that $b_1\cdot e_N < 0 < b_2\cdot e_N$. Of course, even if we do not write it to have simpler notations, (b_1,c_1,l_1) is associated to a control α_1 and (b_2,c_2,l_2) to a control α_2.

For $i = 1,2$, we consider the affine functions

$$\psi_i(\delta) := a - b_i\cdot(p_T + \delta e_N) + c_i u(\bar{x},\bar{t}) - l_i\,.$$

By the above properties we have: (i) ψ_1 is strictly increasing; (ii) ψ_2 is strictly decreasing; (iii) $\mu_1\psi_1(\delta) + \mu_2\psi_2(\delta) = \mathcal{I}$, which is independent of δ.

We argue by contradiction, assuming that $\mathcal{I} > 0$ and choose $\bar{\delta}$ such that $\psi_1(\bar{\delta}) = \psi_2(\bar{\delta})$. Notice that this is possible due to the strict monotonicity properties and the fact that $\psi_1(\mathbb{R}) = \psi_2(\mathbb{R}) = \mathbb{R}$. We have therefore $\psi_1(\bar{\delta}) = \psi_2(\bar{\delta}) = \mathcal{I} > 0$.

Next, for $0 < \varepsilon \ll 1$, we consider the function

$$(x,t) \mapsto u(x,t) - \phi(x',t) - \bar{\delta}x_N - \frac{x_N^2}{\varepsilon^2}\ ,$$

defined in $\mathbb{R}^N \times (0,T_f)$. Since $(\bar{x},\bar{t})$ is a strict local maximum point of $u - \phi$ on $\mathcal{H} \times (0,T_f)$, there exists a sequence $(x_\varepsilon,t_\varepsilon)$ of local maximum point of this function which converges to $(\bar{x},\bar{t})$, with $u(x_\varepsilon,t_\varepsilon)$ converging to $u(\bar{x},\bar{t})$.

Our aim is to show that none of the H_1 or H_2 viscosity inequality holds for u on $\mathcal{H}$, which will contradict the fact that u is a viscosity subsolution. Assume for instance that the the H_1-inequality holds. Then $(x_\varepsilon)_N \geq 0$ and by the regularity of ϕ,

$$a - b_1(x_\varepsilon,t_\varepsilon,\alpha_1)\cdot(p_T + \bar{\delta}e_N + \frac{2(x_\varepsilon)_N}{\varepsilon^2}e_N) + c_1(x_\varepsilon,t_\varepsilon,\alpha_1)u(\bar{x},\bar{t}) - l_1(x_\varepsilon,t_\varepsilon,\alpha_1) \leq o_\varepsilon(1)\ .$$

But since $(x_\varepsilon,t_\varepsilon) \to (\bar{x},\bar{t})$, $b_1(x_\varepsilon,t_\varepsilon,\alpha_1) \to b_1(\bar{x},\bar{t},\alpha_1)$ and therefore $b_1(x_\varepsilon,t_\varepsilon,\alpha_1)\cdot e_N < 0$ for ε small enough. Using that $(x_\varepsilon)_N \geq 0$, this inequality implies

$$a - b_1(x_\varepsilon,t_\varepsilon,\alpha_1)\cdot(p_T + \bar{\delta}e_N) + c_1(x_\varepsilon,t_\varepsilon,\alpha_1)u(\bar{x},\bar{t}) - l_1(x_\varepsilon,t_\varepsilon,\alpha_1) \leq o_\varepsilon(1)\ .$$

By the definition and properties of $\bar{\delta}$ and the fact that $\mathcal{I} > 0$, this inequality cannot hold for ε small enough, showing that the H_1 inequality cannot hold neither. A similar argument being valid for the H_2 inequality, we have a contradiction and therefore $\mathcal{I} \leq 0$, and the proof is finished.

Q.E.D.

Theorem 9.4.2 *Assume that the "standard assumptions in the co-dimension-*1 *case" are satisfied. Then* $\mathbf{U}^+$ *is continuous and it is the maximal Ishii solution of* (7.1).

Proof — Let u be any subsolution of (7.1). We want to show that $u \leq (\mathbf{U}^+)_*$ in $\mathbb{R}^N \times [0,T_f)$ and to do so we first notice that, as we did in the proof of the characterization of $\mathbf{U}^-$ (Theorem 8.4.1), we can reduce the proof to a local comparison argument since (**LOC1**) and (**LOC2**) are satisfied. So, let $Q_{r,h}^{x,t}$ be a cylinder in which we want to perform the **(LCR)** between u and $(\mathbf{U}^+)_*$.

Using again the arguments of the proof of Theorem 8.4.1, we may assume without loss of generality that u is a strict subsolution of (7.1) and in particular a strict subsolution of (8.10). Finally we can regularize u in order that it is C^1 on $\mathcal{H}\times(0,T_f)$.

Using Theorem 5.3.3 to show that u satisfies a sub-dynamic programming principle with trajectories in $\mathcal{T}(x,t)$, we see that we are (almost) in the framework of Lemma 5.4.1, the usual $\mathbb{F}^{\mathcal{M}}$-inequality for u being replaced by (9.10).

Using in an essential way Theorem 9.3.1[(1)], it is easy to see that the result of Lemma 5.4.1 still holds in this slightly different framework and yields

$$\max_{\overline{Q}^{x,t}_{r,h}}(u-(\mathbf{U}^+)_*)\le\max_{\partial Q^{x,t}_{r,h}}(u-(\mathbf{U}^+)_*)\,,$$

and the **(GCR)** follows: $u\le(\mathbf{U}^+)_*$ in $\mathbb{R}^N\times[0,T_f]$.

Concerning the continuity statement, consider $u=(\mathbf{U}^+)^*$. By definition, $(\mathbf{U}^+)^*\ge(\mathbf{U}^+)_*$ but the comparison result above applied to $(\mathbf{U}^+)^*$ which is a subsolution shows that in the end $\mathbf{U}^+=(\mathbf{U}^+)_*=(\mathbf{U}^+)^*$. Hence $\mathbf{U}^+$ is continuous and is maximal amongst Ishii subsolutions.

Q.E.D.

9.5 Appendix: stability of regular trajectories

This appendix is about proving the convergence property of regular trajectories, Lemma 9.2.1. We actually prove a more general result here:

Proposition 9.5.1 *Let $t>0$ be fixed and for each $\varepsilon>0$ let $\mathbf{BCL}^\varepsilon$ be a set-valued map satisfying $(\mathbf{H_{BCL}})_{fund}$. and let $(X,T,D,L)^\varepsilon$ be solution of the differential inclusion*

$$\forall s\in(0,t)\,,\quad (X,T,D,L)^\varepsilon(s)\in\mathbf{BCL}^\varepsilon(X^\varepsilon(s),t-s)\,.$$

(i) If $\mathbf{BCL}^\varepsilon$ converges to $\mathbf{BCL}$ locally unformly in $\mathbb{R}^N\times(0,t)$ (for the Hausdorf distance on sets) and $(X,T,D,L)^\varepsilon(0)\to(x,t,d,l)$, then up to extraction, $(X,T,D,L)^\varepsilon$ converges to some trajectory (X,T,D,L) which satisfies

$$\forall s\in(0,t)\,,\quad (X,T,D,L)(s)\in\mathbf{BCL}(X(s),t-s)$$

with inital value $(X,T,D,L)(0)=(x,t,d,l)$.

[(1)] which replaces the arguments for the supersolution v in the proof of Lemma 5.4.1 (cf. Remark 5.4.2).

(*ii*) *If moreover each trajectory X^ε is regular, then the limit trajectory X is also regular.*

This result is obtained through several lemmas. The first one proves part (i) of the proposition, which is not very difficult.

Lemma 9.5.2 *If* $\mathbf{BCL}^\varepsilon$ *converges to* **BCL** *locally unformly in* $\mathbb{R}^N \times (0,t)$ *(for the Haussdorf distance on sets) and* $(X,T,D,L)^\varepsilon(0) \to (x,t,d,l)$*, then up to extraction,* $(X,T,D,L)^\varepsilon$ *converges to some trajectory* (X,T,D,L) *which is a solution of the differential inclusion associated with* **BCL***, with the corresponding initialization.*

Proof — Notice first that since the $\mathbf{BCL}^\varepsilon$ all satisfy $(\mathbf{H_{BCL}})_{fund}$ with constants independent of ε, and the initial value converges, the trajectories $(X,T,D,L)^\varepsilon$ are equi-Lipschitz and equi-bounded on $[0,t]$. Hence we can extract a subsequence $(X,T,D,L)^{\varepsilon_n}$ converging to (X,T,D,L) uniformly on $[0,t]$. Moreover, for any $\kappa > 0$ small enough, if n is big enough we have

$$\forall s \in (0,t)\,, \quad \mathbf{BCL}^\varepsilon(X^{\varepsilon_n}(s), t-s) \subset \mathbf{BCL}(X(s), t-s) + \kappa B_{N+3}$$

where B_{N+3} is the unit ball of $\mathbb{R}^{N+3}$. Passing to the limit as $\varepsilon_n \to 0$, we deduce that (X,T,D,L) satisfies the differential inclusion associated with **BCL**, and of course its initial data is $(X,T,D,L)(0) = (x,t,d,l)$.

Q.E.D.

Now we need several results in order to prove part (ii) which is much more involved. Before proceeding, let us comment a little bit: using the control representation of the differential inclusion (Lemma 8.1.2), there exist some controls $\alpha_i^\varepsilon, a^\varepsilon$ such that

$$\dot{X}^\varepsilon(s) = \sum_{i=1,2} b_i^\varepsilon\big(X^\varepsilon(s), t-s, \alpha_i^\varepsilon(s)\big)\mathbb{1}_{\{X^\varepsilon \in \Omega_i\}}(s) + b_{\mathcal{H}}^\varepsilon\big(X^\varepsilon(s), t-s, a^\varepsilon(s)\big)\mathbb{1}_{\{X^\varepsilon \in \mathcal{H}\}}(s)\,.$$

Recall that the control a^ε is actually complex since it involves $\alpha_1^\varepsilon, \alpha_2^\varepsilon$ but also α_0^ε. In other words, $b_{\mathcal{H}}$ is a mix of b_0, b_1, b_2 with weights $\mu_0^\varepsilon, \mu_1^\varepsilon, \mu_2^\varepsilon$. However, notice that focusing on regular dynamics, the b_0-term is not a problem since it is already tangential (hence, regular).

In order to send $\varepsilon \to 0$ we face two difficulties: the first one is that we have to deal with weak convergences in the $b_i^\varepsilon, b_{\mathcal{H}}^\varepsilon$-terms. But the problem is increased by the fact that some pieces of the limit trajectory $X(\cdot)$ on $\mathcal{H}$ can be obtained as limits of trajectories $X^\varepsilon(\cdot)$ which lie either on $\mathcal{H}$, Ω_1 or Ω_2. In other words, the indicator functions $\mathbb{1}_{\{X^\varepsilon \in \mathcal{H}\}}(\cdot)$ do not necessarily converge to $\mathbb{1}_{\{X \in \mathcal{H}\}}(\cdot)$, and similarly the $\mathbb{1}_{\{X^\varepsilon \in \Omega_i\}}(\cdot)$ do not converge to $\mathbb{1}_{\{X \in \Omega_i\}}(\cdot)$.

From Lemma 9.5.2 we already know that $\dot{X}^\varepsilon$ converges weakly on $(0,t)$ to some $\dot{X}$ which can be represented as for X^ε above, by means of some controls (α_1, α_2, a). The question is to prove that this control a yields regular dynamics on $\mathcal{H}$. In order to to do, we introduce several tools. The first one is a representation of X by means of some regular controls $(\alpha_1^\sharp, \alpha_2^\sharp, a^\sharp)$. Those controls may differ from (α_1, α_2, a), but they are an intermediate step which will help us to prove the final result.

Lemma 9.5.3 *For any $s \in (0,t)$ there exists three measures $\nu_1(s,\cdot), \nu_2(s,\cdot), \nu_{\mathcal{H}}(s,\cdot)$ on A_1, A_2, A respectively and three controls $(\alpha_1^\sharp(s), \alpha_2^\sharp(s), a^\sharp(s)) \in A_1 \times A_2 \times A$ such that*

(a) *$\nu_1, \nu_2, \nu_{\mathcal{H}} \geq 0$, $\nu_1(s,A_1) + \nu_2(s,A_2) + \nu_{\mathcal{H}}(s,A) = 1$;*

(b) *up to extraction, $b_1^\varepsilon(X^\varepsilon(s), t-s, \alpha_1^\varepsilon) \to b_1(X(s), t-s, \alpha_1^\sharp(s)) \cdot \nu_i(s,A_1)$, and the same holds for $b_2, b_{\mathcal{H}}$ with measures ν_2, ν_H and controls $\alpha_2^\sharp, \alpha_{\mathcal{H}}^\sharp$;*

(c) *for $i = 1, 2$, $b_i(X(s), t-s, \alpha_i^\sharp(s)) \cdot e_N = 0 \quad \nu_i$-a.e. on $\{X(s) \in \mathcal{H}\}$.*

In particular, the dynamic obtained by using $(\alpha_0, \alpha_1^\sharp, \alpha_2^\sharp)$ is regular.

Proof — We use a slight modification of the procedure leading to relaxed control as follows. We write

$$b_1^\varepsilon\big(X^\varepsilon(s), t-s, \alpha_1^\varepsilon(s)\big)\mathbb{1}_{\{X^\varepsilon \in \Omega_1\}}(s) = \int_{A_1} b_1^\varepsilon\big(X^\varepsilon(s), t-s, \alpha\big)\, \nu_1^\varepsilon(s,\, \mathrm{d}\alpha)\,,$$

where $\nu_1^\varepsilon(s,\cdot)$ stands for the measure defined on A_1 by $\nu_1^\varepsilon(s,E) = \delta_{\alpha_1^\varepsilon}(E)\mathbb{1}_{\{X^\varepsilon \in \Omega_1\}}(s)$, for any Borelian set $E \subset A_1$. Similarly we define ν_2^ε and $\nu_{\mathcal{H}}^\varepsilon$ for the other terms. Notice that $\nu_{\mathcal{H}}^\varepsilon$ is a bit more complex measure since it concerns controls of the form $a = (\alpha_1, \alpha_2, \mu)$ on A, but it works as for ν_1^ε so we omit the details.

Note that, for any s, $\nu_1^\varepsilon(s,A_1) + \nu_2^\varepsilon(s,A_2) + \nu_{\mathcal{H}}^\varepsilon(s,A) = 1$ and therefore the measures $\nu_1^\varepsilon(s,\cdot), \nu_2^\varepsilon(s,\cdot), \nu_{\mathcal{H}}^\varepsilon(s,\cdot)$ are uniformly bounded in ε. Up to successive extractions of subsequences, they all converge in $L^\infty(0,T_f;E)$ weak-$*$ (where $E = A_1, A_2, A$) to some measures ν_1, ν_2, $\nu_{\mathcal{H}}$. Since moreover the total mass is 1, we obtain in the limit $\nu_1(s,A_1) + \nu_2(s,A_2) + \nu_{\mathcal{H}}(s,A) = 1$.

Using that up to extraction X^ε converges uniformly on $[0,t]$, using the local uniform convergence of the b_1^ε, we get that

$$\int_{A_1} b_1^\varepsilon\big(X^\varepsilon(s), t-s, \alpha\big)\, \nu_1^\varepsilon(s,\, \mathrm{d}\alpha) \underset{\varepsilon \to 0}{\longrightarrow} \int_{A_1} b_1\big(X(s), t-s, \alpha\big)\, \nu_1(s,\, \mathrm{d}\alpha),$$

weakly in $L^\infty(0,T_f)$. Introducing $\pi_1(s) := \int_{A_1} \nu_1(s,\, \mathrm{d}\alpha)$ and using the convexity of A_1 together with a measurable selection argument (see [11, Theorem 8.1.3]), the last integral can be written as $b_1\big(X(s),\sigma(s),\alpha_1^\sharp(s)\big)\pi_1(s)$ for some control $\alpha_1^\sharp \in L^\infty(0,T_f;A_1)$. The same procedure for the other two terms provides the controls $\alpha_2^\sharp(\cdot)$, $a^\sharp(\cdot)$ and functions $\pi_2(\cdot)$, $\pi_{\mathcal{H}}(\cdot)$, which yields (a) and (b).

We now turn to property (c) that we prove for b_1, the proof being identical for b_2. Since $(X_N^\varepsilon)_+ := \max(X_N^\varepsilon,0)$ is a sequence of Lipschitz continuous functions which converges uniformly to $(X_N)_+$ on $[0,t]$, up to an additional extraction of subsequence, we may assume that the derivatives converge weakly in L^∞ (weak–$*$ convergence). As a consequence, $\frac{d}{ds}\big[(X_N^\varepsilon)_+\big]\mathbb{1}_{\{X\in\mathcal{H}\}}$ converges weakly to $\frac{d}{ds}\big[(X_N)_+\big]\mathbb{1}_{\{X\in\mathcal{H}\}}$.

By Stampacchia's Theorem we have

$$\frac{d}{ds}\big[(X_N^\varepsilon)_+\big] = \dot X_N^\varepsilon(s)\,\mathbb{1}_{\{X^\varepsilon\in\Omega_1\}}(s) \quad \text{ for almost all } s\in(0,t).$$

Therefore, the above convergence reads, in $L^\infty(0,T_f)$weak–$*$

$$\dot X_N^\varepsilon(s)\mathbb{1}_{\{X^\varepsilon\in\Omega_1\}}(s)\mathbb{1}_{\{X\in\mathcal{H}\}}(s) \longrightarrow \dot X_N(s)\mathbb{1}_{\{X\in\Omega_1\}}(s)\mathbb{1}_{\{X\in\mathcal{H}\}}(s) = 0\,.$$

Using the expression of $\dot X^\varepsilon(s)$, $\big(b_1^\varepsilon\big(X^\varepsilon(s),t-s,\alpha_1^\varepsilon(s)\big)\cdot e_N\big)\mathbb{1}_{\{X^\varepsilon\in\Omega_1\}}(s)\mathbb{1}_{\{X\in\mathcal{H}\}}(s)\to 0$ in $L^\infty(0,T_f)$ weak–$*$ which implies that

$$\Big(b_1\big(X(s),t-s,\alpha_1^\sharp(s)\big)\cdot e_N\Big)\,\pi_i(s) = 0 \;\text{ a.e. on } \{X(s)\in\mathcal{H}\}\,, \tag{9.11}$$

which yields property (c). This means that $b_i(X(s),t-s,\alpha_i^\sharp(s))$ is tangential on $\mathcal{H}$ so that combining them with some b_0 (which is tangential by definition), we get a regular dynamic on $\mathcal{H}$.

Q.E.D.

We now want to prove that the controls (α_1,α_2,a) yield regular strategies, not only the $(\alpha_1^\sharp,\alpha_2^\sharp,a^\sharp)$. In order to proceed we introduce the set of regular dynamics:

$$\forall (z,s)\in\mathcal{H}\times[0,t]\,,\quad K(z,s) := \big\{b_{\mathcal{H}}\big(z,s,a_*\big)\,,a_*\in A_0^{\mathrm{reg}}(z,s)\big\}\subset\mathbb{R}^N\,.$$

We notice that, for any $z\in\mathcal{H}$ and $s\in[0,T_f]$, $K(z,s)$ is closed and convex, and the mapping $(z,s)\mapsto K(z,s)$ is continuous on $\mathcal{H}$ for the Hausdorff distance. Then, for any $\eta>0$, we consider the subset of $[0,t]$ consisting of times s for which one has singular (η-enough) dynamics for the control $a(\cdot)$, namely

$$E_{\mathrm{sing}}^\eta := \Big\{s\in[0,t] : X(s)\in\mathcal{H} \text{ and } \operatorname{dist}\Big(b_{\mathcal{H}}\big(X(s),t-s,a(s)\big);K\big(X(s),t-s\big)\Big)\geq\eta\Big\}\,.$$

If $s \in E^\eta_{\rm sing} \neq \emptyset$, since $K(X(s), t-s)$ is closed and convex, there exists an hyperplane separating $b_{\mathcal{H}}\big(X(s), t-s, a(s)\big)$ from $K(X(s), t-s)$ and we can construct an affine function $\Psi_s : \mathbb{R}^N \to \mathbb{R}$ of the form $\Psi_s(z) = A(s)z + B(s)$ such that

$$\Psi_s\Big(b_{\mathcal{H}}\big(X(s), t-s, a(s)\big)\Big) \geq 1 \text{ if } s \in E^\eta_{\rm sing}\,, \quad \Psi_s \leq 0 \text{ on } K\big(X(s), t-s\big)\,.$$

In other words, Ψ_s "counts" the singular dynamics.

Since the mapping $s \mapsto b_{\mathcal{H}}\big(X(s), t-s, a(s)\big)$ is measurable and $s \mapsto K\big(X(s), t-s\big)$ is continuous, we can assume that $s \mapsto A(s), B(s)$ are measurable and bounded (because the distance $\eta > 0$ is fixed), which allows to define the quantity

$$I(\eta) := \begin{cases} \displaystyle\int_0^t \big(\Psi_s(\dot X(s)\big)\mathbb{1}_{E^\eta_{\rm sing}}(s)\,{\rm d}s & \text{if } E^\eta_{\rm sing} \neq \emptyset \\ 0 & \text{if } E^\eta_{\rm sing} = \emptyset\,. \end{cases}$$

By definition, it is clear that $I(\eta) \geq |E^\eta_{\rm sing}|$ (the Lebesgue measure of $E^\eta_{\rm sing}$). The following result gives a converse estimate

Lemma 9.5.4 *For any $\eta > 0$, $I(\eta) \leq 0$.*

Proof — Let $\eta > 0$. If $E^\eta_{\rm sing} = \emptyset$ there is nothing to do so let us assume that this is not the case, and take some $s \in E^\eta_{\rm sing}$. Since Ψ_s is affine, using the weak convergence of $\dot X^\varepsilon$ we know that

$$I(\eta) = \lim_{\varepsilon\to 0} I^\varepsilon(\eta) := \int_0^t \big(\Psi_s(\dot X^\varepsilon(s)\big)\mathbb{1}_{E^\eta_{\rm sing}}(s)\,{\rm d}s\,.$$

The strategy is to use Lemma 9.5.3 to pass to the limit and estimate $I^\varepsilon(\eta)$, knowing that at each level $\varepsilon > 0$, the dynamics are regular. In order to keep this information in the limit, dealing with the b^ε_i-terms is handled by property (c) of Lemma9.5.3. But the $b^\varepsilon_{\mathcal{H}}$-term is more delicate: we need first to fix a regular control independent of ε.

To do so, we start by noticing that for fixed $\varepsilon > 0$ and $s \in [0,t]$, for each $a^\varepsilon(s) \in A_0^{\rm reg}(X^\varepsilon(s), t-s)$ there exists a $\tilde a^\varepsilon(s) \in A_0^{\rm reg}(X(s), t-s)$ such that

$$b^\varepsilon_{\mathcal{H}}(X^\varepsilon(s), t-s, a^\varepsilon(s)) = b_{\mathcal{H}}(X(s), t-s, \tilde a^\varepsilon(s)) + o_\varepsilon(1)\,.$$

Indeed, this comes from a measurable selection argument and the fact that X^ε converges uniformly to X, while $b^\varepsilon_{\mathcal{H}}$ also converges locally uniformly (with respect to its

first variable). So, rewriting the expansion of $\dot{X}^\varepsilon$ and using that Ψ_s is affine we get

$$I^\varepsilon(\eta) = \int_0^t \Psi_s\Big(\sum_{i=1,2} b_i^\varepsilon\big(X^\varepsilon(s), t-s, \alpha_i^\varepsilon(s)\big)\mathbb{1}_{\{X^\varepsilon\in\Omega_i\}}(s)\Big)\mathbb{1}_{E^\eta_{\rm sing}}(s)\,\mathrm{d}s \\ + \int_0^t A(s)\Big(b_{\mathcal{H}}\big(X(s), t-s, \tilde{a}^\varepsilon(s)\big)\,\mathbb{1}_{\{X^\varepsilon\in\mathcal{H}\}}(s)\Big)\mathbb{1}_{E^\eta_{\rm sing}}(s)\,\mathrm{d}s + o_\varepsilon(1)\,.$$

Moreover, by construction and using again a measurable selection argument (see Filippov's Lemma [11, Theorem 8.2.10]), there exists a control $a_\star(s) \in K(X(s), t-s)$ such that

$$A(s)b_{\mathcal{H}}(X(s), t-s, a_\star(s)) = \max_{a\in K(X(s),t-s)} A(s)b_{\mathcal{H}}(X(s), t-s, a).$$

Therefore,

$$I^\varepsilon(\eta) \le \int_0^t \Psi_s\Big\{ \sum_{i=1,2} b_i^\varepsilon\big(X^\varepsilon(s), t-s, \alpha_i^\varepsilon(s)\big)\mathbb{1}_{\{X^\varepsilon\in\Omega_i\}}(s) \\ + b_{\mathcal{H}}\big(X(s), t-s, a_\star(s)\big)\,\mathbb{1}_{\{X^\varepsilon\in\mathcal{H}\}}(s)\Big\}\mathbb{1}_{E^\eta_{\rm sing}}(s)\,\mathrm{d}s + o_\varepsilon(1)\,.$$

Now we pass to the weak limit, using Lemma 9.5.3 but with a constant $b_{\mathcal{H}}$ instead of $b^\varepsilon_{\mathcal{H}}$ and, more importantly, a constant control $a_\star$. In other words, the measure $\nu^\varepsilon_{\mathcal{H}}$ is actually independent of ε in this situation. We get some measures $\nu_1, \nu_2, \nu_{\mathcal{H}}$ and some controls $\alpha_1^\sharp, \alpha_2^\sharp$ and $a^\sharp = a_*$ here, for which

$$\lim_{\varepsilon\to 0} I^\varepsilon(\eta) \le \int_0^t \Psi_s\Big\{ \sum_{i=1,2} b_i\big(X(s), t-s, \alpha_i^\sharp(s)\big)\nu_i(s, A_i) \\ + b_{\mathcal{H}}\big(X(s), t-s, a_\star(s)\big)\nu_H(s, A)\Big\}\mathbb{1}_{E^\eta_{\rm sing}}(s)\,\mathrm{d}s\,.$$

Recall that by construction $b_{\mathcal{H}}(X(s), t-s, a_\star(s)) \in K(X(s), t-s)$ and that $\alpha_1^\sharp, \alpha_2^\sharp$ are regular controls. Therefore, since $\nu_1(s, A_1) + \nu_2(s, A_2) + \nu_{\mathcal{H}}(s, A) = 1$ and the set $K(X(s), t-s)$ is convex, we deduce that the convex combination satisfies

$$\Psi_s\Big\{ \sum_{i=1,2} b_i\big(X(s), t-s, \alpha_i^\sharp(s)\big)\nu_i(s, A_i) + b_{\mathcal{H}}\big(X(s), t-s, a_\star(s)\big)\nu_H(s, A)\Big\} \le 0\,.$$

The conclusion is that $I(\eta) = \lim_{\varepsilon\to 0} I^\eta(\eta) \le 0$ and the result is proved.

Q.E.D.

Proof of Proposition 9.5.1 — The first part (i) is done in Lemma 9.5.2. As for (ii), we proved above that for any $\eta > 0$, $|E^\eta_{\rm sing}| \leq I(\eta) = 0$, so that set $E^\eta_{\rm sing}$ is of zero Lebesgue measure. Hence, using a countable union of negligeable sets we deduce that

$$\left\{ s \in [0,t] : X(s) \in \mathcal{H} \text{ and } b_{\mathcal{H}}\big(X(s), t-s, a(s)\big) \notin K\big(X(s), t-s\big)\Big) \right\}$$

is also of zero Lebesgue measure. This means that for almost any $s \in (0,t)$, the strategy obtained by choosing a as control is regular, which concludes the proof.

Q.E.D.

Chapter 10

Uniqueness and Non-Uniqueness Features

In this chapter, we investigate the question of the uniqueness for Ishii solutions of Problem (7.1), which can be summarized as: when are the value functions $\mathbf{U}^+$, $\mathbf{U}^-$ equal? It is rather clear that, in general, they are different since the restriction to use only regular controls can really penalize the controller, leading to the fact that $\mathbf{U}^+$ is strictly larger than $\mathbf{U}^-$. We give an example of this non-uniqueness situation in the first section of this chapter.

Then we provide some conditions under which uniqueness holds, using a pde point-of-view: as a consequence of Theorem 8.4.1 and Proposition 9.1.3, we know that $\mathbf{U}^+ = \mathbf{U}^-$ if $H_T = H_T^{\mathrm{reg}}$, and we give a simple condition under which this last equality is true.

10.1 A typical example where $\mathbf{U}^+ \not\equiv \mathbf{U}^-$

We consider a one-dimensional finite horizon problem where

$$\Omega_1 = \{x > 0\},\ \Omega_2 = \{x < 0\},\ \mathcal{H} = \{x = 0\}\ .$$

The reader will find in [16] a detailed study of this situation for infinite horizon control problems, a general description of the structure of solutions, the link between the minimal and maximal Ishii solutions with state-constraint solutions as well as several explicit examples. Here we restrict ourselves to exposing an explicit example of non-uniqueness for illustration purposes.

We consider the dynamics

$$\dot X(t) = \alpha_1(t) \text{ in } \Omega_1\,, \quad \dot X(t) = \alpha_2(t) \text{ in } \Omega_2\,,$$

where $\alpha_1(\cdot), \alpha_2(\cdot) \in L^\infty\big(0,+\infty;[-1,1]\big)$ are the controls. In other words, $A_1 = A_2 = [-1,1]$ and $b_1(x,t,\alpha_1) = \alpha_1$, $b_2(x,t,\alpha_2) = \alpha_2$. As for the costs, we choose

$$l_1(x,t,\alpha_1) = 1 - \alpha_1 + \min(|x|,1) \text{ in } \Omega_1\,, \quad l_2(x,t,\alpha_2) = 1 + \alpha_2 + \min(|x|,1) \text{ in } \Omega_2\,.$$

Finally, we set $c_1(x,\alpha_1) = c_2(x,\alpha_2) = 1$ for the discount factor and also $g = \min(|x|,1)$ for the final cost. Therefore,

$$\mathbf{U}^-(x,t) = \inf_{\mathcal{T}(x,t)} \left\{ \int_0^t l(X(s),t-s,a(s))e^{-s}\,\mathrm{d}s + g(X(t))e^{-t} \right\}\,,$$

where l is either l_1, l_2 or a convex combination of both for $x = 0$, and $a(\cdot) = (\alpha_1,\alpha_2,\mu)$ is the extended control. The definition for $\mathbf{U}^+$ is similar, the infimum being taken over $\mathcal{T}^{\mathrm{reg}}(x,t)$.

Computing $\mathbf{U}^-(0,t)$. It is clear that $l_1(x,\alpha_1), l_2(x,\alpha_2) \geq 0$ and these running costs are even strictly positive for $x \neq 0$. Therefore, $\mathbf{U}^-(x,t) \geq 0$ for any $x \in \mathbb{R}$ and $t \geq 0$. On the other hand, for $x = 0$, we have access to a 0-cost strategy by choosing the singular "pull-pull" strategy $a = (\alpha_1,\alpha_2,\mu) = (1,-1,1/2)$ which gives

$$b(0,t-s,a) = \mu\alpha_1 + (1-\mu)\alpha_2 = 0\,,$$

$$l(0,t-s,a) = \mu(1-\alpha_1) + (1-\mu)(1+\alpha_2) = 0\,.$$

As a consequence, it is clear that this is the best strategy for $x = 0$ and $\mathbf{U}^-(0,t) = 0$ for any $t \geq 0$.

Computing $\mathbf{U}^+(0,t)$. For simplicity, we compute it only for $t \leq 1$ here. In this case any trajectory satisfies $|X(s)| \leq 1$ for any $0 \leq s \leq t$ and $\min(|X(s)|,1)$ can be replaced by $|X(s)|$ everywhere (in the running cost and terminal cost).

If X is any trajectory starting from $X(0) = 0$ and associated to a regular control and if $X(s) > 0$, then

$$\begin{aligned} l(X(s),t-s,a(s))e^{-s} =& (1 - \dot X(s) + X(s))e^{-s} \\ =& e^{-s} - (X(s)e^{-s})'\,. \end{aligned}$$

With analogous computations for $X(s) < 0$, we end up with $l(X(s),t-s,a(s))e^{-s} = e^{-s} - [|X|e^{-s}]'(s)$ if $X(s) \neq 0$.

It remains to examine the case when $X(s) = 0$. It is easy to see that, if $b(0,t-s,a) = 0$ is a regular dynamic, then $l(0,t-s,a) \geq 1$ since $\alpha_1 \leq 0, \alpha_2 \geq 0$ and $l(0,t-s,a) = 1$

if and only if $\alpha_1 = \alpha_2 = 0$. Therefore, for $X(s) = 0$, the above formula is changed into $l(X(s), t-s, a(s))e^{-s} \geq e^{-s} - [|X|e^{-s}]'(s)$ since $|X'(s)| = 0$ a.e. on the set $\{X(s) = 0\}$. And actually, equality is attained for the above mentioned choice of a. Therefore

$$\int_0^t l(X(s), t-s, a(s))e^{-s}\,\mathrm{d}s + g(X(t))e^{-t} = \int_0^t \left(e^{-s} - [|X|e^{-s}]'(s)\right)\,\mathrm{d}s + g(X(t))e^{-t}$$
$$= 1 - e^{-t} > 0\,,$$

proving that $\mathbf{U}^+(0,t) = 1 - e^{-t} > \mathbf{U}^-(0,t) = 0$ at least for $0 < t \leq 1$. The conclusion is that $\mathbf{U}^+ \neq \mathbf{U}^-$ and uniqueness does not hold in the class of Ishii solutions.

10.2 Equivalent definitions for H_T and H_T^{reg}

We recall that we defined H_T and H_T^{reg} in Section 8.3, using the subsets $\mathbf{BCL}_T(x,t)$ and $\mathbf{BCL}_T^{\mathrm{reg}}(x,t)$: for $x \in \mathcal{H}$, $t \in (0, T_f)$, $r \in \mathbb{R}$, $p \in \mathbb{R}^N$

$$H_T(x,t,r,p) := \sup_{(b,c,l)\in \mathbf{BCL}_T(x,t)} \left\{ -b\cdot p + cu - l \right\}, \tag{10.1}$$

while the second Hamiltonian is defined similarly but by considering only *regular* tangential dynamics b

$$H_T^{\mathrm{reg}}(x,t,r,p) := \sup_{\mathbf{BCL}_T^{\mathrm{reg}}(x,t)} \left\{ -b\cdot p + cu - l \right\}. \tag{10.2}$$

On the other hand, for any x, t, r, p', the functions $f(s) := H_1(x,t,r,p' + se_N)$ and $g(s) := H_2(x,t,r,p' + se_N)$ are convex and, thanks to Section 6.2, we can introduce the nonincreasing and nondecreasing parts $f^\sharp, f_\flat, g^\sharp, g_\flat$ of f and g. It is easy to see that

$$f^\sharp(s) = \sup_{\substack{(b_1,c_1,l_1)\in \mathbf{BCL}_1(x,t)\\ b_1\cdot e_N \leq 0}} \left\{ -b_1\cdot(p' + se_N) + c_1 u - l_1 \right\} := H_1^-(x,t,r,p' + se_N)\,,$$

and similarly we define "$H_2^- = g^\sharp$", "$H_1^+ = f_\flat$" and "$H_2^+ = g_\flat$" , the choice of "$+$" or "$-$" in $H_i^\pm$ being related to the sign of $b_i \cdot e_N$ in its definition.

In order to provide equivalent definitions of H_T,H_T^{reg}, we follow Section 6.2 where we introduced $M(s) := \max(f(s), g(s))$ and $M^{\mathrm{reg}}(s) := \max(f^\sharp(s), g_\flat(s))$, which leads to consider the Hamiltonians defined for $x \in \mathcal{H}$, $t \in (0, T_f)$, $r \in \mathbb{R}$, $p \in \mathbb{R}^N$ by

$$\tilde{H}(x,t,r,p) := \max\left(H_1(x,t,r,p), H_2(x,t,r,p)\right), \tag{10.3}$$

$$\tilde{H}^{\rm reg}(x,t,r,p) := \max\left(H_1^-(x,t,r,p), H_2^+(x,t,r,p)\right) . \tag{10.4}$$

The following representation holds

Lemma 10.2.1 *For any* $(x,t,r,p') \in \mathcal{H} \times (0,T_f) \times \mathbb{R} \times \mathbb{R}^N$,

$$H_T(x,t,r,p') = \min_{s\in\mathbb{R}} \tilde{H}(x,t,r,p'+se_N)\,, \tag{10.5}$$

$$H_T^{\rm reg}(x,t,r,p') = \min_{s\in\mathbb{R}} \tilde{H}^{\rm reg}(x,t,r,p'+se_N)\,. \tag{10.6}$$

Moreover, there exist $\nu_1 \le \nu_2$ *such that for any* $\lambda \in [\nu_1,\nu_2]$,

$$H_T^{\rm reg}(x,t,r,p') = H_1^-(x,t,r,p'+\lambda e_N) = H_2^+(x,t,r,p'+\lambda e_N)\,. \tag{10.7}$$

Proof — Notice first that (10.7) is a direct consequence of Lemma 6.2.3. Now, concerning (10.5) and (10.6), We only provide the full proof in the case of H_T, the one for $H_T^{\rm reg}$ follows from the same arguments, just changing the sets of (b_1,c_1,l_1), (b_2,c_2,l_2) we consider.

We introduce the function $\varphi : \mathbb{R} \to \mathbb{R}$ defined by

$$\varphi(s) := \max(H_1(x,t,r,p'+se_N), H_2(x,t,r,p'+se_N)) .$$

This function is convex, continuous and coercive since both H_1, H_2 have these properties and therefore there exists $\bar{s} \in \mathbb{R}$ such that $\varphi(\bar{s}) = \min_{s\in\mathbb{R}} \varphi(s)$. As a consequence, $0 \in \partial\varphi(\bar{s})$, the convex subdifferential of φ.

We apply a classical result on the subdifferentials of convex functions defined as suprema of convex (or C^1) functions (cf [150]): here

$$\varphi(s) = \sup\left\{-b_1\cdot(p'+se_N) + c_1 r - l_1; -b_2\cdot(p'+se_N) + c_2 r - l_2\right\} ,$$

where the supremum is taken over all $(b_1,c_1,l_1) \in \mathbf{BCL}_1(x,t)$ and $(b_2,c_2,l_2) \in \mathbf{BCL}_2(x,t)$.

The functions $s \mapsto -b_i\cdot(p'+se_N) + c_i r - l_i$ for $i=1,2$ and $(b_i,c_i,l_i) \in \mathbf{BCL}_i(x,t)$ are all C^1 and $\partial\varphi(\bar{s})$ is the convex hull of their gradients for all the (b_i,c_i,l_i) such that $\varphi(\bar{s}) = -b_i\cdot(p'+se_N) + c_i r - l_i$. Since $\mathbf{BCL}_1(x,t), \mathbf{BCL}_2(x,t)$ are convex, this means that one of the following cases holds

(a) either the above supremum is only achieved at a unique (b_i,c_i,l_i) but then φ is differentiable at $\bar{s}$ and $0 = \varphi'(\bar{s}) = -b_i\cdot e_N$;

(b) or there exists $(b_1, c_1, l_1) \in \mathbf{BCL}_1(x,t)$, $(b_2, c_2, l_2) \in \mathbf{BCL}_2(x,t)$ and $\mu \in [0,1]$ such that

$$\begin{cases} \varphi(\bar{s}) = -b_1 \cdot (p' + \bar{s}e_N) + c_1 r - l_1 = -b_2 \cdot (p' + \bar{s}e_N) + c_2 r - l_2 \\ 0 = \mu(-b_1 \cdot e_N) + (1-\mu)(-b_2 \cdot e_N) \quad \text{i.e. } (\mu b_1 + (1-\mu)b_2) \cdot e_N = 0 \, . \end{cases}$$

In case (b), we deduce that

$$\begin{aligned} \varphi(\bar{s}) &= \mu(-b_1 \cdot (p' + \bar{s}e_N) + c_1 r - l_1) + (1-\mu)(-b_2 \cdot (p' + \bar{s}e_N) + c_2 r - l_2) \\ &= -(\mu b_1 + (1-\mu)b_2) \cdot p' + (\mu c_1 + (1-\mu)c_2) r - (\mu l_1 + (1-\mu) l_2) \\ &\leq \quad H_T(x,t,r,p') \, . \end{aligned}$$

But on the other hand, for any $(\tilde{b}_1, \tilde{c}_1, \tilde{l}_1) \in \mathbf{BCL}_1(x,t)$, $(\tilde{b}_2, \tilde{c}_2, \tilde{l}_2) \in \mathbf{BCL}_2(x,t)$ such that $(\tilde{\mu}\tilde{b}_1 + (1-\tilde{\mu})\tilde{b}_2) \cdot e_N = 0$ for some $\tilde{\mu} \in [0,1]$, the definition of φ implies that

$$\begin{aligned} \varphi(\bar{s}) &\geq \tilde{\mu}(-\tilde{b}_1 \cdot (p' + \bar{s}e_N) + \tilde{c}_1 r - \tilde{l}_1) + (1-\tilde{\mu})(-\tilde{b}_2 \cdot (p' + \bar{s}e_N) + \tilde{c}_2 r - \tilde{l}_2) \\ &= -(\tilde{\mu}\tilde{b}_1 + (1-\tilde{\mu})\tilde{b}_2) \cdot p' + (\mu \tilde{c}_1 + (1-\mu)\tilde{c}_2) r - (\mu \tilde{l}_1 + (1-\mu)\tilde{l}_2), \end{aligned}$$

which, taking the supremum on all such $(\tilde{b}_1, \tilde{c}_1, \tilde{l}_1)$, $(\tilde{b}_2, \tilde{c}_2, \tilde{l}_2)$ and $\tilde{\mu}$, gives $\varphi(\bar{s}) \geq H_T(x,t,r,p')$. Therefore, the equality holds, which gives the result.

Dealing with case (a) follows from the same arguments as in case (b), with $\mu = 0$ or 1. Hence the Lemma is proved.

Q.E.D.

10.3 A sufficient condition to get uniqueness

Applying directly Proposition 6.2.4 yields a condition under which $H_T = H_T^{\text{reg}}$.

Lemma 10.3.1 *We denote by $m_1^+(x,t,r,p')$ the largest minimum point of the function $s \mapsto H_1(x,t,r,p' + se_N)$ and $m_2^-(x,t,r,p')$ the least minimum of the function $s \mapsto H_2(x,t,r,p' + se_N)$. If $m_1^+(x,t,r,p') \leq m_2^-(x,t,r,p')$ for any (x,t,r,p') then $H_T = H_T^{\text{reg}}$ on $\mathcal{H} \times [0,T_f] \times \mathbb{R} \times \mathbb{R}^{N-1}$.*

The importance of this lemma is to give the

Corollary 10.3.2 *If $m_1^+(x,t,r,p') \leq m_2^-(x,t,r,p')$ for any $(x,t,r,p') \in \mathcal{H} \times [0,T_f] \times \mathbb{R} \times \mathbb{R}^{N-1}$, there is a unique solution of (7.1) in the sense of Ishii.*

Therefore we have an easy-to-check sufficient condition in order to have $\mathbf{U}^- = \mathbf{U}^+$, i.e. the uniqueness of the Ishii solution. Moreover this condition can be checked directly on the Hamiltonians H_1, H_2 without coming back to the control problem.

Remark 10.3.3 *In Part III, we consider the more general case when H_1, H_2 are only quasi-convex. We point out that the above results, namely Lemma 10.2.1 and 10.3.1 are of course still valid in the quasi-convex setting (in the codimension 1 case), provided that we use the definition of the $H_i^{\pm}$ through $f^\sharp, f_\flat, g^\sharp, g_\flat$. Indeed, in that way, the definitions do not require a control formulation. We come back later on this.*

10.4 More examples of uniqueness and non-uniqueness

In this section, we give two simple 1-d examples to illustrate Corollary 10.3.2. The first one is

$$\begin{cases} u_t + |u_x - 1| = 0 & \text{in } (-\infty, 0) \times (0, +\infty)\ , \\ u_t + |u_x + 1| = 0 & \text{in } (0, +\infty) \times (0, +\infty)\ , \\ u(x, 0) = |x| & \text{in } \mathbb{R}\ . \end{cases}$$

In this case, $m_1^+(x, t, r, p') = -1 < m_2^-(x, t, r, p') = 1$, uniqueness occurs and it is easy to compute the value functions

$$\mathbf{U}^-(x,t) = \mathbf{U}^+(x,t) = 2(|x| - t)^+ - |x| - (t - |x|)^+ = \begin{cases} 2(|x| - t)^+ - |x| & \text{if } |x| \geq t\ , \\ -t & \text{otherwise.} \end{cases}$$

Next, consider the problem

$$\begin{cases} u_t + |u_x + 1| = 0 & \text{in } (-\infty, 0) \times (0, +\infty)\ , \\ u_t + |u_x - 1| = 0 & (0, +\infty) \times (0, +\infty)\ , \\ u(x, 0) = |x| & \text{in } \mathbb{R}\ . \end{cases}$$

Here, on the contrary, $m_1^+(x, t, r, p') = 1 > m_2^-(x, t, r, p') = -1$, Corollary 10.3.2 does not apply and actually the value functions are different

$$\mathbf{U}^-(x,t) = \begin{cases} |x| & \text{if } |x| \geq t \\ 2|x| - t & \text{if } |x| \leq t \end{cases}$$

while $\mathbf{U}^+(x,t) = |x|$.

Chapter 11

Adding a Specific Problem on the Interface

This chapter is devoted to explain the main adaptations and differences when we consider the more general problem

$$\begin{cases} u_t + H_1(x,t,u,Du) = 0 & \text{for } x \in \Omega_1\,, \\ u_t + H_2(x,t,u,Du) = 0 & \text{for } x \in \Omega_2\,, \\ u_t + H_0(x,t,u,D_T u) = 0 & \text{for } x \in \mathcal{H}\,, \\ u(x,0) = u_0(x) & \text{for } x \in \mathbb{R}^N\,. \end{cases} \tag{11.1}$$

Here, since H_0 is only defined on $\mathcal{H}$, the gradient $D_T u$ consists only on the tangential derivative of u if $x = (x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R}$, $D_T u = D_{x'} u$ (or $(D_{x'} u, 0)$ depending on the convention we choose). In order to simplify some formula, we may write Du instead of $D_T u$ and therefore $H_0(x,t,u,Du)$ instead of $H_0(x,t,u,D_T u)$, keeping in mind that H_0 depends only on $p = Du$ through $p_T = D_T u$.

As we explained in Section 3.1, the conditions on $\mathcal{H}$ for those equations have to be understood in the relaxed (Ishii) sense, namely for (11.1)

$$\begin{cases} \max\Big(u_t + H_0(x,t,u,D_T u), u_t + H_1(x,t,u,Du), u_t + H_2(x,t,u,Du)\Big) \geq 0\,, \\ \min\Big(u_t + H_0(x,t,u,D_T u), u_t + H_1(x,t,u,Du), u_t + H_2(x,t,u,Du)\Big) \leq 0\,, \end{cases} \tag{11.2}$$

meaning that, for the supersolution [*resp.* subsolution] condition, at least one of the inequations has to hold.

In this section, we use the notation with H_0 as a sub/superscript in the mathematical objects to differentiate from the "non"-H_0 case since these are not exactly the

same, in particular of course, the value functions differ whether we have a specific control problem on $\mathcal{H}$ or not.

We say here that the "standard assumptions in the co-dimension-1 case" are satisfied for (11.1) if $(\mathbf{H_{BA-CP}})$ holds for (b_i, c_i, l_i), $i = 0, 1, 2$ and $(\mathbf{NC}_{\mathcal{H}})$ holds for H_1 and H_2.

11.1 The control problem

The control problem is solved exactly as in the case of (7.1) that was considered above. We just need to add a specific control set A_0 and triples (b_0, c_0, l_0), defining $\mathbf{BCL}_0(x,t)$ when $x \in \mathcal{H}$ as for $\mathbf{BCL}_1$ and $\mathbf{BCL}_2$. Since the case $i = 0$ is specific because $\mathcal{H}$ can be identified with $\mathbb{R}^{N-1} \times \{0\}$, we set for all (x, t, α_0), $b_0(x, t, \alpha_0) = (b_0'(x, t, \alpha_0), 0)$ so that $b_0 \cdot p$ reduces to the scalar product of the first $(N-1)$ components.

Using this convention, we define now the new $\mathbf{BCL}$ as

$$\mathbf{BCL}^{H_0}(x,t) := \begin{cases} \mathbf{BCL}_1(x,t) & \text{if } x \in \Omega_1\,, \\ \mathbf{BCL}_2(x,t) & \text{if } x \in \Omega_2\,, \\ \overline{\text{co}}(\mathbf{BCL}_0, \mathbf{BCL}_1, \mathbf{BCL}_2)(x,t) & \text{if } x \in \mathcal{H}\,, \end{cases}$$

where the convex hull takes into account here the three sets $\mathbf{BCL}_i$ for $i = 0, 1, 2$ so that of course, on $\mathcal{H}$ we make a convex combination of all the (b_i, c_i, l_i), $i = 0, 1, 2$.

Lemma 11.1.1 *The set-valued map* $\mathbf{BCL}^{H_0}$ *satisfies* $(\mathbf{H_{BCL}})$.

The proof is an obvious adaptation of Lemma 8.1.1, therefore we skip it.

In order to describe the trajectories of the differential inclusion with $\mathbf{BCL}^{H_0}$, we have to enlarge the control space with A_0 (and introduce a new parameter μ_0 for the convex combination)

$$A^{H_0} := A_0 \times A_1 \times A_2 \times \tilde{\Delta}\,, \quad \text{and} \quad \mathcal{A}^{H_0} := L^\infty(0, T_f; A^{H_0})\,.$$

Here, $\tilde{\Delta} = \{(\mu_0, \mu_1, \mu_2) \in [0,1]^3 : \mu_0 + \mu_1 + \mu_2 = 1\}$, so that the extended control takes the form $a = (\alpha_0, \alpha_1, \alpha_2, \mu_0, \mu_1, \mu_2)$ and if $x \in \mathcal{H}$,

$$(b_{\mathcal{H}}, c_{\mathcal{H}}, l_{\mathcal{H}}) = \mu_0(b_0, c_0, l_0) + \mu_1(b_1, c_1, l_1) + \mu_2(b_2, c_2, l_2)\,,$$

with $\mu_0 + \mu_1 + \mu_2 = 1$.

With this modification, solving the differential inclusion with $\mathbf{BCL}^{H_0}$ and the description of trajectories is similar to that in the $\mathbf{BCL}$-case (see Lemma 8.1.2), except that the control has the form $a(\cdot) = (\alpha_0, \alpha_1, \alpha_2, \mu_0, \mu_1, \mu_2)(\cdot) \in \mathcal{A}^{H_0}$.

Then we define $\mathbf{U}^-_{H_0}$ by

$$\mathbf{U}^-_{H_0}(x,t) := \inf_{\mathcal{T}_{H_0}(x,t)} \left\{ \int_0^t l(X(s), t-s, a(s)) \exp(-D(s))\, \mathrm{d}s + u_0(X(t)) \exp(-D(t)) \right\},$$

where $\mathcal{T}_{H_0}(x,t)$ is the space of trajectories associated with $\mathbf{BCL}^{H_0}$.

11.2 The minimal solution

As far as the value-function $\mathbf{U}^-_{H_0}$ is concerned, only easy adaptations are needed to handle H_0 and the related control problem. Of course we assume that H_0 also satisfies ($\mathbf{H_{Conv}}$), **(NC)**, **(TC)** and (**Mon**-u), as it is the case for H_1 and H_2.

Lemma 8.2.1 holds here with

$$H^{H_0}(x,t,u,p) := \sup_{(b,c,l)\in \mathbf{BCL}^{H_0}(x,t)} \big(-b\cdot p + cu - l \big),$$
$$\mathbb{F}^{H_0}(x,t,u,(p_x,p_t)) := p_t + H^{H_0}(x,t,u,p),$$

and of course we have to add H_0 in the max of the right-hand sides

$$H^{H_0}(x,t,r,p) = \max\Big(H_0(x,t,r,p), H_1(x,t,r,p), H_2(x,t,u,p) \Big),$$

$$\mathbb{F}^{H_0}(x,t,u,(p_x,p_t)) = \max\big(p_t + H_0(x,t,r,p), p_t + H_1(x,t,u,p), p_t + H_2(x,t,u,p) \big).$$

Then, minimality of $\mathbf{U}^-_{H_0}$ follows exactly as in Proposition 8.2.2

Proposition 11.2.1 *Assume that the "standard assumptions in the co-dimension-1 case" are satisfied for (11.1). Then the value function $\mathbf{U}^-_{H_0}$ is an Ishii viscosity solution of (11.1). Moreover $\mathbf{U}^-_{H_0}$ is the minimal supersolution of (11.1).*

Notice that a tangential dynamic $b \in \mathbf{B}^{H_0}_T(x,t)$ is expressed as a convex combination

$$b = \mu_0 b_0 + \mu_1 b_1 + \mu_2 b_2 \tag{11.3}$$

for which $\mu_0+\mu_1+\mu_2 = 1$, $\mu_0,\mu_1,\mu_2 \in [0,1]$ and $(\mu_1 b_1 + \mu_2 b_2)\cdot e_N = 0$ since, here, by definition, $b_0 \cdot e_N = 0$.

Then, all the results of Section 8.3 apply, except that we need a little adaptation for Lemma 8.3.1 in order to take into account the b_0-contribution.

Proof of Lemma 8.3.1 in the $\mathbf{BCL}^{H_0}$*-case* — The only modification consists in rewriting the convex combination as

$$\mu_0 b_0(x,t,\alpha_0) + (1-\mu_0)\left(\frac{\mu_1}{1-\mu_0} b_1(x,t,\alpha_1) + \frac{\mu_2}{1-\mu_0} b_2(x,t,\alpha_2)\right) ,$$

and we apply the arguments of Lemma 8.3.1 to the convex combination

$$\frac{\mu_1}{1-\mu_0} b_1(x,t,\alpha_1) + \frac{\mu_2}{1-\mu_0} b_2(x,t,\alpha_2) .$$

Then, setting

$$\psi^{H_0}(y,s) := \mu_0 b_0(x,t,\alpha_0) + (1-\mu_0)\left(\mu_1^\sharp(y,s)(b_1,c_1,l_1) + \mu_2^\sharp(y,s)(b_2,c_2,l_2)\right) ,$$

it is easy to check that the lemma holds for the $\mathbf{BCL}^{H_0}$-case.

Q.E.D.

Finally, the minimal solution $\mathbf{U}^-_{H_0}$ can also be characterized through $H_T^{H_0}$. The proof follows exactly the "non-H_0" case with obvious adaptations so that we omit it.

Theorem 11.2.2 *Assume that the "standard assumptions in the co-dimension-*1 *case" are satisfied for* (11.1)*. Then* $\mathbf{U}^-_{H_0}$ *is the unique Ishii solution of* (11.1) *such that*

$$u_t + H_T^{H_0}(x,t,u,D_T u) \le 0 \quad \text{on} \quad \mathcal{H}\times(0,T_f) ,$$

where, for $x\in\mathcal{H}$*,* $t\in[0,T_f]$*,* $r\in\mathbb{R}$*,* $p\in\mathbb{R}^{N-1}$*,*

$$H_T^{H_0}(x,t,r,p) := \sup_{(b,c,l)\in \mathbf{BCL}_T^{H_0}(x,t)} \big(-b\cdot p + cu - l\big) ,$$

$\mathbf{BCL}_T^{H_0}(x,t)$ *being the subset of all* $(b,c,l)\in\mathbf{BCL}^{H_0}(x,t)$ *for which* $b\in\mathbf{B}_T^{H_0}(x,t)$*.*

11.3 The maximal solution

Surprisingly, for the maximal solution, the case of (11.1) is very different. And we can see it on the result for subsolutions, analogue to Lemma 9.4.1

Lemma 11.3.1 *If* $u:\mathbb{R}^N\times(0,T_f)\to\mathbb{R}$ *is an u.s.c. subsolution of* (7.1)*, then it satisfies*

$$u_t + \min\big(H_0(x,t,u,D_T u), H_T^{\mathrm{reg}}(x,t,u,D_T u)\big) \le 0 \quad \text{on } \mathcal{H}\times(0,T_f) . \tag{11.4}$$

We omit the proof since it is the same as that of Lemma 9.4.1 (taking into account the b_0-terms), but of course the conclusion is that the H_0-inequality necessarily holds if the $H_T^{\rm reg}$ does not, hence the min.

The important fact in Lemma 11.3.1 is that, while, without H_0, (9.10) keeps the form of an HJB-inequality for a control problem, it is not the case anymore for (11.4) where the min looks more like an Isaacs equation associated to a differential game. As we already mention it in the introduction of this part, this is the analogue for discontinuities of the phenomena which arises in exit time problems/Dirichlet problem where the maximal Ishii subsolution involves a "worse stopping time" on the boundary: we refer to [23] and [31] for details.

As an illustration, let us provide the form of the maximal solution of (11.1) in the particular case when for any $x \in \mathcal{H}$, $t \in (0,T_f)$, $r \in \mathbb{R}$ and $p_T \in \mathbb{R}^{N-1}$

$$H_0(x,t,r,p_T) \le H_T^{\rm reg}(x,t,u,p_T)\ . \tag{11.5}$$

Proposition 11.3.2 *Assume that the "standard assumptions in the co-dimension-*1 *case" are satisfied and assume that* (11.5) *holds. Let* $V : \mathcal{H} \times (0,T_f) \to \mathbb{R}$ *be the unique solution of*

$$u_t + H_0(x,t,u,D_T u) = 0 \quad \text{on } \mathcal{H} \times (0,T_f)\,,$$

with the initial data $(u_0)_{|\mathcal{H}}$. *For* $i = 1,2$, *let* $V_i : \Omega_i \times [0,T_f] \to \mathbb{R}$ *be the unique solutions of the problems*

$$\begin{cases} u_t + H_i(x,t,u,Du) = 0 & \text{on } \Omega_i \times (0,T_f)\ , \\ u(x,t) = V(x,t) & \text{on } \mathcal{H} \times (0,T_f)\ , \\ u(x,0) = (u_0)_{|\overline{\Omega_i}} & \text{on } \overline{\Omega_i}\ . \end{cases}$$

Then the maximal (sub)solution of (11.1) *is given by*

$$\mathbf{U}^+_{H_0}(x,t) = \begin{cases} V_i(x,t) & \text{if } x \in \Omega_i \\ V(x,t) & \text{if } x \in \mathcal{H}\ . \end{cases}$$

Before giving the short proof of Proposition 11.3.2, we examine a simple example in dimension 1 showing the main features of this result. We take

$$\mathbf{BCL}_1(x,t) := \{(\alpha,0,0); \|\alpha| \le 1\}\ ,$$

$$\mathbf{BCL}_2(x,t) := \{(\alpha,0,1);\ |\alpha| \le 1\}\ ,$$

and $\mathbf{BCL}_0(0,t) = \{(0,0,2)\}$. In which case

$$H_1(p) = |p| \ , \ H_2(p) := |p| - 1 \ , \ H_T^{\rm reg} = 0 \ , \ H_0 = -2 \ .$$

Hence (11.5) holds. It is easy to check that, if $u_0(x) = 0$ for all $x \in \mathbb{R}$

$$V(t) = 2t \ , \ V_1(x,t) = 0 \ , \ V_2(x,t) = t \quad \text{for } x \in \mathbb{R}, \ t \geq 0.$$

This example shows several things: first, the value-function $\mathbf{U}^+_{H_0}$ is discontinuous although we have controllability/coercivity for the Hamiltonians H_1 and H_2; it is worth pointing out anyway that the global coercivity is lost since we use the Hamiltonian $\min(H_0, H_1, H_2)$ on $\mathcal{H}$ for the subsolutions instead of $\min(H_1, H_2)$.

Then, the values of $V(t)$ may seem strange since we use the maximal cost 2 but as we mention it above, this phenomena looks like the "worse stopping time" appearing in exit time problems. Finally, and this is even more surprising, the form of $\mathbf{U}^+_{H_0}$ shows that no information is transfered from Ω_1 to Ω_2: indeed, from the control point of view, starting from $x < 0$ where the cost is 1, it would seem natural to cross the border 0 to take advantage of the 0-cost in Ω_1 but this is not the case, even if $x < 0$ is close to 0. We have here two state-constraint problems, both in $\Omega_1 \times [0, T_f]$ and $\Omega_2 \times [0, T_f]$. This also means that the differential games features not only implies that one is obliged to take the maximal cost at $x = 0$ but also may prevent the trajectory to go from a less favourable region to a more favourable region.

Unfortunately we are unable to provide a general formula for $\mathbf{U}^+_{H_0}$, i.e. which would be valid for all cases without (11.5). Of course, trying to define $\mathbf{U}^+_{H_0}$ as in Proposition 11.3.2 but V being the solution of

$$u_t + \min\left\{H_0(x,t,u,D_T u), H_T^{\rm reg}(x,t,u,D_T u)\right\} = 0 \quad \text{on } \mathcal{H} \times (0,T_f)\,, \tag{11.6}$$

does not work as the following example shows. In dimension 1, we take $H_1(p) = H_2(p) = |p|$, $H_0 > 0$ and $u_0(x) = -|x|$ in $\mathbb{R}$. Since $H_T^{\rm reg} = 0$, we have $H_0 > H_T^{\rm reg}$ and solving the above pde gives $V = 0$. Computing V_1 and V_2 as above gives $-|x| - t$ in both cases. Hence V_1 and V_2 are just the restriction to $\Omega_1 \times [0, T_f]$ and $\Omega_2 \times [0, T_f]$ respectively of the solution of

$$u_t + |u_x| = 0 \quad \text{in } \mathbb{R} \times (0, T_f) \ ,$$

with the initial data u_0. Now defining $\mathbf{U}^+_{H_0}$ as in Proposition 11.3.2, we see that we do not have a subsolution: indeed the discontinuity of $\mathbf{U}^+_{H_0}$ at any point $(0,t)$ implies that $(0,t)$ is a maximum point of $\mathbf{U}^+_{H_0} - px$ for any $p \in \mathbb{R}$ and therefore we should have the inequality

$$\min(H_0, |p|, |p|) \leq 0 \ ,$$

which is not the case if $|p| > 0$.

Remark 11.3.3 *Even if we were are able to provide a general formula for* $\mathbf{U}^+_{H_0}$*, we have some (again strange) information on this maximal subsolution: first* $\mathbf{U}^+_{H_0} \geq \mathbf{U}^+$ *in* $\mathbb{R}^N \times (0,T_f)$ *since* $\mathbf{U}^+$ *is a subsolution of* (11.1)*. A surprising result since it shows that adding* H_0 *on* $\mathcal{H} \times (0,T_f)$ *does not decrease the maximal subsolution as it could be thought from the control interpretation. On the other hand, Lemma 11.3.1 provides an upper estimate of* $\mathbf{U}^+_{H_0}$ *on* $\mathcal{H} \times (0,T_f)$*, namely the solution of* (11.6)*.*

Proof of Proposition 11.3.2 — First, by our assumptions, V exists and is continuous, since it is obtained by solving a standard Cauchy problem in $\mathbb{R}^{N-1} \times [0,T_f]$. Next by combining the argument of [23] (See also [31]) with the localization arguments of Section 3.2, V_1 and V_2 exist and are continuous in $\Omega_1 \times [0,T_f]$ and $\Omega_2 \times [0,T_f]$ respectively, with continuous extensions to $\overline{\Omega_1} \times [0,T_f]$ and $\overline{\Omega_2} \times [0,T_f]$.

Considering the Cauchy-Dirichlet problems in Ω_1 and Ω_2, we refer the reader to Proposition 27.1.1-(i) where it is proved that the normal controllability implies

$$V_1(x,t), V_2(x,t) \leq V(x,t) \quad \text{on } \mathcal{H} \times (0,T_f)\ .$$

Hence, defined in that way, $\mathbf{U}^+_{H_0}$ is upper semicontinuous (it may be discontinuous as we already saw above).

It is easy to check that $\mathbf{U}^+$ is a solution of (11.1). Indeed the subsolution properties on $\Omega_1 \times (0,T_f), \Omega_2 \times (0,T_f)$ are obvious. On $\mathcal{H} \times (0,T_f)$ they come from the properties of V since $\mathbf{U}^+_{H_0} = V$ on $\mathcal{H} \times (0,T_f)$; hence the H_0-inequality for V implies the subsolution inequality for $\mathbf{U}^+$.

For the supersolution ones, they comes from the properties of V_1, V_2 and V and the formulation of the Dirichlet problem since $(\mathbf{U}^+_{H_0})_* = \min(V_1, V_2, V) = \min(V_1, V_2)$ on $\mathcal{H} \times (0,T_f)$. Indeed if ϕ is a smooth function in $\mathbb{R}^N \times (0,T_f)$ and if $(\bar{x},\bar{t}) \in \mathcal{H} \times (0,T_f)$ is a minimum point of $(\mathbf{U}^+_{H_0})_* - \phi$, there are several cases:

(a) if $(\mathbf{U}^+_{H_0})_*(\bar{x},\bar{t}) = V_1(\bar{x},\bar{t}) < V(\bar{x},\bar{t})$, then $(\bar{x},\bar{t})$ is a minimum point of $V_1 - \phi$ on $\overline{\Omega_1} \times (0,T_f)$ and, since V_1 is a solution of the Dirichlet problem in $\overline{\Omega_1} \times (0,T_f)$ with the Dirichlet data V, we have

$$\max\big(\phi_t(\bar{x},\bar{t}) + H_1(\bar{x},\bar{t},V_1(\bar{x},\bar{t}), D\phi(\bar{x},\bar{t})), V_1(\bar{x},\bar{t}) - V(\bar{x},\bar{t})\big) \geq 0\ .$$

Hence $\phi_t(\bar{x},\bar{t}) + H_1(\bar{x},\bar{t},V_1(\bar{x},\bar{t}), D\phi(\bar{x},\bar{t})) \geq 0$, which gives the answer we wish.

(b) The case when $(\mathbf{U}^+_{H_0})_*(\bar{x},\bar{t}) = V_2(\bar{x},\bar{t}) < V(\bar{x},\bar{t})$ is treated in a similar way.

(c) Finally if $(\mathbf{U}^+_{H_0})_*(\bar{x},\bar{t}) = V_1(\bar{x},\bar{t}) = V_2(\bar{x},\bar{t}) = V(\bar{x},\bar{t})$, we use that $(\bar{x},\bar{t})$ is a minimum point of $V - \phi$ on $\mathcal{H} \times (0,T_f)$ and therefore

$$\phi_t(\bar{x},\bar{t}) + H_0(\bar{x},\bar{t},V(\bar{x},\bar{t}), D\phi(\bar{x},\bar{t})) \geq 0\ ,$$

implying the viscosity supersolution inequality we wanted.

It remains to prove that any subsolution u of (11.1) is below $\mathbf{U}^+_{H_0}$. This comes from Lemma 11.3.1 which implies, using a standard comparison result on $\mathcal{H} \times [0, T_f]$ that $u(x,t) \leq V(x,t) = \mathbf{U}^+_{H_0}(x,t)$ on $\mathcal{H} \times [0, T_f]$.

Q.E.D.

Chapter 12

Remarks on the Uniqueness Proofs, Problems Without Controllability

12.1 The main steps of the uniqueness proofs and the role of the normal controllability

In this part, we have proved several comparison results showing, on one hand, that $\mathbf{U}^-$ is the minimal supersolution and the unique solution which satisfies the H_T-inequality and, on the other hand, that $\mathbf{U}^+$ is the maximal subsolution and the unique solution which satisfies the H_T^{reg}-inequality.

All the proofs of these results are based on a common strategy which will also be used for stratified problems in Part IV and which can be described in the following "backwards" way

Step 3: The "Magical Lemma" According to Section 3.2 the comparison result is reduced to proving that **(LCR)** holds. For the points located on $\mathcal{H}$, this is a direct consequence of Lemma 5.4.1 if the subsolution is continuous and C^1 in the tangential variables. This tangential regularity allows to use the subsolution as a test-function for the "tangential inequalities" (typically the H_T or H_T^{reg} one), avoiding in particular the usual "doubling of variables" which causes the major problem in the discontinuous setting.

Step 2: Regularization of the subsolution In order to use the "Magical Lemma" to obtain the result for any subsolution, we have to be able to regularize any

subsolution in order that it becomes continuous w.r.t. all the variables, C^1 in the tangent variables, and preserving the subsolution inequalities. This is the role of Propositions 3.4.4 and 3.4.7.

Step 1: Regularity of the subsolution In order to perform the second step in a suitable way, we need at least the subsolution to be regular on $\mathcal{H}$. In particular this is necessary in order that the second step actually provides a subsolution which is continuous on $\mathcal{H}$ (but also on the hyperplanes which are parallel to $\mathcal{H}$).

Going further in the analysis of these three steps, it is clear that the normal controllability assumption **(NC)** plays a crucial role in Step 1 but even more in Step 2. Looking at Proposition 3.4.2, recalling that **(NC)** implies **(NC$_w$)**, Case (a) immediately gives us the complete information we need, even if we can obtain it through Cases (b) and (c) in some situations, see the examples below.

But this is in Step 2 that **(NC)** plays the most important (an maybe unavoidable) role: in order to perform the tangential regularization we have to control, one way or the other, the normal component of the gradient. This is exactly the role of **(NC)**.

This is why we consider **(NC)** as a key "natural" assumption in this type of problems and the fact that the same remarks can be made for stratified problems reinforces this certainty. Being unable to perform the regularization process, the "Magical Lemma" cannot be used and all the proofs collapse.

We also point out that the approach via "Flux-Limited Solutions" described in Part III provides an alternative strategy which seems to avoid some of the above constraints, and in particular **(NC)**. The comparison proof is based on an "almost classical" doubling of variables but the reader can check that this proof actually uses **(NC)** in several ways.

However, some problems without normal controllability can also be treated and we give some examples in the next section.

12.2 Some problems without controllability

In this section, we are not going to examine sophisticated situations: if **(NC)** is not satisfied on $\mathcal{H}$ and if we have a mixture of the different "simple" situations we describe below on $\mathcal{H}$, we are led to problems whose difficulties have to be examined separately. A combination of the arguments which are presented in this book may allow to treat such problems but, in a general framework, this will not be the case.

In the simple situations we are going to emphasize, we examine the situation sep-

arately on both sides of the discontinuity and we respectively denote by u_1 and u_2 the solutions in $\overline{\Omega_1} \times (0, T_f)$ and $\overline{\Omega_2} \times (0, T_f)$. Therefore the value function $\mathbf{U}$ of the control problem in $\mathbb{R}^N \times (0, T_f)$ will be given by

$$\mathbf{U}(x,t) = \begin{cases} u_1(x,t) & \text{if } x \in \Omega_1 \ , \\ u_2(x,t) & \text{if } x \in \Omega_2 \end{cases}$$

while on $\mathcal{H} \times (0, T_f)$, either $\mathbf{U}$ will be the common value of u_1 and u_2 or the l.s.c./u.s.c. envelopes, computed by using values in $\Omega_1 \times (0, T_f)$ and $\Omega_2 \times (0, T_f)$.

The simple situations we have in mind are the following

I. For any $x \in \mathcal{H}$, $t \in [0, T_f]$, $\alpha_2 \in A_2$, $b_2(x, t, \alpha_2) \cdot e_N > 0$. All the dynamics used in Ω_2 are strictly pointing outside $\overline{\Omega_2}$ on $\mathcal{H}$. Here, it is easy to show that H_2 plays the role of a nonlinear Neumann boundary condition on $\mathcal{H}$ for the equation $H_1 = 0$ in $\Omega_1 \times (0, T_f)$[(1)]. Therefore, in order to obtain u_1, we solve this nonlinear Neumann problem in $\Omega_1 \times (0, T_f)$. We obtain a unique continuous solution u_1 (which is continuous up to the boundary). Then, in order to compute u_2, we solve the Dirichlet problem in $\overline{\Omega_2} \times (0, T_f)$ with u_1 as Dirichlet boundary condition on $\mathcal{H} \times (0, T_f)$. This also provides a continuous solution in $\overline{\Omega_2} \times (0, T_f)$ and that way, we have defined a continuous function in $\mathbb{R}^N$ which is the solution of Problem 7.1 and the value function of the associated control problem.

II. By symmetry the situation is the same if, for any $x \in \mathcal{H}$, $t \in [0, T_f]$, $\alpha_1 \in A_1$, $b_1(x, t, \alpha_1) \cdot e_N < 0$.

III. For any $x \in \mathcal{H}$, $t \in [0, T_f]$, $\alpha_2 \in A_2$, $b_2(x, t, \alpha_2) \cdot e_N \leq 0$. Then all the trajectories of the dynamic starting in $\Omega_2 \times (0, T_f)$ stay in $\Omega_2 \times (0, T_f)$. In terms of PDE, the consequence is that all the viscosity inequalities for sub and supersolutions hold up to the boundary of $\Omega_2 \times (0, T_f)$, as soon as these sub and supersolutions are extended up the boundary by upper or lower-semicontinuity. Hence the associated HJB problem is $H_2 = 0$ on $\overline{\Omega_2} \times (0, T_f)$. As in the whole space $\mathbb{R}^N$, this problem enjoys a comparison result and therefore it provides a unique solution $u_2 \in C(\overline{\Omega_2} \times (0, T_f))$. This solution is the value function in $\Omega_2 \times T_f$, extended to $\overline{\Omega_2} \times (0, T_f)$ by continuity[(2)]. Therefore the problem in Ω_2 completely ignores the problem in Ω_1, and we face 3 different cases for the problem in Ω_1

III.1 For any $x \in \mathcal{H}$, $t \in [0, T_f]$, $\alpha_1 \in A_1$, $b_1(x, t, \alpha_1) \cdot e_N < 0$, a case which is already treated in **II** above. But here we are in the case of a simple Dirichlet

[(1)] We give a proof at the end of this section for the reader's convenience

[(2)] Therefore u_2 is equal to the $\mathbb{R}^N$-value function in $\Omega_2 \times T_f$ but maybe not on $\mathcal{H} \times (0, T_f)$.

problem in $\Omega_1 \times T_f$, the Dirichlet boundary condition on $\mathcal{H} \times T_f$ being the value function of the problem in Ω_2. Hence there is a unique continuous solution for Problem 7.1 which is the value function of the control problem in $\mathbb{R}^N$.

III.2 For any $x \in \mathcal{H}$, $t \in [0, T_f]$, there exist $\alpha_1^1, \alpha_1^2 \in A_1$ such that $b_1(x,t,\alpha_1^1) \cdot e_N > 0$ and $b_1(x,t,\alpha_1^2) \cdot e_N < 0$, *i.e.* the normal controllability condition holds. In this case, we also have a Dirichlet problem in $\Omega_1 \times T_f$ with the Dirichlet boundary condition on $\mathcal{H}$ being the value function of the problem in $\Omega_2 \times T_f$. However, while in **III.1** the boundary data is assumed in a classical sense and leads to a continuous solution in $\mathbb{R}^N$, here it is only assumed in the viscosity sense. The value function in $\Omega_1 \times T_f$ being not equal, in general, to the one in $\Omega_2 \times T_f$ in all $\mathcal{H} \times T_f$, the value function of the problem in $\mathbb{R}^N$ may have discontinuities on $\mathcal{H} \times T_f$.

III.3 For any $x \in \mathcal{H}$, $t \in [0, T_f]$, $\alpha_1 \in A_1$, $b_1(x,t,\alpha_1) \cdot e_N \geq 0$: here the problem in $\Omega_1 \times T_f$ and $\Omega_2 \times T_f$ are completely independent. There exists both a unique value function in $\overline{\Omega_1} \times T_f$ and $\overline{\Omega_2} \times T_f$ but their continuous extensions to $\mathcal{H} \times T_f$ are different in general, and the value function in $\mathbb{R}^N$ may have discontinuities on $\mathcal{H} \times T_f$. Anyway the Ishii conditions are satisfied on $\mathcal{H} \times T_f$ since both equations hold up to the boundary.

We conclude this section by proving that, as announced in Case **I** above, dynamics pointing outward generate a nonlinear Neumann boundary condition.

Proposition 12.2.1 *Assume that the "standard assumptions in the co-dimension-1 case" are satisfied and that, for any $x \in \mathcal{H}$, $t \in [0, T_f]$, $\alpha_2 \in A_2$, $b_2(x,t,\alpha_2) \cdot e_N > 0$. Then any locally bounded u.s.c. subsolution [resp. l.s.c. supersolution v] of Problem 7.1 is a subsolution [resp. supersolution] of the nonlinear Neumann problem*

$$\begin{cases} u_t + H_1(x,t,u,D_x u) = 0 & \text{in } \Omega_1 \times (0,T_f) \\ u_t + H_2(x,t,u,D_x u) = 0 & \text{on } \mathcal{H} \times (0,T_f)\,. \end{cases} \tag{12.1}$$

We recall that Neumann boundary conditions for first-order HJ Equations were first studied by Lions [135] and then different comparison results for first and second-order equations were obtained by Ishii [121] and Barles[32]. We refer the reader to the "User's guide to viscosity solutions" of Crandall, Ishii and Lions [67] for a complete introduction of boundary conditions in the viscosity sense and to all these references for checking that the nonlinearity $p_t + H_2(x,t,r,p_x) = 0$ satisfies all the requirement for a nonlinear Neumann boundary condition.

Proof — Of course, we just have to check the boundary condition and we provide the proof only in the subsolution case, the supersolution one being analogous. Let $\phi \in C^1(\mathbb{R}^N \times (0,T_f))$ and let $(\bar{x},\bar{t}) \in \mathcal{H} \times (0,T_f)$ be a strict local maximum point of $u-\phi$. For $0 < \varepsilon \ll 1$, we consider the penalized function

$$(x,t) \mapsto u(x,t) - \phi(x,t) - \frac{[x_N^-]^2}{\varepsilon}$$

An easy application of Lemma 6.4.1 in a compact neighborhood of $(\bar{x},\bar{t})$ shows the existence of a sequence $(x_\varepsilon,t_\varepsilon)$ of maximum points for these functions such that $(x_\varepsilon,t_\varepsilon) \to (\bar{x},\bar{t})$ and $u(x_\varepsilon,t_\varepsilon) \to u(\bar{x},\bar{t})$. If $(x_\varepsilon,t_\varepsilon) \in \overline{\Omega_1} \times T_f$, we have

$$\begin{aligned} &\text{either } \phi_t(x_\varepsilon,t_\varepsilon) + H_1(x_\varepsilon,t_\varepsilon,u(x_\varepsilon,t_\varepsilon),D_x\phi(x_\varepsilon,t_\varepsilon)) \le 0 \\ &\quad \text{or } \phi_t(x_\varepsilon,t_\varepsilon) + H_2(x_\varepsilon,t_\varepsilon,u(x_\varepsilon,t_\varepsilon),D_x\phi(x_\varepsilon,t_\varepsilon)) \le 0\,, \end{aligned}$$

the derivative of the term $\dfrac{[x_N^-]^2}{\varepsilon}$ being 0. Hence the only difficulty is when $(x_\varepsilon,t_\varepsilon) \in \Omega_2 \times T_f$ and

$$\phi_t\Big(x_\varepsilon,t_\varepsilon) + H_2(x_\varepsilon,t_\varepsilon,u(x_\varepsilon,t_\varepsilon),D_x\phi(x_\varepsilon,t_\varepsilon) - \frac{2x_N^-}{\varepsilon}e_N\Big) \le 0\,.$$

But examining H_2 and using the fact that, for any $x \in \mathcal{H}$, $t \in [0,T_f]$, $\alpha_2 \in A_2$, $b_2(x,t,\alpha_2)\cdot e_N > 0$, we see that $\lambda \mapsto H_2(x,t,r,p_x+\lambda e_N)$ is decreasing for all $x \in \mathcal{H}$, $t \in [0,T_f]$, $r \in \mathbb{R}$ and $p_x \in \mathbb{R}^N$. Therefore

$$H_2(x_\varepsilon,t_\varepsilon,u(x_\varepsilon,t_\varepsilon),D_x\phi(x_\varepsilon,t_\varepsilon) - \frac{2x_N^-}{\varepsilon}e_N) \ge H_2(x_\varepsilon,t_\varepsilon,u(x_\varepsilon,t_\varepsilon),D_x\phi(x_\varepsilon,t_\varepsilon))$$

and we also get in this case

$$\phi_t(x_\varepsilon,t_\varepsilon) + H_2(x_\varepsilon,t_\varepsilon,u(x_\varepsilon,t_\varepsilon),D_x\phi(x_\varepsilon,t_\varepsilon)) \le 0\,.$$

In any case

$$\begin{aligned} \min\Big(&\phi_t(x_\varepsilon,t_\varepsilon) + H_1(x_\varepsilon,t_\varepsilon,u(x_\varepsilon,t_\varepsilon),D_x\phi(x_\varepsilon,t_\varepsilon)), \\ &\phi_t(x_\varepsilon,t_\varepsilon) + H_2(x_\varepsilon,t_\varepsilon,u(x_\varepsilon,t_\varepsilon),D_x\phi(x_\varepsilon,t_\varepsilon))\Big) \le 0\,, \end{aligned}$$

and letting $\varepsilon \to 0$, we obtain the desired inequality

$$\min\Big(\phi_t(\bar{x},\bar{t}) + H_1(\bar{x},\bar{t},u(\bar{x},\bar{t}),D_x\phi(\bar{x},\bar{t}))\ ,\ \phi_t(\bar{x},\bar{t}) + H_2(\bar{x},\bar{t},u(\bar{x},\bar{t}),D_x\phi(\bar{x},\bar{t}))\Big) \le 0\,.$$

Q.E.D.

Chapter 13

Further Discussions and Open Problems

13.1 The Ishii subsolution inequality: natural or unnatural from the control point of view?

As it is well-known, the Ishii supersolution inequality is very natural from the control point of view, and even in a very general framework. The reader can be convinced by this claim by looking at Chapter 4, and in particular at Theorem 4.3.4 and Corollary 4.4.3: involving the natural $\mathbb{F}$-Hamiltonian, the proof that the value function is a supersolution—and even the minimal supersolution—is rather easy and reflects as expected the property of the control problem, since it is related to the existence of an optimal trajectory.

On the contrary, the proof of the subsolution inequality—which has to handle $\mathbb{F}_*$—is far more involved, *cf.* Theorem 4.3.5, and no analogue of Corollary 4.4.3 exists. This rises the question: is this Ishii subsolution inequality so natural from the control point of view?

Why the Ishii inequality should not hold — We can provide the beginning of an answer in a rather simple way in the two-domains case. We recall that the role of the subsolution inequality is to reflect the fact that each control (or trajectory) is suboptimal.

If $U = \mathbf{U}^-$ or $\mathbf{U}^+$, if $(x,t) \in \mathcal{H} \times (0,T_f)$ and if α_1 is a control such that $b_1(x,t,\alpha_1) \cdot$

$e_N > 0$, we solve the ode

$$\dot{X}(s) = b_1(X(s), t-s, \alpha_1)\ ,\ X(0) = x\ ,$$

and we remark that, for $s > 0$ small enough, $X(s) \in \Omega_1$. Therefore the trajectory $X(\cdot)$ is admissible and an easy application of the Dynamic Programming Principle (where we assume that we already know that U is continuous for simplicity) implies, for $h > 0$ small enough

$$U(x,t) \le \int_0^h l\big(X(s), t-s, \alpha_1\big)\, e^{-D(s)}\, \mathrm{d}s + U(X(h), t-h)\, e^{-D(h)}\ .$$

We easily deduce that, for such α_1

$$-b_1(x,t,\alpha_1)\cdot DU(x,t) + c_1(x,t,\alpha_1)U(x,t) - l_1(x,t,\alpha_1) \le 0 \tag{13.1}$$

and this inequality can easily be extended to all α_1 such that $b_1(x,t,\alpha_1)\cdot e_N \ge 0$.

On the contrary, if $b_1(x,t,\alpha_1)\cdot e_N < 0$, $X(s) \in \Omega_2$ for $s > 0$ small enough and $X(\cdot)$ is not an admissible trajectory anymore since the dynamic is b_2 in $\Omega_2 \times (0, T_f)$; so there is no reason why (13.1) should hold.

This implies a fortiori that there is no reason why $U_t + H_1(x,t,U(x,t),DU(x,t))$ should be nonpositive and, since we can argue exactly in the same way with control α_2 associated to the control problem in $\Omega_2 \times (0,T_f)$, there is also no reason why $U_t + H_2(x,t,U(x,t),DU(x,t))$ should be nonpositive either. Hence, the Ishii subsolution inequality, namely

$$\min(U_t + H_1(x,t,U,DU), U_t + H_2(x,t,U,DU)) \le 0 \text{ on } \mathcal{H} \times (0,T_f) \tag{13.2}$$

is not natural at all from the control point of view.

Why the Ishii inequality actually holds — The proof of Proposition 9.1.3 gives a first way to answer this puzzle in the case of $\mathbf{U}^+$ (the argument would be exactly the same in the case of $\mathbf{U}^-$).

On one hand, the H_T-inequality is natural since it shows that all the admissible trajectories which stay on $\mathcal{H}$ are suboptimal. On the other hand, if $\mathbf{U}_t^+ + H_1 \le 0$, the Ishii inequality holds while if $\mathbf{U}_t^+ + H_1 > 0$, the inequality $\mathbf{U}_t^+ + H_T \le 0$ implies that necessarily $\mathbf{U}_t^+ + H_2 \le 0$, since the dynamics such that the X-trajectories stay on $\mathcal{H}$ are convex combinations of the b_1 and b_2-ones. In any case we obtain (13.2) for $\mathbf{U}^+$.

Hence, the Ishii subsolution inequality holds on $\mathcal{H} \times (0,T_f)$ as a consequence of the natural H_T-inequality. And one may wonder whether it is not more natural to define subsolution by just imposing the H_T-inequality on $\mathcal{H} \times (0,T_f)$, dropping (13.2).

This is exactly what the notion of Flux-Limited Solutions is doing, *cf.* Chapter 15. Indeed, as a by-product of the argument which leads to (13.1), we have natural H_1^+ and H_2^- inequalities, at least for the value-functions $\mathbf{U}^-, \mathbf{U}^+$.

Moreover it is clear that this last remark remains valid in far more general cases: we have natural subsolution inequalities for the controls for which the dynamics "move away from the discontinuities".

General subsolutions, General discontinuities: the stratified case — Maybe looking only at value-functions is misleading since we know that Theorem 4.3.5 holds and maybe also that the two-domains case is a very particular situation regarding the Ishii subsolution inequality on the discontinuity.

This suggests a more general question: for unnatural reasons, the $\mathbb{F}_* \leq 0$ inequality holds on discontinuities for value functions; does this "little miracle" hold both for general subsolutions and for more complicated discontinuities?

Surprisingly the answer is yes in the stratified framework under suitable assumptions: in Section 20.5, it is a consequence of a **(LCR)** in the case of regular subsolutions and this points out that such inequality always holds *for any regular subsolution* provided a comparison result holds. Hence the $\mathbb{F}_* \leq 0$-inequality on discontinuities appears more as a consequence than as a required inequality in the definition.

This is confirmed by the fundamental Lemma 5.4.1 which is the keystone to prove comparison results: this lemma is based on (i) a "tangential inequality" on the discontinuity (for example the H_T-inequality on $\mathcal{H} \times (0, T_f)$) and (ii) a subdynamic programming principle for the subsolution *outside* the discontinuity (in $\Omega_1 \times (0, T_f)$ and in $\Omega_2 \times (0, T_f)$ here). None of these ingredients uses the Ishii subsolution inequality on $\mathcal{H} \times (0, T_f)$.

As a conclusion of this section, we can remark that, thanks to the above arguments, imposing or not the Ishii subsolution inequality on the discontinuities is not a real issue: one way or the other, it will hold at least in frameworks where a suitable comparison result holds.

But as the reader can notice everywhere in this book, even if it is not the only way to obtain it, the Ishii subsolution inequality on $\mathcal{H} \times (0, T_f)$ provides the regularity of subsolutions, a fundamental ingredient. This is why we make the choice to maintain it most of the time.

13.2 The infinite horizon / stationary equation case

The aim of this section is to briefly describe the analogous results in the infinite horizon case where the HJ-equation is stationary: we will only skim over this problem since all the results are not only straightforward translations and adaptations of the finite horizon/evolution equations case but the proofs are even simpler from a technical point-of-view. We recall that this case was studied in details in the works of Briani and the authors of this book (cf. [16, 17]) and actually almost all the ideas and results of this part appear for the first time in these two articles.

From the control point-of-view, we are given for $x \in \Omega_i$ and for $i = 1, 2$

$$\mathbf{BCL}_i(x) := \{(b_i(x,\alpha), c_i(x,\alpha), l_i(x,\alpha)) \ : \ \alpha \in A\} \ ,$$

where, as above, the control set A is a compact metric space and the (b_i, c_i, l_i) are defined on $\mathbb{R}^N \times A_i$ and satisfy $(\mathbf{H_{BA-CP}})$. We assume, in addition, that there exists $\lambda > 0$ such that, for $i = 1, 2$,

$$c_i(x,\alpha) \geq \lambda \quad \text{in } \mathbb{R}^N \times A \ .$$

As in the finite horizon case, we define $\mathbf{BCL}(x)$ as $\mathbf{BCL}_i(x)$ if $x \in \Omega_i$ and as the closed convex envelope of $\mathbf{BCL}_1(x) \cup \mathbf{BCL}_2(x)$ if $x \in \mathcal{H}$.

Using this $\mathbf{BCL}$, we can solve the differential inclusion equation for (X, D, L)

$$(\dot{X}(s), \dot{D}(s), \dot{L}(s)) \in \mathbf{BCL}(X(s)) \ ,$$

with $(X(0), D(0), L(0)) = (x, 0, 0)$. We can also define "regular" and "singular" dynamics on $\mathcal{H}$, $\mathcal{T}(x)$, $\mathcal{T}^{reg}(x)$ and the tangential Hamiltonians H_T, H_T^{reg}. The associated value functions are

$$\mathbf{U}^-(x) := \inf_{\mathcal{T}(x)} \left\{ \int_0^{+\infty} l(X(s), a(s)) \exp(-D(s)) \, \mathrm{d}s \right\} \ ,$$

$$\mathbf{U}^+(x) := \inf_{\mathcal{T}^{\mathrm{reg}}(x)} \left\{ \int_0^{+\infty} l(X(s), a(s)) \exp(-D(s)) \, \mathrm{d}s \right\} \ ,$$

where $l(X(s), a(s))$ is defined as in Theorem 4.2.1.

From the pde point-of-view, the related problem is

$$\begin{cases} H_1(x, u, Du) = 0 & \text{in } \Omega_1 \ , \\ H_2(x, u, Du) = 0 & \text{in } \Omega_2 \ , \end{cases} \tag{13.3}$$

with the standard Ishii inequalities on $\mathcal{H}$ where, for $i = 1, 2$,

$$H_i(x, r, p) := \sup_{\alpha \in A} \{-b_i(x, \alpha) \cdot p + c_i(x, \alpha)r - l_i(x, \alpha)\} \ .$$

The result is the following

Theorem 13.2.1 *Under the above assumptions*
(i) the value functions $\mathbf{U}^-, \mathbf{U}^+$ *are well-defined and bounded. They are viscosity solutions of* (13.3).
(ii) The value function $\mathbf{U}^-$ *satisfies*

$$H_T(x, u, Du) \leq 0 \quad \text{on } \mathcal{H} \ , \tag{13.4}$$

while the value function $\mathbf{U}^+$ *satisfies*

$$H_T^{reg}(x, u, Du) \leq 0 \quad \text{on } \mathcal{H} \ , \tag{13.5}$$

(iii) The value function $\mathbf{U}^-$ *is the minimal viscosity supersolution (and solution) of* (13.3)*, while* $\mathbf{U}^+$ *is the maximal viscosity subsolution (and solution) of* (13.3).
(iv) The value function $\mathbf{U}^-$ *is the unique viscosity solution of* (13.3) *which satisfies* (13.4).

We leave the proof of this theorem to the reader since, as we already wrote it above, it is a routine adaptation of the ideas described in this part.

13.3 Towards more general discontinuities: a bunch of open problems.

13.3.1 A summary of this part and related questions

A very basic and minimal summary of Part II—including the previous section—can be expressed as follows: for Problem (7.1), we are able to provide an explicit control formula for the minimal supersolution (and solution) $\mathbf{U}^-$, and also an explicit control formula for the maximal (and solution) $\mathbf{U}^+$.

The next natural questions are: is it possible to extend such results to more general type of discontinuities? It can also be thought that some of them are very particular cases which only appear because of the codimension 1 discontinuity and that simpler results may exist for higher codimensions because of some kind of "eliminability

property" (?). This idea can only be reinforced by the fact that, as we will see it in Part III, $\mathbf{U}^+$ is the limit of the vanishing viscosity method.

Before coming back to this question of $\mathbf{U}^+$ or more precisely to the identification of the maximal subsolution, we consider the case of $\mathbf{U}^-$, which may be perhaps considered as being the more natural solution from the control point of view. Here the answer to the above question is yes and this is not so surprising since, by Corollary 4.4.3, we know in a very general framework that $\mathbf{U}^-$ is the minimal viscosity supersolution of the Bellman Equations, therefore we already have a lot of informations on $\mathbf{U}^-$.

In the Part IV, we provide a rather complete study of stratified solutions in $\mathbb{R}^N$ and then, in Part V in general domains, which are the natural generalization of $\mathbf{U}^-$ in the case when the codimension-1 discontinuity is replaced by discontinuities on Whitney stratifications. As in Section 8, we characterize the stratified solution $\mathbf{U}^-$ as the unique solution of a suitable problem with suitable viscosity inequalities. The methods which are used to study Ishii solutions, relying partly on control arguments and partly on pde ones, can be extended to this more general setting and we will emphasize the (even more important) roles of the subsolution inequalities, normal controlability, tangential continuity...etc.

But the case of the maximal subsolution (and solution) $\mathbf{U}^+$ is more tricky and several questions can be asked, in particular

(i) Can one provide an explicit control formula for $\mathbf{U}^+$?

(ii) Is it still true that the vanishing viscosity method converges to $\mathbf{U}^+$?

Before describing the difficulties which appear even for rather simple configurations, we give a simple example which shows that we can definitively forget any hope on "eliminability property"

13.3.2 Non-uniqueness in the case of codimension N discontinuities

We consider the stationary equation

$$|Du - \frac{x}{|x|}| + u = |x| \quad \text{in } \mathbb{R}^N \ , \tag{13.6}$$

for which we have only a discontinuity at $x = 0$. The Ishii inequalities at 0 read

$$\min_{|e|=1} |Du - e| + u(0) \leq 0 \ ,$$

$$\max_{|e|=1} |Du - e| + u(0) \geq 0 \ .$$

A first clear solution is $u_1(x) = |x|$ which is a smooth solution outside 0 and, at 0, the superdifferential of u_1 is empty while the subdifferential is $\overline{B(0,1)}$ and the supersolution inequality obviously holds.

Now we look for an other solution of the form $u_2(x) = \varphi(|x|)$ for a smooth function $\varphi : [0, +\infty) \to \mathbb{R}$. Outside 0, u_2 is smooth and leads to the equation

$$|\varphi'(s) - 1| + \varphi(s) = s \ .$$

And $\psi(s) = \varphi(s) - s$ satisfies $|\psi'(s)| + \psi(s) = 0$. If we assume that $\psi(0) = \lambda$ is given, we have by uniqueness for this $1 - d$ HJ-Equation (assuming that ψ is bounded), $\psi(s) = \lambda e^{-s}$ and this implies that $\lambda \leq 0$. This means that we have a family $(u_2^\lambda)_{\lambda \leq 0}$ of candidates for being solutions of (13.6), where the u_2^λ are given by

$$u_2^\lambda(x) = |x| + \lambda e^{-|x|} \ .$$

First it is clear that u_2^λ is a smooth solution outside 0. At 0, since $\lambda \leq 0$, the superdifferential of u_2^λ is empty while its subdifferential is $\overline{B(0, 1 - \lambda)}$. In particular $p = 0$ is in the subdifferential of u_2^λ and

$$\max_{|e|=1} |0 - e| + \lambda \geq 0 \ .$$

This means that $\lambda \geq -1$ and all $\lambda \in [-1, 0]$ gives a solution.

Hence we do not have uniqueness despite of this very high codimension of the singularity. Examining a little bit more carefully the above argument, it is easy to show that u_1 is the maximal subsolution (and solution) while u_2^{-1} is the minimal supersolution (and solution) of (13.6) in the space of functions with sublinear growth: indeed, it suffices as above to consider that a solution of (13.6) is a solution of the Dirichlet problem

$$|Du - \frac{x}{|x|}| + u = |x| \quad \text{in } \mathbb{R}^N \setminus \{0\} \ , \ u(0) = \lambda \ ,$$

for which we have a comparison result. Then we notice that, by the equation, $\lambda \leq 0$ and the solution of this Dirirchlet problem is necessarely given by u_2^λ for some $\lambda \in [-1, 0]$.

Last but not least, we look at the associated control problem. Outside 0, we have

$$b(x, \alpha) = \alpha \in \overline{B(0,1)} \ , \ c(x, \alpha) = 1 \ , \ l(x, \alpha) = |x| - \alpha \cdot \frac{x}{|x|} \ ,$$

and $\mathbf{BCL}(0)$ is obtained by computing the convex enveloppe. It is worth pointing out that the cost $|x|$ in $l(x,\alpha)$ suggests that the best strategy consists in going to 0 but a direct path from x to 0 would use the control $\alpha = -\dfrac{x}{|x|}$ with a cost $|x|+1$ in $l(x,\alpha)$ because of $-\alpha \cdot \dfrac{x}{|x|}$-term.

This large cost of controls pointing toward 0 is translated in terms of "regular" and "singular" strategies to stay at 0: a "regular" strategy can be thought as a convex combination of controls pointing toward 0, i.e. with $-\alpha\cdot\frac{x}{|x|} \geq 0$. Therefore the minimal cost for a "regular" strategy is 0. But if we accept all convex combination, we may use controls with $-\alpha\cdot\frac{x}{|x|} < 0$ and even $-\alpha\cdot\frac{x}{|x|} = -1$ coming from two opposite directions x and $-x$ at 0.

This explains the extremal value $\lambda = 0$ and $\lambda = -1$ and u_1 is nothing but a $\mathbf{U}^+$ while u_2^{-1} is nothing but $\mathbf{U}^-$.

Last remark: in this case, the convergence of the vanishing viscosity method is easy to establish since u_1 is convex and therefore a subsolution for the vanishing viscosity equation. Hence the two half-relaxed limits for the vanishing viscosity approximation are larger that u_1 but they are also between the maximal subsolution and the minimal supersolution of (13.6), i.e. u_1 and u_2^{-1}. Therefore they are both equal to u_1.

13.3.3 Puzzling examples

In general, we are unable to give a control formula for the maximal subsolution of an HJB-Equation with discontinuities of codimensions > 1, and even in very simple examples. The problem is both to determine what is a "regular" strategy but also to concretely prove that the associated value function is indeed the maximal subsolution.

In order to be more specific and to fix ideas, we consider two interesting examples: the first one is the case when we still have two domains but the interface is not smooth, typically Figure 13.1 below.

Figure 13.1: Two domains with a non-smooth interface

A second very puzzling example is the "cross-case" where $\mathbb{R}^2$ is decomposed into

its four main quadrants, see Figure 13.2 below. And of course, one may also have in mind "triple-junction configurations" in between these two cases.

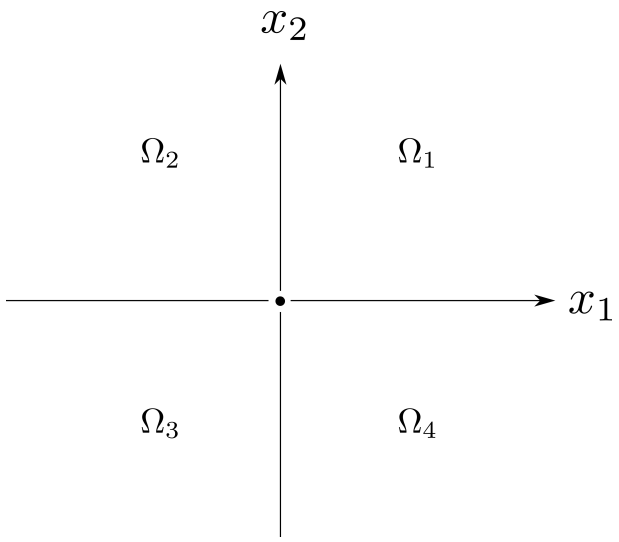


Figure 13.2: The cross-case

The importance of the above questions is due to the numerous applications and we can mention for example front propagations phenomenas or Large Deviations type results: in both case, one has to identify the limit of the vanishing viscosity method and an "action functional" which exactly means to answer the above questions if the diffusions and/or drift involved in these problems are discontinuous.

We refer for example to Souganidis [158] and references therein for the viscosity solutions' approach of front propagations in reactions diffusion equations (like KPP (Kolmogorov-Petrovskii-Piskunov) type equations) and to Bouin [49] and references therein for front propagation in kinetic equations. For the viscosity solutions' approach of Large Deviations problems, we refer to [24] (see also [31]).

Now we turn to the questions (i) and (ii) of Section 13.3.1 which are largely open even in the two simple cases described above. We first remark that most of the results of this part, in particular those obtained by pde methods, use in a crucial way the codimension-1 feature of the problem, via the normal direction which determines which are the inward and outward dynamics to the Ω_i's but also the $H_i^{\pm}$, and therefore the key H_T^{reg} Hamiltonian.

Concerning Question (i), in terms of control, the additional difficulty is to identify the "regular strategies" which allow to stay at the new discontinuity point (0 in the cross-case) and then to show that using only these "regular strategies", $\mathbf{U}^+$ is an Ishii

solution of the problem. For Question (ii), the proofs which are given above use either $\mathbf{U}^+$ (and therefore require an answer to Question (i)) or the codimension-1 feature of the problem via the Kirchhoff condition.

For all these reasons, even in the very simple configurations we propose above, we DO NOT know the right answer... but we hope that some readers will be able to find it!

In order to show the difficulty, we provide a "simple" result in the cross-case in $\mathbb{R}^2$, which DOES NOT give the result we wish but which uses the natural ingredients which should be useful to get it.

We are going to consider the problem

$$u_t + H_i(Du) = 0 \quad \text{in } \Omega_i \times (0, T_f) \ , \ \text{for } i = 1, 2, 3, 4,$$

where the Hamiltonian H_i are given by

$$H_i(p) = \sup_{\alpha_i \in A_i} \{-b_i(\alpha_i) \cdot p - l_i(\alpha_i)\} \, .$$

where A_i are compact metric spaces. We are in a very simplified framework since we do not intend to provide general results, so we also assume that the Hamiltonians H_i are coercive, and even that there exists $\delta > 0$ such that

$$B(0, \delta) \subset \{b_i(\alpha_i); \quad \alpha_i \in A_i\} \quad \text{for any } i = 1, 2, 3, 4 \, .$$

This is natural as a normal controllability assumption.

Of course, these equations in each Ω_i have to be complemented by the Ishii conditions on the two axes: except for $x = 0$, we are in the framework described in this part since we face a co-dimension 1 discontinuity. Therefore we concentrate on the case $x = 0$ where, in order to identify $\mathbf{U}^+$, we have to identify the "H_T^{reg}", i.e. the "regular strategies" which allow to remain at $x = 0$.

In order to do so, we introduce the set $\mathcal{A}$ of controls $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ such that, on one hand, $b_i(\alpha_i) \in D_i$ for $i = 1, 2, 3, 4$ where

$$D_i = \{b_i(\alpha_i); \ b_i(\alpha_i) \cdot x \leq 0 \text{ for all } x \in \Omega_i\} \ ,$$

and, on the other hand, there exists a convex combination of the $b_i(\alpha_i)$ such that $\sum_{i=0}^4 \mu_i b_i(\alpha_i) = 0$. Such a convex combination may not be unique and we denote by Δ the set of all such convex combinations.

Finally we set

$$H_T^{\text{reg}-\text{cross}} := \sup_{\mathcal{A}} \Big\{ \inf_{\Delta} \Big(- \sum_{i=0}^4 \mu_i l_i(\alpha_i) \Big) \Big\} \, .$$

Notice that here, since we consider a zero-dimensional set, the Hamiltonian $H_T^{\mathrm{reg-cross}}$ reduces to a real number. We have the

Lemma 13.3.1 *If $u : \mathbb{R}^2 \times (0, T_f) \to \mathbb{R}$ is an Ishii subsolution of the above problem then*

$$u_t + H_T^{\mathrm{reg-cross}} \leq 0 \quad \text{on } \{0\} \times (0, T_f) \ .$$

Proof — Let ϕ be a C^1 function on $(0, T_f)$ and $\bar{t}$ be a strict local maximum point of $u(0,t) - \phi(t)$. We have to show that $\phi_t(\bar{t}) + H_T^{reg-cross} \leq 0$.

To do so, we consider $(\alpha_i)_i \in \mathcal{A}$ and, for $\delta > 0$ small, we consider the affine functions

$$\psi_i(p) = \phi_t(\bar{t}) - b_i(\alpha_i) \cdot p - l_i(\alpha_i) - \delta \ .$$

Applying Farkas' Lemma, there are two possibilities; the first one is: there exists $\bar{p}$ such that $\psi_i(\bar{p}) \geq 0$ for all i. In that case, we consider the function $(x,t) \mapsto u(x,t) - \psi(t) - \bar{p} \cdot x - \frac{|x|^2}{\varepsilon}$ for $0 < \varepsilon \ll 1$.

Since $\bar{t}$ is a strict local maximum point of $u(0,t) - \phi(t)$, this function has a local maximum point at $(x_\varepsilon, t_\varepsilon)$ and $(x_\varepsilon, t_\varepsilon) \to (0, \bar{t})$ as $\varepsilon \to 0$. Wherever the point x_ε is, we have an inequality of the type

$$\phi_t(t_\varepsilon) + H_i(\bar{p} + \frac{2x}{\varepsilon}) \leq 0 \ .$$

But if such H_i inequality holds, this means that we are on $\overline{\Omega_i}$ and in particular

$$\phi_t(t_\varepsilon) - b_i(\alpha_i) \cdot (\bar{p} + \frac{2x}{\varepsilon}) - l_i(\alpha_i) \leq 0 \ .$$

Recalling that $b_i(\alpha_i) \in D_i$, this implies

$$\phi_t(t_\varepsilon) - b_i(\alpha_i) \cdot \bar{p} - l_i(\alpha_i) \leq 0 \ .$$

For ε small enough, this inequality is a contradiction with $\psi_i(\bar{p}) \geq 0$ and therefore this first case cannot hold.

Therefore, we are always in the second case: there exists a convex combination of the ψ_i, namely $\sum_{i=0}^{4} \mu_i \psi_i$ which gives a negative number. In that case, it is clear that we have

$$\sum_{i=0}^{4} \mu_i b_i(\alpha_i) = 0 \quad \text{and} \quad \phi_t(\bar{t}) - \sum_{i=0}^{4} \mu_i l_i(\alpha_i) - \delta \leq 0 \ .$$

This implies that

$$\phi_t(\bar{t}) + \inf_{\Delta} \Big(- \sum_{i=0}^{4} \mu_i l_i(\alpha_i) \Big) - \delta \leq 0 \ ,$$

and since this is true for any $(\alpha_i)_i \in \mathcal{A}$ and for any $\delta > 0$, we have the result.

Q.E.D.

The interest of this proof is to show the two kinds of arguments which seem useful to obtain an inequality for the subsolutions at 0: (i) to find the suitable set Δ of "regular strategies" which allow to stay fixed at 0; (ii) to have suitable properties on the b_i's which allow to deal with the $2x/\varepsilon$-term in the Hamiltonians, in other words we have to define suitable "outgoing strategies".

Again this result is not satisfactory and we do not think that it leads to the desired result in the cross case.

Part III

Hamilton-Jacobi Equations with Codimension One Discontinuities: the "Network" Point-of view

Chapter 14

Introduction

Contrarily to Part II where the question of a co-dimension 1 discontinuity in Hamilton-Jacobi Equations is mainly addressed in the case of convex Hamiltonians by using control arguments, the aim of this part is to describe several complementary pde points-of-view which allow to obtain more general results, and most of them for non-convex equations. However we often choose to present them in the framework of Part II for justifying the assumptions we use and showing the interest of the results.

14.1 The "network approach": a different point-of-view

In order to present these other pde approaches, let us focus first on a simple 1 dimensional configuration, the terminology "network point-of view" originating from this situation. Considering an Hamilton-Jacobi Equation with a discontinuity at $x = 0$, we have in mind the picture in Fig. 14.1 below

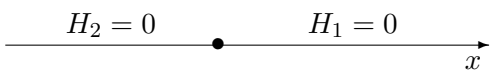


Figure 14.1: The Ishii point of view

Here, Ishii's definition of viscosity solutions in $\mathbb{R}$ is quite natural and involves $\min(H_1, H_2)$ and $\max(H_1, H_2)$ at $x = 0$.

But, since the equations are different in the sets $\{x > 0\}$ and $\{x < 0\}$, we can see as well the picture as two segments joining at $x = 0$:

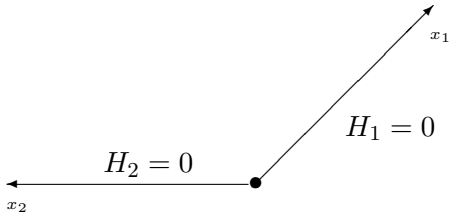


Figure 14.2: The network point of view

Now, $J_1 = \{x > 0\}$ and $J_2 = \{x < 0\}$ become two different branches of a (simple) network and it becomes natural to introduce adapted coordinates on J_1, J_2, which are nothing but $x_1 = x$ on J_1 and $x_2 = -x$ on J_2.

14.1.1 A larger space of test-functions

The first main consequence of this different point of view is that the "natural" test-functions are not the same as in the Ishii approach since they can be chosen differently in J_1 and J_2, with just a continuity assumption at $x = 0$.

In our original framework in $\mathbb{R}^N$ with $\Omega_1, \Omega_2, \mathcal{H}$ introduced in Section 3.1 where an analogous remark holds, just replacing J_1 by Ω_1, J_2 by Ω_2 and 0 by $\mathcal{H}$, this suggests the space of "natural" test-functions as

Definition 14.1.1 *We denote by* $\mathrm{PC}^1(\mathbb{R}^N \times [0,T_f])$ *the space of piecewise* C^1*-functions* $\psi \in C(\mathbb{R}^N \times [0,T_f])$ *such that there exist* $\psi_1 \in C^1(\bar{\Omega}_1 \times [0,T_f])$, $\psi_2 \in C^1(\bar{\Omega}_2 \times [0,T_f])$ *such that* $\psi = \psi_1$ *in* $\bar{\Omega}_1 \times [0,T_f]$ *and* $\psi = \psi_2$ *in* $\bar{\Omega}_2 \times [0,T_f]$.

An important point in this definition is that $\psi = \psi_1 = \psi_2$ on $\mathcal{H} \times [0,T_f]$ and $D_{\mathcal{H}}\psi = D_{\mathcal{H}}\psi_1 = D_{\mathcal{H}}\psi_2$ on $\mathcal{H} \times [0,T_f]$, $\psi_t = (\psi_1)_t = (\psi_2)_t$ on $\mathcal{H} \times [0,T_f]$. We recall here that $D_{\mathcal{H}}$ is the tangential derivative.

This change of test-functions is a first step but it remains of course to examine the kind of "junction condition" we can impose on $\mathcal{H} \times [0,T_f]$, since, contrarily to what happens for the Ishii definition, no obvious choice seems to stand out.

The first attempt could be to try the standard Ishii inequalities with this larger set of test-functions with the convention (since the test-functions are not necessarely

smooth on $\mathcal{H} \times [0, T_f]$) to use the derivatives of ψ_1 in the H_1-inequalities and those of ψ_2 in the H_2-inequalities. On the simplest example where the equations are

$$\begin{cases} u_t + H_1(x, t, u, Du) = 0 & \text{in } \Omega_1 \times (0, T_f) \ , \\ u_t + H_2(x, t, u, Du) = 0 & \text{in } \Omega_2 \times (0, T_f) \ , \end{cases} \tag{HJ-gen}$$

and without additional Hamiltonian on $\mathcal{H}$, these conditions are

$$\begin{cases} \min(u_t + H_1(x, t, u, Du), u_t + H_2(x, t, u, Du)) \leq 0 & \text{on } \mathcal{H} \times (0, T_f) \ , \\ \max(u_t + H_1(x, t, u, Du), u_t + H_2(x, t, u, Du)) \geq 0 & \text{on } \mathcal{H} \times (0, T_f) \ . \end{cases}$$

But it is easy to check that, with test-functions in $\mathrm{PC}^1(\mathbb{R}^N \times [0, T_f])$, there is no subsolutions if H_1, H_2 are both coercive. The argument is the following: if $u - \varphi$ has a maximum at some point $(0, t) \in \mathcal{H} \times (0, T_f)$, then $u - (\varphi + C|x_N|)$ also has a maximum at the same point and since $\varphi_C(x, t) := \varphi(x, t) + C|x_N|$ belongs to $\mathrm{PC}^1(\mathbb{R}^N \times [0, T_f])$ we can use it to test the inequalities. But, since the Hamiltonians are coercive, taking $C > 0$ large enough yields an impossibility since both $|D(\varphi_C)_1(x, t)|$ and $|D(\varphi_C)_2(x, t)|$ can be taken as large as we wish.

14.1.2 Different types of junction conditions

As a consequence of the simple remark above, it is clear that the question of the right junction condition to be imposed on $\mathcal{H}$ becomes crucial. And it obviously depends on the type of applications we have in mind.

(a) *Flux-limited condition* — From Chapter 7, it seems obvious that in the framework of control problems, a natural contidition on $x = 0$ is the following

$$u_t + G(x, t, u, D_{\mathcal{H}} u) = 0 \quad \text{on } \mathcal{H} \times (0, T_f) \ . \tag{FL}$$

Indeed, for applications to optimal control, one may have in mind a specific control on $\mathcal{H}$, i.e. a specific dynamic, discount and cost as in Chapter 11. In the network litterature (cf. Imbert and Monneau [113, 114, 116]), the associated terminology is "flux-limited condition" (See Section 18.2 for a partial justification of this terminology). Concrete modellings and applications lead to a variety of different flux-limited conditions at the boundary, including more general ones

$$G(x, t, u, u_t, D_{\mathcal{H}} u) = 0 \quad \text{on } \mathcal{H} \times (0, T_f) \ , \tag{GFL}$$

where G satisfies: there exists $\gamma > 0$ such that, for any $x \in \mathcal{H}$, $t \in [0, T_f]$, $r \in \mathbb{R}$, $p' \in \mathcal{H}$ and $a_2 \geq a_1$, one has

$$G(x, t, r, a_2, p') - G(x, t, r_2, a_1, p') \geq \gamma(a_2 - a_1) \ . \tag{14.1}$$

In fact, if (14.1) holds, it is a simple exercise to show that there exists $\tilde{G}$ such that $G(x, t, r, a, p')$ and $a + \tilde{G}(x, t, r, p')$ have the same signs. In other words, a general flux-limited condition (GFL) is equivalent to a simple flux-limited condition (FL) (both for the sub and supersolution condition), and pushing the exercise a little bit further, the reader will notice that the assumptions on G can be transfered without any difficulty to $\tilde{G}$.

For this reason, in the sequel we focus on the study of Conditions (FL) but either by doing the above exercise or repeating readily the arguments, it will be clear that all the definitions and results extend without any difficulty to (GFL).

(b) *Kirchhoff type conditions* — This second type of condition involves the normal derivatives of the solution on $\mathcal{H}$. The simplest one, used in various applications and in particular for networks, is the *Kirchhoff condition*

$$\frac{\partial u}{\partial n_1} + \frac{\partial u}{\partial n_2} = 0 \quad \text{on } \mathcal{H} \times (0, T_f) \ , \tag{KC}$$

where, for $i = 1, 2$, $n_i(x)$ denotes the unit normal to $\partial\Omega_i$ pointing outward Ω_i at $x \in \partial\Omega_i$.

(c) *General junction conditions* — More generally, a junction type condition may have the form

$$G\Big(x, t, u, u_t, D_{\mathcal{H}} u, \frac{\partial u}{\partial n_1}, \frac{\partial u}{\partial n_2}\Big) = 0 \quad \text{on } \mathcal{H} \times (0, T_f) \ , \tag{GJC}$$

where $G(x, t, r, a, p', b, c)$ has at least to satisfy the following monotonicty assumption: there exists $\alpha, \beta \geq 0$ such that, for any $x \in \mathcal{H}$, $t \in (0, T_f)$, $r_1 \geq r_2$, $p' \in \mathcal{H}$, $a_1 \geq a_2$, $b_1 \geq b_2$, $c_1 \geq c_2$,

$$\begin{aligned} G(x, t, r_1, a_1, p', b_1, c_1) - G(x, t, r_2, a_2, p', b_2, c_2) \\ \geq \alpha(a_1 - a_2) + \beta(b_1 - b_2) + \beta(c_1 - c_2) \ . \end{aligned} \tag{14.2}$$

In the sequel, we will often drop the dependence in r in junction condition (GJC), just to simplify a little bit the technicalities. But taking into account such dependence

with a suitable monotonicity assumption does not cause major problems. Precise assumptions are given in next section.

Roughly speaking, each of these conditions is treated in the literature by using a different notion of solution. In the case of (FL)-conditions, and in particular if one has in mind applications to control problems, the natural notion of solutions is the "Flux-Limited solutions", which is introduced and extensively studied in [113, 114, 116]. However, this kind of solution is not well-adapted for dealing with Kirchhoff type conditions, where a notion of "Junction viscosity solution" is needed.

This second notion of solution, rather similar to classical viscosity solutions is called "relaxed solutions" in [113] and extensively used in the works of Lions and Souganidis [137, 138].

14.2 The "good assumptions" used in Part III

In this part, most of the results we present are obtained using PDE methods. For this reason, the control interpretation, and therefore the convexity of the Hamiltonians, is not playing a key role. Depending on the chapter or the section, we are going to consider either convex, quasi-convex or merely continuous Hamiltonians. This is why depending on the context we have to translate in this section the "good framework for HJ-Equations with discontinuities" in the particular case of a co-dimension 1 discontinuity already discussed in Section 5.5.3.

We refer first the reader to Section 2.3 where Basic Assumptions ($\mathbf{H_{BA-CP}}$) and ($\mathbf{H_{BA-HJ}}$) are defined. Then, in order to satisfy **(Mon)**, we denote by

($\mathbf{H_{BA-HJ}}$)$^+$: assumption ($\mathbf{H_{BA-HJ}}$) in which we assume $\gamma(R) \geq 0$ for any R.

($\mathbf{H_{BA-CP}}$)$^+$: assumption ($\mathbf{H_{BA-CP}}$) in which we assume $c(x,t,\alpha) \geq 0$ for any x, t, α.

These reductions are only done in order to simplify matters, in any case a change $u \to u\exp(Kt)$ for a suitable constant K allows to reduce to the above assumptions. We also point out that, thanks to Chapter 3.2, ($\mathbf{H_{BA-HJ}}$)$^+$ and ($\mathbf{H_{BA-CP}}$)$^+$ imply (**LOC1**), (**LOC2**) because of the Lipschitz continuity in p of the Hamiltonians.

14.2.1 Good assumptions on H_1, H_2

We need here to translate the normal controllability and tangential continuity assumptions to the case of general Hamiltonians:

(NC-HJ) *Normal controllability for general Hamiltonians* — For any $R > 0$, there exists constants $C_2^R, C_3^R, C_4^R > 0$ such that, for any $(x,t) \in \mathcal{H} \times (0,T_f)$ with $|x| \leq R$, $|u| \leq R$ and $p = (p', p_N)$ with $p' \in \mathbb{R}^{N-1}$, $p_N \in \mathbb{R}$,

$$H(x,t,u,p) \geq C_2^R |p_N| - C_3^R |p'| - C_4^R \,.$$

(TC-HJ) *Tangential Continuity for general Hamiltonians* — for any $R > 0$, there exists $C_1^R > 0$ and a modulus of continuity $m^R : [0,+\infty[\to [0,+\infty[$ such that for any $x = (x', x_N), y = (y', x_N)$ with $|x|, |y| \leq R$, $|x_N| \leq R^{-1}$, $t, s \in [0,T_f]$, $|u| \leq R$, $p = (p', p_N) \in \mathbb{R}^N$,

$$|H(x,t,u,p) - H(y,s,u,p)| \leq C_1^R(|x'-y'| + |t-s|)|p'| + m^R\big(|x'-y'| + |t-s|\big) \,.$$

With these assumptions we can formulate several "good assumptions" depending on the context:

(GA-Gen) *General case* — H_1, H_2 satisfy $(\mathbf{H_{BA-HJ}})^+$ and **(NC-HJ)**.

(GA-Conv) *Convex case* — H_1, H_2 satisfy **(GA-Gen)** and are convex in p.

(GA-QC) *Quasi-convex case* — H_1, H_2 satisfy **(GA-Gen)** and $(\mathbf{H_{QC}})$.

(GA-CC) *Control case* — $(\mathbf{H_{BA-CP}})^+$ and $(\mathbf{NC}_{\mathcal{H}})$ are satisfied.

Remark 14.2.1 *A priori, the variable t being a "tangential variable",* **(TC-HJ)** *should be formulated with a right hand side like $C_1^R(|x'-y'|+|t-s|)(|p'|+|p_t|)$ instead of $C_1^R(|x'-y'|+|t-s|)|p'|$; but since H does not depend on p_t, the above formulation seems more natural. However, using the equation which gives $p_t = -H$, it is probably possible to change this assumption into the more general one, including a term like $C_1^R(|x'-y'|+|t-s|)(|p'|+\max(|H(x,t,u,p)|, |H(y,s,u,p)|))$ in the right-hand side of* **(TC-HJ)**. *We leave this open question to the reader.*

14.2.2 Good assumptions on the interface

We now turn to the assumptions on the function G which appears in (FL) or (GJC), recalling that we are assuming it is independent of r for simplicity. To do so, we first formulate a continuity requirement, where the role of ε_0 will be clear later on.

(GA-ContG) For any $R > 0$, there exist constants C_5^R, C_6^R such that, for any $x, y \in \mathcal{H}$, $t, s \in [0, T_f]$, $|r| \leq R$, $p_1', p_2' \in \mathbb{R}^{N-1}$, $a, b, c, a', b', c' \in \mathbb{R}$

$$|G(x,t,a,p_1',b,c) - G(y,s,a,p_1',b,c)| \leq C_5^R(|x-y|+|t-s|)\big(1+|p_1'|+\varepsilon_0(|a|+|b|+|c|)\big) .$$

$$|G(x,t,a',p_2',b',c') - G(x,t,a,p_1',b,c)| \leq C_6^R(|p_2'-p_1'| + (|a'-a| + |b'-b| + |c'-c|)) .$$

The "Good Assumptions" on G in the various cases are then the following

(GA-G-FL) *Flux-Limiter* — G is independent of a, b, c and **(GA-ContG)** holds with $\varepsilon_0 = 0$.

(GA-G-GKT) *Kirchhoff type* — **(GA-ContG)** holds with $\varepsilon_0 = 0$ and (14.2) holds with $\alpha \geq 0$, $\beta > 0$.

(GA-G-FLT) *Flux-limited type* — $G(x,t,a,p',b,c) = G_1(a,p',b,c) + G_2(x,t,a,p')$ where G_1 is a Lipschitz continuous function which satisfies (14.2) with $\alpha > 0$, $\beta = 0$ while G_2 satisfies **(GA-G-FL)**.

The first two assumptions seem relatively natural, only the third one requires some comments: in order to provide comparison results for the general junction condition (GJC), we are going to present the Lions-Souganidis approach which is based on a "tangential regularization" of both the sub and supersolution in the spirit of Sections 3.4.3 and 3.4.4. While we are able to perform these regularizations in a rather general setting if G is of "Kirchhoff type" since (14.2) holds with $\beta > 0$, this is not the case anymore if (14.2) holds only with $\beta = 0$. For this reason, we need **(GA-G-FLT)** which is (roughly speaking) the analogue of **(TC-s)**.

14.3 What do we do in this part?

In the next two chapters of this part, we successively describe the notions of "Flux-Limited Solutions" and "Junction Viscosity Solutions", and their properties. For each of them, we provide

(*i*) a general comparison result;

(*ii*) a stability result;

(*iii*) a convergence result of the vanishing viscosity method by specific arguments related to the corresponding notion of solution.

Moreover, for "Flux-Limited solutions", we also describe the connections with control problems.

It is worth pointing out that the notion of "Junction Viscosity Solutions" and the arguments of Lions and Souganidis [137, 138] allow to obtain results which are valid without any convexity assumption on the Hamiltonians, and in particular a very general comparison result, despite some limitations due to **(TC)**. The theory for this notion of solutions is quite complete, with very natural stability properties because of a definition which is very similar to the standard viscosity solutions one.

Despite being very different, we prove in Chapter 17 that these notions of solutions are "almost equivalent" in the case of flux-limited conditions (FL), at least in the framework of quasi-convex Hamiltonians. We wrote "almost" because flux-limited subsolutions are automatically regular as an easy consequence of their definition, while this is not the case for junction viscosity subsolutions in general. Hence, complete equivalence holds if we assume that the junction viscosity subsolutions are regula —which is true for instance in the case of Kirchhoff conditions.

In Chapter 17, we provide the characterizations of the maximal and minimal Ishii solutions in terms of other solutions. Last but not least, we show that junction viscosity sub and supersolutions of various general junction conditions (GJC) of Kirchhoff type are flux-limited sub and supersolutions. The associated "flux-limiter" can be identified explicitly in terms of the Hamiltonians H_1, H_2 of the equations in Ω_1, Ω_2 and of the nonlinearity of the general junction conditions. These connections between general junction conditions (GJC) of Kirchhoff type and flux-limited conditions were extensively studied in [113, 114, 116] and they are quite important because they allow to take advantage of the good stability properties of "Junction Viscosity Solutions" and the good connections of "Flux-Limited Solutions" with control problems at the same time. The applications to the vanishing viscosity method and to the KPP problem shows the efficiency of this machinery.

We conclude this part by a chapter describing all the results in a simple 1-d framework very similar to the scalar conservation law and then by various remarks on possible extensions or open problems.

Chapter 15

Flux-Limited Solutions for Control Problems and Quasi-Convex Hamiltonians

In the control case, as it is clear from Chapter 7, one may have in mind a specific control problem on $\mathcal{H}$, i.e. a specific dynamic, discount and cost as in Section 8. In this setting, the most natural condition on $\mathcal{H} \times (0, T_f)$ takes the form

$$u_t + G(x, t, u, D_{\mathcal{H}} u) = 0 \quad \text{on } \mathcal{H} \times (0, T_f) \,, \tag{FL}$$

which is called a "flux-limited condition" in the network litterature (cf. Imbert and Monneau [113, 114, 116]). Concrete modelizations and applications lead to a variety of different flux-limited conditions at the boundary, expressed as specific functions G.

15.1 Definition and first properties

Let us first turn to the definition of "flux-limited sub and supersolutions" which requires the introduction of some notations.

In the case of control problems, for $i = 1, 2$ the Hamiltonians are given by

$$H_i(x, t, r, p) := \sup_{\alpha_i \in A_i} \{-b_i(x, t, \alpha_i) \cdot p + c_i(x, t, \alpha_i) r - l_i(x, t, \alpha_i)\} \,. \tag{15.1}$$

We then set $A_i^- := \{\alpha_i \in A_i \,:\, b_i(x, t, \alpha_i) \cdot e_N \leq 0\}$ and similarly $A_i^+ := \{\alpha_i \in A_i \,:\, b_i(x, t, \alpha_i) \cdot e_N > 0\}$, then

$$H_i^-(x, t, r, p) := \sup_{\alpha_i \in A_i^-} \{-b_i(x, t, \alpha_i) \cdot p + c_i(x, t, \alpha_i) r - l_i(x, t, \alpha_i)\} \,, \tag{15.2}$$

$$H_i^+(x,t,r,p) := \sup_{\alpha_i \in A_i^+} \{-b_i(x,t,\alpha_i)\cdot p + c_i(x,t,\alpha_i)r - l_i(x,t,\alpha_i)\}\ . \tag{15.3}$$

Notice that the $+/-$ notation refers to the sign of $b_i \cdot e_N$ in the supremum, which implies that H_i^- $(i = 1..2)$ is nondecreasing with respect to p_N (the normal gradient variable) while the H_i^+ is nonincreasing with respect to p_N.

Finally, for the specific control problem on $\mathcal{H}$, we define for any $x \in \mathcal{H}$, $t \in [0,T_f]$, $r \in \mathbb{R}$, and $p_{\mathcal{H}} \in \mathbb{R}^{N-1}$

$$G(x,t,r,p_{\mathcal{H}}) := \sup_{\alpha_0 \in A_0} \{-b_0(x,t,\alpha_0)\cdot p_{\mathcal{H}} + c_0(x,t,\alpha_0)r - l_0(x,t,\alpha_0)\}\,. \tag{15.4}$$

For $i = 1...2$, b_i, c_i, l_i are at least bounded continuous functions defined on $\bar{\Omega}_i \times [0,T_f] \times A_i$ and b_0, c_0, l_0 are also bounded continuous functions defined on $\mathcal{H} \times [0,T_f] \times A_0$. Therefore H_1, H_2 and G are continuous.

In the case where the Hamiltonians are quasi-convex in p, Section 6.2.3 provides us with a definition of $H_i^+(x,t,r,p)$, $H_i^-(x,t,r,p)$ and we assume that these functions and G are continuous.

With these notations, we can give the definition of flux-limited viscosity sub and supersolutions —(FLSub) and (FLSuper) in short:

Definition 15.1.1 (Flux-limited solutions for quasi-convex Hamiltonians)
A locally bounded function $u : \mathbb{R}^N \times (0,T_f) \to \mathbb{R}$ *is a* (FLSub) *of* (HJ-Gen)-(FL) *if it is a classical viscosity subsolution of* (HJ-Gen) *and if, for any test-function* $\psi \in \mathrm{PC}^1(\mathbb{R}^N \times [0,T_f])$ *and any local maximum point* $(x,t) \in \mathcal{H} \times (0,T_f)$ *of* $u^* - \psi$ *in* $\mathbb{R}^N \times (0,T_f)$, *at* (x,t) *the following inequality holds*

$$\max\Big(\psi_t + G(x,t,u^*,D_{\mathcal{H}}\psi), \psi_t + H_1^+(x,t,u^*,D\psi_1), \psi_t + H_2^-(x,t,u^*,D\psi_2)\Big) \le 0\ ,$$

where $u^* = u^*(x,t)$.

A locally bounded function $v : \mathbb{R}^N \times (0,T_f) \to \mathbb{R}$ *is a* (FLSuper) *of* (HJ-Gen)-(FL) *if it is a classical viscosity supersolution of* (HJ-Gen) *and if, for any test-function* $\psi \in \mathrm{PC}^1(\mathbb{R}^N \times [0,T_f])$ *and any local minimum point* $(x,t) \in \mathcal{H} \times (0,T_f)$ *of* $v_* - \psi$ *in* $\mathbb{R}^N \times (0,T_f)$, *at* (x,t) *the following inequality holds*

$$\max\Big(\psi_t + G(x,t,v_*,D_{\mathcal{H}}\psi), \psi_t + H_1^+(x,t,v_*,D\psi_1), \psi_t + H_2^-(x,t,v_*,D\psi_2)\Big) \ge 0\ ,$$

where $v_* = v_*(x,t)$.
A locally bounded function is a flux-limited solution if it is both a (FLSub) *and a* (FLSuper).

Several remarks have to be made on this definition which is very different from the classical ones: first we have a "max" both in the definition of supersolutions AND subsolutions; then we do not use the full Hamiltonians H_i in the junction condition on $\mathcal{H}$ but H_1^+ and H_2^-. These changes are justified when looking at the interpretation of the viscosity solutions inequalities in the optimal control framework. Indeed

(*i*) the subsolution inequality means that any control is sub-optimal, *i.e.* if one tries to use a specific control, the result may not be optimal. But, of course, such a control has to be associated with an "admissible" trajectory: for example, if we are on $\mathcal{H}$, a "b_1" pointing towards Ω_2 cannot be associated to a real trajectory, therefore it is not "admissible" and this is why we use H_1^+. And an analogous remark justifies H_2^-. Finally the "max" comes just from the fact that we test all sub-optimal controls.

(*ii*) Analogous remarks hold for the supersolution inequality, except that this inequality is related to the optimal trajectory, which has to be admissible anyway.

With these remarks, the reader may be led to the conclusion that an "universal" definition of solutions of (HJ-Gen) with the condition (FL) can hardly exist: if we look at control problems where the controller tries to maximize some profit, then the analogue of the H_1^+, H_2^- above seem still relevant because of their interpretation in terms of incoming dynamics but the max should be replaced by min in both the definitions of sub and supersolutions. Therefore it seems that such particular definitions have to be used in each case since, again, the Kirchhoff condition does not seem natural in the control framework.

As in the case of classical Ishii sub and supersolutions, we can define (FLSub) and (FLSuper) using the notions of sub and superdifferentials. We refer the reader to Section 3.5 for the introduction of these notions and various properties. Following this section, for $i = 1, 2$, we denote by $Q_i = \Omega_1 \times (0, T_f)$ and $\overline{Q_i}^\ell = \overline{\Omega_1} \times (0, T_f)$. As in Section 3.5, we restrict ourselves to the case of u.s.c. subsolution and l.s.c. supersolutions to simplify the notations but, in the general case, these results have to be reformulated with either the u.s.c. envelope of the subsolution or the l.s.c. envelope of the supersolution.

Proposition 15.1.2 — *Flux-limited viscosity solutions via sub superdifferentials.*

An u.s.c., locally bounded function $u : \mathbb{R}^N \times (0, T_f) \to \mathbb{R}$ *is a* (FLSub) *of* (HJ-Gen)-(FL) *if and only if*

(*i*) *for any* $(x, t) \in Q_i$ $(i = 1, 2)$ *and for any* $(p_x, p_t) \in D^+_{\overline{Q_i}^\ell} u(x, t)$

$$p_t + H_i(x, t, u(x, t), p_x) \leq 0,$$

(*ii*) *for any* $(x,t) \in \mathcal{H} \times (0,T_f)$ *and for any* $p_{\mathcal{H}} \in \mathcal{H}$, $p_1,p_2,p_t \in \mathbb{R}$ *such that* $((p_{\mathcal{H}},p_i),p_t) \in D^+_{\overline{Q_i}^\ell} u(x,t)$ *for* $i=1,2$, *noting* $u=u(x,t)$,

$$\max\Big(p_t+G(x,t,u,p_{\mathcal{H}}),p_t+H_1^+(x,t,u,p_{\mathcal{H}}+p_1e_N),p_t+H_2^-(x,t,u,p_{\mathcal{H}}+p_2e_N)\Big)\le 0\,.$$

A l.s.c., locally bounded function $v:\mathbb{R}^N\times(0,T_f)\to\mathbb{R}$ *is a* (FLSuper) *of* (HJ-Gen)-(FL) *if and only if*

(*i*) *for any* $(x,t)\in Q_i$ $(i=1,2)$ *and for any* $(p_x,p_t)\in D^-_{\overline{Q_i}^\ell} v(x,t)$

$$p_t+H_i(x,t,v(x,t),p_x)\ge 0\,,$$

(*ii*) *for any* $(x,t) \in \mathcal{H} \times (0,T_f)$ *and for any* $p_{\mathcal{H}} \in \mathcal{H}$, $p_1,p_2,p_t \in \mathbb{R}$ *such that* $((p_{\mathcal{H}},p_i),p_t) \in D^-_{\overline{Q_i}^\ell} v(x,t)$ *for* $i=1,2$, *noting* $v=v(x,t)$,

$$\max\Big(p_t+G(x,t,v,p_{\mathcal{H}}),p_t+H_1^+(x,t,v,p_{\mathcal{H}}+p_1e_N),p_t+H_2^-(x,t,v,p_{\mathcal{H}}+p_2e_N)\Big)\ge 0\,.$$

We omit the proof of Proposition 15.1.2 since it is an easy consequence of Lemma 3.5.3 and Lemma 3.5.6. As we already remark after the statement of Lemma 3.5.6, we point out that this equivalent definition via sub and superdifferentials allows to show that, instead of using general PC^1 test-functions, we may consider only test-functions of the form $\chi(x_N)+\varphi(x,t)$ where $\chi\in\mathrm{PC}^1(\mathbb{R})$ and $\varphi\in C^1(\mathbb{R}^N\times(0,T_f))$. The reader will notice that we mainly use test-function of this form in comparison proof, but this property is also useful to simplify the proofs of several results.

Remark 15.1.3 *Definition 15.1.1 provides the notion of "flux-limited viscosity solutions" for a problem with a co-dimension* 1 *discontinuity but it can be used in different frameworks, in particular in problems with boundary conditions: we refer to Guerand [105] for results on state constraints problems and [104] in the case of Neumann conditions where "effective boundary conditions and new comparison results are given, both works being in the case of quasi-convex Hamiltonians.*

We give a first important property of (FLSub)

Proposition 15.1.4 *Assume that* **(GA-QC)** *holds and that the Hamiltonian* G *satisfies* **(GA-G-FL)**. *Any u.s.c.*(FLSub) *is regular on* $\mathcal{H}$.

Proof — It is an immediate application of Proposition 3.4.2 since the Hamiltonian $\mathbb{G}$ defined for $x \in \mathbb{R}^N$, $t \in (0, T_f)$, $r \in \mathbb{R}$, $(p, p_t) \in \mathbb{R}^{N+1}$ by

$$\mathbb{G}(x,t,r,(p,p_t)) := p_t + H_i(x,t,r,p) \text{ if } x \in \Omega_i,$$

$$\mathbb{G}(x,t,r,(p,p_t)) := \max(p_t+H_1^+(x,t,r,p), p_t+H_2^-(x,t,r,p), p_t+G(x,t,r,p')) \text{ if } x \in \mathcal{H},$$

satisfies the assumptions of this proposition with $y = (x', t)$, $z = x_N$, and in particular the normal controllability in the x_N-direction.

Q.E.D.

15.2 Stability of flux-limited solutions

In this section, we provide a result on the stability of flux-limited solutions. As the proof will show it, such result is not an immediate extension of Theorem 3.1.4; indeed, if the change of test-functions does not really cause any problem, the formulation of flux-limited sub and supersolutions with global Hamiltonians which are not l.s.c. or u.s.c. is the source of difficulties.

The result is the

Theorem 15.2.1 *Assume that, for $\varepsilon > 0$, u_ε is a* (FLSub) [*resp.* (FLSuper)] *for the problem with Hamiltonians $H_1^\varepsilon, H_2^\varepsilon, G^\varepsilon$. We assume that $H_1^\varepsilon, H_2^\varepsilon, G^\varepsilon$ are continuous and $H_1^\varepsilon, H_2^\varepsilon$ satisfy* ($\mathbf{H_{QC}}$). *If $H_1^\varepsilon, H_2^\varepsilon, G^\varepsilon$ converge locally uniformly to respectively H_1, H_2, G and if the functions u_ε are uniformly locally bounded on $\mathbb{R}^N$, then $\overline{u} = \limsup^* u_\varepsilon$ [resp. $\underline{u} = \liminf_* u_\varepsilon$] is a* (FLSub) [*resp.* (FLSuper)] *for the problem with Hamiltonians H_1, H_2, G.*

Proof — Due to the dissymmetry in the definitions of (FLSub) and (FLSuper), we have to give the proof in both cases.

(a) We start by the (FLSub) one. Of course, we have just to prove the result on $\mathcal{H}$ since, in Ω_1, Ω_2, the result is an easy application of Theorem 3.1.4. Let $\psi = (\psi_1, \psi_2) \in \mathrm{PC}^1(\mathbb{R}^N \times [0, T_f])$ and let $(x,t) \in \mathcal{H} \times (0, T_f)$ be a strict local maximum point of $\overline{u} - \psi$. We have to show that

$$\max\Big(\psi_t + G(x,t,\overline{u}, D_{\mathcal{H}}\psi), \psi_t + H_1^+(x,t,\overline{u}, D\psi_1), \psi_t + H_2^-(x,t,\overline{u}, D\psi_2)\Big) \le 0\,.$$

By Lemma 3.1.6, there exists a subsequence $(x_{\varepsilon'}, t_{\varepsilon'})$ of maximum point of $u_{\varepsilon'} - \psi$ which converges to (x,t) and such that $u_{\varepsilon'}(x_{\varepsilon'}, t_{\varepsilon'})$ converges to $\overline{u}(x,t)$. To get the

G-inequality, we replace ψ by $\psi + K|x_N|$. Using the quasi-convexity property of H_1 and H_2, for K large enough we get

$$\psi_t + H_1(x,t,\overline{u},D\psi_1) > 0 \quad \text{and} \quad \psi_t + H_2(x,t,\overline{u},D\psi_2) > 0\ .$$

Applying the result of Lemma 3.1.6 to this new ψ, we see that necessarily $x_{\varepsilon'} \in \mathcal{H}$. Then, passing to the limit in the (FLSub) inequality for $(H_1^{\varepsilon'})^+, (H_2^{\varepsilon'})^-, G^{\varepsilon'}$, we end up with $\psi_t + G(x,t,\overline{u},D_{\mathcal{H}}\psi) \le 0$ since the term $K|x_N|$ does not affect $D_{\mathcal{H}}\psi$.

It remains to prove the H_1^+ and H_2^- inequalities and to do so, we come back to the original ψ. We assume that $\psi_t + H_1^+(x,t,\overline{u},D\psi_1) > 0$ and change ψ into $\psi + Kx_N^-$, for K large enough.

For ε' small enough, $x_{\varepsilon'}$ cannot be in Ω_1: since $H_1 \ge H_1^+$ implies $\psi_t + H_1(x,t,u,D\psi_1) > 0$, hence the $H_1^{\varepsilon'}$ inequality cannot hold for ε' small enough. Similarly, $x_{\varepsilon'}$ cannot be on $\mathcal{H}$ because of the H_1^+ inequality. Finally $x_{\varepsilon'}$ cannot be in Ω_2 for K large enough, therefore we reach a contradiction which implies that $\psi_t + H_1^+(x,t,\overline{u},D\psi_1) \le 0$.

Arguing the same way for the case $\psi_t + H_2^-(x,t,\overline{u},D\psi_2) > 0$, the subsolution inequality is proved.

(b) For the (FLSuper) case, again we just have to treat the inequalities on $\mathcal{H}$ and we assume that $(x,t) \in \mathcal{H} \times (0,T_f)$ is a strict local minimum point of $\underline{u} - \psi$ where $\psi = (\psi_1,\psi_2) \in \mathrm{PC}^1(\mathbb{R}^N \times [0,T_f])$. We have to show that

$$\max\Big(\psi_t + G(x,t,\underline{u},D_{\mathcal{H}}\psi), \psi_t + H_1^+(x,t,\underline{u},D\psi_1), \psi_t + H_2^-(x,t,\underline{u},D\psi_2)\Big) \ge 0\ .$$

We argue by contradiction assuming that the three quantities in the max are strictly negative. Similarly to the (FLSub) case, we claim that we can choose $K_1, K_2 \ge 0$ such that

$$\psi_t + H_1(x,t,\underline{u},D\psi_1 - K_1 e_N) < 0 \quad \text{and} \quad \psi_t + H_2(x,t,\underline{u},D\psi_2 + K_2 e_N) < 0\ ,$$

which follows here also from the quasi-convexity of H_1 and H_2. To use it, we change ψ in $\psi - K_1 x_N^+ - K_2 x_N^-$ and notice that (x,t) is still is a strict local minimum point of $\underline{u} - \psi$ for this new ψ.

Applying again Lemma 3.1.6, there exists a subsequence $(x_{\varepsilon'}, t_{\varepsilon'})$ of minimum points of $u_{\varepsilon'} - \psi$ which converges to (x,t) and such that $u_{\varepsilon'}(x_{\varepsilon'}, t_{\varepsilon'})$ converges to $\underline{u}(x,t)$. And we examine the possible inequalities for $(x_{\varepsilon'}, t_{\varepsilon'})$. Clearly $x_{\varepsilon'}$ can be neither in Ω_1 nor in Ω_2 for ε' small enough because of the above property. Hence $x_{\varepsilon'} \in \mathcal{H}$ and the (FLSuper) inequality holds for $(H_1^{\varepsilon'})^+, (H_2^{\varepsilon'})^-, G^{\varepsilon'}$. But passing to the limit as $\varepsilon' \to 0$ in these inequalities yields a contradiction, so the supersolution inequality holds.

Q.E.D.

Remark 15.2.2 *The main weakness of Theorem 15.2.1 is to be strictly restricted to the framework of flux-limited solutions for problems with quasi-convex Hamiltonians. Therefore it is not very flexible, in particular if we compare it with Theorem 16.2.1 in the case of junction viscosity solutions.*

15.3 Comparison results for flux-limited solutions and applications

This section is devoted to prove comparison results for flux-limited solutions; the original proofs given in [113, 114] were based on the rather technical construction of a "vertex function". We present here the simplified proof(s) of [18].

15.3.1 Comparison results in the convex case

The main result here is the following.

Theorem 15.3.1 (Comparison principle)
Assume that either **(GA-Conv)** *or* **(GA-CC)** *holds, that the Hamiltonian* $G(x,t,r,p')$ *is convex in* (r,p') *and satisfies* **(GA-G-FL)**. *If* $u, v : \mathbb{R}^N \times (0,T_f) \to \mathbb{R}$ *are respectively an u.s.c. bounded flux-limited subsolution and a l.s.c. bounded flux-limited supersolution of* (HJ-Gen)-(FL) *and if* $u(x,0) \le v(x,0)$ *in* $\mathbb{R}^N$, *then* $u \le v$ *in* $\mathbb{R}^N \times (0,T_f)$.

Proof — In order to simplify the proof, we provide it only in the case when the Hamiltonians H_1, H_2, G are independent of u; the general case only contains minor additional technical difficulties.

(a) *Reduction of the proof* — First we follow Section 3.2 and check (**LOC1**)-evol: the function $\chi : \mathbb{R}^N \times (0,T_f) \to \mathbb{R}$ defined by

$$\bar{\chi}(x,t) := -Kt - (1+|x|^2)^{1/2} - \frac{1}{T_f - t}\,,$$

is, for $K > 0$ large enough, a strict subsolution of (HJ-Gen)-(FL) with $\bar{\chi}(x,t) \to -\infty$ when $|x| \to +\infty$ or $t \to T_f^-$. We replace u by either $u_\mu := u + (1-\mu)\bar{\chi}$ (a choice which does not use the convexity of the Hamiltonians) or $u_\mu := \mu u + (1-\mu)\bar{\chi}$ (a choice which uses the convexity of the Hamiltonians). Borrowing also the arguments of Section 3.2, (**LOC2**)-evol also holds and therefore we are led to show that **(LCR)**-evol is valid in the case when u is an η-strict subsolution of (HJ-Gen)-(FL).

For a point $(\bar{x},\bar{t})$ where $\bar{x} \in \Omega_1$ or $\bar{x} \in \Omega_2$, the proof of **(LCR)**-evol in $\overline{Q^{\bar{x},\bar{t}}_{r,h}}$ is standard, hence we have just to treat the case when $\bar{x} \in \mathcal{H}$. At this point, we make an other reduction in the proof: using Section 3.4, with $y = (t, x')$ and $z = x_N$, since **(GA-Conv)** or **(GA-CC)** are nothing but Assumptions **(TC)**,**(NC)** and **(Mon)**, Theorem 3.4.4 applies. As a consequence, we can assume *w.l.o.g.* that u is Lipschitz continuous with respect to all variables and semi-convex in the (t, x')-variables. But we may also use the ideas of Proposition 3.4.7 to obtain a subsolution which is C^1 in (t, x') with u_t and $D_{x'}u$ continuous *w.r.t.* all variables: indeed, we can apply the ideas of the proof of Proposition 3.4.7 separately in Ω_1, Ω_2 and $\mathcal{H}$ to obtain the H_1, H_2 and G inequalities for the regularized function, while the H_1^+ and H_2^- ones are deduced from Proposition 3.5.1.

Then we assume that

$$M := \max_{\overline{Q^{x,t}_{r,h}}} (u - v) > 0 \,.$$

If this maximum is achieved on $\partial_p Q^{x,t}_{r,h}$, the result is obvious so we may assume that it is achieved at $(\tilde{x},\tilde{t}) \notin \partial_p Q^{x,t}_{r,h}$. Again, if $\tilde{x} \in \Omega_1$ or $\tilde{x} \in \Omega_2$, we easily obtain a contradiction and therefore we can assume that $\tilde{x} \in \mathcal{H}$.

(b) *Building the test function* — Setting $a = u_t(\tilde{x},\tilde{t})$, $p' = D_{x'}u(\tilde{x},\tilde{t})$, we claim that we can solve the equations

$$a + H_1^-(\tilde{x},\tilde{t},\tilde{p}' + \lambda_1 e_N) = -\eta/2 \quad , \quad a + H_2^+(\tilde{x},\tilde{t},\tilde{p}' + \lambda_2 e_N) = -\eta/2 \,,$$

where we recall that $-\eta$ is the constant which measures the strict subsolution property of function u.

In order to prove the existence of λ_1, we look at maximum points of

$$u(x,t) - \frac{|x-\tilde{x}|^2}{\varepsilon^2} - \frac{|t-\tilde{t}|^2}{\varepsilon^2} - \frac{\varepsilon}{x_N}$$

in $(\overline{Q^{x,t}_{r,h}}) \cap (\Omega_1 \times : [0, T_f])$, and for $0 < \varepsilon \ll 1$. This function achieves its maximum at $(x_\varepsilon, t_\varepsilon)$ which converges to $(\tilde{x},\tilde{t})$ as $\varepsilon \to 0$ and by the semi-convexity of u in t and x', one has

$$u_t(x_\varepsilon,t_\varepsilon) + H_1(x_\varepsilon,t_\varepsilon, D_{x'}u(x_\varepsilon,t_\varepsilon) + \lambda_\varepsilon e_N) \le -\eta \,,$$

for some $\lambda_\varepsilon \in \mathbb{R}$. Moreover, λ_ε is bounded *w.r.t.* ε since u is Lipschitz continuous. Letting ε tend to 0 and using that $u_t(x_\varepsilon,t_\varepsilon) \to a, D_{x'}u(x_\varepsilon,t_\varepsilon) \to p'$ by the semi-convexity property of u, together with the extraction of a subsequence for $(\lambda_\varepsilon)_\varepsilon$, we get a $\bar{\lambda} \in \mathbb{R}$ such that

$$a + H_1(\tilde{x},\tilde{t},\tilde{p}' + \bar{\lambda} e_N) \le -\eta \,.$$

Since $H_1^- \leq H_1$, it follows that $a + H_1^-(\tilde{x}, \tilde{t}, \tilde{p}' + \bar{\lambda} e_N) \leq -\eta$. Then we use the fact that $\lambda \mapsto a + H_1^-(\tilde{x}, \tilde{t}, \tilde{p}' + \lambda e_N)$ is continuous, nondecreasing on $\mathbb{R}$ and tends to $+\infty$ when $\lambda \to +\infty$ to get the existence of $\lambda_1 > \bar{\lambda}$ solving the equation with $-\eta/2$. In this framework, λ_1 is necessarily unique since the convex function $\lambda \mapsto a + H_1^-(\tilde{x}, \tilde{t}, \tilde{p}' + \lambda e_N)$ only has flat parts at its minimum, while clearly λ_1 is not a minimum point for this function. The proof for λ_2 is analogous and we skip it.

In order to build the test-function, we set $h(t) := \lambda_1 t_+ - \lambda_2 t_-$ where $t_+ = \max(t, 0)$, $t_- = \max(-t, 0)$, and

$$\chi(x_N, y_N) := h(x_N) - h(y_N) = \begin{cases} \lambda_1(x_N - y_N) & \text{if } x_N \geq 0\,,\ y_N \geq 0\,, \\ \lambda_1 x_N - \lambda_2 y_N & \text{if } x_N \geq 0\,,\ y_N < 0\,, \\ \lambda_2 x_N - \lambda_1 y_N & \text{if } x_N < 0\,,\ y_N \geq 0\,, \\ \lambda_2(x_N - y_N) & \text{if } x_N < 0\,,\ y_N < 0\,. \end{cases} \tag{15.5}$$

Then, for $0 < \varepsilon \ll 1$ we define a test function as follows

$$\psi_\varepsilon(x, t, y, s) := \frac{|x - y|^2}{\varepsilon^2} + \frac{|t - s|^2}{\varepsilon^2} + \chi(x_N, y_N) + |x - \tilde{x}|^2 + |t - \tilde{t}|^2\,.$$

In view of the definition of h, we see that for any $(x, t) \in \mathbb{R}^N \times [0, T_f]$ the function $\psi_\varepsilon(x, \cdot, t, \cdot) \in \mathrm{PC}^1(\mathbb{R}^N \times [0, T_f])$ and for any $(y, s) \in \mathbb{R}^N \times [0, T_f]$ the function $\psi_\varepsilon(\cdot, y, \cdot, s) \in \mathrm{PC}^1(\mathbb{R}^N \times [0, T_f])$.

We now look at the maximum points of

$$(x, t, y, s) \mapsto u(x, t) - v(y, t) - \psi_\varepsilon(x, t, y, s) \quad \text{in} \quad \left[\overline{Q_{r,h}^{x,t}}\right]^2\,.$$

By standard arguments, this function has maximum points $(x_\varepsilon, t_\varepsilon, y_\varepsilon, s_\varepsilon)$ such that $(x_\varepsilon, t_\varepsilon, y_\varepsilon, s_\varepsilon) \to (\tilde{x}, \tilde{t}, \tilde{x}, \tilde{t})$. Moreover, using the semi-convexity of u, we have

$$p'_\varepsilon = \frac{2(x'_\varepsilon - y'_\varepsilon)}{\varepsilon^2} \to p' \quad \text{and} \quad \frac{2(t_\varepsilon - s_\varepsilon)}{\varepsilon^2} \to a\,,$$

which the Lipschitz continuity of u implies that $(p_\varepsilon)_N = 2((x_\varepsilon)_N - (y_\varepsilon)_N)/\varepsilon^2$ remains bounded.

(c) *Getting contradictions* — We have to consider different cases depending on the position of x_ε and y_ε in $\mathbb{R}^N$. Of course, we have no difficulty for the cases $x_\varepsilon, y_\varepsilon \in \Omega_1$ or $x_\varepsilon, y_\varepsilon \in \Omega_2$, and even less because of the above very precise properties on the derivatives of the test-function; only the cases where x_ε, y_ε are in different domains or on $\mathcal{H}$ cause problem. So, we are left with considering three cases

1. $x_\varepsilon \in \Omega_1$, $y_\varepsilon \in \overline{\Omega_2}$ or $x_\varepsilon \in \Omega_2$, $y_\varepsilon \in \overline{\Omega_1}$.

2. $x_\varepsilon \in \mathcal{H}$, $y_\varepsilon \in (\Omega_1 \cup \Omega_2)$.

3. $x_\varepsilon \in \mathcal{H}$, $y_\varepsilon \in \mathcal{H}$.

Case 1: If $x_\varepsilon \in \Omega_1$, $y_\varepsilon \in \Omega_2 \cup \mathcal{H}$, we use that u is an η-strict H_1-subsolution and taking into account the specific form of the test-function above we get

$$a + o_\varepsilon(1) + H_1(x_\varepsilon, t_\varepsilon, p' + o(1) + \lambda_1 e_N + (p_\varepsilon)_N e_N) \leq -\eta\,. \tag{15.6}$$

Then, using that $H_1 \geq H_1^-$ and the fact that every term in H_1 remains in a compact subset, we also have

$$a + H_1^-(\tilde{x}, \tilde{t}, p' + \lambda_1 e_N + (p_\varepsilon)_N e_N) \leq -\eta + o_\varepsilon(1)\ .$$

Now, since $(p_\varepsilon)_N \geq 0$, thanks to the monotonicity of H_1^- in the e_N-direction we obtain

$$a + H_1^-(\tilde{x}, \tilde{t}, p' + \lambda_1 e_N) \leq -\eta + o_\varepsilon(1)\ ,$$

which is a contradiction with the definition of λ_1. The case $x_\varepsilon \in \Omega_2$, $y_\varepsilon \in \Omega_1 \cup \mathcal{H}$ is of course analogue and we skip it.

Case 2: Since $x_\varepsilon \in \mathcal{H}$, the subsolution inequality holds

$$\max\Big(a + G(\tilde{x}, \tilde{t}, p')\ ;\ a + H_1^+(\tilde{x}, \tilde{t}, p' + \lambda_1 e_N + (p_\varepsilon)_N e_N)\ ;\ a + H_2^-(\tilde{x}, \tilde{t}, p' + \lambda_1 e_N + (p_\varepsilon)_N e_N)\Big) \leq -\eta + o_\varepsilon(1)\ .$$

On the other hand, if $y_\varepsilon \in \Omega_1$, since v is a H_1-supersolution in Ω_1 and of course $(y_\varepsilon, t_\varepsilon) \to (\tilde{x}, \tilde{t})$,

$$a + H_1(\tilde{x}, \tilde{t}, p' + \lambda_1 e_N + (p_\varepsilon)_N e_N) \geq o_\varepsilon(1)\,. \tag{15.7}$$

Now the aim is to show that the same inequality holds for H_1^+ and to do so, we evaluate this quantity for H_1^-: taking into account the fact that here $(p_\varepsilon)_N \leq 0$, the monotonicity of H_1^- in the e_N-direction yields

$$a + H_1^-(\tilde{x}, \tilde{t}, p' + \lambda_1 e_N + (p_\varepsilon)_N e_N) \leq -\eta/2 + O_\varepsilon(1) < 0 \text{ if } \varepsilon \text{ is small enough.}$$

But since $H_1 = \max(H_1^-, H_1^+)$, from (15.7) we actually deduce that

$$a + H_1^+(\tilde{x}, \tilde{t}, p' + \lambda_1 e_N + (p_\varepsilon)_N e_N) \geq o_\varepsilon(1)\ ,$$

which gives a contradiction when compared with the subsolution property on $\mathcal{H}$. The same contradiction is obtained in the case $y_\varepsilon \in \Omega_2$, using λ_2 and H_2^+ instead of λ_1 and H_1^-.

Case 3: If $x_\varepsilon \in \mathcal{H}$, $y_\varepsilon \in \mathcal{H}$, we have viscosity sub and supersolution inequalities for the same Hamiltonian and the contradiction follows easily. So, the proof is complete.

Q.E.D.

15.3.2 A comparison result in the quasi-convex case

In fact, Theorem 15.3.1 extends without difficulties in the "quasi-convex" case and we have the

Theorem 15.3.2 — *Comparison principle in the quasi-convex case.*

The result of Theorem 15.3.1 remains valid if **(GA-QC)** *holds and G satisfies* **(GA-G-FL)**.

Proof — We just sketch it since it follows very closely the proof of Theorem 15.3.1. The only difference here is that Section 3.4 only allows to reduce to the case when the strict subsolution u is Lipschitz continuous and semi-convex in the (t, x')-variables, not C^1. This obliges us to first look at a maximum of

$$(x,t,y,s) \mapsto u(x,t) - v(y,s) - \frac{|x'-y'|^2}{\varepsilon^2} - \frac{|t-s|^2}{\varepsilon^2} \ ,$$

where $x = (x', x_N)$, $y = (y', x_N)$, which is, of course, an approximation of $\max_{\overline{Q}^{x,t}_{r,h}} (u-v)$.

If $(\tilde{x}, \tilde{t}, \tilde{y}, \tilde{s})$ is a maximum point of this function, the semi-convexity of u implies that u is differentiable *w.r.t.* x' and t at $(\tilde{x}, \tilde{t})$ and we have

$$a := \frac{2(\tilde{t}-\tilde{s})}{\varepsilon^2} = u_t(\tilde{x},\tilde{t}) \quad \text{and} \quad p' := \frac{2(\tilde{x}'-\tilde{y}')}{\varepsilon^2} = D_{x'}u(\tilde{x},\tilde{t}) \ .$$

Then we solve the (λ_1, λ_2)-equations with such a and p'; it is worth pointing out that λ_1 and λ_2 are not uniquely defined but this is not important in the proof.

Finally we consider the maxima of the function

$$(x,t,y,s) \mapsto u(x,t) - v(y,s) - \frac{|x'-y'|^2}{\varepsilon^2} - \frac{|t-s|^2}{\varepsilon^2} \\ - \chi(x_N,y_N) - \frac{|x_N-y_N|^2}{\gamma^2} - |x-\tilde{x}|^2 - |t-\tilde{t}|^2 \ ,$$

where $0 < \gamma \ll 1$ is a parameter devoted to tend to 0 first. Using the normal controllability assumption with variables $X = (x', t)$, $Z = x_N$, it is easy to show that

$$|(p_\varepsilon)_N| = \frac{2|(x_\varepsilon)_N - (y_\varepsilon)_N|}{\gamma^2} = O(|p'_\varepsilon| + |a| + 1) \ ,$$

which is bounded since u is Lipschitz continuous in the tangent variables (x', t). This allows to perform all the arguments of the proof as in the convex case. Notice that, even if it is not C^1-smooth, the semi-convexity of u ensures that $u_t(x_\varepsilon, t_\varepsilon) \to a$, $D_{x'}u(x_\varepsilon, t_\varepsilon) \to p'$.

Q.E.D.

15.4 Flux-limited solutions and control problems

In this section, we come back on the control problem of Section 8 which we address here from a different point of view.

In order to do that, we first have to define the admissible trajectories among all the solutions of the differential inclusion: we say that a solution $(X, T, D, L)(\cdot)$ of the differential inclusion starting from $(x, t, 0, 0)$ is an admissible trajectory if

1. there exists a global control $a = (\alpha_1, \alpha_2, \alpha_0)$ with $\alpha_i \in \mathcal{A}_i := L^\infty(0, \infty; A_i)$ for $i = 0, 1, 2$;
2. there exists a partition $\mathbb{I} = (\mathbb{I}_1, \mathbb{I}_2, \mathbb{I}_0)$ of $(0, +\infty)$, where $\mathbb{I}_1, \mathbb{I}_2, \mathbb{I}_0$ are measurable sets, such that $X(s) \in \overline{\Omega_i}$ for any $s \in \mathbb{I}_i$ if $i = 1, 2$ and $X(s) \in \mathcal{H}$ if $s \in \mathbb{I}_0$;
3. for almost every $0 \le s \le t$

$$(\dot{X}, \dot{D}, \dot{L})(s) = \sum_{i=0}^{2} (b_i, c_i, l_i)(X(s), t-s, \alpha_i(s)) \mathbb{1}_{\mathbb{I}_i}(s)\,. \tag{15.8}$$

In Equation (15.8), we have dropped $T(s)$ since we are in the $b^t \equiv -1$ case and therefore $T(s) = t - s$ for $s \le t$. The set of all admissible trajectories $(X, \mathbb{I}, a)$ issued from a point $X(0) = x \in \mathbb{R}^N$ (at $T(s) = t$) is denoted by $\mathcal{T}_x$. Notice that, under the controllability assumption $(\mathbf{NC}_{\mathcal{H}})$, for any point $x \in \overline{\Omega_1}$, there exist trajectories starting from x, which stay in $\overline{\Omega_1}$, and the same remark holds for points in $\overline{\Omega_2}$. These trajectories are clearly admissible (with either $\mathbb{I}_1 \equiv \mathbb{I}$ or $\mathbb{I}_2 \equiv \mathbb{I}$) and therefore $\mathcal{T}_x$ is never void.

Remark 15.4.1 *It is worth pointing out that, in this approach, the partition $\mathbb{I}_0, \mathbb{I}_1, \mathbb{I}_2$ which we impose for admissible trajectories, implies that there is no mixing on $\mathcal{H}$ between the dynamics and costs in Ω_1 and Ω_2, contrarily to the approach of Section 8. A priori, on $\mathcal{H}$, either we have an independent control problem or we can use either (b_1, c_1, l_1) or (b_2, c_2, l_2), but no combination of (b_1, c_1, l_1) and (b_2, c_2, l_2).*

The value function is then defined as

$$\mathbf{U}_G^{\mathrm{FL}}(x,t) := \inf_{(X,\mathbb{I},a)\in\mathcal{T}_x} \left\{ \int_0^t \left(\sum_{i=0}^{2} l_i(X(s), t-s, \alpha_i(s)) \mathbb{1}_{\mathbb{I}_i}(s) \right) e^{-D(s)}\, \mathrm{d}s + u_0(X(t)) \right\}\,,$$

where $u_0 \in C(\mathbb{R}^N)$.

As always, the first key ingredient to go further is the

Lemma 15.4.2 (Dynamic Programming Principle) *Under assumption* **(GA-CC)**, *the value function* $\mathbf{U}_G^{\mathrm{FL}}$ *satisfies: for all* $(x,t) \in \mathbb{R}^N \times (0,T_f]$ *and* $\tau < t$

$$\mathbf{U}_G^{\mathrm{FL}}(x,t) = \inf_{(X,\mathbb{I},a)\in\mathcal{T}_x} \left\{ \int_0^\tau \left(\sum_{i=0}^{2} l_i(X(s), t-s, \alpha_i(s)) \mathbb{1}_{\mathbb{I}_i}(s) \right) e^{-D(s)}\, \mathrm{d}s + \mathbf{U}_G^{\mathrm{FL}}(X(\tau), t-\tau) \right\} .$$

We leave the easy proof of this lemma to the reader, which is standard. Now, using standard arguments based on the Dynamic Programming Principle and the comparison result, we have the

Theorem 15.4.3 *Under assumption* **(GA-CC)** *and if* $u_0 \in C(\mathbb{R}^N)$, *the value function* $\mathbf{U}_G^{\mathrm{FL}}$ *is the unique flux-limited solution of* (HJ-Gen)-(FL) *with* $G = H_0$ *given by*

$$H_0(x,t,r,p) = \sup_{\alpha_0\in A_0} \{-b_0(x,t,\alpha_i)\cdot p + c_0(x,t,\alpha_i) r - l_0(x,t,\alpha_i)\} .$$

Proof — We describe some non-obvious parts of the proof, in particular those to show that the value function $\mathbf{U}_G^{\mathrm{FL}}$ is a flux-limited solution of (HJ-Gen)-(FL). As we will explain at the end of the proof, continuity of $\mathbf{U}_G^{\mathrm{FL}}$ and its uniqueness are an immediate consequence of Theorem 15.3.1.

(a) *Subsolution property.*

Of course, the only difficulty is to prove this property on $\mathcal{H} \times (0,T_f]$, the cases of $\Omega_1 \times (0,T_f]$ and $\Omega_2 \times (0,T_f]$ being classical. To do so, we have to show that

$$(\mathbf{U}_G^{\mathrm{FL}})_t^* - b_i(x,t,\alpha_i)\cdot D(\mathbf{U}_G^{\mathrm{FL}})^* + c_i(x,t,\alpha_i)(\mathbf{U}_G^{\mathrm{FL}})^* - l_i(x,t,\alpha_i) \le 0 , \tag{15.9}$$

for any $i = 0,1,2$ any $\alpha_i \in A_i$ with $b_1(x,t,\alpha_i)\cdot e_N \ge 0$ if $i = 1$ and $b_2(x,t,\alpha_i)\cdot e_N \le 0$ if $i = 2$. The proof of these inequalities is standard once we use the following two remarks:

1. By the arguments of Theorem 21.1.1 which give such result in a more general setting, if $\mathbf{U}_G^{\mathrm{FL}}(x_\varepsilon,t_\varepsilon) \to (\mathbf{U}_G^{\mathrm{FL}})^*(x,t)$, we can assume without loss of generality that $(x_\varepsilon,t_\varepsilon) \in \mathcal{H}\times(0,T_f]$. This first remark allows to prove (15.9) in the case $i = 0$ using classical arguments.

2. The convexity of $\mathbf{BCL}_1(x,t) = \{(b_1(x,t,\alpha_1), c_1(x,t,\alpha_1), l_1(x,t,\alpha_1)) :\ \alpha_1 \in A_1\}$ together with the normal controllability assumption implies that the set

$$\big\{(b_1(x,t,\alpha_1), c_1(x,t,\alpha_1), l_1(x,t,\alpha_1)) :\ b_1(x,t,\alpha_1)\cdot e_N \ge 0,\ \alpha_1 \in A_1\big\}$$

is the closure of the set

$$\left\{(b_1(x,t,\alpha_1), c_1(x,t,\alpha_1), l_1(x,t,\alpha_1)) : \ b_1(x,t,\alpha_1)\cdot e_N > 0, \ \alpha_1 \in A_1\right\},$$

and an analogous property holds for $i = 2$. This remark reduces the proof of (15.9) for α_1 and α_2 such that $b_1(x,t,\alpha_1)\cdot e_N > 0$ and $b_2(x,t,\alpha_2)\cdot e_N < 0$. And this allows to use classical arguments since, for $s \in (0,\tau]$ and τ small enough, trajectories $X(s)$ which are associated to such dynamics with constant controls remains in Ω_1 in the first case and in Ω_2 in the second one.

We point out that Property (a) plays a key role to obtain the three types of inequalities for $i = 0, 1, 2$.

(b) *Supersolution property.*

Again the only non-classical case concerns points of $\mathcal{H}\times(0,T_f]$. Let $(x,t)\in\mathcal{H}\times(0,T_f]$ be a minimum point of $(\mathbf{U}_G^{\mathrm{FL}})_* - \phi$ where $\phi = (\phi_1,\phi_2)\in \mathrm{PC}^1(\mathbb{R}^N\times[0,T_f])$. We assume *w.l.o.g.* that $(\mathbf{U}_G^{\mathrm{FL}})_*(x,t) = \phi(x,t)$.

We first fix $0 < \tau \ll 1$ and, for $0 < \varepsilon \ll 1$, we consider $(x_\varepsilon, t_\varepsilon)$ such that $\mathbf{U}_G^{\mathrm{FL}}(x_\varepsilon,t_\varepsilon) \le (\mathbf{U}_G^{\mathrm{FL}})_*(x,t) + \varepsilon\tau$ with $|(x_\varepsilon,t_\varepsilon)-(x,t)|\le\varepsilon\tau$. Then we choose a global ε-optimal control $a^\varepsilon = (\alpha_1^\varepsilon,\alpha_2^\varepsilon,\alpha_0^\varepsilon)$ and denote by $Z_i^\varepsilon = Z_i^\varepsilon(s) = \big(X^\varepsilon(s), t_\varepsilon - s, \alpha_i^\varepsilon(s)\big)$ for simplicity of notations. In other words,

$$\mathbf{U}_G^{\mathrm{FL}}(x_\varepsilon,t_\varepsilon) \ge \int_0^\tau \left(\sum_{i=0}^2 l_i(Z_i^\varepsilon)\mathbb{1}_{\mathbb{I}_i}(s)\right) e^{-D^\varepsilon(s)}\,\mathrm{d}s + \mathbf{U}_G^{\mathrm{FL}}(X^\varepsilon(\tau), t_\varepsilon-\tau) - \varepsilon\tau\ ,$$

where $X^\varepsilon, D^\varepsilon$ are the trajectory and the discount term computed with the global control a^ε. Using the minimum point property, we have

$$\phi(x_\varepsilon,t_\varepsilon) \ge \int_0^\tau \left(\sum_{i=0}^2 l_i(Z_i^\varepsilon)\mathbb{1}_{\mathbb{I}_i}(s)\right) e^{-D^\varepsilon(s)}\,\mathrm{d}s + \phi(X^\varepsilon(\tau), t_\varepsilon-\tau) - 2\varepsilon\tau\ ,$$

and by classical computations we obtain

$$\int_0^\tau \sum_{i=0}^2 \Big((\phi_i)_t(X^\varepsilon(s),t_\varepsilon-s) - b_i(Z_i^\varepsilon)\cdot D\phi_i(X^\varepsilon(s),t_\varepsilon-s) + c_i(Z_i^\varepsilon)\phi_i(X^\varepsilon(s),t_\varepsilon-s) - l_i(Z_i^\varepsilon)\Big)\mathbb{1}_{\mathbb{I}_i}(s)e^{-D^\varepsilon(s)}\,\mathrm{d}s \ge -2\varepsilon\tau\ ,$$

where, by convention, ϕ_0 denotes $\phi_1 = \phi_2$ on $\mathcal{H}\times(0,T_f]$.

Then, by using the regularity of ϕ_i $(i = 1, 2)$,

$$\int_0^\tau \sum_{i=0}^{2} \Big((\phi_i)_t(x_\varepsilon, t_\varepsilon) - b_i(Z_i^\varepsilon) \cdot D\phi_i(x_\varepsilon, t_\varepsilon) + c_i(Z_i^\varepsilon)\phi_i(x_\varepsilon, t_\varepsilon) - l_i(Z_i^\varepsilon) \Big) \mathbb{1}_{\mathbb{I}_i}(s) e^{-D^\varepsilon(s)} \, \mathrm{d}s \geq -2\varepsilon\tau + o(\tau) \, .$$

In order to conclude, we have to consider several cases

(i) If $\mathbb{I}_0 = (0, \tau)$, the proof just follows classical arguments.

(ii) If $\mathbb{I}_1 = (0, \tau)$, *i.e.* the trajectory X^ε remains in $\overline{\Omega}_1$, we notice that

$$\frac{1}{\tau} \int_0^\tau b_1(Z_i^\varepsilon) ds \cdot e_N = \frac{1}{\tau} (X^\varepsilon(\tau) - x_\varepsilon) \cdot e_N \geq -\varepsilon \, ,$$

because of the choice of $(x_\varepsilon, t_\varepsilon)$. Using the convexity and the compactness of $\mathbf{BCL}_1(x, t)$, we conclude that as $\tau, \varepsilon \to 0$, up to the extraction of a subsequence, we may assume that

$$\frac{1}{\tau} \int_0^\tau \big(b_1(Z_1^\varepsilon), c_1(Z_1^\varepsilon), l_1(Z_1^\varepsilon) \big) \, \mathrm{d}s \to \big(b_1(x, t, \bar{\alpha}_1), c_1(x, t, \bar{\alpha}_1), l_1(x, t, \bar{\alpha}_1) \big)$$

for some $\bar{\alpha}_1 \in A_1$ such that $b_1(x, t, \bar{\alpha}_1) \cdot e_N \geq 0$. From there, one concludes easily that the H_1^+-term is non-negative.

(iii) If $\mathbb{I}_2 = (0, \tau)$, the same arguments allow to conclude that the H_2^--term is non-negative.

(iv) The remaining case is when two of these three sets are non-empty, and the main difficulty is when one of the open sets (or both) $\{s : X^\varepsilon(s) \in \Omega_i\}$ is non-empty. We assume, for example, that it is the case for $i = 1$ and write

$$\{s : X^\varepsilon(s) \in \Omega_1\} = \bigcup_k]s_k, s_{k+1}[\, .$$

If $s_k > 0$ and $s_{k+1} > \tau$, we necessarily $X(s_k) \in \mathcal{H}$ and $X(s_{k+1}) \in \mathcal{H}$, therefore

$$\frac{1}{s_{k+1} - s_k} \int_{s_k}^{s_{k+1}} b_1(Z_1^\varepsilon) \, \mathrm{d}s \cdot e_N = \frac{1}{\tau} (X^\varepsilon(s_{k+1}) - X^\varepsilon(s_k)) \cdot e_N = 0 \, .$$

Using again the convexity and the compactness of $\mathbf{BCL}_1(x,t)$, together with the regularity properties of b_1, c_1, l_1, we deduce that

$$\frac{1}{s_{k+1}-s_k}\int_{s_k}^{s_{k+1}} \Big((\phi_1)_t(x_\varepsilon,t_\varepsilon) - b_1(Z_1^\varepsilon)\cdot D\phi_i(x_\varepsilon,t_\varepsilon) + c_1(Z_1^\varepsilon)\phi_i(x_\varepsilon,t_\varepsilon) - l_1(Z_i^\varepsilon)\Big) e^{-D^\varepsilon(s)}\,\mathrm{d}s \leq (\phi_1)_t(x,t) + H_1^+(x,t,\phi_1(x,t),D\phi_1(x,t)) + 2\varepsilon\tau + o(\tau)\ .$$

To obtain this last inequality, we have used that if

$$H_{1,\eta}^+(x,t,r,p) := \sup_{\alpha_i\in A_{1,\eta}^+} \{-b_i(x,t,\alpha_i)\cdot p + c_i(x,t,\alpha_i)r - l_i(x,t,\alpha_i)\}\ ,$$

where $A_{1,\eta}^+ := \{\alpha_1 \in A_1 : b_1(x,t,\alpha_i)\cdot e_N \geq \eta\}$ and η can be positive or negative, then $H_{1,\eta}^+(x,t,r,p) \to H_1^+(x,t,r,p)$ locally uniformly when $\eta \to 0$, a property which can be easily proved using the normal controllability.

Using similar ideas, one can easily treat the cases $s_k = 0$ or $s_{k+1} = \tau$ and, of course, the case when $\{s : X^\varepsilon(s) \in \Omega_2\}$ is not empty. Gathering all these informations, we end up showing that a convex combination of $\phi_t + H_1^+, \phi_t + H_2^-, \phi_t + H_0$ is non-negative, hence the result.

(c) *Continuity and uniqueness.*

The function $\mathbf{U}_G^{\mathrm{FL}}$ being a discontinuous flux-limited solution of (HJ-Gen)-(FL), Theorem 15.3.1 shows that $(\mathbf{U}_G^{\mathrm{FL}})^* \leq (\mathbf{U}_G^{\mathrm{FL}})_*$ in $\mathbb{R}^N \times [0,T_f]$; indeed it is easy to show that $(\mathbf{U}_G^{\mathrm{FL}})^*(x,0) = (\mathbf{U}_G^{\mathrm{FL}})_*(x,0) = u_0(x)$ in $\mathbb{R}^N$. Therefore $\mathbf{U}_G^{\mathrm{FL}}$ is continuous and the uniqueness comes from the same comparison result.

Q.E.D.

Before considering the connections with the results of Section 8, we want to point out that among all these "flux-limited value-functions", there is a particular one which corresponds to either no specific control on $\mathcal{H}$ (*i.e.* we just consider the trajectories such that $\mathbb{I}_0 \equiv \emptyset$) or, and this is of course equivalent, to a cost $l_0 = +\infty$. This value function is denoted by $\mathbf{U}^{\mathrm{FL}}$.

The aim is to show that the value functions of regional control are flux-limited solutions.

Theorem 15.4.4 *Under the assumptions of Theorem 15.3.1 (comparison result), for any Hamiltonian H_0 we have*

(*i*) $\mathbf{U}^- \leq \mathbf{U}^+ \leq \mathbf{U}^{\mathrm{FL}}$ *in* $\mathbb{R}^N \times [0, T_f]$.

(*ii*) $\mathbf{U}^- = \mathbf{U}^{\mathrm{FL}}_G$ *in* $\mathbb{R}^N \times [0, T_f]$ *where* $G = H_T$ *and* $\mathbf{U}^-_{H_0} = \mathbf{U}^{\mathrm{FL}}_G$ *in* $\mathbb{R}^N \times [0, T_f]$ *where* $G = \max(H_T, H_0)$.

(*iii*) $\mathbf{U}^+ = \mathbf{U}^{\mathrm{FL}}_G$ *in* $\mathbb{R}^N \times [0, T_f]$ *where* $G = H_T^{\mathrm{reg}}$.

This result shows that, by varying the flux-limiter G, we have access to the different value functions described in Section 8.

Proof — For (*i*), the inequalities can just be seen as a consequence of the definition of $\mathbf{U}^-, \mathbf{U}^+, \mathbf{U}^{\mathrm{FL}}$ remarking that we have a larger set of dynamics-costs for $\mathbf{U}^-$ and $\mathbf{U}^+$ than for $\mathbf{U}^{\mathrm{FL}}$. From a more pde point of view, applying Proposition 3.5.1, it is easy to see that $\mathbf{U}^-, \mathbf{U}^+$ are flux-limited subsolutions of (HJ-gen)-(FL) since they are of course subsolutions of

$$u_t + H_1^+(x, t, u, Du) \leq 0 \quad \text{in } \Omega_1 \times [0, T_f] \,,$$

$$u_t + H_2^-(x, t, u, Du) \leq 0 \quad \text{in } \Omega_2 \times [0, T_f] \,.$$

Then Theorem 15.3.1 allows us to conclude.

For (*ii*) and (*iii*), we have to prove respectively that $\mathbf{U}^-$ is a solution of (HJ-gen)-(FL) with $G = H_T$, $\mathbf{U}^-_{H_0}$ is a solution of (HJ-gen)-(FL) with $G = \max(H_T, H_0)$ and $\mathbf{U}^+$ with $G = H_T^{\mathrm{reg}}$. Then the equality is just a consequence of Theorem 15.3.1.

For $\mathbf{U}^-$, the subsolution property just comes from the above argument for the H_1^+, H_2^--inequalities and from Proposition 8.3.2 for the H_T-one. The supersolution inequality is a consequence of the proof of Lemma 5.4.1: alternative **A)** implies that one of the H_1^+, H_2^--inequalities hold while alternative **B)** implies that the H_T-one holds. The same is true for $\mathbf{U}^-_{H_0}$.

For $\mathbf{U}^+$, the subsolution property follows from the same arguments as for $\mathbf{U}^-$, both for the H_1^+, H_2^--inequalities and from Proposition 9.1.3 for the H_T^{reg}-one. The supersolution inequality is a consequence of Theorem 9.3.1: alternative **A)** implies that one of the H_1^+, H_2^--inequalities hold while alternative **B)** implies that the H_T^{reg}-one holds.

And the proof is complete.

Q.E.D.

Notice that inequalities in Theorem 15.4.4-(*i*) can be strict: various examples are given in [16]. The following one shows that we can have $\mathbf{U}^+ < \mathbf{U}^{\mathrm{FL}}$ in $\mathbb{R}$.

Example 15.1 — Let $\Omega_1 = (0, +\infty)$, $\Omega_2 = (-\infty, 0)$. We choose $c \equiv 0$, $u_0(x) = 0$ in $\mathbb{R}$ and

$$b_1(\alpha_1) = \alpha_1 \in [-1, 1] \ , \ l_1(\alpha_1) = \alpha_1 \ ,$$
$$b_2(\alpha_2) = \alpha_2 \in [-1, 1] \ , \ l_1(\alpha_2) = -\alpha_2 \ .$$

It is clear that the best strategy—*i.e.* with the minimal cost—is to use $\alpha_1 = -1$ in Ω_1, $\alpha_2 = 1$ in Ω_2. We can also use these strategies at 0 since

$$\frac{1}{2} b_1(\alpha_1) + \frac{1}{2} b_2(\alpha_2) = 0 \ ,$$

a combination which yields a cost of -1. Therfore, an easy computation gives

$$\mathbf{U}^+(x,t) = \int_0^t -1.dt = -t \ ,$$

other words, the "push-push" strategy at 0 allows to maintain the -1 cost.

But, for $\mathbf{U}^{\mathrm{FL}}$, this "push-push" strategy at 0 is not allowed and, since the optimal trajectories are necessarily monotone, the best strategy when starting at 0 is to stay at 0. Here, the best possible cost is 0.

Hence $\mathbf{U}^{\mathrm{FL}}(0,t) = 0 > \mathbf{U}^+(0,t) = -1$, and in fact it can be shown that

$$\mathbf{U}^{\mathrm{FL}}(x,t) = -|x| > \mathbf{U}^+(x,t) = -t \quad \textit{if} \quad |x| < t \, .$$

On the contrary, for $|x| \geq t$, $\mathbf{U}^{\mathrm{FL}}(x,t) = \mathbf{U}^+(x,t) = -t$ since the above strategy with $\alpha_1 = -1$ in Ω_1, $\alpha_2 = 1$ in Ω_2 can be applied for all time.

Theorem 15.4.4 can be interpreted in several ways but the key point is to chose the kind of controlled trajectories we wish to allow on $\mathcal{H}$. Then, depending on this choice, different formulations have to be used for the associated HJB problem. It could be thought that the flux-limited approach is more appropriate, in particular because of Theorem 15.3.1 which is used intensively in the above proof.

15.5 Convergence of the vanishing viscosity approximation (I): via (FL)-solutions

In the framework of classical viscosity solutions, getting the convergence of the vanishing viscosity method is just a simple exercice done either with a stability result, or the combination of the half-relaxed limit method with a strong comparison result.

However, in the present discontinuous framework, although classical viscosity solutions—(CVS) in short—still have good stability properties as described in Section 3.1, the lack of uniqueness makes this stability far less effective: the two half-relaxed limits are lying between the minimal one $\mathbf{U}^-$ and the maximal one $\mathbf{U}^+$ and one cannot really obtain the convergence in that way, except if $\mathbf{U}^+ = \mathbf{U}^-$.

An interesting idea is to turn to flux-limited solutions for which a general comparison result holds. But, in order to identify the limit of the vanishing viscosity method, a limit flux-limiter is required and to the best of our knowledge, there is no obvious way to determine it. Actually we refer the interested reader to Section 13.3 for a discussion on more general discontinuities where the problem is still open.

We also refer anyway to [113, 114] for general stability results for (FLS) and to Camilli, Marchi and Schieborn [56] for the first results on the convergence of the vanishing viscosity method.

In this book, we give several different proofs of the vanishing viscosity result. Tthe first one below is inspired from [18] and uses only the properties of $\mathbf{U}^+$ as flux-limited solution.

Contrary to the proof relying on (JVS) via the Lions-Souganidis approach, the arguments we use in this section strongly rely on the structure of the Hamiltonians and on the arguments of the comparison proof. It has the advantage anyway to identify the limit in terms of control problems. An other way to do the proof goes through the connections between the Kirchoff condition and Flux-Limited Conditions (See Section 17.3).

Theorem 15.5.1 (Vanishing viscosity limit) *Assume that* **(GA-CC)** *holds. For any $\varepsilon > 0$, let u^ε be a viscosity solution of*

$$u^\varepsilon_t - \varepsilon\Delta u^\varepsilon + H(x,t,u^\varepsilon,Du^\varepsilon) = 0 \quad \text{in} \quad \mathbb{R}^N \times (0,T_f)\,, \tag{15.10}$$

$$u^\varepsilon(x,0) = u_0(x) \quad \text{in} \quad \mathbb{R}^N\,, \tag{15.11}$$

where $H = H_1$ in Ω_1 and H_2 in Ω_2, and u_0 is bounded continuous function in $\mathbb{R}^N$. If the u^ε are uniformly bounded in $\mathbb{R}^N \times (0,T_f)$ and C^1 in x_N in a neighborhood of $\mathcal{H}$, then, as $\varepsilon \to 0$, the sequence $(u^\varepsilon)_\varepsilon$ converges locally uniformly in $\mathbb{R}^N \times (0,T_f)$ to $\mathbf{U}^+$, the maximal Ishii subsolution of (7.1).

Remark 15.5.2 *A priori (15.10)-(15.11) is a uniformly parabolic problem and the regularity we assume on (u^ε) is reasonnable. Indeed the function u^ε is expected to be*

C^1 since it is also expected to be in $W^{2,r}_{\rm loc}$ (for any $r > 1$). On the other hand, it is worth pointing out that, as long as $\varepsilon > 0$, it is not necessary to impose a condition on $\mathcal{H}$ because of the strong diffusion term: a codimension 1 set is not "seen" by the diffusive equation.

Proof — We first recall that, by Theorem 9.4.2, $\mathbf{U}^+$ is the maximal subsolution — and Ishii solution— of (7.1) and we proved in Theorem 15.4.4 that it is the unique flux-limited solution of (HJ-Gen)-(FL) with $G = H_T^{\rm reg}$. We recall that the flux-limited condition consists in complementing (HJ-Gen) with the condition

$$\max\Big(u_t + H_T^{\rm reg}(x,t,D_{\mathcal{H}}u), u_t + H_1^+(x,t,D_x u), u_t + H_2^-(x,t,D_x u)\Big) = 0 \quad \text{on } \mathcal{H}\times(0,T_f)\ ,$$

in the sense of Definition 15.1.1. We refer to Section 3.1 for a definition of the usual half-relaxed limits

$$\underline{u}(x,t) := \liminf_*\, u^\varepsilon(x,t)\ , \quad \overline{u}(x,t) := \limsup^*\, u^\varepsilon(x,t)\ .$$

(a) *Reduction of the proof* — We observe that we only need to prove the following inequality

$$\mathbf{U}^+(x,t) \le \underline{u}(x,t) \quad \text{ in } \mathbb{R}^N \times [0,T_f). \tag{15.12}$$

Indeed, the maximality of $\mathbf{U}^+$ implies $\overline{u}(x,t) \le \mathbf{U}^+(x,t)$ in $\mathbb{R}^N \times [0,T_f)$. Moreover, by definition we have $\overline{u}(x,t) \ge \underline{u}(x,t)$ in $\mathbb{R}^N \times (0,T_f)$, therefore if we prove (15.12) we can conclude that $\mathbf{U}^+(x,t) \le \underline{u}(x,t) \le \overline{u}(x,t) \le \mathbf{U}^+(x,t)$ which implies that $(u^\varepsilon)_\varepsilon$ converges locally uniformly to $\mathbf{U}^+$ in $\mathbb{R}^N \times [0,T_f)$.

In order to prove the inequality, $\mathbf{U}^+ \le \underline{u}$ in $\mathbb{R}^N \times [0,T_f)$, we are going to make several reductions along the lines of Chapter 3 by changing $\mathbf{U}^+$ but we keep the notation $\mathbf{U}^+$ for the changed function for the sake of simplicity of notations. In the same way, we should argue on the interval $[0,T']$ for $0 < T' < T_f$ but we keep the notation T_f for T'.

First, thanks to the localization arguments of Chapter 3, we can assume that $\mathbf{U}^+$ is a strict subsolution such that $\mathbf{U}^+(x,t) \to -\infty$ as $|x| \to +\infty$, uniformly *w.r.t.* $t \in [0,T_f]$. Therefore there exists $(\bar{x},\bar{t}) \in \mathbb{R}^N \times [0,T_f]$ such that

$$M := \mathbf{U}^+(\bar{x},\bar{t}) - \underline{u}(\bar{x},\bar{t}) = \sup_{(x,t)\in\mathbb{R}^N\times[0,T_f]} \big(\mathbf{U}^+(x,t) - \underline{u}(x,t)\big)\ .$$

We assume by contradiction that $M > 0$ and of course this means that $\bar{t} > 0$. The cases when $\bar{x} \in \Omega_1$ or $\bar{x} \in \Omega_2$ can be treated by classical methods, hence we may assume that $\bar{x} \in \mathcal{H}$.

Next, by the regularization arguments of Chapter 3 we can assume in addition that $\mathbf{U}^+$ is C^1-smooth at least in the $t, x_1, \dots, x_{N-1}$ variables. Finally we can suppose that $(\bar{x}, \bar{t})$ is a strict maximum point of $\mathbf{U}^+ - \underline{u}$.

(b) *Construction of the test-function* — Since $\mathbf{U}^+$ is C^1 in the (t, x')-variables, the strict flux-limited subsolution condition can be written as

$$(\mathbf{U}^+)_t(\bar{x}, \bar{t}) + H_T^{\mathrm{reg}}(\bar{x}, \bar{t}, D_{x'}\mathbf{U}^+(\bar{x}, \bar{t})) \leq -\eta\,,$$

where $\eta > 0$ measures the strict subsolution property. Therefore

$$H_T^{\mathrm{reg}}(\bar{x}, \bar{t}, D_{x'}\mathbf{U}^+(\bar{x}, \bar{t})) \leq -(\mathbf{U}^+)_t(\bar{x}, \bar{t}) - \eta\,,$$

and, as in the proof of Theorem 15.3.1, there exist two solutions λ_1, λ_2, with $\lambda_2 < \lambda_1$, of the equation

$$\tilde{H}^{\mathrm{reg}}\Big(\bar{x}, \bar{t}, D_{x'}\mathbf{U}^+(\bar{x}, \bar{t}) + \lambda e_N\Big) = -(\mathbf{U}^+)_t(\bar{x}, \bar{t}) - \eta/2\,.$$

Notice that, since $\bar{x}, \bar{t}$, $a = -(\mathbf{U}^+)_t(\bar{x}, \bar{t})$ and $p' = D_{x'}\mathbf{U}^+(\bar{x})$ are fixed, λ_1, λ_2 are independent of the parameter $\varepsilon > 0$ that is to come below.

We proceed now with the construction of the test-function: let $\chi(x_N, y_N)$ be defined as in (15.5) and

$$\psi_\varepsilon(x, y, t, s) := \frac{|t-s|^2}{\varepsilon^{1/2}} + \frac{|x'-y'|^2}{\varepsilon^{1/2}} + \chi(x, y) + \frac{|x_N - y_N|^2}{\varepsilon^{1/2}}\,.$$

Note that $\psi_\varepsilon(\cdot, y, \cdot, s), \psi_\varepsilon(x, \cdot, t, \cdot) \in \mathrm{PC}^1(\mathbb{R}^N \times [0, T_f])$.

Since $(\bar{x}, \bar{t})$ is a strict global maximum point of $\mathbf{U}^+ - \underline{u}$ while $\underline{u}(\bar{x}, \bar{t}) = \liminf_* u^\varepsilon(\bar{x}, \bar{t})$, the function $\mathbf{U}^+(x, t) - u^\varepsilon(y, s) - \psi_\varepsilon(x, y, t, s)$ has local maximum points $(x_\varepsilon, y_\varepsilon, t_\varepsilon, s_\varepsilon)$ which converge to $(\bar{x}, \bar{x}, \bar{t}, \bar{t})$. For the sake of simplicity of notations, we drop the ε and just denote by (x, y, t, s) such a maximum point.

(c) *Getting a contradiction* — We now consider 3 different cases, depending on the position of (x, y, t, s).

Case 1: $x_N > 0$ and $y_N \leq 0$ (or $x_N < 0$ and $y_N \geq 0$).

We use the subsolution condition for $\mathbf{U}^+$ in Ω_1: recalling that $\mathbf{U}^+$ is C^1-regular in the (t, x')-variables, we write the condition as

$$(\mathbf{U}^+)_t(x, t) + H_1\left(x, t, D_{x'}\mathbf{U}^+(x, t) + \lambda_1 e_N + \frac{2(x_N - y_N)}{\varepsilon^{1/2}}\right) \leq -\eta\,,$$

where we have used the regularity of $\mathbf{U}^+$ to deduce that

$$(\mathbf{U}^+)_t(x,t) = \frac{2(t-s)}{\varepsilon^{1/2}} \quad \text{and} \quad D_{x'}\mathbf{U}^+(x,t) = \frac{2(x'-y')}{\varepsilon^{1/2}}\,. \tag{15.13}$$

Moreover, using further the regularity of $\mathbf{U}^+$ and recalling that $(\mathbf{U}^+)_t$ and $D_{x'}\mathbf{U}^+$ are continuous not only in t, x' but also x_N, we have $(\mathbf{U}^+)_t(x,t) = (\mathbf{U}^+)_t(\bar{x},\bar{t}) + o_\varepsilon(1)$, $D_{x'}\mathbf{U}^+(x,t) = D_{x'}\mathbf{U}^+(\bar{x},\bar{t}) + o_\varepsilon(1)$. Therefore,

$$(\mathbf{U}^+)_t(\bar{x},\bar{t}) + H_1\left(x,t,D_{x'}\mathbf{U}^+(\bar{x},\bar{t}) + \lambda_1 e_N + \frac{2(x_N-y_N)}{\varepsilon^{1/2}}\right) \le -\eta + o_\varepsilon(1)\,.$$

Next, using that H_1^- is non decreasing in p_N, $H_1^- \le H_1$ and $(x_N - y_N) > 0$ we get from the above property

$$\begin{aligned} H_1^-\left(x,t,D_{x'}\mathbf{U}^+(\bar{x},\bar{t}) + \lambda_1 e_N\right) &\le H_1^-\left(x,t,D_{x'}\mathbf{U}^+(\bar{x},\bar{t}) + \lambda_1 e_N + \frac{2(x_N-y_N)}{\varepsilon^{1/2}}\right) \\ &\le -(\mathbf{U}^+)_t(\bar{x},\bar{t}) - \eta + o_\varepsilon(1)\,. \end{aligned}$$

From this inequality, since $D_{x'}\mathbf{U}^+(\bar{x},\bar{t}) + \lambda_1 e_N$ remains bounded with respect to ε, using the continuity of H_1^- yields

$$H_1^-\left(\bar{x},\bar{t},D_{x'}\mathbf{U}^+(\bar{x},\bar{t}) + \lambda_1 e_N\right) \le -(\mathbf{U}^+)_t(\bar{x},\bar{t}) - \eta + o_\varepsilon(1)\,.$$

The contradiction is obtained for ε small enough from the fact that, by construction of λ_1,

$$H_1^-\left(\bar{x},\bar{t},D_{x'}\mathbf{U}^+(\bar{x},\bar{t}) + \lambda_1 e_N\right) = -(\mathbf{U}^+)_t(\bar{x},\bar{t}) - \eta/2\,.$$

The case $x_N < 0$ and $y_N \ge 0$ is completely similar, using H_2 instead of H_1.

Case 2: $x_N = 0$ and $y_N > 0$ (or < 0).

We use the supersolution viscosity inequality for u^ε at (y,t), using (15.13):

$$O(\varepsilon^{1/2}) + (\mathbf{U}^+)_t(x,t) + H_1\Big(y,s,D_{x'}\mathbf{U}^+(x,t) + \lambda_1 e_N + \frac{2(x_N-y_N)}{\varepsilon^{1/2}} + o_\varepsilon(1)\Big) \ge 0\,. \tag{15.14}$$

Notice that, using the arguments of Case 1 and the fact that here $x_N - y_N = -y_N < 0$, we are led by the definition of λ_1 to

$$O(\varepsilon^{1/2}) + (\mathbf{U}^+)_t(x,t) + H_1^-\Big(y,s,D_{x'}\mathbf{U}^+(x,t) + \lambda_1 e_N + \frac{2(x_N-y_N)}{\varepsilon^{1/2}} + o_\varepsilon(1)\Big) < 0\,,$$

from which we deduce that (15.14) holds true with H_1^+.

Moreover, by the subsolution condition of $\mathbf{U}^+$ on $\mathcal{H}$ we have

$$(\mathbf{U}^+)_t(x,t) + H_1^+\Big(x,t,D_{x'}\mathbf{U}^+(x,t) + \lambda_1 e_N + \frac{2(x_N - y_N)}{\varepsilon^{1/2}} + o_\varepsilon(1)\Big) \leq -\eta\,,$$

therefore the conclusion follows by standard arguments putting together the two inequalities for H_1^+ and letting ε tend to zero. If $y_N < 0$, we can repeat the same argument using this time H_2^- instead of H_1^+.

Case 3: $x_N = y_N = 0$.

Let us remark that this case is not possible. Indeed the maximum point property on $\mathbf{U}^+ - u^\varepsilon - \psi_\varepsilon$ implies that 0 is a minimum point of $z_N \mapsto u^\varepsilon((y', z_N), s) + \psi_\varepsilon(x, (y', z_N), t, s))$. But, by definition of ψ_ε and in particular of χ, this also means that we have a minimum point for the function

$$\zeta : z_N \mapsto u^\varepsilon((y', z_N), s) - h(z_N) + \frac{|z_N|^2}{\varepsilon^{1/2}}\,.$$

Both $z_N \mapsto |z_N|^2$ and u^ε are C^1-smooth, but the function h is only Lipschitz continuous at $z_N = 0$. So, using that the left derivative of ζ is negative while the right one is positive leads to $-h'(0^-) \leq -h'(0^+)$, *i.e.* $\lambda_2 \geq \lambda_1$. But this contradicts the construction of function χ which requires $\lambda_2 < \lambda_1$.

Q.E.D.

15.6 Classical viscosity solutions as flux-limited solutions

The aim of this section is to show that, under suitable assumptions, a classical viscosity sub or supersolution of

$$u_t + H(x,t,u,D_x u) = 0 \quad \text{in } \mathbb{R}^N \times (0,T_f)\,, \tag{15.15}$$

where H is a continuous quasi-convex Hamiltonian, is a (FLSub) or (FLSuper) of the problem with $H_1 = H_2 = H$ and $G = H_T$ where, for $x \in \mathcal{H}$, $t \in [0,T_f]$, $r \in \mathbb{R}$ and $p' \in \mathcal{H}$

$$H_T(x,t,r,p') = \min_{s\in\mathbb{R}} H(x,t,r,p' + se_N)\,.$$

We refer the reader to Section 10.2 and in particular to Lemma 10.2.1 for a motivation of the definition of H_T in the convex case but we are going to consider below the more general quasi-convex case.

The precise result is the

Proposition 15.6.1 *Assume that* **(GA-QC)** *holds with* $H_1 = H_2 = H$ *and that* $G = H_T$ *satisfies* **(GA-G-FL)**. *Then* u *is a classical Ishii subsolution* [*resp. supersolution*] *of* (15.15) *if and only if it is a* (FLSub) [*resp.* (FLSuper)] *of* (HJ-Gen)-(FL) *with* $H_1 = H_2 = H$ *and* $G = H_T$.

The interest of this result is to be able to introduce an artificial discontinuity when it is useful. We refer the reader to Section 17.7 for an example of such situation.

Proof — To prove that a (FLSub) (or (FLSuper)) is a classical Ishii subsolution (or supersolution) is easy using that (i) C^1 test-functions are PC^1 test-functions and (ii) $\max(H^+, H^-, H_T) = H$.

We only prove the converse for the subsolution case, the supersolution one being essentially analogous; we just provide below a tiny additional argument to treat this supersolution case. Of course, only the properties on $\mathcal{H} \times (0, T_f)$ are different and therefore we concentrate on this case.

Let u be a classical Ishii subsolution of (15.15) and let $(x,t) \in \mathcal{H} \times (0,T_f)$ be a strict local maximum point of $u - \varphi$ where $\varphi = (\varphi_1, \varphi_2) \in \mathrm{PC}^1(\mathbb{R}^N \times [0,T_f])$. We have to look at two different cases

(i) $\lambda := \dfrac{\partial \varphi_1}{\partial x_N}(x,t) \leq \mu := \dfrac{\partial \varphi_2}{\partial x_N}(x,t)$.

(ii) $\lambda > \mu$.

Case (i) is easy: if $p' = D_{x'}\varphi(x,t)$ and $p_t = \varphi_t(x,t)$ then, for any $\lambda \leq \tau \leq \mu$, $((p',\tau), p_t) \in D^+_{\Omega_i \times (0,T_f)} u(\bar{x},\bar{t})$ for $i = 1$ and $i = 2$; hence $((p',\tau), p_t) \in D^+_{\mathbb{R}^N \times (0,T_f)} u(\bar{x},\bar{t})$ and therefore

$$p_t + H(x,t,u(x,t), p' + \tau e_N) \leq 0 \ .$$

Using that $\max(H^+, H^-, H_T) = H$, we easily obtain the desired inequalities by choosing $\tau = \lambda$ and then $\tau = \mu$.

Case (ii) is more tricky: by Lemma 3.5.6, we can assume without loss of generality that $\varphi = \chi + \psi$ where ψ is C^1 in $\mathbb{R}^N \times (0,T_f)$ and

$$\chi(x_N) := \begin{cases} \lambda x_N & \text{if } x_N \geq 0 \ , \\ \mu x_N & \text{if } x_N \leq 0 \ . \end{cases}$$

We mollify the function χ by using a mollifying kernel with compact support and we obtain a sequence of C^1-functions $(\chi_\varepsilon)_\varepsilon$ and then a sequence $(\varphi_\varepsilon)_\varepsilon$ given by $\varphi_\varepsilon = \chi_\varepsilon + \psi$. Moreover, by standard convolution arguments, we have

$$\mu \leq \frac{\partial \chi_\varepsilon}{\partial x_N}(x_N) \leq \lambda \quad \text{for any } x_N \ .$$

Let $(x_\varepsilon, t_\varepsilon)$ be a sequence of maximum points of $u - \varphi_\varepsilon$ which converges to (x,t) and such that $u(x_\varepsilon, t_\varepsilon) \to u(x,t)$ (such sequence exists since (x,t) is a strict local maximum point of $u-\varphi$ and $\varphi_\varepsilon \to \varphi$ locally uniformly). We have

$$\frac{\partial \psi}{\partial t}(x_\varepsilon, t_\varepsilon) + H(x_\varepsilon, t_\varepsilon, u(x_\varepsilon, t_\varepsilon), D_{x'}\psi(x_\varepsilon, t_\varepsilon) + \frac{\partial \chi_\varepsilon}{\partial x_N}(x_\varepsilon, t_\varepsilon) e_N) \leq 0 \ .$$

Introducing

$$\tilde{H}(\tau) := \frac{\partial \psi}{\partial t}(x,t) + H(x,t,u(x,t), D_{x'}\psi(x,t) + \tau e_N) \ ,$$

and denoting respectively by $\tilde{H}^+$, $\tilde{H}^-$, $\tilde{H}_T$, functions which are defined in the same way, replacing H by H^+, H^- or H_T, we deduce from the continuity of H, the above properties and the C^1 character of ψ, that

$$\tilde{H}(\frac{\partial \chi_\varepsilon}{\partial x_N}(x_\varepsilon, t_\varepsilon)) \leq o_\varepsilon(1) \ .$$

This inequality can be rewritten as

$$\max(\tilde{H}^+, \tilde{H}^-, \tilde{H}_T)(\frac{\partial \chi_\varepsilon}{\partial x_N}(x_\varepsilon, t_\varepsilon)) \leq o_\varepsilon(1) \ ,$$

and using the monotonicity of $\tilde{H}^+, \tilde{H}^-$, we have, because H_T is independent of the x_N-derivative

$$\max(\tilde{H}^+(\lambda),\ \tilde{H}^-(\mu),\ \tilde{H}_T) \leq o_\varepsilon(1) \ ,$$

and we conclude by letting $\varepsilon \to 0$.

For supersolutions, the analogue of Case (ii) is treated exactly in the same way. Case (i)–which is now $\lambda \geq \mu$–required the following additional arguments: with the above notations, we have

$$\max(\tilde{H}^+(\tau),\ \tilde{H}^-(\tau),\ \tilde{H}_T) \geq 0 \ ,$$

for any $\mu \leq \tau \leq \lambda$ and we have three cases

- If $\tilde{H}_T \geq 0$, we are done.
- If $\tilde{H}_T < 0$, by choosing $\tau = \lambda$, we have $\max(\tilde{H}^+(\lambda),\ \tilde{H}^-(\lambda)) \geq 0$. If $\tilde{H}^+(\lambda) \geq 0$, we are done. In the same way, by choosing $\tau = \mu$, we have $\max(\tilde{H}^+(\mu),\ \tilde{H}^-(\mu)) \geq 0$. If $\tilde{H}^-(\mu) \geq 0$, we are done.

- If $\tilde{H}_T < 0$, $\tilde{H}^+(\lambda) < 0$ and $\tilde{H}^-(\mu) < 0$, then necessarely $\tilde{H}^-(\lambda) \geq 0$ and $\tilde{H}^+(\mu) \geq 0$. Hence

$$(\tilde{H}^+ - \tilde{H}^-)(\lambda) < 0 \quad , \quad (\tilde{H}^+ - \tilde{H}^-)(\mu) > 0 \, ,$$

and there exists $\tau \in (\mu, \lambda)$ such that $\tilde{H}^+(\tau) = \tilde{H}^-(\tau)$. But, for such τ, we have $\tilde{H}^+(\tau) = \tilde{H}^-(\tau) = H_T$. Therefore using such τ in the above inequality yields $H_T \geq 0$, a contradiction which means that we are in one of the two first cases.

And the proof is complete.

Q.E.D.

15.7 Extension to second-order equations (I)

In this section, we consider second-order equations of the form

$$u_t + H_i(x, t, Du) - \mathrm{Tr}(a_i(x) D^2 u) = 0 \quad \text{in } \Omega_i \times (0, T_f) \, ,$$

where a_i $(i = 1, 2)$ are continuous functions which are assumed to be on the standard form, *i.e.* $a_i = \sigma_i \cdot \sigma_i^T$ where σ_i^T is the transpose matrix of σ_i. We suppose that the σ_i's are bounded, Lipschitz continuous functions and in order that the definition of flux-limited solutions make sense, the following property has to be imposed

$$\sigma_i((x', 0)) = 0 \quad \text{for } i = 1, 2 \text{ and for all } x' \in \mathbb{R}^{N-1}.$$

The main question we adress here concerns the comparison result in this framework. There are several difficulties that we list below:

(i) in general, we cannot regularize the subsolution as we did above;

(ii) because of the second-order term, the normal controllability cannot be used efficiently outside $\mathcal{H}$;

(iii) a two-parameter proof as in the non-convex case is difficult to handle with the second-order term.

We take this opportunity to remark that the above comparison proofs has several common points with the comparison proof for nonlinear Neumann boundary conditions: in fact, it can be described as a "double Neumann" proof since H_1^- (almost)

plays the role of a Neumann boundary condition for the equation in Ω_2 while conversely H_2^+ (almost) plays the role of a Neumann boundary condition for the equation in Ω_1, see Proposition 12.2.1 for more explanations.

There is anyway a crucial additional difficulty: H_1^-, H_2^+ are NOT strictly monotone functions *w.r.t.* the normal gradient direction. Therefore, if a general "one-parameter proof", avoiding the use of $\gamma \ll \varepsilon$ may be possible, it is probably rather technical and may require additional assumptions on Hamiltonians H_i.

Instead, the following result gives some conditions under which the proof of Theorem 15.3.2 still works.

Theorem 15.7.1 (Comparison principle in the second-order case) *Under the assumptions of Theorem 15.3.2, the result of Theorem 15.3.1 is valid provided that the two following assumptions hold, for $i = 1, 2$, in a neighborhood of $\mathcal{H}$:*

(i) $H_i(x, t, p) = H_{i,1}(x', t, p') + H_{i,2}(x_N, p_N)$,
(ii) $\sigma_i = \sigma_i(x_N)$ *with* $\sigma_i(0) = 0$, σ_i *being locally Lipschitz continuous and bounded.*

It is worth pointing out that this result holds for non-convex Hamiltonians, but requires rather restrictive assumptions on H_i and σ_i. We refer to Imbert and Nguyen [116] for general results for second-order equations in the case of *networks* where not only comparison results are obtained but the notions of (FLS) and (JVS) are discussed and applications are given.

Proof — The proof follows readily the proof of Theorem 15.3.2, we just add here some comments:

- The structure conditions we impose on $(H_i, \sigma_i)_{i=1,2}$ ensures that we can perform a regularization of the subsolution by sup-convolution in the spirit of Proposition 3.4.9: in particular, the Hamiltonians both satisfy **(TC-s)**. This is the first reason to assume *(i)* and *(ii)*.
- Once this regularization is done, we still have to control the dependence in the derivatives in x_N (or all the terms involving the parameter γ): this is where the special dependence in x_N of H_i and σ_i plays a role.
- In all the steps where the properties of λ_1, λ_2 are crucial, the second-order term is small since $|\sigma_i(x_N)| = O(|x_N|)$ and therefore $|a_i(x_N)| = O(x_N^2)$. This can be combined with the facts that

$$\frac{|(x_\varepsilon)_N - (y_\varepsilon)_N|^2}{\gamma^2} \to 0 \quad \text{as } \gamma \to 0\,,$$

  and the second-order derivatives are a $O(\gamma^{-2})$.

Q.E.D.

Remark 15.7.2 *Anticipating the main result of Section 17.3 showing that the Kirchhoff boundary conditions is equivalent to a flux-limited boundary condition with* $G = H_T^{\mathrm{reg}}$ *under the assumptions of Theorems 15.3.1 or 15.3.2, these two results also provide the comparison for the (KC)-condition. The proof(s) would apply readily if we were able to show that we can choose* $\lambda_1 > \lambda_2$ *in the test-function (the function* χ*) but this is not obvious at this point and this property will be clarified in Section 17.3.*

Chapter 16

Junction Viscosity Solutions

Even if flux-limited viscosity solutions have their advantages, it may seem more natural to consider a definition of viscosity solution with a min / max condition on the junction involving H_1 and H_2 instead of their nondecreasing/nonincreasing parts.

In the next sections, we present the general notion of junction viscosity solutions, which is called "relaxed solution" in [113]. However, because of the similarity to the classical notion of viscosity solutions, it seems to us that "junction viscosity solutions" is more appropriate.

16.1 Definition and first properties

We introduce the notion of junction viscosity sub/supersolution for (HJ-Gen) associated with a (GJC) given by a nonlinearity G as follows [(1)]

Definition 16.1.1 (Junction Viscosity Solutions)

A locally bounded function $u : \mathbb{R}^N \times (0,T_f) \to \mathbb{R}$ is a (JVSub) *of* (HJ-Gen)-(GJC) *if it is a classical viscosity subsolution of* (HJ-Gen) *and if, for any test-function $\psi = (\psi_1,\psi_2) \in \mathrm{PC}^1(\mathbb{R}^N \times [0,T_f])$ and any local maximum point $(x,t) \in \mathcal{H} \times (0,T_f)$ of $u^* - \psi$ in $\mathbb{R}^N \times (0,T_f)$,*

$$\min\left(G(x,t,\psi_t,D_{\mathcal{H}}\psi,\frac{\partial\psi_1}{\partial n_1},\frac{\partial\psi_2}{\partial n_2}),\psi_t + H_1(x,t,u^*,D\psi_1),\psi_t + H_2(x,t,u^*,D\psi_2)\right) \le 0\,, \tag{16.1}$$

where u^* and the derivatives of ψ,ψ_1,ψ_2 are taken at (x,t).

[(1)] We recall that we assume that $G(x,t,r,a,p',b,c)$ is independent of r.

A locally bounded function $v : \mathbb{R}^N \times (0,T_f) \to \mathbb{R}$ *is a* (JVSuper) *of* (HJ-Gen)-(GJC) *if it is a classical viscosity supersolution of* (HJ-Gen) *and if, for any test-function* $\psi = (\psi_1, \psi_2) \in \mathrm{PC}^1(\mathbb{R}^N \times [0,T_f])$ *and any local minimum point* $(x,t) \in \mathcal{H} \times (0,T_f)$ *of* $v_* - \psi$ *in* $\mathbb{R}^N \times (0,T_f)$,

$$\max\left(G(x,t,\psi_t, D_{\mathcal{H}}\psi, \frac{\partial\psi_1}{\partial n_1}, \frac{\partial\psi_2}{\partial n_2}), \psi_t + H_1(x,t,v_*, D\psi_1), \psi_t + H_2(x,t,v_*,D\psi_2)\right) \geq 0\,, \tag{16.2}$$

where v_* *and the derivatives of* ψ, ψ_1, ψ_2 *are taken at* (x,t).
A (JVS) *(i.e. a junction viscosity solution) is a locally bounded function which is both* (JVSub) *and* (JVSuper).

As in the case of (FLSub) and (FLSuper), we can define (JVSub) and (JVSuper) using the notions of sub and superdifferentials. With the notations of Proposition 15.1.2, we have the [(2)]

Proposition 16.1.2 — *Junction viscosity solutions via sub superdifferentials.*
An u.s.c., locally bounded function $u : \mathbb{R}^N \times (0,T_f) \to \mathbb{R}$ *is a* (JVSub) *of* (HJ-Gen)-(GJC) *if and only if*

(*i*) *for any* $(x,t) \in Q_i$ $(i = 1,2)$ *and any* $(p_x,p_t) \in D^+_{\overline{Q_i}^\ell} u(x,t)$

$$p_t + H_i(x,t,u(x,t),p_x) \leq 0\,,$$

(*ii*) *for any* $(x,t) \in \mathcal{H} \times (0,T_f)$ *and for any* $p_{\mathcal{H}} \in \mathcal{H}$, $p_1,p_2,p_t \in \mathbb{R}$ *such that* $((p_{\mathcal{H}},p_i),p_t) \in D^+_{\overline{Q_i}^\ell} u(x,t)$ *for* $i = 1,2$, *noting* $u = u(x,t)$,

$$\min\left(G(x,t,p_t,p_{\mathcal{H}},p_1,p_2), p_t + H(x,t,u,p_{\mathcal{H}} + p_1 e_N), p_t + H_2(x,t,u,p_{\mathcal{H}} + p_2 e_N)\right) \leq 0\,.$$

A l.s.c., locally bounded function $v : \mathbb{R}^N \times (0,T_f) \to \mathbb{R}$ *is a* (JVSuper) *of* (HJ-Gen)-(GJC) *if and only if, for* $(x,t) \in \mathbb{R}^N \times (0,T_f)$,

(*i*) *for any* $(x,t) \in Q_i$ $(i = 1,2)$ *and for any* $(p_x,p_t) \in D^-_{\overline{Q_i}^\ell} v(x,t)$

$$p_t + H_i(x,t,v(x,t),p_x) \geq 0\,,$$

[(2)] Again we formulate the result for u.s.c. subsolution and l.s.c. supersolution but the reader can easily transpose it to general sub and supersolutions

(*ii*) *for any* $(x,t) \in \mathcal{H} \times (0,T_f)$ *and for any* $p_{\mathcal{H}} \in \mathcal{H}$, $p_1, p_2, p_t \in \mathbb{R}$ *such that* $((p_{\mathcal{H}}, p_i), p_t) \in D^-_{\overline{Q_i}^\ell} v(x,t)$ *for* $i = 1,2$, *noting* $v = v(x,t)$,

$$\max\Big(G(x,t,p_t,p_{\mathcal{H}},p_1,p_2), p_t + H(x,t,u,p_{\mathcal{H}}+p_1 e_N), p_t + H_2(x,t,u,p_{\mathcal{H}}+p_2 e_N)\Big) \geq 0\,.$$

As for Proposition 15.1.2, we leave the proof of this result to the reader since it is an easy consequence of Lemma 3.5.3 and Lemma 3.5.6. We again point out that this equivalent definition via sub and superdifferentials allows to show that instead of using general PC^1 test-functions, we may only use test-functions of the form $\chi(x_N)+\varphi(x,t)$ where $\chi \in \mathrm{PC}^1(\mathbb{R})$ and $\varphi \in C^1(\mathbb{R}^N \times (0,T_f))$. The reader will notice that we mainly use test-function of this form in the comparison result but this property is also useful to simplify the proofs of several results.

Before considering the regularity properties of (JVSub) and (JVSuper), we point out that one of the advantages of the notion of junction viscosity solution is that it can be applied to a wider class of junction conditions without any convexity/quasi-convexity type assumption. On the other hand, its similarity with the classical notion of viscosity solutions should easily convince the reader that the notion enjoys the stability properties of classical viscosity solutions.

16.1.1 Lack of regularity of subsolutions

This notion has a slight defect since u.s.c. junction viscosity subsolutions are not necessarily regular, contrarily to flux-limited solutions, because of the "min" in the definition. To show it, we consider the following 1-d example

$$\begin{cases} u_t + |u_x| = 0 & \text{in } \mathbb{R}\setminus\{0\} \times (0,+\infty)\,, \\ u_t(0,t) = 0 & \text{in } (0,+\infty)\,, \\ u(x,0) = -|x| & \text{in } \mathbb{R}\,. \end{cases}$$

It is worth pointing out that this problem is far from being pathological since $H_1(p) = H_2(p) = |p|$ satisfy all the "good assumptions", in particular **(NC)**. One checks easily that the expected solution is $U(x,t) = -|x| - t$ but we also have the non-regular subsolution given by

$$u(x,t) = \begin{cases} U(x,t) & \text{if } x \neq 0\,, \\ 0 & \text{for } x = 0\,. \end{cases}$$

It is clear that u is u.s.c. and a subsolution for $x \neq 0$, and it is a subsolution for $x = 0$ because $u_t(0,t) \equiv 0$ and the "min" in the definition allows such inexpected feature.

16.1.2 The case of Kirchhoff-type conditions

We refer to Section 14.1.2 for the complete and precise definitions of different junction conditions on the inferface. Let us just recall here that Kirchhoff-type conditions essentially satisfy (dropping the u-dependance)

$$G(x,t,a_1,p',b_1,c_1) - G(x,t,a_2,p',b_2,c_2) \geq \alpha(a_1-a_2)+\beta(b_1-b_2)+\beta(c_1-c_2) \quad (16.3)$$

for some $\alpha \geq 0$, $\beta > 0$. Of course the typical example is the standard Kirchhoff condition for which $G(x,t,a,p',b,c) = b+c$, encoding $\partial u/\partial n_1 + \partial u/\partial n_2 = 0$.

Proposition 16.1.3 *Assume that H_1, H_2, G are continuous functions and that H_1, H_2 satisfy* **(NC-HJ)**. *Then junction viscosity subsolutions are regular provided* (GJC) *is of Kirchhoff type.*

Proof — We only provide the proof in the subsolution case, the supersolution one being analogous. Assume that $u : \mathbb{R}^N \times (0,T_f) \to \mathbb{R}$ is an u.s.c.(JVSub) of (HJ-Gen)-(GJC) and let (x,t) be a point of $\mathcal{H} \times (0,T_f)$. We argue by contradiction: if, for instance, u is not Ω_1-regular at (x,t), this means that

$$u(x,t) > \limsup_{\substack{(y,s)\to(x,t)\\ y\in\Omega_1}} u(y,s)\,. \quad (16.4)$$

We introduce the function

$$\Psi : (y,s) \mapsto u(y,s) - \frac{|y-x|^2}{\varepsilon^2} - \frac{|s-t|^2}{\varepsilon^2} - C_1 y_N^+ - C_2 y_N^-\,,$$

where $0 < \varepsilon \ll 1$ and $C_1 \in \mathbb{R}$, $C_2 > 0$ are constants to be chosen. Notice that $y \mapsto C_1 y_N^+ + C_2 y_N^-$ belongs to $\mathrm{PC}^1(\mathbb{R}^N \times [0,T_f])$.

Choosing ε small enough and $C_1 = 0$, Ψ has a maximum point $(x_\varepsilon,t_\varepsilon)$ near (x,t) and $(x_\varepsilon,t_\varepsilon) \to (x,t)$, $u(x_\varepsilon,t_\varepsilon) \to u(x,t)$ as $\varepsilon \to 0$. We see that if C_2 is large enough, the H_2-subsolution inequality cannot hold, therefore $(x_\varepsilon,t_\varepsilon) \in \mathcal{H}$. Moreover, if ε is small enough, (16.4) is also true at $(x_\varepsilon,t_\varepsilon)$ and, as a consequence, $(x_\varepsilon,t_\varepsilon)$ is a local maximum point of Ψ for any $C_1 \in \mathbb{R}$.

In the same way, choosing now $C_1 < 0$ large enough implies that the H_1-subsolution inequality cannot hold. But neither can the G-one, provided G is of Kirchhoff-type: since G behaves like $K_\varepsilon + \beta(C_2 - C_1)$ for some constant K_ε, it is strictly positive if C_1 is very negative.

Hence, none of the subsolution inequalities can hold on the interface and we get the desired contradiction.

Q.E.D.

On the contrary, junction viscosity supersolutions are not necessarely regular, even if (GJC) is of Kirchhoff type as shown by the following example.

Example 16.1 — The solution $u : \mathbb{R} \times [0, +\infty[$ of

$$u_t + |u_x| = 0 \quad \text{in } \mathbb{R} \times (0, +\infty) \,,$$

$$u(x, 0) = u_0 \quad \text{in } \mathbb{R} \,,$$

where $u_0(x) := (1 - |x|)^+$ is given by $u(x, t) = (1 - |x| - t)^+$.

Now we look at

$$v(x, t) = \begin{cases} u(x, t) & \text{if } x \leq 0 \,, \\ 1 & \text{if } x > 0 \,. \end{cases}$$

Then one checks easily that v is (i) l.s.c., (ii) a (JVSuper) for (HJ-Gen)-(KC) with $H_1(p) = H_2(p) = |p|$ and (iii) is not regular at $x = 0$. As the reader has probably already noticed it, the (JVSuper) property comes from the fact that u is a solution of the state-constraint problem in $(-\infty, 0] \times (0, +\infty)$, hence $u_t + H_2(u_x) \geq 0$ even on the boundary $\{0\} \times (0, +\infty)$. This may be seen as a little defect of the network approach which, by using PC^1-test-functions, leads to a slight decoupling of the domains $\Omega_1 \times (0, +\infty)$ and $\Omega_2 \times (0, +\infty)$.

16.2 Stability of junction viscosity solutions

Contrarily to the case of flux-limited viscosity solutions, the stability result for (JVS) is almost an immediate extension of Theorem 3.1.4: the change of test-functions, using $\mathrm{PC}^1(\mathbb{R}^N \times [0, T_f])$ implies only minor modifications which is why we skip the proof of the

Theorem 16.2.1 *Assume that*

(*i*) *For any $\varepsilon > 0$, $H_1^\varepsilon, H_2^\varepsilon, G^\varepsilon$ are continuous and converge locally uniformly respectively to H_1, H_2, G .*

(*ii*) *For any $\varepsilon > 0$, u_ε is a* (JVSub) [*resp.* (JVSuper)] *for the problem with Hamiltonians $H_1^\varepsilon, H_2^\varepsilon, G^\varepsilon$.*

(iii) The functions u_ε are uniformly locally bounded on $\mathbb{R}^N \times [0,T_f]$.

Then $\overline{u} = \limsup^* u_\varepsilon$ [resp. $\underline{u} = \liminf_* u_\varepsilon$] is a (JVSub) [*resp.* (JVSuper)] *for the problem with Hamiltonians H_1, H_2, G.*

Remark 16.2.2 *Contrarily to Theorem 15.2.1, Theorem 16.2.1 turns out to be very flexible, without any restriction on the Hamiltonians and with general junction conditions; in particular, it can be used to address the problem of the convergence of the vanishing viscosity method.*

16.3 Comparison results for junction viscosity solutions: the Lions-Souganidis approach

In this section we expose the Lions-Souganidis approach of the comparison proof for (JVS) and apply it to the case of general Kirchhoff conditions, as well as second-order equations.

16.3.1 Preliminary lemmas

We begin with a simple one-dimensional lemma which can be seen as a little bit more precise version of Proposition 3.5.4 in this context

Lemma 16.3.1 *Let $H : \mathbb{R} \to \mathbb{R}$ be a continuous function and $u : [0,r] \to \mathbb{R}$ be a Lipschitz continuous subsolution of $H(u_x) = 0$ in $(0,r)$. Defining*

$$\underline{p} := \liminf_{x\to 0} \left[\frac{u(x)-u(0)}{x}\right] \le \limsup_{x\to 0} \left[\frac{u(x)-u(0)}{x}\right] =: \overline{p}\,,$$

then $H(p) \le 0$ for all $p \in [\underline{p}, \overline{p}]$.

Remark 16.3.2 *In Lemma 16.3.1, the subsolution is assumed to be Lipschitz continuous and this is consistent with the fact that we consider equations with coercive Hamiltonians, or at least satisfying* **(NC)**. *This assumption ensures that $\underline{p}$ and $\overline{p}$ are bounded, but this is not really necessary as the proof will show.*

Without this assumption, we can still prove at least that if $\underline{p} < +\infty$, $H(p) \le 0$ for all $p \in]\underline{p}, \overline{p}[$. The importance of this remark is more for supersolutions: we use below an analogous result for them and it is less natural to assume them to be Lipschitz continuous.

Proof — Let us first notice that since u is assumed to be Lipschitz, both $\underline{p}$ and $\overline{p}$ are not infinite.

(a) We first assume that $\underline{p} < \overline{p}$.

Let $(x_k)_k$ be a sequence of points of $(0,r)$ such that $x_k \to 0$ and $\big(u(x_k)-u(0)\big)/x_k \to \underline{p}$. We pick any $\underline{p} < p < \overline{p}$ and consider the function $\psi(y) = u(y) - u(0) - py$ on the interval $[0, x_k]$. Since

$$\psi(0) = 0\,, \quad \psi(x_k) < 0\,, \quad \limsup_{x\to 0} \frac{\psi(x)}{x} = \overline{p} - p > 0\,,$$

there exists a maximum point $\tilde{x}_k \in (0, x_k)$ of ψ. The subsolution property at $\tilde{x}_k$ yiels the desired inequality: $H(p) \le 0$. Moreover, the continuity of H implies that the same property holds true for all $p \in [\underline{p}, \overline{p}]$.

(b) Now we turn to the case $\underline{p} = \overline{p}$.

For $0 < \varepsilon \ll 1$, we consider $v(x) = u(x) + \varepsilon x \sin(\log(x))$. The function $x \mapsto x\sin(\log(x))$ is C^1 for $x > 0$ and Lipschitz continuous. Therefore, $H(v_x) \le o_\varepsilon(1)$. Moreover

$$\frac{v(x)-v(0)}{x} = \frac{u(x)-u(0)}{x} + \varepsilon \sin(\log(x))\ ,$$

which implies that

$$\liminf_{x\to 0}\left[\frac{v(x)-v(0)}{x}\right] = \overline{p} - \varepsilon < \limsup_{x\to 0}\left[\frac{v(x)-v(0)}{x}\right] = \overline{p} + \varepsilon\ .$$

Since $\overline{p} - \varepsilon < \overline{p} < \overline{p} + \varepsilon$, case **(a)** above implies that $H(\overline{p}) \le o_\varepsilon(1)$ and the conclusion follows by letting ε tend to 0.

Q.E.D.

Remark 16.3.3 *Of course, analogous results hold for supersolutions: if v is a supersolution of $H(v_x) \ge 0$ in $(0,r)$, it suffices to use that u=-v(x) is a subsolution of $-H(-u_x) \le 0$ in $(0,r)$.*

In order to connect the 1-d and multi-dimensional situations and therefore to give a more precise formulation of Proposition 3.5.4 in the framework which is the one of the comparison proof, let us consider a set

$$Q := \big\{(y,x) : y \in \mathcal{V}\,,\ x \in]0,\delta[\big\} \subset \mathbb{R}^{p+1}$$

where $\mathcal{V}$ is a neighborhood of 0 in $\mathbb{R}^p$, and $\delta > 0$. If $w : \overline{Q} \to \mathbb{R}$, we denote by $D^+_{\overline{Q}} w$ and $D^-_{\overline{Q}} w$ the super and sub-differentials of w with respect to both variables (y, x).

If w is differentiable with respect to y at $(0, 0)$, it can be expected that these sub/super-differentials of w in both variables have the forms

$$(D_y w(0,0), D^-_x w(0,0)) \quad \text{and} \quad (D_y w(0,0), D^+_x w(0,0)) .$$

But in view of Lemma 16.3.1 and using Section 6.1, we can give a more precise result

Lemma 16.3.4 *Let $w : \overline{Q} \to \mathbb{R}$ be a function such that the functions $y \mapsto w(y, x)$ are Lipschitz continuous in $\mathcal{V}$ uniformly with respect to $x \in [0, \delta[$ and $y \mapsto w(y, 0)$ is differentiable at 0.*

(*a*) *Superdifferential case*

We assume moreover that w is upper-semicontinuous in Q and that, for any $x \in [0, \delta[$, the function $y \mapsto w(y, x)$ is semi-convex in $\mathcal{V}$. If

$$\overline{p} = \limsup_{x \to 0} \left[\frac{w(0,x) - w(0,0)}{x} \right]$$

exists and is finite, then

$$(D_y w(0,0), p) \in D^+_Q w(0,0) \quad \textit{if and only if} \quad p \geq \overline{p} .$$

(*b*) *Subdifferential case*

We assume moreover that w is lower-semicontinuous in Q and that, for any $x \in [0, \delta[$, the function $y \mapsto w(y, x)$ is semi-concave in $\mathcal{V}$. If

$$\underline{q} = \liminf_{x \to 0} \left[\frac{w(0,x) - w(0,0)}{x} \right]$$

exists and is finite, then

$$(D_y w(0,0), q) \in D^-_Q w(0,0) \quad \textit{if and only if} \quad q \leq \underline{q} \ .$$

The interest of this lemma is clear: under suitable assumptions, we can connect 1-d and multi-d sub or super-differentials. This will be a key step for applying Lemma 16.3.1 to multi-d problems. We point out anyway that Lemma 16.3.1 gives an important additional information on the interval $[\underline{p}, \overline{p}]$ if $\underline{p} \neq \overline{p}$.

Proof — We only do the proof in case (a), the other case working with obvious adaptations.

For $p \geq \bar{p}$, we set

$$\overline{w}(y,x) := w(y,x) - w(0,0) - D_y w(0,0) \cdot y - px\,.$$

If $(D_y w(0,0), p) \in D_Q^+ w(0,0)$, then $\overline{w}(y,x) \leq o(|y|+x)$. Choosing $y=0$ and dividing by $x>0$, we obtain

$$\frac{w(0,x)-w(0,0)}{x} - p \leq o(1)\,,$$

and therefore $\overline{p} \leq p$ by taking the lim sup as $x \to 0$.

Conversely, if $p \geq \bar{p}$, we want to show that $(D_y w(0,0), p) \in D_Q^+ w(0,0)$, *i.e.* $\overline{w}(y,x) \leq o(|y|+x)$. To do so, we argue by contradiction assuming that there exists $\eta > 0$ and a sequence $(y_k, x_k)_k$ converging to $(0,0)$ such that $x_k > 0$ for all k $\overline{w}(y_k,x_k) \geq \eta(|y_k|+x_k)$.

Using the upper semicontinuity of w, hence of $\overline{w}$, we easily deduce that $\overline{w}(y_k,x_k) \to \overline{w}(0,0) = 0$ and the Lipschitz continuity of $y \mapsto \overline{w}(y,x_k)$ implies that $\overline{w}(0,x_k) \to \overline{w}(0,0) = 0$.

Next we use the decomposition

$$\overline{w}(y,x_k) = [\overline{w}(y,x_k) - \overline{w}(0,x_k)] + \overline{w}(0,x_k)\,.$$

By the definition of $\bar{p}$ and $p \geq \bar{p}$, $\overline{w}(0,x_k) = w(0,x_k) - w(0,0) - px_k \leq o(x_k)$. Therefore it remains to estimate the bracket to obtain a contradiction.

To do so, we introduce a regularization by convolution of $y \mapsto \overline{w}(y,x_k)$ for all fixed x_k. Let $(\rho_\varepsilon)_\varepsilon$ be a sequence of approximate identities in $\mathbb{R}^p$, *i.e.* a sequence of positive, C^∞-functions on $\mathbb{R}^p$ with compact support in $B_\infty(0,\varepsilon)$ such that $\int_{\mathbb{R}^p} \rho_\varepsilon(z)dz = 1$. Then we set

$$\overline{w}_\varepsilon(y,x_k) := \int_{\mathbb{R}^p} \overline{w}(y+z,x_k)\rho_\varepsilon(z)\,dz\,.$$

For any k, the functions $y \mapsto \overline{w}_\varepsilon(y,x_k)$ are C^1 in a neighborhood of 0 and therefore

$$\overline{w}_\varepsilon(y,x_k) - \overline{w}_\varepsilon(0,x_k) = \int_0^1 D_y\overline{w}_\varepsilon(sy,x_k)\cdot y\,ds\,.$$

Now we examine $D_y\overline{w}_\varepsilon(sy,x_k)$. Since, for any k, $y \mapsto \overline{w}(y,x_k)$ is Lipschitz continuous, hence differentiable almost everywhere by Rademacher's Theorem, we have

$$D_y\overline{w}_\varepsilon(sy,x_k) = \int_{\mathbb{R}^p} D_y\overline{w}(sy+z,x_k)\rho_\varepsilon(z)\,dz\,.$$

Moreover, using again the Lipschitz continuity of $y \mapsto \overline{w}(y, x_k)$, we have $\overline{w}(y, x_k) \to \overline{w}(0,0) = 0$ when $y \to 0$ and $k \to +\infty$; therefore, by the semi-convexity assumption, it follows from Proposition 6.1.1-(iv) that we have $D_y\overline{w}(y, x_k) = D_y w(y, x_k) - D_y w(0,0) \to 0$ when $(y, x_k) \to (0,0)$. This implies that $D_y\overline{w}_\varepsilon(sy, x_k) = o_k(1) + o_y(1) + o_\varepsilon(1)$ as $(y, x_k, \varepsilon) \to (0,0,0)$, uniformly with respect to $s \in [0,1]$. Therefore $\overline{w}_\varepsilon(y, x_k) - \overline{w}_\varepsilon(0, x_k) = |y|(o_x(1) + o_y(1) + o_\varepsilon(1))$. Letting ε tend to 0, we end up with

$$\overline{w}(y, x_k) - \overline{w}(0, x_k) = o(|y|) + |y| o_x(1) = o(|y|) \quad \text{for } k \text{ large enough},$$

which yields the desired contradiction.

Q.E.D.

16.3.2 A comparison result for the Kirchhoff condition

Before considering other junction conditions, we first provide a comparison result for the problem (HJ-Gen)-(KC), namely

$$\begin{cases} u_t + H_1(x,t,u,Du) = 0 & \text{in } \Omega_1 \times (0,T_f)\,, \\ u_t + H_2(x,t,u,Du) = 0 & \text{in } \Omega_2 \times (0,T_f)\,, \\ \dfrac{\partial u}{\partial n_1} + \dfrac{\partial u}{\partial n_2} = 0 & \text{on } \mathcal{H} \times (0,T_f)\,, \end{cases}$$

where for $i = 1, 2$, $n_i(x)$ denotes the unit outward to $\partial\Omega_i$ at $x \in \partial\Omega_i$. We recall that this Kirchhoff condition has to be taken in the (JVS) sense, namely

$$\min\left(u_t + H_1(x,t,u,Du), u_t + H_2(x,t,u,Du), \frac{\partial u}{\partial n_1} + \frac{\partial u}{\partial n_2}\right) \le 0 \quad \text{on } \mathcal{H} \times (0,T_f)\,,$$

for the subsolution condition and

$$\max\left(u_t + H_1(x,t,u,Du), u_t + H_2(x,t,u,Du), \frac{\partial u}{\partial n_1} + \frac{\partial u}{\partial n_2}\right) \ge 0 \quad \text{on } \mathcal{H} \times (0,T_f)\,,$$

for the supersolution condition, using test-functions in PC^1.

In order to formulate and prove a comparison result for (HJ-Gen)-(KC), we face several difficulties: first, we are not readily in the "good framework for HJ-Equations with discontinuities" because of the min in the subsolution junction condition and the (KC) condition which prevents **(NC)** to be satisfied. As a consequence it is not clear a priori that subsolutions are regular—a general problem for (JVS)—nor supersolutions. This last point is important since, in order to apply the Lions-Souganidis approach,

we have to regularize both the sub and supersolution, needing both to be regular. Fortunately for subsolutions, this problem is solved by Proposition 16.1.3, but not for supersolutions. Last but not least, it is not completely clear that we can apply Proposition 3.2.1 in order to prove only **(LCR)**.

We can overcome all these difficulties under some suitable assumptions

Theorem 16.3.5 *Assume that H_1 and H_2 satisfy* **(GA-Gen)**. *Then the* **(GCR)** *holds for any bounded subsolution u and supersolution v provided*

(i) either v satisfies (3.19), i.e. there exists $C > 0$ such that for all (x', t)

$$|v((x', x_N), t) - v(x', y_N), t)| \leq C|x_N - y_N|\,;$$

(ii) or **(TC-HJ)** *holds for both H_1 and H_2.*

Of course, the main interest of this result is to allow to prove a **(GCR)** which is valid for non convex Hamiltonians H_1 and H_2. In addition, it is easy to see that the proof we give below (and which is almost exactly the Lions-Souganidis one) can provide a comparison result for different types of "junction conditions" on $\mathcal{H}$ and also for more general networks problems; we come back on this point in the next section.

Proof — This proof consists first in reducing to a one-dimensional proof thanks to various reductions and using the preliminary lemmas of the previous section.

(a) *Reduction to a* **(LCR)** *with semiconvex/concave functions.*

Thanks to Section 3.2, we are not going to prove a **(GCR)** but only a **(LCR)**: the results of this section apply since the modifications we perform on the subsolution u are C^1 and therefore do not affect the (KC) condition. The next point concerns the regularity of u and v on $\mathcal{H}$: both are regular by Proposition 16.1.3.

Now, proving the **(LCR)** means that if u is a subsolution and v is a supersolution of (HJ-Gen)-(KC), we want to prove that there exists $r > 0$ and $0 < h < t$ such that, denoting by $\mathcal{K} := \overline{Q^{x,t}_{r,h}}$, if $\max_{\mathcal{K}}(u - v) > 0$, then

$$\max_{\mathcal{K}}(u - v) = \max_{\partial_p \mathcal{K}}(u - v)\,.$$

Considering a point point $(\bar{x}, \bar{t})$ where $\max_{\mathcal{K}}(u - v) > 0$ is achieved, we can assume of course that $\bar{t} > 0$ and $(\bar{x}, \bar{t}) \notin \partial_p \mathcal{K}$ otherwise the result obviously holds. It is also clear that we can assume *w.l.o.g.* that $\bar{x} \in \mathcal{H}$, otherwise only the H_1 or H_2 equation plays a role and we are in the case of a standard proof.

The proof of the theorem is based on the arguments of Section 3.4, and more precisely on Propositions 3.4.4 and 3.4.9: by using Proposition 3.4.4 and adding to u

a term of the form $\eta\chi(x_N)$ where $\chi \in \mathrm{PC}^1(\mathbb{R})$ is a bounded function which satisfies $\chi'(0_+) = 1$ and $\chi'(0_-) = -1$, we can assume *w.l.o.g.* that u is a Lipschitz continuous, η-strict subsolution of the equation and that u is semi-convex in x' and t.

Using similar arguments, it is also possible to assume that v is semi-concave in x' and t but only under the conditions of Proposition 3.4.9, hence assumptions (i) or (ii) above. We point out that these reductions allow to have a supersolution v which is Lipschitz continuous in x' and t, uniformly in x_N, but which can still be discontinuous in x_N, we come back on this point below by modifying v into some supersolution $\tilde{v}$.

A key consequence of the semi-convexity of u and of the semi-concavity of v in the variables x', t is that u, v are differentiable in x' and t at the maximum point $(\bar{x}, \bar{t})$ and

$$D_{x'}u(\bar{x}, \bar{t}) = D_{x'}v(\bar{x}, \bar{t}) \quad \text{and} \quad u_t(\bar{x}, \bar{t}) = v_t(\bar{x}, \bar{t}) \ .$$

For a precise result, see Proposition 6.1.1-(v). Moreover, as a consequence of Remark 6.1.2 (since the semi-convexity of u holds only in the tangent variables), if we denote by (p', p_N, p_t) any element in the superdifferential of u at (y, s) close to $(\bar{x}, \bar{t})$, then $(p', p_t) \to (D_{x'}u(\bar{x}, \bar{t}), u_t(\bar{x}, \bar{t}))$ as (y, s) tends to $(\bar{x}, \bar{t})$.

For the supersolution v however, the same property may not be true for the elements of the subdifferential since v can be discontinuous at $(\bar{x}, \bar{t})$. To turn around this difficulty we introduce

$$\tilde{v}((x', x_N), t) := \min\left(v((x', x_N), t)\ ,\ v((x', 0), t) + K|x_N|\right) \ ,$$

where $K > 0$. If we choose K large enough, function $\tilde{v}$ is a supersolution of the equation for $x_N \neq 0$, as the minimum of two supersolutions and is continuous at $(\bar{x}, \bar{t})$. As a consequence of this continuity property, $\tilde{v}$ being still semi-concave in (x', t) as the minimum of semi-concave functions in (x', t), for any element (p', p_N, p_t) in the subdifferential of $\tilde{v}$ at (y, s) close to $(\bar{x}, \bar{t})$, the following limit holds: $(p', p_t) \to (D_{x'}\tilde{v}(\bar{x}, \bar{t}), \tilde{v}_t(\bar{x}, \bar{t})) = (D_{x'}v(\bar{x}, \bar{t}), v_t(\bar{x}, \bar{t}))$ as (y, s) tends to $(\bar{x}, \bar{t})$.

(c) *Reduction to a stationary, one-dimensional proof.*

These properties of u and $\tilde{v}$ allow us to argue only in the x_N variable since, taking into account the regularity of H_1, H_2, we have

$$\tilde{H}_1(u_{x_N}) \leq -\eta < 0 \leq \tilde{H}_1(\tilde{v}_{x_N}) \quad \text{for } x_N > 0 \ ,$$

$$\tilde{H}_2(u_{x_N}) \leq -\eta < 0 \leq \tilde{H}_2(\tilde{v}_{x_N}) \quad \text{for } x_N < 0 \ ,$$

where for $i = 1, 2$,

$$\tilde{H}_i(p_N) = u_t(\bar{x}, \bar{t}) + H_i(\bar{x}, \bar{t}, u(\bar{x}, \bar{t}), (D_{x'}u(\bar{x}, \bar{t}), p_N)) + o(1) \ ,$$

the $o(1)$ tending to 0 as $\bar{r} \to 0$ if we consider the equations in $B((\bar{x},\bar{t}),\bar{r})$. It is worth pointing out that for the H_i-equations for v, we have used the fact that both $r \mapsto H_i(x,t,r,p)$ are increasing and that $u(\bar{x},\bar{t}) > \tilde{v}(\bar{x},\bar{t}) = v(\bar{x},\bar{t})$.

In order to proceed, we compute the superdifferentials for u in the two directions $x_N > 0$ and $x_N < 0$. We recall that since the test-functions are different in Ω_1 and Ω_2, these superdifferentials are different. For $x_N > 0$, we have $D_1^+u(0) = [\overline{p}_1, +\infty)$ where $\overline{p}_1$ is defined as the $\overline{p}$ in Lemma 16.3.1, but we are referring here to Ω_1. For $x_N < 0$, we have $D_2^+u(0) = [-\infty, -\overline{p}_2)$ where $\overline{p}_2$ is defined as the $\overline{p}$ in Lemma 16.3.1 but for $u(-x_N)$, in Ω_2.

Using the definition of viscosity subsolution together with Lemma 16.3.1 and 16.3.4, we obtain, since $n_1 = -e_N$ and $n_2 = e_N$

$$\min(-p_1 + p_2, \tilde{H}_1(p_1) + \eta, \tilde{H}_2(p_2) + \eta) \leq 0 \ ,$$

for any $p_1 \geq \overline{p}_1$ and $p_2 \leq -\overline{p}_2$; moreover

$$\begin{aligned} &\tilde{H}_1(p_1) + \eta \leq 0 \quad \text{if } p_1 \in [\underline{p}_1, \overline{p}_1] \ , \\ &\tilde{H}_2(p_2) + \eta \leq 0 \quad \text{if } p_2 \in [-\overline{p}_2, -\underline{p}_2] \ . \end{aligned}$$

For the supersolution v, we argue through $\tilde{v}$ but the aim is really to identify the subdifferential of v at $(\bar{x},\bar{t})$. We first notice that, $(\bar{x},\bar{t})$ being a maximum point of $u-v$, then $u(x,t)-v(x,t) \leq u(\bar{x},\bar{t})-v(\bar{x},\bar{t})$ for any (x,t) and the Lipschitz continuity of u implies

$$-C|(x,t) - (\bar{x},\bar{t})| \leq u(x,t) - u(\bar{x},\bar{t}) \leq v(x,t) - v(\bar{x},\bar{t}) \ ,$$

for C large enough, and in particular in the x_N-direction

$$-C|x_N| \leq v((\bar{x}', x_N),\bar{t}) - v(\bar{x},\bar{t}) \ . \tag{16.5}$$

Hence the subdifferentials of v at $(\bar{x},\bar{t})$ in both directions, namely $D_1^-v(0)$ and $D_2^-v(0)$ are non empty.

Moreover, applying Lemma 16.3.1 to $\tilde{v}$, we obtain that

$$\begin{aligned} &\tilde{H}_1(q_1) \geq 0 \quad \text{if } q_1 \in [\underline{q}_1, \overline{q}_1] \ , \\ &\tilde{H}_2(q_2) \geq 0 \quad \text{if } q_2 \in [-\overline{q}_2, -\underline{q}_2] \ , \end{aligned}$$

where $D_1^-\tilde{v}(0) = (-\infty, \underline{q}_1]$ and $D_2^-\tilde{v}(0) = [-\underline{q}_2, +\infty)$.

On the other hand, using Lemma 16.3.4, $(D_{x'}v(\bar{x},\bar{t}), q_1, v_t(\bar{x},\bar{t})) \in D^-_{\overline{Q_1}^\ell}\tilde{v}(\bar{x},\bar{t})$ [(3)] for any $q_1 \leq \underline{q}_1$.

[(3)] We recall that $D^-_{\overline{Q_1}^\ell}\tilde{v}(\bar{x},\bar{t})$ denotes the subdifferential *related to* $\overline{Q_1}^\ell$ of the function $\tilde{v}$ at $(\bar{x},\bar{t})$

In order to connect the sub-differentials of v and $\tilde{v}$, we use the following classical result whose proof is an exercise left to the reader.

Lemma 16.3.6 *Let $w_1, w_2 : A \subset \mathbb{R}^p \to \mathbb{R}$ be two l.s.c. functions such that $w_1(z_0) = w_2(z_0)$ for some $z_0 \in A$. Then*

$$D_A^- \min(w_1, w_2)(z_0) \subset D_A^- w_1(z_0) \cap D_A^- w_2(z_0)\,.$$

Applying the result with $A := \overline{\Omega}_1 \times [0, T_f]$, $z_0 := (\bar{x}, \bar{t})$, $w_1(x,t) = v(x,t)$ and $w_2(x,t) = v((x',0),t) + K|x_N|$, we deduce that $(D_{x'}v(\bar{x},\bar{t}), q_1, v_t(\bar{x},\bar{t})) \in D^-_{\overline{Q_1}^\ell} v(\bar{x},\bar{t})$. Of course, the same arguments can be used for $D^-_{\overline{Q_2}^\ell} v(\bar{x},\bar{t})$.

Hence, for any $q_1 \le \underline{q}_1$ and $q_2 \ge -\underline{q}_2$,

$$\max(-q_1 + q_2, \tilde{H}_1(q_1), \tilde{H}_2(q_2)) \ge 0\,.$$

(c) *Using the viscosity inequalities to get contradictions.*

Case 1 : Either $[\underline{p}_1, \overline{p}_1] \cap [\underline{q}_1, \overline{q}_1] \neq \emptyset$ or $[-\overline{p}_2, -\underline{p}_2] \cap [-\overline{q}_2, -\underline{q}_2] \neq \emptyset$: this means that there exists p such that we have

$$\text{either } \tilde{H}_1(p) + \eta \le 0 \le \tilde{H}_1(p)\,, \text{ or } \tilde{H}_2(p) + \eta \le 0 \le \tilde{H}_2(p)\,,$$

and in each case we reach a contradiction.

Case 2 : Otherwise, since 0 is a maximum point of $u - v$, we have necessarily $\overline{p}_1 \le \overline{q}_1$ and therefore $\underline{p}_1 \le \overline{p}_1 < \underline{q}_1 \le \overline{q}_1$. Considering the function $p \mapsto \tilde{H}_1(p)$ which is less that $-\eta$ in $[\underline{p}_1, \overline{p}_1]$ and positive in $[\underline{q}_1, \overline{q}_1]$, we see that there exists $\overline{p}_1 < r_1 < \underline{q}_1$ such that $\tilde{H}_1(r_1) = -\eta/2$.

Similarly, $-\overline{q}_2 \le -\underline{q}_2 < -\overline{p}_2 \le -\underline{p}_2$ and there exists $-\underline{q}_2 < r_2 < -\overline{p}_2$ such that $\tilde{H}_2(r_2) = -\eta/2$. Then, choosing $\delta > 0$ small enough and $p_1 = r_1 - \delta$, $p_2 = r_2 + \delta$, we have $p_1 \ge \overline{p}_1$ and $p_2 \le -\overline{p}_2$. Therefore the viscosity inequalities give

$$\min(-p_1 + p_2, \tilde{H}_1(p_1) + \eta, \tilde{H}_2(p_2) + \eta) \le 0\,,$$

but with the choice of δ, $\tilde{H}_1(p_1) + \eta > 0$, $\tilde{H}_2(p_2) + \eta > 0$, which implies $-p_1 + p_2 \le 0$, in other words $-r_1 + r_2 + 2\delta \le 0$.

On the other hand, choosing $q_1 = r_1 + \delta$ and $q_2 = r_2 - \delta$ and using $\tilde{H}_1(q_1) < 0$, $\tilde{H}_2(q_2) < 0$, we also get $-q_1 + q_2 \ge 0$ which leads to a contradiction because this implies $-r_1 + r_2 - 2\delta \ge 0$.

The conclusion is that $\max_{\mathcal{K}}(u-v)$, if positive, cannot be reached inside $\mathcal{K}$ but necessarily on $\partial_P\mathcal{K}$, which ends the proof[(4)].

Q.E.D.

16.3.3 Some remarks on the comparison proof and on some possible variants

In the above proof, the following points are crucial

(a) Because of the "normal controllability assumption", the tangential regularization of the subsolution does not cause any problem and provides us with a Lipschitz continuous subsolution to which the Lions-Souganidis Lemma (Lemma 16.3.1) fully applies: as a consequence, we can argue as if the problem was 1-dimensional and we obtain informations on the equation and the junction condition not only for the elements of the superdifferentials $D_1^+u(0) = [\overline{p}_1, +\infty)$ and $D_2^+u(0) = [-\infty, -\overline{p}_2)$, but also on $\tilde{H}_1(p_1)$ for $p_1 \in [\underline{p}_1, \overline{p}_1]$ and $\tilde{H}_2(p_2)$ for $p_2 \in [-\overline{p}_2, \underline{p}_2]$, *i.e.* a priori on larger intervals than the expected ones.

(b) The situation is completely different for the supersolution, mainly because the "normal controllability assumption" cannot play the same role. This first generates Assumptions (i) or (ii) in order to be able to do the tangential regularization of the supersolution but we end up with a function which is Lipschitz continuous and semi-concave in the tangential variables but which can still be discontinuous in x_N. In particular at $x_N = 0$. This is a difficulty to apply the Lions-Souganidis approach because we cannot prove that $\tilde{H}_i(v_{x_N}) \geq 0$ for any x_N since we cannot use Proposition 6.1.1-(iv) which provides some kind of "continuity" for the tangential derivatives but only if we have the right continuity property on v. In the above proof, we have chosen a strategy which consists in fully applying Lions-Souganidis Lemma to the supersolution but this requires the introduction of $\tilde{v}$. With this trick, whose aim is to superimpose the continuity of the supersolution at $x_N = 0$, the case of the supersolution can be treated as the subsolution one.

(c) Is this trick necessary/unavoidable? This question is important to extend the result of Theorem 16.3.5 to more general junction conditions and in particular to (FL) or (GJC) of flux-limited types for which such trick may not work. A first natural

[(4)]The authors wish to thank Peter Morfe for pointing out several unclear points in this proof which led us to several improvements, in particular the statements of Lemma 16.3.4 and 16.3.6.

reaction could be to accept the discontinuity in x_N of the regularized supersolution and to try to prove Lemma 16.3.1 in this framework. This may be feasible but a natural assumption to do that would be (at least) to have a regular supersolution, an assumption that we would like to avoid. Before going further, we remark that the proof of Theorem 16.3.5 consists in examining carefully the properties of the elements of the different super and subdifferentials of the sub and supersolution (and even more for the subsolution as we mention it above). At this point, it is worth mentionning that the maximum point property (16.5) yields

$$u(x_N) - u(0) \leq v(x_N) - v(0)\ , \tag{16.6}$$

and the first consequence of this inequality and of the Lipschitz continuity of u is that $D_1^- v(0)$ and $D_2^- v(0)$ are non-empty. Hence we have, for any $q_1 \in D_1^- v(0)$ and $q_2 \in D_2^- v(0)$

$$\max(-q_1 + q_2, \tilde{H}_1(q_1), \tilde{H}_2(q_2)) \geq 0\ . \tag{16.7}$$

Then we argue in the following way

- If $D_1^- v(0) = (-\infty, \underline{q}_1]$, then (16.7) holds but Lemma 16.3.4 applies and gives, in addition, $\tilde{H}_1(\underline{q}_1) \geq 0$. And, of course, we may use a similar argument if $D_2^- \tilde{v}(0) = [-\underline{q}_2, +\infty)$.

- Or $D_1^- v(0) = \mathbb{R}$ – this is the case, in particular, if v is not $\Omega_1 \times (0, T_f)$-regular at $(\bar{x}, \bar{t})$ and (16.7) holds for any $q_1 \in \mathbb{R}$ and $q_2 \in D_2^- v(0)$. In addition, $\tilde{H}_1(q_1) \geq 0$ if $q_1 \geq \underline{q}_1$ for some $\underline{q}_1$ by the coercivity of $\tilde{H}_1$. And, of course, a similar argument holds if $D_2^- v(0) = \mathbb{R}$.

In summary, by applying Lemma 16.3.4 instead of Lemma 16.3.1, not only we have more informations than we could have obtained by applying the Lions-Souganidis Lemma but we can also avoid introducing $\tilde{v}$.

(d) Last very important point: in Equation 16.6, for $x_N > 0$, we can divide by x_N and take the lim inf: by using Lemma 16.3.1 for u and Lemma 16.3.4 for v, we obtain the inequality $\underline{p}_1 \leq \underline{q}_1$ where the information on $\underline{p}_1$ becomes crucial (we again insist on the fact that $\underline{p}_1 \notin D_1^+ u(0)$).

With all these ingredients, we will be able in the next section to extend the result of Theorem 16.3.5 to a large class of (GJC) without assuming any regularity on the supersolution.

16.3.4 Comparison results for more general junction conditions of (FL) or (GJC) type

Theorem 16.3.5 can be generalized for (FL) and (GJC) conditions under some hypotheses:

Theorem 16.3.7 *Assume that H_1 and H_2 satisfy* **(GA-Gen)** *and* **(TC-HJ)**. *Then* **(GCR)** *holds*

- *in the case of* (FL)*: for any bounded REGULAR subsolution u and supersolution v if G satisfies* **(GA-G-FL)**.
- *in the case of* (GJC) *of "Kirchhoff type", i.e. if G satisfies* **(GA-G-GKT)***: for any bounded subsolution u and supersolution v.*
- *in the case of* (GJC) *of "Flux-limited type", i.e. if G satisfies* **(GA-G-FLT)***: for any bounded REGULAR subsolution u and supersolution v.*

Moreover, if v is locally Lipschitz continuous in x_N, uniformly in x', t, then these three results hold true without assuming **(TC-HJ)** *for H_1, H_2 and, in the last case by assuming only that G satisfies* **(GA-ContG)** *with $\varepsilon_0 = 1$ and* (14.2) *holds with $\alpha > 0$, $\beta = 0$.*

Several remarks can be made on Theorem 16.3.7. First, for (FL) type conditions, we face the difficulty that sub and supersolutions may not be regular and we have to add these properties as an assumption.

But the main one follows along the lines of the remark we made when introducing the assumptions **(GA-G-FL)**, **(GA-G-GKT)**, **(GA-G-FLT)**. Using the Lions-Souganidis approach as in the proof of Theorem 16.3.5 requires a "tangential regularization" both for the sub and the supersolution in the spirit of Sections 3.4.3 and 3.4.4 in order to reduce to a 1-dimensional proof. While this regularization does not cause much problem in the first case of (FL)—except that we have to impose **(TC-HJ)** for H_1 and H_2—, it requires a particular treatment when G is of "Kirchhoff type", and a particular form of G when it is of "flux-limited type". For this reason, we need **(GA-G-FLT)** which is roughly speaking the analogue of **(TC-s)**.

Proof — We are not going to give the full proof of Theorem 16.3.7 since most of the arguments are those of the proof of Theorem 16.3.5, using in an essential way the Lions-Souganidis approach. We just indicate the additional arguments which are needed.

We first comment (**LOC1**),(**LOC2**): in the case of (FL) or when (GJC) is of "Flux-limited type", the checking can be made exactly as in Section 3.2.2, the u_t-term playing the main role. In the case when (GJC) is of "Kirchhoff type", one has to add a $\alpha\varphi(x_N)$ (or $\delta\varphi(x_N)$) term where $\varphi \in \mathrm{PC}^1(\mathbb{R})$ is a bounded function which behaves like $L|x_N|$ in a neighborhood of $x_N = 0$. With this additional argument, (**LOC1**),(**LOC2**) hold in the three cases and we can reduce the proof to a **(LCR)**.

The next important point concerns the regularization of both the sub and supersolution which we examine case by case.

(a) (FL) case — the regularity of the sub and supersolution is a key assumption since we have seen that (JVSub) and (JVSuper) are not necessarely regular but the rest of the proof follows readily the arguments of Sections 3.4.3 and 3.4.4 because **(TC-s)** is ensured by **(TC-HJ)** or **(GA-G-FL)**, even if, for the subsolution, the equation does not satisfy **(NC)** on $\mathcal{H}$.

(b) "Flux-limited type" (GJC)— The same comments are true here and explain the restrictive assumptions we have to impose in this case.

(c) "Kirchhoff type" (GJC)— This is the most complicated case where we need the

Lemma 16.3.8 *Under the assumptions of Theorem 16.3.7 in the case of* (GJC) *of "Kirchhoff type", if u and v are respectively a sub and supersolution of* (HJ-Gen)-(GJC)*, then, for K_2 large enough and K_1 large compared to K_2, the functions*

$$u^\varepsilon(x', x_N, t) := \max_{(y',s)} \left\{ u(y', x_N, s) - \exp(K_1 t)\exp(-K_2|x_N|)\Big(\frac{|x'-y'|^2}{\varepsilon^2} + \frac{|t-s|^2}{\varepsilon^2}\Big)\right\},$$

$$v^\varepsilon(x', x_N, t) := \min_{(y',s)} \left\{ v(y', x_N, s) + \exp(K_1 t)\exp(-K_2|x_N|)\Big(\frac{|x'-y'|^2}{\varepsilon^2} + \frac{|t-s|^2}{\varepsilon^2}\Big)\right\},$$

are respectively approximate (JVSub) *and* (JVSuper) *of* (HJ-Gen)-(GJC).

Proof of Lemma 16.3.8 — The proof follows essentially the proofs of the regularization results of Sections 3.4.3 and 3.4.4 with the following additional arguments: first, we notice that we can use a PC^1-term $\exp(-K_2|x_N|)$. If $x_N > 0$ or $x_N < 0$, this term produces a "bad term" in the x_N-derivative which has to be controlled by the t-derivative coming from the $\exp(K_1 t)$-term.

At $x_N = 0$, if the G-inequality holds, the u_t-term cannot control the "bad terms" anymore but the $\exp(K_1 t)$-term has the "good sign" and, since $\beta > 0$, the derivatives of $\exp(-K_2|x_N|)$ allow to control all the error terms.

Q.E.D.

Once this regularization is done, we can in each case transform the approximate subsolution into a strict subsolution using either a $-\eta t$ term or a $\eta\varphi(x_N)$ term where φ is the function we already used at the beginning of the proof.

From this point, we can follow readily the proof of Theorem 16.3.5 to conclude.

Q.E.D.

16.3.5 Extension to second-order problems (II)

The same approach allows to deal with second-order problems, with similar structure assumptions on the Hamiltonians.

Theorem 16.3.9 (Comparison in the second-order case – LS-version) *Under the assumptions of Theorem 16.3.7, the comparison result remains valid for Lipschitz continuous sub and supersolutions of second-order equations of the form*

$$u_t - \mathrm{Tr}(a_i(x)D^2u) + H_i(x,t,u,Du) = 0 \quad \text{in } \Omega_i \times (0,T_f)\ , \tag{16.8}$$

provided that, for $i = 1,2$, $a_i = \sigma_i\sigma_i^T$ for a bounded, Lipschitz continuous function σ_i, depending only on x_N in a neighborhood of $\mathcal{H}$.

Of course, the most restrictive assumption in Theorem 16.3.9 is the Lipschitz continuity of the sub and supersolutions to be compared. But if we examine the proof in the first-order case, we remark that the tangential regularization provides a Lipschitz continuous subsolution because of the normal controllability and the regularized supersolution "behaves" like a Lipschitz continuous function because of the maximum point property in the proof of the **(LCR)**. All these arguments, and in particular the first one, fail here because of the second-order term and we need to replace them by the *ad hoc* assumption.

The proof follows the arguments of the proof of Theorem 16.3.7, except that we need the following extension of the Lions-Souganidis Lemma (Lemma 16.3.1).

Lemma 16.3.10 *We assume that for some $r > 0$, $u : B(0,r)\times[0,r] \subset \mathbb{R}^{N-1}\times\mathbb{R} \to \mathbb{R}$ is an u.s.c. subsolution [resp. l.s.c. supersolution] of*

$$-\mathrm{Tr}(a(x_N)D^2w) + \tilde{H}(w_{x_N}) = 0 \quad \text{in } B(0,r)\times(0,r)\ ,$$

where $\tilde{H}$ is a continuous function and $a(x_N) = \sigma(x_N)\sigma^T(x_N)$ for some bounded Lipschitz continuous function σ. If moreover

(*i*) $u(x', x_N)$ *is Lipschitz continuous and semi-convex* [*resp. semi-concave*] *in* x', *uniformly for* $x_N \in [0, r]$,

(*ii*) *there exists a constant* $k > 0$, *such that* $u(0, x_N) - u(0, 0) \leq k x_N$ [*resp.* $u(0, x_N) - u(0, 0) \geq k x_N$] *for* $0 \leq x_N \leq r$,

(*iii*) *the function* u *is differentiable w.r.t.* x' *at* $(0, 0)$,

then $\tilde{H}(p) \leq 0$ [*resp.* $\tilde{H}(p) \geq 0$] *for all* $p \in [\underline{p}, \overline{p}]$ *where*

$$\underline{p} := \liminf_{x_N \to 0} \left[\frac{u(0, x_N) - u(0, 0)}{x_N} \right] \leq \overline{p} := \limsup_{x_N \to 0} \left[\frac{u(0, x_N) - u(0, 0)}{x_N} \right] .$$

We first point out that the assumption on u implies that it is continuous at $(0, 0)$ but may still have discontinuities outside $(0, 0)$.

The additional difficulty in this lemma (compared to Lemma 16.3.1) is the $x' \in \mathbb{R}^{N-1}$-dependence in the $D^2 u$-term which cannot be dropped by the semi-convex (or semi-concave) assumption on u.

Proof — We only give the proof in the subsolution case, the supersolution one being analogous.

Replacing $u(x', x_N)$ by the subsolution $u(x', x_N) - u(0, 0) - D_{x'} u(0, 0) \cdot x'$, we can assume *w.l.o.g.* that $u(0, 0) = 0$ and $D_{x'} u(0, 0) = 0$. Also, as in the proof of Lemma 16.3.1, we first assume that $\underline{p} < \overline{p}$.

(a) The first step consists in giving some estimates on u for (x', x_N) close to $(0, 0)$. By the semi-convexity assumption, if u is differentiable in x' at (x', x_N) and if (x', x_N) is close to $(0, 0)$, then $|D_{x'} u(x', x_N)|$ is also close to $|D_{x'} u(0, 0)| = 0$ and we deduce from this property that

$$u(x', x_N) - u(0, x_N) = |x'| \varepsilon(x', x_N) ,$$

where $\varepsilon(x', x_N) \to 0$ when $(x', x_N) \to (0, 0)$.

(b) For $r' \leq r$, we consider the domain $D_{r'} := \{(x', x_N) : \ |x'| \leq x_N, \ 0 \leq x_N \leq r'\}$ and the function

$$\psi(x', x_N) := u(x', x_N) - \frac{|x'|^4}{x_N^4} - p \cdot x_N , \quad \text{where} \quad \underline{p} < p < \overline{p} .$$

This function ψ is defined in $D_{r'} \setminus \{(0, 0)\}$ but we can extend it at $(0, 0)$ by setting $\psi(0, 0) = 0$, which yields an u.s.c. function on $D_{r'}$.

(c) We consider a sequence $(s_k)_k$ such that $s_k > 0$ for all k, $s_k \to 0$ and

$$\frac{u(0,s_k)-u(0,0)}{s_k} = \frac{u(0,s_k)}{s_k} \to \underline{p}\ ,$$

and we choose $r' = s_k$. We claim that the maximum of ψ on D_{s_k} is achieved in the interior of the domain.

Indeed, if $|x'| = x_N$, $\psi(x',x_N) = u(x',x_N) - 1 - px_N \le -1 + o(1)$, while for $x_N = r' = s_k$, by the estimate in **(a)** above,

$$\begin{aligned}\psi(x',s_k) \le& |x'|\varepsilon(x',s_k) + u(0,s_k) - ps_k\\ \le& (\underline{p}-p)s_k + o(s_k) < 0\ .\end{aligned}$$

Finally $\psi(0,0) = 0$ and we conclude that $\psi \le 0$ on ∂D_{s_k}.

On the other hand, there exists a sequence $(r_l)_l$ such that $r_l > 0$ for all l, $r_l \to 0$ and

$$\frac{u(0,r_k)-u(0,0)}{r_k} = \frac{u(0,r_k)}{r_k} \to \overline{p}\ .$$

For l large enough, we have $r_l < s_k$ and

$$\psi(0,r_k) \le u(0,r_k) - pr_k = (\overline{p}-p)r_k + o(r_k) > 0\ .$$

Therefore the maximum of ψ is achived in the interior of D_{s_k}.

(d) We can now apply the viscosity subsolution inequality at the maximum point (x',x_N) of ψ. To do so, we first remark that u is differentiable *w.r.t.* x' at this maximum point by the semi-convexity property and

$$D_{x'}u(x',x_N) = \frac{4|x'|^2x'}{x_N^4}\ ,$$

therefore $4|x'|^3/x_N^4 \to 0$ as $s_k \to 0$. On the other hand, the term corresponding to the second derivative, taking into account that $a(x_N) = O(x_N^2)$ is estimated by

$$-\mathrm{Tr}(a(x_N)D^2\psi) = O(x_N^2)\cdot O\left(\frac{|x'|^2}{x_N^4} + \frac{|x'|^3}{x_N^5} + \frac{|x'|^4}{x_N^6}\right) \to 0 \quad \text{as } s_k \to 0\ .$$

We deduce that

$$\tilde{H}\Big(p - \frac{4|x'|^4}{x_N^5}\Big) \le o(1)\ ,$$

but since $4|x'|^4/x_N^5 \to 0$ as $s_k \to 0$, the conclusion follows: $\tilde{H}(p) \le 0$.

(e) The case when $p = \underline{p}$ or $\overline{p}$ follows from the continuity of $\tilde{H}$, and the case $\underline{p} = \overline{p}$ is treated exactly as in the first-order case, so the proof is complete.

Q.E.D.

Using Lemma 16.3.10, the proof of Theorem 16.3.9 follows along the lines of Theorem 16.3.7, the Lipschitz continuity of u and v ensuring that the regularization process allows to use the lemma.

16.4 Convergence of the vanishing viscosity approximation (II): via (JVS)

In this section, we use the Lions-Souganidis comparison result to show that the vanishing viscosity approximation converges to the unique solution of the Kirchhoff problem; this gives another version of Theorem 15.5.1 in a non-convex setting.

Theorem 16.4.1 *Assume that, for any $\varepsilon > 0$, u^ε is a continuous viscosity solution of*

$$u^\varepsilon_t - \varepsilon \Delta u^\varepsilon + H(x,t,Du^\varepsilon) = 0 \quad \text{in} \quad \mathbb{R}^N \times (0,T_f)\,, \tag{16.9}$$

$$u^\varepsilon(x,0) = u_0(x) \quad \text{in} \quad \mathbb{R}^N\,, \tag{16.10}$$

where $H = H_1$ in Ω_1 and H_2 in Ω_2, and u_0 is bounded continuous function in $\mathbb{R}^N$. Under the assumptions of Theorem 16.3.5 and if the sequence $(u^\varepsilon)_\varepsilon$ is uniformly bounded in $\mathbb{R}^N \times (0,T_f)$, C^1 in x_N in a neighborhood of $\mathcal{H}$, then, as $\varepsilon \to 0$, the sequence $(u^\varepsilon)_\varepsilon$ converges locally uniformly to the unique solution of the Kirchhoff problem in $\mathbb{R}^N \times (0,T_f)$.

This second result on the convergence of the vanishing viscosity approximation may appear as being more general than Theorem 15.5.1 since it covers the case of non-convex Hamiltonians. But we point out that Theorem 16.3.5 requires either (3.19) or **(TC-s)** which limit its range of applications. This also suggests connections between solutions with (FL) and (KC) junctions conditions: we study these connections in the next section.

The proof is almost standard since we use the half-relaxed limits method to pass to the limit, coupled with a strong comparison result to conclude, here Theorem 16.3.5. So, the only difficulty consists in proving lemma 16.4.2 below which, despite its very classical formulation, is not standard at all: the formulation involves test-function which are not smooth across $\mathcal{H}$. Therefore, this is not an usual stability result for viscosity solutions.

Lemma 16.4.2 *The half-relaxed limits* $\overline{u} = \limsup^* u^\varepsilon$ *and* $\underline{u} = \liminf_* u^\varepsilon$ *are respectively sub and supersolution of the Kirchhoff problem.*

Proof — We prove the result for $\overline{u}$, the one for $\underline{u}$ being analogous. Let $\phi \in \mathrm{PC}^1(\mathbb{R}^N \times [0, T_f])$ be a test-function and let $(\bar{x}, \bar{t})$ be a strict local maximum point of $\overline{u} - \phi$. The only difficulty is when $\bar{x} \in \mathcal{H}$ and therefore we concentrate on this case.

By standard arguments, $u^\varepsilon - \phi$ has a local maximum point at $(x_\varepsilon, t_\varepsilon)$ and $(x_\varepsilon, t_\varepsilon) \to (\bar{x}, \bar{t})$ as $\varepsilon \to 0$.

a) If there exists a subsequence $(x_{\varepsilon'}, t_{\varepsilon'})$ with $x_{\varepsilon'} \notin \mathcal{H}$, the classical arguments can be applied and passing to the limit (along another subsequence) in the inequality

$$\phi_t(x_\varepsilon, t_\varepsilon) - \varepsilon \Delta \phi(x_\varepsilon, t_\varepsilon) + H(x_\varepsilon, t_\varepsilon, u^\varepsilon(x_\varepsilon, t_\varepsilon), D\phi(x_\varepsilon, t_\varepsilon)) \leq 0$$

yields the result.

(b) The main difficulty is when $x_\varepsilon \in \mathcal{H}$ for all ε small enough since ϕ is not smooth at $(x_\varepsilon, t_\varepsilon)$. Here we use

$$a := \frac{\partial \phi}{\partial x_N}((\bar{x}', 0+), \bar{t}) = \lim_{x_N \to 0, x_N > 0} \frac{\partial \phi}{\partial x_N}((\bar{x}', x_N), \bar{t})$$

and

$$b := \frac{\partial \phi}{\partial x_N}((\bar{x}', 0-), \bar{t}) = \lim_{x_N \to 0, x_N < 0} \frac{\partial \phi}{\partial x_N}((\bar{x}', x_N), \bar{t}) \ .$$

If $-a + b \leq 0$, the Kirchoff subsolution condition is satisfied and the result holds.

(c) The last possibility is that $-a + b > 0$, and since u_ε is smooth, the maximum point property at $(x_\varepsilon, t_\varepsilon)$ implies that

$$\frac{\partial u_\varepsilon}{\partial x_N}((x'_\varepsilon, 0), \bar{t}) \leq \frac{\partial \phi}{\partial x_N}((x'_\varepsilon, 0+), \bar{t}) \ ,$$

and

$$\frac{\partial u_\varepsilon}{\partial x_N}((x'_\varepsilon, 0), \bar{t}) \geq \frac{\partial \phi}{\partial x_N}((x'_\varepsilon, 0-), \bar{t}) \ .$$

Therefore

$$-\frac{\partial \phi}{\partial x_N}((x'_\varepsilon, 0+), \bar{t}) + \frac{\partial \phi}{\partial x_N}((x'_\varepsilon, 0-), \bar{t}) \leq 0 \ ,$$

and using that both partial derivatives are continuous in x', we reach a contradiction for ε small enough (remember that $-a + b > 0$ here).

Q.E.D.

Chapter 17

From One Notion of Solution to the Others

The aim of this chapter is to connect the three notions of solutions we have at hand: Ishii solutions, flux-limited solutions and junction viscosity solutions.

We first show that Ishii solutions for (7.1) can be seen as (FLS) associated with an H_T or H_T^{reg} flux-limiter in the case of quasi-convex Hamiltonians. We point out that the definitions of H_T and H_T^{reg} are extended to the case of quasi-convex Hamiltonians by (10.5) and (10.6), and we refer the reader to Section 6.2 for useful results on them. Through this (FLS) interpretation, we complement the results of Part II both by taking into account more general Hamiltonians but also by considering a notion of solution which allows a pure pde approach of the problem.

Then we compare (FLS) and (JVS) in the context of flux-limited conditions: here the formulation of the notion of solutions on $\mathcal{H}$ is the key point.

Finally, we prove that junction viscosity solutions associated with Kirchhoff conditions are flux-limited solutions for a specific flux-limiter which we identify explicitly. We do the analysis first for the most classical Kirchhoff condition and then for generalized ones.

17.1 Ishii solutions and (FLS)

The main result of this section is the

Proposition 17.1.1 *Assume that* $(\mathbf{H_{QC}})$ *holds. Then*

(*i*) *An u.s.c. function* $u : \mathbb{R}^N \times (0,T_f) \to \mathbb{R}$ *is an Ishii subsolution of* (7.1) *if and only if it is a* (FLSub) *of* (HJ-Gen)-(FL) *with the flux-limiter* $H_T^{\rm reg}$.

(*ii*) *A l.s.c. function* $v : \mathbb{R}^N \times (0,T_f) \to \mathbb{R}$ *is an Ishii supersolution of* (7.1) *if and only if it is a* (FLSuper) *of* (HJ-Gen)-(FL) *with the flux-limiter* H_T.

An immediate corollary of this result is the

Corollary 17.1.2 *Under the assumptions of Theorem 15.3.2 with* $G = H_T$ *or* $H_T^{\rm reg}$
(i) If $\mathbf{U}^+$ *is the unique* (FLS) *of* (HJ-Gen)-(FL) *with the flux-limiter* $H_T^{\rm reg}$, *it is the maximal Ishii subsolution of* (7.1).
(ii) If $\mathbf{U}^-$ *is the* (FLS) *of* (HJ-Gen)-(FL) *with the flux-limiter* H_T, *it is a minimal Ishii supersolution of* (7.1).

Proof of Proposition 17.1.1 — Of course, only the viscosity inequalities on $\mathcal{H}\times(0,T_f)$ are different and therefore we concentrate on them.

(a) *Generalities* — Throughout the proof we consider elements of the superdifferential of u or the subdifferential of v at $(x,t) \in \mathcal{H} \times (0,T_f)$ of the form $(p_x + \lambda e_N, p_t)$, and inequalities for H_1, H_2 and H_T or $H_T^{\rm reg}$. In all these inequalities, only the dependence in λ is going to play a role and, in order to simplify the notations, we set, for $i = 1, 2$

$$F_i(\lambda) := p_t + H_i(x,t,r,p_x + \lambda e_N)\ ,$$

where x, t, p_x and $r = u(x,t)$ or $v(x,t)$ are assumed to be fixed. We define analogously $F_i^{\pm}$. We also use the notations

$$F_T = p_t + H_T(x,t,r,p_x)\ ,\ F_T^{\rm reg} = p_t + H_T^{\rm reg}(x,t,r,p_x)\ ,$$

for, again, x, t, $r = u(x,t)$ or $v(x,t)$ and p_x being fixed. We also recall that, in the quasi-convex case,

$$H_T(x,t,r,p_x) = \min_{\lambda\in\mathbb{R}} \max(H_1(x,t,r,p_x+\lambda e_N), H_2(x,t,r,p_x+\lambda e_N))\ ,$$

$$H_T^{\rm reg}(x,t,r,p_x) = \min_{\lambda\in\mathbb{R}} \max(H_1^-(x,t,r,p_x+\lambda e_N), H_2^+(x,t,r,p_x+\lambda e_N))\ ,$$

i.e. $F_T = \min_\lambda \max(F_1(\lambda), F_2(\lambda))$ and similarly $F_T^{\rm reg} = \min_\lambda \max(F_1^-(\lambda), F_2^+(\lambda))$. From these representations we deduce easily from Section 6.2.2 the existence of $(\underline{\lambda}, \overline{\lambda}) \in \mathbb{R}^2$ such that $F_T^{\rm reg} = F_1^-(\overline{\lambda}) = F_2^+(\overline{\lambda})$ and

$$F_T = \begin{cases} \text{either } F_1(\underline{\lambda}) = F_2(\underline{\lambda}) \\ \text{or } F_1(\underline{\lambda}) = \min F_1(\lambda) \text{ if } F_1 \geq F_2 \\ \text{or } F_2(\underline{\lambda}) = \min F_2(\lambda) \text{ if } F_2 \geq F_1\,. \end{cases} \tag{17.1}$$

(b) *Subsolution case* — Before providing the proof, we point out that ($\mathbf{H_{QC}}$) implies that ($\mathbf{NC}_w$) holds for all the Hamiltonians involved in the Ishii and flux-limited formulations, *cf.* Remark 3.4.3. Hence all the subsolutions we are going to consider are regular on $\mathcal{H}$.

If u is an Ishii subsolution of (7.1) and if we pick $(p_x + \lambda_1 e_N, p_t) \in D^+_{\overline{Q_1}} u(x,t)$, $(p_x + \lambda_2 e_N, p_t) \in D^+_{\overline{Q_2}} u(x,t)$, we have to show that

$$\max(F_1^+(\lambda_1), F_2^-(\lambda_2), F_T^{\mathrm{reg}}) \leq 0 \, .$$

Notice that Ishii subsolutions are Lipschitz continuous. Therefore, not only are they regular on $\mathcal{H}$ but the normal components of their superdifferentials are bounded from below in $\overline{Q_1}$ and from above in $\overline{Q_2}$. Proposition 3.5.4 can then be applied, which gives existence of $\mu_1 \leq \lambda_1$ and $\mu_2 \geq \lambda_2$ such that $(p_x + \mu_1 e_N, p_t) \in D^+_{\overline{Q_1}} u(x,t)$, $(p_x + \mu_2 e_N, p_t) \in D^+_{\overline{Q_2}} u(x,t)$ and

$$F_1(\mu_1) \leq 0 \, , \; F_2(\mu_2) \leq 0 \, .$$

Since these inequalities imply $F_1^+(\mu_1) \leq 0$ and $F_2^-(\mu_2) \leq 0$, we deduce from the monotonicity of F_1^+ and F_2^- that $F_1^+(\lambda_1) \leq 0$ and $F_2^-(\lambda_2) \leq 0$.

It remains to prove that $F_T^{\mathrm{reg}} \leq 0$. To do so, we argue by contradiction: if $F_T^{\mathrm{reg}} > 0$, then $F_1^-(\overline{\lambda}) = F_2^+(\overline{\lambda}) > 0$. Since $F_1^-(\mu_1) \leq 0$ and $F_2^+(\mu_2) \leq 0$, we deduce from the monotonicity of F_1^- and F_2^+ that $\mu_1 < \overline{\lambda} < \mu_2$. But this double inequality implies that

$$(p_x + \overline{\lambda} e_N, p_t) \in D^+_{\mathbb{R}^N} u(x,t)$$

and therefore $\min(F_1(\overline{\lambda}), F_2(\overline{\lambda})) \leq 0$ because u is an Ishii subsolution of (7.1). This is a contradiction with the fact that $F_1^-(\overline{\lambda}) = F_2^+(\overline{\lambda}) > 0$, and we deduce that the (FLSub) condition hold son $\mathcal{H}$.

Conversely, if u is (FLSub) of (HJ-Gen)-(FL) associated with the flux-limiter H_T^{reg} and if $(p_x + \lambda e_N, p_t) \in D^+_{\mathbb{R}^N} u(x,t)$, we want to show that

$$\min(F_1(\lambda), F_2(\lambda)) \leq 0 \, .$$

By the (FLSub) property, we know that

$$\max(F_1^+(\lambda), F_2^-(\lambda), F_T^{\mathrm{reg}}) \leq 0 \, ,$$

and therefore it remains to prove that $\min(F_1^-(\lambda), F_2^+(\lambda)) \leq 0$. We argue by contradiction assuming that this min is strictly positive. Using again the monotonicity of F_1^-, F_2^+ and the fact that $F_T^{\mathrm{reg}} \leq 0$, we deduce that $\lambda > \overline{\lambda}$ and $\overline{\lambda} > \lambda$ which is clearly

a contradiction, proving that u is an Ishii subsolution. The proof for the subsolution case is then complete.

(c) *Supersolution case* — Contrarily to the case of subsolutions, the supersolutions are not necessarily regular on $\mathcal{H}$ but as we see below, this does not pose any problem.

If v is an Ishii supersolution of (7.1) and if $(p_x + \lambda_1 e_N, p_t) \in D^-_{\overline{Q_1}} v(x,t)$, $(p_x + \lambda_2 e_N, p_t) \in D^-_{\overline{Q_2}} v(x,t)$, we want to show that

$$\max(F_1^+(\lambda_1), F_2^-(\lambda_2), F_T) \geq 0 \, .$$

In order to apply Proposition 3.5.4, we consider several cases

1. If the set $J_1 = \{\lambda \in \mathbb{R} : (p_x + \lambda Dd(x), p_t) \in D^-_{\overline{Q_1}^l} v(x,t)\}$ is bounded from above, there exists $\mu_1 \geq \lambda_1$ such that $(p_x + \mu_1 e_N, p_t) \in D^-_{\overline{Q_1}} v(x,t)$ and $F_1(\mu_1) \geq 0$.

2. If the set $J_2 = \{\lambda \in \mathbb{R} : (p_x + \lambda Dd(x), p_t) \in D^-_{\overline{Q_2}^l} v(x,t)\}$ is bounded from below, there exists $\mu_2 \leq \lambda_2$ such that $(p_x + \mu_2 e_N, p_t) \in D^-_{\overline{Q_2}} v(x,t)$ and $F_2(\mu_2) \geq 0$.

3. Otherwise, for any $\mu_i \in \mathbb{R}$, $(p_x + \mu_i e_N, p_t) \in D^-_{\overline{\Omega_i}} v(x,t)$ and therefore, by the coercivity of F_i, $F_i(\mu_i) \geq 0$ for some μ_i such that $\mu_i \geq \lambda_1$ if $i = 1$ or $\mu_i \leq \lambda_2$ if $i = 2$.

In any case, if either $F_1^+(\mu_1) \geq 0$ or $F_2^-(\mu_2) \geq 0$, we are done by using the monotonicity of F_1^+ and F_2^-. Therefore we can assume without loss of generality that $F_1^-(\mu_1), F_2^+(\mu_2) \geq 0$ and we have to prove that $F_T \geq 0$.

We argue by contradiction assuming that $F_T < 0$ and using (17.1), we see that there are three options.

(*i*) If there exists $\underline{\lambda}$ such that $F_T = F_1(\underline{\lambda}) = F_2(\underline{\lambda}) < 0$, then by the monotonicity of F_1^- and F_2^+, it follows that $\mu_1 > \underline{\lambda} > \mu_2$. Hence $(p_x + \underline{\lambda} e_N, p_t)$ is both in $D^-_{\overline{Q_1}^l} v(x,t)$ and in $D^-_{\overline{Q_2}^l} v(x,t)$, and therefore in $D^-_{\mathbb{R}^N} v(x,t)$ and by the Ishii supersolution property, we get $\max(F_1(\underline{\lambda}), F_2(\underline{\lambda})) \geq 0$, a contradiction.

(*ii*) If $F_T = F_1(\underline{\lambda}) = \min F_1(\lambda) < 0$ with $F_1 \geq F_2$, we still have $F_2(\underline{\lambda}) < 0$ and we conclude in the same way.

(*iii*) Of course this is also the case when $F_T = F_2(\underline{\lambda}) = \min F_2(\lambda) < 0$ with $F_2 \geq F_1$.

In conclusion, $F_T \geq 0$ and v is (FLSuper) associated to the flux-limiter H_T.

Conversely, if v is (FLSuper) of (HJ-Gen)-(FL) with the flux-limiter H_T and if $(p_x + \lambda e_N, p_t) \in D^-_{\mathbb{R}^N} v(x,t)$, we have to show that

$$\max(F_1(\lambda), F_2(\lambda)) \geq 0 \ .$$

By the (FLSuper) property, we already know that

$$\max(F_1^+(\lambda), F_2^-(\lambda), F_T) \geq 0 \ ,$$

which implies that: (i) either $F_1(\lambda) \geq F_1^+(\lambda) \geq 0$ or $F_2(\lambda) \geq F_2^-(\lambda) \geq 0$, in which case we are done; (ii) or $F_T \geq 0$, but using that $\max(F_1^-(\lambda), F_2^+(\lambda)) \geq F_T \geq 0$ implies that $\max(F_1(\lambda), F_2(\lambda)) \geq 0$ and we also get the conclusion. The proof for the supersolution case is then complete.

Q.E.D.

17.2 (FLS) and (JVS) for flux-limited conditions

We now prove the equivalence of both notions of solutions in the case of Flux-Limited conditions. We point out that, since (JVSub) are not necessarily regular, we have to make this non-trivial assumption. However, as we saw in Proposition 16.1.3, this assumption is automatically satisfied in the case of Kirchhoff-type conditions.

Proposition 17.2.1 *Assume* **(GA-QC)** *and that G satisfies* $(\mathbf{H_{BA-HJ}})$. *Then*

(i) *an u.s.c., locally bounded function $u : \mathbb{R}^N \times (0,T_f) \to \mathbb{R}$ is a flux-limited subsolution of* (HJ-Gen)-(FL) *with flux-limiter G if and only if it is a regular G-*(JVSub).

(ii) *a l.s.c., locally bounded function $v : \mathbb{R}^N \times (0,T_f) \to \mathbb{R}$ is a flux-limited supersolution of* (HJ-Gen)-(FL) *if it is a G-*(JVSuper).

Proof — In all this proof, ψ is always a generic test-function in $\mathrm{PC}^1(\mathbb{R}^N \times [0,T_f])$ and the maximum or minimum of $u - \psi$ in $\mathbb{R}^N \times (0,T_f)$ is always denoted by (x,t), which we assume to be located on $\mathcal{H} \times (0,T_f)$.

(a) *Subsolutions* – We just sketch the proof here since this case is easy. If u is a flux-limited subsolution, it clearly satisfies (16.1). Indeed, if $u - \psi$ has a maximum at (x,t), then $\psi_t + G(x,t,v,D_{\mathcal{H}}\psi) \leq 0$ because of the "max" in the definition of flux-limited

subsolutions. To prove the converse, we use in an essential way the regularity of the (JVSub): using Proposition 3.5.1 with $L = H_1^+$ or H_2^-, we see that $(x,t) \in \mathcal{H} \times (0,T_f)$, local maximum point of $u - \psi$ in $\mathbb{R}^N \times (0,T_f)$ then

$$\psi_t + H_1^+(x,t,u,D\psi_1) \le 0 \quad , \quad \psi_t + H_2^-(x,t,u,D\psi_2) \le 0 \, .$$

It remains to prove that $\psi_t + G(x,t,v,D_{\mathcal{H}}\psi) \le 0$, which is done as follows: for any $C > 0$, $u - (\psi + C|x_N|)$ has also a maximum at $(x,t) \in \mathcal{H} \times (0,T_f)$ but taking $C > 0$ large enough in (16.1) yields that the min cannot be reached by the H_1/H_2-terms since both Hamiltonians are coercive. Thus necessarily, the non-positive min is given by the junction condition and the result follows.

(b) *Supersolutions* – This case is a little bit more delicate. Of course, a flux-limited supersolution v satisfies (16.2) since $H_1 \ge H_1^+$ and $H_2 \ge H_2^-$. The main point is then to prove that supersolutions of (16.2) are flux-limited supersolutions.

If $(x,t) \in \mathcal{H} \times (0,T_f)$ is a local maximum point of $u - \psi$, (16.2) holds and we wish to show that

$$\max\Big(\psi_t + G(x,t,v,D_{\mathcal{H}}\psi), \psi_t + H_1^+(x,t,v,D\psi_1), \psi_t + H_2^-(x,t,v,D\psi_2)\Big) \ge 0 \, .$$

Assuming this is not the case, then necessarily all three quantities above are strictly negative and (16.2) implies

$$\max\Big(\psi_t + H_1^-(x,t,v,D\psi_1), \psi_t + H_2^+(x,t,v,D\psi_2)\Big) \ge 0 \, .$$

Let us assume for example that $\psi_t + H_1^-(x,t,v,D\psi_1) \ge 0$, the other case being treated similarly.

Refering the reader to Section 10.2 and Remark 10.3.3 where the properties of H_1^+, H_2^- are described we see that, if $D\psi_i = p_T^i + p_N^i e_N$ for $i = 1,2$, where $p_T^i \in \mathcal{H}$ and $p_N^i \in \mathbb{R}$, then these inequalities imply for instance

$$-\psi_t(x,t) > H_1^+(x,t,v,p_T^1 + p_N^1 e_N) \, , \text{ therefore } -\psi_t(x,t) > \min_s (H_1(x,t,v,p_T^1 + s e_N)) \, .$$

Denoting by $s_* \in \mathbb{R}$ a real such that $-\psi_t(x,t) = H_1^-(x,t,v,p_T^1 + s e_N)$, we deduce that $s_* > m_1^+(x,t,v,p_T^1)$, the largest point of where $s \mapsto H_1(x,t,v,p_T^1 + s e_N)$ reaches its minimum. On the other hand, the inequality $\psi_t + H_1^-(x,t,v,D\psi_1) \ge 0$ implies that $p_N^1 \ge s_*$, so that finally $p_N^1 > m_1^+(x,t,v,p_T^1)$.

There are now two cases. In the first case $\psi_t + H_2^+(x,t,v,D\psi_2) \ge 0$ and similarly as above, $p_N^2 < m_2^-(x,t,v,p_T^1)$, the least minimum point for H_2. Here, we set

$$\tilde{\psi}(x,t) := \begin{cases} \tilde{\psi}_1(x,t) = \psi_1(x,t) + (m_1^+(x,t,v,p_T^1) - p_N^1)x_N & \text{if } x_N > 0 \\ \tilde{\psi}_2(x,t) = \psi_2(x,t) + (m_2^-(x,t,v,p_T^2) - p_N^2)x_N & \text{if } x_N < 0 \, . \end{cases}$$

This new test-function still belongs to $\mathrm{PC}^1(\mathbb{R}^N \times [0,T_f])$ and $v-\tilde{\psi}$ has still a minimum point at (x,t), therefore (16.2) holds with $\tilde{\psi}$. But, since by construction $D\tilde{\psi}_1(x,t) = m_1^+(x,t,v,p_T^1)$ while $D\tilde{\psi}_2(x,t) = m_2^-(x,t,v,p_T^2)$, it follows that for $i=1,2$,

$$\tilde{\psi}_t + H_i(x,t,u,D\tilde{\psi}_i) = \tilde{\psi}_t + \min_s(H_i(x,t,v,p_T^i + se_N)) < 0\ .$$

Therefore $\tilde{\psi}_t + G(x,t,v,D_{\mathcal{H}}\tilde{\psi}) \geq 0$, which obviously implies $\psi_t + G(x,t,v,D_{\mathcal{H}}\psi) \geq 0$, so that the flux-limited condition holds.

If, on the contrary, $\psi_t + H_2^+(x,t,v,D\psi_2) < 0$, then $\psi_t + H_2(x,t,v,D\psi_2) < 0$ and the change of test-function reduces to

$$\tilde{\psi}(x,t) := \begin{cases} \tilde{\psi}_1(x,t) = \psi_1(x,t) + (m_1^+(x,t,v,p_T^1) - p_N^1)x_N & \text{if } x_N > 0 \\ \tilde{\psi}_2(x,t) = \psi_2(x,t) & \text{if } x_N < 0\,, \end{cases}$$

but we conclude as in the first case, which ends the proof.

Q.E.D.

17.3 The Kirchhoff condition and flux-limiters

Here we compare the sub/supersolution of (HJ-Gen) associated with the Kirchhoff condition (KC) on one hand, and (FL)-conditions on the other hand in the framework of quasi-convex Hamiltonians. We also consider the cases of more general Kirchhoff type conditions. To simplify matter, we also drop here the dependence of the Hamiltonians in u since this does not create much more difficulty in the proofs.

The results of this section are based on the analysis of various properties of the Hamiltonians (in particular H_T^{reg}) which first appear in Section 10.2, taking into account Remark 10.3.3. We again recall that the definitions of H_T and H_T^{reg} are extended to the case of quasi-convex Hamiltonians by (10.5) and (10.6) and we refer the reader to Section 6.2 for useful results on them. Notice that these sections are written in a slightly more general form, where the Hamiltonians depend on u for the sake of completeness but the results apply here, of course.

Our main result is

Proposition 17.3.1 *Assume* **(GA-QC)**.

(*i*) *An u.s.c. function u is a* (JVSub) *of* (HJ-Gen)-(KC) *if and only if u is a* (FLSub) *with $G = H_T^{\mathrm{reg}}$.*

(*ii*) *A l.s.c. function v is a* (JVSuper) *of* (HJ-Gen)-(KC) *if and only if v is a* (FLSuper) *with* $G = H_T^{\rm reg}$.

It is worth pointing out that this result holds both in the convex and non-convex case, provided that (**H$_{\mathbf{QC}}$**) is satisfied.

Proof — Of course, in both results, only the viscosity inequalities on $\mathcal{H}$ are different and therefore we concentrate on this case. Again we are going to use the results of Section 10.2 in light of Remark 10.3.3.

(a) We begin with the simpler implication that a (FLSub) [*resp.* (FLSuper)] with $G = H_T^{\rm reg}$ is a (JVSub) [*resp.* (JVSuper)] of (HJ-Gen)-(KC). This is a consequence of the properties

$$\begin{aligned} H_T^{\rm reg}(x,t,p') &= \min_{s\in\mathbb{R}} \max\left(H_1^-(x,t,p'+se_N), H_2^+(x,t,p'+se_N)\right) \\ &= \max_{s\in\mathbb{R}} \min\left(H_1^-(x,t,p'+se_N), H_2^+(x,t,p'+se_N)\right)\,, \end{aligned} \tag{17.2}$$

the first equality being the definition of $H_T^{\rm reg}$, the second one being and easy consequence of the monotonicity property of H_1^-, H_2^+.

We just sketch the proof dropping the variables x, t, p' and keeping only the one corresponding to the x_N-derivative for the sake of clarity and denote by a the u_t-variable.

For the (FLSub) case, we start from

$$\max(a + H_1^+(p_1), a + H_2^-(p_2), a + H_T^{\rm reg}) \le 0 \tag{17.3}$$

and using (17.2) in the max min form, we get both

$$H_T^{\rm reg} \ge \min\left(H_1^-(p_1), H_2^+(p_1)\right) \text{ and } \min\left(H_1^-(p_2), H_2^+(p_2)\right)\,.$$

Now, there are two possibilities:

1. Either $-p_1 + p_2 \le 0$, in which case we clearly get the (KC) condition: $\max(a + H_1(p_1), a + H_2(p_2), -p_1 + p_2) \le 0$.
2. Or $p_1 > p_2$ and the monotonicity of H_2^+ implies $H_T^{\rm reg} \ge \min\left(H_1^-(p_1), H_2^+(p_2)\right)$, which leads to
$$\max\left(a + H_1^-(p_1), a + H_2^+(p_2)\right) \le 0\,.$$
But combining this inequality with (17.3), we also end up with the (KC) condition:
$$\min(a + H_1(p_1), a + H_2(p_2), -p_1 + p_2) \le 0\,.$$

The (FLSuper) case is done by similar arguments, using this time (17.2) in the min max form. Now we turn to the proofs that (JVSub) [*resp.* (JVSuper)] of (HJ-Gen)-(KC) are (FLSub) [*resp.* (FLSuper)] with $G = H_T^{\rm reg}$.

(b) *Subsolutions* – We first recall that, by Proposition 16.1.3, the (JVSub) of (HJ-Gen)-(KC) are regular on $\mathcal{H}$. As a consequence, the H_1^+, H_2^- inequalities clearly hold on $\mathcal{H}$ thanks to Proposition 3.5.1 with $L = H_1^+$ or H_2^-.

Hence we just have to prove that, if $(\bar{x}, \bar{t}) \in \mathcal{H} \times (0, T_f)$ is a strict local maximum point of $u - \psi$ for some function $\psi = (\psi_1, \psi_2) \in \mathrm{PC}^1(\mathbb{R}^N \times [0, T_f])$, then

$$\psi_t(\bar{x}, \bar{t}) + H_T^{\rm reg}\big(\bar{x}, \bar{t}, D_{\mathcal{H}}\psi(\bar{x}, \bar{t})\big) \le 0\ . \tag{17.4}$$

In particular, $(\bar{x}, \bar{t})$ is a strict local maximum point of $((x', 0), t) \mapsto u((x', 0), t) - \psi((x', 0), t)$ on $\mathcal{H}$. Now, in order to build a specific test-function, we consider for some small $\kappa > 0$

$$\chi(y_N) := \begin{cases} (\lambda - \kappa)y_N & \text{if } y_N \ge 0\,, \\ (\lambda + \kappa)y_N & \text{if } y_N < 0\,, \end{cases}$$

where, referring to Lemma 10.2.1, λ is a minimum point of the coercive, continuous function $s \mapsto \max\big(H_1^-(\bar{x}, \bar{t}, D_{x'}\psi(\bar{x}, \bar{t}) + se_N), H_2^+(\bar{x}, \bar{t}, D_{x'}\psi(\bar{x}, \bar{t}) + se_N)\big)$. Notice that by this lemma,

$$H_T^{\rm reg}(\bar{x}, \bar{t}, D_{x'}\psi(\bar{x}, \bar{t})) = H_1^-(\bar{x}, \bar{t}, D_{x'}\psi(\bar{x}, \bar{t}) + \lambda e_N) = H_2^+(\bar{x}, \bar{t}, D_{x'}\psi(\bar{x}, \bar{t}) + \lambda e_N)\,, \tag{17.5}$$

By standard arguments, the following function

$$(x, t) \mapsto u(x, t) - \psi((x', 0), t) - \chi(x_N) - \frac{(x_N)^2}{\varepsilon^2} \tag{17.6}$$

has a maximum point $(x_\varepsilon, t_\varepsilon)$ near $(\bar{x}, \bar{t})$ and $(x_\varepsilon, t_\varepsilon) \to (\bar{x}, \bar{t})$ as ε tends to 0 since $(\bar{x}, \bar{t})$ is a strict local maximum point of $(x, t) \mapsto u(x, t) - \psi((x', 0), t)$ on $\mathcal{H}$.

Notice that since $\kappa > 0$, choosing χ as above prevents the (KC)-condition to hold on $\mathcal{H}$, hence the condition on $\mathcal{H}$ reduces to "$\min\big(\psi_t + H_1, \psi_t + H_2\big) \le 0$". Now we examine the quantity

$$Q_\varepsilon := H_1\Big(x_\varepsilon, t_\varepsilon, D_{x'}\psi((x'_\varepsilon, 0), t_\varepsilon) + (\lambda - \kappa)e_N + \frac{2(x_\varepsilon)_N}{\varepsilon^2})\Big)\,,$$

defined only if $(x_\varepsilon)_N \ge 0$. Since $H_1 \ge H_1^-$ and H_1^- is increasing in the e_N-direction,

it follows that

$$\begin{aligned} Q_\varepsilon &\geq H_1^-\Big(x_\varepsilon, t_\varepsilon, D_{x'}\psi((x'_\varepsilon,0),t_\varepsilon) + (\lambda-\kappa)e_N + \frac{2(x_\varepsilon)_N}{\varepsilon^2})\Big) \\ &\geq H_1^-\big(x_\varepsilon, t_\varepsilon, D_{x'}\psi((x'_\varepsilon,0),t_\varepsilon) + (\lambda-\kappa)e_N)\big) \\ &\geq H_1^-\big(\bar{x},\bar{t}, D_{x'}\psi(\bar{x},\bar{t}) + (\lambda-\kappa)e_N) + o_\varepsilon(1) \\ &= H_T^{\rm reg}(\bar{x},\bar{t}, D_{x'}\psi(\bar{x},\bar{t}) + \lambda e_N) + o_\varepsilon(1) + O(\kappa)\ . \end{aligned}$$

An analogous inequality holds if $(x_\varepsilon)_N \leq 0$ with H_2 and H_2^+ and we deduce (17.4) necessarily holds on $\mathcal{H}$.

(c) *Supersolutions* – Let v be a (JVSuper) of (HJ-Gen)-(KC): we have to prove that v is a flux-limited supersolution with $G = H_T^{\rm reg}$.

To do so, we consider a test-function $\psi = (\psi_1, \psi_2) \in {\rm PC}^1(\mathbb{R}^N \times [0,T_f])$ such that $v - \psi$ reaches a local strict minimum at $(\bar{x},\bar{t}) \in \mathcal{H} \times (0,T)$. For $i = 1,2$, we use the notations

$$a = \psi_t(\bar{x},\bar{t})\ ,\ p' = D_{x'}\psi(\bar{x},\bar{t})\ ,\ \lambda_i = \frac{\partial \psi_i}{\partial x_N}(\bar{x},\bar{t})\ .$$

By the supersolution property of v, dropping the dependence in $\bar{x},\bar{t},p'$ to simplify the notations,

$$\max\Big(-\lambda_1+\lambda_2, a + H_1(\lambda_1), a + H_2(\lambda_2)\Big) \geq 0\ ,$$

and we want to prove that

$$\max\Big(a + H_T^{\rm reg}, a + H_1^+(\lambda_1), a + H_2^-(\lambda_2)\Big) \geq 0\ .$$

We argue by contradiction assuming that in the latter inequality, each term is strictly negative.

With the notations of Section 3.5, we look at the subdifferential of v at $(\bar{x},\bar{t})$, restricted to each domain $\overline{Q_i}^\ell := \overline{\Omega_i} \times (0,T_f)$ for $i = 1,2$, and see that $((p',\lambda_i),a) \in D^-_{\overline{Q_i}^\ell} v(\bar{x},\bar{t})$. Now we apply Proposition 3.5.4, denoting by

$$\begin{aligned} \tilde{\lambda}_1 &:= \sup\big\{\lambda \in \mathbb{R} : ((p',\lambda),a) \in D^-_{\overline{Q_1}^l} v(\bar{x},\bar{t})\big\}\ , \\ \tilde{\lambda}_2 &:= \inf\big\{\lambda \in \mathbb{R} : ((p',\lambda),a) \in D^-_{\overline{Q_2}^l} v(\bar{x},\bar{t})\big\}\ , \end{aligned}$$

and we point out that $Dd(x) = e_N$ on Ω_1 while $Dd(x) = -e_N$ on Ω_2, which explains the difference supremum-infimum. We assume that both quantities are finite and explain at the end of the proof that the other cases can be treated by similar and simpler arguments.

The fact that both $\tilde{\lambda}_1, \tilde{\lambda}_2$ are finite implies that v is regular at $(\bar{x}, \bar{t})$ and Proposition 3.5.4 implies

$$a + H_1(\tilde{\lambda}_1) \geq 0 \quad \text{and} \quad a + H_2(\tilde{\lambda}_2) \geq 0 \,. \tag{17.7}$$

Recall that $s \mapsto H_1^+(s)$ is nonincreasing and we are assuming $a + H_1^+(\lambda_1) < 0$. Therefore, $\tilde{\lambda}_1 \geq \lambda_1$ implies that also $a + H_1^+(\tilde{\lambda}_1) < 0$. In the same way, $a + H_2^-(\tilde{\lambda}_2) < 0$ which both imply that

$$a + H_1^-(\tilde{\lambda}_1) \geq 0 \quad \text{and} \quad a + H_2^+(\tilde{\lambda}_2) \geq 0 \,. \tag{17.8}$$

Taking into account the definition of ν_1, ν_2 in Lemma 10.2.1 and the fact that we assume $a + H_T^{\text{reg}}(p') < 0$, the monotonicity properties of H_1^- and H_2^+ imply that $\tilde{\lambda}_2 < \nu_1 \leq \nu_2 < \tilde{\lambda}_1$. Moreover, since $a + H_1^-(\nu_2) = a + H_T^{\text{reg}}(p') < 0$ and $a + H_1^-(\tilde{\lambda}_1) \geq 0$, there exists $\delta_2 \in (\nu_2, \tilde{\lambda}_1)$ such that

$$a + H_1^-(\delta_2) = \frac{1}{2}\Big(a + H_T^{\text{reg}}(p')\Big) \,.$$

Since $H_1^-(\delta_2) > H_1^-(\nu_2)$, it follows that $\delta_2 > m_1^-(x, t, p')$, in other words δ_2 belongs to the region where $s \mapsto H_1^-(x, t, p' + se_N)$ is increasing, and as a consequence, $a + H_1^-(\delta_2) = a + H_1(\delta_2)$

Similarly, there exists $\delta_1 \in (\tilde{\lambda}_2, \nu_1)$ such that $H_2^+(\tilde{\delta}_1) = H_2(\tilde{\delta}_1) = (a + H_T^{\text{reg}}(p'))/2$ and by Proposition 3.5.4 on the structure of the sub-differential, we see that

$$((p', \delta_2), a) \in D^-_{\overline{Q_1}^l} v(\bar{x}, \bar{t}) \,, \qquad ((p', \delta_1), a) \in D^-_{\overline{Q_2}^l} v(\bar{x}, \bar{t}) \,,$$

which leads to

$$\max\big(-\delta_2 + \delta_1, a + H_1(p' + \delta_2), a + H_2(p' + \delta_1)\big) \geq 0 \,.$$

But we reach a contradiction here: clearly $-\delta_2 + \delta_1 < 0$, and the other terms are obviously negative by the construction of δ_1, δ_2.

We finally remark that the key property we use in the proof is (17.8), *i.e.* roughly speaking, the existence of $\tilde{\lambda}_1, \tilde{\lambda}_2$ in the subdifferential for which such inequalities hold. If v is not regular on one side (either on $\overline{Q_1}$ or $\overline{Q_2}$), then any $\lambda \in \mathbb{R}$ is in the corresponding subdifferential and therefore (17.8) is a consequence of the coercivity of either H_1^- near $+\infty$, or H_2^+ near $-\infty$.

Q.E.D.

We conclude this section by a caracterization of the solution associated to H_T^{reg} in the non-convex case.

Proposition 17.3.2 *Under the assumptions of Proposition 17.3.1, an u.s.c. function u is an Ishii subsolution of* (HJ-Gen) *if and only if it is a* (FLSub) *associated to the flux-limiter $H_T^{\rm reg}$.*

Proof — Of course, we are just interested in the inequalities on $\mathcal{H} \times (0, T_f)$.

(a) We first show that an Ishii subsolution of (HJ-Gen) is necessarily a subsolution of (HJ-Gen)-(FL) for the flux limiter $H_T^{\rm reg}$.

Let u be an Ishii subsolution of (HJ-Gen)-(FL); by Proposition 3.5.1, we already know that the H_1^+ and H_2^- inequalities hold on $\mathcal{H} \times (0, T_f)$ and therefore we have just to check the $H_T^{\rm reg}$-one.

To do so, we pick a test-function $\psi : \mathbb{R}^{N-1} \times (0, T_f) \to \mathbb{R}$ and assume that $x' \mapsto u((x', 0), t) - \psi(x', t)$ has a strict, local maximum point at $(\bar{x}, \bar{t}) = ((\bar{x}', 0), t) \in \mathcal{H} \times (0, T_f)$. Then, for $0 < \varepsilon \ll 1$, we consider the function

$$(x, t) = ((x', x_N), t) \mapsto u(x, t) - \psi(x', t) - \lambda x_N - \frac{x_N^2}{\varepsilon^2} \ ,$$

where $\lambda \in [\nu_1, \nu_2]$ is fixed, ν_1, ν_2 being defined in Lemma 10.2.1 at the point $(\bar{x}, \bar{t})$ with $p' = D_{x'}\psi(\bar{x}, \bar{t})$. This function has a local maximum point at a point $(x_\varepsilon, t_\varepsilon)$ which converges to $(\bar{x}, \bar{t})$.

If $(x_\varepsilon, t_\varepsilon) \in \Omega_1 \times (0, T_f)$, it follows that

$$\psi_t(x_\varepsilon, t_\varepsilon) + H_1\Big(x_\varepsilon, t_\varepsilon, D_{x'}\psi(x_\varepsilon, t_\varepsilon) + \lambda e_N + \frac{2x_N}{\varepsilon^2} e_N\Big) \leq 0 \ .$$

Using that $H_1 \geq H_1^-$, the monotonicity property of H_1^- (which allows to drop the $2x_N\varepsilon^{-2}$-term), together with the continuity of both H_1^- and the derivatives of ψ, we obtain

$$\psi_t(\bar{x}, \bar{t}) + H_1^-(\bar{x}, \bar{t}, D_{x'}\psi(\bar{x}, \bar{t}) + \lambda e_N) \leq o_\varepsilon(1) \ ,$$

and since $\lambda \in [\nu_1, \nu_2]$, we get

$$\psi_t(\bar{x}, \bar{t}) + H_T^{\rm reg}(\bar{x}, \bar{t}, D_{x'}\psi(\bar{x}, \bar{t}) + \lambda e_N) \leq o_\varepsilon(1) \ .$$

The conclusion follows by letting ε tend to 0. The two other cases $(x_\varepsilon, t_\varepsilon) \in \Omega_2 \times (0, T_f)$ and $(x_\varepsilon, t_\varepsilon) \in \mathcal{H} \times (0, T_f)$ can be treated similarly.

(b) Conversely, assuming that u is a subsolution with the flux limiter $H_T^{\rm reg}$, we have to show that it satisfies the right Ishii subsolution inequalites on $\mathcal{H}$. Let φ be a smooth function and $(\bar{x}, \bar{t}) \in \mathcal{H} \times (0, T_f)$ be a maximum point of $u - \varphi$, we have to show that

$$\min\Big(a + H_1(\bar{x}, \bar{t}, p' + \lambda e_N), a + H_2(\bar{x}, \bar{t}, p' + \lambda e_N)\Big) \leq 0 \ , \tag{17.9}$$

where $a = \varphi_t(\bar{x},\bar{t})$, $p' = D_{x'}\varphi(\bar{x},\bar{t})$, $\lambda = \dfrac{\partial\varphi}{\partial x_N}(\bar{x},\bar{t})$. Since the flux-limited condition on $\mathcal{H}\times(0,T_f)$ reads

$$\max\Big(a + H_1^+(\bar{x},\bar{t},p'+\lambda e_N)\,, a + H_2^-(\bar{x},\bar{t},p'+\lambda e_N)\,, a + H_T^{\rm reg}(\bar{x},\bar{t},p'+\lambda e_N)\Big) \le 0\,,$$

it is enough to prove either $a + H_1^-(\bar{x},\bar{t},p'+\lambda e_N) \le 0$ or $a + H_2^+(\bar{x},\bar{t},p'+\lambda e_N) \le 0$ in order to deduce (17.9).

Now, if $\nu_1 = \nu_1(\bar{x},\bar{t},p')$ and $\nu_2 = \nu_2(\bar{x},\bar{t},p')$ are given by Lemma 10.2.1, the result is obvious if $\nu_1 \le \lambda \le \nu_2$. On the other hand, if $\lambda < \nu_1$,

$$a + H_1^-(\bar{x},\bar{t},p'+\lambda e_N) \le a + H_1^-(\bar{x},\bar{t},p'+\nu_1 e_N) = a + H_T^{\rm reg}(\bar{x},\bar{t},p'+\lambda e_N) \le 0\,,$$

while if $\lambda > \nu_2$,

$$a + H_2^+(\bar{x},\bar{t},p'+\lambda e_N) \le a + H_2^+(\bar{x},\bar{t},p'+\nu_2 e_N) = a + H_T^{\rm reg}(\bar{x},\bar{t},p'+\lambda e_N) \le 0\,.$$

Hence in any case, (17.9) holds and the proof is complete.

Q.E.D.

17.4 General Kirchhoff coondition and flux-limiters

The aim of this section is to give an extension of Proposition 17.3.1 to the case of general Kirchhoff conditions. The identification of the flux-limited condition leads to a (GFL) given by the function

$$\begin{aligned}
&A(x,t,a,p') := \min_{s_1,s_2}\Phi(s_1,s_2) = \max_{s_1,s_2}\tilde{\Phi}(s_1,s_2) \quad \text{where}\\
&\Phi(s_1,s_2) := \max\Big(a + H_1^-(x,t,p'+s_1e_N)\,, a + H_2^+(x,t,p'+s_2e_N)\,, G(x,t,a,p',-s_1,s_2)\Big)\,,\\
&\tilde{\Phi}(s_1,s_2) := \min\Big(a + H_1^-(x,t,p'+s_1e_N)\,, a + H_2^+(x,t,p'+s_2e_N)\,, G(x,t,a,p',-s_1,s_2)\Big)\,.
\end{aligned} \tag{17.10}$$

Theorem 17.4.1 *Assume* **(GA-QC)** *and that G is a* (GJC) *of Kirchhoff type. Then u is a regular* (JVSub) [*resp.* (JVSuper)] *of* (HJ-Gen)-(GJC) *if and only if it is a* (FLSub) [*resp.* (FLSuper)] *of* (HJ-Gen)-(GFL) *with general flux-limiter $A(x,t,a,p')$ given by* (17.10).

Here we face a general flux-limited condition, namely

$$A(x,t,u_t,D_{\mathcal{H}}u)=0 \quad \text{on } \mathcal{H}\times(0,T_f)\ . \tag{17.11}$$

and to show that we have indeed a (GFL), we prove below that we are in the framework described in Section 14.1.2, *i.e.* (14.1) holds.

Proof — First we leave out the proof of (FLSub) [*resp.* (FLSuper)] implies (JVSub) [*resp.* (JVSuper)] since, as in the proof of Proposition 17.3.1, it relies on easy manipulations of the definitions.

On the other hand, since in all the proof, the dependence in x, t, p' does not play a role, we drop these arguments in H_1, H_2 and G. In other words, we essentially provide the proof in dimension 1 because there is no additional difficulty in higher dimension. Notice however that these dependences may generate some smaller terms $o_\varepsilon(1)$ as $\varepsilon\to 0$ below.

(a) *Subsolution case* — If u is a (JVSub) for the generalized Kirchhoff condition G, we have to show that it is a (FLSub) with the general flux limiter A, *i.e.* if $\varphi=(\varphi_1,\varphi_2)\in \mathrm{PC}^1(\mathbb{R}^N\times[0,T_f])$ and $(x,t)=((x',0),t)$ is a strict local maximum point of $u-\varphi$ then setting

$$a=\varphi_t(x,t)\ ,\ p_1=\frac{\partial\varphi_1}{\partial x_N}(x,t)\ ,\ p_2=\frac{\partial\varphi_2}{\partial x_N}(xs,t)\ ,$$

we have to deduce that $\max(a+H_1^+(p_1),a+H_2^-(p_2),A(a))\le 0$ from the the (JVSub) property, namely

$$\min(a+H_1(p_1),a+H_2(p_2),G(a,-p_1,p_2))\le 0\ .$$

The inequalities $a+H_1^+(p_1)\le 0, a+H_2^-(p_2)\le 0$ are direct consequences of Proposition 3.5.1, therefore we have just to show that $A(a)\le 0$.

Let us assume by contradiction that $A(a)>0$ and denoting by

$$f(t)=a+H_1^-(t)\ ,\ g(s)=a+H_2^+(s)\ ,\ h(t,s)=G(a,-t,s)\,,$$

let us use Lemma 6.3.1, which states that

$$A(a)=\min_{t,s}\{\max(f(t),g(s),h(t,s))\}=\max_{t,s}\{\min(f(t),g(s),h(t,s))\}\ ,$$

that both the min and max are achieved at the same point which we denote by $(\bar p_1,\bar p_2)$, and finally that $A(a)=a+H_1^-(\bar p_1)=a+H_2^+(\bar p_2)=G(a,-\bar p_1,\bar p_2)$.

We now consider the PC^1-function

$$\psi(y_N) := \begin{cases} \bar{p}_1 y_N & \text{if } y_N \geq 0\,, \\ \bar{p}_2 y_N & \text{if } y_N \leq 0\,, \end{cases}$$

and we look at maximum points of

$$\chi(y,s) = u(y,s) - \varphi((y',0),s) - \psi(y_N) - \frac{y_N^2}{\varepsilon^2}\,.$$

Since on $\mathcal{H} \times (0,T_f)$, (x,t) is a strict local maximum point of $u(y,t) - \varphi((y',0),s)$, there exists a sequence $(y_\varepsilon, s_\varepsilon)$ of maximum points of χ which converges to $((x',0),t)$. Examining the (JVSub) inequality at $(y_\varepsilon, s_\varepsilon)$, we see that, if $y_\varepsilon \in \Omega_1$, then

$$a + H_1\Big(\bar{p}_1 + \frac{2(y_\varepsilon)_N}{\varepsilon^2}\Big) \leq 0\,.$$

But, for ε small enough

$$a + H_1\Big(\bar{p}_1 + \frac{2(y_\varepsilon)_N}{\varepsilon^2}\Big) \geq a + H_1^-\Big(\bar{p}_1 + \frac{2(y_\varepsilon)_N}{\varepsilon^2}\Big) \geq a + H_1^-(\bar{p}_1) + o_\varepsilon(1) > 0\,,$$

since H_1^- is increasing in the normal direction e_N and because $a + H_1^-(\bar{p}_1) = A(a) > 0$ (we recall that the $o_\varepsilon(1)$-term reflects the dependence on $(x_\varepsilon, t_\varepsilon, p'_\varepsilon)$).

Therefore y_ε cannot be in Ω_1, nor Ω_2 by a similar argument using H_2^+. Hence $y_\varepsilon = x$ but here also we get a contradiction: using as above that $H_1 \geq H_1^-, H_2 \geq H_2^+$ we obtain

$$\min(a + H_1(\bar{p}_1), a + H_2(\bar{p}_2), G(a, -\bar{p}_1, \bar{p}_2)) = A(a) > 0\,.$$

This proves that $A(a) \leq 0$ and the proof is complete in the subsolution case.

(b) *Supersolution case* — If v is a (JVSuper) for the generalized Kirchhoff condition G, we have to show that it is a (FLSuper) with the flux limiter A, *i.e.* if $\varphi = (\varphi_1, \varphi_2) \in \mathrm{PC}^1(\mathbb{R}^N \times [0,T_f])$ and if $(x,t) = ((x',0),t)$ is a strict local minimum point of $v - \varphi$ then, with the same notations as above, we have to deduce that

$$\max(a + H_1^+(p_1), a + H_2^-(p_2), A(a)) \geq 0\,,$$

from the (JVSuper) property, namely

$$\max\Big(a + H_1(p_1), a + H_2(p_2), G(a, -p_1, p_2)\Big) \geq 0\,.$$

We argue by contradiction assuming that $a + H_1^+(p_1) < 0$, $a + H_2^-(p_2) < 0$ and $A(a) < 0$. Repeating exactly the arguments of the proof of Proposition 17.3.1, we

voluntarily shorten some passages below. Notice that a key ingredient in the proof is Proposition 3.5.4 which describes the structure of sub and superdifferentials on $\mathcal{H} \times (0, T_f)$, on both side.

Using the same notations as in Proposition 3.5.4 and assuming also that $\tilde{\lambda}_1, \tilde{\lambda}_2$ are both finite, the arguments in Proposition 17.3.1 first yield

$$a + H_1(\tilde{\lambda}_1) \geq 0 \quad \text{and} \quad a + H_2(\tilde{\lambda}_2) \geq 0 \, ,$$

and then

$$a + H_1^-(\tilde{\lambda}_1) \geq 0 \quad \text{and} \quad a + H_2^+(\tilde{\lambda}_2) \geq 0 \, .$$

Now, since

$$a + H_1(\bar{p}_1) = a + H_2(\bar{p}_2) = G(a, -\bar{p}_1, \bar{p}_2)) = A(a) < 0 \, ,$$

we get $a + H_1^-(\bar{p}_1), a + H_2^+(\bar{p}_2) < 0$ and therefore $\bar{p}_1 < \tilde{\lambda}_1$, $\bar{p}_2 > \tilde{\lambda}_2$. Moreover, there exists $\bar{p}_1 < s_1 < \tilde{\lambda}_1$ and $\tilde{\lambda}_2 < s_2 < \bar{p}_2$ such that

$$a + H_1^-(s_1) = a + H_2^+(s_2) = A(a)/2 \, .$$

The inequality $a + H_1^-(s_1) > a + H_1^-(\bar{p}_1)$ implies that s_1 belongs necessarily to the interval where $H_1 = H_1^-$, and a similar argument being also true for s_2 we arrive at

$$a + H_1^-(s_1) = a + H_1(s_1) \quad \text{and} \quad a + H_2^+(s_2) = a + H_2(s_2) \, .$$

But the fact that $s_1 < \tilde{\lambda}_1$ and $\tilde{\lambda}_2 < s_2$ means that s_1, s_2 are respectively in the subdifferential relatively to $\overline{Q_1}$ and $\overline{Q_2}$, hence

$$\max(a + H_1(s_1), a + H_2(s_2), G(a, -s_1, s_2)) \geq 0 \, .$$

However, each terms of the max is strictly negative: this is clear for the two first ones, and for the last one we use that, by the monotonicity properties of G,

$$G(a, -s_1, s_2) \leq G(a, -\bar{p}_1, \bar{p}_2)) = A(a) < 0 \, .$$

So, we reach a contradiction and the proof is then complete.

Q.E.D.

Now we show that the function $A(x, t, a, p')$ given by Theorem 17.4.1 is equivalent to a (FL) condition, since it is strictly monotone in a.

Proposition 17.4.2 *Under the assumptions of Theorem 17.4.1, there exists* $\bar{\gamma} > 0$ *such that, for any* $x \in \mathcal{H}, t \in [0, T_f], p' \in \mathbb{R}^{N-1}$ *and* $a_2 > a_1$

$$A(x,t,a,p') - A(x,t,a,p') \geq \bar{\gamma}(a_2 - a_1) \ .$$

Moreover, junction condition (17.11) *is equivalent to* (FL) *for a function* $\mathbb{G}$ *which satisfies* $(\mathbf{H_{BA-HJ}})$.

Proof — In order to prove the first part of the result, we drop the variable x, t, p' which are fixed for the sake of simplicity of notations and therefore we assume that $H_1^-(x,t,p' + s_1 e_N), H_2^+(x,t,p' + s_2 e_N), G(x,t,a,p',-s_1,s_2)$ and A are functions of s_1, s_2 and a only.

(a) By Lemma 6.3.1, for any $a \in \mathbb{R}$ there exists $s_1(a), s_2(a)$ such that

$$A(a) = a + H_1^-(s_1(a)) = a + H_2^+(s_2(a)) = G(a, -s_1(a), s_2(a)) \ .$$

In fact, this lemma does not apply readily since H_1^- is not increasing but only non-decreasing and H_2^+ is not decreasing but only non-increasing. However, this property remains true by easy approximations arguments, using the linear growth of H_1^- at $+\infty$ and H_2^+ at $-\infty$ coming from **(NC)**, to keep $s_1(a), s_2(a)$ bounded.

Examining $A(a_2) - A(a_1)$ there are three cases.

(i) If $s_1(a_2) \geq s_1(a_1)$, then

$$A(a_2) - A(a_1) = a_2 - a_1 + H_1^-(s_1(a_2)) - H_1^-(s_1(a_1)) \geq a_2 - a_1 \ ,$$

since H_1^- is non-decreasing and the desired property is satisfied with $\bar{\gamma} = 1$.

(ii) If $s_2(a_2) \geq s_2(a_1)$, then

$$A(a_2) - A(a_1) = a_2 - a_1 + H_2^+(s_2(a_2)) - H_2^+(s_1(a_1)) \geq a_2 - a_1 \ ,$$

since H_2^+ is non-decreasing and the desired property is satisfied with $\bar{\gamma} = 1$.

(iii) If $s_1(a_2) < s_1(a_1)$ and $s_2(a_2) < s_2(a_1)$, then we use the three above representations for $A(a_2), A(a_1)$: if C is the Lipschitz constant of H_1, H_2 in p and using the

monotonicity of G in a, s_1, s_2

$$\begin{aligned}(2\alpha + C)(A(a_2) - A(a_1)) &= \alpha\Big(a_2 - a_1 + H_1^-(s_1(a_2)) - H_1^-(s_1(a_1))\Big)\\ &\quad + \alpha\Big(a_2 - a_1 + H_2^+(s_1(a_2)) - H_2^+(s_1(a_1))\Big)\\ &\quad + C\Big(G(a_2, -s_1(a_2), s_2(a_2)) - G(a_1, -s_1(a_1), s_2(a_1))\Big)\\ &\geq \alpha\Big(a_2 - a_1 - C|s_1(a_2) - s_1(a_1)|\Big)\\ &\quad + \alpha\Big(a_2 - a_1 - C|s_1(a_2) - s_1(a_1)|\Big)\\ &\quad - \alpha C\Big((s_1(a_2) - s_1(a_1)) + (s_2(a_2)) - s_2(a_1))\Big)\\ &\geq 2\alpha(a_2 - a_1)\ .\end{aligned}$$

Gathering the three cases, we see that the result holds with $\bar{\gamma} = 2\alpha/(2\alpha + C)$.

(b) This monotonicity property implies that that there exists $\mathbb{G}(x, t, p')$ such that

$$A(x, t, a, p') = 0 \quad \Leftrightarrow \quad a + \mathbb{G}(x, t, p') = 0\ .$$

And the fact that $\mathbb{G}$ satisfies ($\mathbf{H_{BA-HJ}}$) can easily be proved by using the definition of A— which implies that A satisfies ($\mathbf{H_{BA-HJ}}$)—and the monotonicity of A in a.

Q.E.D.

17.5 Convergence of the vanishing viscosity approximation (III)

In this section, we revisit the convergence of the vanishing viscosity method in the cases of quasi-convex Hamiltonians. By using the connections between flux-limited and junction viscosity solutions of problems with (FL) and (KC), we are able to obtain more general results for this type of Hamiltonians, with more complete formulations and more natural proofs. Indeed, we can combine the advantages of these two notions of solutions, the (JVS) being more flexible in terms of stability while more general comparison results are available for (FLS) (as far as quasi-convex Hamiltonians are concerned) since they do not require the restrictive assumption **(TC-s)**.

The result is the

Theorem 17.5.1 *For any $\varepsilon > 0$, let $u^\varepsilon \in C(\mathbb{R}^N \times [0,T))$ be a viscosity solution of*

$$u_t^\varepsilon - \varepsilon \Delta u^\varepsilon + H(x,t,u^\varepsilon, Du^\varepsilon) = 0 \quad \text{in} \quad \mathbb{R}^N \times (0,T_f)\,, \tag{17.12}$$

with the initial data

$$u^\varepsilon(x,0) = u_0(x) \quad \text{in} \quad \mathbb{R}^N\,, \tag{17.13}$$

where $H(x,t,r,p) = H_1(x,t,r,p)$ if $x \in \Omega_1$ and $H(x,t,r,p) = H_2(x,t,r,p)$ if $x \in \Omega_2$ and u_0 is bounded continuous function in $\mathbb{R}^N$. We assume that both Hamiltonians H_1, H_2 satisfy **(GA-QC)**.

If the u^ε are uniformly bounded in $\mathbb{R}^N \times (0,T_f)$ and C^1 in x_N in a neighborhood of $\mathcal{H}$, then, as $\varepsilon \to 0$, the sequence $(u^\varepsilon)_\varepsilon$ converges locally uniformly in $\mathbb{R}^N \times (0,T_f)$ to a continuous function u which is at the same time

(i) the maximal Ishii subsolution of (7.1),
(ii) the unique (JVS) *of the Kirchhoff problem,*
(iii) the unique (FL) *associated to the flux-limiter H_T^{reg}.*

Proof — It consists in the following steps.

1. We use the stability result of Lemma 16.4.2: $\overline{u} = \limsup^* u^\varepsilon$ and $\underline{u} = \liminf_* u^\varepsilon$ are respectively (JVSub) and (JVSuper) of the Kirchhoff problem.

2. By Proposition 17.3.1, $\overline{u}$ and $\underline{u}$ are (FLSub) and (FLSuper) with the flux-limiter $G = H_T^{\mathrm{reg}}$.

3. By the comparison result for (FLS) in the quasi-convex setting (Theorem 15.3.2), $\overline{u} \le \underline{u}$ in $\mathbb{R}^N \times [0,T_f))$

4. By the usual argument, we deduce that $u^\varepsilon \to u := \overline{u} = \underline{u}$ in $C(\mathbb{R}^N \times [0,T_f))$.

We conclude the proof by remarking that Proposition 17.3.1 provides the equivalence of properties (*ii*) and (*iii*) while (*i*) comes from the fact that an Ishii subsolution of (7.1) is also a subsolution with (KC), hence a (FLSub) with the flux-limiter $G = H_T^{\mathrm{reg}}$. Again the comparison comes from Theorem 15.3.2.

Q.E.D.

Remark 17.5.2 *The above proof shows how much we can take advantage of Proposition 17.3.1 and more generally of all the results of Chapter 17 in order to use all the different qualities of* (FLS) *and* (JVS).

17.6 A few words about existence

In general, existence of viscosity solutions is not an issue: the *Perron method* of Ishii [119] (see also the User's guide [67]) provides existence of solutions in such a general framework that adressing the question of existence has quickly become irrelevant. On the contrary, when applying Perron method, strong comparison results are crucial in order to obtain the existence of *continuous* viscosity solutions: indeed, the basic arguments of this method consists in building an u.s.c. subsolution u such that u_* is a supersolution and then the **(SCR)** implies the continuity of u since it gives $u \leq u_*$, hence $u = u_*$ since of course $u_* \leq u$ by definition. Therefore $u = u_*$ is both u.s.c. and l.s.c., hence continuous. Of course, this general argument is valid for equations with discontinuous Hamiltonians (or with junctions), which yields another reason why it is important to extend such **(SCR)** to more and more general contexts.

As we know, **(SCR)** holds both for (FLS) and (JVS) but is it so clear that the basic arguments of the Perron method work in these frameworks? The answer is yes but with some difficulties, which is the reason why this section exists.

To be more precise we formulate the

Proposition 17.6.1 (Existence of solutions)

(*i*) *Under the assumptions of Theorem 15.3.2, if u_0 is a bounded continuous function, there exists a unique bounded, continuous solution of (7.1) with the flux-limited condition given by the flux-limiter G.*

(*ii*) *Under the assumptions of Theorem 16.3.7, if u_0 is continuous there exists a unique bounded, continuous solution of (7.1) both for* (GJC) *of Kirchhoff type and for* (FL) *conditions.*

Proof — Here we just sketch the proof since it readily follows the "classical Perron method" approach and only focus on some specificities below. To simplify the presentation, we assume that u_0 is C^1 with a bounded gradient: in fact, once this particular case is treated, the general case follows by standard approximation arguments and stability, using in a crucial way a **(SCR)** to conclude.

We first consider the (FL) case and we introduce $\underline{u}(x,t) := u_0(x) - Ct$, $\overline{u}(x,t) := u_0(x) + Ct$. If $C > 0$ is large enough, these functions are respectively (FLSub) and (FLSuper) of (7.1). We then introduce the function $u_{\rm FL} : \mathbb{R}^N \times [0,T_f] \to \mathbb{R}$ defined at each point (x,t) by

$$u_{\rm FL}(x,t) := \sup\left\{w(x,t) : \underline{u} \leq w \leq \overline{u}\,,\ w \text{ is an (FLSub)}\right\}.$$

Similarly, we define u_{GJC} for the (GJC) case and, in the rest of the proof, u denotes either u_{FL} or u_{GJC} since many arguments work equally for both. Notice that the subsolution property is checked using u^* and the supersolution uses u_* because u is not continuous a priori.

(a) *The subsolution property* — This part is easy and follows the standard procedure, whether in the (FL) or (JVS) case. It is done in three steps

1. The maximum of two subsolutions is a subsolution: a result which does not cause any problem in the discontinuous framework using the following property which is analogous to the one given in Lemma 16.3.6: for any u.s.c. functions $u_1, u_2 : \mathbb{R}^N \times [0,T_f] \to \mathbb{R}$, for any $(x,t) \in \mathcal{H} \times (0,T_f)$ such that $u_1(x,t) = u_2(x,t)$ and $i = 1,2$ we have

$$D^+_{\overline{\Omega_i}\times[0,T_f]} \max(u_1,u_2)(x,t) \subset D^+_{\overline{\Omega_i}\times[0,T_f]} u_1(x,t) \cap D^+_{\overline{\Omega_i}\times[0,T_f]} u_2(x,t)\ .$$

   A similar property holds if $(x,t) \in \Omega_1 \times (0,T_f)$ or $(x,t) \in \Omega_2 \times (0,T_f)$.

2. The supremum of a countable number of subsolutions is a subsolution: this is a consequence of Theorem 15.2.1 or Theorem 16.2.1. Indeed, if $(u_n)_n$ is a sequence of u.s.c. subsolutions[(1)] then $v_n := \max_{k\le n} u_k$ is a sequence of subsolutions by Point 1. Then, it is a simple exercice to show that since (v_n) is non-decreasing, setting $u := \sup_{n\ge 0} v_n$ yields $u^* = \limsup^*_n v_n$, were we recall that the relaxed limsup is given by

$$\limsup^*_n v_n = \limsup_{\substack{n\to\infty \\ (y,s)\to(x,t)}} v_n(y,s)\,.$$

3. The supremum of any set of subsolutions (possibly not countable) is a subsolution: indeed, for each $(x,t) \in \mathbb{R}^N \times [0,T_f]$, there exists a sequence $(u_n)_n = (u_n^{(x,t)})_n$ of subsolutions, whether (FLSub) or (JVSub), such that $u^*(x,t) = \limsup^*_n u_n(x,t)$. For this specific sequence $(u_n^{(x,t)})_n$, if we set

$$\tilde{u}(y,s) := \limsup^*_n u_n^{(x,t)}(y,s)\,, \quad (y,s) \in \mathbb{R}^N \times [0,T_f]\,,$$

   the following holds: (i) $\tilde{u}$ is a subsolution by point 2.; (ii) $\tilde{u} \le u^*$ everywhere and $u^*(x,t) = \tilde{u}(x,t)$; (iii) by a similar property as the one used in point 1.,

$$D^+_{\overline{\Omega_i}\times[0,T_f]} u^*(x,t) \subset D^+_{\overline{\Omega_i}\times[0,T_f]} \tilde{u}(x,t)\,, \quad \text{for any } (x,t) \in \overline{\Omega_i} \times [0,T_f]\,.$$

   Hence the subsolution property of $\tilde{u}$ is automatically transfered to u.

(1) We may assume that they are u.s.c. by replacing u_n by u_n^* if necessary.

As a by-product of the above arguments, u^* is a subsolution which satisfies $\underline{u} \le u^* \le \overline{u}$, hence $u \ge u^*$, which means that $u = u^*$, *i.e.* u is u.s.c..

(b) *The* (JVS) *case* — Proving that the maximal subsolution u is also a supersolution is done via a "bump function" argument. The reader can easily check that this argument applies without any difficulty in the case of (ii), *i.e.* for (JVS), when the junction condition is of Kirchhoff type.

The reason is the following: if u_* is not a supersolution, this is of course because of the junction condition. Indeed, elsewhere classical Ishii's arguments apply. This means that there exist $(x,t) \in \mathcal{H} \times (0,T_f)$ and a test-function $\psi = (\psi_1, \psi_2) \in \mathrm{PC}^1(\mathbb{R}^N \times [0,T_f])$ such that $u_* - \psi$ has a strict local minimum point at (x,t) and

$$\max(\psi_t + H_1(x,t,u_*, D_x\psi_1), \psi_t + H_2(x,t,u_*,D_x\psi_2), G(\cdots)) < 0 \ ,$$

where all functions are evaluated at (x,t) and G at $x,t,u_*(x,t),\psi_t(x,t), D_{\mathcal{H}}\psi(x,t)$, $\frac{\partial \psi_1}{\partial n_1}(x,t), \frac{\partial \psi_2}{\partial n_2}(x,t)$. We may also assume that $u_*(x,t) = \psi(x,t)$.

The first consequence of this property is that $u_*(x,t) < \overline{u}(x,t)$; otherwise, u_* would satisfy the supersolution requirement at (x,t) by the same argument as Point 3. above since we would have $u_* \le \overline{u}$ and $u_*(x,t) = \overline{u}(x,t)$, hence, for $i = 1,2$

$$D^-_{\overline{\Omega_i}\times[0,T_f]} u_*(x,t) \subset D^+_{\overline{\Omega_i}\times[0,T_f]} \overline{u}(x,t) \ .$$

The second consequence is that that ψ is a (JVSub) in a neighborhood of (x,t) since in particular

$$\psi_t + H_1(x,t,u_*,D_x\psi_1) < 0 \quad \text{and} \quad \psi_t + H_2(x,t,u_*,D_x\psi_2) < 0 \ ,$$

the fact that $G < 0$ giving the subsolution property on $\mathcal{H} \times (0,T_f)$. Hence, using also the strict minimum point property, there exists a small neighborhood $\mathcal{V}$ of (x,t) such that, for $\varepsilon > 0$ small enough, $\psi + \varepsilon$ is a (JVSub) in $\mathcal{V}$ and $\psi + \varepsilon < u$ in a neighborhood of $\partial\mathcal{V}$. If we set $u_\varepsilon := \max(u, \psi + \varepsilon)$ in $\mathcal{V}$ and $u_\varepsilon = u$ on the complementary of $\mathcal{V}$, then u_ε is a (JVSub) and, for ε small enough, we have $\underline{u} \le u_\varepsilon \le \overline{u}$.

To get a contradiction, we have to show that there exists at least one point (y,s) where $u_\varepsilon(y,s) > u(y,s)$ since this will be a contradiction with the definition of u. But, by definition of u_*, there exists a sequence $(y_k,s_k)_k$ converging to (x,t) such that $u(y_k,s_k) \to u_*(x,t) = \psi(x,t)$. Hence

$$u(y_k,s_k) - (\psi+\varepsilon)(y_k,s_k) \to -\varepsilon < 0 \ ,$$

and therefore $u(y_k,s_k) < (\psi+\varepsilon)(y_k,s_k)$ if k is large enough. Finally $u_\varepsilon(y_k,s_k) = (\psi+\varepsilon)(y_k,s_k) > u(y_k,s_k)$, a contradiction which implies that u_* is a supersolution.

Finally, since subsolutions are regular when (GJC) is of Kirchhoff type—*cf.* Proposition 16.1.3—, Theorem 16.3.7 shows that $u \le u_*$ in $\mathbb{R}^N \times [0, T_f]$, proving the continuity of u and showing that u is the unique solution of (7.1) with the (GJC) junction condition.

(c) *The* (FL) *case* — On the contrary, in case (i) of the argument by contradiction leads to

$$\max(\psi_t + H_1^+(x,t,u_*,D_x\psi_1), \psi_t + H_2^-(x,t,u_*,D_x\psi_2), G(\cdots)) < 0\ ,$$

which does not imply the same H_1, H_2 inequalities. In other words, it is not clear that ψ is a subsolution in a neighborhood of (x,t) and therefore we cannot apply the "bump function" argument directly.

To turn around this difficulty we use Proposition 17.2.1 back and forth, being a little bit careful with the regularity. Since $u = u_{\rm FL}$ is a (FLSub), it is regular on $\mathcal{H} \times (0, T_f)$ and therefore it is a (JVSub) for the (FL) condition. The "bump function" argument, used exactly in the same way as above in the (JVS) formulation, shows that u_* is also a (JVSuper) for the (FL) condition. Indeed, this argument consists in building a (JVSub) which is strictly larger that u at some point and the construction *preserves the regularity of subsolutions*. Hence this (JVSub) is also a (FLSub) by Proposition 17.2.1.

By the same argument as above, this shows that u is a continuous (JVS) of (7.1) with the (FL) junction condition (by Theorem 16.3.7), hence a continuous (FLS) by applying again Proposition 17.2.1.

Q.E.D.

17.7 Where the equivalence helps to pass to the limit

The aim of this section is to describe an example where using at the same time several notions of solutions helps to pass to the limit in an asymptotic problem.

To fix ideas and to simplify matters, we consider an example which looks like the one we study in Part II but with two "close" hyperplanes instead of one. The reader may have in mind a control problem where we only allow regular strategies on one of the hyperplanes and all the strategies, including singular ones, on the other one. But, in the sequel, we consider general flux-limiter on each hyperplane.

In terms of pdes, for $0 < \varepsilon \ll 1$, we consider the solution $u_\varepsilon \in C(\mathbb{R}^N \times [0, T_f])$ of

$$u_t + H_2(x,t,u_\varepsilon, D_x u_\varepsilon) = 0 \quad \text{in } \{x_N < -\varepsilon\} \times (0, T_f)\ ,$$

$$u_t + H_0(x,t,u_\varepsilon, D_x u_\varepsilon) = 0 \quad \text{in } \{-\varepsilon < x_N < \varepsilon\} \times (0,T_f)\ ,$$
$$u_t + H_1(x,t,u_\varepsilon, D_x u_\varepsilon) = 0 \quad \text{in } \{x_N > \varepsilon\} \times (0,T_f)\ ,$$

with a flux limiter G_2 on the hyperplane $\{x_N = -\varepsilon\}$ and G_1 on the hyperplane $\{x_N = \varepsilon\}$. Taking into account the results and methods of Chapter 15, both the pde and control ones, using also the equivalence results of Chapter 17, the associated value-function is the unique (FLS) or (JVS) solution of the problem with the flux-limiters G_1 and G_2. We point out that most of the arguments being local, in particular the **(LCR)**, taking into account these two hyperplanes case is not more difficult than to consider only one hyperplane.

Our result is the following

Proposition 17.7.1 *Assume that H_0, H_1, H_2 satisfy $(\mathbf{H_{BA-HJ}})^+$ and* **(NC-HJ)** *and G_1, G_2 satisfy* **(GA-G-FL)**. *Then u_ε converges locally uniformly to the unique solution u of* (HJ-Gen)-(FL) *with the flux limiter $G := \max(G_1, G_2, (H_0)_T)$ where*

$$(H_0)_T(x,t,r,p') = \min_{s\in\mathbb{R}} H_0(x,t,r,p' + se_N)\ .$$

Proof — We first recall that, by Proposition 17.2.1, u_ε is either a (FLS) or (JVS) solution of the associated flux-limiter problem and the natural idea is to use the half-relaxed limits method for the (JVS) formulation which has the most general and flexible stability result. If $\overline{u} = \limsup^* u_\varepsilon$ and $\underline{u} = \liminf_* u_\varepsilon$, we easily obtain the H_2-inequality in $\Omega_2 \times (0,T_f)$, the H_1-inequality in $\Omega_1 \times (0,T_f)$ and, dropping the arguments in the Hamiltonians for the sake of notational simplicity

$$\min(\overline{u}_t + H_0, \overline{u}_t + H_1, \overline{u}_t + H_2, \overline{u}_t + G_1, \overline{u}_t + G_2) \le 0\ ,$$

$$\max(\underline{u}_t + H_0, \underline{u}_t + H_1, \underline{u}_t + H_2, \underline{u}_t + G_1, \underline{u}_t + G_2) \ge 0\ .$$

But none of these inequalities is satisfactory since they are very far from the result we wish to prove. In particular, using the normal controllability, the first one implies

$$\min(\overline{u}_t + G_1, \overline{u}_t + G_2) \le 0\ ,$$

while we need (at least) a max.

To improve these results, we first consider the case of $\overline{u}$. We suppose that $(\bar{x},\bar{t}) \in \mathcal{H} \times (0,T_f)$ is a strict local maximum point of $u - \varphi$ where $\varphi \in \mathrm{PC}^1(\mathbb{R}^N \times [0,T_f])$. We are going to consider, for $C > 0$, the following functions

$$(x,t) \mapsto u_\varepsilon(x,t) - \varphi((x', x_N + \varepsilon), t) - C|x_N + \varepsilon|\ ,$$

$$(x,t) \mapsto u_\varepsilon(x,t) - \varphi((x',x_N),t) - C|x_N| \ ,$$

$$(x,t) \mapsto u_\varepsilon(x,t) - \varphi((x',x_N-\varepsilon),t) - C|x_N-\varepsilon| \ .$$

For each of these functions, there exists a subsequence $(x_{\varepsilon'}, t_{\varepsilon'})$ of maximum points converging to $(\bar{x},\bar{t})$ such that $u_{\varepsilon'}(x_{\varepsilon'},t_{\varepsilon'}) \to \overline{u}(\bar{x},\bar{t})$. Now we examine the possible viscosity inequalities at $(x_{\varepsilon'},t_{\varepsilon'})$ and to do so, we use that $u_{\varepsilon'}$ is a (FLS) subsolution on the hyperplanes $\{x_N = -\varepsilon\}$ and $\{x_N = \varepsilon\}$ but also on the hyperplane $\{x_N = 0\}$ with the flux-limiter $(H_0)_T$ by Proposition 15.6.1.

By the normal controllability, if we choose C large enough, it is clear that, for the first function, $x_{\varepsilon'}$ is necessarely on $\{x_N = -\varepsilon\}$ and the G_2-inequality holds, while for the second one, $x_{\varepsilon'}$ is necessarely on $\{x_N = 0\}$ and the $(H_0)_T$-inequality holds, and the third one leads to the G_1-inequality. Hence

$$\max(\overline{u}_t + G_1, \overline{u}_t + G_2, \overline{u}_t + (H_0)_T) \le 0 \quad \text{on } \mathcal{H}\times(0,T_f).$$

The next step consists in proving that $\overline{u}$ is regular on $\mathcal{H}\times(0,T_f)$: indeed this information is crucial, on one hand, to show that the H_2^- and H_1^+ inequalities hold by using Proposition 3.5.1 and, on the other hand, to be able to use Theorem 16.3.7 later to get the full result.

If this is not the case, there exists $(\bar{x},\bar{t}) \in \mathcal{H}\times(0,T_f)$ such that

$$\text{either} \quad \overline{u}(\bar{x},\bar{t}) > \limsup_{\substack{(y,s)\to(\bar{x},\bar{t})\\(y,s)\in\Omega_1\times(0,T_f)}} \overline{u}(y,s) \quad \text{or} \quad \overline{u}(\bar{x},\bar{t}) \ge \limsup_{\substack{(y,s)\to(\bar{x},\bar{t})\\(y,s)\in\Omega_2\times(0,T_f)}} \overline{u}(y,s) \ .$$

We assume, for example, that $\overline{u}(\bar{x},\bar{t}) \ge \limsup_{\substack{(y,s)\to(\bar{x},\bar{t})\\(y,s)\in\Omega_1\times(0,T_f)}} \overline{u}(y,s) + \eta$ for some $\eta > 0$, the other case being treated similarly.

For $0 < \beta \ll 1$ and some large $C > 0$, we introduce the function

$$\psi_{\beta,C}(y,s) = \overline{u}(x,t) - \frac{|x-\bar{x}|^2}{\beta} - \frac{|t-\bar{t}|^2}{\beta} + Cx_N \ .$$

We first consider this function in $\overline{\Omega_2}\times(0,T_f)$: if β is small enough, $\psi_{\beta,C}$ achieves its maximum at some point (x_β,t_β) close to $(\bar{x},\bar{t})$ and, if C is chosen large enough compared to β^{-1}, we even have $(x_\beta,t_\beta) \in \mathcal{H}\times(0,T_f)$ by the normal controllability assumption because the H_2 inequality cannot hold. And, by subtracting a term like $|x-x_\beta|^2 + |t-t_\beta|^2$, we can even assume that it is a strict local maximum point in $\overline{\Omega_2}\times(0,T_f)$.

On the other hand, if $(y,s) \in \Omega_1 \times (0,T_f)$ is close enough to (x_β, t_β), hence to $(\bar{x},\bar{t})$, we have

$$\begin{aligned}\psi_{\beta,C}(y,s) &= \overline{u}(y,s) - \frac{|y-\bar{x}|^2}{\beta} - \frac{|s-\bar{t}|^2}{\beta} + Cy_N \\ &\leq \overline{u}(\bar{x},\bar{t}) - \frac{\eta}{2} + Cy_N \\ &< \overline{u}(\bar{x},\bar{t}) = \psi_{\beta,C}(\bar{x},\bar{t}) \quad \text{if } Cy_N < \eta/2\ ,\end{aligned}$$

and therefore $\psi_{\beta,C}(y,s) < \psi_{\beta,C}(x_\beta,t_\beta)$. Hence (x_β,t_β) is a strict local maximum point in $\mathbb{R}^N \times (0,T_f)$

Now, for fixed β and C, we consider the functions $\psi_\varepsilon(y,s) := u_\varepsilon(x,t) - \frac{|x-\bar{x}|^2}{\beta} - \frac{|t-\bar{t}|^2}{\beta} + Cx_N$: there exists a subsequence $(x_{\varepsilon'}, t_{\varepsilon'})$ of maximum points of $\psi_{\varepsilon'}$ converging to $(\bar{x},\bar{t})$ such that $u_{\varepsilon'}(x_{\varepsilon'},t_{\varepsilon'}) \to \overline{u}(\bar{x},\bar{t})$. If C is chosen large enough compared to β^{-1}, a case-by-case study, using the (FLS) formulation and the normal controllability, leads to a contradiction since no subsolution inequality can hold at $(x_{\varepsilon'},t_{\varepsilon'})$ if ε' is small enough[(2)], wherever $x_{\varepsilon'}$ is because of the coercivity of the H_i's or the fact that the H_i^+ are positive thanks to the $-Ce_N$-term in the derivative of the test-function. This shows that we cannot have $\overline{u}(\bar{x},\bar{t}) > \limsup_{\substack{(y,s)\to(\bar{x},\bar{t}) \\ (y,s)\in\Omega_1\times(0,T_f)}} \overline{u}(y,s)$. The proof showing that we cannot have $\overline{u}(\bar{x},\bar{t}) > \limsup_{\substack{(y,s)\to(\bar{x},\bar{t}) \\ (y,s)\in\Omega_2\times(0,T_f)}} \overline{u}(y,s)$ can be done analogously and the proof of the regularity is complete.

As we explain it above, this implies that $\overline{u}$ is a (JVS)-Sub with the flux-limiter $\max(G_1,G_2,(H_0)_T)$ and the proof for $\overline{u}$ is complete.

Now we turn to the supersolution properties for $\underline{u}$. We have to prove that $\underline{u}$ satisfies

$$\max(\underline{u}_t + H_1, \underline{u}_t + H_2, \underline{u}_t + G) \geq 0\ .$$

As above, we suppose that $(\bar{x},\bar{t}) \in \mathcal{H} \times (0,T_f)$ is a strict local minimum point of $u-\varphi$ where $\varphi = (\varphi_1,\varphi_2) \in \mathrm{PC}^1(\mathbb{R}^N \times [0,T_f])$. We argue by contradiction assuming that

$$\max(\varphi_t + H_1, \varphi_t + H_2, \varphi_t + G) = -\eta < 0\ .$$

We consider the function

$$(x,t) \mapsto u_\varepsilon(x,t) - \varphi(x,t) - \varepsilon\chi\left(\frac{x_N}{\varepsilon}\right)\ ,$$

[(2)] in order to have $C(x_{\varepsilon'})_N < \eta/2$.

where $\chi : \mathbb{R} \to \mathbb{R}$ is defined in the following way

$$\chi(y) = \begin{cases} -\delta_2 & \text{if } y \leq -1 \ , \\ \delta_2 y & \text{if } -1 \leq y \leq 0 \ , \\ \delta_1 y & \text{if } 0 \leq y \leq 1 \ , \\ \delta_1 & \text{if } y \geq 1 \ , \end{cases}$$

where δ_1, δ_2 will be chosen later on.

As above, there exists a subsequence $(x_{\varepsilon'}, t_{\varepsilon'})$ of minimum points of this function converging to $(\bar{x}, \bar{t})$ such that $u_{\varepsilon'}(x_{\varepsilon'}, t_{\varepsilon'}) \to \underline{u}(\bar{x}, \bar{t})$. In order to examine the possible viscosity inequalities at $(x_{\varepsilon'}, t_{\varepsilon'})$, we set for $F = H_0, H_1, H_2, G_1, G_2$

$$\tilde{F}(\tau) := \varphi_t(\bar{x}, \bar{t}) + F(\bar{x}, \bar{t}, \underline{u}(\bar{x}, \bar{t}), D_{x'}\varphi(\bar{x}, \bar{t}) + \tau e_N) \ .$$

By assumption, we have

$$\tilde{H}_1(\frac{\partial \varphi_1}{\partial x_N}) \leq -\eta < 0 \quad , \quad \tilde{H}_2(\frac{\partial \varphi_2}{\partial x_N}) \leq -\eta < 0 \ ,$$

and the constants (in τ) $\tilde{G}_1, \tilde{G}_2, (\tilde{H}_0)_T$ are also less than $-\eta < 0$.

Now we examine the different possibilities

(a) $(x_{\varepsilon'})_N < -\varepsilon'$: then, by the continuity of H_2 and the fact that φ_2 is C^1, we should have $\tilde{H}_2(\frac{\partial \varphi_2}{\partial x_N}) \geq o(1)$ but clearly this inequality cannot hold for ε' small enough.

(b) $(x_{\varepsilon'})_N = -\varepsilon'$: using again the continuity of the Hamiltonians and of φ_2, the (FLSuper) inequality should read

$$\max((\tilde{H}_2)^-(\frac{\partial \varphi_2}{\partial x_N}), (\tilde{H}_0)^+(\frac{\partial \varphi_2}{\partial x_N} + \delta_2), \tilde{G}_2) \geq o(1) \ .$$

Here $\tilde{G}_2 < 0$, $(\tilde{H}_2)^- \leq \tilde{H}_2 < 0$ and we choose δ_2 in order that $\dfrac{\partial \varphi_2}{\partial x_N} + \delta_2$ is a minimum point of $\tilde{H}_0$, hence

$$(\tilde{H}_0)^+(\frac{\partial \varphi_2}{\partial x_N} + \delta_2) = (\tilde{H}_0)_T < 0 \ .$$

With this choice of δ_2, this second case turns out to be impossible.

(c) $-\varepsilon' < (x_{\varepsilon'})_N < 0$: with our choice of δ_2, the H_0-inequality cannot hold for ε' small enough and this case cannot happen neither.

(d) $(x_{\varepsilon'})_N = 0$: we choose δ_1 such that $\dfrac{\partial \varphi_1}{\partial x_N} + \delta_1$ is a minimum point of $\tilde{H}_0$, hence

$$(\tilde{H}_0)^+(\frac{\partial \varphi_1}{\partial x_N} + \delta_1) = (\tilde{H}_0)_T < 0 \ .$$

With this choice, the (FLSuper) inequality which reads

$$\max((\tilde{H}_0)^-(\frac{\partial \varphi_2}{\partial x_N} + \delta_2), (\tilde{H}_0)^+(\frac{\partial \varphi_1}{\partial x_N} + \delta_1), (\tilde{H}_0)_T) \geq 0 \ ,$$

cannot hold for ε' small enough.

(e) $0 < (x_{\varepsilon'})_N < \varepsilon'$: this case is the exact symmetric of (c),

(f) $(x_{\varepsilon'})_N = \varepsilon'$: this case is the exact symmetric of (b),

(g) $(x_{\varepsilon'})_N > \varepsilon'$: this case is the exact symmetric of (a),

and in the three cases (e), (f), (g), we also conclude that the (FLSuper) inequality cannot hold for ε' small enough. Hence, whereever $x_{\varepsilon'}$ is, the (FLSuper) inequality cannot hold. This gives a contradiction and prove that

$$\max(\varphi_t + H_1, \varphi_t + H_2, \varphi_t + G) \geq 0 \ .$$

Hence $\underline{u}$ is a (JVSuper) with the flux-limiter G.

The classical arguments of the half-relaxed limits method to gether with the comparison result for (JVS) solutions (Theorem 16.3.7), taking into account that $\overline{u}$ is a regular subsolution, implies $\overline{u} \leq \underline{u}$ in $\mathbb{R}^N \times [0, T_f]$. Hence $u = \overline{u} = \underline{u}$ is continuous and the unique (JVS) with the flux-limiter G. And the local uniform convergence of u_ε to u follows by classical arguments.

Q.E.D.

Chapter 18

Applications and Emblematic Examples

The aim here is to give an overview of the results of Part II in the context of Hamilton-Jacobi equations corresponding to 1D scalar conservations laws with a discontinuous flux. This chapter is especially intented for the (partial) reader of this book who wishes

(i) to get an idea of what can be done in this context ...

(ii) ... without reading the totality of this book!

Thus, the reader will find here some redundancy concerning definitions, results, ideas... with respect to the previous sections. We try to keep it simple and refer to those previous sections for more precise results and proofs.

18.1 HJ analogue of a discontinuous 1D-scalar conservation law

The starting point here is the problem

$$u_t + H(x, u_x) = 0 \quad \text{in } \mathbb{R} \times (0, T_f)\,, \tag{18.1}$$

where Hamiltonian H is given by

$$H(x,p) = \begin{cases} H_1(p) & \text{if } x > 0\,, \\ H_2(p) & \text{if } x < 0\,. \end{cases}$$

Equation (18.1) has to be complemented by an initial datum

$$u(x,0) = u_0(x) \quad \text{in } \mathbb{R}\,, \tag{18.2}$$

where u_0 is assumed to be bounded and continuous in $\mathbb{R}$.

In this definition of H, H_1, H_2 are continuous functions which are coercive, *i.e.*

$$H_1(p), H_2(p) \to +\infty \quad \text{as } |p| \to +\infty\,,$$

and we consider two main cases: the "Lipschitz case" where both Hamiltonians are supposed to be Lipschitz continuous in $\mathbb{R}$ and the "convex case" where they are supposed to be convex, but not necessarily Lipschitz continuous, even if this case is not completely covered by the results of Part II [(1)].

In the "Lipschitz case", a natural sub-case is the one when the H_i $(i = 1, 2)$ are *quasi-convex* [(2)], *i.e.* built as the maximum of an increasing and a decreasing function. For this reason, we write

$$H_1 = \max(H_1^+, H_1^-) \quad \text{and} \quad H_2 = \max(H_2^+, H_2^-)\,,$$

where H_1^+, H_2^+ are the decreasing parts of H_1, H_2 respectively and H_1^-, H_2^- their increasing parts. Using these notations for the monotone Hamiltonians may seem strange but the reader has to keep in mind that *characteristics*—or dynamics in terms of control problems—play a key role in these problems. A way to better understand this remark is to consider the convex control case where the dynamic is given by b_1 and where $H_1(p) = \sup_{b_1 \in B}\{-b_1 \cdot p - l_1\}$; in this case

$$H_1^+(p) = \sup_{b_1 \geq 0}\{-b_1 \cdot p - l_1\}\,,$$

which means that we keep in H_1^+ only the dynamics pointing toward the positive direction, explaining the "+".

Of course, the case of quasi-concave (or concave) Hamiltonians can be treated in the same way since, by changing u in $-u$, we change $H_1(p), H_2(p)$ in $-H_1(-p), -H_2(-p)$, the latter being quasi-convex (or convex) if the former are quasi-concave (or concave).

18.1.1 On the condition at $x = 0$

Of course, the first key question is: what kind of condition has to be imposed at $x = 0$ where the Hamiltonian H is discontinuous?

(1) but we trust the reader to be able to fill up the gaps!

(2) We refer the reader to Section 6.2 for a short presentation of the notion of quasi-convexity and for the related properties we use throughout this book.

Viscosity solutions theory provides a default answer which is the notion of *Classical Viscosity Solutions* ((CVS) in short) introduced by by Ishii [118]. These conditions are

$$\min(u_t + H_1(u_x), u_t + H_2(u_x)) \leq 0 \quad \text{on} \quad \{0\} \times (0, T_f)\,,$$
$$\max(u_t + H_1(u_x), u_t + H_2(u_x)) \geq 0 \quad \text{on} \quad \{0\} \times (0, T_f)\,.$$

These sub and supersolutions properties have to be tested with test-functions which are C^1 in $\mathbb{R} \times (0, T_f)$. We do not detail them here, referring the reader to Section 3.1 for more informations.

Unfortunately (or fortunately?), this classical notion of solution has two main defects: on one hand, (CVS) are not unique in general and, on the other hand, in concrete applications, the modelling may lead to other "transfer conditions" at $x = 0$. To be convinced by this claim, it suffices to look at the well-known Kirchhoff condition

$$-u_x(0^+, t) + u_x(0^-, t) = 0 \quad \text{on} \quad \{0\} \times (0, T_f)\,, \tag{18.3}$$

for which testing with $C^1(\mathbb{R} \times (0, T_f))$-test-functions is of course meaningless, this condition being automatically satisfied for smooth test-functions. Clearly we need a larger set of testing possibilities in order to take into account in a right way such conditions and to have a hope for a well-posed problem (in particular, comparison and uniqueness results).

18.1.2 Network viscosity solutions

For the Kirchhoff condition but also for more general conditions like

$$G(u_t, -u_x(0^+, t), u_x(0^-, t)) = 0 \quad \text{on} \quad \{0\} \times (0, T_f)\,, \tag{18.4}$$

where $G(a, b, c)$ is a continuous function which is increasing in a, b and c[(3)], one has to use a notion of *"Network viscosity solution"* based on testing the viscosity properties with continuous, "piecewise C^1"-test-functions, denoted by PC^1. More precisely $\phi \in C(\mathbb{R} \times (0, T_f))$ is a suitable test-function if there exists two functions ϕ_1, ϕ_2 which are C^1 in $\mathbb{R} \times (0, T_f)$ such that

$$\phi(x, t) = \begin{cases} \phi_1(x, t) & \text{if } x > 0, \\ \phi_2(x, t) & \text{if } x < 0, \end{cases}$$

with $\phi_1(0, t) = \phi_2(0, t)$ for any $t \in (0, T_f)$. In order to define "Network viscosity solutions" in the viscosity properties at a point $(0, t)$ we use the derivatives of ϕ_1 for

[(3)] Precise assumptions will be given later on.

the H_1 and $u_x(0^+,t)$-term, and the derivatives of ϕ_2 for the H_2 and $u_x(0^-,t)$-term. Notice that both time-derivatives $(\phi_1)_t$ and $(\phi_2)_t$ coincide on $x=0$.

However, the notion of "Network viscosity solution" with condition at $x=0$ can be used in at least two slightly different ways.

(a) *The "flux limited"* notion of solutions of Imbert-Monneau—(FL) in short—which is valid in the quasi-convex case, *i.e.* in a more general framework than the "convex case". A general flux limited condition at $x=0$ takes the form

$$u_t + A = 0 \quad \text{on} \quad \{0\}\times(0,T_f)\,, \tag{18.5}$$

where A is a real constant called the *flux limiter*. In terms of viscosity inequalities at $x=0$, the condition reads [(4)]

$$\begin{cases} \max\left(u_t + H_1^+(u_x), u_t + H_2^-(u_x), u_t + A\right) \le 0 & \text{on} \quad \{0\}\times(0,T_f)\,, \\ \max\left(u_t + H_1^+(u_x), u_t + H_2^-(u_x), u_t + A\right) \ge 0 & \text{on} \quad \{0\}\times(0,T_f)\,. \end{cases}$$

Why using only H_1^+ and H_2^-? As we already explain it above, the most (vague but) convincing answer is probably through the *characteristics*, or dynamics in the control viewpoint: we use inequalities which test characteristics entering each domain, *i.e.* $[0,+\infty)$ for H_1 and $(-\infty,0]$ for H_2. We respectively call these conditions the sub and supersolution (FL) conditions. In the definition above, we can replace the u_t+A-term by a more general $\chi(u_t)$-term where the function $\tau\mapsto\chi(\tau)$ is strictly increasing.

(b) *The notion of "junction viscosity solutions"* (JVS) which is closer to the Ishii formulation since the inequalities for $x=0$ read

$$\begin{cases} \min\left(u_t + H_1(u_x), u_t + H_2(u_x), G(u_t, -u_x(0^+,t), u_x(0^-,t))\right) \le 0\,, \\ \max\left(u_t + H_1(u_x), u_t + H_2(u_x), G(u_t, -u_x(0^+,t), u_x(0^-,t))\right) \ge 0\,. \end{cases}$$

We refer to Section 15.1 and Section 16.1 for a more precise definition of (FLS) and (JVS). Let us point out three key differences between these notions of solutions:

(i) the notion of (JVS) can take into account both general Kirchhoff type conditions like Equation 18.4 but also flux-limited conditions by assuming that $G(a,b,c) = a + A$ in Equation 18.4. On the contrary, the notion of (FLS) is restricted to flux-limited conditions;

(ii) while the notion of (FL) solutions requires the Hamiltonians to be quasi-convex, the (JVS) notion is valid for any continuous Hamiltonians;

[(4)] with the above mentioned conventions

(*iii*) while the (FLS) one uses a pair of "max-max" inequalities, the (JVS) one uses a classical "min-max" ones.

These three differences seem to indicate that the notion of (JVS) is more general and more adapted than the (FLS) one but the notion of (FLS) is more natural to address control problems (see Section 15.4) and therefore is useful in order to obtain explicit formulas à la Oleinik-Lax.

18.1.3 Main results

We now expose briefly the main results and connections between the different notions of solutions, (CVS), (FLS), (JVS).

The convergence of the vanishing viscosity method is a natural entrance door since, in the classical framework, it selects the "right solution".

(a) *On the vanishing viscosity method and the Kirchhoff junction solution* — In the absence of discontinuities, passing to the limit in this method simply relies on the stability properties of classical viscosity solutions. However here, in presence of discontinuous Hamiltonians, we need to identify the right condition on the interface. The result is the

Theorem 18.1.1 (Convergence of the vanishing viscosity method)

For each $\varepsilon > 0$, let u^ε be a continuous viscosity solution of

$$u^\varepsilon_t - \varepsilon u^\varepsilon_{xx} + H(x, u^\varepsilon_x) = 0 \quad \text{in} \quad \mathbb{R} \times (0, T_f)\,, \tag{18.6}$$

$$u^\varepsilon(x, 0) = u_0(x) \quad \text{in} \quad \mathbb{R}\,. \tag{18.7}$$

If the u^ε are uniformly bounded in $\mathbb{R} \times [0, T_f)$ and C^1 in x in a neighborhood of $x = 0$ for $t > 0$, then, as $\varepsilon \to 0$, the sequence $(u^\varepsilon)_\varepsilon$ converges locally uniformly to the unique (JVS) *solution of the Kirchhoff problem* (18.1)-(18.2)-(18.3).

We first point out that Theorem 18.1.1 is valid for any continuous Hamiltonians H_1, H_2 without any type of convexity (or concavity) assumption.

The formal idea to prove this result is straightforward: u^ε being C^1 in x in a neighborhood of $x = 0$ for $t > 0$, it satisfies the Kirchhoff condition

$$-u^\varepsilon_x(0^+, t) + u^\varepsilon_x(0^-, t) = 0 \quad \text{on} \quad \{0\} \times (0, T_f)\,,$$

and it suffices to pass to the limit using the good stability properties of viscosity solutions, but written with piecewise C^1 test-functions and then to use an adapted comparison result.

This formal proof can be justified using the notion of (JVS) solutions via Lions-Souganidis arguments for the comparison result. Indeed, on one hand, this notion of solutions allows to extend the classical stability argument for viscosity solutions: the half-relaxed limits of u^ε are "junction sub and supersolution" of the Kirchhoff problem, *i.e.*

$$\begin{cases} \min(u_t + H_1(u_x), u_t + H_2(u_x), -u_x(0^+,t) + u_x(0^-,t)) \leq 0\ , \\ \max(u_t + H_1(u_x), u_t + H_2(u_x), -u_x(0^+,t) + u_x(0^-,t)) \geq 0\ . \end{cases}$$

Hence we have a stability result which is as similar as it could be to the classical one, despite of the different spaces of test-functions. It is worth pointing out that the notion of (JVS) is not only necessary to define properly the Kirchhoff condition but it also plays a key role here via this stability result. And then the Lions-Souganidis arguments provide the "Strong Comparison Result" which is needed to conclude.

We actually provide three different proofs of the convergence of the vanishing viscosity method in Part II: a first one via (FLS) solutions, the above one via (JVS) solutions and a last one which combines both notions in order to identify the limit, in particular by giving an explicit formula in the control case.

(b) *On Classical Viscosity Solutions and the Kirchhoff condition* — The result of Theorem 18.1.1 suggests two natural questions

1. Is it possible to characterize the unique (JVS) of (18.1)-(18.2)-(18.3), *i.e.* the Kirchhoff solution, in terms of classical viscosity solutions (CVS)?

2. In the "convex case", is it possible to write down an explicit formula for solutions of the Kirchhoff problem? (à la Oleinik-Lax). In other words, is there an underlying control problem which gives a control formula for this solution?

Our second result answers these questions, of course in the "convex case" since we are looking for explicit formulas. We point out that the results are unavoidably a little bit vague to avoid long statements but precise results can be found in Chapter 7.

Theorem 18.1.2 — *Classical Viscosity Solutions.*

In the "quasi-convex or convex case",

(*i*) *Classical Viscosity Solutions of* (18.1)-(18.2) *with the natural Ishii conditions at* $x = 0$ *are not unique in general. There is a minimal* (CVS) *denoted by* $\mathbf{U}^-$ *and a maximal* (CVS) *denoted by* $\mathbf{U}^+$. *In the convex case, they are both given explicitly as value functions of suitable control problems.*

(*ii*) *If* m_1 *is the largest minimum point of* H_1 *and* m_2 *the least minimum of* H_2, *a sufficient condition in order to get* $\mathbf{U}^- = \mathbf{U}^+$ *is* $m_2 \geq m_1$.

(*iii*) *The solution of the Kirchhoff problem is* $\mathbf{U}^+$. *Hence the vanishing viscosity method converges to the maximal* (CVS).

This result shows the weakness of (CVS) for equations with discontinuities: although they are very stable because of the half-relaxed limits method, they are not unique in this framework and this is, of course, more than a problem. Result (*ii*) is a last desesperate attempt to maintain uniqueness in a rather general case but it seems to be a little bit anecdotic.

Result (*iii*) is a first bridge between the notions of (CVS) and "junction solutions" and it is proved using in a key way the notion of "flux-limited solutions".

We refer to Chapter 17 for various results comparing the (CVS), (FLS) and (JVS) notions of solutions.

(c) *On the characterization via flux limited solutions* — The previous results open the way to the next questions which can be formulated in several different ways, but which all concern the relations between different notions of solutions.

1. In the case of control problems, two particular value-functions appear in Theorem 18.1.2, $\mathbf{U}^-$ and $\mathbf{U}^+$. Both may be interesting for some particular application but clearly, the characterization as (CVS) is not appropriate. Is there any other way to identify them uniquely?

2. From the pde point of view, Result (*iii*) gives a connection between the "junction solution" for the Kirchhoff condition and a value-function of some control problem. But is it possible to prove some similar connexion for more general conditions (18.4) with a rather explicit way?

The answer is provided in the following result which relies on the notion of (FLS).

Theorem 18.1.3 — *Characterizations with flux limiters.*

A. In the quasi-convex case

(*i*) *For any A, there exists a unique flux limited solution of* (18.1)-(18.2)-(18.5). *Moreover a comparison principle holds result for this flux limited problem.*

(*ii*) *If G satisfies: there exists $\alpha, \beta \geq 0$ with $\beta > 0$such that for any $a_1 \geq a_2$, $b_1 \geq b_2$, $c_1 \geq c_2$*

$$G(a_1, b_1, c_1) - G(a_2, b_2, c_2) \geq \alpha(a_1 - a_2) + \beta(b_1 - b_2) + \beta(c_1 - c_2) \ ,$$

then any JVS subsolution [resp. supersolution] of (18.1)-(18.2)-(18.4) *is a* (FLS) *subsolution [resp. supersolution] with flux limiter*

$$\chi(a) = \max_{p_1,p_2} \Big(\min \big(a + H_1^-(p_1), a + H_2^+(p_2), G(a, -p_1, p_2) \big) \Big) \ .$$

B. In the convex case

(*i*) *The value function $\mathbf{U}^-$ is associated to the flux-limiter*

$$A^- = \min_s \big(\max(H_1(s), H_2(s)) \big) \ .$$

(*ii*) *The value function $\mathbf{U}^+$ is associated to the flux-limiter*

$$A^+ = \min_s \big(\max(H_1^-(s), H_2^+(s)) \big) \ .$$

The second part of this result shows that value-functions of control problems can be caracterized as a "flux limited solution" of (18.1)-(18.2) with the right flux limiter at $x = 0$. Contrarily to (CVS), a uniqueness result holds but, as the vanishing viscosity method shows, stability becomes a problem since one has to identify the right flux limiter for the limiting problem.

Remark 18.1.4 *The case of more general junction conditions like* (18.1)-(18.2)-(18.4) *can be treated by the Lions-Souganidis approach: in particular, we have a comparison result for* (18.1)-(18.2)-(18.4) *in the case of general Hamiltonians H_1, H_2 without assuming them to be quasi-convex. Of course, the monotonicity properties of G are necessary not only for having such a comparison result but even for the notion of "junction solution" to make sense.*

We conclude this section by the extension of the Oleinik-Lax formula to our discontinuous framework. To do so, we set $\Omega_1 = \{x > 0\}$ and $\Omega_2 = \{x < 0\}$ and we denote by H_i^* the Fenchel conjugate of H_i for $i = 1, 2$.

Proposition 18.1.5 (Oleinik-Lax Formula)
Under the assumptions of Theorem 18.1.3, we assume moreover that H_1, H_2 are convex coervive, continuous and set, for $x \in \overline{\Omega_i}$ and $t > 0$,

$$\mathbf{U}^i(x,t) := \inf_{z \in \overline{\Omega_i}} \left(u_0(z) + tH_i^*\left(\frac{x-z}{t}\right) \right) .$$

Then the following formulas hold:

$$\mathbf{U}^+(x,t) = \min \left(\mathbf{U}^i(x,t), \inf_{\substack{j=1,2,\ z\in\Omega_j \\ 0\le t_1\le t_2\le t}} \left\{ u_0(z) + t_1 H_i^*\left(\frac{x}{t_1}\right) - A^+(t_2-t_1) + (t-t_2)H_j^*\left(\frac{-z}{t-t_2}\right) \right\} \right) ,$$

and

$$\mathbf{U}^-(x,t) = \min \left(\mathbf{U}^i(x,t), \inf_{\substack{j=1,2,\ z\in\Omega_j \\ 0\le t_1\le t_2\le t}} \left\{ u_0(z) + t_1 H_i^*\left(\frac{x}{t_1}\right) - A^-(t_2-t_1) + (t-t_2)H_j^*\left(\frac{-z}{t-t_2}\right) \right\} \right) ,$$

with the convention that $(t-t_2)H_j^*(\frac{-z}{t-t_2}) = 0$ if $z = 0$ and $t - t_2 = 0$.

In order to apply these Oleinik-Lax formulas, we come back on the examples of Section 10.4. In the first one,

$$H_1(p) = |p+1| \quad , \quad H_2(p) = |p-1| ,$$

therefore $A^+ = A^- = 1$ and it follows that uniqueness holds in the Ishii class of solutions, $\mathbf{U}^+ \equiv \mathbf{U}^-$.

Now, $H_1^*(p) = -p$ if $|p| \le 1$ and $+\infty$ otherwise, while $H_2^*(p) = p$ if $|p| \le 1$ and $+\infty$ otherwise. Hence, since $u_0(x) = |x|$ in $\mathbb{R}$, we see that

$$\begin{cases} \mathbf{U}^1(x,t) := \inf\limits_{z\ge 0,\ |z-x|\le t} (z - (x-z)) = 2(|x|-t)^+ - |x| , \\ \mathbf{U}^2(x,t) := \inf\limits_{z\le 0,\ |z-x|\le t} (-z + (x-z)) = 2(|x|-t)^+ - |x| . \end{cases}$$

Then, in order to compute $\mathbf{U}^+ = \mathbf{U}^-$, we face several cases

(*i*) If $|x| > t$, then $H_i^*(\frac{x}{t_1}) = +\infty$ so $\mathbf{U}^+(x,t) = \mathbf{U}^i(x,t) = 2(|x|-t)^+ - |x|$.

(*ii*) If $|x| \le t$, for the second part of the min, if $x, z \in \overline{\Omega_1}$ we get

$$\inf_{\substack{z\le t-t_2, x\le t_1 \\ 0\le t_1\le t_2\le t}} \{z - x - (t_2 - t_1) + z\} = -t ,$$

since $z = 0$, $t_1 = x$ and $t_2 = t$ is clearly optimal. If $x, z \in \overline{\Omega_2}$, an analogous result holds.

(*iii*) If $|x| \le t$, for the second part of the min, if $x \in \overline{\Omega_1}$, $z \in \overline{\Omega_2}$ we get

$$\inf_{\substack{z \le t-t_2, x \le t_1 \\ 0 \le t_1 \le t_2 \le t}} \{-z - x - (t_2 - t_1) - z\} = -t \ ,$$

since $z = 0$, $t_1 = x$ and $t_2 = t$ is clearly optimal. The case $x \in \overline{\Omega_2}$, $z \in \overline{\Omega_1}$ gives an analogous result.

Finally, since for $|x| \le t$, we have $2(|x| - t)^+ - |x| \ge -t$, we conclude that

$$\mathbf{U}^+(x,t) = \mathbf{U}^-(x,t) = \begin{cases} 2(|x| - t)^+ - |x| & \text{if } |x| \ge t \ , \\ -t & \text{otherwise.} \end{cases}$$

In the second example

$$H_1(p) = |p - 1| \quad , \quad H_2(p) = |p + 1| \ ,$$

and therefore $A^+ = 0$, $A^- = 1$. This time, the solution $\mathbf{U}^+$ and $\mathbf{U}^-$ are different. We leave the checking of their formulas to the reader (they are given in Section 10.4).

18.2 Traffic flow models with a fixed or moving flow constraint

Traffic flows can be studied at the micro or macroscopic level, leading to different, yet complementary models. Here we focus only on the macroscopic scale, looking at the density $\rho(x,t)$ of vehicles at each point x of a one-dimensional infinite highway modelled by $\mathbb{R}$ and any time t. It is often more convenient to use the "renormalized density", *i.e.* the ratio between the actual density and a maximal density, therefore assuming that $0 \le \rho(x,t) \le 1$ for any x and t.

18.2.1 The LWR model

In the context of simple traffic flow without constraints, one of the most famous macropscopic models is the LWR model, originated in the works of Lighthill and Whithan [131] and Richards [149]. It consists in describing the evolution of ρ through a scalar conservation law in $\mathbb{R} \times (0, T_f)$, namely

$$\rho_t + \partial_x(f(\rho)) = 0 \quad \text{in } \mathbb{R} \times (0, T_f) \ , \tag{18.8}$$

where the flux $f : \mathbb{R} \to \mathbb{R}$ is given in the simplest case by $f(\rho) = \rho(1-\rho)$.

In one space dimension, a rather easy way to tackle Equation (18.8) is to use the connections with the Hamilton-Jacobi formulation, *cf.* Corrias, Falcone and Natalini [65] and Aaibid and Sayah [1]: if $u : \mathbb{R} \times (0, T_f) \to \mathbb{R}$ is the unique Lipschitz continuous viscosity solution of

$$u_t + f(u_x) = 0 \quad \text{in } \mathbb{R} \times (0, T_f) \,. \tag{18.9}$$

with a Lipschitz continuous initial data u_0 such that $0 \le u_0'(x) \le 1$ in $\mathbb{R}$, then $\rho = u_x$ is the unique entropy solution of (18.8).

At this point, it is worth pointing out that here f is concave, not convex. However, using that $-u$ is a solution of (18.9) with f being replaced by $h(p) = -f(-p)$ which is convex, one may use the Oleinik-Lax formula to obtain the explicit form of the solution:

$$u(x,t) := \sup_{y \in \mathbb{R}} \left\{ u_0(y) - \frac{t}{4} \left| \frac{x-y}{t} - 1 \right|^2 \right\} .$$

In particular, if $u_0(y) = \rho_0 y$ in $\mathbb{R}$ for some $0 < \rho_0 < 1$, corresponding to a constant density ρ_0 at time $t = 0$, then

$$u(x,t) = \rho_0 x - \rho_0(1-\rho_0)t \quad \text{for any } (x,t) \in \mathbb{R} \times (0, T_f) \,. \tag{18.10}$$

18.2.2 Constraints on the flux

A more interesting question in the context of this book is to investigate the question of traffic reduction, due to a car crash or traffic lights located at one point, $x = 0$. Such problems were first studied by Colombo and Rosini [64] and Colombo and Goatin [63] and they lead to a constraint of the type

$$f(\rho)\Big|_{x=0} \le \delta \,,$$

where we we choose here to consider a *constant* flux limiter $\delta > 0$.

To the best of our knowledge, there is no rigourous result connecting such problems with Hamilton-Jacobi ones in this framework. We can just guess that the corresponding constraint in the Hamilton-Jacobi case takes the form

$$u_t \ge -\delta \quad \text{at } x = 0$$

since at least formally, $f(\rho) = f(u_x) = -u_t$. This formulation corresponds to a flux limiter $G = +\delta$ at $x = 0$ coupled with Hamiltonians $H_1 = H_2 = f$ but we recall

that f being concave we have either to change u in $-u$ or to adapt the results of the previous sections. In any case, we point out that all these results apply and the theory can be applies without any difficulty.

Now, let us go back to the same initial data as above: $u_0(y) = \rho_0 y$ in $\mathbb{R}$ for some $0 < \rho_0 < 1$, associated to a flux $\delta < \rho_0(1-\rho_0) < 1/4$ otherwise the flux is not really limited, but moreover we assume that δ is close to 0 for simplicity. This corresponds to a strong limitation on the crossing at $x = 0$.

Using the control formulation, it is clear that $u(0,t) = -\delta t$ since the reward $-\delta$ for $x = 0$ is maximal—recall that we maximize the reward since f is concave. Since u is identified on $x = 0$, it only remains to solve a Dirichlet problem for the HJ-Equation in the domains $x > 0$ and $x < 0$ separately. We can also guess that the solution is piecewise affine,

$$u(x,t) = at + bx \quad \text{with } a = -b(1-b)\ ,$$

and by using this ansatz we solve the equations separately in different regions. Matching everything is done by finding suitable lines originating from $(0,0)$ so that u is globally continuous.

(a) If $x \neq 0$ and t is close to 0 the flux limitation is not interacting yet, so we use formula (18.10) which yields here also

$$u(x,t) = \rho_0 x - \rho_0(1-\rho_0)t\,.$$

(b) If $t > 0$ and x close to > 0 the flux limitation is acting on the solution and using the boundary value $u(0,t) = -\delta$, we define ρ_1 and ρ_2 as the two solutions of the equation $-\delta = -b(1-b)$, namely

$$\rho_1 = \frac{1 + (1-4\delta)^{1/2}}{2} \quad \text{close to } 1\ ,$$

$$\rho_2 = \frac{1 - (1-4\delta)^{1/2}}{2} \quad \text{close to } 0\ .$$

In this region, the solution is given by $u(x,t) = \rho_1 x - \delta t$ if $x < 0$ and $\rho_2 x - \delta t$ if $x > 0$.

(c) It remains to match everything by continuity. For example, the continuity condition

$$\rho_0 x - \rho_0(1-\rho_0)t = \rho_1 x - \delta t \quad \text{implies } x = \frac{\delta - \rho_0(1-\rho_0)}{\rho_1 - \rho_0}t\ ,$$

where the coefficient of t is strictly negative since the denominator is positive while for δ small enough, $\rho_1 > \rho_0$. This defines a first line Δ_1, located on the left. Similarly,

Δ_2 is defined by the matching condition using ρ_0 and ρ_2, here the coefficient of t is positive since for δ small enough, $\rho_2 < \rho_0$.

Computing $\rho = u_x$, we find that $\rho(0_-, t) = \rho_1$ is close to 1, which is reasonnable since the limited flux implies an accumulation of cars for $x < 0$ close to 0. On the contrary, $\rho(0_+, t) = \rho_2$ is close to 0 since the flux of cars is limited, therefore only a few cars go through $x = 0$. Figure 18.1 gives a typical picture when $\rho_0 > 1/2$.

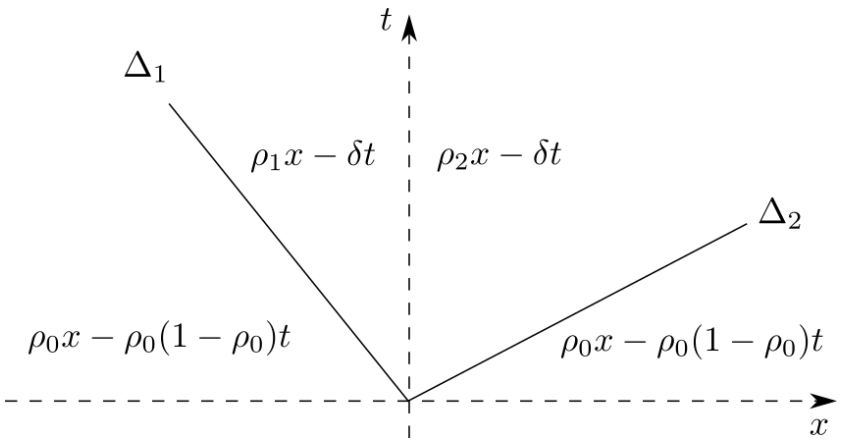


Figure 18.1: The solution u

We conclude this section by mentioning the case of moving constraints

$$f\big(\rho(y(t), t)\big) - \dot{y}(t)\rho(y(t), t) \leq g(t) \ ,$$

where y and g are given functions. Looking at the new function

$$w(x, t) = u(x + y(t), t) \ ,$$

we end up being in an analogous situation where again the theory of the previous chapter applies. But, of course, we also use a completely formal argument to connect this problem with the HJ-one.

Chapter 19

Further Discussions and Open Problems

As the title indicates, the aim of this chapter is both to summarize and comment the results we have provided in the case of co-dimension 1 discontinuities in Part II and III but also to mention puzzling questions.

Let us first examine the three approaches we have described.

The first one, using *Ishii's notion of viscosity solutions*, has the advantage to be very stable and universal in the sense that it can be formulated for any type of Hamiltonians, convex or not. But Chapter 7 shows that it has poor uniqueness properties in the present situation. In the simple case of the optimal control framework we have considered, with a discontinuity on an hyperplane $\mathcal{H}$ and with perhaps a specific control on $\mathcal{H}$, we are able to identify the minimal solution ($\mathbf{U}^-$) and the maximal solution ($\mathbf{U}^+$): if $\mathbf{U}^-$ is a natural value-function providing the minimal cost over all possible controls, $\mathbf{U}^+$ completely ignores some controls and in particular all the specific control on $\mathcal{H}$.

Why can $\mathbf{U}^+$ be an Ishii viscosity solution of the Bellman Equations anyway? The answer is that the Ishii subsolution condition on $\mathcal{H}$ is not strong enough in order to force the subsolutions to see all the particularities of the control problem on $\mathcal{H}$. This generates unwanted (or not?) subsolutions. We point out that, as all the proofs of Chapter 7 show, there is a complete disymmetry between the sub and supersolutions properties in this control setting: this fact is natural and well-known due to the form of the problem but it is accentuated in the discontinuous framework.

This lack of uniqueness properties for Ishii viscosity solutions leads to consider different notions of solutions but, in some interesting applications, one may recover

this uniqueness since $\mathbf{U}^- = \mathbf{U}^+$. We point out Lemma 10.3.1 below which provides a condition under which $H_T = H_T^{\text{reg}}$ and therefore $\mathbf{U}^- = \mathbf{U}^+$. This condition is formulated directly on the Hamiltonians and can sometimes be easy to check (see for example, Section 30.5).

In the *Network Approach*, one can either use the notion of *flux-limited solutions* or the notion of *junction viscosity solutions*. The first one is particularly well-adapted to control problems and has the great advantage to reinforce the subsolutions conditions on $\mathcal{H}$ and, through the flux-limiter, to allow to consider various control problems at the same time by just varying this flux-limiter. The value-functions $\mathbf{U}^-$ and $\mathbf{U}^+$ are reinterpreted in this framework as value-functions associated to particular flux-limiters.

But we are very far from the universality of the definition of viscosity solutions since this "max-max" definition in the case of convex Hamiltonians has to be replaced by a "min-min" one in the case of concave ones, and it has no analogue for general ones. On the other hand, this notion of solution is less flexible in terms of stability properties compared to Ishii solutions.

The notion of *junction viscosity solution* tries to recover all the good properties of Ishii solutions for general Hamiltonians: it is valid for any kind of "viscosity solutions compatible" junction conditions, it is stable and the Lions-Souganidis proof (even if there are some limitations in Theorem 16.3.5) is the only one which is valid for general Hamiltonians with Kirchhoff's boundary conditions. Though this approach is not as well-adapted to control problems as the flux-limiter one, it gives however a common formulation for problems when the controller wants to minimize some cost (which leads to convex Hamiltonians) or maximize it (which leads to concave Hamiltonians).

The Kirchhoff boundary condition is one of the most natural "junction condition" in the networks theory but a priori, it has no connection with control problems. However, as it is shown by Proposition 17.3.1 together with Theorem 15.4.4, this boundary condition is associated $\mathbf{U}^+$. The explanation is maybe in the next paragraph.

In fact, the main interest of the approach by junction solution, using the Lions-Souganidis comparison result, is to provide the convergence of the vanishing viscosity method in the most general framework (with the limitations of Theorem 16.3.5), without using some convexity or quasi-convexity assumption on the Hamiltonians. In the convex setting, we have several proofs of the convergence to $\mathbf{U}^+$ which shows that it is the most stable value-function if we add a stochastic noise on the dynamic.

In the next parts, we examine stratified solutions in $\mathbb{R}^N$ or in general domains, *i.e.* essentially the generalization of $\mathbf{U}^-$ which we aim at characterizing as the unique solution of a suitable problem with the right viscosity inequalities. And we will emphasize

the (even more important) roles of the subsolution inequalities, normal controllability, tangential continuity...etc. But we will not consider questions related to $\mathbf{U}^+$ and the vanishing viscosity method, even if some of these questions are really puzzling.

This analysis generates a lot of questions.

The first one may concerns the limitations due to the assumptions of Theorem 16.3.5: it is not completely clear that **(TC-s)** is really necessary; maybe a different proof, avoiding the tangential regularization, can handle general Hamiltonians without this superfluous hypothesis.

All the other questions concern the extensions to higher codimension discontinuities of the notions of flux-limited and junction viscosity solutions. Clearly the first step should be to have the right space of test-functions (like $\mathrm{PC}^1(\mathbb{R}^N \times [0,T_f])$ above). It is not very difficult to guess what this space could be: in the stratified case, i.e. if the discontinuities for a stratification $\mathbb{M} = (\mathbf{M}^k)_k$, one may perhaps use continuous functions those restrictions to each $\mathbf{M}^k$ are C^1 and with derivatives which have continuous extensions to $\overline{\mathbf{M}^k}$. To write that is one point, to make a concrete proof is an other one.

We point out anyway how Lemma 5.4.1 is closely related to flux-limited solutions by looking at trajectories which either leave $\mathcal{M}$ (suggesting a $H_1^+ - H_2^-$-type inequality) or stay on $\mathcal{M}$. It seems clear that a pde analogue of this lemma should exists and allow to obtain a comparison result for—at least—HJB-equations by a pure pde method, even in the stratified case of Part IV.

Finally we come back to the question which is clearly the most puzzling for us: one of the main result of this part is the convergence of the vanishing viscosity method to $\mathbf{U}^+$, the maximal viscosity subsolution in the control framework. What is the analogue of this result in the case of higher codimension discontinuities? Is the convergence to the maximal viscosity subsolution always true? And of course, can we identify this maximal viscosity subsolution in the control framework via an explicit formula?

Part IV

General Discontinuities: Stratified Problems

Chapter 20

Stratified Solutions

20.1 Introduction

Throughout Part IV, we consider Hamilton-Jacobi-Bellman Equations with more general discontinuities than hyperplanes. Those discontinuities can be of any codimension but with the restriction that they form a "Whitney stratification", *cf.* Section 3.3.

This generality is at the expense of considering only equations which are closely related to control problems (hence with convex Hamiltonians) but with the advantage that we do not have to deal with existence results: as can be expected, the value function of the associated control problem is a solution, even if this fact will not be completely obvious, *cf.* Chapter 21.

We always assume that we are in the "good framework for discontinuities": even if some of these assumptions can certainly be weakened, this general framework seems the most natural for us since, as we have already pointed out several times, the basic hypothesis we impose are useful—if not unavoidable—in the proof of *any* results.

This chapter is devoted to introduce the notion of *Stratified Solutions* in this framework and to present a comparison result which is valid under "natural" assumptions. We also show that the stratified solution corresponds to the minimal Ishii supersolution (see Section 21.2).

We give here two notions of stratified solution: a *weak* one and a *strong* one, the strong notion involving additional inequalities with respect to the weaker one. The difference between these notions can be understood in a better way after reading Section 13.1: for the strong one, we impose the $\mathbb{F}_* \leq 0$-inequality on the discontinuities while, for the weak one, we just impose "tangential inequalities". Each notion may

have a specific interest, in particular for stability results but also for further developments, but they turn out to be the same in the "good framework" which we always use, see ($\mathbf{H_{BA-SF}}$) below (see Section 20.5).

More concretely, we are given a general HJB Equation of the form

$$\mathbb{F}(x,t,U,DU) = 0 \quad \text{in} \quad \mathbb{R}^N \times [0,T_f]\,, \tag{20.1}$$

where $DU = (D_xU, D_tU)$ and

$$\mathbb{F}(x,t,r,p) := \sup_{(b,c,l)\in \mathbf{BCL}(x,t)} \big\{ -b\cdot p + cr - l \big\}\,, \tag{20.2}$$

where $\mathbf{BCL} : \mathbb{R}^N \times [0,T] \to \mathbb{R}^{N+3}$ is a set-valued map (*cf.* Section 4.1). And we define the initial Hamiltonian as

$$\mathbb{F}_{init}(x,r,p_x) = \sup_{((b^x,0),c,l)\in \mathbf{BCL}(x,0)} \big\{ -b^x\cdot p_x + cr - l \big\}\,. \tag{20.3}$$

The fundamental assumptions we make in this part are the following

($\mathbf{H_{BA-SF}}$) *Basic Assumptions on the Stratified Framework*

(*i*) There exists a stratification $\mathbb{M} = (\mathbf{M}^k)_{k=0\dots(N+1)}$ of $\mathbb{R}^N \times (0,T_f)$ such that $(x,t) \mapsto \mathbb{F}(x,t,r,p)$ is continuous on $\mathbf{M}^{N+1}$ and may be discontinuous on $\mathbf{M}^0 \cup \mathbf{M}^1 \cup \cdots \cup \mathbf{M}^N$. Moreover $(0_{\mathbb{R}^N},1) \notin (T_{(x,t)}\mathbf{M}^k)^\perp$ for any $(x,t) \in \mathbf{M}^k$ and for any $k = 1...N$ [(1)]. In the same way, there exists a stratification $\mathbb{M}_0 = (\mathbf{M}_0^k)_{k=0\dots N}$ of $\mathbb{R}^N$ such that the Hamiltonian $x \mapsto \mathbb{F}_{init}(x,r,p_x)$ is continuous on $\mathbf{M}_0^N$ and may be discontinuous on $\mathbf{M}_0^0 \cup \mathbf{M}_0^1 \cup \cdots \cup \mathbf{M}_0^{N-1}$.

(*ii*) The "good framework for HJB Equations with discontinuities" holds for Equation (20.1) in $\mathcal{O} = \mathbb{R}^N \times (0,T_f)$ associated to the stratification $\mathbb{M}$.

(*iii*) The "good framework for HJB Equations with discontinuities" holds for the equation $\mathbb{F}_{init} = 0$ in $\mathcal{O} = \mathbb{R}^N$, associated to the stratification $\mathbb{M}_0$.

We recall that the assumptions for a "Good Framework for HJ Equations with Discontinuities" are that ($\mathbf{H_{BCL}}$), **(TC-BCL)** and **(NC-BCL)** hold. We refer to Section 5.5 where the connections with Hamiltonian assumptions **(Mon)**, **(TC)**, **(NC)** are described.

[(1)] This assumption, whose aim is to avoid "flat part" of $\mathbf{M}^k$ in time, will be redundant to the normal controllability assumption in $\mathbb{R}^N \times (0,T_f)$.

20.2 Definition and comparison principle

In order to state a definition, we introduce Hamiltonians $\mathbb{F}^k$, defined as follows: if $(x,t) \in \mathbf{M}^k$, $r \in \mathbb{R}$ and $p \in T_{(x,t)}\mathbf{M}^k$, we set

$$\mathbb{F}^k(x,t,r,p) := \sup_{\substack{(b,c,l)\in \mathbf{BCL}(x,t) \\ b\in T_{(x,t)}\mathbf{M}^k}} \big\{ -b\cdot p + cr - l \big\}. \tag{20.4}$$

Similarly, for $t = 0$ we define

$$\mathbb{F}^k_{init}(x,r,p_x) := \sup_{\substack{((b^x,0),c,l)\in \mathbf{BCL}(x,0) \\ b^x\in T_x\mathbf{M}^k_0}} \big\{ -b^x\cdot p_x + cr - l \big\}. \tag{20.5}$$

We may also use these definitions for $p \in \mathbb{R}^{N+1}$ or $p_x \in \mathbb{R}^N$ since it is clear that there is no contribution from the $(T_{(x,t)}\mathbf{M}^k)^\perp$ or $(T_{(x,t)}\mathbf{M}^k_0)^\perp$ part of p or p_x.

In the framework of Part II, as the reader may guess, we have $\mathbf{M}^N = \mathcal{H}\times(0,T_f)$, $\mathbf{M}^{N+1} = (\Omega_1\cup\Omega_2)\times(0,T_f)$ and H_T is exactly $\mathbb{F}^N$, while $\mathbf{M}^0, \cdots \mathbf{M}^{N-1} = \emptyset$.

In the sequel, the notation (HJB-S) refers to problem (20.1), seen in the context of stratified solutions, that we detail below. The notion of stratified supersolution, denoted by (S-Super), is nothing but the usual Ishii supersolution definition involving $\mathbb{F}^* = \mathbb{F}$. On the other hand, we introduce two notions of weak and strong subsolutions, respectively denoted by (w-S-Sub) and (s-S-Sub), recalling that, because of $(\mathbf{H_{BA-SF}})$, the Hamiltonians $\mathbb{F}^k, \mathbb{F}^k_{init}$ are continuous for all k.

Definition 20.2.1 — *Stratified sub and supersolutions of* (HJB-S).

1. — (S-Super): *A locally bounded function $v : \mathbb{R}^N\times[0,T_f[\to\mathbb{R}$ is a stratified supersolution of* (HJB-S) *if it is an Ishii supersolution of* (20.1).

2. — (w-S-Sub): *A locally bounded function $u : \mathbb{R}^N\times[0,T_f[\to\mathbb{R}$ is a* weak *stratified subsolution of* (HJB-S) *if*

(a) *for any $k = 0,...,(N+1)$, u^* is a viscosity subsolution of*

$$\mathbb{F}^k\big(x,t,u^*,Du^*\big) \le 0 \quad \text{on } \mathbf{M}^k,$$

(b) *similarly, for $t = 0$, and $k = 0..N$, $u^*(x,0)$ is a viscosity subsolution of*

$$\mathbb{F}^k_{init}(x,u^*(x,0),D_xu^*(x,0)) \le 0 \quad \text{on } \mathbf{M}^k_0.$$

3. — (s-S-Sub): *A locally bounded function* $u : \mathbb{R}^N \times [0, T_f[\to \mathbb{R}$ *is a* strong *stratified subsolution of* (HJB-S) *if it is a* (w-S-Sub) *and satisfies additionally*

(a) $\mathbb{F}_*\big(x, t, u^*, Du^*\big) \leq 0 \quad$ *in* $\mathbb{R}^N \times (0, T_f)$,

(b) $(\mathbb{F}_{init})_*(x, u^*(x, 0), D_x u^*(x, 0)) \leq 0 \quad$ *in* $\mathbb{R}^N$.

4. — *A weak or strong stratified solution is a function which is both a* (S-Super) *and either a* (w-S-Sub) *or a* (s-S-Sub).

As usual, we will say that u is an η-strict (weak or strong) stratified subsolution if the various subsolution inequalities of the type "$\mathbb{G} \leq 0$" are replaced by a "$\mathbb{G} \leq -\eta$" inequality, the constant $\eta > 0$ being independent of (x, t).

The difference between weak and strong stratified solutions can be better understood through the discussion in Section 13.1, let us comment on this now.

The notion of "strong" stratified (sub)solution which was used in [35] is the easiest to interpret and maybe the more natural one from the viscosity solutions—or pde—point of view: Part II teaches us that a subsolution inequality is missing on the discontinuity $\mathcal{H} \times (0, T_f)$ in order to get a uniqueness property, and that adding the right one solves this problem. Therefore, it is not surprising to introduce the concept of "stratified solution" by super-imposing *additional* subsolutions inequalities on each set of discontinuity $\mathbf{M}^k$, including those at time $t = 0$. We point out that these additional subsolution conditions are real "$\mathbf{M}^k$-inequalities", *i.e.* they are obtained by looking at maximum points of $u^* - \varphi$ on $\mathbf{M}^k$ where φ is a test-function which is smooth on $\mathbf{M}^k$.

Of course, at first glance, removing the $\mathbb{F}_* \leq 0$-inequality does not seem to go in the right direction since some subsolution property is clearly missing. This is why, in [35], looking for "additional" (and not different) subsolution properties seems more natural; and Section 20.5 gives an other a posteriori justification of the interest of "strong" stratified subsolutions.

But, on the other hand, from the control point of view, the $\mathbb{F}_*$-inequality (as well as the $(\mathbb{F}_{init})_*$-one) is not so natural for reasons explained in Section 13.1. In that sense, a controller may understand in a better way the notion of "weak" stratified (sub)solution since all the inequalities have a clear sense in terms of control.

However, the defect of the notion of "weak" stratified (sub)solution is that it completely decouples the subsolution inequalities on the different $\mathbf{M}^k$ and by doing so, opens the possiblity to generate "artificial" subsolutions with uncorrelated values on these discontinuities.

The reconciliation of these two points of view involves in a central way the question of the regularity of subsolutions on the various $\mathbf{M}^k$. Spoiling the results of Section 20.5, we can summarize the answer as

$$\text{if } (\mathbf{H_{BA-SF}}) \text{ holds, then regular (w-S-Sub)} = \text{(s-S-Sub)}.$$

Roughly speaking, this means that under Assumption $(\mathbf{H_{BA-SF}})$, the $\mathbb{F}_*$ and $(\mathbb{F}_{init})_*$ inequalities are just used to obtain the regularity of subsolutions and, once this regularity is obtained, only the "weak" notion of stratified subsolution plays a role. Actually, as far as the regularity issue is concerned, $\mathbb{F}_*$ and $(\mathbb{F}_{init})_*$ are not playing any special role in the definition of (s-S-Sub); in fact they could be replaced by other Hamiltonians satisfying **(NC)** which is the real key point in order to get regularity. This remark may seem anecdotical here but it will play a real role in Part V.

In the rest of the book, since we will always use Assumption $(\mathbf{H_{BA-SF}})$, we will always be in a context where "weak" and "strong" stratified solutions coincide and we will use mainly *strong stratified solutions* (or we will mention the difference if there is one). Therefore the terminologies "sub/supersolution of (HJB-S)" or "stratified sub/supersolution of (20.1)" refer to the above definition combining (s-S-Sub) and (S-Super). We also sometimes use the terminology *Standard Stratified Problem* (SSP) referring to a problem in the form of (20.1)–(20.2)–(20.3) satisfying $(\mathbf{H_{BA-SF}})$, understanding sub and supersolutions in the strong sense of Definition 20.2.1.

The next section is devoted to show that (s-S-Sub) are regular, being more precise about the term "regular" itself. Then we will show that, under Assumption $(\mathbf{H_{BA-SF}})$, a comparison holds between regular (w-S-Sub) and (S-Super), which will allow us to prove also that regular (w-S-Sub) and (s-S-Sub) are the same.

20.3 The regularity of strong stratified subsolutions

We recall that the regularity of discontinuous functions is defined in Definition 3.4.1; based on it, we define the regularity of stratified subsolutions (weak or strong).

Definition 20.3.1 *Let $u : \mathbb{R}^N \times [0, T_f] \to \mathbb{R}$ be a u.s.c. weak or strong stratified subsolution of (20.1). We say that u is a regular subsolution if*

(*i*) *for any $k < N + 1$, u is ω^k-regular on $\mathbf{M}^k$, where*

$$\omega^k = \mathbf{M}^{k+1} \cup \mathbf{M}^{k+2} \cup \cdots \cup \mathbf{M}^{N+1}\,.$$

In other words, for any $(x,t) \in \mathbf{M}^k$ *and* $k < N+1$,

$$u(x,t) = \limsup\{u(y,s),\ (y,s) \to (x,t),\ (y,s) \in \omega^k\}\ . \tag{20.6}$$

Moreover, for the special case where $(x,t) \in \mathbf{M}^N$, *we also have*

$$\begin{aligned} u(x,t) &= \limsup\{u(y,s),\ (y,s) \to (x,t),\ (y,s) \in M_+^{(x,t)}\} \\ &= \limsup\{u(y,s),\ (y,s) \to (x,t),\ (y,s) \in M_-^{(x,t)}\}, \end{aligned} \tag{20.7}$$

where, for r *small enough,* $M_+^{(x,t)}, M_-^{(x,t)} \subset \mathbf{M}^{N+1} \cap B((x,t),r)$ *are the locally disjoint connected components of* $\big((\mathbb{R}^N \times (0,T_f)) \setminus \mathbf{M}^N\big) \cap B((x,t),r)$.

(*ii*) *For* $t = 0$, *for any* $k < N$, *the function* $x \mapsto u(x,0)$ *is* ω_0^k*-regular on* $\mathbf{M}_0^k$ *where*

$$\omega_0^k = \mathbf{M}_0^{k+1} \cup \mathbf{M}_0^{k+2} \cup \cdots \cup \mathbf{M}_0^N$$

and, in the special case where $(x,0) \in \mathbf{M}_0^{N-1}$, *we also have*

$$\begin{aligned} u(x,0) &= \limsup\{u(y,0),\ y \to x,\ (y,0) \in M_{0,+}^{(x,t)}\} \\ &= \limsup\{u(y,s),\ y \to x,\ (y,0) \in M_{0,-}^{(x,t)}\}, \end{aligned} \tag{20.8}$$

where, for $r > 0$ *small enough,* $M_{0,+}^{(x,t)}, M_{0,-}^{(x,t)} \subset \mathbf{M}_0^N \cap (B(x,r) \times \{0\})$ *are the locally disjoint connected components of* $\big((\mathbb{R}^N \times \{0\}) \setminus \mathbf{M}_0^{N-1}\big) \cap (B(x,r) \times \{0\})$.

The result on strong stratified subsolutions is the following.

Proposition 20.3.2 *Assume that* $(\mathbf{H_{BA-SF}})$ *holds. Then any strong stratified subsolution is regular.*

We leave the proof of this important proposition to the reader since it is a routine application of Proposition 3.4.2 after a suitable flattening of the $\mathbf{M}^k$ we are interested in, using the definition of a (RS).

Remark 20.3.3 *Again we insist on the very anecdotical role played by the Hamiltonians* $\mathbb{F}_*, (\mathbb{F}_{init})_*$ *and the* $\mathbb{F}_*, (\mathbb{F}_{init})_* \le 0$ *inequalities in this result: only Assumption* **(NC-BCL)** *is playing a key role. As a consequence, if we replace the* $\mathbb{F}_*, (\mathbb{F}_{init})_* \le 0$ *inequalities in the definition of* (s-S-Sub), *by some other ones like* $\mathbb{G}_*, (\mathbb{G}_{init})_* \le 0$ *with* $\mathbb{G}_*, (\mathbb{G}_{init})_*$ *satisfying* **(NC-BCL)**, *we would still get regular subsolutions. This remark is important when dealing with stability results: given a sequence* $(u_\varepsilon)_\varepsilon$ *of* (s-S-Sub) *for Hamiltonians* $(\mathbb{F}_\varepsilon)_\varepsilon$, *then* $\limsup^*\, u_\varepsilon$ *is a subsolution for* $\underline{\mathbb{F}} := \liminf_*\, \mathbb{F}_\varepsilon$.

With suitable assumptions, $\underline{\mathbb{F}}$ may satisfy **(NC-BCL)**, *while, in general, it is not clear that $\underline{\mathbb{F}} = (\overline{\mathbb{F}})_*$, where $\overline{\mathbb{F}} = \limsup^* \mathbb{F}_\varepsilon$. Hence a stability result for "strong" solutions is far more delicate to obtain than for "weak" solution.*

This remark yields another justification for introducing the notion of (w-S-Sub), *apart from being natural from the control viewpoint: while their needed regularity may come indeed from $\mathbb{F}_*, (\mathbb{F}_{init})_*$ inequalities—referring to strong solutions—it may also come from other inequalities coming for Hamiltonians satisfying* **(NC-BCL)**. *But also, it can derive from a particular situation where* **(NC-BCL)** *may not even be satisfied. We refer the reader to Section 26.4 where the connections between the regularity of subsolutions and the existence of certain viscosity inequalities on the boundary are discussed for state-constraint problems; such arguments can also be applied on $\mathbf{M}^k$ for standard stratified problems.*

As an example, the reader may consider the cases when all the dynamics are pointing toward $\mathbf{M}^k$ for some k: clearly **(NC-BCL)** *is not satisfied but the subsolutions are expected to be regular on $\mathbf{M}^k$. Under suitable assumptions, it should be possible to handle such cases by solving a problem on $\mathbf{M}^k$, and then by using the solution on $\mathbf{M}^k$ as a Dirichlet data for the problem in $\mathbb{R}^N \times (0, T_f)$.*

Despite we are not going to use it in this part, let us mention an immediate consequence of Proposition 20.3.2 which will be useful in the case of state-constraint problems.

Corollary 20.3.4 *Assume that* $(\mathbf{H_{BA-SF}})$ *holds and let $u : \mathbb{R}^N \times [0, T_f] \to \mathbb{R}$ be an u.s.c. strong stratified subsolution of* (20.1). *Then, for any $k < N + 1$, u is $\mathbf{M}^{N+1}$-regular on $\mathbf{M}^k$. Similarly for $t = 0$, for any $k < N$, the function $x \mapsto u(x, 0)$ is $\mathbf{M}_0^N$-regular on $\mathbf{M}_0^k$.*

Proof — We just sketch it in the case of $\mathbf{M}^k$ (*i.e.* for $t > 0$) because it is an easy consequence of Proposition 20.3.2 by backward induction. The case $t = 0$ is of course similar. Notice first that the result clearly holds for $k = N$ as a direct application of Proposition 20.3.2.

Now, assume that the result holds for $k = N, \cdots, (N-l)$ and take $(x, t) \in \mathbf{M}^{N-l-1}$. By Proposition 20.3.2, $u(x, t) = \limsup_\varepsilon u(x_\varepsilon, t_\varepsilon)$ with $(x_\varepsilon, t_\varepsilon) \in \mathbf{M}^{N-l} \cup \cdots \cup \mathbf{M}^{N+1}$.

But we can use the $\mathbf{M}^{N+1}$-regularity at $(x_\varepsilon, t_\varepsilon) \in \mathbf{M}^{N-l} \cup \cdots \cup \mathbf{M}^{N+1}$ to build a new sequence $(x'_\varepsilon, t'_\varepsilon) \in \mathbf{M}^{N+1}$ such that $u(x, t) = \limsup_\varepsilon u(x'_\varepsilon, t'_\varepsilon)$, implying that the induction works since the regularity result holds true for $k = N - l - 1$.

Q.E.D.

20.4 The comparison result

The main advantage of the concept of (weak and strong) stratified solutions is reflected in the comparison principle which we state now.

Theorem 20.4.1 — *Comparison result for stratified solutions.*

(*i*) *Assuming* ($\mathbf{H_{BA-SF}}$), *a comparison result holds between bounded regular* (w-S-Sub) *and bounded* (S-Super) *of Equation* (20.1).

(*ii*) *Assuming* ($\mathbf{H_{BA-SF}}$), *a comparison result holds between bounded* (s-S-Sub) *and bounded* (S-Super) *of Equation* (20.1).

Proof — Of course, the second part of the result is an immediate consequence of the first one because of Proposition 20.3.2.

Now we turn to the proof of the first part. Essentially the proof follows the main steps as the proof of Theorem 8.4.1 where it is shown that $\mathbf{U}^-$ is the unique solution of the Bellman Equation with the H_T-complemented inequality, which turns out to be an $\mathbf{M}^N$-inequality in the stratified setting. The only difference is that we have to use the more sophisticated form of Theorem 5.3.1.

Before describing these main steps, let us introduce some notations and perform some reductions. Let $u, v : \mathbb{R}^N \times [0, T_f[\to \mathbb{R}$ be respectively a bounded u.s.c. regular stratified subsolution and a bounded l.s.c. stratified supersolution of Equation (20.1). Our aim is to show that $u \le v$ in $\mathbb{R}^N \times [0, T_f[$[(2)].

This inequality is proved via two successive comparison results: first, one has to show that $u(x, 0) \le v(x, 0)$ in $\mathbb{R}^N$ which derives from a comparison result associated to the stationary equation $\mathbb{F}_{init} = 0$. Then, to prove that $u \le v$ in $\mathbb{R}^N \times]0, T_f[$, using a comparison for the evolution problem. The global strategy to obtain the comparison is the same in both cases and the changes to pass from one to the other are minor. Therefore we are going to provide the full proof only in the evolution case, admitting that $u(x, 0) \le v(x, 0)$ in $\mathbb{R}^N$.

Reductions — In order to prove these comparison results, we perform the following changes which are based on Assumption $(\mathbf{H_{BCL}})_{struct}$. We first use the by-now classical change

$$\bar{u}(x,t) = \exp(-Kt)u(x,t) \quad \text{and} \quad \bar{v}(x,t) = \exp(-Kt)v(x,t)\ ,$$

[(2)] The reason why we do not include T_f in the comparison will be clarified later on.

which, according to $(\mathbf{H_{BCL}})_{struct}$-$(ii)$, allows to reduce to the case when $c \geq 0$ for any $(b, c, l) \in \mathbf{BCL}(x, t)$ and $(x, t) \in \mathbb{R}^N \times [0, T_f]$. We may also assume that $c \geq 1$ if $-b^t \geq \underline{c}$ given by $(\mathbf{H_{BCL}})_{struct}$-$(iv)$. Notice that Assumption $(\mathbf{H_{BCL}})_{struct}$-$(ii)$ implies that $c \geq 0$ if $t = 0$ and $b^t = 0$, hence the Hamiltonian $\mathbb{F}_{init}(x, r, p_x)$ is increasing in r.

Next, adding $C_1 t + C_2$ to $\bar{u}$ and $\bar{v}$ and using $(\mathbf{H_{BCL}})_{struct}$-$(iv)$, we can assume without loss of generality that $l \geq \underline{c}$ for any element $(b, c, l) \in \mathbf{BCL}(x, t)$, for any $(x, t) \in \mathbb{R}^N \times [0, T_f]$.

In the comparison proofs, both for $t \in (0, T_f)$ and $t = 0$, we use in a key way that $c \geq 0$ and $l \geq \underline{c}$ for any element $(b, c, l) \in \mathbf{BCL}(x, t)$, any $(x, t) \in \mathbb{R}^N \times [0, T_f]$. Indeed, these properties together with the convexity of $\mathbb{F}$ and $\mathbb{F}_{init}$ allow us to reduce to the case of *strict subsolutions*, a favorable situation both in the stationary and evolution case.

Then, the comparison proof in $\mathbb{R}^N \times]0, T_f[$ is done in five steps.

Step 1: *Reduction to a local comparison result* **(LCR)**-*evol* – Using the assumptions on the **BCL**, one easily proves that

$$\psi(x, t) := -\delta(1 + |x|^2)^{1/2}$$

is a smooth stratified subsolution for δ is small enough, and even a η-strict subsolution for some $\eta > 0$: indeed, by $(\mathbf{H_{BCL}})_{fund}$ and the above reductions, if δ is small enough

$$-b \cdot (D_x \psi(x, t), D_t \psi(x, t)) + c\psi(x, t) - l \leq \delta M + 0 - \underline{c} < -\underline{c}/2.$$

Therefore changing $\bar{u}(x, t)$ into

$$\bar{u}_\mu(x, t) = \mu \bar{u}(x, t) + (1 - \mu)\psi(x, t) \ ,$$

for $0 < \mu < 1$, we are left to the case of comparing a $(1-\mu)\eta$-strict subsolution $\bar{u}_\mu$ and a supersolution v such that $\bar{u}_\mu(x, t) - v(x, t) \to -\infty$ when $|x| \to +\infty$. In other words, **(LOC1)** is satisfied. And so is **(LOC2)** by considering $\bar{u}_\mu(x, t) - \delta'(|x - \bar{x}|^2 + |t - \bar{t}|^2)$ where $(\bar{x}, \bar{t})$ is the point where we wish to check **(LOC2)**, $\delta' > 0$ being a small enough constant.

Thanks to Section 3.2, and since all the above reductions does not affect the regularity of the subsolution, we are reduced to prove local comparison results between regular subsolutions and supersolutions. For the sake of simplicity of notations, we just denote by u a strict regular stratified subsolution and v a stratified supersolution in the following.

Step 2: *Local comparison and argument by induction* – In order to prove **(LCR)**-evol we argue by induction. But using Theorem 5.3.1 we have to show, at the same time

a local comparison result not only for Equation (20.1) but also for equations of the type $\max(\mathbb{F}(x,t,w,Dw), w-\psi)=0$ where ψ is a continuous function. In fact, with the assumptions we use, there is no difference when proving **(LCR)**-evol for these two slightly different equations but, in order to be rigourous, we have to consider the "obstacle" one, which reduces to the $\mathbb{F}$–one if we choose $\psi(x)=K$ where the constant K is larger than $\max(||u||_\infty, ||v||_\infty)$.

For the sake of simplicity, we use below the generic expression ψ–Equation for the equation $\max(\mathbb{F}(x,t,w,Dw), w-\psi)=0$ and we will always assume that ψ is a continuous function, at least in a neighborhood of the domain we consider.

We are then reduced now to show that, for any $(\bar{x},\bar{t})\in\mathbb{R}^N\times(0,T_f)$:

$\mathbf{LCR}^\psi(\bar{x},\bar{t})$: *There exists* $r=r(\bar{x},\bar{t})>0$ *and* $h=h(\bar{x},\bar{t})\in(0,\bar{t})$ *such that, if* u *and* v *are respectively a strict regular stratified subsolution*[(3)] *and a stratified supersolution of some* ψ*–Equation in* $Q_{r,h}^{\bar{x},\bar{t}}$ *and if* $\max\limits_{\overline{Q_{r,h}^{\bar{x},\bar{t}}}}(u-v)>0$, *then*

$$\max_{\overline{Q_{r,h}^{\bar{x},\bar{t}}}}(u-v)\le \max_{\partial_p Q_{r,h}^{\bar{x},\bar{t}}}(u-v)\,,$$

where we recall that $\partial_p Q_{r,h}^{\bar{x},\bar{t}}$ *stands for the parabolic boundary of* $Q_{r,h}^{\bar{x},\bar{t}}$, *namely here* $\partial B(\bar{x},r)\times[\bar{t}-h,\bar{t}]\cup\overline{B(\bar{x},r)}\times\{\bar{t}-h\}$.

It is clear that $\mathbf{LCR}^\psi(\bar{x},\bar{t})$ holds in $\mathbf{M}^{N+1}$ since $\mathbb{F}^{N+1}$ and all the ψ–Equations satisfy all the property ensuring a standard comparison result in the open set $\mathbf{M}^{N+1}$; therefore $\mathbf{LCR}^\psi(\bar{x},\bar{t})$ is satisfied for r and h small enough—see Section 3.2.4.

In order that it holds for $(\bar{x},\bar{t})$ in any $\mathbf{M}^k$, we use a (backward) induction on k and more precisely, we introduce the property

$$\mathbf{P}(k):=\left\{\ \mathbf{LCR}^\psi(\bar{x},\bar{t})\ \textit{holds for any}\ (\bar{x},\bar{t})\in\mathbf{M}^k\cup\mathbf{M}^{k+1}\cup\cdots\cup\mathbf{M}^{N+1}\right\}.$$

Since $\mathbf{P}(N+1)$ is true, the core of the proof consists in showing that $\mathbf{P}(k+1)$ implies $\mathbf{P}(k)$ for $0\le k\le N$. To do so, we assume that $(\bar{x},\bar{t})\in\mathbf{M}^k$ and want to prove that $\mathbf{LCR}^\psi(\bar{x},\bar{t})$ holds provided $\mathbf{P}(k+1)$ is satisfied.

Step 3: *Regularization of the subsolution* – In order to apply the ideas of Section 3.4.3,

(3) According to the type of obstacle ψ we have to use in the proof of Theorem 5.3.1, we can assume *w.l.o.g.* that $u\le\psi-\delta$ for some $\delta>0$ in $\overline{Q_{r,h}^{\bar{x},\bar{t}}}$ and therefore a strict subsolution of $\mathbb{F}=0$ or of the ψ-Equation have essentially the same meaning.

we use the definition of a regular stratification which allows us to assume that $\bar{x} = 0$, $\bar{t} > 0$ and that we are in the case of a flat stratification in a neighborhood of $(\bar{x}, \bar{t}) = (0, \bar{t})$. We can also assume that $\mathbf{M}^k$ is the k-dimensional manifold parametrized by $(t, x_1, \cdots, x_{k-1})$, given by the equations $x_k = x_{k+2} =, \cdots, = x_N = 0$. This reduction is based on a $W^{2,\infty}$-change of variable in x which is done *only for the regularization step* and then we come back to the initial framework by the inverse of the change.

In the new setting, we keep the notations $\mathbb{F}$, $\mathbb{F}^j$ (for all j) and u. We just point out here that the t-variable is always part of the tangent variables which explains some restriction in the assumption concerning the behavior of $\mathbb{F}^l$ in t, *cf.* **(TC)**. Before proceeding, we emphasize the fact that, since r and h may depend on $(\bar{x}, \bar{t})$, we can handle without any difficulty the localization to reduce to the case of a flat stratification.

Since $(\bar{x}, \bar{t}) = (0, \bar{t}) \in \mathbf{M}^k$, we may assume that $Q^{x,t}_{r,h}$ only contains points of $\mathbf{M}^k, \mathbf{M}^{k+1}, \cdots, \mathbf{M}^{N+1}$ and, by assumption[(4)], we know that the subsolution u is ω^k-regular on $\mathbf{M}^k$ where $\omega^k = \mathbf{M}^{k+1} \cup \mathbf{M}^{k+2} \cup \cdots \cup \mathbf{M}^{N+1}$.

In order to regularize the subsolution and apply Proposition 3.4.4, we make the change of functions

$$\tilde{u}(x,t) = -\exp(-\alpha u(x,t)) \quad \text{and} \quad \tilde{v}(x,t) = -\exp(-\alpha v(x,t)) \ .$$

Indeed, a priori the initial $\mathbb{F}$ and $\mathbb{F}^j$ do not satisfy **(Mon)** while the new Hamiltonians obtained after this exponential change satisfy (**Mon**-u) if α is small enough. However, these new Hamiltonians are not necessarily convex in r and p.

Hence we first apply Proposition 3.4.4 to regularize the strict subsolution in $Q^{x,t}_{r,h}$, using the variables $y = (t, x_1, \cdots, x_{k-1})$, $z = (x_k, x_{k+2}, \cdots, x_N)$ and the $G((y,z), u, p)$ corresponding to $\max(\mathbb{F}_*(x,t,r,p), \mathbb{F}^j(x,t,r,p), u - \psi)$ but with the new Hamiltonian obtained with the above change of variable. This approximation by a sup-convolution in the tangential variables leads to a Lipschitz continuous subsolution which is semi-convex in the tangential variables y.

To proceed in order to obtain a sequence of strict stratified subsolutions which are C^1 in the variables $y = (t, x_1, \cdots, x_{k-1})$, there are two options: either we use Proposition 3.4.7 with Remark 3.4.8 since the new Hamiltonians satisfy (**Mon**-u) but are not necessarily convex in r or we make the change back and we use Lemma 3.4.6 to avoid assumption (**Mon**-u).

In any case, applying back the change of variables if necessary, and using that the above procedure gives a strict stratified subsolution in a neighborhood of $(\bar{x}, \bar{t}) =$

[(4)] or by using using Proposition 20.3.2 for the reader who is just interested in strong stratified subsolutions.

$(0,\bar t)$, we find that there exists $r,h>0$, $t'>\bar t$ and a sequence $(u_\varepsilon)_\varepsilon$ of subsolutions of the stratified problem in $Q_{r,h}^{\bar x,t'}$, which are in $C^0\left(\overline{Q_{r,h}^{\bar x,t'}}\right)\cap C^1\left(\mathbf{M}^k\cap\overline{Q_{r,h}^{\bar x,t'}}\right)$ and are all $(\eta/2)$-strict subsolutions of Equation (4.8) in $Q_{r,h}^{\bar x,t'}$. Moreover, because of Remark 3.2.5, we can assume as well that each u_ε is a $(\eta/2)$-strict subsolution on $Q_{r,h}^{\bar x,\bar t}$ [(5)].

Step 4: *Properties of the regularized subsolution* – Step 3 has two consequences

(*a*) for any $\varepsilon>0$ small enough, $\mathbb{F}^k(x,t,u_\varepsilon,Du_\varepsilon)\le -\eta/2<0$ on $\mathbf{M}^k\cap\overline{Q_{r,h}^{\bar x,\bar t}}$ in a classical sense;

(*b*) since u_ε is an $(\eta/2)$-strict (w-S-Sub) of the ψ-Equation in $\mathcal{O}:=Q_{r,h}^{\bar x,\bar t}\setminus\mathbf{M}^k$ and since **(LCR)** holds there because $\mathbf{P}(k+1)$ holds, we use the subdynamic programming principle for subsolutions (cf. Theorem 5.3.1) which implies that each u_ε satisfies an $(\eta/2)$-strict dynamic programming principle in $\mathcal{O}$[(6)].

These two properties allow us to have **(LCR)**-evol in $Q_{r,h}^{\bar x,\bar t}$ in the final step.

Step 5: *Performing the local comparison* – From the previous step we know that for each $\varepsilon>0$, u_ε satisfies the hypotheses of Lemma 5.4.1 and we deduce from this lemma that

$$\forall (y,s)\in Q_{r,h}^{\bar x,\bar t},\quad (u_\varepsilon-v)(y,s)<\max_{\partial_p Q_{r,h}^{\bar x,\bar t}}(u_\varepsilon-v)\,.$$

Using that $u=\limsup^*u_\varepsilon$, this yields a local comparison result (with inequality in the large sense) between u and v as $\varepsilon\to0$.

Therefore we have shown that $\mathbf{P}(k+1)$ implies $\mathbf{P}(k)$, which ends the proof.

Q.E.D.

Remark 20.4.2 *As it is clear in the above proof, the special structure of* $\mathbb{M}$ *does not play any role and time-dependent stratifications do not differ so much from time-independent ones. We remark anyway that a difference is hidden in the normal controllability assumption is that we cannot have a normal direction of the form* $(0_{\mathbb{R}^N},\pm1)$ *for* $\mathbf{M}^k$ *and this, for any* k.

[(5)] This regularization step cannot be done if $\bar t=T_f$: this is why the comparison may only be proved on $\mathbb{R}^N\times[0,T_f-\delta]$ for any $\delta>0$.

[(6)] We leave to the reader the careful checking that the proof of Theorem 5.3.1 uses only $\mathbf{P}(k+1)$ in $\mathcal{O}$ and never the $\mathbb{F}_*$-inequalities.

20.5 Regular weak stratified subsolutions are strong stratified subsolutions

As the title of the section indicates it, the main result is the

Proposition 20.5.1 *If* $(\mathbf{H_{BA-SF}})$ *holds, then any regular weak stratified subsolution is a strong stratified subsolution.*

It is a little bit surprising to see that, provided $(\mathbf{H_{BA-SF}})$ holds, the "unnatural" $\mathbb{F}_*, (\mathbb{F}_{init})_*$ inequalities necessarily hold for regular weak stratified subsolution. Besides, the rather indirect proof of Proposition 20.5.1 below—via the comparison result—confirms how artificial these inequalities are.

But, for the purpose of this book, this has a clear consequence: since $(\mathbf{H_{BA-SF}})$ is a basic assumption which is supposed to hold everywhere in this book, regular weak or strong stratified subsolutions make no difference. For this reason, in the sequel, we almost only use the notion of (s-S-Sub).

Proof — We only show that a regular weak stratified subsolution satisfies the $\mathbb{F}_*$ inequality, the $(\mathbb{F}_{init})_*$ one being similar.

Let u be a regular weak stratified subsolution and $(x,t) \in \mathbb{R}^N \times (0,T_f)$ be a strict local maximum point of $u-\phi$ where ϕ is a C^1-function in $\mathbb{R}^N \times [0,T_f]$. We may assume *w.l.o.g.* that $u(x,t) = \psi(x,t)$. If $\mathbb{F}_*(x,t,\phi(x,t),D\phi(x,t)) \leq 0$, we are done. Hence we assume by contradiction that $\mathbb{F}_*(x,t,\phi(x,t),D\phi(x,t)) = 2\delta > 0$.

But $\mathbb{F}_*$ being l.s.c. and ϕ being smooth, $\mathbb{F}_*(y,s,\phi(y,s),D\phi(y,s)) \geq \delta > 0$ in $Q^{x,t}_{r,h}$ for $r,h > 0$ small enough and we may also assume because of the strict local maximum point property that $u-\phi < 0$ on $\partial_p Q^{x,t}_{r,h}$. Since $\mathbb{F}_* \leq \mathbb{F}$, ϕ is a (S-Super) in $Q^{x,t}_{r,h}$ and the **(LCR)** for the equation which holds as a by-product of the proof of Theorem 20.4.1—choosing perhaps smaller r,h—leads to

$$0 = u(x,t) - \psi(x,t) \leq \max_{\partial_p Q^{x,t}_{r,h}} (u-\phi) < 0\ ,$$

i.e. a contradiction to the maximum point property. Hence $\mathbb{F}_*(x,t,\phi(x,t),D\phi(x,t)) \leq 0$ and the result is proved.

Q.E.D.

Chapter 21

Connections with Control Problems and Ishii Solutions

In this chapter we connect the notions of stratified solution to control problems and classical Ishii solutions. Given a set-valued map, we first identify the natural value function as the unique (strong) stratified solution of the associated Bellman equation. As can be expected, we prove that it corresponds to the minimal Ishii solution. We also present some partial results connecting Ishii and stratified subsolutions.

21.1 The value-function as a stratified solution

In Section 4.2, it is already shown that the value-function U defined by

$$U(x,t) := \inf_{\mathcal{T}(x,t)} \Big\{ \int_0^{+\infty} l\big(X(s), T(s)\big)\, e^{-D(s)}\, \mathrm{d}s \Big\}$$

is an Ishii supersolution of $\mathbb{F} = 0$, therefore it is a stratified supersolution. But in order to get the subsolution properties, the behaviour of the dynamic is playing a key role via Assumptions **(TC-BCL)** and **(NC-BCL)**.

In the sequel, we treat in details the subsolution properties on $\mathbf{M}^k$ $(0 \leq k \leq N+1)$, *i.e.* those for $t > 0$. The case $t = 0$ and the corresponding inequalities on the $\mathbf{M}_0^k$ follow readily from the same arguments.

Theorem 21.1.1 — *Subsolution Properties.*

Under Assumption $(\mathbf{H_{BA-SF}})$*, the value-function U is a regular weak stratified subsolution. More precisely,*

(*i*) *For any* $k = 0..N$, $U^* = (U|_{\mathbf{M}^k})^*$ *on* $\mathbf{M}^k$;

(*ii*) *for any* $k = 0..(N+1)$, U^* *is a Ishii subsolution of*

$$\mathbb{F}^k(x,t,U^*,DU^*) = 0 \quad \text{on } \mathbf{M}^k\,.$$

In this result, we recall again that, for $k = 0..N$, (*ii*) is a viscosity inequality for an equation *restricted to* $\mathbf{M}^k$, which means that if ϕ is a smooth function on $\mathbf{M}^k$ (or equivalently on $\mathbb{R}^N \times (0,T_f)$ by extension) and if $(x,t) \in \mathbf{M}^k$ is a local maximum point of $U^* - \phi$ on $\mathbf{M}^k$, then

$$\mathbb{F}^k(x,t,U^*(x,t),D\phi(x,t)) \leq 0 \quad .$$

This is why point (*i*) is an important fact since it allows to restrict everything (including the computation of the u.s.c. envelope of U) to $\mathbf{M}^k$.

Of course, the case $k = N+1$ is particular since $\mathbb{F}^{N+1} = \mathbb{F}_* = \mathbb{F}$ on $\mathbf{M}^{N+1}$, because $(\mathbf{H_{BA-SF}})$ implies that $\mathbb{F}$ is continuous on $\mathbf{M}^{N+1}$.

This result already shows that U^* is a weak stratified subsolution of the problem. But the reader has probably already understood that, since **(NC-BCL)** holds, U^* is also going to be a strong stratified subsolution by Proposition 20.5.1.

Proof — Since all the results are local, we can assume *w.l.o.g.* that we are in the case of an (AFS), a complete proof being obtained via a simple change of variable.

(a) *Proof of* (*i*) — For $k = 0..N$, we consider $(x,t) \in \mathbf{M}^k$ and a sequence $(x_\varepsilon,t_\varepsilon) \to (x,t)$ such that

$$U^*(x,t) = \lim_{\varepsilon\to 0} U(x_\varepsilon,t_\varepsilon)\,.$$

We have to show that we can assume that $(x_\varepsilon,t_\varepsilon) \in \mathbf{M}^k$. In all the sequel, we assume that $\varepsilon > 0$ is small enough so that all the points remain in $B((x,t),r)$, the ball given by **(NC-BCL)**.

Given a sequence $(x_\varepsilon,t_\varepsilon)$, we build a sequence $(\bar{x}_\varepsilon,\bar{t}_\varepsilon)_\varepsilon$ such that $(\bar{x}_\varepsilon,\bar{t}_\varepsilon) \in \mathbf{M}^k$ for any ε and with $U^*(x,t) = \lim_\varepsilon U(\bar{x}_\varepsilon,\bar{t}_\varepsilon)$. Notice that of course if $(x_\varepsilon,t_\varepsilon)$ already belongs to $\mathbf{M}^k$ we can set $(\bar{x}_\varepsilon,\bar{t}_\varepsilon) = (x_\varepsilon,t_\varepsilon)$ so let us assume that this is not the case.

By Theorem 4.3.3, for any solution (X,T,D,L) of the differential inclusion starting from $(x_\varepsilon,t_\varepsilon,0,0)$ and any $\theta > 0$,

$$U(x_\varepsilon,t_\varepsilon) \leq \int_0^\theta l\big(X(s),T(s)\big)\exp(-D(s))ds + U\big(X(\theta),T(\theta)\big)\exp\big(-D(\theta)\big)\,.$$

Now let $(\tilde{x}_\varepsilon,\tilde{t}_\varepsilon)$ be the projection of $(x_\varepsilon,t_\varepsilon)$ onto $\mathbf{M}^k$ and let us denote by n_ε the vector $n_\varepsilon := (\tilde{x}_\varepsilon,\tilde{t}_\varepsilon) - (x_\varepsilon,t_\varepsilon) \in V_k^\perp$. Using **(NC-BCL)** we know that, for any $(y,s) \in$

$B((x,t),r)$, there exists $b \in \mathbf{B}(y,s)$ with normal component $2^{-1}\delta.n_\varepsilon|n_\varepsilon|^{-1} \in B(0,\delta)$. More precisely, there exist b which can be decomposed as $b = b_\top + b_\perp$ with $b_\top \in V_k$, $b_\perp \in V_k^\perp$, and $b_\perp := 2^{-1}\delta.n_\varepsilon|n_\varepsilon|^{-1}$. We denote by $\widetilde{\mathbf{BCL}}(y,s)$ the set of all $(b,c,l) \in \mathbf{BCL}(y,s)$ for which b is of this form.

Clearly, the map $(x,t) \mapsto \widetilde{\mathbf{BCL}}(y,s)$ has compact, convex images and is upper semi-continuous. Solving the associated differential inclusion starting from $(x_\varepsilon,t_\varepsilon)$, we get a solution (X,T,D,L) such that $(X(s),T(s)) \in B((x,t),r)$ for s small enough, independent of ε. Moreover, for $s_\varepsilon = 2|n_\varepsilon|/\delta$,

$$(\bar{x}_\varepsilon,\bar{t}_\varepsilon) = (X(s_\varepsilon),T(s_\varepsilon)) = (x_\varepsilon,t_\varepsilon) + s_\varepsilon b = (\tilde{x}_\varepsilon + y_\varepsilon, \tilde{t}_\varepsilon + \tau_\varepsilon)\ ,$$

where $(y_\varepsilon,\tau_\varepsilon) \in V_k$, $|(y_\varepsilon,\tau_\varepsilon)| = O(|\tilde{x}_\varepsilon - x_\varepsilon| + |\tilde{t}_\varepsilon - t_\varepsilon|)$. Indeed, $s_\varepsilon b_\perp = n_\varepsilon$ and therefore $(x_\varepsilon,t_\varepsilon) + s_\varepsilon b \in (\tilde{x}_\varepsilon,\tilde{t}_\varepsilon) + V_k$.

Therefore, $(\bar{x}_\varepsilon,\bar{t}_\varepsilon) \in \mathbf{M}^k$ by Lemma 3.3.3 and using the Dynamic Programming Principle above with $\theta = s_\varepsilon$ yields

$$U(x_\varepsilon,t_\varepsilon) \le O(s_\varepsilon) + U\big(X(s_\varepsilon),T(s_\varepsilon)\big)\exp(-D(s_\varepsilon)) = O(s_\varepsilon) + U\big(\bar{x}_\varepsilon,\bar{t}_\varepsilon\big)(1+O(s_\varepsilon))\,.$$

Finally since $s_\varepsilon \to 0$ as $\varepsilon \to 0$, we deduce that

$$\limsup_{\varepsilon\to 0} U\big(\bar{x}_\varepsilon,\bar{t}_\varepsilon\big) \ge \limsup_{\varepsilon\to 0}\ U(x_\varepsilon,t_\varepsilon) = U^*(x,t)\ ,$$

which shows (i) since $\big(\bar{x}_\varepsilon,\bar{t}_\varepsilon\big) \in \mathbf{M}^k$.

(b) *Proof of* (ii) – As we already mentioned above, the result for $k = N+1$ is given by Theorem 4.3.5. Hence it remains to examine the cases $k = 0..N$.

For such k, let ϕ be a smooth function on $\mathbf{M}^k$ and let $(x,t) \in \mathbf{M}^k$ be a local maximum point of $U^* - \phi$ on $\mathbf{M}^k$, we have to show that

$$\mathbb{F}^k(x,t,U^*(x,t),D\phi(x,t)) \le 0\ \ .$$

Using (i), we can consider a sequence $(x_\varepsilon,t_\varepsilon) \in \mathbf{M}^k$ such that $U(x_\varepsilon,t_\varepsilon) \to U^*(x,t)$ and use Theorem 4.3.3, which implies

$$U(x_\varepsilon,t_\varepsilon) \le \int_0^\theta l\big(X(s),T(s)\big)\exp(-D(s))ds + U\big(X(\theta),T(\theta)\big)\exp\big(-D(\theta)\big)\ , \quad (21.1)$$

for any solution (X,T,D,L) of the differential inclusion starting from $(x_\varepsilon,t_\varepsilon,0,0)$.

But now we can use the result of Lemma 5.5.2: for any $(b,c,l) \in \mathbf{BCL}^k(x,t)$ and $\eta > 0$, $\mathbf{BCL}^k(y,s) \cap B((b,c,l),\eta) \neq \emptyset$ if (y,s) is close enough to (x,t). Solving locally

the differential inclusion with $\mathbf{BCL}^k(y,s) \cap B((b,c,l),\eta)$ instead of $\mathbf{BCL}$ and using the associated solution in (21.1) allows to obtain the viscosity inequality for (b,c,l) as in the standard case.

Since this is true for any $(b,c,l) \in \mathbf{BCL}^k(x,t)$, the result is complete.

Q.E.D.

An immediate consequence of Theorem 21.1.1—and of its analogue for $t = 0$—, using also Theorem 20.4.1 and Proposition 20.5.1, is the

Corollary 21.1.2 *Under the assumptions of Theorem 21.1.1, the value-function U is continuous in $\mathbb{R}^N \times [0,T_f[$ and is the unique (strong) stratified solution of the Bellman Equation.*

21.2 Stratified solutions and classical Ishii viscosity solutions

The aim of this section is to compare the two notions of solutions, in particular under the assumptions of Theorem 20.4.1. Of course, (weak or strong) stratified solutions and classical Ishii viscosity solutions can coincide only when the latter are uniquely identified and the case of codimension-1 discontinuities shows that this clearly requires some additional assumptions, *cf.* Part II.

We present two kind of results in this section: the first one, which is just an easy remark, is that the stratified solution is the minimal Ishii (super)solution; the second one provides a particular case where we can show that Ishii subsolutions are (strong) stratified subsolutions.

21.2.1 The stratified solution as the minimal Ishii solution

Before addressing the question of identifying conditions under which classical Ishii viscosity solution and stratified solution coincide, we begin with an easy consequence of Theorem 20.4.1.

Corollary 21.2.1 *If $(\mathbf{H_{BA-SF}})$ holds, the unique stratified solution of Equation (20.1) is also the minimal Ishii viscosity supersolution and solution of Equation (20.1).*

The proof of this result is obvious since Ishii viscosity supersolutions and stratified supersolutions are the same; therefore Corollary 21.2.1 is a straightforward application of Theorem 20.4.1.

In the case of codimension-1 discontinuities (see Part II), Corollary 21.2.1 implies that $\mathbf{U}^-$ is the unique stratified solution and actually the reader can check that Theorem 8.4.1 is nothing but the first uniqueness (and comparison) result for a stratified solution in this book, H_T providing the subsolution inequality on $\mathbf{M}^N$.

21.2.2 Ishii subsolutions as stratified subsolutions

The next very natural question is: under which conditions can it be proved that a classical Ishii viscosity subsolution is a stratified (strong) subsolution? Of course, this question is meaningful only for subsolutions since the supersolutions are the same. Notice that when this is the case, we conclude that uniqueness holds for the Ishii formulation since the unique stratified subsolution is also the unique classical Ishii viscosity solution.

This question then appears as a generalization in the direction of looking for conditions which ensure, in one dimension, that $\mathbf{U}^+ \equiv \mathbf{U}^-$. A partial but rather general answer is given by Lemma 10.3.1. The reader can check on examples that this lemma is of a rather simple use as it can be seen on Chapter 30.

In the more complicated framework of stratified problems, we are also looking for simple conditions which can easily be checked for more general types of discontinuities. The ones we propose in this section are unavoidably rather restrictive but they cover anyway some interesting cases as we will illustrate below by several examples.

As in the previous section, we treat only the case of the subsolution inequalities on the $\mathbf{M}^k$, *i.e.* those for $t > 0$, but similar arguments gives the same results for the $\mathbf{M}_0^k$ if $t = 0$.

A way to obtain the $\mathbb{F}^k$-subsolution inequalities by using the Ishii subsolution condition $\mathbb{F}_* \leq 0$ consists in

(i) "shifting" $\mathbf{M}^k$ into some $\mathbf{M}_\varepsilon^k \subset \mathbf{M}^{N+1}$ such that $\mathbf{M}_\varepsilon^k \to \mathbf{M}^k$;

(ii) using the Ishii inequality $\mathbb{F}_* \leq 0$ on $\mathbf{M}_\varepsilon^k$, since it is contained in $\mathbf{M}^{N+1}$, in order to obtain a $\mathbb{F}_\varepsilon^k \leq 0$-inequality on $\mathbf{M}_\varepsilon^k$;

(iii) passing to the limit through a stability property to get $\mathbb{F}^k \leq 0$ on $\mathbf{M}^k$ as $\varepsilon \to 0$.

In order to perform this stability strategy in a quite general way we need to take into account the fact that $\mathbf{M}^k$ may be approached by several $\mathbf{M}_\varepsilon^{k,i} \subset \mathbf{M}^{N+1}$, and of course we only need to choose one which yields the result. We refer to the example below to better understand this remark. Notice that using stability in the stratified setting is by no means a routine exercise as it is in the classical continuous case, *cf.* Chapter 22, but here we use a "simple" stability result since the $\mathbb{F}_\varepsilon^k$-inequalities are

set on the $\mathbf{M}^k_\varepsilon$, which are just copies of $\mathbf{M}^k$ and which converge to $\mathbf{M}^k$ in a strong enough way.

To be more precise, let us assume that $\mathcal{O} \subset \mathbb{R}^{N+1}$ is an open set and that for $(x,t) \in \mathcal{O}$,

$$\mathbb{F}(x,t,r,p) = \sup_{(b,c,l)\in \mathbf{BCL}(x,t)} \big\{ -b\cdot p + cr - l \big\}$$

for some set-valued map $\mathbf{BCL} : \mathcal{O} \to \mathcal{P}(\mathbb{R}^{N+3})$.

Definition 21.2.2 *For $k = 0..N$, we say that $\mathbf{M}^k$ is locally $\mathbf{M}^{N+1}$-approached by a family of k-dimensional manifolds $(\mathbf{M}^{k,i}_\varepsilon)_{i\in\mathcal{I}}$ at $(x,t) \in \mathbf{M}^k$ if there exists $r > 0$ such that $B((x,t),\bar r) \subset \mathcal{O}$ and for all $i \in \mathcal{I}$,*

$$\begin{cases} \mathbf{M}^{k,i}_\varepsilon \cap B((x,t),\bar r) \subset \mathbf{M}^{N+1}\,, \\ \mathbf{M}^{k,i}_\varepsilon \cap B((x,t),\bar r) \to \mathbf{M}^k \cap B((x,t),\bar r) \text{ in the } C^1\text{-topology}\,. \end{cases}$$

In such a situation, for $(y,s) \in \mathbf{M}^{k,i}_\varepsilon$, we set

$$\mathbb{F}^{k,i}_\varepsilon(y,s,r,p) := \sup_{\substack{(b,c,l)\in \mathbf{BCL}(y,s) \\ b\in T_{(y,s)}\mathbf{M}^{k,i}_\varepsilon}} \big\{ -b\cdot p + cr - l \big\}\,.$$

Proposition 21.2.3 *Let us assume that $(\mathbf{H_{BA-SF}})$ holds and that u be an u.s.c. classical Ishii viscosity subsolution of*

$$\mathbb{F}(x,t,u,Du) \le 0 \quad \text{in } \mathcal{O} \subset \mathbb{R}^{N+1}\,.$$

We also assume that $\mathbf{M}^k$ is $\mathbf{M}^{N+1}$-approached by $(\mathbf{M}^{k,i}_\varepsilon)$ at (x,t). If

$$\mathbb{F}^k \le \max_{i\in\mathcal{I}} \big(\liminf \mathbb{F}^{k,i}_\varepsilon \big) \text{ in } \mathbf{M}^k \cap B((x,t),\bar r)\,, \tag{21.2}$$

then

$$\mathbb{F}^k(x,t,u,Du) \le 0 \text{ in } \mathbf{M}^k \cap B((x,t),\bar r)\,.$$

The idea of this proposition is very simple and follows the above described program: if, on the "shifted" $\mathbf{M}^{k,i}_\varepsilon$, the (approximate) $\mathbb{F}^{k,i}_\varepsilon$-inequalities follow from the $\mathbb{F}_*$-one, we can conclude by "stability" that $\liminf \mathbb{F}^{k,i}_\varepsilon \le 0$ on $\mathbf{M}^k$. Then by using any possible choice of these "shifted" families $(\mathbf{M}^{k,i}_\varepsilon)_{i,\varepsilon}$, it is enough to have (21.2) in order to conclude that the $\mathbb{F}^k$-inequality holds.

We point out that this result takes a simpler form in the (AFS) case since we can typically use manifolds $\mathbf{M}_\varepsilon^{k,i}$ which are nothing but $e_i^\varepsilon + \mathbf{M}^k$ for some suitable choice of $e_i^\varepsilon \in \mathbb{R}^{N+1}$, *i.e.* on some copy of $\mathbf{M}^k$ which is *included in* $\mathbf{M}^{N+1}$. Typically, we need at least one of these copies on each connected components of $\mathbf{M}^{N+1} \setminus \mathbf{M}^k$.

From a control point-of-view, the interpretation of Proposition 21.2.3 is the following: the best strategy to stay on $\mathbf{M}^k$ is to use tangential dynamics which already exist in one of the connected components of $\mathbf{M}^{N+1}$, without combining incoming or outgoing dynamics coming from several of these connected components. Such situation clearly leads to $\mathbf{U}^+ = \mathbf{U}^-$ in the two-domains case.

We also insist on the fact that we have treated the case of $\mathbb{F}$ but an analogous result also holds for $\mathbb{F}_{init}$.

Proof — Using Proposition 22.2.1 later in this book, the $\mathbb{F}_\varepsilon^{k,i}$-inequalities on the $\mathbf{M}_\varepsilon^{k,i}$ are direct consequences of the $\mathbb{F}$-one on $\mathbf{M}^{N+1}$. Hence $\mathbb{F}_\varepsilon^{k,i} \leq 0$ on $\mathbf{M}_\varepsilon^{k,i} \cap B((x,t),\bar{r})$ and, by stability we deduce that $\liminf \mathbb{F}_\varepsilon^{k,i} \leq 0$ on $\mathbf{M}^k \cap B((x,t),\bar{r})$. This implies that $\max_i(\liminf \mathbb{F}_\varepsilon^{k,i}) \leq 0$ on $\mathbf{M}^k \cap B((x,t),\bar{r})$ and the result follows.

Q.E.D.

An interesting direct consequence is the

Corollary 21.2.4 *If the assumptions of Proposition 21.2.3 hold for any $(x,t) \in \mathbf{M}^k$ for $k = 1..N$, then any Ishii subsolution in $\mathbb{R}^N \times (0,T_f)$ is a stratified subsolution. As a consequence, Ishii sub and supersolutions are the same as stratified sub and supersolutions in $\mathbb{R}^N \times (0,T_f)$.*

We have decided to restrict Proposition 21.2.3 and this corollary to the domain $\mathbb{R}^N \times (0,T_f)$ but, of course, similar results can be obtained at $t = 0$ and for general stationary problems. We leave these generalization to the reader.

The example below shows a simple situation where the result can be applied, leading to uniqueness for the Ishii problem.

Example 21.1 — We consider the equation

$$u_t + a(x)|Du| = g(x) \quad \text{in } \mathbb{R}^2 \times (0,T_f)\ ,$$

where $a = a_i$ and $g = g_i$ in Ω_i where the Ω_i are in Figure 13.2 and the functions a_i, g_i are continuous. Of course, we assume that $a_i(x) \geq 0$ for any $x \in \Omega_i$, $1 \leq i \leq 4$.

(a) Let us first consider $\mathbf{M}^2$ and the part $\{x_1 = 0, x_2 > 0\}$. Here

$$\mathbb{F}^2(x,t,(p_x,p_t)) = p_t + \sup\ \{-(\theta a_1 v_1 + (1-\theta)a_2 v_2)\cdot p_x - (\theta g_1 + (1-\theta)g_2)\}\ ,$$

where the supremum is taken on all $|v_1|, |v_2| \leq 1$ and all $0 \leq \theta \leq 1$ such that $(\theta a_1 v_1 + (1-\theta) a_2 v_2) \cdot e_1 = 0$.

It is obvious that $\mathbb{F}^2$ can be computed by choosing v_1, v_2 such that $v_1 \cdot e_1 = v_2 \cdot e_1 = 0$, *i.e.* by taking dynamics which are in the direction of $\mathbf{M}^2$ and writing

$$-(\theta a_1 v_1 + (1-\theta) a_2 v_2).p_x - (\theta g_1 + (1-\theta) g_2) = \theta(-a_1 v_1 \cdot p_x - g_1) + (1-\theta)(-a_2 v_2 \cdot p_x - g_2) \ ,$$

it follows that

$$\mathbb{F}^2(x, t, (p_x, p_t)) = \max(p_t + a_1 |(p_x)_2| - g_1, p_t + a_2 |(p_x)_2| - g_2) \ ,$$

where $(p_x)_2$ is the second component of p_x, *i.e.* the tangential part of the gradient in space.

(b) Examining the condition to be checked for Proposition 21.2.3, we see that we can choose $\mathbf{M}^{2,i} = (-1)^{i+1} \varepsilon e_1 + \mathbf{M}^2$ and for $\mathbb{F}^{2,i}_\varepsilon$, we have

$$\mathbb{F}^{2,i}_\varepsilon(x + (-1)^{i+1} \varepsilon e_1, t, (p_x, p_t)) = p_t + a_i(x + (-1)^{i+1} \varepsilon e_1) |(p_x)_2| - g_i(x + (-1)^{i+1} \varepsilon e_1) \ ,$$

and $\mathbb{G}_i(x, t, (p_x, p_t)) = p_t + a_i(x) |(p_x)_2| - g_i(x)$. Therefore we have $\mathbb{F}^2 = \max(G_1, G_2)$, we can apply the result and of course the same property holds for the three other parts of $\mathbf{M}^2$.

(c) For $\mathbf{M}^1 := \{(0,0)\} \times (0, T_f)$, the checking is even simpler by considering $(0,0) \pm \varepsilon e_1 \pm \varepsilon e_2$, one can easily check that the $\mathbb{F}^1$ condition, *i.e.* $p_t - \min(g_i) \leq 0$, is satisfied.

Remark 21.2.5 *As the above example shows, the result of Proposition 21.2.3 is not very sophisticated but it has the advantage to be very simple to apply.*

21.3 Concrete situations that fit into the stratified framework

In this section, we consider stratified problems through a different point of view, maybe closer to concrete applications. We give general frameworks in which ($\mathbf{H_{BA-SF}}$), **(NC)**/**(NC-BCL)** and **(TC)**/**(TC-BCL)** are satisfied so that the connections between the control and pde approaches are satisfied.

In the following two subsections, we assume that we are given a general regular stratification $(\mathbf{M}^k)_k$ of $\mathbb{R}^N$ and each manifold $\mathbf{M}^k$ is written as the union of its connected components $\mathbf{M}^{k,j}$

$$\mathbf{M}^k = \bigcup_{j=1}^{J(k)} \mathbf{M}^{k,j} \ ,$$

where $J(k) \in \mathbb{N} \cup \{+\infty\}$.

21.3.1 A general control-oriented framework

Here we start from a collection of specific control problems on each $\mathbf{M}^{k,j}$.

The control problems — On each $\mathbf{M}^{k,j}$, we are given a space of control $A_{k,j}$ and functions $(b^{k,j}, c^{k,j}, l^{k,j})$ representing the dynamic, discount factor and cost for a control problem on $\mathbf{M}^{k,j}$. For the sake of simplicity, we assume that all these function are defined in $\mathbb{R}^N \times [0, T_f] \times A_{k,j}$ with the condition $b^{k,j}(x, t, \alpha_{k,j}) \in T_x\mathbf{M}^k$ for any $(x, t) \in \mathbf{M}^{k,j}$ and $\alpha_k \in A_k$ in order that the dynamic preserves $\mathbf{M}^{k,j}$ at least for a short time.

The Hamiltonians — If $(x, t) \in \mathbf{M}^{k,j}$, we introduce the associated Hamiltonian

$$\tilde{H}^{k,j}(x, t, r, p) := \sup_{\alpha_{k,j} \in A_{k,j}} \left\{ -b^{k,j}(x, t, \alpha_{k,j}) \cdot p + c^{k,j}(x, t, \alpha_{k,j})r - l^{k,j}(x, t, \alpha_{k,j}) \right\} \ ,$$

which is defined for $r \in \mathbb{R}$ and a priori only for $p \in T_x\mathbf{M}^k$ but we can as usual extend this definition for $p \in \mathbb{R}^N \times \mathbb{R}$.

If $(x, t) \in \mathbb{R}^N \times (0, T_f)$, setting $L(x, t) := \{(k, j);\ (x, t) \in \overline{\mathbf{M}^{k,j}}\}$ and define

$$\mathbf{BCL}(x, t) = \overline{\text{Conv}} \left\{ \bigcup_{(k,j) \in L(x,t)} \{(b^{k,j}, c^{k,j}, l^{k,j})(x, t, \alpha_{k,j}),\ \alpha_{k,j} \in A_{k,j}\} \right\} \ ,$$

$$\mathbb{F}(x, t, r, p) = \sup_{\substack{(k,j) \in L(x,t), \\ \alpha_{k,j} \in A_{k,j}}} \left\{ -b^{k,j}(x, t, \alpha_{k,j}) \cdot p + c^{k,j}(x, t, \alpha_{k,j})r - l^{k,j}(x, t, \alpha_{k,j}) \right\} .$$

Assumptions — In order to have Assumption **(TC)** satisfied, it is enough that each $(b^{k,j}, c^{k,j}, l^{k,j})$ satisfies $(\mathbf{H_{BACP}})$ and for **(NC)**, we have to assume that if $(x, t) \in \mathbf{M}^{\bar{k}}$, then the set

$$\overline{\text{Conv}} \left(\bigcup_{\substack{(k,j) \in L(x,t), \\ k > \bar{k}}} \{(b^{k,j}, c^{k,j}, l^{k,j})(x, t, \alpha_{k,j}),\ \alpha_{k,j} \in A_{k,j}\} \right) \ ,$$

satisfies **(NC-BCL)** (instead of **B**).

21.3.2 A general pde-oriented framework

On the contrary, here, we start from a general equation and define all the $\mathbb{F}^k$ by induction. Unfortunately this pde-oriented example will not be completely formulated in terms of pde and Hamiltonians, the difficulty being analogous to defining H_T in Part II. To simplify, we treat the case when the stratification does not depend on times, *i.e.* $\mathbf{M}^{k+1} = \tilde{\mathbf{M}}^k \times (0,T_f)$ for all $0 \le k \le N$, where $(\tilde{\mathbf{M}}^k)_k$ is a stratification of $\mathbb{R}^N$.

The case $k = N+1$ — We start from $\mathbf{M}^{N+1}$ which we write as the union of its connected components

$$\mathbf{M}^{N+1} = \bigcup_{j=1}^{J(N+1)} \tilde{\mathbf{M}}^{N,j} \times (0,T_f) \ .$$

We consider the case when

$$\mathbb{F}^{N+1}(x,t,r,(p_x,p_t)) = p_t + \tilde{H}^{N,j}(x,t,r,p_x) \quad \text{in } \tilde{\mathbf{M}}^{N,j} \times (0,T_f) \ ,$$

for all j where the Hamiltonians $\tilde{H}^{N,j}$ are defined by

$$\tilde{H}^{N,j}(x,t,r,p) = \sup_{\alpha_{N,j} \in A_{N,j}} \left\{ -b^{N,j}(x,t,\alpha_{N,j}) \cdot p + c^{N,j}(x,t,\alpha_{N,j}) r - l^{N,j}(x,t,\alpha_{N,j}) \right\} \ ,$$

where the control sets $A_{N,j}$ are compact metric spaces. A simple but natural situation is when all these Hamiltonians can be extended as continuous in $\mathbb{R}^N \times [0,T_f]$ functions satisfying $(\mathbf{H_{BA-HJ}})$. These Hamiltonians are the analogues of H_1, H_2 in Part II.

Induction for $k < N+1$ — It remains to define $\mathbb{F}$ and $\mathbb{F}^{k+1}$ on all $\tilde{\mathbf{M}}^k \times (0,T_f)$ for $k < N$ and this has to be done by induction. For $k = N-1$, if

$$\mathbf{M}^N = \bigcup_{j=1}^{J(N)} \tilde{\mathbf{M}}^{N-1,j} \times (0,T_f) \ ,$$

we can assume that, on each $\tilde{\mathbf{M}}^{N-1,j} \times (0,T_f)$, we have an Hamiltonian $\tilde{H}^{N-1,j}$ and we have, for any $(x,t) \in \tilde{\mathbf{M}}^{N-1,j} \times (0,T_f)$

$$\mathbb{F}(x,t,r,(p_x,p_t)) = \max_{l \in L(x,t)} \left(p_t + \tilde{H}^{N,l}(x,t,r,p_x), p_t + \tilde{H}^{N-1,j}(x,t,r,p_x) \right) \ ,$$

with $L(x,t) := \{l;\ (x,t) \in \overline{\tilde{\mathbf{M}}^{N,l}} \times (0,T_f)\}$. On the other hand, $\mathbb{F}^N$ may be decomposed into two parts: the analogue of the H_T-one in Part II coming from $\mathbb{F}^{N+1}$ and the specific $\tilde{H}^{N-1,j}$-one reflecting a particular control problem on $\tilde{\mathbf{M}}^{N-1,j} \times (0,T_f)$. This means

$$\mathbb{F}^N(x,t,r,(p_x,p_t)) = \max\left(\mathbb{F}_T^{N+1}(x,t,r,(p_x,p_t)), p_t + \tilde{H}^{N-1,j}(x,t,r,p_x)\right),$$

where $\mathbb{F}_T^{N+1}(x,t,r,(p_x,p_t))$ is built in the following way: as in the previous section, we set

$$\overline{\mathrm{Conv}}\Big(\bigcup_{l\in L(x,t)} \{(b^{N,j},c^{N,j},l^{N,j})(x,t,\alpha_{N,j}),\ \alpha_{N,j} \in A_{N,j}\}\Big),$$

and, for $(x,t) \in \tilde{\mathbf{M}}^{N-1,j} \times (0,T_f)$ we denote by $\mathbf{BCL}_T^{N-1}(x,t)$ the subset of (b,c,l) in this closed convex envelope such that $b \in T_x\tilde{\mathbf{M}}^{N-1,j}$. Then

$$\mathbb{F}_T^{N+1}(x,t,r,(p_x,p_t)) = p_t + \sup_{\mathbf{BCL}_T^{N-1}(x,t)} \{-b\cdot p_x + cr - l\}\ .$$

For any k, the construction is analogous. For any connected component of $\tilde{\mathbf{M}}^{k,j} \times (0,T_f)$ of $\tilde{\mathbf{M}}^k \times (0,T_f)$, $\mathbb{F}$ and $\mathbb{F}^{k+1}$ are constructed in the same way by using, for $\mathbb{F}$, a maximum of the $\mathbb{F}^{k+2},\mathbb{F}^{k+3},\cdots,\mathbb{F}^{N+1}$ nearby and of $p_t+\tilde{H}^{k,j}(x,t,r,p_x)$ where $\tilde{H}^{k,j}$ is a specific Hamiltonian on $\mathbf{M}^{k,j}\times(0,T_f)$, while for $\mathbb{F}^{k+1}$, one has to built a tangential Hamiltonian $\mathbb{F}_T^{k+2}$ and take the maximum with $p_t+\tilde{H}^{k,j}(x,t,r,p_x)$. The construction of $\mathbb{F}_T^{k+2}$ is the same as in the previous section and is based on computing the element of $\mathbf{BCL}(x,t)$ for $(x,t)\in\tilde{\mathbf{M}}^{k,j}\times(0,T_f)$ coming from $\tilde{\mathbf{M}}^{\bar{k},l}\times(0,T_f)$ for $\bar{k}>k$ and for the nearby connected components of the $\tilde{\mathbf{M}}^{\bar{k}}\times(0,T_f)$.

Chapter 22

Stability Results

Stability results are, of course, a fundamental feature of viscosity solutions. A general stability result for solutions in the Ishii sense (*cf.* Theorem 3.1.4) is readily available for Hamiltonians with any type of discontinuities. Therefore it can be used for supersolutions in the stratified case, stratified supersolutions being nothing but ordinary Ishii supersolutions.

But clearly the case of subsolutions is far more complicated: passing to the limit in all the viscosity inequalities $\mathbb{F}^k \leq 0$ on $\mathbf{M}^k$ for all $k = 0, .., N+1$ creates difficulties both at the level of the Hamiltonians and the stratification.

For the Hamiltonians, if we assume that we have a sequence of Hamiltonians $(\mathbb{F}_\varepsilon)_\varepsilon$ which are all associated to the same stratification and if $\limsup^* \mathbb{F}_\varepsilon = \mathbb{F}$, first it is not clear that $\liminf_* \mathbb{F}_\varepsilon = \mathbb{F}_*$ but it is even less clear that for $k = 0..N$, $\liminf_* \mathbb{F}^k_\varepsilon = \mathbb{F}^k$, or at least $\liminf_* \mathbb{F}^k_\varepsilon \geq \mathbb{F}^k$, which would be sufficient to get standard subsolution inequalities for the limiting problem. We point out here that the notion of (w-S-Sub) may drop the the first difficulty—which is already an important result—but the second one is, of course, unavoidable.

For the stratification, there are two levels of difficulties: either we just want to take into account cases for which the local structure of the stratification is unchanged, *i.e.* the discontinuities are the same, they are just slightly moved; or we wish to treat cases where some parts of the stratification are created or deleted, *i.e.* some discontinuities may appear or disappear in the Hamiltonians.

In this chapter, we address all these difficulties: we first provide a basic stability result in the case where the structure of the stratification is unchanged, without solving the difficulty related to the convergence of the Hamiltonians. Then we show how the difficulty related to the convergence of the Hamiltonians can be treated and

finally we show how to take into account some modifications in the structure of the stratification, both when new parts appear and when some parts disappear in the limit.

22.1 Strong convergence of stratifications when the local structure is unchanged

As it is clear from the above introduction, a stability result for a stratified problem requires two ingredients; first a suitable notion of convergence for regular stratifications and then some assumptions on the convergence of the Hamiltonians. But, of course, these ingredients should be compatible enough to lead to a stability result.

Let us start from a definition given in [35] for the convergence of regular stratifications which we adapt to the case of stratifications of $\mathbb{R}^N \times (0,T_f)$.

Definition 22.1.1 *We say that a sequence $(\mathbb{M}_\varepsilon)_\varepsilon$ of regular stratification of $\mathbb{R}^N \times (0,T_f)$ converges to a regular stratification $\mathbb{M}$ if, for each $(x,t) \in \mathbb{R}^N \times (0,T_f)$, there exists $r>0$, an* (AFS) *$\mathbb{M}^\star = \mathbb{M}^\star((x,t),r)$ in $\mathbb{R}^N$, a change of coordinates $\Psi^{x,t}$ as in Definition 3.3.5 and, for any $\varepsilon>0$, a family of changes of coordinates $\Psi^{x,t}_\varepsilon$ as in Definition 3.3.5 [(1)] satisfying*

(i) for any $0 \le k \le N$, if $\mathbf{M}^k \cap B((x,t),r) \neq \emptyset$, then $\Psi^{x,t}(\mathbf{M}^k \cap B((x,t),r)) = \mathbb{M}^\star \cap \Psi^{x,t}(B((x,t),r))$ and, for any $\varepsilon>0$, $\Psi^{x,t}_\varepsilon(\mathbf{M}^k_\varepsilon \cap B((x,t),r)) = \mathbb{M}^\star \cap \Psi^{x,t}_\varepsilon(B((x,t),r))$.

(ii) the changes of coordinates $\Psi^{x,t}_\varepsilon$ converge in $C^1(B((x,t),r))$ to $\Psi^{x,t}$ and their inverses $(\Psi^{x,t}_\varepsilon)^{-1}$ defined on $\Psi^{x,t}(B((x,t),r))$ also converge in C^1 to $(\Psi^{x,t})^{-1}$.

We denote this convergence by $\mathbb{M}_\varepsilon \xrightarrow{\text{RS}-s} \mathbb{M}$ where RS *stands for Regular Stratification and "s" for "strong" convergence.*

This definition essentially means that a sequence $(\mathbb{M}_\varepsilon)_\varepsilon$ of stratification converges to $\mathbb{M}$ if the $\mathbb{M}_\varepsilon$ are locally just *smooth*, little deformations of $\mathbb{M}$. Indeed,

$$\mathbf{M}^k_\varepsilon \cap B((x,t),r) = [\Psi^{x,t}_\varepsilon]^{-1} \circ \Psi^{x,t}\Big(\mathbf{M}^k \cap B((x,t),r)\Big) \quad \text{and} \quad [\Psi^{x,t}_\varepsilon]^{-1} \circ \Psi^{x,t} \to \text{Id in } C^1\,.$$

Technically, this allows to work locally with a fixed stratification $\mathbb{M}^\star$, removing completely the difficulty of the convergence of stratification which is easily described by the convergence of $\Psi^{x,t}_\varepsilon$ to $\Psi^{x,t}$.

[(1)] without imposing $\Psi^{x,t}_\varepsilon(x,t) = (x,t)$ and $\Psi^{x,t}(x,t) = (x,t)$

Drawbacks — Unfortunately this definition excludes a lot of interesting cases, the first one being the regularization of a corner, see Figure 22.1: in $\mathbb{R}^3$, we define $\mathbb{M}$ by

$$\mathbf{M}^1 := \{(0,0,x_3),\, x_3 \in \mathbb{R}\}\,,\ \mathbf{M}^2 := \{(x_1,|x_1|,x_3),\, x_1 \in \mathbb{R}\setminus\{0\},\, x_3 \in \mathbb{R}\}\,,$$

and $\mathbf{M}^0 = \emptyset$, $\mathbf{M}^3 = \mathbb{R}^3 \setminus (\mathbf{M}^1 \cup \mathbf{M}^2)$. Defining $\mathbb{M}_\varepsilon$ through

$$\mathbf{M}^2_\varepsilon := \{(x_1,(x_1^2+\varepsilon^2)^{1/2}-\varepsilon,x_3),\, x_1 \in \mathbb{R}\setminus\{0\},\, x_3 \in \mathbb{R}\}\,,$$

and with $\mathbf{M}^0_\varepsilon = \mathbf{M}^0$, $\mathbf{M}^1_\varepsilon = \mathbf{M}^1$ and $\mathbf{M}^3_\varepsilon = \mathbb{R}^3 \setminus (\mathbf{M}^1 \cup \mathbf{M}^2_\varepsilon)$, we see that we cannot expect the convergence of $\mathbb{M}_\varepsilon$ in the sense of the above definition. Indeed, the dashed axis on Figure 22.1 which should converge to the x_3-axis of the limiting stratification does not exist in the approximating stratifications.

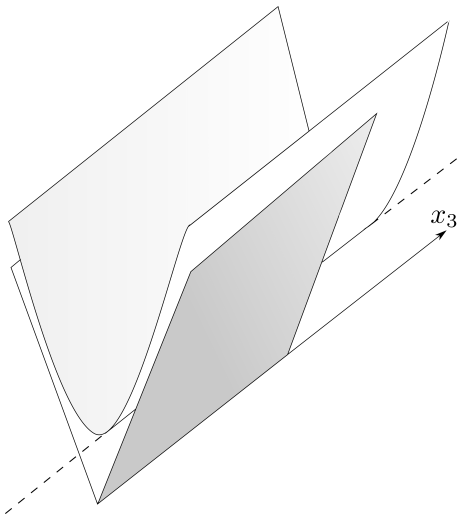


Figure 22.1: The "book" approximation

Another approach uses the following approximation of $\mathbb{M}$: we set $\mathbf{M}^0_\varepsilon = \emptyset$,

$$\mathbf{M}^1_\varepsilon := \{(\varepsilon,0,x_3),\, x_3 \in \mathbb{R}\} \cup \{(-\varepsilon,0,x_3),\, x_3 \in \mathbb{R}\}\,,$$

$\mathbf{M}^2_\varepsilon := \{(x_1+\varepsilon,x_1-\varepsilon,x_3),\, x_1>0,\, x_3 \in \mathbb{R}\} \cup \{(x_1-\varepsilon,x_1+\varepsilon,x_3),\, x_1<0,\, x_3 \in \mathbb{R}\}\,,$

and $\mathbf{M}^3_\varepsilon = \mathbb{R}^3 \setminus (\mathbf{M}^1 \cup \mathbf{M}^2_\varepsilon)$. But this other sequence of stratification $\mathbb{M}_\varepsilon$ does not converge either to $\mathbb{M}$ in the sense of the above definition. This second example is a bit trickier since the limiting $\mathbf{M}^1$ is obtained by merging the two connected components of the $\mathbf{M}^1_\varepsilon$, a case which is again clearly excluded by definition 22.1.1.

22.1.1 Weak convergence of stratifications and the associated stability result

The aim of this section is to provide a notion of convergence of stratifications which partially corrects the defects above and allows to take into account the second above approximation of $\mathbb{M}$ (but not the first one yet). This notion of convergence allows the "merging" of different connected components of $\mathbf{M}^k_\varepsilon$ but does not permit the emergence of new parts of the stratification (*i.e.*, no creation of new discontinuities for the equation) nor the disappearance of some of them (no elimination of discontinuities). We address these questions later in this chapter.

To do so, we concentrate on the equation in $\mathbb{R}^N \times (0, T_f)$, the case $t = 0$ being treated analogously. In order to formulate the stability result, a notion of convergence of stratifications of [35] is changed into the more general following definition.

Definition 22.1.2 *We say that a sequence $(\mathbb{M}_\varepsilon)_\varepsilon$ of regular stratifications of $\mathbb{R}^N \times (0, T_f)$ converges to a regular stratification $\mathbb{M}$ if*

(*i*) *for any $k = 0 \dots N+1$, $\mathbf{M}^k_\varepsilon \to \mathbf{M}^k$ for the Hausdorff distance;*

(*ii*) *for any $k = 1 \dots N+1$, for any $(x,t) \in \mathbf{M}^k$, there exists $r > 0$ and $J \geq 1$ such that, for ε small enough*

- (*a*) *there exists J connected components $(\mathbf{M}^k_{j,\varepsilon})_{j=1..J}$ of $\mathbf{M}^k_\varepsilon$ such that $\mathbf{M}^k_\varepsilon \cap B((x,t),r) = \cup_j \mathbf{M}^k_{j,\varepsilon} \cap B((x,t),r)$;*
- (*b*) *for any $j = 1..J$ and $\varepsilon > 0$, there exists a C^1-change of coordinates $\Psi^{x,t}_{j,\varepsilon}$: $B((x,t),r) \to \mathbb{R}^N \times (0,T_f)$ such that $\Psi^{x,t}_{j,\varepsilon}(\mathbf{M}^k \cap B((x,t),r)) = \mathbf{M}^k_{j,\varepsilon} \cap B((x,t),r)$;*
- (*c*) *the family of changes of coordinates $\Psi^{x,t}_{j,\varepsilon}$ and their inverses $(\Psi^{x,t}_{j,\varepsilon})^{-1}$ converge in C^1 to identity in a neighborhood of (x,t) as $\varepsilon \to 0$.*

We denote this convergence by $\mathbb{M}_\varepsilon \xrightarrow{\mathrm{RS}-w} \mathbb{M}$ where RS *stands for Regular Stratification and "w" for "weak" convergence.*

In the previous definition, we were using local changes of coordinates which transform globally one stratification into an other one, typically into an (AFS). Here, on the contrary, the changes act only on a single $\mathbf{M}^k$ with no information on their effects on the other parts of the stratification. As we mentioned above, this formulation allows the merging of several connected components of the $\mathbf{M}^k_\varepsilon$. The reader can easily

check that it applies without any difficulty to the second approximation of $\mathbb{M}$ in the example of the previous section.

But again we recall that no new part of the stratification (with a dimension $l < k$) can be created in this passage to the limit. Similarly, no part of the stratification can really disappear, except through the merging of connected components with similar dimension.

Now, for each $\varepsilon > 0$, we consider the associated Hamilton-Jacobi-Bellman problem in the stratified domain $\mathbb{M}_\varepsilon$. The meaning of sub and supersolutions is the one that is introduced in Definition 20.2.1, with the family of Hamiltonians $\mathbb{F}_\varepsilon$ and $(\mathbb{F}^k_\varepsilon)$ that are constructed from $\mathbb{M}_\varepsilon$ and some family $\mathbf{BCL}_\varepsilon$. We write $(\text{HJB-S})_\varepsilon$ for the stratified problem associated to $\mathbb{F}_\varepsilon$ and $(\mathbb{F}^k_\varepsilon)$.

In order to formulate the following stability result, we have to define limiting Hamiltonians for the $\mathbb{F}^k_\varepsilon(x,t,r,p)$, which are defined only if $p \in T_{(x,t)}\mathbf{M}^k$. The definition of the RS-convergence gives us the right way to do so: if $(x,t) \in \mathbf{M}^k$, we set

$$\liminf_*\, \mathbb{F}^k_\varepsilon(x,t,r,p) = \liminf_{\substack{(x_\varepsilon,t_\varepsilon)\in\mathbf{M}^k_\varepsilon\to(x,t),\ r_\varepsilon\to r\\ p_\varepsilon\in T_{(x_\varepsilon,t_\varepsilon)}\mathbf{M}^k_\varepsilon\to p,\ \varepsilon\to 0}} \mathbb{F}^k(x_\varepsilon,t_\varepsilon,r_\varepsilon,p_\varepsilon)\ .$$

Notice that this definition is consistent with Definition 22.1.2 since if $p_\varepsilon \in T_{(x_\varepsilon,t_\varepsilon)}\mathbf{M}^k_\varepsilon \to p$ then $p \in T_{(x,t)}\mathbf{M}^k$.

Theorem 22.1.3 *Let $\mathbb{S}_\varepsilon = (\mathbf{M}^k_\varepsilon, \mathbb{F}^k)_{\varepsilon,k}$ be a sequence of* (SSP) *in $\mathbb{R}^N \times (0,T_f)$ such that $\mathbb{M}_\varepsilon \xrightarrow{\text{RS}-w} \mathbb{M}$. Then the following holds:*

(*i*) *if for all $\varepsilon > 0$, v_ε is a l.s.c. supersolution of* $(\text{HJB-S})_\varepsilon$*, then $\underline{v} = \liminf_* v_\varepsilon$ is a l.s.c. supersolution of* (HJB-S)*, associated to $\mathbb{F} = \limsup^*\ \mathbb{F}_\varepsilon$;*

(*ii*) *if for $\varepsilon > 0$, u_ε is an u.s.c. subsolution of* $(\text{HJB-S})_\varepsilon$ *and the Hamiltonians $(\mathbb{F}^k_\varepsilon)_{k=0..N}$ satisfy* **(NC)** *and* **(TC)** *with uniform constants on a uniform neighborhood of $\mathbb{M}$, then $\bar{u} = \limsup^* u_\varepsilon$ is an u.s.c. subsolution of* (HJB-S) *associated to $\mathbb{G}^k := \liminf_* \mathbb{F}^k_\varepsilon$ for any $k = 0..N$.*

Of course, the "strong" convergence implies the "weak" one and therefore Theorem 22.1.3 a fortiori holds if we replace "$\mathbb{M}_\varepsilon \xrightarrow{\text{RS}-w} \mathbb{M}$" by "$\mathbb{M}_\varepsilon \xrightarrow{\text{RS}-s} \mathbb{M}$".

Important — In the statement of Theorem 22.1.3, we have used the notation $\mathbb{G}^k$ for $\liminf_* \mathbb{F}^k_\varepsilon$ because it is not clear a priori that the limit problem is a consistent stratified problem, *i.e.* that there exists **BCL** such that $\mathbb{F}$ is given by (20.2) and $\mathbb{G}^k = \mathbb{F}^k$ is given by (20.4). We refer to the next section for sufficient conditions that allow to get this property.

Proof — Result (i) is standard since only the $\mathbb{F}_\varepsilon/\mathbb{F}$-inequalities are involved and therefore (i) is nothing but the standard stability result for discontinuous viscosity solutions with discontinuous Hamiltonians, see [118]. We now focus on getting (ii).

(a) We assume that $(x_0,t_0) \in \mathbf{M}^k$ is a *strict* local maximum point of $\bar{u}-\phi$ on $\mathbf{M}^k$ where ϕ is a C^1 function in $\mathbb{R}^N \times (0,T_f)$ and we want to show that

$$\mathbb{G}^k(x_0,t_0,\bar{u}(x_0,t_0),D\phi(x_0,t_0) \le 0\,.$$

Since we are going to argue locally, we may assume without loss of generality that $\mathbb{M}$ is an (AFS). In particular, $\mathbf{M}^k = (x_0,t_0)+V_k$ where V_k is a k-dimensional subspace of $\mathbb{R}^{N+1}$.

We consider, in a small neighborhood of (x_0,t_0),

$$\chi_\varepsilon : (x,t) \mapsto u_\varepsilon(x,t) - \phi(x,t) - L\psi_\varepsilon(x,t)\,,$$

where $\psi_\varepsilon(x,t) = \mathrm{dist}((\Psi^{x,t}_{1,\varepsilon})^{-1}(x,t),\mathbf{M}^k)$, the function $\mathrm{dist}(\cdot,\mathbf{M}^k)$ denoting the distance to $\mathbf{M}^k$ which is smooth in a neighborhood of $\mathbf{M}^k$, except on $\mathbf{M}^k$. We point out that we have chosen arbitrarily the change $(\Psi^{x,t}_{j,\varepsilon})^{-1}$ of Definition 22.1.2 with $j=1$ to fix ideas.

(b) For $\varepsilon>0$ small enough, χ_ε has a maximum point $(x_\varepsilon,t_\varepsilon)$ near (x_0,t_0). From the definition of an (AFS), we can find a small neighborhood of (x_0,t_0) excluding any point of $\mathbf{M}^l$ for $l<k$, and also from connected components of $\mathbf{M}^k$ than the one of (x_0,t_0) itself. So, for $\varepsilon>0$ small enough, we know that $(x_\varepsilon,t_\varepsilon) \in \mathbf{M}^l_\varepsilon$ for some $l \ge k$ not depending on ε.

If $(x_\varepsilon,t_\varepsilon) \in \mathbf{M}^l_\varepsilon$ for $l>k$, $(\Psi^{x,t}_{1,\varepsilon})^{-1}(x_\varepsilon,t_\varepsilon)$ does not belong to $\mathbf{M}^k$ (otherwise we would have $(x_\varepsilon,t_\varepsilon) \in \mathbf{M}^{k,\varepsilon}_1$) and therefore ψ_ε is C^1 in a neighborhood of $(x_\varepsilon,t_\varepsilon)$. Since u_ε is an u.s.c. subsolution of $(\text{HJB-S})_\varepsilon$, we deduce that

$$\mathbb{F}^l_\varepsilon\Big(x_\varepsilon,t_\varepsilon,u_\varepsilon(x_\varepsilon,t_\varepsilon),D\phi(x_\varepsilon,t_\varepsilon)+LD\psi_\varepsilon(x_\varepsilon,t_\varepsilon)\Big) \le 0\,.$$

Next we remark that, on the one hand, $D\big[\,\mathrm{dist}((x,t),\mathbf{M}^k)\big]$ is orthogonal to V_k and on the other hand $\big|D\big[\,\mathrm{dist}((x,t),\mathbf{M}^k)\big]\big| = 1$ where the distance function is differentiable. Therefore, by Definition 22.1.2 and the convergence of $(\Psi^{x,t}_{1,\varepsilon})^{-1}$ to identity in C^1, $D\psi_\varepsilon(x_\varepsilon,t_\varepsilon)$ is a transverse vector to $\mathbf{M}^k_\varepsilon$, hence to all the $\mathbf{M}^k_{j,\varepsilon}$. Moreover, recalling that we are in the flat case, it is easy to see that

$$|[D\psi_\varepsilon(x_\varepsilon,t_\varepsilon)]_\perp| \ge \kappa > 0\,,$$

for some $\kappa \in (0,1)$ which does not depend neither on ε nor on j. Here we have strongly used that the distance to $\mathbf{M}^k$ is smooth if we are not on $\mathbf{M}^k$.

Using **(NC)** which holds in an uniform neighborhood of $\mathbf{M}^k$ by assumptions, we deduce that the $\mathbb{F}^l_\varepsilon$-inequality cannot hold if we chosen L large enough, and of course, L can be chosen independently of ε since it depends only on κ.

We deduce that $(x_\varepsilon, t_\varepsilon) \in \mathbf{M}^k_\varepsilon$, *i.e.* $(x_\varepsilon, t_\varepsilon) \in \mathbf{M}^k_{j,\varepsilon}$ for some $j = 1..J$, and not necessarily for $j = 1$.

(c) Using that $(x_\varepsilon, t_\varepsilon)$ is a local maximum point of χ_ε on $\mathbf{M}^k_{j,\varepsilon}$, we get

$$\mathbb{F}^k_\varepsilon\Big(x_\varepsilon, t_\varepsilon, u_\varepsilon(x_\varepsilon, t_\varepsilon), D\phi(x_\varepsilon, t_\varepsilon) + LD\psi_\varepsilon(x_\varepsilon, t_\varepsilon)\Big) \leq 0 \, .$$

Now, let us examine $D\psi_\varepsilon(x_\varepsilon, t_\varepsilon)$. Of course, this term does not exist if $j = 1$ since $\psi_\varepsilon \equiv 0$ on $\mathbf{M}^k_{1,\varepsilon}$. Now, assuming that $j \neq 1$ we claim that

$$b \cdot D\psi_\varepsilon(x_\varepsilon, t_\varepsilon) = o_\varepsilon(1) \, ,$$

uniformly for b bounded in $T_{(x_\varepsilon,t_\varepsilon)}\mathbf{M}^k_{j,\varepsilon}$. Indeed, we use here that $D\big[\,\mathrm{dist}((x,t), \mathbf{M}^k)\big]$ is orthogonal to V_k if $(x,t) \notin \mathbf{M}^k$ while if b is tangent to $\mathbf{M}^k_{j,\varepsilon}$, $D(\Psi^{x,t}_{1,\varepsilon})^{-1}b \to b \in V_k$.

This leads to the inequality

$$\mathbb{F}^k_\varepsilon\Big(x_\varepsilon, t_\varepsilon, u_\varepsilon(x_\varepsilon, t_\varepsilon), D\phi(x_\varepsilon, t_\varepsilon)\Big) \leq o_\varepsilon(1) \, .$$

But using that $\bar{u} = \limsup{}^* u_\varepsilon$ and that (x_0, t_0) is a strict local maximum point of $\bar{u} - \phi$ on $\mathbf{M}^k$, classical arguments imply that $(x_\varepsilon, t_\varepsilon) \to (x_0, t_0)$ and the conclusion of the proof follows as in the standard case.

Q.E.D.

22.1.2 An example showing the difficulty connected to the convergence of Hamiltonians

We drop here the time dependence for the sake of simplicity and investigate the following example: If (e_1, e_2) is the canonical basis of $\mathbb{R}^2$, *i.e.* $e_1 = (1, 0), e_2 = (0, 1)$, we consider the stratification $\mathbb{M}$ defined by

$$\mathbf{M}^1 = \mathbb{R}e_1 \quad , \quad \mathbf{M}^2 = \mathbb{R}^2 \setminus \mathbf{M}^1 \, .$$

Next we introduce $\mathbf{BCL}(x_1, x_2)$ defined in the following way: $(b_x, c, l) \in \mathbf{BCL}(x_1, x_2)$ if $c = 1$, $l = 1$ and $b_x \in \{e_1\} \times [-1, 1]$ if $x_2 > 0$, $b_x \in \{-e_1\} \times [-1, 1]$ if $x_2 < 0$. Hence, by the assumptions on $\mathbf{BCL}$, we have

$$\mathbf{BCL}(x_1, 0) = ([-1, 1] \times [-1, 1]) \times \{1\} \times \{1\} \, ,$$

and, if $p = (p_1, p_2)$, $\mathbb{F}^1$ is given on $\mathbf{M}^1$ by

$$\mathbb{F}^1(x_1, r, p) = \sup_{(b_1,0)\in[-1,1]\times\{0\}} \{-b_1 p_1 - b_2 p_2 + r - 1\} = |p_1| + r - 1 \ .$$

Now we consider the approximation of $\mathbb{M}$ and **BCL** by $\mathbb{M}_\varepsilon = \mathbb{M}$ and

$$\mathbf{BCL}_\varepsilon(x_1, x_2) = (\{(b^\varepsilon_x)_1\} \times [-1, 1]) \times \{1\} \times \{1\} \ ,$$

where, if $\chi : \mathbb{R} \to \mathbb{R}$ is the Lipschitz continuous function given by $\chi(t) = 0$ if $t \le 0$, $\chi(t) = t$ if $0 \le t \le 1$ and $\chi(t) = 1$ if $t \ge 1$

$$(b^\varepsilon_x)_1 = \chi(\frac{x_1}{\varepsilon})e_1 - \chi(-\frac{x_1}{\varepsilon})e_1 \ .$$

Admittedly we are in a continuous setting for $\varepsilon > 0$ but we ignore this point on purpose, this regularization yielding a smooth transition between $-e_1$ and e_1.

Here, specifically at $x_1 = 0$ we see that $\mathbb{F}^1_\varepsilon(0, r, p) = r - 1$, which has nothing to do with $\mathbb{F}^1$, although it is also clear that

$$\mathbb{F}^1(x_1, r, p) = \limsup_{y_1 \to x_1} \mathbb{F}^1_\varepsilon(y_1, r, p) \, .$$

This example shows that, in general, the $\mathbb{G}^k$ in Theorem 22.1.3 are different from $\mathbb{F}^k$ and this is a clear problem for the applications. If we want to correct this flaw, we need additional assumptions.

22.1.3 Sufficient conditions for stability

We conclude this first part devoted to the basic stability results with some sufficient conditions on **BCL** correcting the above defect and implying a real stability of solutions.

Lemma 22.1.4 *For any $\varepsilon > 0$, we assume that $\mathbf{BCL}_\varepsilon$ satisfies ($\mathbf{H_{BCL}}$), **(TC-BCL)** and **(NC-BCL)** on a uniform neighborhood of a regular stratification $\mathbb{M}_\varepsilon$ with constants independent of ε. Moreover, we assume that there exists a regular stratification $\mathbb{M}$ such that $\mathbb{M}_\varepsilon \xrightarrow{\mathrm{RS}-s} \mathbb{M}$.*

(i) If the following condition holds

$$\mathbf{BCL}(x, t) = \limsup_{\varepsilon \to 0}{}^* \ \mathbf{BCL}_\varepsilon(x, t) = \bigcap_{\delta>0} \bigcap_{\varepsilon>0} \left(\overline{K(x, t, \delta, \varepsilon)}\right) \ , \tag{22.1}$$

where

$$K(x,t,\delta,\varepsilon) := \bigcup_{\substack{|(y,s)-(x,t)|\le\delta \\ 0<\tilde\varepsilon\le\varepsilon}} \mathbf{BCL}_{\tilde\varepsilon}(y,s)\ ,$$

then $\mathbb{F} = \limsup^*\ \mathbb{F}_\varepsilon$ *and the stability result for supersolutions holds.*

(*ii*) *If* (22.1) *holds and if, for any* $k = 0,..,(N+1)$ *and any* $(x,t) \in \mathbf{M}^k$,

$$\mathbf{BCL}_\varepsilon([\Psi_\varepsilon^{x,t}]^{-1}(y,s)) \to \mathbf{BCL}([\Psi^{x,t}]^{-1}(y,s))$$

for any $(y,s) \in \Psi^{x,t}(B((x,t),r))$ *in the sense of the Haussdorff distance where* $r, \Psi_\varepsilon^{x,t}, \Psi^{x,t}$ *are as in Definition 22.1.1, then Theorem 22.1.3 holds true for subsolutions with* $\mathbb{G}^k = \mathbb{F}^k$.

Proof — We treat successively (*i*) and (*ii*).

The supersolution case — If $(b,c,l) \in \mathbf{BCL}(x,t)$, (22.1) implies that, for all $\delta,\varepsilon > 0$ small enough, there exists $|(y,s)-(x,t)| \le \delta$, $0 < \tilde\varepsilon \le \varepsilon$ and $(\tilde b,\tilde c,\tilde l) \in \mathbf{BCL}_{\tilde\varepsilon}(y,s)$ such that $|b-\tilde b|+|c-\tilde c|+|l-\tilde l| \le \varepsilon$. Therefore, if $|p|+|r| \le R$ and $|\tilde p-p|+|\tilde r-r| \le 1$,

$$-b\cdot p + cr - l \le -\tilde b\cdot\tilde p + \tilde c\tilde r - \tilde l + \varepsilon(2R+1) \le \mathbb{F}_{\tilde\varepsilon}(y,s,\tilde r,\tilde p) + \varepsilon(2R+1)\ .$$

Taking the lim sup in $\delta,\varepsilon \to 0$ but also on $\tilde r \to r$, $\tilde p \to p$ and using the definition of the $\limsup^*$, we deduce that

$$-b\cdot p + cr - l \le \limsup^*\ \mathbb{F}_\varepsilon(x,t,r,p)\ .$$

Since this is true for any $(b,c,l) \in \mathbf{BCL}(x,t)$, we get that for any (x,t,r,p),

$$\mathbb{F}(x,t,r,p) \le \limsup^*\ \mathbb{F}_\varepsilon(x,t,r,p)\ .$$

To get the conversely inequality, we consider a sequence $(x_{\tilde\varepsilon},t_{\tilde\varepsilon},r_{\tilde\varepsilon},p_{\tilde\varepsilon}) \to (x,t,r,p)$ such that

$$\mathbb{F}_{\tilde\varepsilon}(x_{\tilde\varepsilon},t_{\tilde\varepsilon},r_{\tilde\varepsilon},p_{\tilde\varepsilon}) \to \limsup^*\ \mathbb{F}_\varepsilon(x,t,r,p)\ .$$

Since the sets $\mathbf{BCL}_\varepsilon$ are compact, there exists $(b_{\tilde\varepsilon},c_{\tilde\varepsilon},l_{\tilde\varepsilon}) \in \mathbf{BCL}_{\tilde\varepsilon}(x_{\tilde\varepsilon},t_{\tilde\varepsilon})$ such that

$$\mathbb{F}_{\tilde\varepsilon}(x_{\tilde\varepsilon},t_{\tilde\varepsilon},r_{\tilde\varepsilon},p_{\tilde\varepsilon}) = -b_{\tilde\varepsilon}\cdot p_{\tilde\varepsilon} + c_{\tilde\varepsilon}r_{\tilde\varepsilon} - l_{\tilde\varepsilon}\ . \tag{22.2}$$

Now we pick $\delta,\varepsilon > 0$. It is clear that, for $\tilde\varepsilon$ small enough, $(b_{\tilde\varepsilon},c_{\tilde\varepsilon},l_{\tilde\varepsilon}) \in \overline{K(x,t,\delta,\varepsilon)}$. But, since $\overline{K(x,t,\delta,\varepsilon)}$ is compact, we can assume without loss of generality that

$$(b_{\tilde\varepsilon},c_{\tilde\varepsilon},l_{\tilde\varepsilon}) \to (b,c,l) \in \overline{K(x,t,\delta,\varepsilon)}\ .$$

This property being true for all δ and ε, we have by assumption $(b,c,l) \in \mathbf{BCL}(x,t)$. Letting $\tilde{\varepsilon} \to 0$ in (22.2), we get

$$\limsup{}^{*}\ \mathbb{F}_\varepsilon(x,t,r,p) = -b\cdot p + cr - l \leq \mathbb{F}(x,t,r,p)\ ,$$

which proves that (i) holds: $\mathbb{F} = \limsup{}^{*}\ \mathbb{F}_\varepsilon$.

The subsolution case — For the proof of (ii), we only have to examine the convergence of the Hamiltonians $\mathbb{F}^k_\varepsilon$, and not $\liminf_{*} F_\varepsilon$. We recall that this is a consequence of "weak subsolutions=strong subsolutions" in this framework.

Because of the assumptions, we can assume *w.l.o.g.* that we are in a flat (and static) situation and that all the Hamiltonians $\mathbb{F}^k_\varepsilon$ are all defined on the same set. On the other hand, by **(TC-BCL)**, all these Hamiltonians are equicontinuous on $\mathbf{M}^k = \mathbf{M}^k_\varepsilon$ for any ε. Combining the convergence of $\mathbf{BCL}_\varepsilon$ to $\mathbf{BCL}$ with **(NC-BCL)** implies that $(\mathbf{BCL}_\varepsilon)|_k$ (the restriction to $\mathbf{M}^k \times [0,T_f]$) converges to $\mathbf{BCL}|_k$. It follows directly that

$$\mathbb{F}^k_\varepsilon(x,r,p) := \sup_{\substack{(b,c,l)\in\mathbf{BCL}_\varepsilon(x,t)\\ b\in T_x\mathbf{M}^k}} \left\{-b\cdot p+cr-l\right\} \longrightarrow \sup_{\substack{(b,l)\in\mathbf{BCL}(x,t)\\ b\in T_x\mathbf{M}^k}} \left\{-b\cdot p+cr-l\right\} = \mathbb{F}^k(x,r,p)\,.$$

Combining this pointwise convergence with Ascoli's Theorem, we obtain the local uniform convergence of the $\mathbb{F}^k_\varepsilon$ to $\mathbb{F}^k$ on $\mathbf{M}^k$, and the result is proved.

Q.E.D.

Corollary 22.1.5 *Under the assumptions of Lemma 22.1.4, for any $\varepsilon > 0$ let U_ε is the unique solution of* (HJB-S)$_\varepsilon$. *If the functions U_ε are uniformly bounded, then*

$$U_\varepsilon \to U \quad \textit{locally uniformly in } \mathbb{R}^N \times [0,\infty)\,,$$

where U is the unique solution of the limit problem (HJB-S) *associated to* $(\mathbb{F}^k)_{k=0..N}$.

Proof — The proof is immediate: by Lemma 22.1.4, the half-relaxed limits of the U_ε are sub and supersolutions of the limit problem (HJB-S) thanks to Theorem 22.1.3. Then, the comparison result—Theorem 20.4.1—implies that $\liminf_{*} U_\varepsilon = \limsup{}^{*} U_\varepsilon$, so that all the sequence converges to the common limit, U, locally uniformly by the classical half-relaxed limits method, see Lemma 3.1.7.

Q.E.D.

22.2 Stability under structural modifications of Standard Stratified Problems

In the previous section, we have provided a stability result in the case when the structure of the stratification remains unchanged. On the contrary, in this section, we consider cases where this structure can be changed by the appearance of new discontinuity sets or the disappearance of existing ones. Anyway, the first stability property shown in the previous section is be the keystone of this improved result. So, we have to show how to introduce a new part of $\mathbf{M}^k$ or remove an existing one in order to manage these changes of stratifications. Again we only treat the case of $\mathbb{R}^N \times (0, T_f)$, the case $t = 0$ following similar principles.

It is important to notice that here, such structural modifications of the stratification have an impact on the associated Hamiltonians and conversely. So, a generalized stability result necessarily implies considering both at the same time.

22.2.1 Introducing new parts of the stratification

The result is the

Proposition 22.2.1 *Let $\mathbb{S} = (\mathbf{M}^k, \mathbb{F}^k)_k$ be a* (SSP) *and $u : \mathbb{R}^N \times (0, T_f) \to \mathbb{R}$ an u.s.c. subsolution of this problem. If $\mathcal{M}$ is a C^1-smooth l-dimensional submanifold of $\mathbf{M}^k$ for some $l < k$ and if the normal controlability assumption is satisfied in a neighborhood of $\mathcal{M}$, then*

$$\mathbb{F}^{\mathcal{M}}(x, t, u, Du) \leq 0\,,$$

where for $x \in \mathcal{M}$, $t \in (0, T_f)$, $r \in \mathbb{R}$, $p = (p_x, p_t) \in \mathbb{R}^N \times \mathbb{R}$

$$\mathbb{F}^{\mathcal{M}}(x, t, r, p) := \sup_{\substack{(b,c,l)\in \mathbf{BCL}(x,t) \\ b\in T_{(x,t)}\mathcal{M}}} \big\{ -b \cdot p + cr - l \big\}\,.$$

This result means that, a priori, we can create an artificial $\mathbf{M}^l$-component in $\mathbb{M}$ since $\mathcal{M}$ can be seen as some new part of $\mathbf{M}^l$.

But of course, for a concrete use, there are conditions in order that replacing $\mathbf{M}^l$ by $\mathbf{M}^l \cup \mathcal{M}$ in $\mathbb{M}$ leads to a new, consistant (SSP): $\mathcal{M}$ may have a boundary which has to be taken into account, and new viscosity inequalities have also to be checked on this boundary.

Example 22.1 — $\mathcal{M} = (-1, 1) \times \{0\}$ in the whole space $\mathbb{R}^2$ generates a new $\mathbf{M}^1$-part but also a $\mathbf{M}^0$-set with $(\{-1\} \times \{0\}) \cup (\{1\} \times \{0\})$. Moreover, for the equation, one also has to examine the $\mathbb{F}^0$-inequalities at these two points.

Proof — Since the result is local, we can assume without loss of generality that $\mathbf{M}^k = \mathbb{R}^k$ and that $\mathcal{M}$ is an affine subspace of $\mathbb{R}^k$. If $\phi : \mathbb{R}^N \times [0, T_f] \to \mathbb{R}$ is a smooth function and $(\bar{x}, \bar{t}) \in \mathcal{M}$ is a strict, local maximum point of $u - \phi$ on $\mathcal{M}$, we have to show that

$$\mathbb{F}^{\mathcal{M}}(\bar{x}, \bar{t}, u(\bar{x}, \bar{t}), D\phi(\bar{x}, \bar{t})) \leq 0 \ .$$

To do so, for $0 < \varepsilon \ll 1$, we consider the function defined for $(x, t) \in \mathbf{M}^k = \mathbb{R}^k$

$$(x, t) \mapsto u(x, t) - \phi(x, t) - \frac{[d(x, t)]^2}{\varepsilon} \ ,$$

where $d(x, t) = d((x, t), \mathcal{M})$ is the distance function to $\mathcal{M}$ which is C^1 outside $\mathcal{M}$ but not on $\mathcal{M}$. On the contrary, $(x, t) \mapsto [d(x, t)]^2$ is C^1 even on $\mathcal{M}$.

By standard arguments, this function has a maximum point at $(x_\varepsilon, t_\varepsilon)$ and

$$(x_\varepsilon, t_\varepsilon) \to (\bar{x}, \bar{t}) \quad u(x_\varepsilon, t_\varepsilon) \to u(\bar{x}, \bar{t}) \quad \text{and} \quad \frac{[d(x_\varepsilon, t_\varepsilon)]^2}{\varepsilon} \to 0 \quad \text{as } \varepsilon \to 0 \ .$$

Since u is a subsolution of the stratified problem,

$$\mathbb{F}^k\Big(x_\varepsilon, t_\varepsilon, u(x_\varepsilon, t_\varepsilon), D\phi(x_\varepsilon, t_\varepsilon) + \frac{2d(x_\varepsilon, t_\varepsilon) Dd(x_\varepsilon, t_\varepsilon)}{\varepsilon}\Big) \leq 0 \ .$$

In order to deduce the result from this inequality, we use the tangential continuity property on $\mathbf{M}^k$: if $(y_\varepsilon, s_\varepsilon)$ is the unique projection of $(x_\varepsilon, t_\varepsilon)$ on $\mathcal{M}$ (recall that locally we are reduced to consider affine subspaces), then $|y_\varepsilon - x_\varepsilon| + |t_\varepsilon - s_\varepsilon| = d(x_\varepsilon, t_\varepsilon)$ and

$$\mathbb{F}^k\Big(y_\varepsilon, s_\varepsilon, u(x_\varepsilon, t_\varepsilon), D\phi(x_\varepsilon, t_\varepsilon) + \frac{2d(x_\varepsilon, t_\varepsilon) Dd(x_\varepsilon, t_\varepsilon)}{\varepsilon}\Big) \leq o_\varepsilon(1) \ .$$

On the other hand, if $b^1 \in T_{(y_\varepsilon, s_\varepsilon)}\mathcal{M}$ then $b^1 \cdot Dd(x_\varepsilon, t_\varepsilon) = 0$ because $(y_\varepsilon, s_\varepsilon)$ is the unique projection of $(x_\varepsilon, t_\varepsilon)$ on $\mathcal{M}$. Therefore, restricting the above inequality to sich vectors b^1, it follows that

$$\mathbb{F}^{\mathcal{M}}\Big(y_\varepsilon, s_\varepsilon, u(x_\varepsilon, t_\varepsilon), D\phi(x_\varepsilon, t_\varepsilon)\Big) \leq o_\varepsilon(1) \ .$$

In order to conclude, we use again the tangential continuity on $\mathbf{M}^k = \mathbb{R}^k$ combined with the normal controllability: if $(b, c, l) \in \mathbf{BCL}(\bar{x}, \bar{t})$ with $b \in T_{(\bar{x}, \bar{t})}\mathcal{M}$, there exists $(b^1_\varepsilon, c^1_\varepsilon, l^1_\varepsilon) \in \mathbf{BCL}(y_\varepsilon, s_\varepsilon)$ with $b^1_\varepsilon \in T_{(y_\varepsilon, s_\varepsilon)}\mathcal{M}$ and such that $(b^1_\varepsilon, c^1_\varepsilon, l^1_\varepsilon) \to (b, c, l)$ as $\varepsilon \to 0$. Using this property, the result is obtained by letting ε tend to 0.

Q.E.D.

22.2.2 Eliminable parts of the stratification

In this section, the aim is to remove "artificial" parts of the stratification, that is, parts on which there is no real discontinuity and the viscosity inequalities are just a consequence of those coming from lower codimensions manifolds. Our result is the

Proposition 22.2.2 *Let $\mathbb{S} = (\mathbf{M}^k, \mathbb{F}^k)_k$ be a* (SSP) *and $u : \mathbb{R}^N \times (0, T_f) \to \mathbb{R}$ an u.s.c. subsolution of this problem. Let $\mathcal{M} \subset \mathbf{M}^k$ be a C^1-smooth submanifold such that*

(*i*) *$\mathcal{M} \subset \overline{\mathbf{M}^l}$ for some $l > k$;*

(*ii*) *$\mathcal{M} \cup \mathbf{M}^l$ is a l-dimensional submanifold of $\mathbb{R}^N$;*

(*iii*) **BCL** *satisfies the tangential continuity assumption on $\mathcal{M} \cup \mathbf{M}^l$.*

Then u is a subsolution of

$$\tilde{\mathbb{F}}^l(x, t, u, Du) \leq 0 \quad \text{on } \mathcal{M} \cup \mathbf{M}^l \,,$$

where, for $x \in \mathcal{M} \cup \mathbf{M}^l$, $t \in (0, T_f)$, $r \in \mathbb{R}$, $p = (p_x, p_t) \in \mathbb{R}^N \times \mathbb{R}$

$$\tilde{\mathbb{F}}^l(x, t, r, p) := \sup_{\substack{(b,c,l) \in \mathbf{BCL}(x,t) \\ b \in T_{(x,t)}(\mathcal{M} \cup \mathbf{M}^l)}} \big\{ -b \cdot p + cr - l \big\} \,.$$

In other words, this proposition means that $\mathbf{M}^l$ can be replaced by $\mathcal{M} \cup \mathbf{M}^l$: the higher co-dimension discontinuity manifold $\mathcal{M}$ can be removed and integrated into $\mathbf{M}^l$. Such a result can be used when a standard continuous HJ-Equation is approximated by a problem with discontinuities: to recover the right equation at the limit, one has to remove the artificial discontinuities created by the approximation.

Concerning assumption (*ii*), notice that of course, including $\mathcal{M}$ into $\mathbf{M}^l$ may completely change its decomposition into connected components: for instance, adding $\{0\}$ to $\mathbf{M}^1 = (-\infty; 0) \cup (0; +\infty)$ leads to a unique connected component, $\mathbb{R}$ itself.

Example 22.2 — we consider a case similar to the one described at the beginning of the stability chapter: in $\mathbb{R}^3$ we define $\mathbb{M}$ by

$$\mathbf{M}^1 := \{(0, 0, x_3),\, x_3 \in \mathbb{R}\} \,,\ \mathbf{M}^2 := \{(x_1, x_1^2, x_3),\, x_1 \in \mathbb{R} \setminus \{0\},\, x_3 \in \mathbb{R}\} \,,$$

and $\mathbf{M}^0 = \emptyset$, $\mathbf{M}^3 = \mathbb{R}^3 \setminus (\mathbf{M}^1 \cup \mathbf{M}^2)$. In this setting, it seems relevant to remove $\mathbf{M}^1$ and see if we can replace $\mathbf{M}^2$ by $\{(x_1, x_1^2, x_3),\, x_1 \in \mathbb{R},\, x_3 \in \mathbb{R}\}$. This can be done provided a suitable continuity of the Hamiltonian (assumption (*iii*) above) holds.

Proof — Again we can assume without loss of generality that $\mathbf{M}^l = \mathbb{R}^l$ and that $\mathcal{M}$ is an affine subspace of $\mathbb{R}^l$. If $\phi : \mathbb{R}^N \times [0, T_f] \to \mathbb{R}$ is a smooth function and $(\bar{x}, \bar{t}) \in \mathcal{M}$ is a strict, local maximum point of $u - \phi$ on $(\mathcal{M} \cup \mathbf{M}^l)$, we have to show that

$$\tilde{\mathbb{F}}^l(\bar{x}, \bar{t}, u(\bar{x}, \bar{t}), D\phi(\bar{x}, \bar{t})) \leq 0 \ .$$

Here the difficulty is that the set $(b, c, l) \in \mathbf{BCL}(x, t)$ with $b \in T_{(x,t)}(\mathcal{M} \cup \mathbf{M}^l)$ is larger than the set for which $b \in T_{(x,t)}\mathcal{M}$.

If $b \in T_{(x,t)}\mathcal{M}$, the desired inequality is nothing but a consequence of the $\mathbb{F}^k$-inequality on $\mathcal{M}$, therefore we can assume *w.l.o.g.* that $b \notin T_{(x,t)}\mathcal{M}$. We decompose

$$b = b^\top + b^\perp \quad \text{with } b^\top \in T_{(x,t)}\mathcal{M},\ b^\perp \text{ in its orthogonal space.}$$

For $0 < \varepsilon \ll 1$, we consider on $D = \left\{(x, t) \in \mathbf{M}^l = \mathbb{R}^l;\ (x - \bar{x}, t - \bar{t}) \cdot b^\perp > 0\right\}$ the function

$$(x, t) \mapsto u(x, t) - \phi(x, t) - \frac{\varepsilon}{(x - \bar{x}, t - \bar{t}) \cdot b^\perp} \ .$$

We first remark that the normal controllability assumption on $\mathbf{M}^k$ (and therefore on $\mathcal{M}$) implies the regularity property

$$u(\bar{x}, \bar{t}) = \limsup_{\substack{(x,t) \to (\bar{x}, \bar{t}) \\ (x,t) \in D}} u(x, t) \ ,$$

and because of this property, standard arguments show that this function has a maximum point at $(x_\varepsilon, t_\varepsilon) \in D$ satisfying

$$(x_\varepsilon, t_\varepsilon) \to (\bar{x}, \bar{t}) \, , \quad u(x_\varepsilon, t_\varepsilon) \to u(\bar{x}, \bar{t}) \quad \text{and} \quad \frac{\varepsilon}{(x_\varepsilon - \bar{x}, t_\varepsilon - \bar{t}) \cdot b^\perp} \to 0 \quad \text{as } \varepsilon \to 0 \ .$$

Using assumption (iii), there exists $(b^1_\varepsilon, c^1_\varepsilon, l^1_\varepsilon) \in \mathbf{BCL}(x_\varepsilon, t_\varepsilon)$ with $b^1_\varepsilon \in T_{(x_\varepsilon, t_\varepsilon)}\mathbf{M}^l$ such that $(b^1_\varepsilon, c^1_\varepsilon, l^1_\varepsilon) \to (b, c, l)$ as $\varepsilon \to 0$. The $\mathbb{F}^l$-inequality for such triplet yields

$$-b^1_\varepsilon \cdot \left(D\phi(x_\varepsilon, t_\varepsilon) - \frac{\varepsilon b^\perp}{((x_\varepsilon - \bar{x}, t_\varepsilon - \bar{t}) \cdot b^\perp)^2} \right) + c^1_\varepsilon u(x_\varepsilon, t_\varepsilon) - l^1_\varepsilon \leq 0 \ .$$

But $(-b^1_\varepsilon) \cdot (-b^\perp) \to |b^\perp|^2 > 0$ as $\varepsilon \to 0$ and therefore the corresponding term is positive for ε small enough. We deduce that for such ε,

$$-b^1_\varepsilon \cdot D\phi(x_\varepsilon, t_\varepsilon) + c^1_\varepsilon u(x_\varepsilon, t_\varepsilon) - l^1_\varepsilon \leq 0 \ ,$$

and the conclusion follows by letting ε tend to 0.

Q.E.D.

22.2.3 Sub/Super-stratified problems and a generalized stability result

The two preceeding sections lead us to introduce the following definition

Definition 22.2.3 *Let* $\mathbb{S} = (\mathbf{M}^k, \mathbb{F}^k)_k$, $\tilde{\mathbb{S}} = (\tilde{\mathbf{M}}^k, \tilde{\mathbb{F}}^k)_k$ *be two* (SSP) *associated with the same* **BCL** *set.*

(*i*) $\tilde{\mathbb{S}}$ *is said to be a super-stratified problem of* $\mathbb{S}$ *if it can be deduced from* $\mathbb{M}$ *by applying a finite (or countable) number of time Proposition 22.2.1.*

(*ii*) $\tilde{\mathbb{S}}$ *is said to be a sub-stratified problem of* $\mathbb{S}$ *if it can be deduced from* $\mathbb{M}$ *by applying a finite (or countable) number of time Proposition 22.2.2.*

Before commenting these definitions, we use them to extend the notion of convergence of stratified problems.

Theorem 22.2.4 *Let* $\mathbb{S}_\varepsilon = (\mathbf{M}^k_\varepsilon, \mathbb{F}^k_\varepsilon)_{k,\varepsilon}$ *be a sequence of standard stratified problems such that there exists* $\mathbb{S} = (\mathbf{M}^k, \mathbb{F}^k)_k$, *a sequence* $\tilde{\mathbb{S}}_\varepsilon = (\tilde{\mathbf{M}}^k_\varepsilon, \tilde{\mathbb{F}}^k_\varepsilon)_{k,\varepsilon}$ *and* $\tilde{\mathbb{S}} = (\tilde{\mathbf{M}}^k, \tilde{\mathbb{F}}^k)_k$ *such that*

(*i*) *for any* $\varepsilon > 0$, $\tilde{\mathbb{S}}_\varepsilon$ *is a super-stratified problem of* $\mathbb{S}_\varepsilon$;

(*ii*) $\tilde{\mathbb{M}}_\varepsilon \xrightarrow{\text{RS}-w} \tilde{\mathbb{M}}$;

(*iii*) $\mathbb{S}$ *is a sub-stratified problem of* $\tilde{\mathbb{S}}$.

Then the stability results of Theorem 22.1.3 remain valid, taking into account the addition and removal of subsolution inequalities due to the super/sub stratification induced by $\tilde{\mathbb{S}}$ *and* $\tilde{\mathbb{S}}_\varepsilon$.

Theorem 22.2.4 makes precise a very simple and natural idea: of course, the conditions imposed by Theorem 22.1.3 on the convergence of stratified problems are very restrictive and do not cover (for example) the convergence of problems without discontinuities (like, for instance, Fillipov's approximation) to a problem with discontinuities.

To correct this defect, it suffices to introduce suitable "artificial" elements of stratification, using Proposition 22.2.1 (thus creating a super-stratified problem) then to use Theorem 22.1.3 and, at the end, we can drop some useless part of the obtained stratification using the elimination result of Proposition 22.2.2. Of course, all these

operations require suitable tangential continuity or normal controllability assumptions.

Example 22.3 — Denoting by $x = (x_1, x_2)$ the points in $\mathbb{R}^2$, let us consider a stationary discontinuous problem along the curve

$$\mathbf{M}^1_\varepsilon := \left\{x_2 = \gamma_\varepsilon(x_1) := \sqrt{x_1^2 + \varepsilon^2} : x_2 \in \mathbb{R}\right\},$$

the set $\mathbf{BCL}_\varepsilon$ being given by:

$$\mathbf{BCL}_\varepsilon(x) := \begin{cases} B(0,1) \times \{1\} \times \{1\} & \text{if } x_2 > \gamma_\varepsilon(x_1)\,, \\ B(0,1) \times \{1\} \times \{0\} & \text{if } x_2 < \gamma_\varepsilon(x_1)\,, \\ B(0,1) \times \{1\} \times [0,1] & \text{if } x_2 = \gamma_\varepsilon(x_1)\,. \end{cases} \tag{22.3}$$

The associated Hamiltonians are easy to compute: $\mathbb{F}^2_\varepsilon(x, r, p) = r + |p| - \mathbb{1}_{\{x_2 > \gamma_\varepsilon(x_1)\}}$ and $\mathbb{F}^1_\varepsilon(x, r, p') = r + |p'| - 1$. We recall that in $\mathbb{F}^1$, p' is the tangential component of the gradient.

The singular point $\{0,0\}$ appears as a specific singularity in the limit stratification $\mathbb{M}$, which is given by $\mathbf{M}^1 = \{x_2 = -x_1 : x_1 < 0\} \cup \{x_2 = x_1 : x_1 > 0\}$, $\mathbf{M}^0 = \{(0,0)\}$ and $\mathbb{M}^2 = \mathbb{R}^2 \setminus (\mathbf{M}^1 \cup \mathbf{M}^0)$. So, in order to understand the limit as $\varepsilon \to 0$, we creat an artificial singularity $\tilde{\mathbf{M}}^0_\varepsilon = \{(0,0)\}$ in $\mathbb{M}_\varepsilon$, respecting the structure of $\mathbb{M}$. Let also $\tilde{\mathbf{M}}^1_\varepsilon := \{x_2 = \gamma_\varepsilon(x_2) : x_1 < 0\} \cup \{x_2 = \gamma_\varepsilon(x_1) : x_1 > 0\}$ and $\tilde{\mathbf{M}}^2 = \mathbb{R}^2 \setminus (\tilde{\mathbf{M}}^1 \cup \tilde{\mathbf{M}}^0)$. The associated set of Hamiltonians is essentially the same, except that there is a new one: $\mathbb{F}^0_\varepsilon((0,0), r, p) = r - 1$.

Now, passing to the limit we get

$$\begin{cases} \limsup{}^*\, \mathbb{F}_\varepsilon(x, r, p) = r + |p| - \mathbb{1}_{\{x_2 > |x_1|\}} & \text{in } \mathbb{R}^2\,, \\ \liminf{}_*\, \mathbb{F}_\varepsilon(x, r, p) = r + |p| - \mathbb{1}_{\{x_2 \geq |x_1|\}} & \text{in } \mathbb{R}^2\,, \\ \liminf{}_*\, \tilde{\mathbb{F}}^2_\varepsilon(x, r, p) = r + |p| - \mathbb{1}_{\{x_2 > |x_1|\}} & \text{in } \tilde{\mathbf{M}}^2\,, \\ \liminf{}_*\, \tilde{\mathbb{F}}^1_\varepsilon(x, r, p) = r + |p| - 1 & \text{on } \tilde{\mathbf{M}}^1\,, \\ \liminf{}_*\, \tilde{\mathbb{F}}^0_\varepsilon(0, r, p) = r - 1 & \text{at } \tilde{\mathbf{M}}^0\,. \end{cases}$$

On the other hand, the limit **BCL** is given by (22.3) with $\gamma(x_1) := |x_1|$ instead of γ_ε. So, we see that the $\liminf_*$ above coincide with the various Hamiltonians associated with **BCL** and the stability property works. Notice that here we do need to perform a substratification (step (*iii*) in Theorem 22.2.4).

Chapter 23

Applications

23.1 A crystal growth model – where the stratified formulation is needed

The following problem concerns a model of 2-d nucleation in crystal growth phenomenon. In [101], Giga and Hamamuki use concave Hamiltonians but we re-formulate the equations with convex ones to be in the framework of this book. Moreover, we consider the problem in $\mathbb{R}^N$ instead of $\mathbb{R}^2$ since this does not create any additional difficulty.

The simplest equation takes the form

$$u_t + |D_x u| = I(x) \quad \text{in } \mathbb{R}^N \times (0, T_f) \tag{23.1}$$

where the function $I : \mathbb{R}^N \to \mathbb{R}$ is given by

$$I(x) = \begin{cases} 1 & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

This equation is associated with a bounded, continuous initial data

$$u(x, 0) = u_0(x) \quad \text{in } \mathbb{R}^N . \tag{23.2}$$

23.1.1 Ishii solutions

Of course, the key difficulty in this problem comes from the discontinuity of I. In terms of classical viscosity solutions' theory, Ishii's definition yields the subsolution

condition

$$u_t + |Du| \leq I^*(x) = 1 \quad \text{in } \mathbb{R}^N \times (0, T_f) \,,$$

and the important information that $I(0) = 0$ completely disappears here. As a consequence, one easily checks that $u(x,t) = t$ is an Ishii subsolution associated to the initial data $u_0(x) = 0$ in $\mathbb{R}^N$.

On the other hand, and formally for the time being, the classical control interpretation of (23.1) is that the system can evolve at any velocity b^x with $|b^x| \leq 1$, with cost $l = 1$ outside 0 and $l = 0$ at 0. In the case $u_0 = 0$, the natural value function is $U(x,t) = \min(|x|, t)$ in $\mathbb{R}^N \times [0, T_f]$ by adopting the strategy to go as quickly as possible to $x = 0$ and then to stay there.

Clearly $u(x,t) > U(x,t)$ if $|x| < t$ although U should be the "good solution" and u is a subsolution. Therefore, we cannot expect any comparison result in this framework. But it is also clear that u is a kind of "unnatural" subsolution, due to the fact that Ishii's definition erases the value 0 of I at $x = 0$ as we saw it above, which is undoubtedly an important information.

23.1.2 The stratified formulation

In this context, the stratified approach could certainly be simplified but let us stick to our framework: if $t > 0$, taking into account the upper semi-continuity and convexity of **BCL**, we introduce

$$\mathbf{BCL}(x,t) = \mathbf{BCL}(x) = \begin{cases} \left\{ \big((b^x, -1), 0, 1\big);\ |b^x| \leq 1 \right\} , & \text{if } x \neq 0 \,, \\ \left\{ \big((b^x, -1), 0, l\big);\ |b^x| \leq 1,\ 0 \leq l \leq 1 \right\} , & \text{if } x = 0 \,. \end{cases}$$

And if $t = 0$, $\mathbf{BCL}(x,0)$ is the convex hull of $\mathbf{BCL}(x) \cup \left\{ \big((0,0), 1, u_0(x)\big) \right\}$.

The stratification of $\mathbb{R}^N \times (0, T_f)$ just contains $\mathbf{M}^1 = \{0\} \times (0, T_f)$ and $\mathbf{M}^{N+1} = (\mathbb{R}^N \setminus \{0\}) \times (0, T_f)$ and, since $I(x) = 1$ in $\mathbf{M}^{N+1}$,

$$\mathbb{F}^{N+1}(x,t,p) = p_t + |p_x| - 1 \,.$$

While, since $b = (b^x, -1) \in T_{(0,t)}\mathbf{M}^1$ is equivalent to $b^x = 0$, it follows that

$$\mathbb{F}^1(t,p) = \max_{\substack{(b,c,l) \in \mathbf{BCL}(0) \\ b^x = 0}} \{p_t - l\} = p_t \,.$$

For $t = 0$, we just get the classical initial condition since $b^t \equiv -1$ for any $(b,c,l) \in \mathbf{BCL}(x)$ and for any x.

Therefore, a subsolution[(1)] of the problem is an u.s.c. function $u : \mathbb{R}^N \times [0, T_f] \to \mathbb{R}$ satisfying

$$u_t + |D_x u| \leq 1 \quad \text{in } \mathbb{R}^N \times (0, T_f) \,, \tag{23.3}$$

$$u_t \leq 0 \quad \text{on } \mathbf{M}^1 \,, \tag{23.4}$$

this last subsolution inequality being understood as a 1-d inequality which is obtained by looking at maxima of $u(0,t) - \phi(t)$ for smooth functions ϕ, while the first one is just the classical Ishii subsolution definition.

A supersolution of the problem is a l.s.c. function $v : \mathbb{R}^N \times [0, T_f] \to \mathbb{R}$ which satisfies

$$v_t + |D_x v| \geq I(x) \quad \text{in } \mathbb{R}^N \times (0, T_f) \,. \tag{23.5}$$

As we developed in the previous chapters, the stratified formulation consists in super-imposing the right subsolution inequalities on $\mathbf{M}^1$, while the supersolution condition is nothing but the classical Ishii conditions. Finally it is easy to see that the $\mathbb{F}_{init}$-conditions reduce to

$$u(x,0) \leq u_0(x) \leq v(x,0) \quad \text{in } \mathbb{R}^N \,. \tag{23.6}$$

In this framework, several results hold

Theorem 23.1.1 — *Crystal growth problem.*

(*i*) *A comparison result between stratified sub and supersolutions of* (23.1)–(23.2)*, i.e. sub and supersolutions which satisfy* (23.3)–(23.4) *and* (23.5) *respectively, with* (23.6)*.*

(*ii*) *There exists a unique stratified solution of* (23.1)–(23.2)*, which is given by*

$$U(x,t) = \inf \left\{ \int_0^t I(X(s))ds + u_0(X(t));\ X(0) = x, \quad |\dot{X}(s)| \leq 1 \right\} \,.$$

(*iii*) *This solution is the minimal Ishii viscosity solution.*

(*iv*) *Finally, if* $(I_k)_k$ *is a sequence of continuous functions such that*

$$\liminf_k{}_* \, I_k(x) = I(x) \quad \textit{and} \quad \limsup_k{}^* \, I_k(x) = I^*(x) = 1 \,,$$

then the unique (classical) viscosity solutions u_k *associated to* I_k *converges locally uniformly to* U*.*

(1) Here we use the notion of "strong stratified subsolution" to have the $\mathbb{F}_* \leq 0$-inequlity at $x = 0$.

Proof — The proof just consists in applying the result of Chapters 20, 21 and 22, and therefore in checking the normal controllability and tangential regularity assumptions, which are obvious here. Then, comparison result (i) is just a very particular case of Theorem 20.4.1, (ii) is obtained by examining carefully the value function of the stratified problem.

For (iii), it is enough to remark that any Ishii supersolution is a supersolution of the stratified problem, as it was done in Corollary 21.2.1.

Finally, (iv) is a straightforward adaptation of Chapter 22: indeed, there exists a sequence $x_k \to 0$ such that $I_k(x_k) \to 0$ and using the stratification $\mathbf{M}_k^1 = \{x_k\} \times (0, T_f)$ and $\mathbf{M}_k^N = \left(\mathbb{R}^N \times (0, T_f)\right) \setminus \mathbf{M}_k^1$, Proposition 22.2.1 shows that

$$(u_k)_t \leq I_k(x_k) \quad \text{in } \mathbf{M}_k^1 \, .$$

Using the stability result (Corollary 22.1.5) and part (i) of Theorem 23.1.1 lead directly to (iv).

Q.E.D.

Remark 23.1.2 *In [101], Giga and Hamamuki tested several notions of solutions for (23.1)–(23.2) and remarked that most of them were not completely adapted: for the notion of D or $\bar{D}$-solutions, they tried to impose on $\mathbf{M}^1$ an Ishii subsolution inequality with $I(x)$, not $I^*(x)$. But this was a $\mathbb{R}^N \times (0, T_f)$- inequality, not a $\mathbf{M}^1$-one. Although imposing a stronger subsolution condition on $\mathbf{M}^1$ was going in the right direction, this inequality was too strong compared to (23.4), at least the $\bar{D}$-ones, and they found that the problem has no $\bar{D}$-solution in general. They ended up considering enveloppe solutions, i.e. using Result (iii) of Theorem 23.1.1.*

23.1.3 Generalization

Of course, the simplest case we study above can be generalized in several ways, even if we wish to stay in a similar context: it is clear enough that the case when I vanishes at several points instead of one can be treated exactly in the same way, just changing $\mathbf{M}^1$. A more intriguing case which is considered in [101] is when

$$I(x) = \begin{cases} 1 & \text{if } x \notin \mathcal{S} \, , \\ 0 & \text{if } x \in \mathcal{S} \, , \end{cases}$$

for some closed subset $\mathcal{S}$ of $\mathbb{R}^N$.

Giga and Hamamuki aim at treating the case of very general closed subsets $\mathcal{S}$, which does not seem possible in our framework—though maybe we are missing something

here. A natural assumption for us is the following: there exists a stratification $\tilde{\mathbf{M}} = (\tilde{\mathbf{M}}^k)_k$ of $\mathbb{R}^N$ such that

$$\tilde{\mathbf{M}}^N = \mathcal{S}^c \cup \mathrm{Int}\,(\mathcal{S})\ ,$$

where $\mathrm{Int}\,(\mathcal{S})$ denotes the interior of $\mathcal{S}$, and

$$\partial\mathcal{S} = \tilde{\mathbf{M}}^{N-1} \cup \tilde{\mathbf{M}}^{N-2} \cdots \cup \tilde{\mathbf{M}}^0\ .$$

Once this hypothesis holds, we then set $\mathbf{M}^k = \tilde{\mathbf{M}}^{k-1} \times (0, T_f)$ for $1 \le k \le N+1$.

Clearly this assumption on $\mathcal{S}$ implies that $\partial\mathcal{S}$ has some regularity properties but, at least, it allows to use all the stratification arguments and therefore all the above results can be extended thoroughly.

23.2 Combustion – where the stratified formulation may unexpectedly help

In [39], motivated by a model of solid combustion in heterogeneous media, Roquejoffre and the first author studied the time-asymptotic behaviour of flame fronts evolving with a periodic space-dependent normal velocity. By using the "level-set approach", the authors introduce an Eikonal Equation

$$u_t + R(x)|Du| = 0 \quad \text{in } \mathbb{R}^N \times (0, +\infty)\ , \tag{23.7}$$

where, in the most standard case, $R : \mathbb{R}^N \to \mathbb{R}$ is a positive, Lipschitz continuous function.

In [39], results on the propagation are given, in particular on the asymptotic velocity but only in the case of Lipschitz continuous functions R. However, an interesting case—which is the purpose of an entire but formal section in [39]—concerns the case when R is discontinuous, given in $\mathbb{R}^2$ by

$$R(x) = R(x_1, x_2) = \begin{cases} M & \text{if } x_1 \in \mathbb{Z} \\ m & \text{otherwise,} \end{cases}$$

where m, M are positive constants. The interesting case is when $m \ll M$ for which we have "lines with maximal speed".

The stratified approach allows to bridge the gap between the formal results in [39] in the discontinuous case, and detailed proofs. This section is devoted to expose such content and we hope this will help the reader be convinced that the classical proofs for the homogenization of Hamilton-Jacobi Equations extend easily to the discontinuous case provided the right stratified formulation is used.

23.2.1 The level-set approach

We recall that the "level-set approach" consists in identifying a moving front Γ_t with the zero-level-set of a solution u of a "geometric type" equation, for which one has a unique viscosity solution, *i.e.* $\Gamma_t = \{x \in \mathbb{R}^N : u(x,t) = 0\}$,

Based on an idea appearing in Barles [28] for constant normal velocity, the "level-set approach" was first used for numerical computations by Osher and Sethian [142] who did these computations for more general normal velocities, in particular curvature dependent ones. Then Evans and Spruck [78], Chen, Giga and Goto [59] developed the theoretical basis. We also refer to Souganidis [159, 158] and to [40] for a complete description of the "level-set approach" but also for applications to the study of moving fronts in reaction-diffusion equations.

As we already mentioned above, the key idea here is to represent the moving front $t \mapsto \Gamma_t$ using the level-set, and in general the 0-level-set, of a continuous function $u : \mathbb{R}^N \times [0, +\infty) \to \mathbb{R}$, typically a solution of Equation (23.7) or a more general parabolic equation

$$u_t + F(x, t, D_x u, D^2_{xx} u) = 0 \quad \text{in } \mathbb{R}^N \times (0, T)\ . \tag{23.8}$$

where F satisfies suitable properties. But, in fact, one remarks that a more adapted way of describing things consists in saying that the "level-set approach" actually describes the evolution of a domain $t \mapsto \Omega_t$, whose boundary is precisely Γ_t. In combustion, Ω_t typically represents the "burnt region" while Γ_t is the flame front and $\mathbb{R}^N \setminus (\Omega_t \cup \Gamma_t)$ is the "unburnt region".

The key result of the "level-set approach" can be described in the following way: suppose that we can solve (23.8) to gether with any initial data

$$u(x, 0) = u_0(x) \quad \text{in } \mathbb{R}^N\ , \tag{23.9}$$

where $u_0 \in C(\mathbb{R}^N)$ represents the front at time $t = 0$ in the sense that $\Gamma_0 = \{x : u_0(x) = 0\}$ and, for example, $\Omega_0 = \{x : \ u_0(x) < 0\}$ and $\mathbb{R}^N \setminus (\Omega_0 \cup \Gamma_0) = \{x : u_0(x) > 0\}$. Then the sets

$$\Omega_t = \{x : \ u(x,t) < 0\},\ \Gamma_t = \{x : \ u(x,t) = 0\} \ \text{ and } \ \mathbb{R}^N \setminus (\Omega_t \cup \Gamma_t) = \{x : \ u(x,t) > 0\}$$

are independent of the choice of u_0 satisfying the above conditions, but they depend only on Ω_0, Γ_0 and F. Of course, opposite signs can be chosen for u_0 in Ω_0 and $\mathbb{R}^N \setminus (\Omega_0 \cup \Gamma_0)$: a similar result holds and we come back on the effect of this change later.

Without entering into details, the above result is based on two key properties of the equation: first a comparison result for bounded continuous sub and supersolutions

and then the fact that (23.8) is invariant by change of unknown function $u \to \varphi(u)$, for all C^1-change φ such that $\varphi' > 0$ in $\mathbb{R}$.

Clearly Equation (23.7) satisfies these two conditions when R is a positive, Lipschitz continuous function since the classical existence and uniqueness theory applies. This allows to define $t \mapsto \Gamma_t$ as the level-set evolution of Γ_0 with normal velocity R. In addition, the solution u is given by the control formula

$$u(x,t) = \inf\left\{u_0(\gamma(t)) : \gamma(0) = x,\ |\dot\gamma(s)| \le R(\gamma(s))\right\} \tag{23.10}$$

where γ is taken among all piecewise C^1 curves.

On this example, the role of the choice of the signs of u_0 is clear: by Equation (23.7), $u_t \le 0$ and therefore, if the burnt region is defined by $\Omega_t = \{x :\ u(x,t) < 0\}$, it increases, an expected phenomena. With the choice of the other sign, the unburnt region would increase, which would be unsatisfactory from the modelling point of view.

Hence, the choice of the signs of u_0 in Ω_0 and $\mathbb{R}^N \setminus (\Omega_0 \cup \Gamma_0)$, to gether with the equation, gives the direction of propagation of the front by implying the expansion or shrinking of Ω_t, the direction of propagation for Γ_t being either outward or inward to Ω_t in one or the other case. Such property holds in general for level-sets equations, even if, for the Mean Curvature Equation,

$$u_t - \Delta u + \frac{\langle D^2_{xx} u D_x u, D_x u\rangle}{|D_x u|^2} = 0 \quad \text{ in } \mathbb{R}^N \times (0,T)\ ,$$

the signs of u_0 are irrelevant.

As we said, the reader will find in [39] results on this propagation and on the asymptotic velocity in the case of Lipschitz continuous functions R, the discontinuous case being only considered formally.

23.2.2 The stratified formulation

We extend the discontinuous $\mathbb{R}^2$-framework to a $\mathbb{R}^N$-one by setting

$$R(x) = R(x', x_N) = R(x') = \begin{cases} M & \text{if } x' \in \mathbb{Z}^{N-1} \\ m & \text{otherwise,} \end{cases}$$

where, as usual $x = (x', x_N)$ with $x' \in \mathbb{R}^{N-1}$, addressing the problem through the stratified formulation. More precisely, we consider the stratification $\mathbb{R}^N \times (0, +\infty) =$

$\mathbf{M}^2 \cup \mathbf{M}^{N+1}$ where $\mathbf{M}^2 = (\mathbb{Z}^{N-1} \times \mathbb{R}) \times (0, +\infty)$, and $\mathbf{M}^{N+1}$ is its complementary set in $\mathbb{R}^N \times (0, +\infty)$. Next, let

$$\mathbf{BCL}(x,t) = \mathbf{BCL}(x) = \begin{cases} \{((mv,-1),0,0);\ v \in \mathbb{R}^N,\ |v| \le 1\} & \text{if } x \in \mathbf{M}^{N+1}\,, \\ \{((Mv,-1),0,0);\ v \in \mathbb{R}^N,\ |v| \le 1\} & \text{if } x \in \mathbf{M}^2\,. \end{cases}$$

Notice that, since $M > m$, **BCL** is actually uppersemi-continuous on $\mathbf{M}^2$. Therefore a (strong) stratified subsolution $u : \mathbb{R}^N \times (0, +\infty)$ of (23.7) is an u.s.c. function with satisfies

$$u_t + m|Du| \le 0 \quad \text{in } \mathbb{R}^N \times (0, +\infty)\,, \tag{23.11}$$

$$u_t + M|Du| \le 0 \quad \text{in } \mathbf{M}^2 \times (0, +\infty)\,, \tag{23.12}$$

while a stratified supersolution $v : \mathbb{R}^N \times (0, +\infty)$ of (23.7) is a l.s.c. function satisfying

$$v_t + R(x)|Dv| \ge 0 \quad \text{in } \mathbb{R}^N \times (0, +\infty)\,. \tag{23.13}$$

Using results of Section 20.4, one can easily prove the

Theorem 23.2.1 *For any $u_0 \in C(\mathbb{R}^N)$, problem (23.7)-(23.9) has a unique stratified solution given by (23.10). Moreover, a comparison result holds for this problem.*

We leave the proof to the reader since it comes from a simple checking of the assumptions required in Section 20.4.

23.2.3 Asymptotic analysis

The next question concerns the asymptotic velocity when $t \to +\infty$. A classical method consists in looking first at initial data of the form $u_0(x) = p \cdot x$ for some $p \in \mathbb{R}^N$, in order to obtain the velocity when the normal direction is p.

The classical hyperbolic scaling $(x,t) \to (x/\varepsilon, t/\varepsilon)$, which preserves velocities, allows to reduce to finite times the asymptotic behaviour, leading to study the equation satisfied by the rescaled function $u_\varepsilon(x,t) := \varepsilon u(x/\varepsilon, t/\varepsilon)$, namely

$$(u_\varepsilon)_t + R(\frac{x}{\varepsilon})|Du_\varepsilon| = 0 \quad \text{in } \mathbb{R}^N \times (0, +\infty)\,. \tag{23.14}$$

We also notice that the initial data is unchanged by the scaling, *i.e.* $u_\varepsilon(x,0) = p \cdot x$. We can formulate the result in the following simple form

Theorem 23.2.2 *The following limit holds*

$$\lim_{\varepsilon\to 0} u_\varepsilon(x,t) = p\cdot x - t\bar H(p)$$

where, for $p=(p_1,p_2,\cdots,p_N)$, $\bar H(p)=\max(M|p_N|,m|p|)$.

This theorem implies in particular that if $|p|=1$, $\bar H(p)$ is the velocity of the front in the direction p. Let us first remark that, by Theorem 23.2.1, since $m\le R(x)\le M$ in $\mathbb{R}^N$,

$$p\cdot x - Mt \le u_\varepsilon(x,t) \le p\cdot x - mt \quad \text{ in } \mathbb{R}^N\times(0,+\infty)\ .$$

Therefore u_ε is uniformly locally bounded.

Now, in order to prove the convergence result we provide two proofs. The more general consist in following the method of Lions, Papanicolaou and Varadhan [132] together with the perturbed test-function method of Evans [79, 80] as in the article of Briani, Tchou and the two authors [34]. These arguments allows to treat far more general problems but here we can also provide simplified arguments.

Proof of Theorem 23.2.2: the common ingredients —

The first step is the

Lemma 23.2.3 *For any* $p\in\mathbb{R}^N$, *there exists a unique constant* $\bar H(p)$ *such that the equation*

$$R(x)|p+D_xw| = \bar H(p) \quad \text{in } \mathbb{R}^N \tag{23.15}$$

has a bounded, Lipschitz continuous stratified solution $w=w(x,p)$. *Moreover,* $\bar H(p)=\max(M|p_N|,m|p|)$.

Proof — This lemma is classical and so is its proof, except that, in our case, R is discontinuous but the method remains the same.

(a) for $0<\alpha\ll 1$, we consider the equation

$$R(x)|p+D_xw^\alpha| + \alpha w^\alpha = 0 \quad \text{in } \mathbb{R}^N\ . \tag{23.16}$$

Borrowing arguments in Section 20.4 and Chapter 22, it is easy to prove that (23.16) has a unique stratified solution: if R is Lipschitz continuous, such result is standard and can easily be obtained by the Perron method of Ishii [119]. Here we can use an approximation of R by Lipschitz continuous functions from above since R is u.s.c. and then to use the stability results of Chapter 22.

Now, w^α depends only on x' since R depends only on x' and it is $\mathbb{Z}^{N-1}$-periodic since R is $\mathbb{Z}^{N-1}$-periodic: indeed, $w^\alpha(x', x_N)$ and $w^\alpha(x'+k, x_N+h)$ are solutions of the same equation for any $k \in \mathbb{Z}^{N-1}$ and $h \in \mathbb{R}$ and therefore they are equal. Hence, for $k = 0$, $w^\alpha(x', x_N) = w^\alpha(x', x_N + h)$ for any $h \in \mathbb{R}$ and for $h = 0$, $w^\alpha(x', x_N) = w^\alpha(x' + k, x_N)$ for any $k \in \mathbb{Z}^{N-1}$, proving the claims.

Moreover, thanks again to the comparison results, $-M|p| \leq \alpha w^\alpha(x) \leq -m|p|$ in $\mathbb{R}^N$ since $-M|p|/\alpha$ and $-m|p|/\alpha$ are respectively sub and supersolution of (23.16). Finally, the w^α are equi-Lipschitz continuous since αw^α is uniformly bounded and the term $R(x)|p+q|$ is coercive in q, uniformly in x. We point out that, in all the proof, we use extensively the comparison result for stratified solutions of (23.16).

(b) Applying Ascoli's Theorem to the sequence $(w^\alpha(\cdot) - w^\alpha(0))_\alpha$ which is equi-Lipschitz continuous and equi-bounded by the periodicity of each w^α, we can extract a subsequence $(w^{\alpha'}(\cdot) - w^{\alpha'}(0))_{\alpha'}$ which converges uniformly in $\mathbb{R}^N$ (by periodicity) to a periodic, Lipschitz continuous function w. Moreover, we can assume that $\alpha' w^{\alpha'}(0)$ converges to a constant $-\lambda$. By the stability result for stratified solutions, w is a stratified solution of

$$R(x)|p + D_x w| = \lambda \quad \text{in } \mathbb{R}^N \ .$$

In order to prove that λ is unique, we assume by contradiction that there exists a bounded stratified solution w' of

$$R(x)|p + D_x w'| = \lambda' \quad \text{in } \mathbb{R}^N \ ,$$

for some different constant λ'.

Since the functions $(x,t) \mapsto w(x) - \lambda t$ and $(x,t) \mapsto w'(x) - \lambda' t$ are stratified solutions of the same equation, therefore for any $t > 0$

$$||(w(x) - \lambda t) - (w'(x) - \lambda' t)||_\infty \leq ||w(x) - w'(x)||_\infty \ ,$$

an inequality which can hold for large t only if $\lambda = \lambda'$, proving the claim about the uniqueness of λ.

(c) It remains to show that $\lambda = \bar{H}(p)$ is given by $\max(M|p_N|, m|p|)$. By the Dynamic Programming Principle, we have, for any $\theta > 0$

$$w(x) = \inf \left\{ \int_0^\theta (b(s) \cdot p + \bar{H}(p))ds + w(X(\theta)); \ X(0) = x, \ \dot{X}(s) = b(s) \in B(X(s)) \right\} \ ,$$

where $\mathbf{B}(y) = MB(0,1)$ if $y \in \mathbb{Z}^{N-1}$ and $\mathbf{B}(y) = mB(0,1)$ if $y \notin \mathbb{Z}^{N-1}$. Here we trust the reader will be able to translate in this setting the framework of Chapters 20 and 21 without any difficulty, even if we have dropped the b^t-term since $b^t \equiv -1$.

In order to compute the infimum in the above formula, there are several choice for $b(s)$. First, at any point of $\mathbb{R}^N$, one can choose $|b(s)| = m$ which comes associated to a minimal cost $b(s) \cdot p = -m|p|$; if $X(s) \in \mathbb{Z}^{N-1}$, we can choose $b(s) = +/- Me_N$ to stay on $\mathbb{Z}^{N-1}$ and then the minimal cost becomes $b(s) \cdot p = -M|p_N|$. The optimal choice, at least if $x \in \mathbf{M}^2$ is $\min(-m|p|, -M|p_N|) = -\max(m|p|, M|p_N|)$ since, by the above choice of $b(s)$, we have $X(s) \in \mathbf{M}^2$ if the maximum is $M|p_N|$. Choosing this strategy for $x \in \mathbf{M}^2$, we see that for any $\theta > 0$,

$$w(x) \leq \theta(-\max(m|p|, M|p_N|) + \bar{H}(p)) + w(X(\theta))$$

and therefore $\bar{H}(p) \geq \max(m|p|, M|p_N|)$.

To prove the equality, we examine the two different cases: if the maximum is $m|p|$ and $p \neq 0$ (the case $p = 0$ is obvious and $\bar{H}(0) = 0$ since w can be taken as a constant), we notice that, for any $b(s)$, $b(s) \cdot p \geq -m|p|$. Hence $b(s) \cdot p + \bar{H}(p) \geq -m|p| + \bar{H}(p)$ and if $-m|p| + \bar{H}(p) \geq \eta > 0$, we get for any choice of $b(s)$

$$\int_0^\theta (b(s) \cdot p + \bar{H}(p))ds \geq \theta\eta,$$

which leads to a clear contradiction with the boundedness of w.

If the maximum is $M|p_N|$, we cannot have $\bar{H}(p) \geq M|p_N| + \eta \geq m|p| + \eta$ exactly by the same argument: either $X(s) \in \mathbb{Z}^{N-1}$ and the minimal cost is $b(s) = -M|p_N|$, while if $X(s) \notin \mathbb{Z}^{N-1}$, it is $-m|p|$. In any case, $b(s) \cdot p + \bar{H}(p) \geq \eta$ and we conclude as above.

Q.E.D.

We now continue by the

A.– Simplified proof.

Because of the very simple form of the initial data for u_ε and even more, the simple form of the limit of the $\{u_\varepsilon\}$, there is a very quick proof to conclude. Indeed the function $\chi_\varepsilon(x,t) := p \cdot x - t\bar{H}(p) - \varepsilon w\,(x/\varepsilon, p)$ is a solution of (23.14) and moreover

$$\chi_\varepsilon(x,0) - \varepsilon||w(\cdot,p)||_\infty \leq u_\varepsilon(x,0) \leq \chi_\varepsilon(x,0) + \varepsilon||w(\cdot,p)||_\infty\,.$$

Therefore, using that $\chi_\varepsilon + C$ is also a solution for any constant C, by the comparison result we get

$$\chi_\varepsilon(x,t) - \varepsilon||w(\cdot,p)||_\infty \leq u_\varepsilon(x,t) \leq \chi_\varepsilon(x,t) + \varepsilon||w(\cdot,p)||_\infty\,.$$

Taking into account the form of χ_ε and the boundedness of w, this gives the result.

Now we turn to

B.– A more general proof.

Now we provide more general arguments, which allow to take care of more general initial data and limits. Here, proving the convergence of the sequence $\{u_\varepsilon\}$ relies on the perturbed test-function method of Evans [79, 80] as in the article of Briani, Tchou and the two authors [34].

Let us introduce the half-relaxed limits $\overline{u} = \limsup^* u_\varepsilon$ and $\underline{u} = \liminf_* u_\varepsilon$ which are well-defined since the sequence $\{u_\varepsilon\}$ is locally uniformly bounded. We recall that for each $p \in \mathbb{R}^N$, Lemma 23.2.3 provides a unique real denoted by $\bar{H}(p)$ such that there exists a solution w of ergodic problem (23.15). The key step is the

Lemma 23.2.4 *The functions $\overline{u}$ and $\underline{u}$ are respectively (classical) viscosity sub and supersolution of*

$$u_t + \bar{H}(Du) = 0 \quad \text{in } \mathbb{R}^N \times (0, +\infty)\,, \tag{23.17}$$

$$u(x,0) = p \cdot x \quad \text{in } \mathbb{R}^N\,. \tag{23.18}$$

Proof — We provide the proof only for $\overline{u}$, the one for $\underline{u}$ being analogous.

(a) Let $\phi : \mathbb{R}^N \times (0,+\infty) \to \mathbb{R}$ be a smooth test-function and let $(\bar{x},\bar{t})$ be a strict local maximum point of $\overline{u} - \phi$. Since we may assume *w.l.o.g.* that $(\overline{u} - \phi)(\bar{x},\bar{t}) = 0$, we know that for $r, h > 0$ small enough, $(\overline{u} - \phi)(x,t) \le 0$ in $Q_{r,h}^{\bar{x},\bar{t}}$. Moreover, there exists some $\delta = \delta(r,h) > 0$ such that $(\overline{u} - \phi)(x,t) \le -2\delta$ on $\partial_p Q_{r,h}^{\bar{x},\bar{t}}$.

We want to show that $\phi_t(\bar{x},\bar{t}) + \bar{H}(D\phi(\bar{x},\bar{t})) \le 0$ and to do so, we argue by contradiction, assuming on the contrary that $\phi_t(\bar{x},\bar{t}) + \bar{H}(D\phi(\bar{x},\bar{t})) > 0$.

(b) The first step consists in considering the *perturbed test-function*

$$\phi_\varepsilon(x,t) := \phi(x,t) + \varepsilon w\Big(\frac{x}{\varepsilon}, D\phi(\bar{x},\bar{t})\Big)$$

where w is defined in Lemma 23.2.3, and to look at $(\phi_\varepsilon)_t(x,t) + R(\frac{x}{\varepsilon})|D\phi_\varepsilon(x,t)|$ in $Q_{r,h}^{\bar{x},\bar{t}}$. Formally, using the equation satisfied by $w(\cdot, D\phi(\bar{x},\bar{t}))$, we have

$$\begin{aligned}
(\phi_\varepsilon)_t(x,t) + R(\tfrac{x}{\varepsilon})|D\phi_\varepsilon(x,t)| &= \phi_t(x,t) + R(\tfrac{x}{\varepsilon})\Big|D\phi(x,t) + D_x w(\tfrac{x}{\varepsilon}, D\phi(\bar{x},\bar{t}))\Big| \\
&= \phi_t(\bar{x},\bar{t}) + R(\tfrac{x}{\varepsilon})\Big|D\phi(\bar{x},\bar{t}) + D_x w(\tfrac{x}{\varepsilon}, D\phi(\bar{x},\bar{t}))\Big| + O(r) + O(h) \\
&= \phi_t(\bar{x},\bar{t}) + \bar{H}(D\phi(\bar{x},\bar{t})) + O(r) + O(h),
\end{aligned}$$

the terms $O(r), O(h)$ coming from the replacement of $\phi_t(x,t)$ by $\phi_t(\bar{x},\bar{t})$ and of $D\phi(x,t)$ by $D\phi(\bar{x},\bar{t})$. Therefore, taking potentially r, h, δ smaller, we have

$$(\phi_\varepsilon)_t(x,t) + R(\frac{x}{\varepsilon})|D\phi_\varepsilon(x,t)| \geq \delta > 0 \quad \text{in } Q^{\bar{x},\bar{t}}_{r,h}.$$

The formal computations above can be justified by looking carefully at the stratification formulation but such checking does not present any difficulty, it only consists in adding the specific tangential inequality on the lines in $\mathbf{M}^2 = \mathbb{Z}^{N-1} \times \mathbb{R} \times (0, +\infty)$.

(c) From the first part of this proof, we know that $\phi(x,t) \geq \overline{u} + 2\delta$ on $\partial_p Q^{\bar{x},\bar{t}}_{r,h}$. Therefore, by the definition of $\overline{u}$, it follows that for ε small enough $\phi_\varepsilon(x,t) \geq u_\varepsilon + \delta$ on $\partial_p Q^{\bar{x},\bar{t}}_{r,h}$.

Using the local comparison result for stratified solutions, we conclude that for any $\varepsilon > 0$ small enough, $\phi_\varepsilon(x,t) \geq u_\varepsilon + \delta$ in $Q^{\bar{x},\bar{t}}_{r,h}$. Then, passing to the $\limsup^*$ yields

$$\phi(x,t) \geq \overline{u} + \delta \quad \text{in } Q^{\bar{x},\bar{t}}_{r,h}\,,$$

which contradicts the fact that $(\overline{u} - \phi)(\bar{x},\bar{t}) = 0$. Hence we conclude that $\overline{u}$ is indeed a (classical) subsolution of (23.17)–(23.18).

Q.E.D.

Notice that in the above proof we use the classical notion of viscosity solutions for the limit problem (23.17) while we use stratifed solutions for the construction of the perturbed test-function at level $\varepsilon > 0$. The proof now follows easily

Since $\chi(x,t) := p \cdot x - t\bar{H}(p)$ is an explicit solution of (23.17)-(23.18) for which a (classical) comparison result holds, we deduce that

$$\overline{u}(x,t) \leq p \cdot x - t\bar{H}(p) \leq \underline{u}(x,t) \quad \text{in } \mathbb{R}^N \times [0, +\infty)\,.$$

Using the usual arguments of the half-relaxed limits method *cf.* Section 3.1.2, we conclude that $u_\varepsilon \to u = \overline{u} = \underline{u}$ and the result is proved.

Q.E.D.

23.3 Large time behavior

This section enters a little bit more in the description of open problems, that we consider mainly in Chapter 24. However here we give substantially more information and some partial result. Before considering the case where discontinuities occur, let us (very) briefly recall the situation in the simple periodic framework.

23.3.1 A short overview of the periodic case

We consider here Hamilton-Jacobi Equations of the form

$$u_t + H(x, Du) = 0 \quad \text{in } \mathbb{R}^N \times (0, +\infty) \ ,$$

where $H(x,p)$ is convex and coercive in p, and periodic in x; for example, let us assume that it is $\mathbb{Z}^N$-periodic in x.

In this framework, the expected large time behavior of the solution $u(x,t)$ is an *ergodic behavior*, *i.e.*

$$u(x,t) = \lambda t + \bar{\phi}(x) + o(1) \quad \text{as } t \to +\infty,$$

where λ, the *ergodic constant*, is the unique constant such that the following *ergodic or additive eigenvalue problem* has a periodic solution ϕ

$$H(x, D\phi) = -\lambda \quad \text{in } \mathbb{R}^N \ ,$$

and $\bar{\phi}$ is one of the solutions of this ergodic problem[(2)].

We refer to [132] for the introduction of the ergodic problem which is nothing but the "cell problem" in homogenization and for the proof of the uniqueness of λ. Actually we already met this type of ergodic problem in Section 23.2: some of the basic arguments to study it are given in the proof of Lemma 23.2.3, while a concrete use of its solution is done in the proof of Lemma 23.2.4.

The proof of such ergodic behaviors can be done in two different ways: either by the "Weak KAM" theory initiated by Fathi [82, 83], based on dynamical systems arguments using control formulas; or by PDE-type methods.

The first results in this direction were obtained by Namah and Roquejoffre [139] for equations with a particular structure. In order to be a little bit more specific on the kind of results they obtained and the methods they used, we consider the simplest example where we can describe them, namely the case when the Hamiltonian is given by

$$H(x,p) = |p| - f(x) \quad \text{for } x \in \mathbb{R}^N,\ p \in \mathbb{R}^N,$$

the function f being periodic, continuous, positive and $K = \{x : f(x) = 0\}$ being a non-empty set. In this case, it can easily be proved that $\lambda = 0$ and the large time behavior is obtained by noticing first that $u(x,t)$ is decreasing in t on K; therefore it

(2) We remark that, if ϕ is a solution of the ergodic problem then $\phi + C$ is also a solution for any constant $C \in \mathbb{R}$; hence the ergodic problem has always many solutions and this invariance by the addition of constants is not, in general, the only cause of non-uniqueness.

converges on K. On the other hand, the half-relaxed limit method associated with a comparison result allows to deduce the convergence to a function $\bar{\phi}$ on $\mathbb{R}^N \setminus K$; here the key point to have such a comparison result is the fact that 0 is (locally) a strict subsolution. We conclude this short description by pointing out that here, periodicity does not play an important role, nor the convexity of H in p but only the fact that

$$H(x,p) \geq H(x,0) \quad \text{in } \mathbb{R}^N \times \mathbb{R}^N \text{ and } \max_{\mathbb{R}^N} H(x,0) = 0^{(3)} .$$

Results in the general framework of strictly convex Hamiltonians were then obtained by Fathi [82, 83] using the "Weak KAM" theory. In fact, they were not generalizations of the Namah and Roquejoffre results since, as the reader can see on the example we have chosen above, their result does not require strict convexity.

To give an idea of the "Weak KAM" theory, we first notice that we can assume *w.l.o.g.* that $\lambda = 0$ by replacing u by $u(x,t) - \lambda t$ and H by $H + \lambda$. Using such reduction, we then recall that when H is strictly convex, we can introduce the Lagrangian L, which is the Legendre-Fenchel transform of H, namely

$$L(x,v) = \sup_{q \in \mathbb{R}^N} \left\{ v \cdot q - H(x,q) \right\} ,$$

and the solution of the evolution equation can be written in terms of the Lagrangian as

$$u(x,t) = \inf \left\{ \int_0^t L(\gamma(s), \dot{\gamma}(s)) ds + u_0(\gamma(0));\ \gamma(t) = x \right\} ,$$

where u_0 is the initial data of u.

Furthermore, the solution of the ergodic problem satisfies some dynamical programming property like

$$\phi(x) = \inf \left\{ \int_0^t L(\gamma(s), \dot{\gamma}(s)) ds + \phi(\gamma(0));\ \gamma(t) = x \right\} \text{ for any } t > 0 .$$

In the "Weak KAM" theory, the main point is to identify the large time behavior of geodesics for the action functional

$$\int_0^t L(\gamma(s), \dot{\gamma}(s)) ds ,$$

the main result being that the large time behavior of $(\gamma, \dot{\gamma})$ is described in terms of the Aubry-Mather set. Here "large time" means both that t is large AND $(t - s)$ is

[3] We recall that $H(x,0)$ is $\mathbb{Z}^N$-periodic.

large too: more precisely, if we set $\tilde{\gamma}(s) = \gamma(s+t)$ for $-t \le s \le 0$, we are interested in the behavior of $(\tilde{\gamma}(s), \dot{\tilde{\gamma}}(s))$ as $s \to -\infty$.

One of the key results of Fathi was to prove that, for t and $t-s$ large enough (for example if s is bounded), the geodesics satisfy[(4)]

$$H\left(\gamma(s), \frac{\partial L}{\partial v}(\gamma(s), \dot{\gamma}(s))\right) \simeq 0\,.$$

One way or the other, this property is a cornerstone to obtain the large time behavior of u using the "Weak KAM" theory.

This approach was then revisited, simplified, developed in several directions and generalized by Roquejoffre [151], Davini and Siconolfi [69], Fathi and Siconolfi [85]. Of course, this short list of references is far from being complete. We refer to the book of Fathi [81] and his survey [84] for a more satisfactory one.

This general case can also be treated by PDE methods, with slightly more general assumptions than convexity, which was first done in Barles and Souganidis [27]. Here the idea was to show that $(u_t)^- \to 0$ as $t \to +\infty$. Roughly speaking, the consequence is that the solution looks like a subsolution of $H = 0$ for large t: indeed, if $(u_t)^- \to 0$, then $u_t \ge -o(1)$ and, using the equation $u_t + H = 0$, we deduce that $H \le o(1)$.

But, if the initial data is a subsolution, the behavior is well-known since $u(x,t)$ is increasing in t. Using this argument together with the "compactness" given by the periodic feature of the problem, one concludes easily. Such compactness is crucial and we refer to [41] for counter-examples in the case where we still have $(u_t)^- \to 0$ as $t \to +\infty$ but without "compactness".

Before considering the discontinuous case, we point out that both approaches can be extended to problems with boundary conditions, *cf.* for example Ishii [122] Barles, Mitake and Ishii [37].

23.3.2 The discontinuous framework

After this quick overview of the "continuous" theory, the question is: what could we expect to be easily extendable to the "discontinuous" case? Of course, because of the above framework where we aim at treating discontinuities of $H(x,p)$ in x, the natural stratifications of $\mathbb{R}^N \times \mathbb{R}$ we have to deal with take the form

$$\mathbf{M} = \tilde{\mathbf{M}} \times \mathbb{R} \quad \text{where } \tilde{\mathbf{M}} \text{ is a stratification of } \mathbb{R}^N\,,$$

hence we are not looking at general stratifications in x and t.

[(4)] we recall that $\lambda = 0$, otherwise we would have $-\lambda$ at the right-hand side here.

Let us give some ideas for the PDE approach and let people who are more experts than us to have a look at the "Weak KAM" theory in the discontinuous case [(5)]

(*i*) One point is clear: under suitable assumptions, we do not see any major difficulty to extend Namah-Roquejoffre type results in the discontinuous framework: they only rely on the half-relaxed limit method and comparison results, both ingredients which are available in the stratified case.

(*ii*) The Barles-Souganidis approach is more tricky since—even if it is completely transparent in the continuous case—the method involves at the same time sub and supersolutions properties, *i.e.* H_* and H^*, and, in general, a tripling of variables. This does not seem very convenient in the discontinuous case where doubling variables is already a major difficulty.

(*iii*) We believe that the $(u_t)^-$-estimate can be done by rewriting completely the Barles-Souganidis proof in terms of a comparison result but this is not completely straightforward and far beyond the scope of this book.

Let us check anyway that a relatively easy proof can be done under the stronger assumption [(6)]

($\mathbf{H_{asymp}}$) *For any $\varepsilon > 0$, there exists a bounded, C^1-function $(\phi_\varepsilon)_\varepsilon$ such that*

$$H^*(x, D\phi_\varepsilon(x)) \le \varepsilon \quad \text{in } \mathbb{R}^N \,. \tag{23.19}$$

This assumption may seem too strong because of the use of H^* (instead of H_*) but it allows to use the arguments as in [36] to obtain the $(u_t)^-$-estimate through a simple inf-convolution in t. More precisely, the idea is to introduce

$$v(x,t) := \inf_{s \ge t} \left\{ u(x,s) + (s-t)\eta(s) \right\} ,$$

for some suitable function η such that $\eta(s) \to 0$ as $s \to +\infty$, and then to use the stratified comparison result. Under the above assumption and with suitable hypothesis on H, this argument yields the estimate

$$u_t(x,t) \ge -\eta(t) \to 0 \,.$$

We point out that the proof uses a classical inf-convolution in a direction which is parallel to the discontinuities of H, and is therefore not affected by them. But again, this proof requires the above unnatural assumption to be useful in this context.

[(5)] a not so easy task since now L is discontinuous, at least on an hyperplane...

[(6)] We also assume here that $\lambda = 0$

23.3.3 An example

We consider the 1-d example

$$u_t + |u_x + p|^2 = V(x) \quad \text{in } \mathbb{R} \times (0, +\infty) \,,$$

where $p \in \mathbb{R}$ is a parameter and $V(x) = 0$ if $x \in \mathbb{Z}$ and $V(x) = 1$ if $x \in \mathbb{R} \setminus \mathbb{Z}$. Applying the above approach, the 1-d computations are easy

$$\lambda(p) = \begin{cases} 0 & \text{if } |p| \le 1, \\ 1 - |p|^2 & \text{otherwise.} \end{cases}$$

Concerning ϕ, it is given (up to an additive constant) by

- if $|p| \le 1$, and $x \in [0,1]$,

$$\phi(x) = \begin{cases} (1-p)x & \text{if } 0 \le x \le \frac{1+p}{2} \,, \\ (1+p)(1-x) & \text{if } \frac{1+p}{2} < x \le 1 \,, \end{cases}$$

  and then this function is extended by periodicity for $x \notin [0,1]$.

- if $|p| \ge 1$, $\phi(x) = 0$ for any $x \in \mathbb{R}$.

For the large time behavior, the following remarks can be made

(*i*) The case $p = 0$ is the Namah-Roquejoffre case which can be handled without any difficulty.

(*ii*) On the other hand, if $p \neq 0$, we are not anymore in the Namah-Roquejoffre framework and assumption (23.19) requires the existence of C^1-functions ϕ_ε such that

$$|\phi_\varepsilon'(x) + p|^2 \le 1 + \varepsilon \quad \text{on } \mathbb{R} \quad \text{and} \quad |\phi_\varepsilon'(x) + p| \le \varepsilon \quad \text{if } x \in \mathbb{Z} \,.$$

If $|p| < 1$ it is easy to check that the assumption is satisfied. Actually, since the Hamiltonian is independent of x and since ϕ has a particular form, we can even take a single function obtained by smoothing in a suitable way the solution of the ergodic problem which satisfies the inequality with $\varepsilon = 0$. But, for $|p| \ge 1$, the two properties which are required on ϕ_ε are incompatible with its boundedness.

(*iii*) If $|p| \geq 1$, one can conclude by the following arguments: let w be the unique solution of

$$w_t + |w_x + p|^2 = 1 \quad \text{in } \mathbb{R} \times (0, +\infty) \ ,$$
$$w(x, 0) = u_0(x) \quad \text{in } \mathbb{R} \ ,$$

where u_0 is a continuous, periodic initial data. Notice that we have replaced $V(x)$ by 1 in the equation.

For this equation, the ergodic problem has exactly the same ergodic constant $\lambda(p) = 1 - |p|^2$ and the same periodic solutions (the constant functions, this will be proved below). Since this equation is now continuous, we know that, as $t \to +\infty$,

$$w(x, t) = \lambda(p)t + \phi(x) + o(1) \ ,$$

where ϕ is a solution of the ergodic problem and the $o(1)$ is uniform on $\mathbb{R}$.

The equation for ϕ reads $|\phi_x + p|^2 = |p|^2$ and rewriting it as $2p\phi_x + |\phi_x|^2 = 0$, one proves easily that the periodic function $p\phi$ is decreasing and therefore $p\phi$ (hence ϕ) is a constant function. We claim that $\phi = \min_{\mathbb{R}} u_0$.

Indeed, applying the Oleinik-Lax formula to $\tilde{w} = w - \lambda(p)t$ yields

$$\tilde{w}(x, t) = \min_{y \in \mathbb{R}} \left(u_0(y) + \frac{1}{4t} (x - y - 2pt)^2 \right) \ ,$$

and therefore $\tilde{w}(x + 2pt, t) \leq u_0(x)$ for any x[(7)].

Choosing x such that $u_0(x) = \min_{\mathbb{R}} u_0$ and noticing that, by comparison, $\min_{\mathbb{R}} u_0 \leq \tilde{w}$ in $\mathbb{R} \times (0, +\infty)$, we have $\tilde{w}(x + 2pt, t) \equiv \min_{\mathbb{R}} u_0$. But the uniform convergence of $\tilde{w}$ to ϕ on $\mathbb{R}$ immediately yields that $\phi = \min_{\mathbb{R}} u_0$.

To conclude, we remark that, by comparison results

$$\min_{\mathbb{R}} u_0 \leq u(x, t) - \lambda(p)t \leq \tilde{w}(x, t) \quad \text{in } \mathbb{R} \times (0, +\infty) \ .$$

Indeed, the constant $\min_{\mathbb{R}} u_0$ is a subsolution of the equation satisfied by $u(x, t) - \lambda(p)t$, which is itself a subsolution for the $\tilde{w}$-equation. The conclusion immediately follows from the uniform convergence of $\tilde{w}$ to $\min_{\mathbb{R}} u_0$.

This last case mixes (in some sense) "weak KAM" arguments and pde ones: indeed we point out the important role of the geodesic $\gamma(t) = x + 2pt$ and of the behavior of the different solutions along the geodesic (of course we are here in a very simple framework). The key point in this case is that these geodesics cross the discontinuity in a transversal way, making it irrelevant. This is why u and w have the same behavior.

(7) The same result can be obtained by a careful examination of the pde satisfied by $\tilde{w}$.

23.4 Lower semicontinuous solutions à la Barron-Jensen

The extension of the Barron-Jensen approach to the stratified case requires a change of definition since it is based on the fact that, when considering equations with a convex Hamiltonian, one can just look at minimum points when testing both the sub and supersolutions properties. Of course, the same is true for stratified problems and leads to a new definition.

23.4.1 A typical l.s.c. Eikonal example

Before providing precise definitions and a comparison result, we want to examine a key example in order to recall the difficulties which are solved by the Barron-Jensen approach. We consider the Eikonal Equation

$$u_t + |D_x u| = 0 \quad \text{in } \mathbb{R}^N \times (0, T_f) \ ,$$

with a l.s.c. initial data

$$u(x,0) = g(x) = \begin{cases} 1 & \text{if } x \neq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Using—at least formally to begin with—the Oleinik-Lax formula, the "natural solution" is given by

$$u(x,t) := \min_{|y-x| \leq t} (g(y)) = \begin{cases} 0 & \text{if } |x| \leq 1 \ , \\ 1 & \text{if } |x| > 1 \ . \end{cases}$$

Therefore the solution is discontinuous and the approach via a **(SCR)** is useless in this l.s.c. framework. One wishes to prove, anyway, that u is the unique solution of the above problem.

Ishii's notion of viscosity solution is not well-adapted, in particular for subsolutions: for example, $w(x,t) = 1$ in $\mathbb{R}^N \times [0, T_f)$ is an Ishii viscosity subsolution because it satisfies the equation and $w^*(x,0) \leq g^*(x) = 1$ in $\mathbb{R}^N$. But we are far from having the expected inequality $w \leq u$ in $\mathbb{R}^N \times [0, T_f)$ that a comparison result would give.

On this example, it is clear that the problem comes from the initial data, and more precisely the way it is taken, since, with the classical viscosity solutions definition, the upper-semicontinuous enveloppe erases the value $g(0) = 0$. Similarly to the difficulties in the stratified framework, the subsolution inequality has to be reinforced at $t = 0$ but here we cannot impose the "stratification-like" inequality $w^*(0,0) \leq 0$ because

this inequality for an u.s.c. subsolution is clearly too strong, it is not even satisfied by u.

Therefore, if we wish to take into account l.s.c. initial data, we have to argue only with l.s.c. enveloppes and then super-impose subsolution inequalities at $t = 0$ in a *suitable way* in order to be sure that this initial data will be seen. Indeed, the lower continuous function

$$\tilde{u}(x,t) = \begin{cases} 1 & \text{if } t > 0 \\ g(x) & \text{if } t = 0, \end{cases}$$

is a l.s.c. subsolution of the problem but $\tilde{u}$ "does not see the initial data enough" since

$$\tilde{u}(0,0) < \liminf\{\tilde{u}(y,t),\ (y,t) \to (0,0) \text{ with } t > 0\}\ .$$

Besides formulating the notions of viscosity sub and supersolution both for l.s.c. functions (or their l.s.c. enveloppes), an important assumption in the Barron-Jensen approach is to avoid such problems with the initial data, hence the initial regularity assumption (23.22) below.

23.4.2 Definition and regularity of subsolutions

We use below the acronym (SBJ) for *Stratified Barron-Jensen* subsolutions, supersolutions and solutions of the general equation

$$\mathbb{F}\big(x,t,u,(u_t,Du)\big) = 0 \text{ in } \mathbb{R}^N \times [0,T_f]\,. \tag{23.20}$$

In order to get a reasonnable comparison result for (23.20), we restrict ourselves to the following set of assumptions

($\mathbf{H_{SBJ}}$) *Assumptions for the Stratified Barron-Jensen framework.*

(i) The stratification does not depend on time: for any $k = 0..N$,

$$\mathbf{M}^{k+1} = \tilde{\mathbf{M}}^k \times \mathbb{R}\ ,$$

where $(\tilde{\mathbf{M}}^k)_k$ is a stratification of $\mathbb{R}^N$.

(ii) We are given a classical l.s.c. and bounded initial data g, *i.e.* we assume the l.s.c. sub and supersolutions u and v we are considering satisfy

$$u(x,0) \le g(x) \le v(x,0) \quad \text{in } \mathbb{R}^N\ .$$

(*iii*) Hamiltonian $\mathbb{F}$ is a classical Hamiltonian of the form

$$\mathbb{F}\big(x,t,r,(p_t,p_x)\big) = p_t + \tilde{\mathbb{F}}(x,t,r,p_x)\ ,$$

and there exists $0 < \tilde{T}_f \le T_f$ such that $\tilde{\mathbb{F}}$ is independent of t if $0 \le t \le \tilde{T}_f$ and coercive, *i.e.* there exists $\nu > 0$ such that

$$\tilde{\mathbb{F}}(x,t,r,p_x) = \tilde{\mathbb{F}}(x,\tilde{T}_f,r,p_x) \ge \nu|p_x| - M|r| - M\ ,$$

for any $x \in \mathbb{R}^N$, $t \in [0,\tilde{T}_f]$, $r \in \mathbb{R}$ and $p_x \in \mathbb{R}^N$, M being the constant appearing in the assumptions for **BCL**.

(*iv*) The "good framework for stratified solutions" is satisfied.

We can now give some precise definitions for the (SBJ) problem.

Definition 23.4.1 — *Stratified Barron-Jensen sub and supersolutions.*

(*i*) *A locally bounded, l.s.c. function $v : \mathbb{R}^N \times [0,T_f[\to \mathbb{R}$ is a* (SBJ) *supersolution of Equation (23.20) iff it is an Ishii supersolution of this equation on $[0,T_f]$.*

(*ii*) *A locally bounded, l.s.c. function $u : \mathbb{R}^N \times [0,T_f[\to \mathbb{R}$ is a* (SBJ) *subsolution of Equation (23.20) iff*

(*a*) *it is a Barron-Jensen subsolution of this equation, i.e. for any smooth function φ, at any minimum point (x,t) of $u-\varphi$,*

$$\mathbb{F}\big(x,t,u(x,t),(D_t\varphi(x,t),D_x\varphi(x,t))\big) \le 0\ ;$$

(*b*) *for any $k = 0,...,(N+1)$, for any smooth function φ, at any minimum point (x,t) of $u-\varphi$ on $\mathbf{M}^k$,*

$$\mathbb{F}^k\big(x,t,u(x,t),(D_t\varphi(x,t),D_x\varphi(x,t))\big) \le 0\ ;$$

(*iii*) *In addition, we will say that u is an η-strict* (SBJ) *subsolution if the various "$\mathbb{F} \le 0$" or "$\mathbb{F}^k \le 0$" subsolution inequalities are replaced by strict "$\mathbb{F} \le -\eta < 0$" "$\mathbb{F}^k \le -\eta < 0$" inequalities, $\eta > 0$ being independent of x and t*[(8)].

[(8)] A notion that we will use only locally.

We point out that, in this definition, the notion of subsolution is in the spirit of (s-S-Sub) but we could as well choose to present a notion of "weak Barron-Jensen subsolution". However, we have decided not to do so since the aim of this section is just to present few ideas for the extension of stratified solutions in the case of l.s.c. data and we do not intend to go too far in this direction. Of course, it is not difficult to imagine that, in order to deal with "weak Barron-Jensen subsolution", we have to assume them to be "regular", a notion which has to be properly redefined here and this is the purpose of the next paragraphs.

As we already noticed in the standard stratified framework and throughout this book, comparison results require some regularity property of the subsolution with respect to the stratification. Whether this property follows automatically from the specific structure of the equation or it has to be imposed, we cannot escape it.

In the standard stratified case, since subsolutions are u.s.c., the regularity takes the form of a limsup property, which is also linked to a regularization by sup-convolution in a first step. We refer to Section 3.4 for the details.

Here, since subsolutions are l.s.c., we have to change the strategy by using the inf-convolution tool. The consequence is also that the regularity property for subsolutions has to be expressed in terms of liminf. But, as we already noticed in Remark 3.4.3-(ii), such property holds provided the normal controlability assumption is satisfied, which is the case under $(\mathbf{H_{SBJ}})$ above. More precisely, we have the

Proposition 23.4.2 *Assume that* $(\mathbf{H_{SBJ}})$ *holds and that* u *is a bounded, l.s.c.,* (SBJ) *subsolution. Then* u *is regular: for any* $(x,t) \in \tilde{\mathbf{M}}^k \times (0,T_f)$ *and* $0 \le k < N$,

$$u(x,t) = \liminf\{u(y,s)\ ;(y,s) \to (x,t),\ \ y \in \tilde{\mathbf{M}}^{k+1} \cup \cdots \cup \tilde{\mathbf{M}}^N\}\ . \tag{23.21}$$

Moreover, if $k = N-1$, *then locally* $\mathbb{R}^N \setminus \tilde{\mathbf{M}}^{N-1}$ *has two connected components* $(\tilde{\mathbf{M}}^{N-1})_+$, $(\tilde{\mathbf{M}}^{N-1})_-$ *and the above result is valid imposing to* y *to be either in* $(\tilde{\mathbf{M}}^{N-1})_+$ *or in* $(\tilde{\mathbf{M}}^{N-1})_-$.

Notice that this result does not provide any similar regularity property as $t \to 0$. In the standard case of u.s.c. subsolutions, this is not needed: the fact that $u-v$ is u.s.c. implies that if a maximizing sequence (x_k,t_k) for $\max(u-v) > 0$ is such that $t_k \to 0$, using that $\limsup(u-v)(x_k,t_k) \le (u-v)(x,0) \le 0$ easily yields a contradiction.

On the contrary, if u is l.s.c. instead of u.s.c., the argument obviously fails and we have seen in Section 23.4.1 above that simple counter-examples to uniqueness can be built because of this. Hence, in order to get a comparison result, a specific regularity requirement has to be made on the subsolution as $t \to 0$:

Definition 23.4.3 *A* (SBJ) *subsolution is initially regular if, for any* $x \in \mathbb{R}^N$,

$$u(x,0) = \liminf \left\{ u(y,t),\ (y,t) \to (x,0) \text{ with } t > 0 \right\} . \tag{23.22}$$

23.4.3 The (SBJ) comparison result

In the (SBJ) approach we described above, we are able to present very general results but we just provide here a uniqueness result using $(\mathbf{H_{SBJ}})$, a framework which slightly generalizes the one of Ghilli, Rao and Zidani [99].

Theorem 23.4.4 *Assume that* $(\mathbf{H_{SBJ}})$ *holds. Let* u *and* v *be two bounded, l.s.c.,* (SBJ) *sub and supersolution of* (23.20) *respectively such that* u *is initially regular, i.e. it satisfies* (23.22)*. Then, the comparison result holds*

$$u(x,t) \le v(x,t) \quad \text{in } \mathbb{R}^N \times [0,T_f) .$$

Proof — Of course, the approach of Section 3.2 has to be slightly modified. The quantities $\max_{\mathcal{K}}(u-v)^+$ and $\max_{\partial_p \mathcal{K}}(u-v)^+$ where $\mathcal{K} = \overline{Q^{x,t}_{r,h}[\mathcal{F}]}$ have to be replaced by $\max_{\mathcal{K}}[(u-v)^+]^*$ and $\max_{\partial_p \mathcal{K}}[(u-v)^+]^*$. Inded, since $u-v$ is not u.s.c. anymore there is no reason why $(u-v)^+$ would achieve its supremum. But, with ad hoc modifications, the ideas of Section 3.2 still apply; we skip these modifications here, trusting the reader will be able to perform them.

We point out anyway, that we face two different situations: with a standard localization argument, we can assume that $(u-v)(x,t) \to -\infty$ when $|x| \to +\infty$ or $t \to T_f$, and there exists maximizing sequences $(x_k,t_k)_k$ which are bounded, t_k remaining away from T_f; then either, at least along a subsequence, we have $t_k \to t > 0$ and an analogue of a **(LCR)** is needed, or $t_k \to t = 0$ and we face the difficulty connected to the way the initial data is assumed and how the initial regularity of u can be used.

(a) *The case* $t > 0$ — Here we can argue in a similar way to the standard stratified case with the help of the following result, which is an easy adaptation Proposition 3.4.4

Proposition 23.4.5 *Under the assumptions of Theorem 23.4.4, if* u *is a bounded l.s.c., stratified Barron-Jensen subsolution of* (23.20)*, then for any* $(x,t) \in \tilde{\mathbf{M}}^k \times (0,T_f)$*, there exists a sequence of Lipschitz continuous functions* $(u^{\varepsilon,\alpha})_{\varepsilon,\alpha}$ *defined in a neighborhood* $\mathcal{V}$ *of* (x,t) *such that*

(*i*) *each* $u^{\varepsilon,\alpha}$ *is a stratified Barron-Jensen subsolutions of* (23.20) *in* $\mathcal{V}$ *;*

(*ii*) *each* $u^{\varepsilon,\alpha}$ *is semi-concave and* C^1 *on* $\tilde{\mathbf{M}}^k \times (0, T_f)$,

(*iii*) $\sup u^{\varepsilon,\alpha} = \lim_{\varepsilon,\alpha\to 0} u^{\varepsilon,\alpha} = u$ *in* $\mathcal{V}$.

Proposition 23.4.5 is proved exactly as Proposition 3.4.4 except that we use an inf-convolution instead of a sup-convolution and we treat differently the tangent space variable (with the parameter ε) and the t-variable (with parameter α). Of course, the regularity of u in terms of liminf, Proposition 23.4.2, is used to proceed here.

With this adaptation, we get a contradiction in the case $t > 0$ exactly as in the standard stratified case.

(b) *The case* $t = 0$ — By the coercivity assumption on the time interval $(0, \tilde{T}_f)$, u is a Barron-Jensen subsolution of the (continuous) equation

$$u_t + \nu|D_x u| - M(||u||_\infty + 1) = 0 \quad \text{in } \mathbb{R}^N \times (0, T_f) \ .$$

Therefore, using (23.22), by the uniqueness property for this problem and the Oleinik-Lax (or control) formula,

$$u(x,t) \le \inf_{|y-x|\le \nu t} (u(y,0)) + M(||u||_\infty + 1)t \ .$$

On the other hand, a similar (yet reversed) inequality for v holds, either by the same arguments or using the Dynamic Programming Principle

$$v(x,t) \ge \inf_{|y-x|\le Mt} (u(y,0)) - M(||v||_\infty + 1)t \ .$$

From these two inequalities we deduce that if $\delta > 0$ is a small constant,

$$u(x, \delta + t) \le v(x,t) + K\delta \ ,$$

for any $0 \le t \le \dfrac{\nu\delta}{M-\nu}$ and for some constant K which can be computed explicitly.

But the problem is that this inequality is valid for t in a time interval which depends on δ. To get rid of this dependence, we remark that, thanks to the assumptions on $\mathbb{F}$ on the time interval $[0, \tilde{T}_f]$, the function $u(x, \delta + t)$ is a subsolution of the problem. On the other hand, because of the Lipschitz continuity of $\mathbb{F}(x,t,r,(p_t,p_x))$ in r, $v(x,t) + K\delta\exp(\tilde{K}t)$ is also a supersolution of the problem for $\tilde{K} > 0$ large enough. By using the argument of Step (a) and the fact that $u(x,\delta+t) \le v(x,t) + K\delta\exp(\tilde{K}t)$ for $0 \le t \le \dfrac{\nu\delta}{M-\nu}$, we can compare them; hence

$$u(x, \delta + t) \le v(x,t) + K\delta\exp(\tilde{K}t) \quad \text{in } \mathbb{R}^N \times [0, \tilde{T}_f - \delta) \ ,$$

for any $0 < \delta \ll \tilde{T}_f$. For fixed $(x,t) \in [0, \tilde{T}_f/2]$, we can send δ to 0 using the lower semi-continuity of u: this yields $u \leq v$ in $\mathbb{R}^N \times [0, \tilde{T}_f/2]$. And the proof is complete.
Q.E.D.

Remark 23.4.6 *We point out that the case we treat above is the stratified analogue of the Barron-Jensen approach with continuous Hamiltonians. For example, the case of obstacle problems with l.s.c. obstacles ψ, i.e.*

$$\max\left(\mathbb{F}\big(x,t,u,(u_t,Du)\big); u-\psi\right) = 0 \ \text{in } \mathbb{R}^N \times [0,T_f]\,,$$

does not enter into the ($\mathbf{H_{SBJ}}$) *framework since the tangential continuity may not be satisfied. But since the functions $u^{\varepsilon,\alpha}$ built in Proposition 23.4.5 through an inf-convolution procedure satisfy $u^{\varepsilon,\alpha} \leq \psi^{\varepsilon,\alpha} \leq \psi$ where the $\psi^{\varepsilon,\alpha}$ are built by using exactly the same procedure, the result extends to this more general case under suitable assumptions on the initial data.*

Of course, we think that the Barron-Jensen approach can be extended to a more general framework; in particular, it is perhaps possible to remove the restrictive assumption on $\mathbb{F}$ near $t = 0$. But it is worth pointing out that the role of the inf-convolution in the classical Barron-Jensen argument—typically an inf-convolution in x on the solution in order to treat the lower-semi-continuity of the initial data—and the inf-convolution which takes care of the stratification are not completely compatible. This is what is generating these strong and restrictive assumptions.

Chapter 24

Further Discussions and Open Problems

We start this section by recalling the main ideas of a comparison proof for stratified solutions

(i) We localize, *i.e.* we reduce the proof of a GCR to the proof of a LCR.

(ii) In order to show that the LCR holds, we first regularize the subsolution by a partial sup-convolution procedure using the tangential continuity and the normal controllability and then (still tangentially) with a standard convolution with a smoothing kernel.

(iii) After Step **(ii)** the subsolution is Lipschitz continuous *w.r.t.* all variables and C^1 *w.r.t.* the tangent variable and we use the "magical lemma" (Lemma 5.4.1) to conclude.

Analyzing these 3 steps in conjunction with Section 3.2 and the examples therein, it seems rather clear that the localisation procedure can be made via various arguments and is not a limiting step—even if we agree that there are more complicated situations where this might become a problem. In the same way, Step **(iii)** is not really a limiting step, especially the way we use it in the proof by induction.

Hence, in the generalizations we wish to present here, the main issue comes from Step **(ii)** and more precisely from the first part, *i.e.* the tangential sup-convolution procedure. This is why we mainly insist on this point.

24.1 More general dependence in time

A quite restrictive—or at least unusual—assumption we have used so far concerns the time dependence of the Hamiltonians and on the dynamics of the control problems. In general, it is well-known that a simple continuity assumption is a sufficient requirement.

In stratified problems however, we face two main cases: the general case when the stratification may depend on time for which space and time play a similar role; and the case when the stratification does not depend on time. While, in the first case, it seems natural to impose similar assumptions on x and t for the Hamiltonians, this is no longer the case for the second one and actually this particular structure allows to weaken the assumptions on the time dependence.

Indeed, in this second case, we can write the stratification as

$$\mathbf{M}^{k+1} = \tilde{\mathbf{M}}^k \times \mathbb{R}\ ,$$

where $\tilde{\mathbb{M}} = (\tilde{\mathbf{M}}^k)_k$ is a stratification of $\mathbb{R}^N$ and $\mathbb{M} = (\mathbf{M}^k)_k$ is the resulting one in $\mathbb{R}^N \times [0,T_f]$, which is here presented as the trace on $\mathbb{R}^N \times [0,T_f]$ of a stratification on $\mathbb{R}^N \times \mathbb{R}$.

As far as Section 3.4 is concerned, the t-variable is always a tangent variable — this is the main difference with the general case— and we can use, as it is classical in all the comparison proofs in viscosity solutions' theory, a "double parameters sup-convolution". More precisely, if $u : \mathbb{R}^N \times [0,T_f] \to \mathbb{R}$ is a sub-solution, $\tilde{\mathbf{M}}^k$ is identified with $\mathbb{R}^k$ and $x = (y,z)$ with $y \in \mathbb{R}^k$ and $z \in \mathbb{R}^{N-k}$, we set

$$u^{\varepsilon,\beta}(x,t) := \max_{y' \in \mathbb{R}^k, s \in [0,T_f]} \left\{ u((y',z),s) - \frac{(|y-y'|^2+\varepsilon^4)^{\alpha/2}}{\varepsilon^\alpha} - \frac{(|t-s|^2+\beta^4)^{\alpha/2}}{\beta^\alpha} \right\},$$

where the parameter β governing the regularization in time satisfies $0 < \beta \ll \varepsilon$.

We drop all the details here but we are sure that they will cause no problem to the reader.

24.2 Unbounded control problems

In the case of unbounded control problems we face two difficulties: (i) the localization that we treat—probably in a non-optimal way—in Section 3.2, *cf.* the "convex case'; (ii) the sup-convolution regularization.

In order to treat this difficulty, we refer the reader to Section 3.2.4, in particular to Theorem 3.2.8 and Assumption ($\mathbf{H_{BA-HJ-U}}$). Indeed, in the sup-convolution procedure, if we examine the proof of Theorem 3.4.4, we have to manage the error made by replacing y by y' and this is done by using the dependence in u of the Hamiltonian. This is exactly what Assumption ($\mathbf{H_{BA-HJ-U}}$) means: performing the Kruzkov change of variable $u \to -\exp(-u)$, one compensates the large terms in "$D_x H$" by large terms in "$D_u H$".

The same ideas can be used in the stratified framework: we drop the details here since a lot of very different situations can occur. It would be impossible and maybe useless to try to describe all of them.

We refer to [147] and [148] where unbounded control problems are studied in the hyperplane case under the assumption

$$\lim_{|\alpha|\to+\infty} \frac{l(x,\alpha)}{1+|b(x,\alpha)|} = +\infty\,,$$

locally uniformly in x. This assumption which appears in [29] allows to recover some compactness of trajectories since fast-moving trajectories get associated with high costs.

24.3 Large deviations type problems

The sections of this book in which we consider KPP-type problems give an idea of what can be done in the context of Large Deviations, but also of the limitations: in the cases where only codimension-1 discontinuities are present, Part II provides all the needed tools to completely analyze the problem. We point out that, as it was already remarked in Imbert and Nguyen [116], this allows not only to treat in a rather easy way the problem of Boué, Dupuis and Ellis [48] but even to generalize it, by allowing the diffusion matrix to be discontinuous on the hyperplane, *cf.* Section 30.5.

For more general discontinuities, the situation is not so well understood. Section 30.5 only gives few arguments to treat very particular cases. We can summarize the difficulty in one sentence: we have learned from the codimension-1 case that the vanishing viscosity method converges to the maximal Ishii subsolution (and solution) of the limiting Hamilton-Jacobi Equation. Though we think that it is still the case for any type of discontinuities, we are unable to identify this maximal subsolution, which implies TWO open problems: the identification of the maximal subsolution and the convergence of the vanishing viscosity method.

Most of Large Deviations problems involve boundary conditions and for these prob-

lems, there are two different cases: either there is no specific difficulty with the boundary conditions (as it is mainly the case in the four examples presented in [31]) and we believe that the above mentioned tools apply; or there is some specific difficulties with the boundary conditions. In this latter case, the problem and its solution may not only be related to discontinuities in the Hamiltonians and/or boundary conditions, see for example [33].

24.4 Homogenization

We first point out that the arguments which are used in Section 23.2, which are strongly inspired by those appearing in Barles, Briani, Chasseigne and Tchou [34], are very flexible: the identification of the effective Hamiltonian and the application of the perturbed test-function method of Evans [79, 80] rely on basic results of the theory (existence of solutions, comparison results and stability). They can therefore be used in a very general framework.

Among all possible applications, the first one we have in mind concerns homogenization in a chessboard-type configuration, this problem is treated in Forcadel and Rao [89]. The approach we describe above together with the results of this book lead to more general results with simpler proofs; typically the case of all periodic stratified domains can be addressed without additional difficulties, of course under suitable assumptions.

Let us conclude this section with some additional references. Some of them may have been put in the networks section but we think that they are relevant here since some methods are either similar or share common features. In addition to [34], the most specific one on HJ Equations with discontinuities is Achdou, Oudet and Tchou[5] for the two-domain case, while in the networks configurations, the reader can check Achdou and Tchou [7], Galise, Imbert and Monneau [96], Forcadel and Salazar [90].

24.5 Convergence of numerical schemes and estimates

For first-order Hamilton-Jacobi Equations, the convergence of numerical schemes is usually obtained by using the half-relaxed limit method and a comparison result. Therefore we seem to have the key tools in the stratified framework.

Actually, in [54], Cacace and Camilli introduce a semi-Lagrangian approximation scheme for a general stratified problem and prove the convergence by using these

tools, and in particular the comparison result.

The estimates are generally obtained by a comparison result, combined with the consistency of the scheme and the regularity of the solution. Here the difficulty may come from the different nature of the equation and the scheme which may appear as being more problematic than in the continuous case. Maybe the scheme has, in some sense, to "respect" the discontinuities and it does not seem so easy to produce a general theory.

We did not find so many specific references—we apologize if we have missed some works—but the work of Guerand and Koumaiha [106] addresses the key difficulties we have in mind.

24.6 About Ishii inequalities and weak stratified solutions

We show in Section 20.5 that, roughly speaking, Ishii subsolutions' inequalities are a consequence of the **(LCR)** for weak stratified solutions. One way or the other, this type of property is connected to several existing results in the viscosity solutions literature which show the links between this notion of solutions and monotonicity.

For example, Alvarez, Guichard, Lions & Morel [9] (see also Biton [45]) prove under suitable assumptions that a monotone semi-group acting on a space of continuous functions is necessarily the semi-group of viscosity solutions for a possibly fully nonlinear parabolic equation. In a different framework, the "geometrical approach to front propagation problems" of Souganidis and the first author [40] allows to define a weak motion of subsets of $\mathbb{R}^N$ which is almost equivalent to the Level-Set Approach by using: (i) the monotonicity property of sets for the inclusion relation; (ii) the use of suitable smooth moving sets, which can be seen as the analogue of test-functions in the geometrical framework.

This second example is closer in the spirit to what is done in Section 20.5 and it seems interesting to re-formulate the idea of Section 20.5 in a more general, abstract way, even if we are going to do so a little bit formally. We consider here a "stationary" framework which, as it is the case in Section 3.2, is easier to describe, but we trust the reader to be able to extend the following to the evolution case.

We assume that, for a local equation $\mathbb{F}(x, u, D_x u) = 0$ in $\mathcal{O}$, we are given two "abstract" sets of functions: a set of locally bounded, u.s.c. "subsolutions" $\mathcal{S}^{sub}$ and a set of locally bounded, l.s.c. "supersolutions" $\mathcal{S}^{sup}$ with the following properties

(*sub*) — For any $x \in \mathcal{O}, r > 0$ such that $B(x,r) \subset \mathcal{O}$, any smooth function ϕ in $\mathcal{O}$ such that $\mathbb{F}_*(x, \phi, D_x\phi) \geq 0$ in $B(x,r)$, we have, for any $u \in \mathcal{S}^{sub}$,

$$u(y) - \phi(y) \leq \max_{\partial B(x,r)} (u - \phi) \quad \text{for any } y \in B(x,r).$$

(*sup*) — For any $x \in \mathcal{O}, r > 0$ such that $B(x,r) \subset \mathcal{O}$, any smooth function ϕ in $\mathcal{O}$ such that $\mathbb{F}^*(x, \phi, D_x\phi) \leq 0$ in $B(x,r)$, we have, for any $v \in \mathcal{S}^{sub}$,

$$\phi(y) - v(y) \leq \max_{\partial B(x,r)} (\phi - v) \quad \text{for any } y \in B(x,r).$$

We point out that properties (*sub*) and (*sup*) can be interpreted as **(LCR)** between either subsolutions and smooth local supersolutions or supersolutions and smooth local subsolutions. In this context, it follows that the Ishii inequalities are satisfied. More precisely

(*i*) for any $u \in \mathcal{S}^{sub}$, $\mathbb{F}_*(x, u, D_x u) \leq 0$ in $\mathcal{O}$ in the viscosity sense;

(*ii*) for any $v \in \mathcal{S}^{sup}$, $\mathbb{F}^*(x, v, D_x v) \geq 0$ in $\mathcal{O}$ in the viscosity sense.

This result is an easy consequence of the arguments of Section 20.5 by looking at *strict* local maxima and minima. Take for instance $u \in \mathcal{S}^{sub}$ and suppose that $\mathbb{F}_*(x, u, Du) \leq 0$ does not hold in the viscosity sense. Then, there exists a test-function ϕ such that $u - \phi$ has a strict local maximum at x in $B(x,r)$ and $\mathbb{F}_*(x, \phi, D\phi) > 0$. But using property (*sub*) above we get a contradiction with the fact that $u - \phi$ has a strict local maximum at x.

Part V

State-Constraint Problems

Chapter 25

Introduction to State-Constraint Problems

In this part we extend the results of Part IV to the case of problems set in a bounded or unbounded domain of $\mathbb{R}^N$ with state-constraint boundary conditions.

As we are going to see it, this readily includes the classical Dirichlet, Neumann, Robin etc. boundary conditions in a unique framework. But more importantly, the state-constraint stratified approach allows to deal at the same time with (i) singular (discontinuous) boundary value problems; (ii) non-smooth boundaries; (iii) a mix of various boundary conditions on different portions of the boundary.

25.1 Why only state-constraint problems?

To begin with, let us first recall that in Chapter 4 on "Control Tools", we have already presented finite horizon control problems in a state-constraint framework. Indeed, the space-time trajectory (X, T) has to satisfy the constraint $T(s) \in [0, T]$ for any $s \geq 0$. As a consequence of this general framework, the usual initial data (the terminal cost) was not given but it has to be computed by solving the $\mathbb{F}_{init}$-equation. It is therefore natural to investigate problems for which this constraint in T is complemented by a constraint in X, like $X(s) \in \overline{\Omega}$ for some domain Ω of $\mathbb{R}^N$.

However, this "natural extension"argument does not explain the "only" part in the title of this section.

Traditionally, Dirichlet, Neumann, Robin, state-constraint problems etc. are considered as separate, different problems with specific boundary conditions. But in fact,

the main reason for choosing to consider only state-constraint boundary condition problems in the stratified approach is because it allows to treat within the same global framework all these different types of boundary conditions—almost all the usual ones—both for smooth and non-smooth domains, as well as combinations of them even in rather singular settings.

To convince the reader and to give a more concrete idea of what we mean in the previous paragraph, we describe in the next section a deterministic control problem proposed by P.L. Lions [136] in one of his lessons at the Collège de France in 2016—the "Tanker problem"—which was one of our main motivation to look at such formulations.

A second reason for considering such state-constraints boundary conditions for stratified problems is that they can be treated with the methods of Part IV with only few additional difficulties, and more precisely with ONE additional difficulty: the regularity of subsolutions on the boundary.

Boundary regularity — To be more precise, we recall that the question of the "regularity" of subsolutions is crucial in the stratified approach: either as an assumption for *weak* stratified solutions or as a property for the *strong* ones, this regularity is used on each part of the stratification in the comparison proof and it is a key property to make the comparison proof work. However checking this regularity becomes trickier on the boundary.

Indeed, for strong stratified subsolutions, the regularity property in the $\mathbb{R}^N \times (0, T_f)$-case is a consequence of the $\mathbb{F}_* \leq 0$ inequality (or maybe of a similar inequality with a suitable Hamiltonian in the cases of weak ones) and the "normal controllability" assumption. We recall that the subsolution inequalities can be interpreted by "all trajectories are sub-optimal" or equivalently "all choices of the dynamic are sub-optimal" from the control point-of-view. But, on the boundary, we cannot use all the trajectories, only those which stay in $\overline{\Omega} \times (0, T_f)$. Hence the $\mathbb{F}_* \leq 0$ inequality does not hold on $\partial\Omega \times (0, T_f)$, and we have to find a suitable substitute providing the regularity of subsolutions.

Of course, the first natural idea is to use an Hamiltonian built out of all the dynamics pointing inside Ω but this a priori optimal choice may be complicated in general since Ω is not necessarily smooth so that the definition of "pointing inside Ω" may be rather complicated. Another choice is to use only ONE dynamic pointing inside Ω but we still need to make precise the sense of this property. In the sequel, we mainly use this second option.

We point out that such difficulty with the regularity of sub and supersolutions on

the boundary already appears when studying state-constraints or Dirichlet boundary conditions, even in the most standard continuous cases. It is clear that such boundary conditions in the viscosity sense allow the sub and supersolutions to have "artificial values" on $\partial\Omega$ (they may be non-regular in the language of this book). In particular, this is obviously the case for the subsolutions of "classical" state-constraints problems since they do not satisfy anything on the boundary. Therefore, one way or the other, some additional properties have to be imposed to solve this difficulty.

In the pionneering works of Soner [154, 155], the "cone condition" appears involving both some regularity of the boundary (interior cone regularity) but also some property of the dynamic (one of the control fields has to enter in this cone). From these first articles on the subject, it was clear that a comparison result holds if the subsolution is not only Ω-regular at each point of the boundary but is also K-regular where K is the interior cone.

Then, in their systematic study of Dirichlet problems, Perthame and the first author [22, 23, 24] obtain comparison results avoiding the direct use of a cone condition by showing, under **(NC)** type conditions, that on some parts of the boundary, these sub and supersolutions are regular while on other parts, one can redefine their values on the boundary in order to transform them in regular sub and supersolutions.

Finally, and perhaps closer in the spirit of what we suggest above, Ishii and Koike [123] have formulated the state-constraint boundary condition in a different way, with an unusual subsolution condition on the boundary, by looking only at dynamics which are pointing inside the domain on the boundary: as can be guessed, their boundary condition "$u_t + H_{in} \leq 0$" avoids non-regular subsolution provided there is an inner dynamic and Lemma 26.4.1 below justifies this natural idea.

In all this part, we are going to avoid this difficulty connected to the regularity of sub and supersolutions by proving several results only for regular ones, *i.e.* for sub and/or supersolutions which boundary values are essentially limits of their values inside Ω. Of course, the next natural question is to identify some stable viscosity inequalities implying that, in particular, subsolutions are "regular" : we refer the reader to Section 26.4 for a discussion. For supersolutions, this regularity is treated in a more classical way.

Boundary conditions — Coming back on the connections between general state-constraint problems with different boundary conditions, in this part we show that the case of control problems, classical Dirichlet, Neumann or even mixed boundary conditions for the associated Hamilton-Jacobi Bellman Equation can be reformulated as stratified problems with state-constraints boundary conditions. Moreover, this can

be done also in the case of non-smooth domains since globally a stratification is not smooth—it may contains crosses, for example.

This remark allows us to revisit Dirichlet and Neumann boundary conditions in deterministic control problems and extend some results to far more general frameworks: discontinuous Hamiltonians of course, non-smooth boundary conditions, mixing of boundary conditions and treatment of rather singular cases (including the above example).

We conclude this introduction by pointing out that other approaches for treating state-constraint problems in stratified situations appear in Hermosilla and Zidani [112], Hermosilla, Wolenski and Zidani [111], Hermosilla, Vinter and Zidani [110].

25.2 A tanker problem mixing boundary conditions

In this situation, a controller has to manage a tanker: the aim is to decide when and where it will unload its cargo depending typically on the market price for the goods in the cargo. Of course, this price may depend on the location—typically the country—therefore to the harbour where the unloading takes place.

In the simplest modelling, the sea is identified with a smooth domain $\Omega \subset \mathbb{R}^2$ and the harbours are isoled points $P_1, P_2, \cdots, P_L$ on the boundary $\partial\Omega$. The tanker has to be controlled in such a way that it stays far from the coast and keeps its cargo if prices are low or, on the contrary, comes to one of the harbours, unloads and sells its cargo when they become higher at this harbour. The choice of the harbour is clearly part of the problem and there is no reason why all harbours should be equivalent. Of course, there is an underlying state constraint boundary condition on $\partial\Omega$ outside $P_1, P_2, \cdots, P_L$ since the tanker cannot accost where no harbour exists!

In terms of boundary conditions, we are facing a non-standard and rather singular problem involving a state-constraint boundary condition on $\partial\Omega \setminus \{P_1, P_2, \cdots, P_L\}$ and P.L. Lions suggested Neumann boundary conditions for the harbours to model the flux of goods which are sold, leading to a mathematical fomulation as follows:

$$\begin{aligned} &u_t + H(x,t,Du) = 0 \text{ in } \Omega \times (0,T_f) \ , \\ &u_t + H(x,t,Du) \geq 0 \text{ on } \partial\Omega \setminus \{P_1, P_2, \cdots, P_L\} \times (0,T_f) \ , \\ &\frac{\partial u}{\partial n} = g_i(t) \text{ at } P_i \ , \text{ for } i = 1, \cdots, L. \end{aligned} \tag{25.1}$$

To the best of our knowledge, there is no work on such type of boundary condi-

tions: here the mixing of state-constraint and Neumann boundary conditions (which is already not so standard) is even more complicated since the Neumann boundary conditions take place only at isolated points. In fact, even if one can give a sense to such problems using viscosity solutions' theory, these problems are ill-posed in the sense that no uniqueness result holds in general, *cf.* Section 25.3 for a counter-example.

The important point is that the Neumann boundary conditions, imposed only at isolated points, are "not sufficiently seen" to give sufficient constraints on solutions to provide a uniqueness result.

To overcome this difficulty, we use below a re-formulation in terms of stratified problems, allowing discontinuities in the Hamiltonians as well as in the boundary conditions as we will develop here. The point is also that the definition of viscosity solutions for stratified problem consists in "super-imposing" some (subsolutions) inequalities on the discontinuity sets of the Hamiltonians, which can be not only of codimension 1 but also of higher codimension. This is exactly what is lacking for obtaining uniqueness, as described in the previous paragraph.

25.3 A counter-example for the tanker problem

Let us examine problem (25.1) in the following case: $\Omega = \{x_N > 0\} \subset \mathbb{R}^N$; there is only one harbour $P_1 = 0 \in \partial\Omega$; the equation is given by

$$u_t + |Du| = 1 \quad \text{in } \Omega \times (0, +\infty) \,,$$

and the Neumann boundary condition is

$$\frac{\partial u}{\partial n} = g \quad \text{at } 0 \text{ for all } t \in (0, +\infty) \,,$$

for some constant $g \in \mathbb{R}$. For the initial data, we choose $u(x, 0) = 0$ on $\overline{\Omega}$.

To compute a solution, we argue formally: the associated control problem is a problem with a reflection at 0 and the controlled trajectory is given by [(1)]

$$\dot{X}(s) = \alpha(s)\,\mathrm{d}s - 1_{\{X(s)=0\}} n(X(s))\,\mathrm{d}|k|_s \,, \;\; X(0) = x \in \overline{\Omega} \,,$$

where $\alpha(\cdot)$ is the control taking values in $B(0, 1)$. The term $-1_{\{X(s)=0\}} n(X(s))\, d|k|_s$ is the reflection at 0, $(|k|_s)_s$ being the intensity of the reflection and $n(X(s)) = -e_N$ is the outward unit normal vector to $\partial\Omega$ at $X(s)$. The value function is

$$U(x, t) = \inf_{\alpha(\cdot)} \left\{ \int_0^t 1\,\mathrm{d}s + \int_0^t g 1_{\{X(s)=0\}}\,\mathrm{d}|k|_s \right\} \,.$$

[(1)] We give here a general formula which the reader will recognize for a reflection term

In this case, the term $1_{\{X(s)=0\}}\,\mathrm{d}|k|_s$ is nothing but $1_{\{X(s)=0\}}\alpha(s)\cdot n(0)\,\mathrm{d}s$.

If $g<0$—a favorable case to unload the cargo—the clear strategy to minimize the cost is to maximize the integral of $|g|1_{\{X(s)=0\}}\,\mathrm{d}|k|_s$. Therefore, the strategy is to reach 0 as soon as possible and then to have $\alpha(s)\cdot n(0)=1$, *i.e.* $\alpha(s)=n(0)$. Since $|x|$ is the time which is necessary to reach 0 from x and then we integrates g till time t, this gives the solution:

$$U(x,t)=t+g(t-|x|)^+\,,$$

Now take $g<g'<0$ and consider $V(x,t)=t+g'(t-|x|)^+$. We claim that V is still a subsolution of (25.1): indeed, since changing g into g', we just have to check the inequality at $x=0$, for $t>0$. But, if $(y,s)\sim(0,t)$, then $(s-|y|)^+>0$ and $V(y,s)=s+g'(s-|y|)$. Now, since $g'<0$ the super-differential of V is empty at $(0,t)$, leaving us with no subsolution inequality to check.

Therefore V is a subsolution of the problem but clearly $V>U$ for $t>|x|$ and this shows that no comparison result can hold.

The interpretation of this counter-example is that the Neumann boundary condition at only one point (or at isolated points) is not seen enough by the notion of viscosity solution, at least not sufficiently to imply comparison/uniqueness. This defect will be corrected by the stratified formulation which superimposes an inequality at 0 for all t.

Chapter 26

Stratified Solutions for State-Constraint Problems

26.1 Control problems, stratifications and state-constraint conditions

In this section, we consider finite horizon, deterministic control problems with state-constraints in $\overline{\Omega}\times[0,T_f]$ where Ω is a domain in $\mathbb{R}^N$ which a priori is neither bounded nor regular. To formulate it, we are going to assume that the dynamics, discounts and costs are defined in $\mathbb{R}^N\times[0,T_f]$ (this is not a loss of generality) and may be discontinuous on subsets $\mathbf{M}^k\subset\overline{\Omega}$ for $k<N$ where $\mathbf{M}^k$ is a collection of k-dimensional submanifolds of $\mathbb{R}^N$. More precise assumptions will be given later on.

Following Section 4.2, we first define a general control problem associated to a differential inclusion. As we mention it above, at this stage, we do not need any particular assumption concerning the structure of the stratification, nor on the control sets. We also use the same notations and assumptions as in Section 4.2.

The control problem — as we said, we embed the accumulated cost in the trajectory by solving a differential inclusion in $\mathbb{R}^N\times\mathbb{R}$, namely (4.1) and we introduce the value function which is defined only on $\overline{\Omega}\times[0,T_f]$ by

$$U(x,t)=\inf_{\mathcal{T}(x,t)}\Big\{\int_0^{+\infty} l\big(X(s),T(s)\big)\exp(-D(s))ds\Big\},$$

where $\mathcal{T}(x,t)$ stands for all the Lipschitz trajectories (X,T,D,L) of the differential inclusion which start at $(x,t)\in\overline{\Omega}\times[0,T_f]$ and such that $(X(s),T(s))\in\overline{\Omega}\times[0,T_f]$ for all $s>0$.

Contrarily to Section 4.2, we point out that assumptions are needed in order to have $\mathcal{T}(x,t) \neq \emptyset$ for all $(x,t) \in \mathbb{R}^N \times (0,T_f]$: indeed, if there is no problem with the boundary $\{t = 0\}$, there is a priori no reason why there exists trajectories X satisfying the constraint to remain in $\overline{\Omega}$ for any $x \in \overline{\Omega}$ and $t \in [0,T_f]$. Therefore, the fact that $\mathcal{T}(x,t)$ is non-empty will be an assumption in all this part: we will say that $(\mathbf{H_U})$ is satisfied if the value-function U is locally bounded on $\overline{\Omega} \times [0,T_f]$ which is almost equivalent.

A first standard result gathers Theorem 4.3.3 and 4.3.4

Theorem 26.1.1 (Dynamic Programming Principle and Supersolution's Property) *Under Assumptions* $(\mathbf{H_{BCL}})$ *and* $(\mathbf{H_U})$*, then* U *satisfies*

$$U(x,t) = \inf_{\mathcal{T}(x,t)} \left\{ \int_0^\theta l\big(X(s),T(s)\big) \exp(-D(s))ds + U\big(X(\theta),T(\theta)\big) \exp(-D(\theta))\big) \right\},$$

for any $(x,t) \in \mathbb{R}^N \times (0,T_f]$, $\theta > 0$. *Moreover, if* $\mathbb{F}$ *is defined by* (20.2)*, then the value function* U *is a viscosity supersolution of*

$$\mathbb{F}(x,t,U,DU) = 0 \quad \text{on } \overline{\Omega} \times [0,T_f]\,, \tag{26.1}$$

where we recall that $DU = (D_xU, D_tU)$.

We point out that, in the same way as Theorem 4.3.3 and 4.3.4, Theorem 26.1.1 holds in a complete general setting, independently of the stratification we may have in mind.

We conclude this first part by the analogue of Lemma 4.4.2 showing that supersolutions always satisfy a super-dynamic programming principle, even in this constrainted setting: again we remark that this result is independent of the possible discontinuities for the dynamic, discount and cost.

Lemma 26.1.2 *Under Assumptions* $(\mathbf{H_{BCL}})$, $(\mathbf{H_U})$ *and* $(\mathbf{H_{Sub}})$*, if* v *is a bounded l.s.c. supersolution of* $\mathbb{F}(x,t,v,Dv) = 0$ *on* $\overline{\Omega} \times (0,T_f]$*, then, for any* $(\bar{x},\bar{t}) \in \overline{\Omega} \times (0,T_f]$ *and any* $\sigma > 0$*,*

$$v(\bar{x},\bar{t}) \geq \inf_{\mathcal{T}(\bar{x},\bar{t})} \left\{ \int_0^\sigma l\big(X(s),T(s)\big) \exp(-D(s))\,\mathrm{d}s + v\big(X(\sigma),T(\sigma)\big) \exp(-D(\sigma)) \right\} \tag{26.2}$$

Proof — The idea is to use Lemma 4.4.2 with a penalization type argument.

To do so, as in the proof of Lemma 4.4.2, we are going to prove Inequality (26.2) for fixed $(\bar{x},\bar{t})$ and σ, and to argue in the domain $B(\bar{x},M\sigma)\times[0,\bar{t}]$ where M is given by $(\mathbf{H_{BCL}})$, thus in a bounded domain. Next, for $\delta>0$ small, we set

$$v_\delta(x,t):=\begin{cases} v(x,t) & \text{if } x\in\overline{\Omega}\\ \delta^{-1} & \text{otherwise}\end{cases}$$

Since we argue in $B(\bar{x},M\sigma)\times[0,\bar{t}]$, v_δ is l.s.c. in $B(\bar{x},M\sigma)\times[0,\bar{t}]$.

Next we change **BCL** into $\mathbf{BCL}_\delta$ in the following way: if $x\in\Omega$, $\mathbf{BCL}_\delta(x,t)=\mathbf{BCL}(x,t)$, while if $x\notin\Omega$, then $(b_\delta,c_\delta,l_\delta)\in\mathbf{BCL}_\delta(x,t)$ iff, either $(b_\delta,c_\delta,l_\delta)=(b,c,l+\delta^{-1}d(x,\overline{\Omega}))$ where $(b,c,l)\in BCL(x,t)$ and $d(\cdot,\overline{\Omega})$ denotes the distance to $\overline{\Omega}$, or $(b_\delta,c_\delta,l_\delta)=(0,1,\delta^{-1})$.

If we set for $(x,t)\in B(\bar{x},M\sigma)\times[0,\bar{t}]$

$$\mathbb{F}_\delta(x,t,r,p):=\sup_{(b_\delta,c_\delta,l_\delta)\in\mathbf{BCL}_\delta(x,t)}\left\{-b_\delta\cdot p+c_\delta r-l_\delta\right\},$$

then v_δ is a l.s.c. supersolution of $\mathbb{F}_\delta(x,t,v_\delta,Dv_\delta)=0$ in $B(\bar{x},M\sigma)\times(0,\bar{t})$. Indeed, we have, at the same time $\mathbb{F}_\delta\geq\mathbb{F}$ if $x\in\overline{\Omega}$ and $\mathbb{F}_\delta(x,t,r,p)\geq r-\delta^{-1}$ if $x\notin\Omega$.

Therefore Lemma 4.4.2 implies

$$v_\delta(\bar{x},\bar{t})\geq\inf\left\{\int_0^\sigma l_\delta\big(X_\delta(s),t-s\big)\exp(-D_\delta(s))\,\mathrm{d}s+v_\delta\big(X_\delta(\sigma),T_\delta(\sigma)\big)\exp(-D_\delta(\sigma))\right\},$$

the infimum being taken on all the solutions $(X_\delta,D_\delta,L_\delta)$ of the $\mathbf{BCL}_\delta$ differential inclusion.

To conclude the proof, we have to let δ tend to 0 in the above inequality where we can notice that $v_\delta(\bar{x},\bar{t})=v(\bar{x},\bar{t})$. To do so, we pick an optimal or δ-optimal trajectory $(X_\delta,D_\delta,L_\delta)$.

By the uniform bounds on $\dot{X}_\delta,\dot{D}_\delta,\dot{L}_\delta$, Ascoli-Arzela' Theorem implies that up to the extraction of a subsequence, we may assume that $X_\delta D_\delta,L_\delta$ converges uniformly on $[0,\sigma]$ to (X,D,L). And we may also assume that they derivatives converge in L^∞ weak-* (in particular $\dot{L}^\delta=l^\delta$).

We use the above property for the δ-optimal trajectory, namely

$$\int_0^\sigma l_\delta\big(X_\delta(s),t-s\big)\exp(-D_\delta(s))\,\mathrm{d}s+v_\delta\big(X_\delta(\sigma),T_\delta(\sigma)\big)\exp(-D_\delta(\sigma))-\delta\leq v(\bar{x},\bar{t}),$$

in two ways: first by multiplying by δ, using that $l_\delta\geq-M+\delta^{-1}d(x,\overline{\Omega}))$ and the definition of v_δ outside $\overline{\Omega}$, we get

$$\int_0^\sigma d(X_\delta(s),\overline{\Omega}))\exp(-Ms)ds+\mathbb{1}_{X_\delta(\sigma)\notin\overline{\Omega}}\exp(-M\sigma)=O(\delta).$$

The uniform convergence of X_δ implies that both terms in the left-hand side tend to 0, meaning that $X(s) \in \overline{\Omega}$ for any $s \in [0, \sigma]$. And the proof is complete.

Q.E.D.

26.2 Admissible stratifications for state-constraint problems

In this section, we extend the notions of *admissible stratification* for a Bellman Equations set on $\overline{\Omega} \times (0, T_f)$.

Definition 26.2.1 (Admissible Stratification)
We say that a family of subsets $\mathbf{M}^0, \mathbf{M}^1, \cdots, \mathbf{M}^{N+1}$ *of* $\overline{\Omega} \times (0, T_f)$ *is an* Admissible Stratification of $\overline{\Omega} \times (0, T_f)$ *if*

$$\overline{\Omega} \times (0, T_f) = \mathbf{M}^0 \cup \mathbf{M}^1 \cup \cdots \cup \mathbf{M}^{N+1} \ ,$$

$$\partial\Omega \times (0, T_f) \subset \mathbf{M}^0 \cup \mathbf{M}^1 \cup \cdots \cup \mathbf{M}^N \ ,$$

and the family $\tilde{\mathbb{M}} = (\tilde{\mathbf{M}}^k)_k$ *defined by* $\tilde{\mathbf{M}}^0 = \mathbf{M}^0$, $\tilde{\mathbf{M}}^1 = \mathbf{M}^1, \cdots, \tilde{\mathbf{M}}^N = \mathbf{M}^N$ *and* $\tilde{\mathbf{M}}^{N+1} = (\mathbf{M}^{N+1} \cup [\overline{\Omega}^c \times (0, T_f)])$ *is an admissible stratification of* $\mathbb{R}^N \times (0, T_f)$.

In this definition, the only difference comes from the boundary $\partial\Omega \times (0, T_f)$ and the following result explains the structure of the $\mathbf{M}^k$'s on $\overline{\Omega} \times (0, T_f)$.

Proposition 26.2.2 (Structure of the $\mathbf{M}^k$)
For any $0 \le k \le N$, *if* $\mathbf{M}_i^k$ *is a connected component of* $\mathbf{M}^k$, *we have either* $\mathbf{M}_i^k \subset \partial\Omega \times (0, T_f)$ *or* $\mathbf{M}_i^k \subset \Omega \times (0, T_f)$.

As a consequence, there is no interaction between $\Omega \times (0, T_f)$ and $\partial\Omega \times (0, T_f)$ through the $\mathbf{M}^k$'s: no connected component can have some part in $\Omega \times (0, T_f)$ and the complementary in $\partial\Omega \times (0, T_f)$. This does not mean that the closure of some $\mathbf{M}_i^k \subset \Omega \times (0, T_f)$ cannot contain some points of $\partial\Omega \times (0, T_f)$ but then they are in some $\mathbf{M}^l$ for some $l < k$. As an example of such situation, where we drop the time-variable in order to simplify the example, we can have:

$$\Omega := \{(x_1, x_2) \in \mathbb{R}^2; \ |x_1| + |x_2| < 1\} \ ,$$

and we have a stratification of $\overline{\Omega}$ by setting

$$\mathbf{M}^0 = \{(0, -1), (1, 0), (0, 1), (-1, 0), (0, 0)\} \ ,$$

$$\mathbf{M}^1 = \{(x_1,0);\ 0<|x_1|<1\} \cup \{(0,x_2);\ 0<|x_2|<1\} \cup \left(\partial\Omega \setminus \mathbf{M}^0\right)\ ,$$
$$\mathbf{M}^2 = \overline{\Omega} \setminus \left(\mathbf{M}^1 \cup \mathbf{M}^0\right)\ .$$

Of course, Proposition 26.2.2 applies but the two first connected components of $\mathbf{M}^1$ are not bounded away from $\partial\Omega$.

As we will see it later on, this will have a key importance in the definition of stratified subsolutions since either we will consider interior points and, of course, this will be analogous to the $\mathbb{R}^N \times (0,T_f)$ case, or we will consider $\mathbb{F}^k$- inequalities at points of the boundary and we will not see any influence from $\Omega \times (0,T_f)$ since, $\mathbf{M}^k$ being included in $\partial\Omega$ in a neighborhood of these points, these inequalities are just "tangent" inequalities.

Proof of Proposition 26.2.2 — It is an easy consequence of the

Lemma 26.2.3 *If $(x,t) \in \mathbf{M}^k \cap \partial\Omega \times (0,T_f)$ for some $0 \le k \le N$, then for r small enough, $\mathbf{M}^k \cap B((x,t),r) \subset \partial\Omega \times (0,T_f)$.*

Indeed, if $\mathbf{M}^k_i$ is a connected component of $\mathbf{M}^k$, we have two cases

– either $\mathbf{M}^k_i \subset \Omega \times (0,T_f)$ and we are done.

– Or there exists $(x,t) \in \mathbf{M}^k_i \cap \partial\Omega \times (0,T_f)$. If $\mathbf{M}^k_i$ is not entirely contained in $\partial\Omega \times (0,T_f)$, then the two subsets of $\mathbf{M}^k_i$ defined by

$$\mathbf{M}^k_{i,1} = \mathbf{M}^k_i \cap \partial\Omega \times (0,T_f)\ ,\ \mathbf{M}^k_{i,2} = \mathbf{M}^k_i \cap \Omega \times (0,T_f)\ ,$$

are both non-empty, open (by Lemma 26.2.3 for $\mathbf{M}^k_{i,1}$) and we have $\mathbf{M}^k_i = \mathbf{M}^k_{i,1} \cup \mathbf{M}^k_{i,2}$. A situation which is a contradiction with the connectedness of $\mathbf{M}^k_i$. Hence $\mathbf{M}^k_i \subset \partial\Omega \times (0,T_f)$ and the proof is complete.

Q.E.D.

Now we turn to the proof of Lemma 26.2.3.

Proof — The result being local, we can assume without loss of generality that we are in the (AFS) case, i.e. there exists $r>0$ such that $\mathbf{M}^k \cap B((x,t),r) = [(x,t)+V_k] \cap B((x,t),r)$ where V_k is a k dimensional vector space.

Lemma 26.2.3 is a consequence of Lemma 3.3.3 using the $\tilde{\mathbb{M}}$-stratification. If, for some $v \in V_k$, $(x,t)+v \in [\Omega \times (0,T_f)] \cap B((x,t),r)$, then there exists $0<\delta \ll r$ such that $B((x,t)+v,\delta) \subset [\Omega \times (0,T_f)] \cap B((x,t),r)$. On the other hand, $B((x,t),\delta) \cap [\Omega^c \times (0,T_f)] \neq \emptyset$ and if $(x_\delta,t_\delta) \in B((x,t),\delta) \cap [\Omega^c \times (0,T_f)] \subset B((x,t),r) \cap \tilde{\mathbf{M}}^{N+1}$, we have $(x_\delta,t_\delta) \in \tilde{\mathbf{M}}^{N+1}$. By Lemma 3.3.3, $(x_\delta,t_\delta)+V_k \subset \tilde{\mathbf{M}}^{N+1}$ but $(x_\delta,t_\delta) \in \tilde{\mathbf{M}}^{N+1} \cap [\Omega^c \times (0,T_f)]$ and $(x_\delta,t_\delta)+v \in \tilde{\mathbf{M}}^{N+1} \cap [\Omega \times (0,T_f)]$ since $(x_\delta,t_\delta)+v \in$

$B((x,t)+v,\delta) \subset [\Omega \times (0,T_f)]$. Therefore $(x_\delta,t_\delta)+V_k$ has a point in $\partial\Omega \times (0,T_f)$ which is a contradiction since there is no point of $\tilde{\mathbf{M}}^{N+1}$ on $\partial\Omega \times (0,T_f)$.

Q.E.D.

26.3 Stratified solutions and comparison result

Now we turn to the notion of stratified solution; as in the $\mathbb{R}^N$-case, we present both a weak and a strong notion.

Definition 26.3.1 (Stratified sub and supersolutions for state-constraint problems)
(*i*) *A locally bounded function* $v : \overline{\Omega} \times [0,T_f[\to \mathbb{R}$ *is a stratified supersolution of*

$$\mathbb{F}(x,t,w,Dw) = 0 \quad \text{on } \overline{\Omega} \times [0,T_f[\,, \tag{26.3}$$

if and only if it is an Ishii supersolution of this equation on $\overline{\Omega} \times [0,T_f[$.

(*ii*) *A locally bounded function* $u : \overline{\Omega} \times [0,T_f[$ *is a* weak *stratified subsolution of Equation (26.3) if, for any* $k = 0,...,(N+1)$, *it is a viscosity subsolution of*

$$\mathbb{F}^k(x,t,u^*,(D_t u^*, D_x u^*)) \le 0 \quad \text{on } \mathbf{M}^k, \text{ for } t > 0,$$

and it is a viscosity subsolution of

$$\mathbb{F}^k_{init}(x,u^*,D_x u^*) \le 0 \quad \text{on } \mathbf{M}^k_0.$$

The function u *is a* strong *stratified subsolution of Equation (26.3) if it is a weak stratified subsolution and if it is a viscosity subsolution of this equation in* $\Omega \times (0,T_f)$ *and it also satisfies*

$$(\mathbb{F}_{init})_*(x,u^*,D_x u^*) \le 0 \quad \text{in } \Omega\,.$$

In addition, we will say that u *is a strict (weak or strong) stratified subsolution if the* ≤ 0*-inequalities are replaced by* $a \le -\eta < 0$*-inequality where* $\eta > 0$ *is independent of* x *and* t.

Several remarks on the definition: for the supersolution, we have the classical Ishii's inequality, and up to the boundary $\partial\Omega \times (0,T_f)$ as it is classical for state-constraint problems. Of course at time $t = 0$, the analogue of Proposition 5.1.1 implies that $\mathbb{F}$ can be replaced by $\mathbb{F}_{init}$.

For the subsolution case, there is no change in $\Omega \times (0,T_f)$, the main feature of stratified subsolutions are preserved, i.e. we have to super-impose $\mathbb{F}^k$-inequalities on

all $\mathbf{M}^k$ (including at time $t = 0$). What may be more suprising and unusual in this state-constraint framework is the fact that there are subsolutions inequalities on $\partial\Omega \times (0, T_f)$ but, on one hand, these inequalities are on $\mathbf{M}^k \cap [\partial\Omega \times (0, T_f)]$ for $k = 0, ..., N$ and therefore they take into account only the dynamics which stay on $\mathbf{M}^k$, i.e. on $\partial\Omega \times (0, T_f)$; on the other hand, to take into account these inequalities on the boundary is not a real difficulty here as long as we deal with "regular subsolutions on the boundary", a notion that we define precisely below but which is a natural extension of the notion of regular subsolutions that we have seen in Part IV.

In fact, the new difficulty which is caused by the boundary is the following: in the $\mathbb{R}^N$-case, the regularity of subsolutions is ensured by the inequality $\mathbb{F}_* \leq 0$ and **(NC)**. This is one of the reasons why weak=strong. But here, on $\partial\Omega \times (0, T_f)]$, the $\mathbb{F}_* \leq 0$ inequality cannot hold and the regularity of subsolutions is a real issue: we have to find a way to prove that subsolutions are "regular" on the $\mathbf{M}^k$ which lies on the boundary. This is THE additional difficulty for state-constraint problems.

To address this question, we need to introduce a notion of boundary regularity for subsolutions.

Definition 26.3.2 *Given an admissible stratification $\mathbb{M}$ of $\overline{\Omega} \times (0, T_f)$, we say that an u.s.c. function $u : \overline{\Omega} \times [0, T_f[\to \mathbb{R}$ is regular at the boundary (with respect to the stratification $\mathbb{M}$) if it satisfies the following conditions:*

(i) for any $(x, t) \in [\partial\Omega \times (0, T_f)] \cap \mathbf{M}^k$,

$$u(x,t) = \limsup \left\{ u(y,s);\ (y,s) \to (x,t),\ (y,s) \in \mathbf{M}^{k+1} \cup \cdots \cup \mathbf{M}^{N+1} \right\} ; \quad (26.4)$$

(ii) for any $x \in \partial\Omega \cap \mathbf{M}_0^k$,

$$u(x,0) = \limsup \left\{ u(y,0);\ (y,0) \to (x,0),\ (y,0) \in \mathbf{M}_0^{k+1} \cup \cdots \cup \mathbf{M}_0^{N+1} \right\} . \quad (26.5)$$

We can then conclude this part by the comparison result.

Theorem 26.3.3 *In the "good framework for HJ Equations with discontinuities" which is described above, the comparison results between bounded stratified weak subsolutions and supersolutions of Equation (26.3) holds provided that the subsolution is regular in $\Omega \times (0, T_f)$, $u(x, 0)$ is regular in Ω and both are regular at the boundary. In the case of strong subsolutions, the result holds for subsolutions which are regular at the boundary.*

As we point out above, to have "strong subsolutions" implies that they are regular in $\Omega \times (0, T_f)$ and in, for $t = 0$, in Ω but we have to keep the assumption that they are regular at the boundary.

As the reader may guess, the proof is (almost) exactly the same as the proof of Theorem 20.4.1 and this is easy to explain why: the fact that some parts of the stratification are on the boundary does not cause any problem and the key ingredients were already used in the $\mathbb{R}^N$-case. The only difference comes from the regularity of the subsolution at the boundary whose aim is, of course, to eliminate "artificial values" on the boundary.

As we will see below, this condition is analogous to the "cone condition" which is used in state-constraint or Dirichlet problems for standard continuous equations. We will see in Section 26.4 how an analogue of the $\mathbb{F}_*$-inequality and Proposition 3.4.2 for boundary points can be used to obtained (26.4) and/or (26.5); this point may be important for stability reasons.

26.4 On the boundary regularity of subsolutions

As we keep pointing it out, the "regularity" of subsolutions plays a central role since this is a keystone argument of the comparison result in the stratified case when we deal with subsolutions. We recall that such regularity allows to obtain continuous subsolutions after "tangential regularization" by sup-convolution. In $\mathbb{R}^N$, this property is, in general, a consequence of the standard Ishii subsolution inequality $\mathbb{F}_* \leq 0$, provided that the normal controllability assumption is satisfied (cf. Proposition 3.4.2); in other words, strong stratified subsolutions are regular weak subsolutions (cf. Section 20.3). Of course, in the present context, the same is true if we consider parts of the stratification which are inside the domain $\Omega \times (0, T_f)$.

However, the situation is a completely different on $\partial\Omega \times (0, T_f)$ since the subsolution inequality $\mathbb{F}_* \leq 0$ does not hold, in general, on the boundary. It is replaced by the $\mathbb{F}^k$-ones, involving only tangential dynamics which, therefore, cannot give information on the values of the subsolution in $\Omega \times (0, T_f)$ near the boundary. Actually, it is well-known that, even in classical cases, (sub)solutions of the Dirichlet problems may have "artificial" values on the boundary which have no connections with interior ones (cf. [22, 23, 24] or [31]). We come back on that classical case below.

We propose two ways to circumvent this difficulty: the first one is when some ad hoc inequalities on the boundary play the role of the "$\mathbb{F}_* \leq 0$"-one, allowing to prove the regularity of subsolutions. The second one, inspired by the "continuous case", is completely different, which consists in redefining the subsolution on the different portions $\mathbf{M}^k$ of the boundary in order to have (26.4). Of course this second way is far more restrictive since it requires that no real discontinuity (in terms of **BCL**) is present on the boundary. But it may be useful though, since the stratified approach

allows non-smooth boundaries.

Before providing our results, let us come back to the classical case and point out that such type of difficulty is classical in state-constraint or Dirichlet problems, even if it is in general formulated in a slightly different way: from the very first articles of Soner [154, 155] on state-constraint problems, the "cone condition" (or related properties) is known to play a role in comparison results for such problems. Here also, one needs to get some kind of continuity property of the subsolution on the boundary, at least to avoid artificial values.

Concerning Dirichlet problems, Perthame and the first author [22, 23, 24] have worked on this difficulty by either showing such continuity property (even in a weaker sense) or by redefining the subsolution on the boundary in order to get it, two possibilities that we investigate below. Ishii and Koike [123] have formulated the state-constraint boundary condition in a different way, with an unusual subsolution condition on the boundary, by looking only at dynamics which are pointing inside the domain on the boundary: Lemma 26.4.1 below shows that their boundary condition "$u_t + H_{in} \leq 0$" avoids non-regular subsolution if there is an inner dynamic.

Finally we point out that some results for first but also second-order equations are obtained by Katsoulakis [125] or Rouy and the first author [25]: in [25], a blow-up argument allows to show that the cone condition holds under suitable assumptions for first-order equations and that we have a related property for the second-order case.

Our first result is the

Lemma 26.4.1 *Assume that $\overline{\Omega} \times (0, T_f)$ is a stratified domain and that $(x_0, t_0) \in \mathbf{M}^k \cap [\partial\Omega \times (0, T_f)]$. Assume also that there exist $r, M, \bar{\tau}, \delta > 0$ and a continuous function $b : [\overline{\Omega} \times (0, T_f)] \cap B((x_0, t_0), r) \to \mathbb{R}^N$ such that, for any $y \in \partial\Omega \cap B(x_0, r)$ and $0 < \tau < \bar{\tau}$,*

$$B(y + \tau b(x_0, t_0), \delta\tau) \subset \Omega \, . \tag{26.6}$$

If u is a subsolution of

$$u_t - b(x,t) \cdot Du \leq M \quad \text{on } [\partial\Omega \times (0, T_f)] \cap B((x_0, t_0), r) \ ^{(1)} \, , \tag{26.7}$$

then (26.4) holds at (x_0, t_0), more precisely

$$u(x_0, t_0) = \limsup\{u(y, s), \ (y, s) \to (x_0, t_0), \ y \in \mathbf{M}^{k+1} \cup \cdots \mathbf{M}^N\} \, .$$

Before proving this lemma, we want to point out that, as the proof is going to show, this is a very basic result ; a more interesting point is to give general (and if possible,

(1) in the sense that, for any smooth test-function ϕ if $u - \phi$ has a local maximum point relatively to $\overline{\Omega} \times (0, T_f)$ at (x, t) then $\phi_t(x, t) - b(x, t) \cdot D\phi(x, t) \leq M$.

natural) conditions under which a subsolution of the stratified problem is a viscosity subsolution of an equation like (26.7). Of course, but this has to be formulated a little bit more precisely, such property is, in general, a consequence of (i) an interior cone condition like (26.6) and (ii) the normal controllability, with a suitable compatibility between them.

Proof — Throughout the proof, we assume that we are in the flat case, namely $\mathbf{M}^k \cap B((x_0,t_0),r) = [(x_0,t_0)+V_k]\cap B((x_0,t_0),r)$, where V_k is a k-dimensional vector space in $\mathbb{R}^N \times (0,T_f]$, taking perhaps r smaller in (26.6). We first claim that $(b(x_0,t_0),-1)$ cannot be in $T_{x_0}\mathbf{M}^k$.

This property is an easy consequence of Lemma 26.2.3: indeed, otherwise we would have that the distance from $(x_0+\tau b(x_0,t_0),t_0-\tau)$ to $\mathbf{M}^k$ would be a $o(\tau)$ which would contradict the assumption on b which says that the distance to $\partial\Omega\times(0,T_f)$—and therefore to $\mathbf{M}^k$—is at least $\delta\tau$ $(\delta>0)$.

The above property on $b(x_0,t_0)$ implies that there exists a vector $e\in\mathbb{R}^N$, such that $(e,0)$ is orthogonal to $T_{x_0}\mathbf{M}^k$, such that $b(x_0,t_0)\cdot e>0$. Then, in $\overline{\Omega}\times(0,T_f)$, we consider the function

$$(x,t)\mapsto u(x,t)-\frac{|x-x_0|^2}{\varepsilon^2}+\frac{2}{\varepsilon}e\cdot(x-x_0)-\frac{|t-t_0|^2}{\varepsilon}\ .$$

Using the properties satisfied by a stratification, if (26.4) does not hold then, for $0<\varepsilon\ll 1$, this function necessarily achieves its maximum on $\mathbf{M}^k$ at $(x_\varepsilon,t_\varepsilon)$ and, as a consequence of the maximum point property we have

$$\begin{aligned} u(x_0,t_0) &\le u(x_\varepsilon,t_\varepsilon)-\frac{|x_\varepsilon-x_0|^2}{\varepsilon^2}-\frac{|t_\varepsilon-t_0|^2}{\varepsilon}+\frac{1}{\varepsilon}e\cdot(x_\varepsilon-x_0)\\ &= u(x_\varepsilon,t_\varepsilon)-\frac{|x_\varepsilon-x_0|^2}{\varepsilon^2}-\frac{|t_\varepsilon-t_0|^2}{\varepsilon}\ , \end{aligned}$$

the equality coming from the definition of e and the fact that we have a flat stratification.

By Lemma 6.4.1 (or at least by using the underlying arguments), this implies that the penalisation terms $\dfrac{|x_\varepsilon-x_0|^2}{\varepsilon^2}, \dfrac{|t_\varepsilon-t_0|^2}{\varepsilon}$ tend to 0 when $\varepsilon\to 0$ and in particular, we have $(x_\varepsilon,t_\varepsilon)\to(x_0,t_0)$. Writing the viscosity subsolution inequality yields

$$\frac{2(t_\varepsilon-t_0)}{\varepsilon}-b(x_\varepsilon,t_\varepsilon)\cdot\Big(\frac{2(x_\varepsilon-x_0)}{\varepsilon^2}-\frac{2}{\varepsilon}e\Big)\le M\ ,$$

which gives, thanks to the previous properties

$$\frac{o(1)}{\varepsilon}+\frac{2}{\varepsilon}b(x_\varepsilon,t_\varepsilon)\cdot e\le M\ .$$

But, by the continuity of b, $b(x_\varepsilon, t_\varepsilon)\cdot e \to b(x_0,t_0)\cdot e > 0$, and we get a contradiction in this above inequality for ε small enough.

Q.E.D.

The next result shows how (26.7) can be obtained and the kind of compatibility conditions which are needed to get it, combining the cone condition (26.6) and the dynamic in the control problem.

Lemma 26.4.2 *Assume that $\overline{\Omega}\times(0,T_f)$ is a stratified domain, that* **(NC-BCL)**, **(TC-BCL)** *hold and that $(x_0,t_0)\in \mathbf{M}^k\cap[\partial\Omega\times(0,T_f)]$. Assume also that there exist $r, M, \bar\tau, \delta > 0$ and a continuous function b defined on $[\overline{\Omega}\times(0,T_f)]\cap B((x_0,t_0),r)$ and taking values in $\mathbb{R}^N$ such that, for any $y\in\partial\Omega\cap B(x_0,r)$ and $0<\tau<\bar\tau$, (26.6) holds. Assume, in addition, that there exists $\kappa>0$ such that, for any $(x,t)\in[\overline{\Omega}\times(0,T_f)]\cap B((x_0,t_0),r)$, $\kappa(b(x,t),-1)\in\mathbf{B}(x,t)$.*

If u is a subsolution of the stratified state constraint problem in $[\overline{\Omega}\times(0,T_f)]\cap B((x_0,t_0),r)$ and if (26.4) holds for any $(x,t)\in[\partial\Omega\times(0,T_f)]\cap B((x_0,t_0),r)$ then u is a subsolution of (26.7).

This corollary means that, in some sense, the property (26.4) is equivalent to a natural "control" inequality (as it is the case in Ω) and that such inequality should be automatically extended to the boundary if the boundary values are the limit of the interior ones.

Proof — Using the regularization procedure of Section 3.4, we can assume without loss of generality that u is Lipschitz continuous on $[\overline{\Omega}\times(0,T_f)]\cap B((x_0,t_0),r)$. We point out that (26.4) plays a key role in this property to avoid any discontinuity on the boundary.

If ϕ is a smooth test-function and if $(\bar x,\bar t)\in[\partial\Omega\times(0,T_f)]\cap B((x_0,t_0),r)$ is a strict local maximum point of $u-\phi$ in $[\overline{\Omega}\times(0,T_f)]\cap B((x_0,t_0),r)$, we consider the function

$$\Psi(x,t,y,s)=u(x,t)-\phi(y,s)-\frac{|x-y-\varepsilon b(x_0,t_0)|^2}{\varepsilon^2}-\frac{|t-s|^2}{\varepsilon}\ .$$

which achieves its maximum at $(x_\varepsilon,t_\varepsilon,y_\varepsilon,s_\varepsilon)$. Since u is Lipschitz continuous, we have $u(\bar x+\varepsilon b(x_0,t_0),\bar t)=u(\bar x,\bar t)+o_\varepsilon(1)$ and therefore

$$u(\bar x,\bar t)-\phi(\bar x,\bar t)+o_\varepsilon(1)\le\Psi(\bar x+\varepsilon b(x_0,t_0),\bar t,\bar x,\bar t)\le\Psi(x_\varepsilon,t_\varepsilon,y_\varepsilon,s_\varepsilon)\le u(\bar x,\bar t)-\phi(\bar x,\bar t)+o_\varepsilon(1)\ ,$$

the last inequality coming from the fact that $x_\varepsilon-y_\varepsilon, t_\varepsilon-s_\varepsilon$ are $O(\varepsilon)$ and therefore $(x_\varepsilon,t_\varepsilon)=(y_\varepsilon,s_\varepsilon)+O(\varepsilon)$.

By Lemma 6.4.1 (or at least by using the underlying arguments), we deduce that, not only $(x_\varepsilon, t_\varepsilon), (y_\varepsilon, s_\varepsilon) \to (\bar{x}, \bar{t})$ but we also have

$$\frac{|x_\varepsilon - y_\varepsilon - \varepsilon b(x_0,t_0)|^2}{\varepsilon^2} + \frac{|t_\varepsilon - s_\varepsilon|^2}{\varepsilon} \to 0 \ ,$$

as $\varepsilon \to 0$. In particular $x_\varepsilon \in B(y_\varepsilon + \varepsilon b(x_0,t_0), k\varepsilon) \subset \Omega$ since $|x_\varepsilon - y_\varepsilon - \varepsilon b(x_0,t_0)| = o(\varepsilon)$ if ε is small enough.

We can write down the viscosity subsolution inequality for u

$$\mathbb{F}_*(x_\varepsilon, t_\varepsilon, u(x_\varepsilon,t_\varepsilon), (p_\varepsilon, \alpha_\varepsilon)) \leq 0 \ ,$$

where $\alpha_\varepsilon = \dfrac{2(t_\varepsilon - s_\varepsilon)}{\varepsilon} = \phi_t(y_\varepsilon, s_\varepsilon)$ by the maximum point property in s, while $p_\varepsilon = \dfrac{2(x_\varepsilon - y_\varepsilon - \varepsilon b(x_0,t_0))}{\varepsilon^2} = D\phi(y_\varepsilon, s_\varepsilon)$ if $y_\varepsilon \in \Omega$ but not necessarely if $y_\varepsilon \in \partial\Omega$.

In order to estimate $\mathbb{F}_*$, we recall that

$$\mathbb{F}(x,t,u,p) \geq -b \cdot p + cr - l \ ,$$

where $(b,c,l) \in \mathbf{BCL}(x,t)$. In particular, by the assumption on $(b(x,t), -1)$, we have $\mathbb{F}(x,t,u,p) \geq \kappa p_t - \kappa b(x,t) \cdot p - \tilde{M}$ for some constant $\tilde{M}$ since c, u, l are bounded. And therefore $\mathbb{F}_*(x,t,u,p) \geq \kappa p_t - \kappa b(x,t) \cdot p - \tilde{M}$.

Using this estimate, we conclude easily if we know that $y_\varepsilon \in \Omega$ at least for a subsequence of ε tending to 0, with $M = \tilde{M}/\kappa$.

If $y_\varepsilon \in \partial\Omega$, we write for $0 < \tau \ll 1$

$$\Psi(x_\varepsilon, t_\varepsilon, y_\varepsilon, y_\varepsilon + \tau b(y_\varepsilon), t_\varepsilon) \leq \Psi(x_\varepsilon, t_\varepsilon, y_\varepsilon, s_\varepsilon) \ ,$$

and this variation gives $-b(y_\varepsilon, t_\varepsilon) \cdot D\phi(y_\varepsilon, s_\varepsilon) \leq -b(y_\varepsilon, t_\varepsilon) \cdot p_\varepsilon$. The conclusion follows easily. And the proof is complete.

Q.E.D.

Now we turn to the second possibility which is a little bit more restrictive.

Lemma 26.4.3 *Assume that $\overline{\Omega} \times (0, T_f)$ is a stratified domain and that* **(NC)**, **(TC)** *hold. Let $(x_0,t_0) \in \mathbf{M}^k$ be a point such that there exists $r > 0$ such that $\mathbf{M}^k \cap B((x_0,t_0),r) = \partial\mathbf{M}^{k+1} \cap B((x_0,t_0),r)$. If, for any $(x,t) \in \mathbf{M}^k \cap B((x_0,t_0),r)$ and any $(y,s) \in \mathbf{M}^{k+1} \cap B((x_0,t_0),r)$, we have the following property*

for any $(b,c,l) \in \mathbf{BCL}(x,t)$ such that $b \in T_{(x,t)}\mathbf{M}^k$, there exists $(b',c',l') \in \mathbf{BCL}(y,s)$ with $b' \in T_{(x,t)}\mathbf{M}^{k+1}$ and $|b' - b| \leq C(|x-y| + |t-s|)$ for some constant C,

$$|c-c'|+|l-l'|=o(1) \text{ as } (y,s)-(x,t)\to 0$$

and if u is a subsolution of the $\mathbb{F}^{k+1}$- equation in $\mathbf{M}^{k+1}\cap B((x_0,t_0),r)$ then the function $\tilde{u}$ defined on $\mathbf{M}^k\cap B((x_0,t_0),r)$ by

$$\tilde{u}(x,t)=\limsup\{u(y,s),\ (y,s)\to(x,t)\ y\in\mathbf{M}^{k+1}\}$$

satisfies the $\mathbb{F}^k$-inequality in $\mathbf{M}^k\cap B((x_0,t_0),r)$.

Proof — We consider a smooth test-function ϕ and $(x,t)\in\mathbf{M}^k\cap B((x_0,t_0),r)$, a strict local maximum point of $\tilde{u}-\phi$ on $\mathbf{M}^k\cap B((x_0,t_0),r)$. By definition of $\tilde{u}$, we can approximate this maximum by maximum points on $\mathbf{M}^{k+1}\cap B((x_0,t_0),r)$

$$(y,s)\mapsto u(y,s)-\phi(y,s)-\frac{\alpha}{d(y,\mathbf{M}^k)}-\frac{d(y,\mathbf{M}^k)}{\varepsilon}\ ,$$

by choosing in a proper way the parameters ε,α which are devoted to tend to 0.

If the maximum is achieved at $(\bar{x},\bar{t})$ depending on α and ε, the $\mathbb{F}^{k+1}$-inequality has to be written as a supremum for all $(b,c,l)\in\mathbf{BCL}(\bar{x},\bar{t})$ with $b\in T_{(\bar{x},\bar{t})}\mathbf{M}^{k+1}$ but we are going to argue in a different way: if $(\bar{b},\bar{c},\bar{l})\in\mathbf{BCL}(\bar{y},\bar{s})$ where $(\bar{y},\bar{s})$ is the projection of $(\bar{x},\bar{t})$ on $\mathbf{M}^k$ then, on one hand, there exists $(b,c,l)\in\mathbf{BCL}(\bar{x},\bar{t})$ with $|b-\bar{b}|\ \le C(|\bar{x}-\bar{y}|+|\bar{t}-\bar{s}|)=Cd((\bar{x},\bar{t}),\mathbf{M}^k)$ and $|\bar{c}-c|,|\bar{l}-l|$ are $o(1)$ in α,ε if we let first α and then ε tend to 0. On the other hand, the $\mathbb{F}^{k+1}$-inequality reads

$$-b\cdot(D_x\phi(\bar{x},\bar{t})+P_{\alpha,\varepsilon},\phi_t(\bar{x},\bar{t}))+cu(\bar{x},\bar{t})-l\le 0\ ,$$

where $P_{\alpha,\varepsilon}$ denotes the derivatives of the two last terms, for which we have $|P_{\alpha,\varepsilon}|d((\bar{x},\bar{t}),\mathbf{M}^k)\to 0$ as $\alpha,\varepsilon\to 0$ (again with the proper choice we have to make). Taking into account the fact that $u(\bar{x},\bar{t})\to u(x,t)$, we are lead to

$$-\bar{b}\cdot(D_x\phi(\bar{x},\bar{t}),\phi_t(\bar{x},\bar{t}))+\bar{c}u(x,t)-\bar{l}\le o(1)\ ,$$

and since this is true for any $(\bar{b},\bar{c},\bar{l})\in\mathbf{BCL}(\bar{y},\bar{t})$ and therefore for any $(\bar{b},\bar{c},\bar{l})\in\mathbf{BCL}(x,t)$ by tangential continuity, the result follows.

Q.E.D.

The above lemma suggests the following procedure if $\mathbf{M}^k\subset\partial\mathbf{M}^{k+1}$ for any k: we first redefine u on $\mathbf{M}^{N-1}$ and, after this step, (26.4) holds on $\mathbf{M}^{N-1}$. Then we repeat the same operation on $\mathbf{M}^{N-2}$ and inductively until $\mathbf{M}^0$.

Chapter 27

Classical Boundary Conditions and Stratified Formulation

In this chapter, we are going to investigate the connections between stratified problems with state-constraints and classical (or almost classical) problems with boundary conditions (Dirichlet, Neumann or mixed boundary conditions). Of course, the interest of the stratified formulation is to allow to treat cases where the boundary is not smooth or the boundary conditions may present discontinuities, and also both at the same time.

Clearly our aim cannot be to give extremely general results: this would be unreadable and of a poor interest. In fact, what is done in Section 27.3.2 for the Tanker Problem shows that the stratified formulation allows to treat very general problems with even exotic "boundary conditions", if we can still call them like that since, in this framework, there is no main difference between the equation and the boundary conditions. As a consequence, most of the Dirichlet, Neumann, oblique derivatives and mixed problems we are going to consider have a unique stratified solution provided that we formulate them in the right way and that the "natural assumptions" are satisfied.

Instead, we address the following two complementary questions, mainly in very simple frameworks, whose answers may emphasize the role and the interest of the stratified formulation

(i) in which cases classical Ishii viscosity solutions and stratified solutions are the same? Of course, in such cases, the theory which is developed in the previous chapter provides complete comparison results;

(ii) on the contrary, in which cases is the stratified formulation needed because the

Ishii formulation is not precise enough to identify the "good" solution?

In order to do so, we are just going to consider standard problems where the difficulty only comes from the boundary and boundary data, and for which the equation inside the domain is continuous. More precisely, we are interested throughout this chapter in a standard Hamilton-Jacobi-Bellman Equations associated with a classical initial condition, namely

$$\begin{cases} u_t + H(x,t,D_x u) = 0 & \text{in } \Omega \times (0,T_f) \\ u(x,0) = u_0(x) & \text{in } \Omega \end{cases} \tag{27.1}$$

where Ω is a domain in $\mathbb{R}^N$ and where H has the form

$$H(x,t,p) := \sup_{\alpha \in A} \left\{ -b(x,t,\alpha) \cdot p - l(x,t,\alpha) \right\} \ , \tag{27.2}$$

for any $x \in \overline{\Omega}$, $t \in [0,T_f]$, $p \in \mathbb{R}^N$, where A is a compact metric space and b, l are continuous functions on $\overline{\Omega} \times [0,T_f] \times A$ taking respectively values in $\mathbb{R}^N$, $\mathbb{R}$ and $\mathbb{R}$. We assume that b, l and u_0 satisfy $(\mathbf{H_{BA-CP}})$ which implies that H satisfies $(\mathbf{H_{BA-HJ}})$.

Clearly, our results would need to be extended to treat problems where we have also discontinuities inside: some of these extensions are easy using some ideas of this chapter (typically if the discontinuities of H inside $\Omega \times [0,T_f]$ are apart from the boundary) but some other ones are more delicate (if the discontinuities of H inside $\Omega \times [0,T_f]$ interfere with the boundary).

On the other hand, and this is obvious from the definition of stratified solutions, we are going to concentrate on subsolution properties since a stratified supersolution is nothing but a classical viscosity supersolution in the sense of Ishii. Of course, we are going to place ourselves in the "good framework for HJ Equations with discontinuities", in particular **(NC)** will hold on the boundary), which implies that most of the time we will be able to assume without loss of generality that the subsolutions are Lipschitz continuous by the by-now standard tangential regularization procedure.

27.1 On the Dirichlet problem

We are interested in this section in the Dirichlet problem for Hamilton-Jacobi-Bellman Equations, namely (27.1) associated with the boundary condition

$$u(x,t) = \varphi(x,t) \quad \text{on } \partial\Omega \times (0,T_f) \ . \tag{27.3}$$

where we first assume that φ is a continuous function which satisfy the compatibility condition

$$u_0(x) = \varphi(x,0) \quad \text{on } \partial\Omega \ .$$

In this classical case, there are two kinds of results which are described in the book [31] and are originated from the works of Perthame and the first author [22, 23, 24].

- The first one is the "discontinuous approach" where one tries to determine the minimal and maximal solution of (27.1)-(27.3) in full generality, i.e. without any particular additional assumption on the dynamic and cost, and without assuming the boundary of Ω to be smooth. The result is that there exist a minimal solution U^- and a maximal solution U^+ which are value-functions of exit time problems, U^- being associated to the best stopping time on the boundary, while U^+ is associated to the worst stopping time on the boundary.
- The second one is the "continuous approach" in which one looks for conditions under which the value function is continuous and the unique solution of (27.1)-(27.3): in [24], the result is obtained under classical assumptions on the dynamics and cost, plus an hypothesis of normal controllability on the boundary which looks like very much **(NC)**. This second type of results required some regularity of the boundary ($W^{2,\infty}$ in general).

In this section, our aim is to reformulate the Dirichlet problem in the "stratified" framework, in order to investigate the cases when it is equivalent to the classical viscosity solutions formulation and then to examine the type of extensions that we can get in that way.

27.1.1 The stratified formulation for the most classical case

For the reformulation, the idea is very clear and classical: if $(x,t) \in \overline{\Omega} \times [0,T_f]$, we set

$$\mathbf{BCL}_e(x,t) := \{((b(x,t,\alpha),-1),0,l(x,t,\alpha));\ \alpha \in A\} \ ,$$

"e" for "equation" and, if $(x,t) \in \partial\Omega \times [0,T_f]$, we introduce

$$\mathbf{BCL}_{bc}(x,t) := \{((0,0),1,\varphi(x,t))\} \ ,$$

"bc" for "boundary condition". Indeed, at the level of the general Hamiltonian $\mathbb{F}$, this produces the expected term on the boundary, namely

$$-b \cdot p + cu - l = u - \varphi(x,t) \ ,$$

and, for the control point of view, this provides a 0-dynamic allowing to stop at the point (x,t) and pay a cost which is $\varphi(x,t)$, the discount factor being 1.

Of course $\mathbf{BCL}(x,t) = \mathbf{BCL}_e(x,t)$ if $(x,t) \in \Omega \times (0,T_f]$ and it is the closure of the convex enveloppe of $\mathbf{BCL}_e(x,t) \cup \mathbf{BCL}_{bc}(x,t)$ if $(x,t) \in \partial\Omega \times (0,T_f]$. For $t=0$, but this is by now classical, we have also to add a term $(b,c,l) = ((0,0),1,u_0(x))$ to take into account the initial data and $\mathbf{BCL}(x,0)$ is given by the closure of the convex enveloppe of either $\mathbf{BCL}_e(x,0) \cup \{((0,0),1,u_0(x))\}$ or $\mathbf{BCL}_e(x,0) \cup \mathbf{BCL}_{bc}(x,0) \cup \{((0,0),1,u_0(x))\}$ depending whether x is in Ω or $\partial\Omega$.

With this point of view, we have just a state-constraint problem since the trajectory (X,T) exists for all times and stays on $\overline{\Omega}\times[0,T_f]$, the Dirichlet condition just allowing a choice $b=0$ on the boundary, exactly as it is the case for the initial data at $t=0$ since $b^t = -1$ for any (b,c,l) in $\mathbf{BCL}_e(x,0)$ for which $b \neq 0$.

Since H is continuous on $\overline{\Omega}\times[0,T_f]$, the stratified approach consists in considering, for $t>0$, the stratification $\mathbf{M}^{N+1} = \Omega \times (0,T_f)$ and $\mathbf{M}^N = \partial\Omega \times (0,T_f)$. In order to apply the above results, we have to impose

(*i*) some regularity of $\partial\Omega$: here $W^{2,\infty}$ (exactly as in [24]) is natural in general since we have to flatten $\mathbf{M}^N$, by keeping the needed property on H (in particular **(TC)**). But this can be reduced to C^1 if H is coercive, to the cost of sophisticating a little bit our arguments, treating differently the variables t and x.

(*ii*) some normal controlability assumptions which turn out to be also the same as in [24], namely, for any $(x,t) \in \partial\Omega \times [0,T_f]$, the existence of two controls $\alpha_i = \alpha_i(x,t)$ for $i=1,2$ such that

$$b(x,t,\alpha_1)\cdot n(x) < 0 \quad , \quad b(x,t,\alpha_2)\cdot n(x) > 0 \, , \tag{27.4}$$

where $n(x)$ is the unit, outward normal vector to $\partial\Omega$ at x.

We come back later on the advantages of this new approach but let us examine first the boundary condition from the stratified point of view. To do so, we have to compute $\mathbb{F}^N(x,t,p)$ if $(x,t) \in \mathbf{M}^N$ where we recall that $p = (p_x, p_t)$.

Taking into account the form of $\mathbf{BCL}(x,t)$ if $(x,t) \in \partial\Omega \times (0,T_f]$ which is obtained by considering the convex enveloppe of elements of the form

$$(b,c,l) = ((b(x,t,\alpha),-1),0,l(x,t,\alpha)) \in \mathbf{BCL}_e(x,t)$$

[associated to the Hamiltonian H] and of $((0,0),1,\varphi(x,t))$ associated to the Dirichlet boundary condition; therefore we have to consider all the

$$(\mu(b(x,t,\alpha),-1),(1-\mu),\mu l(x,t,\alpha) + (1-\mu)\varphi(x,t))$$

for $0 \leq \mu \leq 1$ but with

$$\mu b = \mu(b^x, -1) \in T_{(x,t)}\mathbf{M}^N \,,$$

in other words $b^x \in T_x\partial\Omega$.

In order to compute $\mathbb{F}^N(x,t,r,p)$, we have to look at the supremum in μ and $(b,0,l) \in \mathbf{BCL}(x,t)$ with $b^x \in T_x\partial\Omega$, of

$$-\mu b \cdot p + (1-\mu)u - (\mu l + (1-\mu)\varphi(x,t)) = \mu(b\cdot p - l) + (1-\mu)(u - \varphi(x,t))\,,$$

and clearly this supremum is achieved either for $\mu = 0$, or $\mu = 1$. Hence the Dirichlet boundary condition

$$\max(u_t + H^N(x,t,D_x u), u - \varphi(x,t)) \leq 0 \quad \text{on } \mathbf{M}^N \,, \tag{27.5}$$

where

$$H^N(x,t,p_x) = \sup_{b(x,t,\alpha)\in T_x\partial\Omega} \{-b(x,t,\alpha)\cdot p_x - l(x,t,\alpha)\}\,.$$

This is a rather unusual inequality which, to the best of our knowledge, never appears in the study of Dirichlet boundary conditions for HJ-Equations. But, on the other hand, it is rather natural from the control point of view; indeed the inequality $u_t + H^N(x,t,D_x u) \leq 0$ just says that the tangent dynamics are sub-optimal and, in the same way, the inequality $u - \varphi(x,t) \leq 0$ reflects the sub-optimality of the strategy consisting in stopping at (x,t) and paying the cost $\varphi(x,t)$. We point out anyway that the normal controllability plays a role here since such strategy is available to the controller as soon as the state (X,T) comes close to $\partial\Omega \times (0,T_f)$ since he can choose to quickly exit the domain via a dynamic pointing outward $\Omega \times (0,T_f)$; then, when (X,T) is on $\partial\Omega \times (0,T_f)$, he can either stop and pay the φ-cost or continue on the boundary with the dynamic $b(x,t,\alpha) \in T_x\partial\Omega$, waiting a better stopping time on $\partial\Omega \times (0,T_f)$.

It is also worth remarking that the non-tangential dynamics are taken into account in the Ishii viscosity subsolution inequality

$$\min(u_t + H(x,t,D_x u), u - \varphi(x,t)) \leq 0 \quad \text{on } \partial\Omega \times (0,T_f)\,.$$

Now we turn to the first key question: does a classical viscosity subsolution always satisfy such H^N-inequality in the stratified framework? And, in the case of a less regular boundary -but still in a stratified framework-, does an analogous one hold on $\mathbf{M}^k$ for $1 \leq k \leq N$?

27.1.2 Ishii and stratified solutions for the Dirichlet problem

In this section, we address the above questions in a full generality under the assumptions on H which are given at the beginning of the chapter (in particular H is continuous) but $\partial\Omega \times (0, T_f)$ is not smooth anymore. More precisely, we assume that there is a stratification $\mathbb{M} = (\mathbf{M}^k)_k$ of $\overline{\Omega} \times \mathbb{R}$ such that

$$\mathbf{M}^{N+1} = \Omega \times \mathbb{R} \quad , \quad \mathbf{M}^1 \cup \cdots \mathbf{M}^N = \partial\Omega \times \mathbb{R} \, .$$

Under these conditions, we will say that we have a *stratified domain*.

We begin with the

Proposition 27.1.1 *Assume that the above assumption on $\mathbb{M}$ holds, that φ is continuous on $\mathbf{M}^N$ and that we are in the good framework for stratified problem. Then,*

(i) if u is any viscosity subsolution of the Dirichlet problem, $u \leq \varphi$ on $\mathbf{M}^N$.

(ii) If $\tilde{u}$ is given by $\tilde{u} = u$ in $\Omega \times (0, T_f)$ and

$$\tilde{u}(x,t) = \limsup_{\substack{(y,s)\to(x,t) \\ y\in\Omega}} u(y,s) \, ,$$

then $\tilde{u}$ is still a classical viscosity subsolution of the Dirichlet problem and (27.5) holds for $\tilde{u}$ on $\mathbf{M}^N$.

We first want to point out that, in Proposition 27.1.1, the essential consequence of the "good framework for stratified problem" is that (27.4) holds on $\mathbf{M}^N$.

The introduction of the function $\tilde{u}$ in order to redefine u on the boundary is classical: in fact, it is needed because the viscosity subsolution inequality is not strong enough to avoid artificial values of u on the boundary. Indeed since the viscosity subsolution property is ensured by the fact that $u \leq \varphi$ on $\mathbf{M}^N$, u could be changed into any u.s.c. function which lies below φ on $\mathbf{M}^N$, with no link whatsoever with the values inside $\Omega \times (0, T_f)$. The introduction of $\tilde{u}$ consists in imposing the "natural" values of the subsolution on $\mathbf{M}^N$ since they are consistent with those in $\Omega \times (0, T_f)$. Once this "cleaning" of the boundary values is done, then we have the desired result, namely that viscosity subsolutions are stratified subsolutions.

A different point of view is the regularity of subsolutions on the boundary: we have insisted, since the beginning of Part V, that this is a key difficulty in state-constraint problem. Here we face it and Proposition 27.1.1 solves it in the case when

$\mathbf{M}^N = \partial\Omega\times(0,T_f)$ by replacing the non-regular subsolution by a regular one, changing only its values on the boundary.

Proof — We start proving that, if u is an u.s.c. viscosity subsolution of the Dirichlet problem then $u \le \varphi$ on $\mathbf{M}^N$. We can argue locally and therefore assume that $\partial\Omega \times (0,T_f) = \mathbf{M}^N$ is smooth, hence $\partial\Omega$ is smooth. If d denotes the distance to $\partial\Omega$, d is at least C^1 and we recall that $Dd(y) = -n(y)$ if $y \in \partial\Omega$.

If $(x,t) \in \mathbf{M}^N$, we consider the function

$$(y,s) \mapsto u(y,s) - \frac{(s-t)^2}{\varepsilon^2} - \frac{|y-x|^2}{\varepsilon^2} - C_\varepsilon d(y)\ ,$$

where $C_\varepsilon > 0$ is a large constant to be chosen later. This function has a maximum point $(y_\varepsilon, t_\varepsilon)$ near (x,t) and, by classical arguments, we have $(y_\varepsilon,t_\varepsilon) \to (x,t)$ and $u(y_\varepsilon,t_\varepsilon) \to u(x,t)$. If the H-inequality holds at $(y_\varepsilon,t_\varepsilon)$, we would have

$$\frac{2(s_\varepsilon - t)}{\varepsilon^2} + H(y_\varepsilon, s_\varepsilon, \frac{2(y_\varepsilon - x)}{\varepsilon^2} + C_\varepsilon D_x d(y_\varepsilon)) \le 0\ .$$

But $D_x d(y_\varepsilon) = D_x d(x) + o(1) = -n(x) + o(1)$ and, by the normal controllability assumption (cf. (27.4)), this inequality cannot hold if we choose C_ε large enough. As a consequence, we have both $(y_\varepsilon,t_\varepsilon) \in \partial\Omega \times (0,T_f)$ and $u(y_\varepsilon,t_\varepsilon) \le \varphi(y_\varepsilon,t_\varepsilon)$.

Letting $\varepsilon \to 0$, we obtain the desired result since φ is continuous on $\mathbf{M}^N$.

As we already mentioned it above, the viscosity subsolution inequality being reduced to $u \le \varphi$ on $\mathbf{M}^N$, since $\tilde{u} \le u$, it follows that $\tilde{u}$ is also a viscosity subsolution of the Dirichlet problem.

Next we have to show that the $\mathbb{F}^N$-inequality holds for $\tilde{u}$. We may assume without loss of generality that $\tilde{u}$ is Lipschitz continuous because we can perform the regularization in the tangent variables (including t), and then use the normal controllability property. In the same way, we can assume that the boundary is flat and use the definition of H^N not only when $x \in \partial\Omega$ but also for $x \in \Omega$. We notice that $H^N(x,t,p) \le H(x,t,p)$ if $x \in \partial\Omega$ since the supremum is taken on a smaller set than **BCL** and, if n is the unit outward normal to $\partial\Omega$ (which is flat), $H^N(x-\varepsilon n,t,p) \to H^N(x,t,p)$ when $\varepsilon \to 0$ as a consequence on the $(\mathbf{H_{BA-CP}})$-assumptions and of the normal controllability. Hence $H^N(x-\varepsilon n,t,p) \le H(x,t,p) + o_\varepsilon(1)$ where the $o_\varepsilon(1)$ is uniform for bounded p.

As in Proposition 21.2.3, it is clear that $\tilde{u}^\varepsilon(x,t) := \tilde{u}(x+\varepsilon n,t)$ is a subsolution of

$$\tilde{u}^\varepsilon_t + H^N(x+\varepsilon n,t,D_x\tilde{u}^\varepsilon) \le o_\varepsilon(1) \quad \text{on } \mathbf{M}^N\ ,$$

and passing to the limit by a standard stability result (since $\tilde{u}^\varepsilon$ converges to $\tilde{u}$ uniformly and since the $o_\varepsilon(1)$ is uniform for bounded p), we obtain (27.5).

Q.E.D.

Remark 27.1.2 *Two remarks can be made on the above proof.*

(i) The first one concerns the inequality $u(x,t) \leq \varphi(x,t)$. In fact, even if φ is discontinuous, the inequality $u(x,t) \leq \varphi^*(x,t)$ (with the u.s.c. enveloppe of φ) can be proved not only for points in $\mathbf{M}^N$ but for any point where the exterior sphere condition holds, i.e. there exists $\bar{x} \in \mathbb{R}^N, \bar{r} > 0$ such that

$$\overline{B(\bar{x},\bar{r})} \cap \overline{\Omega} = \{x\} \ .$$

Indeed, it is enough to reproduce the above proof replacing the function $d(y)$ by $\chi(y) := |y - \bar{x}| - \bar{r}$. Indeed, if x is a minimum point of χ on $\partial\Omega$ and therefore, if $(x,t) \in \mathbf{M}^k$, $(D\chi(x), 0)$ is orthogonal to $T_{(x,t)}\mathbf{M}^k$, allowing to use **(NC)**. *This inequality is therefore a general fact, but unfortunately not convenient for the stratification formulation which requires the more restrictive inequality $u(x,t) \leq \varphi_*(x,t)$.*

(ii) On the other hand, in order to obtain the H^N-inequality, we use very few properties, namely the characterization of the stratification (the fact that we have a parallel hyperplane to $\mathbf{M}^N$ inside $\mathbf{M}^{N+1}$) and the inequality $H^N \leq H$, both arguments being true for any $\mathbf{M}^k$.

In order to go further we introduce the

Definition 27.1.3 *Assume that $\overline{\Omega} \times (0,T_f)$ is a stratified domain and $\varphi : \partial\Omega \times (0,T_f) \to \mathbb{R}$ is a lower-semicontinuous function. We say that φ is adapted to the stratification if for any $1 \leq k \leq N$, $\varphi|_{\mathbf{M}^k}$ is continuous. Moreover, φ is said to be W-adapted ("well-adapted") to the stratification if, in addition, for any $x \in \mathbf{M}^k$ and for any k*

$$\varphi(x,t) = \liminf_{\substack{(y,s)\to(x,t)\\(y,s)\in\mathbf{M}^N}} \varphi(y,s) \ .$$

The result for W-adapted boundary conditions is the following

Proposition 27.1.4 *Assume that the above assumption on $\mathbb{M}$ holds and that we are in the good framework for stratified problems. If $\varphi : \partial\Omega \times [0,T_f) \to \mathbb{R}$ is a lower-semicontinuous function which is W-adapted to the stratification and if u is*

viscosity subsolution of the Dirichlet problem, then $\tilde{u} : \overline{\Omega} \times [0, T_f] \to \mathbb{R}$ *defined by* $\tilde{u}(x,t) = u(x,t)$ *if* $x \in \Omega$ *and*

$$\tilde{u}(x,t) = \limsup_{\substack{(y,s)\to(x,t)\\ y\in\Omega}} u(y,s) \quad \text{if } x \in \partial\Omega \,,$$

is a stratified subsolution of the problem.
If, in addition, we assume that

(*i*) $u_0 \in C(\overline{\Omega})$ *and* $u_0(x) \le \varphi_*(x,0)$ *on* $\partial\Omega$,

(*ii*) *For any* $x \in \partial\Omega$, *there exists a* C^1*-function* ϕ *defined in a neighborhood* V *of* x *such that* $\phi(y) = 0$ *if* $y \in \partial\Omega \cap V$, $\phi(y) > 0$ *if* $y \in \Omega \cap V$, *and*

$$\sup_{\mathbf{BCL}(y,s)} \{b^x \cdot D_x\phi(y)\} \ge 0 \,,$$

for any $y \in \Omega \cap V$ *and for small* s,

then, for any viscosity supersolution of the Dirichlet problem, we have

$$\tilde{u} \le v \quad \text{on } \overline{\Omega} \times [0, T_f) \,.$$

In particular, there exists a unique continuous viscosity solution of the Dirichlet problem (up to a modification of its values on the boundary).

The first part of this proposition says that, under suitable "standard" assumptions and modification of the subsolution on the boundary, then Ishii viscosity subsolutions and stratified subsolution are the same. For a complete application of this first result, one needs to treat the initial data and, as it will be clear in the proof, Conditions (*i*) and (*ii*) imply that

$$\tilde{u}(x,0) \le u_0(x) \le v(x,0) \quad \text{on } \overline{\Omega}\,.$$

Notice that this double inequality prevents maximum points of $u - v$ to be achieved on $\partial\Omega \times \{0\}$ if this maximum is assumed to be strictly positive.

Example 27.1 — A standard example where Proposition 27.1.4 can be applied is the square $[0,1] \times [0,1]$ in $\mathbb{R}^2$, with

$$\varphi(x) = \varphi_i(x,t) \quad \text{on } S_i,$$

where $S_1 =]0,1[\times\{0\}$, $S_2 = \{1\}\times]0,1[$, $S_3 =]0,1[\times\{1\}$ $S_4 = \{0\}\times]0,1[$, each φ_i being continuous on S_i. Of course, in order to have a function φ which is W-adapted to the

stratification, the values at the four corners are imposed by the values on each S_i and obtained by computing their lower semi-continuous extensions. For example, at $(0,0)$ we have $\min(\varphi_1(0,t),\varphi_4(0,t))$. We point out that φ is still adapted if the values at the four corners are below these values.

If H satisfies all the controllability conditions, then the first part Proposition 27.1.4 applies.

For the second one, the compatibility condition on $\partial\Omega\times\{0\}$ should hold and for ϕ, we can choose the distance to the boundary if x is not located on one of the corners. In case of a corner, say $(0,0)$, we may choose, noting $x=(x_1,x_2)$, the function $\phi(x)=x_1x_2$, while for $(0,1)$, we may choose $\phi(x)=x_1(1-x_2)$, i.e. in each case the product of the distances to the adjacent sides. The controllability condition ensures that the requirement on $D_x\phi$ is satisfied.

Proof of Proposition 27.1.4 — The first part of the result is easy: by Proposition 27.1.1, $\tilde{u}$ is a stratified solution on $\mathbf{M}^N$ and it remains to show that it is true on any $\mathbf{M}^k$.

If $(x,t)\in\mathbf{M}^k$, using a tangential regularization in a neighborhood of (x,t), we obtain a Lipschitz continuous function which is below φ on each connected component of $\mathbf{M}^N$ and therefore $\tilde{u}(x,t)\le\varphi(x,t)$, since (as in the above example) the lower semicontinuous enveloppe of φ can just be computed using points of $\mathbf{M}^N$.

The $\mathbb{F}^k$-inequality of $\mathbf{M}^k$ can be obtained exactly as for the $\mathbb{F}^N$-one in the proof of Proposition 27.1.1 (cf. the second part of Remark 27.1.2): we can also here assume without loss of generality that $\mathbf{M}^k$ is flat and consider inequalities on $\mathbf{M}^k+\varepsilon e\subset\mathbf{M}^{N+1}$, where e is a suitable vector, normal to $\mathbf{M}^k$.

For the comparison, the only additional difficulty is $t=0$ and more precisely the points of $\partial\Omega\times\{0\}$ where we have to show that $\tilde{u}\le u_0$ and $v\ge u_0$.

The proof for v is easy since the viscosity supersolution inequality reads $v\ge\max(u_0,\varphi_*)\ge u_0$ on $\partial\Omega\times\{0\}$.

But for the subsolution case, we only have $\tilde{u}\le\max(u_0,\varphi^*)$ on $\partial\Omega\times\{0\}$ and this is not sufficient. To turn around this difficulty and to show that the right inequality holds at $(x,0)$, $x\in\partial\Omega$, we introduce the function

$$(y,s)\mapsto\tilde{u}(y,s)-\frac{s}{\varepsilon}-\frac{|y-x|^2}{\varepsilon}-\frac{\alpha}{\phi(y)}\,,$$

where $0<\alpha\ll\varepsilon\ll 1$ are parameters devoted to tend to 0 and ϕ is the function associated to x in Assumption (ii).

By classical arguments, this function has a local maximum point $(y_\varepsilon, s_\varepsilon)$ in a neighbordhood of $(x,0)$ and $(y_\varepsilon, s_\varepsilon) \to (x,0)$ with $\tilde{u}(y_\varepsilon, s_\varepsilon) \to \tilde{u}(x,0)$ at least if $\alpha, \varepsilon \to 0$ with $\alpha \ll \varepsilon$ [(1)].

Because of the ϕ-term, $y_\varepsilon \in \Omega$. If $s_\varepsilon > 0$, the H-inequality holds and we have

$$\frac{1}{\varepsilon} + H\left(y_\varepsilon, s_\varepsilon, p_\varepsilon - \frac{\alpha D_x\phi(y_\varepsilon)}{[\phi(y_\varepsilon)]^2}\right) \leq 0 \ ,$$

where $p_\varepsilon = \dfrac{2(y_\varepsilon - x)}{\varepsilon} = \dfrac{o(1)}{\varepsilon}$. Examining the H-term, it can be estimated by

$$\frac{1}{\varepsilon} - M(\frac{o(1)}{\varepsilon} + 1) + \sup_{\mathbf{BCL}(y_\varepsilon, s_\varepsilon)} \left\{ b^x \cdot \frac{\alpha D_x\phi(y_\varepsilon)}{[\phi(y_\varepsilon)]^2} \right\} \leq 0 \ ,$$

where M takes into account the Lipschitz constant of $H(x,t,p)$ in p (coming from boundedness of b) and the boundedness of l.

By the assumption on ϕ the supremum is positive and therefore this inequality cannot hold for ε small enough. Therefore we necessarily have $s_\varepsilon = 0$ and $\tilde{u}(y_\varepsilon, s_\varepsilon) \leq u_0(y_\varepsilon)$. And letting $\alpha, \varepsilon \to 0$ with $\alpha \ll \varepsilon$, we obtain $\tilde{u}(x,0) \leq u_0(x)$.

These inequalities at time $t = 0$ being proved, we have just to apply the comparison result for the stratified problem and the proof is complete.

Q.E.D.

Remark 27.1.5 *We are not going to push further away the question of the existence of the functions ϕ which play the role of a distance function and which are used in a key way to obtain the desired property at time 0. But we think that the existence of such functions is not a problem in general, even if it might be difficult to provide a very general result regarding such existence.*

A convincing example is the case when $\overline{\Omega}$ is a convex set given by

$$\overline{\Omega} := \bigcap_i \{x : \ p_i \cdot x \geq q_i\} \ ,$$

where the p_i are in $\mathbb{R}^N$ and the q_i in $\mathbb{R}$. The example of the square above can be generalized in the following way: if $x \in \partial\Omega$ and if $I(x)$ is the set of indices i for which $p_i \cdot x = q_i$, then one can choose

$$\phi(y) := \prod_{i \in I(x)} (p_i \cdot x - q_i) \ .$$

[(1)] By the definition of $\tilde{u}$, the values of $\tilde{u}$ on the boundary are the limits of the values of $\tilde{u}$ in $\Omega \times (0, T_f)$ and for α small enough, we keep track of the boundary values of $\tilde{u}$

It is easy to check that the condition on $D_x\phi$ is satisfied as an easy consequence of the normal controllability since the p_i's are clearly orthogonal to the space of $\mathbf{M}^k$ at x is in $\mathbf{M}^k$.

This first part where we describe a general framework for which the stratified formulation and the classical viscosity solutions' one are (in some sense) equivalent, also suggests the cases when the stratified formulation is unavoidable: if φ is a l.s.c. function which is adapted but not W-adapted to the stratification, i.e. if, on some connected component of some $\mathbf{M}^k$, we have

$$\varphi(x,t) < \liminf_{\substack{(y,s)\to(x,t)\\(y,s)\in\mathbf{M}^N}} \varphi(y,s) \ ,$$

for some $(x,t) \in \mathbf{M}^k$, there is no way that a subsolution (even after "cleaning" it) should satisfy $u \le \varphi$ on $\mathbf{M}^k$. This property has to be superimposed through the stratification formulation since the Ishii one (using φ^*) will simply erase the small values of φ.

In this case we have the

Proposition 27.1.6 *Assume that the above assumption on $\mathbb{M}$ holds and that we are in the good framework for stratified problem. Let $\varphi : \partial\Omega \times [0,T_f) \to \mathbb{R}$ be a lower-semicontinuous function which is adapted to the stratification.*

If u is an u.s.c. viscosity subsolution of the Dirichlet problem such that

$$u(x,t) \le \varphi(x,t) \quad \text{for any } (x,t) \in \mathbf{M}^{N-1} \cup \cdots \cup \mathbf{M}^1 \ , \tag{27.6}$$

then $\tilde{u} : \overline{\Omega} \times [0,T_f) \to \mathbb{R}$ defined by $\tilde{u}(x,t) = u(x,t)$ if $x \in \Omega$ and

$$\tilde{u}(x,t) = \limsup_{\substack{(y,s)\to(x,t)\\y\in\Omega}} u(y,s) \quad \text{if } x \in \partial\Omega \ ,$$

is a stratified subsolution of the problem.
If, in addition, we have

(*i*) *$u_0 \in C(\overline{\Omega})$ and $u_0(x) \le \varphi_*(x,0)$ on $\partial\Omega$,*

(*ii*) *For any $x \in \partial\Omega$, there exists a C^1-function ϕ defined in a neighborhood V of x such that $\phi(y) = 0$ if $y \in \partial\Omega \cap V$, $\phi(y) > 0$ if $y \in \Omega \cap V$, and*

$$\sup_{\mathbf{BCL}(y,s)} \{b^x \cdot D_x\phi(y)\} \ge 0 \ ,$$

for any $y \in \Omega \cap V$ and for small s,

then, for any viscosity supersolution of the Dirichlet problem, we have

$$\tilde{u} \leq v \quad \text{on } \overline{\Omega} \times [0, T_f) \ .$$

In particular, there exists a unique continuous viscosity solution of the Dirichlet problem which satisfies (27.6).

As we already explain it above, the key difference between Propositions 27.1.4 and 27.1.6 is that the first one applies to all Ishii viscosity solutions while, in the second case, Condition 27.6 has to be imposed.

The *Proof of Proposition 27.1.6* follows the ideas of the proof of Proposition 27.1.4, namely

(i) For any k, the condition on $\mathbf{M}^k$, i.e.

$$\max(u_t + H^k(x, t, D_x u), u - \varphi(x, t)) \leq 0 \quad \text{on } \mathbf{M}^k \ ,$$

where

$$H^k(x, t, p_x) = \sup_{b \in T_x \mathbf{M}^k} \{-b^x \cdot p_x - l\} \ ,$$

is obtained by combining (27.6) with an approximation "from inside", following Remark 27.1.2.

(ii) The comparison result follows from the stratified formulation, while the existence is provided by the value-function of the associated control problem.

We conclude by an example showing the interest of the stratified formulation and related to Proposition 27.1.6.

Example 27.2 — We come back to an example in the square $S = [0, 1] \times [0, 1] \subset \mathbb{R}^2$. The equation is

$$u_t + |Du| = 1 \quad \text{in } S \times (0, 1) \ ,$$

with the initial data $u(x, 0) = 0$ in S and the Dirichlet boundary condition

$$\varphi(x) = 1 \quad \text{on } \partial S \setminus \{0\} \text{ and } \varphi(0) = 0 \ .$$

Since $\varphi^*(x) \equiv 1$ on ∂S, it is easy to check that $u_1(x, t) = t$ is a classical viscosity solution of this problem for $0 \leq t \leq 1$. This first solution completely ignores the fact that $\varphi(0) = 0$.

On the other hand, we can consider $u_2(x, t) = \min(t, |x|)$. It is also easy to check that u_2 is an other solution of our problem but it also satisfies $u_2(0, t) \leq 0$, i.e. Condition (27.6). On this example, one can verify that Condition (27.6) is nothing

but the main missing stratified inequality on $\mathbf{M}^1$, the other ones on $\mathbf{M}^2$ being also satisfied. We also point out that, at time $t = 0$, it is important to have the stratified inequality $u \leq \min(u_0, \varphi)$ on ∂S to recover the correct initial data, solving the $\mathbb{F}_{init}$ equation.

27.2 On the Neumann problem

27.2.1 The case of continuous data

As for the Dirichlet problem, we begin with the most standard framework: an oblique derivative problem in a smooth enough domain, namely (27.1) associated with the boundary condition

$$\frac{\partial u}{\partial \gamma} = g(x,t) \quad \text{on } \partial\Omega \times (0,T_f) \ , \tag{27.7}$$

where γ and g are, at least, continuous functions and γ satisfies

$$\gamma(x,t) \cdot n(x) \geq \nu > 0 \ ,$$

for some $\nu > 0$ and for any $(x,t) \in \partial\Omega \times (0,T_f)$, where $n(x)$ is the unit outward normal to $\partial\Omega$ at x. As in Section 27.1, H is given by (27.2) with b, l satisfying $(\mathbf{H_{BA-CP}})$ and therefore H satisfies $(\mathbf{H_{BA-HJ}})$. We assume also that the normal controllability assumption (27.4) holds. Because of these hypothesis, we have $\mathbf{M}^{N+1} = \Omega \times (0,T_f)$ and $\mathbf{M}^N = \partial\Omega \times (0,T_f)$.

The first key difference with the Dirichlet problem is that viscosity subsolutions are regular at the boundary and therefore we do not need to redefine them on the boundary. More precisely

Proposition 27.2.1 *Under the above assumptions on H, γ and g and if Ω is a C^1-domain, subsolutions of (27.1)-(27.7) are regular at the boundary for $t > 0$.*

Proof — Let u be an u.s.c. subsolution of (27.1)-(27.7) and $(x,t) \in \partial\Omega \times (0,T_f)$. If u is not regular at (x,t) this means that

$$u(x,t) > \limsup_{\substack{(y,s)\to(x,t) \\ (y,s)\in\mathbf{M}^{N+1}}} u(y,s) \ .$$

We consider, for $0 < \varepsilon \ll 1$, the function defined on $\mathbf{M}^N$ by

$$(y,s) \mapsto u(y,s) - \frac{(s-t)^2}{\varepsilon^2} - \frac{|y-x|^2}{\varepsilon^2} \ .$$

This function has a local maximum point at $(y_\varepsilon, s_\varepsilon)$ near (x,t) and $u(y_\varepsilon, s_\varepsilon) \to u(x,t)$ as $\varepsilon \to 0$. But the jump of u on the boundary implies that necessarely $(y_\varepsilon, s_\varepsilon) \in \partial\Omega \times (0, T_f)$ and$(y_\varepsilon, s_\varepsilon)$ is also a local maximum point of the function defined on $\overline{\Omega} \times (0, T_f)$

$$(y,s) \mapsto u(y,s) - \frac{(s-t)^2}{\varepsilon^2} - \frac{|y-x|^2}{\varepsilon^2} - \lambda d(y) \ ,$$

FOR ANY $\lambda \in \mathbb{R}$, where d denotes the distance function to the boundary $\partial\Omega$ which is C^1 in a neighborhood of $\partial\Omega$ by the assumption on the regularity of Ω. Hence we have

$$\min(a_\varepsilon + H(y_\varepsilon, s_\varepsilon, p_\varepsilon - \lambda n(y_\varepsilon)), (p_\varepsilon - \lambda n(y_\varepsilon)) \cdot \gamma(y_\varepsilon) - g(y_\varepsilon, s_\varepsilon)) \leq 0 \ ,$$

where

$$a_\varepsilon := \frac{2(s_\varepsilon - t)}{\varepsilon^2} \quad \text{and} \quad p_\varepsilon := \frac{2(y_\varepsilon - x)}{\varepsilon^2} \ .$$

Of course, for $\lambda < 0$ large enough, we have a contradiction because of the normal controllability assumption on H and the assumption on γ.

Q.E.D.

In this simple case, it remains to identify the $\mathbb{F}^N$-inequality on $\mathbf{M}^N$ and to show the equivalence between Ishii viscosity (sub)solutions and stratified (sub)solutions. As we did for the Dirichlet case, we enlarge the set **BCL** on the boundary to take into account the boundary condition. Here, the enlargement consists in adding triplets of the form $((-\gamma(x,t),0),0,g(x,t))$, assigning the cost $g(x,t)$ to a reflection-type boundary dynamic $-\gamma(x,t)$ on $\partial\Omega$.

The result is the following

Proposition 27.2.2 *Assume that $\partial\Omega$ is a C^1-domain and that the above assumptions on H, γ and g hold. Then if u is a viscosity subsolution of the oblique derivative problem, it is a stratified subsolution of the problem with*

$$\mathbb{F}^N(x,t,(p_x,p_t)) = \sup\{\theta p_t - (\theta b^x - (1-\theta)\gamma) \cdot p_x - (\theta l + (1-\theta)g)\} \ \textit{on} \ \mathbf{M}^N \ ,$$

where the supremum is taken on all $(b,0,l) \in \mathbf{BCL}(x,t)$ such that there exists $\theta \in (0,1)$ such that $(\theta b^x - (1-\theta)\gamma) \cdot n(x) = 0$, where $n(x)$ is the unit outward normal to $\partial\Omega$ at x.

In Proposition 27.2.2, we have used lighter notations but it is clear that $b^x = b(x,t,\alpha)$ for some $\alpha \in A$, $l = l(x,t,\alpha)$ and $b = (b^x, -1)$.

Proof — We have to show that, if ϕ is a smooth function and if $(x,t) \in \mathbf{M}^N$ is a strict local maximum point of $u - \phi$ then

$$\theta\phi_t(x,t) - (\theta b^x - (1-\theta)\gamma) \cdot D_x\phi(x,t) - (\theta l + (1-\theta)g) \ \leq 0 \ ,$$

for any b, l, θ satisfying the conditions of Proposition 27.2.2.

To do so, we introduce $\lambda \in \mathbb{R}$ which is the unique solution of the equation

$$\gamma(x,t) \cdot (D_x\phi(x,t) - \lambda n(x)) = g(x,t) \, , \tag{27.8}$$

which is well-defined since $\gamma(x,t) \cdot n(x) \neq 0$. Then, we consider the function

$$(y,s) \mapsto u(y,s) - \phi(y,s) - (\lambda - \delta)d(y) - \frac{[d(y)]^2}{\varepsilon^2} \ ,$$

for $0 < \varepsilon \ll 1$ and for some small $\delta > 0$. We recall that, as above, d denotes the distance function to the boundary $\partial\Omega$ and we recall that $D_x d(x) = -n(x)$ on $\partial\Omega$; we will use the notation $n(x)$ for $-D_x d(x)$ even if x is not on the boundary.

We first fix δ. If ε is small enough, this function has a local maximum point at $(x_\varepsilon, t_\varepsilon)$ and $(x_\varepsilon, t_\varepsilon) \to (x,t)$ as $\varepsilon \to 0$ by the maximum point property of (x,t).

If $(x_\varepsilon, t_\varepsilon) \in \partial\Omega \times (0, T_f)$, we claim that the γ-inequality cannot hold. Indeed otherwise

$$\gamma(x_\varepsilon, t_\varepsilon) \cdot (D_x\phi(x_\varepsilon, t_\varepsilon) - (\lambda - \delta)n(x_\varepsilon)) \leq g(x_\varepsilon, t_\varepsilon) \ ,$$

but this inequality cannot be valid for ε small enough because of the definition of λ and the facts that $\delta > 0$ and $\gamma(x_\varepsilon, t_\varepsilon) \cdot n(x_\varepsilon) \geq \nu > 0$. Hence the H-inequality holds.

For any $(b, 0, l) \in \mathbf{BCL}(x,t)$, we have $b^x = b(x,t,\alpha)$ and $l = l(x,t,\alpha)$ and choosing $(b_\varepsilon, 0, l_\varepsilon) \in \mathbf{BCL}(x_\varepsilon, t_\varepsilon)$ such that $(b^x_\varepsilon, 0, l_\varepsilon) = ((b(x_\varepsilon, t_\varepsilon, \alpha), -1), 0, l(x_\varepsilon, t_\varepsilon, \alpha)$, we have, as a particular case of the H-inequality

$$\phi_t(x_\varepsilon, t_\varepsilon) - b^x_\varepsilon \cdot [D_x\phi(x_\varepsilon, t_\varepsilon) - (\lambda - \delta)n(x_\varepsilon) - \frac{2d(x_\varepsilon)}{\varepsilon^2} n(x_\varepsilon)] - l_\varepsilon \ \leq 0 \ .$$

If we take $(b, 0, l) \in \mathbf{BCL}(x,t)$ and θ such that the property $(\theta b^x - (1-\theta)\gamma) \cdot n(x) = 0$ holds, we first deduce that $\theta > 0$ since $\gamma(x,t) \cdot n(x) > 0$ and then that $b^x \cdot n(x) > 0$. Therefore, if ε is small enough, $b^x_\varepsilon \cdot n(x_\varepsilon) > 0$ and we can drop the term $\dfrac{2d(x_\varepsilon)}{\varepsilon^2} b^x_\varepsilon \cdot n(x_\varepsilon)$ in the above inequality. Letting $\varepsilon \to 0$, this yields

$$\phi_t(x,t) - b^x \cdot [D_x\phi(x,t) - (\lambda - \delta)n(x)] - l \ \leq 0 \ .$$

Next we let $\delta \to 0$ in this inequality and the conclusion follows by using the θ-convex combination of this inequality with (27.8).

Q.E.D.

Several remarks after this result.

(*i*) It is clear enough from the proof that the case of *sliding boundary conditions*, i.e.

$$u_t + \frac{\partial u}{\partial \gamma} = g(x,t) \quad \text{on } \partial\Omega \times (0,T_f) \ ,$$

can be treated exactly in the same way.

(*ii*) Less obviously (but this is still easy), the case where there is a control on the reflection

$$\sup_\beta \{\gamma_\beta \cdot D_x u - g_\beta\} = 0 \quad \text{on } \partial\Omega \times (0,T_f) \ ,$$

where the set of (γ_β, g_β) is convex and continuous in (x,t), can also be treated[(2)]. It is easy to check that one has just to repeat the above arguments for (b,l) and (γ_β, g_β) such that $(\theta b^x - (1-\theta)\gamma_\beta) \cdot n(x) = 0$.

(*iii*) But, as it may be expected, the stratified formulation does not bring new results as long as all data are continuous...

Remark 27.2.3 *For a Neumann boundary condition of the form*

$$\frac{\partial u}{\partial n} = g(x,t) \quad \textit{on } \partial\Omega \times (0,T_f) \ ,$$

we have

$$\mathbb{F}^N(x,t,(p_x,p_t)) = \sup\{\theta p_t - (\theta b^x - (1-\theta)n(x)) \cdot p_x - (\theta l + (1-\theta)g)\} \ \textit{on } \mathbf{M}^N \ ,$$

where the supremum is taken on all $(b,0,l) \in \mathbf{BCL}(x,t)$ *such that there exists* $\theta \in (0,1)$ *such that* $(\theta b^x - (1-\theta)n(x)) \cdot n(x) = 0$.

This means that we have

$$\theta b^x \cdot n(x) - (1-\theta) = 0 \ .$$

and therefore $b^x \cdot n(x) \geq 0$ *and* $\theta = (1 + b^x \cdot n(x))^{-1}$. *If* $b^x = b^{x,\perp} + b^{x,\top}$ *where* $b^{x,\perp}$ *is projection of* b^x *on the normal direction and* $b^{x,\top}$ *on the tangent space of* $\partial\Omega$ *at* x, *we have to look at the supremum of* $\theta(p_t - b^{x,\top} \cdot p_x - (l + b \cdot ng(x,t)))$ *for* $b^x \cdot n \geq 0$ *since* $(1-\theta) = \theta b^x \cdot n(x)$. *But* θ *cannot vanish and the condition reduces to*

$$u_t + \sup_{b^x \cdot n(x) \geq 0} \left(b^{x,\top} \cdot D_x u - (l + b \cdot ng(x,t))\right) \leq 0 \ .$$

[(2)] In general this set is not convex but we can take a convex enveloppe and this does not change the "sup" in the boundary condition.

When looking at the reflected trajectory for a control problem, one has to solve an ode like

$$\dot{X}(s) = b^x(s) - 1_{\{X(s)\in\partial\Omega\}} n(X(s)).d|k|_s \ ,$$

where $|k|_s$ *is the process with bounded variation which keeps the trajectory inside* $\overline{\Omega}$ *and the associated cost is*

$$\int_0^t l(s)ds + \int_0^t g(X(s),s)d|k|_s \ .$$

It is easy to see that $d|k|_s = 1_{\{X(s)\in\partial\Omega\}} b(s).n(X(s))ds$ *if* $b(s)^x \cdot n(X(s)) \geq 0$ *and the cost becomes*

$$\int_0^t (l(s) + 1_{\{X(s)\in\partial\Omega\}} b^x(s).n(X(s)))ds \ ,$$

which is exactly what the stratified formulation is seeing on the boundary.

27.2.2 Oblique derivative problems in stratified domains

In [74], Dupuis and Ishii prove comparison results for oblique derivative problems in non-smooth domains: roughly speaking, they treat the case of a smooth enough direction of reflection in domains which satisfy only an interior-exterior cone condition. Our aim is to provide here a stratified version of their result.

In the following result, we assume that $\overline{\Omega} \times (0,T_f)$ is a stratified domain with a t-independent stratification such that

$$\mathbf{M}^{N+1} = \Omega \times (0,T_f) \quad \text{and} \quad \mathbf{M}^k \subset \partial\Omega \times (0,T_f) \text{ for } 1,2,\cdots,N.$$

We assume that $\gamma = \gamma(x) \in C^1(\mathbb{R}^N)$ and that, for any $x \in \partial\Omega$, there exists $\bar{\delta}, \eta > 0$ such that, for any $0 < \delta \leq \bar{\delta}$

$$B(x + \delta\gamma(x), \eta\delta) \subset \Omega^c \quad \text{and} \quad B(x - \delta\gamma(x), \eta\delta) \subset \Omega \ .$$

It is worth pointing out that this condition on γ provides a regularity property on Ω since it implies an exterior-interior cone condition.

Proposition 27.2.4 *Under the above assumptions and if* g *is a continuous function, a classical viscosity subsolution of* (27.1)-(27.7) *is also a stratified subsolution of* (27.1)-(27.7)*. Therefore we have a comparison result for classical viscosity sub and supersolutions of* (27.1)-(27.7)*.*

Proof — Let u be an Ishii subsolution of (27.1)-(27.7). We have to prove that if $(x,t) \in \mathbf{M}^k$, for any smooth function, if $u-\phi$ has a strict local maximum point at $(\bar{x},\bar{t})$ on $\mathbf{M}^k$ then

$$\theta\phi_t(\bar{x},\bar{t}) - (\theta b^x - (1-\theta)\gamma(\bar{x})) \cdot D_x\phi(\bar{x},\bar{t}) \leq (\theta l + (1-\theta)g(\bar{x},\bar{t}))\ ,$$

for any $((b^x,-1),0,l) \in \mathbf{BCL}(\bar{x},\bar{t})$ and $0 \leq \theta \leq 1$ such that $(\theta b^x - (1-\theta)\gamma(\bar{x})) \in T_{(\bar{x},\bar{t})}\mathbf{M}^k$.

Because of our assumptions on the stratification, we can assume that $\mathbf{M}^k$ is an affine space but taking also into account the fact that $(b^x,l) = (b(\bar{x},\bar{t},\bar{\alpha}), l(\bar{x},\bar{t},\bar{\alpha}))$ and that we can use $(b(x,t,\bar{\alpha}), l(x,t,\bar{\alpha}))$ for nearby points, it is easy to see the coordinates on $\mathbf{M}^k$ are not playing a key role, nor the term $(\theta b^x - (1-\theta)\gamma(\bar{x})) \cdot D_x\phi(\bar{x},\bar{t})$.

In order to simplify the proof and to emphasize the main ideas, we just examine the case when $k=1$, $\bar{x}=0$, $\gamma(0,\bar{t}) = -e_N$ and $\mathbf{M}^k = \mathbf{M}^1$ is $\{0\} \times (0,T_f)$. With this reduction, the inequality we want to obtain is

$$\theta\phi_t(\bar{t}) \leq (\theta l + (1-\theta)g(0,\bar{t}))\ ,$$

for any smooth function ϕ of t such that $u(0,t)-\phi(t)$ has a strict local maximum point at $\bar{t}$, and for any $((b^x,-1),0,l) \in \mathbf{BCL}(0,\bar{t})$, $0 \leq \theta \leq 1$ such that $(\theta b^x - (1-\theta)\gamma(\bar{x})) = 0$.

We borrow the arguments Dupuis and Ishii [74] for proving this inequality. By the conditions on γ and denoting $x = (x', x_N)$ with $x' \in \mathbb{R}^{N-1}$, we have

$$\{(x',x_N):\ x_N > 0,\ |x'| \leq \eta x_N\} \subset \Omega \quad \text{and} \{(x',x_N):\ x_N < 0,\ |x'| \leq -\eta x_N\} \subset \Omega^c\ .$$

We consider the function

$$u(x,t) - \phi(t) - \frac{1}{\varepsilon}\psi_\varepsilon(x',x_N) + (g(0,\bar{t}) - \delta)x_N\ ,$$

for $0 < \varepsilon \ll 1$, where $x = (x', x_N)$ and

$$\psi_\varepsilon(x',x_N) = \sqrt{[(\eta^2 x_N^2 - |x'|^2)^+]^2 + \varepsilon^4} + |x'|^2\ .$$

This function is an ad hoc regularized version of $[\max(|x'|, \eta x_N)]^2$. Since, for all x, $\psi_\varepsilon(x',x_N) \geq \nu|x|^2 + \varepsilon^2$ for ν small enough, the above function has a local maximum point at $(x_\varepsilon, t_\varepsilon)$ near $(0,\bar{t})$ and $(x_\varepsilon,t_\varepsilon) \to (0,\bar{t})$ as $\varepsilon \to 0$.

Now we examine the different possibilities for the subsolution inequality: if $x_\varepsilon \in \partial\Omega$ then, the condition on γ implies that $|(x_\varepsilon)'| > \eta|(x_\varepsilon)_N|$ and therefore the space-derivative of the test-function is nothing but $2(x_\varepsilon)'/\varepsilon - (g(0,\bar{t}) - \delta)e_N$. Hence, if the

oblique derivative inequality were satisfied, we would get

$$\left(\frac{2(x_\varepsilon)'}{\varepsilon} - (g(0,\bar t) - \delta)e_N\right)\cdot\gamma(x_\varepsilon) \le g(x_\varepsilon, t_\varepsilon)\ .$$

But, using the Lipschitz continuity of γ, the continuity of g and $\gamma(0) = e_N$, we have

$$\begin{aligned}\frac{2(x_\varepsilon)'}{\varepsilon}\cdot\gamma(x_\varepsilon) &= \frac{2(x_\varepsilon)'}{\varepsilon}\cdot(\gamma(x_\varepsilon) - \gamma(0)) + \frac{2(x_\varepsilon)'}{\varepsilon}\cdot\gamma(0)\\ &= \frac{2(x_\varepsilon)'}{\varepsilon}\cdot(\gamma(x_\varepsilon) - \gamma(0))\\ &= O\left(\frac{|x_\varepsilon|^2}{\varepsilon}\right) = o(1)\ ,\end{aligned}$$

while

$$-(g(0,\bar t) - \delta)e_N\cdot\gamma(x_\varepsilon, t_\varepsilon) = g(x_\varepsilon, t_\varepsilon) + \delta + o(1)\ .$$

Therefore such γ-inequality cannot hold and necessarily the H-one does. In particular, we see that, for $(b_\varepsilon, l_\varepsilon) = (b(x_\varepsilon, t_\varepsilon, \bar\alpha), l(x_\varepsilon, t_\varepsilon, \bar\alpha))$

$$\phi'(t_\varepsilon) - \frac{1}{\varepsilon}D_x\psi_\varepsilon\cdot b^x_\varepsilon + (g(0,\bar t) - \delta)b^x_\varepsilon\cdot e_N \le l_\varepsilon\ .$$

Assume for the moment that

$$-\frac{1}{\varepsilon}D_x\psi_\varepsilon\cdot b^x_\varepsilon \ge o(1)\ ,$$

letting $\varepsilon\to 0$, we obtain

$$\phi'(\bar t) + (g(0,\bar t) - \delta)b^x\cdot e_N \le l\ ,$$

which is the desired inequality (up to letting $\delta\to 0$) since $\theta b^x = -(1-\theta)e_N$: in order to conclude, it is enough to multiply the above inequality by θ[(3)].

It remains to prove the claim. For the $|x'|^2$-term in ψ_ε, the proof is already done above while checking that the oblique derivative inequality cannot hold: we get a $o(1)$ as $\varepsilon\to 0$.

It remains to treat the other term (with the root) when $\eta^2 x_N^2 > |x'|^2$. Notice that the sign of $-D_x\psi_\varepsilon\cdot b^x_\varepsilon$ is the same as the one of $-(2\eta^2 x_N e_N - 2x')\cdot b^x_\varepsilon$ and therefore the same as for $(2\eta^2 x_N e_N - 2x')\cdot e_N$. Now, on this last formula, it appears clearly that this sign is positive, which concludes the proof.

Q.E.D.

[(3)] We point out that θ is bounded away from 0 since $\theta b^x\cdot e_N = -(1-\theta)$ and therefore $\theta\neq 0$, $b^x\cdot e_N\le 0$ and $\theta = (1 - b^x\cdot e_N)^{-1}$.

27.2.3 Discontinuities in the direction of reflection and domains with corners: the $\mathbb{R}^2$-case.

We start by a standard evolution problem in 2-d described by Fig. 27.1

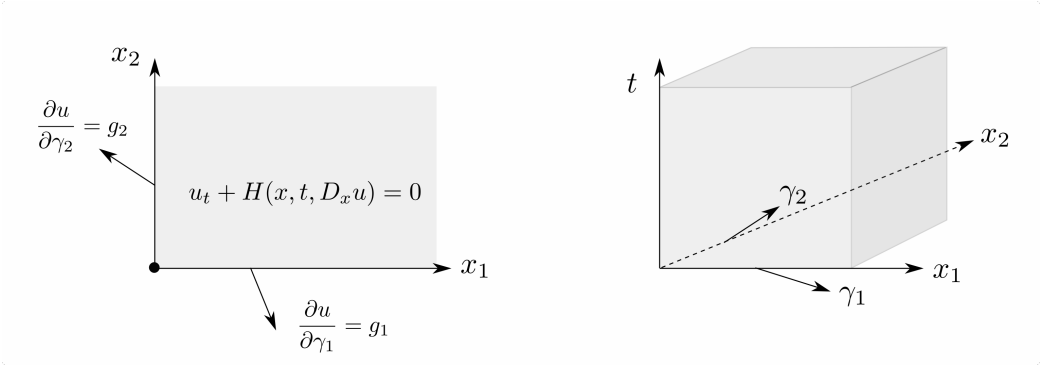


Figure 27.1: Standard Neuman problem with corner

Here we have

$$\mathbf{M}^3 = \{(x_1, x_2);\ x_1 > 0,\ x_2 > 0\} \times (0, T_f)\ ,$$
$$\mathbf{M}^2 = \{(x_1, x_2);\ x_1 = 0, x_2 > 0 \text{ or } x_1 > 0, x_2 = 0\} \times (0, T_f)\ ,$$
$$\mathbf{M}^1 = \{(0,0)\} \times (0, T_f)\ .$$

Of course the analysis of the previous section gives the stratified formulation on all the boundary except on $\mathbf{M}^1$, i.e at the points $((0,0), t)$ for $t \in (0, T_f)$, which require a specific treatment.

For $\mathbf{M}^1$, the answer is given by the following result in which we denote by **BCL** the set of dynamic-discount factor and cost related to H. We also point out that, in order to simplify, we argue as if $\gamma_1, \gamma_2, g_1, g_2$ were constants but the reader can check that all the arguments work if they are continuous functions of x and t.

Proposition 27.2.5 *We assume that*

(*i*) *either* $\gamma_1 \cdot e_1 = \gamma_2 \cdot e_2 = 0$

(*ii*) *or* $\gamma_1 \cdot e_1$, $\gamma_2 \cdot e_2$ *have the same strict sign and* $\det(\gamma_1, \gamma_2) < 0$.

If u is a viscosity subsolution of the above oblique derivative problem, it is a stratified subsolution of the problem with

$$\mathbb{F}^1((p_x, p_t)) = \sup\{\theta_3 p_t - (\theta_3 l - \theta_1 g_1 - \theta_2 g_2)\} \ \textit{on}\ \mathbf{M}^1\ ,$$

where the supremum is taken on all $(b,0,l) \in \mathbf{BCL}(0,t)$ *such that there exists* $\theta_1,\theta_2,\theta_3 \in (0,1)$ *such that* $\theta_1+\theta_2+\theta_3=1$ *and* $\theta_1\gamma_1+\theta_2\gamma_2-\theta_3 b^x=0$.

Proof — Let ϕ be a C^1-function on $\mathbb{R}$ and t be a strict local maximum point of the function $s \mapsto u(0,s)-\phi(s)$. We have to show that, if $\theta_1,\theta_2,\theta_3,\gamma_1,\gamma_2,b$ satisfy the property which is required in Proposition 27.2.5, then

$$\theta_3\phi'(t)-(\theta_3 l-\theta_1 g_1-\theta_2 g_2)\le 0\ .$$

It is worth pointing out that we can do that only if $(b,0,l)$ is in the interior of $\mathbf{BCL}(0,t)$, a point that we will use in the proof.

To do so, we first construct p_δ such that

$$p_\delta\cdot\gamma_1=g_1+\delta \quad , \quad p_\delta\cdot\gamma_2=g_2+\delta\ , \tag{27.9}$$

notice that such a p_δ exists because of the assumptions on γ_1,γ_2.

Next, we introduce the function

$$(y,s)\mapsto u(y,s)-\phi(s)-p_\delta\cdot y-\frac{Ay\cdot y}{\varepsilon^2}\ .$$

where A is a symmetric, positive definite matrix A. Additional properties on A will be needed and described all along the proof. At the end, we will show that such a matrix exists.

This function has a maximum point at $(x_\varepsilon,t_\varepsilon)$ and $(x_\varepsilon,t_\varepsilon)\to(0,t)$ as $\varepsilon\to 0$ by the maximum point property of $(0,t)$. Now we examine the different possibilities: if $x_\varepsilon=((x_\varepsilon)_1,0)=(x_\varepsilon)_1e_1$ with $(x_\varepsilon)_1\ge 0$ and $e_1=(1,0)$, we have

$$\begin{aligned}\Big[p_\delta+\frac{2Ax_\varepsilon}{\varepsilon^2}\Big]\cdot\gamma_1 &= g_1+\delta+\frac{2x_\varepsilon}{\varepsilon^2}\cdot A\gamma_1\\ &= g_1+\delta+\frac{2(x_\varepsilon)_1e_1}{\varepsilon^2}\cdot A\gamma_1\ .\end{aligned}$$

Hence if $A\gamma_1\cdot e_1\ge 0$, the inequality "$\dfrac{\partial u}{\partial\gamma_1}\le g_1$" cannot hold.

In the same way, if $x_\varepsilon=(0,(x_\varepsilon)_2)=(x_\varepsilon)_2e_2$, $(x_\varepsilon)_2\ge 0$ and $e_2=(0,1)$, the inequality "$\dfrac{\partial u}{\partial\gamma_2}\le g_2$" cannot hold provided $A\gamma_2\cdot e_2\ge 0$.

Therefore, whereever x_ε is, the H-inequality holds and, since $(b,0,l)$ is in the interior of $\mathbf{BCL}(0,t)$, for ε small enough, $(b,0,l)\in\mathbf{BCL}(x_\varepsilon,t_\varepsilon)$ and we have

$$\phi'(t_\varepsilon)-b^x\cdot\Big[p_\delta+\frac{2Ax_\varepsilon}{\varepsilon^2}\Big]\le l\ .$$

Now we examine the b^x-term, remarking that θ_3 cannot be 0

$$\begin{aligned} -b^x \cdot \frac{2Ax_\varepsilon}{\varepsilon^2} &= -\frac{1}{\theta_3}(\theta_1\gamma_1 + \theta_2\gamma_2) \cdot \frac{2Ax_\varepsilon}{\varepsilon^2} \\ &= -\frac{1}{\theta_3}(\theta_1 A\gamma_1 + \theta_2 A\gamma_2) \cdot \frac{2x_\varepsilon}{\varepsilon^2} \ , \end{aligned}$$

(recall that, being a symmetric matrix, the transpose of A is A itself). Since we want this term to be positive for any $x_\varepsilon = ((x_\varepsilon)_1, (x_\varepsilon)_2)$ with $(x_\varepsilon)_1, (x_\varepsilon)_2 \geq 0$, we have to require that all the coordinates of the vector $\theta_1 A\gamma_1 + \theta_2 A\gamma_2$ be negative.

If these properties hold true, we end up with

$$\phi'(t_\varepsilon) - b^x \cdot p_\delta \leq l \ .$$

Letting ε tend to 0, and using a convex combination with (27.9) provides the answer, after letting δ tend to 0.

It remains to show that such matrix A exists under the conditions of Proposition 27.2.5. We point out that this matrix may depend on the convex combination, hence on $\theta_1, \theta_2, \theta_3$ since the above proof is done for any fixed such convex combination.

We recall that the three conditions are

$$A\gamma_1 \cdot e_1 \geq 0 \quad , \quad A\gamma_2 \cdot e_2 \geq 0 \ ,$$

$$\theta_1 A\gamma_1 + \theta_2 A\gamma_2 \leq 0 \ ,$$

this last condition meaning that all the components of the vector are negative.

Looking at these conditions, a natural choice would be

$$A\gamma_1 = -\lambda_1 e_2 \quad \text{and} \quad A\gamma_2 = -\lambda_2 e_1 \ .$$

where λ_1, λ_2 are non-negative constants which have to be chosen properly, or equivalently

$$A^{-1}e_1 = -(\lambda_2)^{-1}\gamma_2 \quad \text{and} \quad A^{-1}e_2 = -(\lambda_1)^{-1}\gamma_1 \ .$$

Therefore

$$A^{-1} = \begin{pmatrix} -(\lambda_2)^{-1}\gamma_{2,1} & -(\lambda_1)^{-1}\gamma_{1,1} \\ -(\lambda_2)^{-1}\gamma_{2,2} & -(\lambda_1)^{-1}\gamma_{1,2} \end{pmatrix} \ .$$

In order that A satisfies the required condition to be a symmetric, positive definite matrix, we have just to check that A^{-1} satisfies these properties. Recalling that $\gamma_1 \cdot e_2 = \gamma_{1,2} < 0$ and $\gamma_2 \cdot e_1 = \gamma_{2,1} < 0$ by the oblique derivatives conditions, this leads to the following conditions

(*i*) A^{-1} is symmetric if either $\gamma_{1,1} = \gamma_{2,2} = 0$ or $\gamma_{1,1}, \gamma_{2,2}$ have the same strict sign. Then we can choose $\lambda_1 = |\gamma_{1,1}|$ and $\lambda_2 = |\gamma_{2,2}|$.

(*ii*) The trace of A is non-negative since $\gamma_{2,1}, \gamma_{1,2} < 0$ by the conditions on the directions of reflection.

(*iii*) $\det(A^{-1}) = (\lambda_1\lambda_2)^{-1} \det(\gamma_2, \gamma_1) > 0$ by assumption.

Hence we can conclude if one of the two conditions holds
1. $\gamma_{1,1} = \gamma_{2,2} = 0$ with $A = Id$.
2. $\gamma_{1,1}, \gamma_{2,2}$ have the same strict sign and $\det(\gamma_1, \gamma_2) < 0$.

In order to investigate the other cases and to show that A does not exist in these cases, we assume (without loss of generality) that $\gamma_1 = (\gamma_{1,1}, -1)$, $\gamma_2 = (-1, \gamma_{2,2})$ and we write A as

$$A = \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix} ,$$

where β can be chosen as $0, 1$ or -1 since A can be replaced by λA for $\lambda > 0$.

The constraint can be written as

$$\alpha\gamma_{1,1} - \beta \geq 0 ,$$

$$-\beta + \gamma\gamma_{2,2} \geq 0 ,$$

$$\theta_1(\alpha\gamma_{1,1} - \beta) + \theta_2(-\alpha + \beta\gamma_{2,2}) \leq 0 ,$$

$$\theta_1(\beta\gamma_{1,1} - \gamma) + \theta_2(-\beta + \gamma\gamma_{2,2}) \leq 0 .$$

We begin with the case when $\gamma_{1,1} \geq 0$, $\gamma_{2,2} \leq 0$. In this case, the (necessary) choice $\beta = -1$ yields

$$\alpha\gamma_{1,1} + 1 \geq 0 ,$$

$$1 + \gamma\gamma_{2,2} \geq 0 ,$$

$$\theta_1(\alpha\gamma_{1,1} + 1) + \theta_2(-\alpha - \gamma_{2,2}) \leq 0 ,$$

$$\theta_1(-\gamma_{1,1} - \gamma) + \theta_2(1 + \gamma\gamma_{2,2}) \leq 0 .$$

The first constraint gives no limitation on α, while the second one imposes (a priori) γ to be small enough. For the two next ones we recall that $\theta_3 b = \theta_1\gamma_1 + \theta_2\gamma_2$ and therefore

$$\theta_3 b_1 = \theta_1\gamma_{1,1} - \theta_2 \quad , \quad \theta_3 b_2 = -\theta_1 + \theta_2\gamma_{2,2} < 0 .$$

Hence the two last constraints can be written as

$$\alpha b_1 - b_2 \leq 0 \ ,$$

$$-b_1 + \gamma b_2 \leq 0 \ .$$

Clearly one can conclude only if $b_1 < 0$ by choosing α large and with

$$\frac{b_1}{b_2} \leq \gamma \leq -\frac{1}{\gamma_{2,2}} \ .$$

We have indeed the existence of such γ since

$$\frac{b_1}{b_2} \leq -\frac{1}{\gamma_{2,2}}$$

is equivalent to $\det(b, \gamma_2) \leq 0$ and $\det(b, \gamma_2) = \dfrac{\theta_1}{\theta_3} \det(\gamma_1, \gamma_2) \leq 0$.

But, given γ_1, γ_2, in order to have $b_1 < 0$ for any choice of the convex combination, the only possibility is $\gamma_{1,1} = 0$. And the same conclusion holds in the case $\gamma_{1,1} \leq 0$, $\gamma_{2,2} \geq 0$.

The proof is then complete.

Q.E.D.

Example 27.1 — We consider the following problem where $Q = (0,1) \times (0,1)$

$$\begin{cases} u_t + a(x,t)|D_x u| = f(x) & \text{in } Q \times (0, T_f) \\ u(x,0) = u_0(x) & \text{in } \Omega \\ \dfrac{\partial u}{\partial n_i} = g_i(x,t) & \text{on } \partial Q_i \times (0, T_f) \ \ , \end{cases} \tag{27.10}$$

where $\partial Q_1 = (0,1) \times \{0\}$, $\partial Q_2 = \{1\} \times (0,1)$, , $\partial Q_3 = (0,1) \times \{1\}$, ,$\partial Q_4 = \{1\} \times (0,1)$ and n_i is the exterior unit normal vector to ∂Q_i.

If a is a Lipschitz continuous function (in particular in x) satisfying $a(x,t) > 0$ on $\overline{Q} \times [0, T_f]$, and $u_0, f, g_1, \cdots g_4$ are continuous, there exists a unique viscosity solution of this problem which coincides with the stratified solution. This result is a straightforward consequence of the preceeding results which shows that the notions of viscosity solutions and stratified solutions are the same. It is worth remarking on this example that, despite we did not insist above on that point, the Hamiltoniant $\mathbb{F}^1, \mathbb{F}^2$ satisfy the right conditions: indeed these Hamiltonians fullfill the required continuity assumptions because in the above convex combinations like $\theta_1 + \theta_2 + \theta_3 = 1$ and $\theta_1 \gamma_1 + \theta_2 \gamma_2 - \theta_3 b^x = 0$, θ_3 is bounded away from 0.

In the present exemple, on ∂Q_i

$$\mathbb{F}^2(x,t,(p_x,p_t)) = \max(\theta(p_t - a(x,t)v\cdot p_x - f(x)) + (1-\theta)(n_i\cdot p_x - g_i)) \ ,$$

the maximum being taken on all $|v| \le 1$ and $\theta \in [0,1]$ such that $[\theta a(x,t)v - (1-\theta)n_i]\cdot n_i = 0$

Writing $v = v^\perp + v^\top$, where $v^\perp$ is the normal part of v (i.e. the part which is colinear to n_i) and $v^\top$ the tangent part, we have $\theta a(x,t)v^\perp \cdot n_i = (1-\theta)$ and we take divide by θ to have

$$\mathbb{F}^2(x,t,(p_x,p_t)) = \max_{|v^\top|^2+|v^\perp|^2=1} (p_t - a(x,t)v^\top\cdot p_x - f(x) + a(x,t)v^\perp\cdot n_i g_i) \ .$$

On an other hand, at $x=0$, a simple computation gives

$$\mathbb{F}^1(0,t,p_t) = \max(p_t - f(x) - g_1; p_t - f(x) - g_4) \ .$$

Remark 27.2.6 *We do not know if the conditions given in Proposition 27.2.5 are optimal or not. Clearly they are stronger than those given in Dupuis and Ishii [74, 73] inspired by those of Harrison and Reiman [109] and Varadhan and Williams [160]. Maybe a different choice of test-function, namely the term* $\dfrac{Ay\cdot y}{\varepsilon^2}$*, could lead to more general cases but we have no idea how to build such a function which necessarely will be* C^1 *but not* C^2 *at* 0*.*

Now we turn to the case of a "flat" discontinuity in $\mathbb{R}^2$, as depicted in Fig. 27.2 below

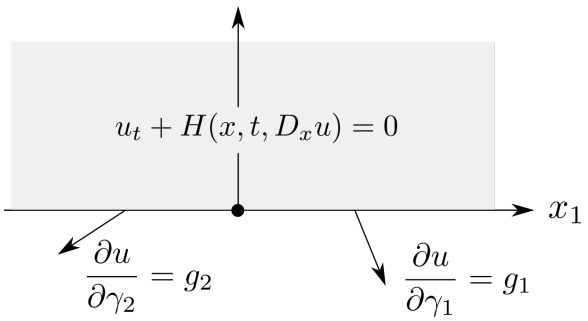


Figure 27.2: Flat discontinuous Neuman problem

Proposition 27.2.7 *Assume that* γ_1, γ_2 *satisfy* $\det(\gamma_1, \gamma_2) \leq 0$. *Then the same result as in Proposition 27.2.5 holds.*

Proof — The proof follows exactly along the same lines, only the constraints on A are different. Indeed, we should have

$$A\gamma_1 \cdot e_1 \geq 0 \ , \ A\gamma_2 \cdot e_1 \leq 0 \ ,$$

and $Ab = -\lambda e_2$ for some $\lambda > 0$. In this last property, we can take $\lambda = 1$ without loss of generality and therefore the symmetric matrix A^{-1} has the form

$$\begin{pmatrix} \alpha & -b_1 \\ -b_1 & -b_2 \end{pmatrix} \ ,$$

for some suitable parameter $\alpha > 0$. Moreover,

$$A = [\det(A^{-1})]^{-1} \begin{pmatrix} -b_2 & b_1 \\ b_1 & \alpha \end{pmatrix} \ .$$

Hence $A\gamma_1 \cdot e_1 = \gamma_1 \cdot Ae_1$ has the same sign as $\det(b, \gamma_1)$ and $A\gamma_2 \cdot e_1 = \gamma_2 \cdot Ae_1$ has the same sign as $\det(b, \gamma_2)$.

Therefore A exists (taking $\alpha > 0$ large enough) if $\det(b, \gamma_1) \geq 0$ and $\det(b, \gamma_2) \leq 0$. But $\theta_3 b = \theta_1\gamma_1 + \theta_2\gamma_2$ and therefore these conditions reduce to $\det(\gamma_1, \gamma_2) \leq 0$.

Q.E.D.

Remark 27.2.8 *If we assume that* $\gamma_1 \cdot e_2 = \gamma_2 \cdot e_2 = -1$, *then the condition* $\det(\gamma_1, \gamma_2) \leq 0$ *reduces to the tangential components inequality* $\gamma_{2,1} \leq \gamma_{1,1}$. *Fig. 27.3 below shows different types of situation where* γ_1, γ_2 *and their oppposite (in dashed lines) are shown, those opposites being involved in the reflexion process that occurs on the boundary.*

From the first two examples it could be guessed that the trajectories in the good case are more of a "regular" type. However, the other examples show that we can also allow some cases where the reflexions go in the same direction, provided some "squeezing" effect holds.

27.2.4 Discontinuities in the direction of reflection and domains with corners: the $\mathbb{R}^N$-case.

Of course, in $\mathbb{R}^N$, there exists a lot of possibilities and we are going to investigate the following three situations

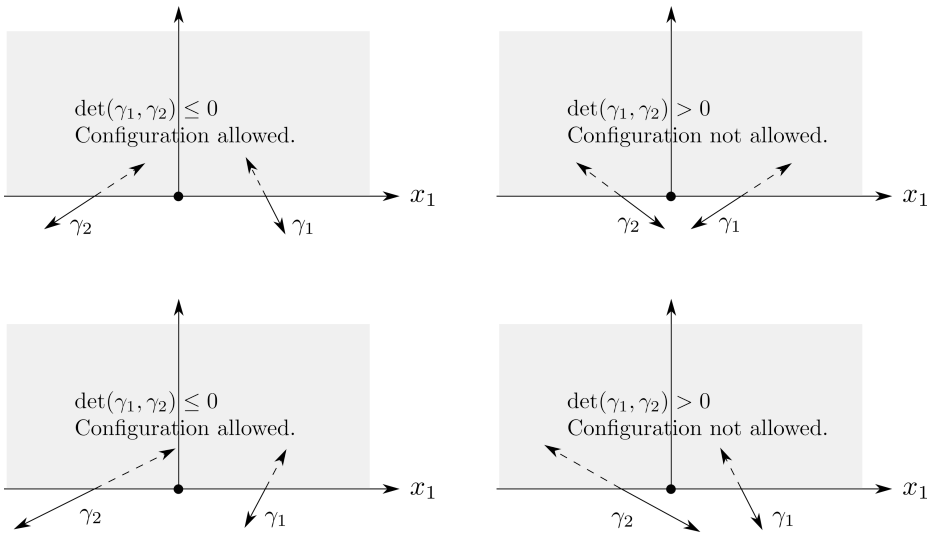


Figure 27.3: Configurations for discontinuous Neuman problem

Case 1: a simple 1-dimensional corner

$$\begin{aligned}
\mathbf{M}^{N+1} &= \{(x_1, \cdots, x_N);\ x_1 > 0, x_2 > 0\} \times (0, T_f), \\
\mathbf{M}^{N} &= \{(x_1, \cdots, x_N);\ x_1 = 0, x_2 > 0 \text{ or } x_1 > 0, x_2 = 0\} \times (0, T_f)\ , \\
\mathbf{M}^{N-1} &= \{(x_1, \cdots, x_N);\ x_1 = 0, x_2 = 0\} \times (0, T_f)\ .
\end{aligned}$$

Case 2: a simple discontinuity in the oblique derivative boundary condition

$$\begin{aligned}
\mathbf{M}^{N+1} &= \{(x_1, \cdots, x_N);\ x_2 > 0\} \times (0, T_f), \\
\mathbf{M}^{N} &= \{(x_1, \cdots, x_N);\ x_1 \neq 0, x_N = 0\} \times (0, T_f)\ , \\
\mathbf{M}^{N-1} &= \{(x_1, \cdots, x_N);\ x_1 = 0, x_2 = 0\} \times (0, T_f)\ .
\end{aligned}$$

Case 3: a multi-dimensional corner

$$\begin{aligned}
\mathbf{M}^{N+1} &= \{(x_1, \cdots, x_N);\ x_1 > 0 \cdots x_N > 0\} \times (0, T_f), \\
\mathbf{M}^{N} &= \bigcup_i \{(x_1, \cdots, x_N);\ x_1 \geq 0 \cdots x_N \geq 0, x_i = 0\} \times (0, T_f)\ , \\
\mathbf{M}^{N-1} &= \{(x_1, \cdots, x_N);\ x_{N-1} = 0, x_N = 0\} \times (0, T_f)\ .
\end{aligned}$$

In each case, the question is: when is the classical notion of subsolution equivalent to the stratification formulation?

The answer is simple in the two first cases. Let us write

$$\gamma_1 = (\gamma_1^{(1)}, \gamma_2^{(1)}, \cdots, \gamma_N^{(1)}) \quad \text{and} \quad \gamma_2 = (\gamma_1^{(2)}, \gamma_2^{(2)}, \cdots, \gamma_N^{(2)})\ ,$$

and introduce the two vectors of $\mathbb{R}^2$

$$\tilde{\gamma}_1 = (\gamma_1^{(1)}, \gamma_2^{(1)}) \quad \text{and} \quad \tilde{\gamma}_2 = (\gamma_1^{(2)}, \gamma_2^{(2)}) \, .$$

The result is

Proposition 27.2.9 *In Case 1 and 2, the classical viscosity formulation and the stratified formulation are equivalent if $\tilde{\gamma}_1, \tilde{\gamma}_2$ satisfy the condition of Proposition 27.2.5 in Case 1 and Proposition 27.2.7 in Case 2.*

Proof — In Case 1, we have to show that a viscosity subsolution u is also a stratified subsolution on $\mathbf{M}^{N-1}$. To do so, we denote any $x \in \mathbb{R}^N$ by (x_1, x_2, x') where $x' = (x_3, \cdots, x_N)$.

If $(\bar{x}, \bar{t}) \in \mathbf{M}^{N-1}$ is a maximum point of $x' \mapsto u(0, 0, x', t) - \phi(x', t)$ where ϕ is a smooth function, we have to show that, if we have a convex combination $(-\theta_1\gamma_1 - \theta_2\gamma_2 + \theta_3 b^x, -\theta_3) \in T_{(\bar{x},\bar{t})}\mathbf{M}^{N-1}$ with $(b, 0, l) \in \mathbf{BCL}(\bar{x}, \bar{t})$ and $b = (b^x, -1)$. Then

$$\theta_3\phi_t + (\theta_1\gamma_1 + \theta_2\gamma_2 - \theta_3 b^x) \cdot D_x\phi(\bar{x}', \bar{t}) \leq -\theta_1 g_1 - \theta_2 g_2 + \theta_3 l \, .$$

As in the proof of Proposition 27.2.5, we introduce p_δ such that

$$p_\delta \cdot \gamma_1 = g_1 + \delta \quad , \quad p_\delta \cdot \gamma_2 = g_2 + \delta \, ,$$

and the function

$$(y, s) \mapsto u(y, s) - \phi(y', s) - p_\delta \cdot y - \frac{A\tilde{y} \cdot \tilde{y}}{\varepsilon^2} \, ,$$

where $\tilde{y} = (y_1, y_2)$ and A is a 2×2 symmetric, positive definite matrix.

It is clear on this formulation that, only the $\tilde{y}$ terms plays a real role and we are in the same situation as in $\mathbb{R}^2$. This explains the statement of the result and shows that the proof for Case 2 follows from the same arguments.

Q.E.D.

Example 27.2 — A standard example for Case 2 is the case when we look at an oblique derivative problem in a smooth domain $\Omega \subset \mathbb{R}^N$ whose boundary is splitted into three components

$$\partial\Omega = \partial\Omega_1 \cup \partial\Omega_2 \cup \Gamma \, ,$$

where $\partial\Omega_1, \partial\Omega_2$ are smooth $(N-1)$-dimensional manifolds and Γ a smooth $(N-2)$-dimensional manifold. The idea is to have the oblique derivative boundary condition

$$\frac{\partial u}{\partial \gamma_i} = g_i \quad \text{on } \partial\Omega_i \times (0, T_f) \, ,$$

for $i = 1, 2$.

The question is when the classical viscosity solution coincides with the stratified one and therefore is unique?

To answer to this question is not completely obvious since we have to apply the above result for Case 2 in the right way on Γ. To do so, we consider $x \in \Gamma$ and we introduce two unit vectors: n the unit outward normal to $\partial\Omega$ at x and $r \in T_x\partial\Omega$ a unit vector which is normal to $T_x\Gamma$ and which is pointing toward Ω_1.

With these notations, the answer to the above question is yes if the determinant

$$\begin{vmatrix} \gamma_1 \cdot r & \gamma_2 \cdot r \\ -\gamma_1 \cdot n & -\gamma_2 \cdot n \end{vmatrix} \leq 0 \ .$$

For Case 3, we introduce the $N \times N$-matrix Γ whose columns are given by $\gamma_1, \gamma_2, \cdots \gamma_n$ and we formulate the

Proposition 27.2.10 *In Case 3, the classical viscosity formulation and the stratified formulation are equivalent if there exists a $N \times N$-diagonal matrix D with strictly positive diagonal terms such that $\Gamma.D^{-1}$ is a symmetric, negative definite matrix.*

Proof — The proof follows along the arguments of the proof of Proposition 27.2.5: the key (and only) point is to find a symmetric, positive definite matrix A such that

$$A\gamma_i = -d_i e_i \quad \text{with } d_i > 0, \quad \text{for any } 1 \leq i \leq N \ .$$

This property can be written as $A.\Gamma = -D$ and therefore $A = -D\Gamma^-1 = -(\Gamma.D^{-1})^{-1}$. The assumption ensures the existence of A.

Q.E.D.

This result can, of course, be extended to the case of more general convex domains like

$$\Omega := \bigcap_i \{x : \ n_i \cdot x < q_i\} \ ,$$

with a direction of reflection γ_i on $\{x : \ n_i \cdot x = q_i\}$ by the

Proposition 27.2.11 *The classical viscosity formulation and the stratified formulation are equivalent if there exists a $N \times N$ symmetric, positive definite matrix A such that*

$$A\gamma_i = d_i n_i \quad \textit{with } d_i > 0, \quad \textit{for any } 1 \leq i \leq N \ .$$

.

Remark 27.2.12 *Clearly, as in the case of the* 2*-dimensional corner we have no idea if these conditions are optimal or not but, at least, they are obviously satisfied if* $\gamma_i = n_i$ *with* $A = Id$ *and all* $d_i = 1$.

We conclude this section by an open question in the case of a non-convex domain, the model case being in 2-d

$$\Omega = \{(x_1, x_2) : \ x_1 > 0 \text{ or } x_2 > 0\} \ ,$$

with normal reflection on the two parts of the boundary, $\{(x_1, 0) : \ x_1 > 0\}$ and $\{(0, x_2) : \ x_2 > 0\}$, or different oblique derivative boundary conditions.

The strategy we follow above clearly fails due to the non-convexity of the domain and,maybe surprisingly, we were unable to obtain any general result in this case (some particular cases can, of course, be treated). We do not know if this is just a technical problem or if really they are counterexample where Ishii solutions are not unique since, otherwise, they coincide with the unique stratified solution.

27.3 Mixing the Dirichlet and Neumann problems

27.3.1 The most standard case

The most standard case is the case when the boundary $\partial\Omega$ is smooth and can be written as

$$\partial\Omega = \partial\Omega_1 \cup \partial\Omega_2 \cup \mathcal{H} \ ,$$

where $\partial\Omega_1, \partial\Omega_2$ are open sets of $\partial\Omega$ and $\mathcal{H}$ is a $(N-2)$ submanifold of $\partial\Omega$, and where the boundary condition is of the type

$$\begin{cases} u = \varphi \quad \text{on } \partial\Omega_1 \times (0, T_f) \ , & (27.11) \\ \dfrac{\partial u}{\partial \gamma} = g \quad \text{on } \partial\Omega_2 \times (0, T_f) \ , & (27.12) \end{cases}$$

where φ, γ and g are continuous functions on the boundary, γ satisfying the usual condition, i.e.

$$\gamma(x,t) \cdot n(x) \geq \nu > 0 \ ,$$

for some $\nu > 0$ and for any $(x,t) \in \partial\Omega \times (0, T_f)$, where $n(x)$ is the unit outward normal to $\partial\Omega$ at x.

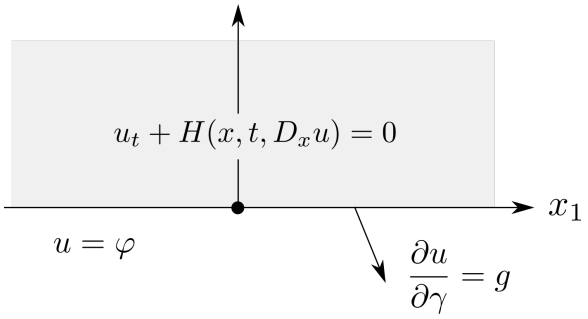


Figure 27.4: Flat Dirichlet-Neuman problem

Locally, after flattening the boundary in the neighborhood of a point in $\mathcal{H}$, we have a picture like Fig. 27.4 below.

But we may also be in a slightly less standard case if $\partial\Omega$ is not smooth and if, in terms of stratification, we have

$$\mathbf{M}^N = \partial\Omega_1 \cup \partial\Omega_2 \times (0,T_f) \quad , \quad \mathbf{M}^{N-1} = \mathcal{H} \times (0,T_f)\ ,$$

typically a picture like Fig. 27.5

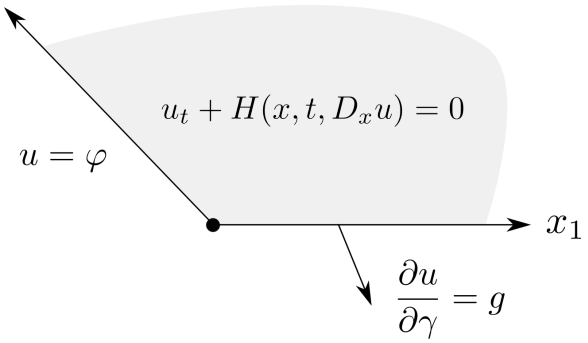


Figure 27.5: Angular Dirichlet-Neuman problem

Maybe surprisingly both cases can be treated in the same way and the main property needed is the

Lemma 27.3.1 *Assume that $\overline{\Omega} \times (0,T_f)$ is a stratified domain with the above described stratification and that we are in the good framework for stratified problems. Then*

$$u \le \varphi \quad \text{on } \mathbf{M}^{N-1} = \mathcal{H} \times (0,T_f)\ .$$

Proof — Let $(\bar{x},\bar{t}) \in \mathbf{M}^{N-1}$; we want to prove the inequality $u(\bar{x},\bar{t}) \le \varphi(\bar{x},\bar{t})$.

We first remark that $u(x,t) \leq \varphi(x,t)$ if $x \in \partial\Omega_1$, $t > 0$ as a consequence of the results for the Dirichlet problem. Hence, if we assume by contradiction that $u(\bar{x},\bar{t}) > \varphi(\bar{x},\bar{t})$ and if we redefine u on $\partial\Omega_1$ by introducing

$$\tilde{u}(x,t) = \limsup_{\substack{(y,s)\to(x,t)\\ y\in\Omega}} u(y,s) \quad \text{if } x \in \partial\Omega_1 \ ,$$

$\tilde{u}$ being equal to u otherwise, then $\tilde{u}$ is still an u.s.c. subsolution of the problem.

Next, following the arguments of the Neumann part, we have

$$\tilde{u}(x,t) = \limsup_{\substack{(y,s)\to(x,t)\\ y\in\Omega}} \tilde{u}(y,s) \quad \text{if } x \in \partial\Omega_2 \ .$$

With all these properties, the regularization procedure of Section 3.4.3 together with normal controllability properties of H implies that the partial sup-convolution procedure in the $\mathbf{M}^{N-1}$-direction provides a function, still denoted by $\tilde{u}$ which is continuous except perhaps on $\mathbf{M}^{N-1}$ and such that

$$\limsup_{\substack{(y,s)\to(\bar{x},\bar{t})\\ y\in\Omega\cup\partial\Omega_1\cup\partial\Omega_2}} \tilde{u}(y,s) \leq \varphi(\bar{x},\bar{t}) < \tilde{u}(\bar{x},\bar{t}) \ .$$

Now we consider the function $(x,t) \mapsto \tilde{u}(x,t) - \dfrac{|x-\bar{x}|}{\varepsilon^2} - \dfrac{|t-\bar{t}|}{\varepsilon^2}$: for $\varepsilon > 0$ small enough, this function has a maximum point at $(x_\varepsilon, t_\varepsilon)$ near $(\bar{x},\bar{t})$ and the above properties implies:

(i) $(x_\varepsilon, t_\varepsilon) \in \mathbf{M}^{N-1}$,

(ii) $\tilde{u}(x_\varepsilon, t_\varepsilon) \to \tilde{u}(\bar{x},\bar{t})$ and $\dfrac{|x_\varepsilon - \bar{x}|}{\varepsilon^2} + \dfrac{|t_\varepsilon - \bar{t}|}{\varepsilon^2} \to 0$ as $\varepsilon \to 0$

(iii) for any vector p which is normal to $\mathbf{M}^{N-1}$, $(x_\varepsilon, t_\varepsilon)$ is still a maximum point of

$$(x,t) \mapsto \tilde{u}(x,t) - \frac{|x-\bar{x}|}{\varepsilon^2} - \frac{|t-\bar{t}|}{\varepsilon^2} - \frac{p\cdot(x-\bar{x})}{\varepsilon} \ .$$

Choosing $p = n_2(\bar{x})$ the outward unit normal vector to $\partial\Omega_2$ at $\bar{x}$ and using the normal controlability assumption, it is clear that the viscosity subsolution inequality at $(\bar{x},\bar{t})$ cannot hold for ε small enough, giving the desired contradiction.

Q.E.D.

The above lemma gives us the

Proposition 27.3.2 *Assume that $\overline{\Omega} \times (0,T_f)$ is a stratified domain with the above described stratification and that we are in the good framework for stratified problem. Assume that φ, γ, g are continuous functions on $\partial\Omega \times [0,T_f)$ such that*

(*i*) $\varphi(x,0) = u_0(x)$ *on* $\partial\Omega \times \{0\}$,

(*ii*) γ *is Lipschitz continuous on* $\partial\Omega \times [0,T_f)$.

If u *is an u.s.c. viscosity subsolution of the mixed problem, then the function* $\tilde{u} : \overline{\Omega} \times [0,T_f) \to \mathbb{R}$ *defined by* $\tilde{u}(x,t) = u(x,t)$ *if* $x \in \Omega \cup \partial\Omega_2 \cup \mathcal{H}$ *and*

$$\tilde{u}(x,t) = \limsup_{\substack{(y,s)\to(x,t)\\ y\in\Omega}} u(y,s) \quad \text{if } x \in \partial\Omega_1 \ ,$$

is a stratified subsolution of the associated stratified problem with the Hamiltonian defined on $\mathbf{M}^{N-1}$

$$\mathbb{F}^{N-1}(x,t,Du) := \max\Big(u-\varphi(x,t)\,;\,\sup\{\theta p_t - (\theta b^x - (1-\theta)\gamma)\cdot p_x - (\theta l + (1-\theta)g)\}\Big)\,,$$

where the supremum is taken on all $(b,0,l) \in \mathbf{BCL}(x,t)$ *such that there exists* $\theta \in (0,1)$ *such that* $(\theta b^x - (1-\theta)\gamma) \in T_{(x,t)}\mathbf{M}^{N-1}$.

As a consequence, up to a modification of the value of the subsolution on $\partial\Omega_1$*, one has a comparison result for the mixed problem and therefore there exists a unique continuous viscosity solution of the mixed problem up to this modification.*

The proof of this result is simple since we use both the ingredients for the Dirichlet and Neumann problems in $\partial\Omega_1$ and $\partial\Omega_2$, the only difficulty being $\mathcal{H}\times(0,T_f) = \mathbf{M}^{N-1}$.

The lemma provides the inequality $\tilde{u} \le \varphi$ on $\mathbf{M}^{N-1}$, while the other part is obtained by a stability result from "inside" $\mathbf{M}^N$ in the spirit of Remark 27.1.2 or Proposition 21.2.3.

We leave these details to the reader but we provide the following result result which treats the difficulty connected to the initial data.

Lemma 27.3.3 *Under the assumptions of Proposition 27.3.2, if* u *and* v *are respectively an u.s.c. viscosity subsolution and a l.s.c. supersolution of the mixed problem, we have*

$$u(x,0) \le u_0(x) \le v(x,0)\ .$$

Proof — Of course, the difficulty comes from the points of $\partial\Omega \times \{0\}$ which are on $\mathcal{H}$. We only prove the result for the subsolution, the proof for the supersolution being analogous.

If $\bar{x} \in \mathcal{H}$, we want to prove that $u(\bar{x},0) \leq u_0(\bar{x})$. Since we are in a stratified framework, we can assume that $\partial\Omega_2 \subset \{x : \ (x-\bar{x})\cdot n_2 = 0\}$ where $n_2 \cdot \gamma > 0$ on $\partial\Omega_2$.

For $0 < \varepsilon \ll 1$, we consider the function

$$(x,t) \mapsto u(x,t) - \frac{|x-\bar{x}|^2}{\varepsilon^2} - C_\varepsilon t - \varepsilon\psi\left(\frac{(x-\bar{x})\cdot n_2}{\varepsilon^2}\right) .$$

where $C_\varepsilon > \varepsilon^{-1}$ is a large constant to be chosen later on and $\psi : \mathbb{R} \to \mathbb{R}$ is a C^1, increasing function such that $\psi(t) = -1$ if $t \leq -1$, $\psi(t) = 1$ if $t \geq 1$ and, $\psi(0) = 0$, $\psi'(0) = 1$.

This function has a maximum point at $(x_\varepsilon, t_\varepsilon)$ near $(\bar{x}, 0)$ and we have $(x_\varepsilon, t_\varepsilon) \to (\bar{x}, 0)$, $u(x_\varepsilon, t_\varepsilon) \to u(\bar{x}, 0)$ and $\dfrac{|x_\varepsilon - \bar{x}|}{\varepsilon^2} + C_\varepsilon t_\varepsilon \to 0$ as $\varepsilon \to 0$.

If $x_\varepsilon \in \partial\Omega_2$, then the oblique derivative inequality cannot holds since in the viscous sense,

$$Du \cdot \gamma = \frac{2(x-\bar{x})}{\varepsilon^2} + \frac{1}{\varepsilon}\psi'(0) = \frac{o(1)}{\varepsilon} + \frac{1}{\varepsilon} > 0 ,$$

for ε small enough. On the other hand, by choosing C_ε large enough, the H-inequality cannot hold wherever is $(x_\varepsilon, t_\varepsilon)$. Hence one of the inequalities $u(x_\varepsilon, t_\varepsilon) \leq \varphi(x_\varepsilon, t_\varepsilon)$ or $u(x_\varepsilon, t_\varepsilon) \leq u_0(x_\varepsilon)$ holds and the conclusion follows by letting ε tend to 0.

Q.E.D.

27.3.2 The tanker problem

As an example where the classical Ishii viscosity solutions formulation cannot be sufficient for treating singular discontinuities, we come back to the example given by P.L. Lions in his course at the Collège de France, namely the problem (25.1).

At P_1, P_2, P_L, one would like to impose Neumann boundary conditions

$$\frac{\partial u}{\partial n} = g_i(t) \quad \text{at } P_i \ ,$$

(see Fig. 27.6) but such a boundary condition is far from being classical. However, we can handle it through the stratified formulation by setting $\mathbf{M}^{N+1} = \Omega \times (0, T_f)$, $\mathbf{M}^1 = \{P_1, P_2, \cdots, P_L\} \times (0, T_f)$ and $\mathbf{M}^N = (\partial\Omega \setminus \{P_1, P_2, \cdots, P_L\}) \times (0, T_f)$.

The only point is compute $\mathbb{F}^1$ and the computation is done as in the previous section, except that we are in $\mathbf{M}^1$ and we look for dynamics consisting in staying at P_i for some i.

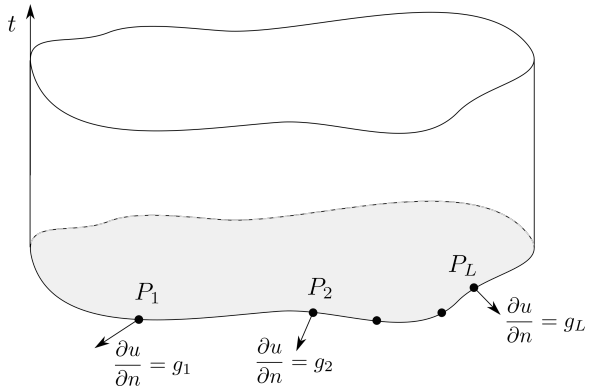


Figure 27.6: The tanker problem

At any P_i, we have to consider the convex combinations of $(b, c, l) = ((b^x, -1), 0, l) \in \mathbf{BCL}(x, t)$ and $((-n(P_i), 0), 0, g_i(t))$, i.e $(\mu b - (1-\mu)(n(P_i), 0), 0, \mu l + (1-\mu)g(x, t))$ for $0 \leq \mu \leq 1$ with (since we are on $\mathbf{M}^1$) the constraint $\mu b^x - (1-\mu)n(P_i) = 0$ which leads to

$$\mu b^x = (1-\mu)n(P_i) \ .$$

To compute $\mathbb{F}^1(x, t, p_t)$, we have to look at the supremum of $\mu p_t - (\mu l + (1-\mu)g_i(t))$ but taking into account the fact that $1 - \mu = \mu b.n(P_i)$ and since μ cannot vanish and the condition reduces to

$$u_t + \sup_{b^x = \lambda n,\ \lambda \geq 0} \Big\{ -(l + b^x \cdot n(P_i)g_i(t)) \Big\} \leq 0 \ .$$

Adequate controllability assumptions yield uniqueness of the stratified solution for such problem and of course, we can weaken the regularity assumptions on Ω which can be a square in $\mathbb{R}^2$ if the corners are harbours.

Chapter 28

Stability for Singular Boundary Value Problems

This section is concerned with some stability results involving Dirichlet and Neuman problems in presence of singularities. The aim here is to mix results from Sections 22 and 27. However, we do not try to cover all the possible results here since such formulations would imply a lot of technicalities and make everything very difficult to read.

We prefer instead to present a few illustrative examples that highlight the important ideas and results.

So, the main theme of this section is the following: we start from standard, continuous boundary condition problems and we want to pass to the limit when singularities are created (in the limit) at the boundary.

28.1 Singularites in Dirichlet problems

Starting from a sequence of smooth domain $(\Omega_\varepsilon)_{\varepsilon>0}$, and smooth (say continuous) boundary data $(\varphi_\varepsilon)_{\varepsilon>0}$, there are essentially two mechanisms that can yield some singularities as $\varepsilon \to 0$:

(a) the boundaries $\partial\Omega_\varepsilon$ converge to a boundary $\partial\Omega$ presenting a non-trivial stratification, ie. $\partial\Omega$ is not a C^1 hypersurface of $\mathbb{R}^N$;

(b) the sequence φ_ε converges to a discontinuous limit boundary condition φ.

Of course, a general problem may involve both mechanisms and even at the same

location in the limit. But here we are going to separate both cases in order to keep things as simple as possible. The good news being that both situations can be handled by the stratified framework.

28.1.1 Non-smooth domains

We consider here the case of a square $\Omega := [0,1]^2 \subset \mathbb{R}^2$, already considered in Example 1 and a stratified problem

$$\begin{cases} u_t + H(x,u,Du) = 0 & \text{in } \Omega \times (0,T_f)\,, \\ u(x,t) = \varphi(x,t) & \text{on } \partial\Omega \times (0,T_f)\,, \\ u(x,0) = u_0(x) & \text{in } \Omega\,. \end{cases} \tag{28.1}$$

where φ and u_0 are W-adapted to the natural stratification of $\partial\Omega \times (0,T_f)$. For simplicity here, let us even assume that φ, u_0 are continuous and satisfy the compatibility condition $\varphi(x,0) = u_0(x)$ for $x \in \partial\Omega$.

Here the problem is singular due to the geometric nature of Ω wich is clearly only a Lipschitz domain. As we have already seen, this singular Dirichlet problem can be handled in the stratified framework under some quite general hypotheses. We refer to the corresponding Section for all the details. Let us just mention here that there is an underlying set-valued map **BCL** defined on $\bar{\Omega} \times [0,T_f]$, taking into account the Dirichlet boundary data φ on $\partial\Omega \times (0,T_f)$ as well as the initial data u_0 on $\Omega \times \{0\}$, which allows to get a unique stratified solution of (28.1).

Of course, the stratification $\mathbb{M}$ of $\Omega \times [0,T_f]$ is given by $\mathbb{M}^3 := \Omega \times [0,T_f]$, $\mathbb{M}^1 := \{\{P_i\} \times (0,T_f); i = 1..4\}$ and $\mathbb{M}^2$ is the collection of lateral surfaces of $\partial\Omega \times (0,T_f)$ defined by removing the edges.

The question is, can we approximate the stratified solution by a sequence of "'standard" problems, defined in smooth domains. Or equivalently, can we identify the limit of such regular problems when the geometry of the boundary yields some singularities in the limit ?

In order to answer the question, we approximate the square by a sequence of smooth domains (Ω_ε) "converging" to Ω. For simplicity, we choose here

$$\Omega_\varepsilon := \{x \in \Omega : \mathrm{dist}(x, \partial\Omega) > \varepsilon\}\,.$$

This specific sequence has some advantages like preserving the symmetries of the problem, being convex, monotone, included in Ω, see Fig. 28.1. However, we could use more general approximations provided they satisfy some properties, the most important one being of course the convergence to Ω in the sense of Definition 22.2.4.

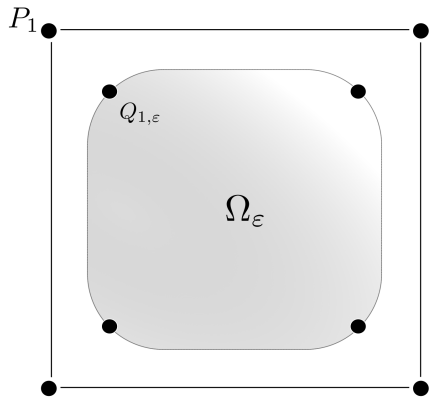


Figure 28.1: Approximation of the square

Concerning the Dirichlet boundary data, we introduce a sequence of continuous function φ_ε defined on $\partial\Omega_\varepsilon \times (0,T_f)$, satisfying the following convergence property: for any sequence of points $(x_\varepsilon,t_\varepsilon) \in \partial\Omega_\varepsilon \times (0,T_f)$ converging to a point $(x,t) \in \partial\Omega \times (0,T_f)$,

$$\lim_{\varepsilon\to 0} \varphi_\varepsilon(x_\varepsilon,t_\varepsilon) = \varphi(x,t)\,.$$

Moreover, we assume that for each $\varepsilon > 0$, the compatibility condition $\varphi_\varepsilon(x,0) = u_0(x)$ is valid on Ω_ε. Under these assumptions, we know that for any $\varepsilon > 0$ there is a unique viscosity solution u_ε of the following problem

$$\begin{cases} (u_\varepsilon)_t + H(x,u_\varepsilon,Du_\varepsilon) = 0 & \text{in } \Omega_\varepsilon \times (0,T_f)\,,\\ u_\varepsilon(x,t) = \varphi_\varepsilon(x,t) & \text{on } \partial\Omega_\varepsilon \times (0,T_f)\,,\\ u_\varepsilon(x,0) = u_0(x) & \text{in } \Omega_\varepsilon\,. \end{cases}$$

The main result on the convergence is the

Theorem 28.1.1 *As $\varepsilon \to 0$, the sequence (u_ε) converges locally uniformly in $\Omega \times (0,T_f)$ to the stratified solution u of problem (28.1).*

Proof — The proof consists just in assembling various results that appear in the previous parts of this book. However, there are a few things to do before that.

A. Stratifying the problem — Thanks to our assumption on Ω_ε, we can easily define four points $Q_{i,\varepsilon} \in \Omega_\varepsilon$, located on the square diagonals. Notice that $Q_{i,\varepsilon}$ actually minimizes the distance to P_i from Ω_ε, so that $Q_{i,\varepsilon} \to P_i$ for each $i = 1..4$ as $\varepsilon \to 0$.

Those points allow us to define a super-stratification $\mathbb{M}_\varepsilon$ of $\Omega_\varepsilon \times [0,T_f]$ where $\mathbf{M}^1_\varepsilon = \{Q_{i,\varepsilon} \times [0,T_f]; i = 1..4\}$ and the four others parts of $\partial\Omega_\varepsilon \times [0,T_f]$ are elements of $\mathbf{M}^2_\varepsilon$.

B. Defining the sets $\mathbf{BCL}_\varepsilon$ and the Hamiltonians — here we follow exactly Section 27.1: $\mathbf{BCL}_\varepsilon$ is just a modification of $\mathbf{BCL}$ on $\partial\Omega_\varepsilon \times (0,T_f)$, where we extend it as the convex enveloppe of

$$\mathbf{BCL}(x,t) \cup \{(0,1,\varphi_\varepsilon(x,t))\}\,.$$

Of course this new $\mathbf{BCL}_\varepsilon$ is $\mathbb{M}_\varepsilon$-adapted.

Using Proposition 22.2.1, we introduce three Hamiltonians: first, $\mathbb{F}^3_\varepsilon(x,t,u,p) := p_t + H(x,t,u,p_x)$ defined in $\Omega_\varepsilon \times (0,T_f)$, which is just related to the equation inside Ω_ε. Then, on each point $Q_{i,\varepsilon}$ of $\mathbf{M}^1_\varepsilon$, we set

$$\mathbb{F}^1_\varepsilon(x,t,u,p) := u(x,t) - \varphi_\varepsilon(x,t)\,,$$

and similarly, we set $\mathbb{F}^2_\varepsilon(x,t,u,p) = u(x,t) - \varphi_\varepsilon(x,t)$ when x belongs to $\partial\Omega_\varepsilon \setminus \{Q_{i,\varepsilon}; i = 1..4\}$.

In this setting, u_ε can be seen as a the stratified solution of $\mathbb{F}_\varepsilon(x,t,u,D_x u,u_t) = 0$, meaning that for each $k = 1..3$, a $\mathbb{F}^k_\varepsilon$-subsolution inequality holds on $\mathbf{M}^k_\varepsilon$ while u_ε is a (viscosity) $\mathbb{F}^3$-supersolution.

C. Passing to the limit — In the sense of Definition 22.2.4, $\mathbb{M}_\varepsilon$ converges to $\mathbb{M}$, the stratification of $\partial\Omega$.

In order to apply Theorem 22.2.4, we just notice that by construction, $\mathbf{BCL}_\varepsilon$ converges to $\mathbf{BCL}$ in the sense of Lemma 22.1.4 (because of our assumption on the convergence of φ_ε). This allows us to use the theorem, and conclude as in Corollary 22.1.5.

Q.E.D.

28.1.2 Non-smooth data

Here we assume that Ω is a fixed smooth domain in $\mathbb{R}^2$ and that we have a sequence of boundary data φ_ε continuous on $\partial\Omega \times [0,T_f]$, converging to some φ which may be discontinuous at some isolated points of $\partial\Omega \times [0,T_f]$. For simplicity here, we assume that there is only one point $P_0 = (x_0,t_0)$ with $t_0 > 0$ at which φ is not continuous.

We assume that $\mathbf{BCL}(x,t)$ is defined in $\Omega \times [0,T_f]$ and satisfies the standard assumptions. In order to deal with the boundary data, we extend $\mathbf{BCL}$ up to the boundary, defining $\mathbf{BCL}_\varepsilon$ and $\mathbf{BCL}_*$ as above: for any $x \in \partial\Omega$ and $t \geq 0$,

$$\mathbf{BCL}(x,t) = \overline{\mathrm{co}}\Big(\limsup_{y\to x} \mathbf{BCL}(y,t) \cup (0,1,\varphi_\varepsilon(x,t))\Big)\,.$$

Similarly, we define $\mathbf{BCL}_*$ on $\partial\Omega\times[0,T_f]$ by inserting the boundary data $(0,1,\varphi_*(x,t))$ before taking the convex enveloppe. Here,

$$\varphi_*(x,t) := \liminf_{\substack{\varepsilon\to 0, y\to x\\ y\in\partial\Omega}} \varphi_\varepsilon(y,t)\,,$$

and of course in this simple situatioh $\varphi_*(x,t)=\varphi(x,t)$ for any $(x,t)\neq(x_0,t_0)$.

From the pde viewpoint, we are considering the following boundary value problem

$$\begin{cases} (u_\varepsilon)_t + H(x,u_\varepsilon,Du_\varepsilon)=0 & \text{in } \Omega\times(0,T_f)\,,\\ u_\varepsilon(x,t)=\varphi_\varepsilon(x,t) & \text{on } \partial\Omega\times(0,T_f)\,,\\ u_\varepsilon(x,0)=u_0(x) & \text{in } \Omega \end{cases}$$

which has a unique viscosity solution since φ_ε is continuous, and the singular limit problem

$$\begin{cases} u_t + H(x,u,Du)=0 & \text{in } \Omega\times(0,T_f)\,,\\ u(x,t)=\varphi_*(x,t) & \text{on } \partial\Omega\times(0,T_f)\,,\\ u(x,0)=u_0(x) & \text{in } \Omega\,. \end{cases}$$

Since φ is discontinuous at P_0, we have to consider a stratification of $\partial\Omega\times(0,T_f)$ involving $\mathbb{M}^1:=\{\{P_0\}\times(0,T_f)\}$. The result is the following

Theorem 28.1.2 *As $\varepsilon\to 0$, the sequence of viscosity solutions (u_ε) converges to the stratified solution of the limit problem, associated with the boundary data φ_*.*

Proof — The strategy is essentially the same as for the case of a singular boundary: we introduce a super-stratification $\mathbb{M}$ which takes into account the discontinuity at P_0.

With this new stratification (which does not depend on ε), (u_ε) can be seen as the stratified solution of $\mathbb{F}_\varepsilon(x,u_\varepsilon,Du_e,(u_\varepsilon)_t)=0$, meaning $\mathbb{F}^k_\varepsilon$-subsolution inequalities $(k=1..3)$ coupled with a $\mathbb{F}^3_\varepsilon$-supersolution inequality.

We do not detail every Hamiltonian since they are similar to those that we used in Theorem 28.1.1. Notice that we only get $\limsup^*\mathbf{BCL}_\varepsilon(x,t)\supset\mathbf{BCL}_*(x,t)$ because the limsup contains any possible relaxed limit of φ_ε, not only φ_*.

Using the convergence results of Theorem 22.2.4 and Corollary 22.1.5 we see that u_ε converges locally uniformly to the stratifed solution u, associated with the set $\limsup^*\mathbf{BCL}_\varepsilon$.

However, when minimizing the cost functional, it is clear that the relaxed limits of φ_ε which lie above φ_* are not taken into account since they induce a greater cost. It

follows that u can be seen as the stratified solution associated with the set $\mathbf{BCL}_*$, which implies the result.

Q.E.D.

28.2 Singularities in Neuman problems

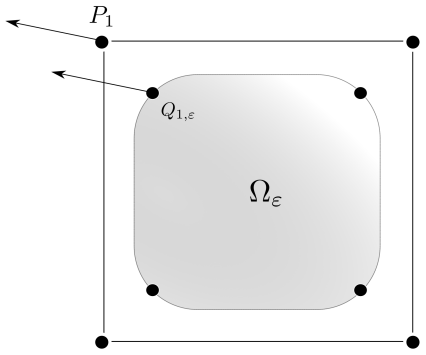


Figure 28.2: Approximation of Neuman problem

Chapter 29

Further Discussions and Open Problems

Part VI

Investigating Other Applications

29.1 Introduction

The aim of this part is double: on one hand, we are going to investigate as completely as possible KPP-type problems and show to obtain general results by using all the ressources of this book; this means both to look at the case of codimension 1 discontinuities but also more general ones where the stratified approach is needed. On the other hand, we are going to outline further developments in the directions of problems with jumps or networks: this will be the opportunity to explain what kind of jumps can be taken into account by our general assumptions on the control problems (on the **BCL**) and the possible (almost free) extensions of the stratified approach.

Chapter 30

KPP-Type Problems with Discontinuities

30.1 Introduction: KPP equations and front propagations

In this chapter, we are interested in Kolmogorov-Petrovsky-Piskunov [126] type equations (KPP in short), whose simplest form is

$$u_t - \frac{1}{2}\Delta u = cu(1-u) \quad \text{in } \mathbb{R}^N \times (0,+\infty)\ , \tag{30.1}$$

where c is a nonnegative constant.

Such reaction-diffusion equation appears in several different models in Physics (combustion for example) and Biology (typically for the evolution of population) and, in all these applications, one of the main interest comes front the large time behavior of the solutions which is mainly described in terms of front propagations. One of the main ingredients to understand this behavior is the study of the existence of *travelling waves solutions*, i.e. solutions which can be written as

$$u(x,t) := q(x \cdot e - \alpha t)\ ,$$

where $q : \mathbb{R} \to [0,1]$ is a smooth enough function, $e \in \mathbb{R}^N$ is such that $|e| = 1$, and $\alpha \in \mathbb{R}$. The travelling wave connects the instable equilibrium $u = 0 = q(-\infty)$ with the stable one $u = 1 = q(+\infty)$.

The connection between these travelling waves solutions and front propagation phenomenon is clear: the existence of such a solution implies that the hyperplanes

$x \cdot e = constant$ propagate with a normal velocity α. And clearly, understanding the propagation of such flat fronts is a key step towards dealing with more complicated fronts.

The case of KPP Equations is complicated in terms of travelling waves: while for other nonlinearities (for example cubic non linearities like $f(u) = (u-\mu)(1-u^2)$) there exists a unique characteristic velocity, KPP Equations admit a critical velocity $\alpha^* > 0$ such that travelling waves solutions exist for all $\alpha \geq \alpha_*$. And it is well-known that the large time behavior of the solutions (in particular the choice(s) of the velocity) depends on the behavior at infinity of the initial data. Actually this large time behavior can be rather complicated since it can be explained by the "mixing" of several different travelling waves as explained in Hamel and Nadirashvili [108].

We are going to concentrate here on the case where the minimal velocity α_* is selected. In this case it is known that $\alpha_* = \sqrt{2c}$ and that the large time behavior of the solutions of the KPP Equation is described by a front propagating with a $\sqrt{2c}$ normal velocity, where the front separates the regions where u is close to 0 and to 1.

In order to prove this result, Freidlin [95] introduced a scaling in space and time $(x,t) \to (\frac{x}{\varepsilon}, \frac{t}{\varepsilon})$ which has the double advantage to preserve the velocities and to allow to observe in finite times the large time behavior of the solution by examining the behavior of the scaled solution as $\varepsilon \to 0$. Hence one has to study the behavior when $\varepsilon \to 0$ of

$$u_\varepsilon(x,t) = u\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon}\right) ,$$

which solves the singular perturbation problem

$$(u_\varepsilon)_t - \frac{\varepsilon}{2}\Delta u_\varepsilon = \frac{c}{\varepsilon} u_\varepsilon(1-u_\varepsilon) \quad \text{in } \mathbb{R}^N \times (0,+\infty) .$$

We complement this pde with the initial data

$$u_\varepsilon(x,0) = g(x) \quad \text{in } \mathbb{R}^N ,$$

where $g : \mathbb{R}^N \to \mathbb{R}$ is a compactly supported continuous function satisfying $0 \leq g(x) \leq 1$ in $\mathbb{R}^N$.

The reader might be surprised by this unscaled initial data but, in this approach, the role of g is just to initialize the position of the front, given here by the boundary of the support of g, $\Gamma_0 := \partial\,\mathrm{supp}(g)$.

In this context, the following properties can be proved:

$$u_\varepsilon(x,t) = \exp\left(-\frac{I(x,t)+o(1)}{\varepsilon}\right) ,$$

where I is the unique viscosity solution of the variational inequality

$$\min\left(I_t + \frac{1}{2}|DI|^2 + c, I\right) = 0 \quad \text{in } \mathbb{R}^N \times (0, +\infty) \ ,$$

with

$$I(x,0) = \begin{cases} 0 & \text{if } x \in G_0 \\ +\infty & \text{if } x \in \mathbb{R}^N \backslash G_0. \end{cases}$$

oreover $I = \max(J, 0)$ where J is the unique viscosity solution of

$$J_t + \frac{1}{2}|DJ|^2 + c = 0 \quad \text{in } \mathbb{R}^N \times (0, +\infty) \ .$$

The importance of this second part of the result is to allow for an easy computation of J, and therefore I, through the Oleinik-Lax formula

$$J(x,t) = \frac{[d(x, G_0)]^2}{t} - ct \ ,$$

where $G_0 = \text{supp}(g)$. Hence $u_\varepsilon(x,t) \to 0$ in the domain $\{I > 0\} = \{J > 0\} = \{d(x, G_0) > \sqrt{2c}t\}$ and it can be shown that $u_\varepsilon(x,t) \to 1$ in the interior of the set $\{I = 0\} = \{J \leq 0\} = \{d(x, G_0) \leq \sqrt{2c}\, t\}$. Therefore the propagating front is $\Gamma_t = \{d(x, G_0) = \sqrt{2c}\, t\}$ which means a propagation with normal velocity $\sqrt{2c}$ as predicted by the travelling waves.

Such kind of results, in the more general cases of x, t dependent velocities $c(x,t)$, diffusion and drift terms, were obtained by Freidlin [95] using probabilistic Large Deviation type methods and later pdes' proofs, based on viscosity solutions' arguments, were introduced by Evans and Souganidis [77, 76]. They were then developed not only for KPP Equations but for other reaction-diffusion equations by Barles, Evans and Souganidis [20], Barles, Bronsard and Souganidis [19], Barles, Georgelin and Souganidis [21]. Later, these front propagation problems were considered in connections with the "levet-set approach": one of the first articles in this direction was the one by Evans, Soner and Souganidis [75] (see also Barles, Soner and Souganidis [26]). The most general results in this direction are obtained through the "geometrical approach" of Barles and Souganidis [40]. A complete overview of all these developments can be found in the CIME course of Souganidis [159] where a more complete list of references is given.

Of course, the aim of the section is to extend the results for KPP Equations to the case of discontinuous diffusions, drifts and reaction terms.

Before doing so, we come back to the main steps of the above mentioned result

1. Introduce the change of variable $I_\varepsilon := -\varepsilon \log(u_\varepsilon)$ and show that I_ε is uniformly locally bounded.

2. Pass to the limit by using the half-relaxed limit method in the equation satisfied by I_ε.

3. Prove a strong comparison result for the variational inequality which allows to prove that $I_\varepsilon \to I$ locally uniformly in $\mathbb{R}^N \times (0, +\infty)$.

4. Show that $I = \max(J, 0)$, when this is true (see just below).

All these steps are classical, except perhaps the last one which is related to the *Freidlin condition*: J is given by a formula of representation given by the associated control problem and Freidlin's condition holds if the optimal trajectories for points (x, t) such that $J(x, t) > 0$ remain in the domain $\{J > 0\}$.

It is worth pointing out that this condition is not always satisfied, but keep in mind that this fourth step is only used to give a simplest form to the result.

30.2 A simple discontinuous example

In order to introduce discontinuities in the KPP Equation, but also to point out an interesting feature of the fronts associated to this equation, let us consider a 1-d example borrowed from Freidlin's book [95] in which we have

$$u_t - \frac{1}{2}\Delta u = c(x)u(1-u) \quad \text{in } \mathbb{R} \times (0, +\infty)\ ,$$

where $c(x) = c_1$ if $x < 1$ and $c(x) = c_2$ if $x \geq 1$. We also assume that $G_0 = (-\infty, 0)$, i.e. the front is located at $x = 0$ initially.

For the control formulation for the function J, we follow the approach of Part II: for $x \in \Omega_1 := \{x < 1\}$, we set

$$\mathbf{BCL}_1(x, t) := \left\{ \Big(v_1\ ,\ 0\ ,\ -c_1 \Big);\ v_1 \in \mathbb{R}^N \right\}.$$

and for $x \in \Omega_2 := \{x > 1\}$, we set

$$\mathbf{BCL}_2(x, t) := \left\{ \Big(v_1\ ,\ 0\ ,\ -c_2 \Big);\ v_2 \in \mathbb{R}^N \right\}.$$

Therefore the cost $-c$ is discontinuous at $x = 1$.

The following formula allows to compute explicitly function J

$$J(x,t) = \inf\left\{ \int_0^t \left(\frac{|\dot{y}(s)|^2}{2} - c(y(s)) \right) ds \ ; \ y \in H^1(0,t),\ y(0) = x,\ y(t) \leq 0 \right\} .$$

Notice that, a priori, we should have been careful with this formal formula since the function c is discontinuous at $x = 1$ but, at this point of the book, it should be clear for the reader that the present situation is quite easy to handle. Indeed, if the trajectory stays on the line $x = 1$, we are just going to choose $c(y(s)) = \max(c_1, c_2)$, which yields the minimal possible cost, and with simply a tangential dynamic.

From now on, we assume for our purpose that $c_2 > c_1$. And we adress the question: when does the front, starting from $x = 0$ reach the value 1? If we just consider the domain $x < 1$, the answer should be $t_1 = \left(\sqrt{2c_1}\right)^{-1}$ since the velocity of the front is $\sqrt{2c_1}$ in this domain.

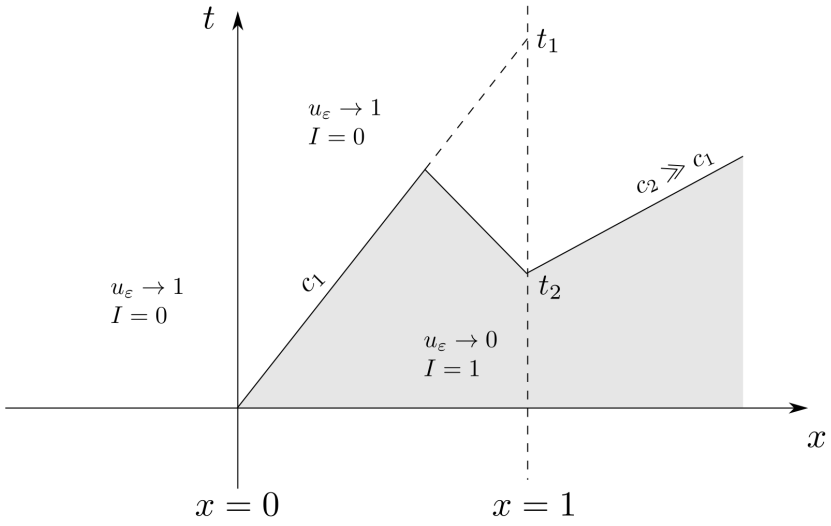
But we may also examine $J(1,t)$ and compute the smallest t for which it is 0 (which indeed corresponds to $u = 1$). It is clear that an optimal trajectory should stay at $x = 1$ on an interval $[0,h]$ and then a straight line to reach $x = 0$. Therefore

$$J(1,t) = \min_{0 \leq h \leq t} \left(-c_2 h + \frac{1}{2(t-h)} - c_1(t-h) \right) .$$

An easy computation gives

$$J(1,t) = \begin{cases} \dfrac{1}{2t} - c_1 t & \text{if } t \leq \dfrac{1}{\sqrt{2(c_2 - c_1)}} \\ \sqrt{2}\sqrt{c_2 - c_1} - c_2 t & \text{otherwise} \end{cases} ,$$

and the front reaches 1 either at time $t_1 = \left(\sqrt{2c_1}\right)^{-1}$ or $t_2 = \sqrt{2}\dfrac{\sqrt{c_2 - c_1}}{c_2}$, depending which one is the smallest one. Hence, an easy computation shows that, if $c_2 > 2c_1$, we have $t_2 < t_1$ and, for the front, a picture like



We observe at time t_2 a strange phenomenon: a new front is created at $x = 1$, ahead of the front travelling in Ω_1 with velocity $\sqrt{2c_1}$. This kind of phenomenon can arise even if $c(x)$ is continuous but the computations are easier to describe in the discontinuous setting. We also point out that Freidlin's condition holds true in this example.

In the next sections, we first provide results for general KPP Equations in the framework of Part II, i.e. in the case when we have discontinuities on an hyperplane. Then we consider some extensions to more general type of discontinuities which uses some particular features of the KPP Equations.

30.3 KPP-type problems with discontinuities (I): the co-dimension 1 case

With the notations of Part II, we consider the problem

$$(u_\varepsilon)_t - \frac{\varepsilon}{2}\mathrm{Tr}(a(x)D^2u_\varepsilon) - b(x)\cdot Du_\varepsilon = \frac{1}{\varepsilon}f(x,u_\varepsilon) \quad \text{in } \mathbb{R}^N \times (0,+\infty)\,, \tag{30.2}$$

where, in Ω_i, $a = a^{(i)}$, $b = b^{(i)}$, $f = f^{(i)}$ for $i = 1, 2$, where $a^{(i)}, b^{(i)}, f^{(i)}$ are bounded Lipschitz continuous functions taking values respectively in $\mathcal{S}^N$, $\mathbb{R}^N$ and $\mathbb{R}$. We assume that the following additional properties hold

(Uniform ellipticity) there exists $\nu > 0$ such that

$$a^{(i)}(x)p \cdot p \geq \nu |p|^2 \quad \text{for any } x, p \in \mathbb{R}^N \, . \tag{30.3}$$

(KPP-nonlinearity) For $i = 1, 2$ and for any $x \in \Omega_i$: $u \mapsto f^{(i)}(x, u)$ is differentiable at 0 and for any $u \in [0, 1]$

$$\begin{cases} f^{(i)}(x, 0) = f^{(i)}(x, 1) = 0, \quad f^{(i)}(x, u) > 0 \quad \text{if } 0 < u < 1 \\ c^{(i)}(x) = \dfrac{\partial f^{(i)}}{\partial u}(x, 0) = \sup\limits_{0<u<1} \left(\dfrac{f^{(i)}(x, u)}{u} \right), \end{cases} \tag{30.4}$$

with $c^{(i)}$ being bounded Lipschitz continuous on $\overline{\Omega_i}$.

Of course, the prototypal example of $f^{(i)}$ is $f^{(i)}(x, u) = c^{(i)}(x)u(1 - u)$ which is not a globally Lipschitz continuous function of u but since all the solutions u_ε will take values in $[0, 1]$, this is not a problem.

Next we complement (30.2) with the initial data

$$u_\varepsilon(x, 0) = g(x) \quad \text{in } \mathbb{R}^N \, , \tag{30.5}$$

where $g : \mathbb{R}^N \to \mathbb{R}$ is a compactly supported continuous function such that $0 \leq g(x) \leq 1$ in $\mathbb{R}^N$. As above we denote by G_0 the support of g which is assumed to be a non-empty compact subset of $\mathbb{R}^N$ with

$$\overline{\mathrm{Int}(G_0)} = G_0 \, .$$

In order to formulate the result, we introduce the Hamiltonians defined for $i = 1, 2$ by

$$H_i(x, p) := \frac{1}{2} a^{(i)}(x)p \cdot p - b^{(i)}(x) \cdot p + c^{(i)}(x) \, .$$

As we already noticed in the previous subsection, keep in mind that in the control viewpoint, the cost is $l^{(i)} = -c^{(i)}$.

Theorem 30.3.1 *As $\varepsilon \to 0$, we have*

$$-\varepsilon \log(u_\varepsilon) \to I \quad \textit{locally uniformly in } \mathbb{R}^N \times (0, +\infty) \, ,$$

where I can be seen as either the unique solution of

$$\begin{cases} \min(I_t + H_i(x, DI), I) = 0 \qquad \textit{in } \Omega_i \times (0, +\infty) \, , \\ I(x, 0) = \begin{cases} 0 & \textit{if } x \in G_0, \\ +\infty & \textit{otherwise} \, , \end{cases} \end{cases} \tag{30.6}$$

associated to the Kirchhoff condition

$$\frac{\partial I}{\partial n_1} + \frac{\partial I}{\partial n_2} = 0 \quad \text{on } \mathcal{H} \times (0, +\infty) \,, \tag{30.7}$$

or equivalently, the maximal Ishii solution of variational inequality (30.6) in $\mathbb{R}^N \times (0, +\infty)$. *oreover we have*

$$u_\varepsilon(x,t) \to \begin{cases} 0 & \text{in } \{I > 0\}, \\ 1 & \text{in the interior of the set } \{I = 0\}. \end{cases}$$

Finally if Freidlin's condition holds then $I = \max(J, 0)$ *where* J *is either the unique solution of*

$$\begin{cases} J_t + H_i(x, DJ) = 0 & \text{in } \Omega_i \times (0, +\infty) \,, \\ J(x,0) = \begin{cases} 0 & \text{if } x \in G_0, \\ +\infty & \text{otherwise} \,, \end{cases} \end{cases} \tag{30.8}$$

associated to the Kirchhoff condition, or equivalently the maximal Ishii solution of (30.8) in $\mathbb{R}^N \times (0, +\infty)$. *Function* J *is given by the following representation formula*

$$J(x,t) = \inf \left\{ \int_0^t l(y(s), \dot{y}(s)) ds; \ y(0) = x, \ y(t) \in G_0, \ y \in H^1(0,t) \right\} \,,$$

where $l(y(s), \dot{y}(s)) = \dfrac{1}{2}[a^{(i)}(y(s))]^{-1}(\dot{y}(s) - b^{(i)}(y(s))) \cdot (\dot{y}(s) - b^{(i)}(y(s))) - c^{(i)}(y(s))$ *if* $y(s) \in \Omega_i$ *and with the regular control procedure on* $\mathcal{H} \times (0, +\infty)$.

We can summarize this result by saying that the "usual" KPP-result holds true provided that the "action functional" J is suitably defined, taking only regular controls on $\mathcal{H} \times (0, +\infty)$, using the links between the maximal Ishii viscosity solution, flux-limited solutions and junction viscosity solutions for the Kirchhoff condition.

Remark 30.3.2 *As the previous paragraph suggests, the proof of Theorem 30.3.1 uses the most sophisticated results and tools of this part, combining the different approaches and their connections. We refer the reader to Section 30.5 where a different point of view is described with the aim of treating more general discontinuities. That point of view consists in checking whether it is possible to conclude by using only the notion of Ishii viscosity solution.*

Proof — The proof relies on classical arguments which remains valid because of the results of Theorem 30.4.2 given in Section 30.4.

The aim is make the change of variable

$$I_\varepsilon(x,t) = -\varepsilon \log(u_\varepsilon(x,t))$$

and to show that $I_\varepsilon \to I$ locally uniformly in $\mathbb{R}^N \times (0,+\infty)$. But, in order to do so, we first need local uniform bounds on I_ε.

We first notice that, by the Maximum Principle, we have

$$0 \leq u_\varepsilon(x,t) \leq 1 \quad \text{in } \mathbb{R}^N \times (0,+\infty)\ ,$$

and therefore $I_\varepsilon(x,t) \geq 0$ in $\mathbb{R}^N \times (0,+\infty)$. In addition, I_ε is well-defined because $u_\varepsilon(x,t) > 0$ in $\mathbb{R}^N \times (0,+\infty)$ by the Strong Maximum Principle.

Getting an upper bound on I_ε is done by using the trick introduced in [23, 24] (the reader can look in those references for the details which follow): we set

$$I_\varepsilon^A(x,t) = -\varepsilon \log\left(u_\varepsilon(x,t) + \exp(-A/\varepsilon)\right)\ ,$$

where $A \gg 1$. Then $o(1) \leq I_\varepsilon^A(x,t) \leq A$, and it is easy to show that

$$\limsup{}^*\ I_\varepsilon^A = \min(\limsup{}^*\ I_\varepsilon, A)\ .$$

Therefore controlling I_ε^A uniformly in A provides the same control on I_ε.

Next, using that $f^{(i)}(x,u_\varepsilon) \geq 0$ in $\mathbb{R}^N \times (0,+\infty)$, the function I_ε^A satisfies

$$(I_\varepsilon^A)_t - \frac{\varepsilon}{2}\mathrm{Tr}(a^{(i)}(x)D^2 I_\varepsilon^A) + \frac{1}{2}a^{(i)}(x)DI_\varepsilon^A \cdot DI_\varepsilon^A - b^{(i)}(x)\cdot DI_\varepsilon^A \leq 0 \quad \text{in } \Omega_i \times (0,+\infty)\ ,$$

and the ellipticity assumption together with a Cauchy-Schwartz inequality on the $b^{(i)}$-term leads to

$$(I_\varepsilon^A)_t - \frac{\varepsilon}{2}\mathrm{Tr}(a^{(i)}(x)D^2 I_\varepsilon^A) + \frac{1}{2}\nu|DI_\varepsilon^A|^2 \leq k(\nu) \quad \text{in } \Omega_i \times (0,+\infty)\ ,$$

for some constant $k(\nu)$ large enough, depending only on $\|b^{(i)}\|_\infty$ and ν.

Passing to the limit through the half-relaxed limit method and setting $\bar{I}_A = \limsup{}^*\ I_\varepsilon^A$, we get the following inequality for $\bar{I}_A$:

$$\begin{cases} (\bar{I}_A)_t + \frac{1}{2}\nu|D\bar{I}_A|^2 \leq k(\nu) & \text{in } \mathbb{R}^N \times (0,+\infty)\ , \\ \bar{I}_A(x,0) = \begin{cases} 0 & \text{if } x \in G_0, \\ A & \text{otherwise}\ . \end{cases} \end{cases}$$

The Oleinik-Lax formula then implies

$$\bar{I}_A(x,t) \leq \frac{[d(x,G_0)]^2}{2\nu t} + k(\nu)t \quad \text{in } \mathbb{R}^N \times (0,+\infty)\ ,$$

which is the desired uniform bound.

Therefore we can perform the I_ε change of function and we obtain

$$(I_\varepsilon)_t - \frac{\varepsilon}{2}\mathrm{Tr}(a^{(i)}(x)D^2 I_\varepsilon) + \frac{1}{2}a^{(i)}(x)DI_\varepsilon \cdot DI_\varepsilon - b^{(i)}(x)\cdot DI_\varepsilon \le -\frac{f^{(i)}(x,u_\varepsilon)}{u_\varepsilon} \quad \text{in } \Omega_i \times (0,+\infty)\ ,$$

where we have kept the notation u_ε in the right-hand side to emphasize the role of the quantity $f^{(i)}(x,u_\varepsilon)/u_\varepsilon$. Indeed we have both

$$-\frac{f^{(i)}(x,u_\varepsilon)}{u_\varepsilon} \ge -c^{(i)}(x) \quad \text{for any } x\ ,$$

and

$$-\frac{f^{(i)}(x,u_\varepsilon)}{u_\varepsilon} \to -c^{(i)}(x) \quad \text{if } u_\varepsilon(x,t) \to 0\ ,$$

and this last case occurs if $I_\varepsilon(x,t)$ tends to a strictly positive quantity.

Using these properties, Theorem 17.5.1 implies that $\overline{I} = \limsup^* I_\varepsilon$ and $\underline{I} = \liminf_* I_\varepsilon$ are respectively sub and supersolutions of the variational inequality (30.6) associated with Kirchhoff condition on $\mathcal{H}$.

In order to conclude, we have just to use Theorem 30.4.2: with the notations of this result, we have

$$\overline{I}(x,t) \le I^+(x,t) \le \underline{I}(x,t) \quad \text{in } \mathbb{R}^N \times (0,+\infty)\ ,$$

and, I^+ being continuous, this implies that $I_\varepsilon \to I^+$ locally uniformly in $\mathbb{R}^N \times (0,+\infty)$.

The proof is complete since the other results can be obtained exactly as in the standard KPP case.

Q.E.D.

30.4 The variational inequality in the case of a codimension 1 discontinuity

In this section, we study the control/game problems related to the functions I and J arising in the statement of Theorem 30.3.1, together with the properties of the associated Bellman equation or variational inequality.

To do so, we follow the approach of Part II: for $x \in \Omega_i$, we set

$$\mathbf{BCL}_i(x,t) := \left\{ \left(v_i\ ,\ 0\ ,\ \frac{1}{2}[a^{(i)}(x)]^{-1}(v_i - b^{(i)}(x)) \cdot (v_i - b^{(i)}(x)) - c^{(i)}(x) \right);\ v_i \in \mathbb{R}^N \right\}.$$

This definition can be seen as using v as a control, hence authorizing any possible dynamic $v \in \mathbb{R}^N$ at any point (x,t), but with a cost $l(x,v) = l_i(x,v) = \frac{1}{2}[a^{(i)}(x)]^{-1}(v - b^{(i)}(x)) \cdot (v - b^{(i)}(x)) - c^{(i)}(x)$ if $x \in \Omega_i$.

Of course, we are in an unbounded control framework but this does not create any major additional difficulty.

It remains to define the dynamic/cost on $\mathcal{H}$ (regular or not) and for $x \in \mathcal{H}$, we have: $(v,0,l) \in \mathbf{BCL}_T(x,t)$ if $v \in \mathcal{H}$, $v = \alpha v_1 + (1-\alpha)v_2$ and

$$l = \alpha l_1(x,v_1) + (1-\alpha) l_2(x,v_2) \ ,$$

where as above, $l_i(x,v_i) = \frac{1}{2}[a^{(i)}(x)]^{-1}(v_i - b^{(i)}(x)) \cdot (v_i - b^{(i)}(x)) - c^{(i)}(x)$. The set $\mathbf{BCL}_T^{\mathrm{reg}}(x,t)$ is defined in the same way, adding the condition $v_1 \cdot e_N \leq 0$, $v_2 \cdot e_N \geq 0$.

If $I_0 \in C_b(\mathbb{R}^N)$, we introduce

$$J^-(x,t) = \inf_{\mathcal{T}_{x,t}} \left\{ \int_0^t l(X(s), \dot{X}(s)) ds + I_0(X(t)) \right\} \ ,$$

$$J^+(x,t) = \inf_{\mathcal{T}_{x,t}^{\mathrm{reg}}} \left\{ \int_0^t l(X(s), \dot{X}(s)) ds + I_0(X(t)) \right\} \ ,$$

where, in these formulations, we have replaced $v_i(s)$ $(i = 1,2)$ or $v(s)$ by $\dot{X}(s)$.

In the same way, we introduce

$$I^-(x,t) = \inf_{\mathcal{T}_{x,t,\theta}} \left\{ \int_0^{t\wedge\theta} l(X(s), \dot{X}(s)) ds + \mathbb{1}_{t<\theta} I_0(X(t)) \right\} \ ,$$

$$I^+(x,t) = \inf_{\mathcal{T}_{x,t}^{\mathrm{reg}},\theta} \left\{ \int_0^{t\wedge\theta} l(X(s), \dot{X}(s)) ds + \mathbb{1}_{t<\theta} I_0(X(t)) \right\} \ .$$

Following the methods of Part II, it is easy to show the following result

Theorem 30.4.1

(*i*) *The value-functions J^- and J^+ are continuous and respectively the minimal Ishii supersolution (and solution) and maximal Ishii subsolution (and solution) of the equation*

$$J_t + H(x, DJ) = 0 \quad \text{in } \mathbb{R}^N \times (0, +\infty) \ , \tag{30.9}$$

where $H = H_i$ in $\Omega_i \times (0,+\infty)$ with the initial data

$$J(x,0) = I_0(x) \quad \text{in } \mathbb{R}^N \ .$$

(*ii*) **(SCR)** *holds for the flux-limited problems for Equation* (30.9) *with flux-limiters* H_T *and* $H_T^{\rm reg}$ *on* $\mathcal{H}$*;* J^- *is the unique flux-limited solution associated to the flux-limiter* H_T *and* J^+ *is the unique flux-limited solution associated to the flux-limiter* $H_T^{\rm reg}$*.* J^+ *is also the unique solution associated to the Kirchhoff condition on* $\mathcal{H}$*.*

(*iii*) *The value-functions* I^- *and* I^+ *are continuous and respectively the minimal Ishii supersolution (and solution) and maximal Ishii subsolution (and solution) of the equation*

$$\min(I_t + H(x, DI), I) = 0 \quad \text{in } \mathbb{R}^N \times (0, +\infty)\ , \tag{30.10}$$

where $H = H_i$ *in* $\Omega_i \times (0, +\infty)$ *with the initial data*

$$I(x, 0) = I_0(x) \quad \text{in } \mathbb{R}^N\ .$$

(*iv*) **(SCR)** *holds for the flux-limited problems for the variational inequality* (30.10) *with flux-limiters* H_T *and* $H_T^{\rm reg}$*;* I^- *is the unique flux-limited solution associated to the flux-limiter* H_T *and* I^+ *is the unique flux-limited solution associated to the flux-limiter* $H_T^{\rm reg}$*.* I^+ *is also the unique solution associated to the Kirchhoff condition on* $\mathcal{H}$*.*

In order to treat the KPP problem, we have to extend this result to the case of discontinuous I_0, with possibly infinite values. Of course, stricto sensu, a **(SCR)** cannot hold in this case. Indeed, if u and v are respectively a sub and supersolution of either (30.9) or (30.10) with initial data I_0, the inequalities at time $t = 0$ are

$$u(x, 0) \le I_0^*(x) \quad \text{and} \quad v(x, 0) \ge (I_0)_*(x) \quad \text{in } \mathbb{R}^N\ ,$$

and it is false in general that $u(x, 0) \le v(x, 0)$ in $\mathbb{R}^N$. Therefore we have to extend the meaning of **(SCR)** by saying that a **(SCR)** holds in this context if we have

$$u(x, t) \le v(x, t) \quad \text{in } \mathbb{R}^N \times (0, +\infty)\ ,$$

hence for all $t > 0$.

With this modified definition, we can formulate a simple result which is exactly what we need (we do not try to reach the full generality here):

Theorem 30.4.2 *Assume that* $\overline{\mathrm{Int}(G_0)} = G_0$*, then the results of Theorem 30.4.1 remain true if* $I_0(x) = A\mathbb{1}_{G_0}$ *for some* $A > 0$*, and even if* $A = +\infty$*.*

Proof — We begin with the case when $A < +\infty$ and we provide the full proof only in the I-case, the J-one being obtained by similar and even simpler arguments.

Step 1: *Approximation of the data.*

In order to prove the analogue of (*iii*), we can approximate I_0 by above and below by sequences $((I_0)^A)_A$ and $((I_0)_A)_A$ of bounded continuous initial data such that

$$(I_0)^A \downarrow I_0^* \quad \text{and} \quad (I_0)_A \uparrow (I_0)_* \, .$$

We denote by $(I^A)^{\pm}$ and $(I_A)^{\pm}$ the minimal and maximal solutions given by Theorem 30.4.1 with these intial data.

If u, v are respectively a subsolution and a supersolution of the variational inequality with initial data I_0, they are respectively subsolution with $(I_0)^A$ and supersolution with $(I_0)_A$. Therefore

$$u \leq (I^A)^+ \quad \text{and} \quad (I_A)^- \leq v \quad \text{in } \mathbb{R}^N \times (0, +\infty) \, .$$

It remains to pass to the limit in the variational formulas for $(I^A)^+$ and $(I_A)^-$. This step is easy for $(I^A)^-$ by the stability of solutions of differential inclusion (one has just to be careful of the fact that we obtain $(I_0)_*$ in the formula at the limit).

For $(I^A)^+$, things are more delicate since we have to deal with regular trajectories. But here, we can take advantage of the inequality we wish to show and first argue with a FIXED trajectory (here also one has to be careful because we obtain $(I_0)^*$ in the formula at the limit).

Step 2: *Both functions $(I^A)^+$ and $(I^A)^-$ are continuous.*

In order to prove the claim, we can use the approach of the authors in [36], showing that $I = (I^A)^-$ or $(I^A)^+$ both satify

$$-\eta(t) \leq I_t(x,t) \leq C \, .$$

for some positive function η which may tend to $+\infty$ when $t \to 0$ and for some constant C. This inequality is obtained by using the arguments of [36]: we just use a sup-convolution in time

$$\sup_{0 \leq s \leq t} \left(I(x,s) - \eta(s)(t-s) \right) ,$$

and combine it with a comparison result for flux-limited solutions (with the suitable flux-limiter for $(I^A)^-$ and $(I^A)^+$).

This argument shows that $(I^A)^-$ and $(I^A)^+$ are Lipschitz continuous in x (for $t > 0$) where they are strictly positive. Indeed, if $I > 0$, variational inequality (30.10) implies that $H(x, DI) = -I_t \leq \eta(t)$, and the coercivity of H implies a bound on DI. Then,

it is a simple exercice to extend it to all points in $\mathbb{R}^N \times (0, +\infty)$, whether $I > 0$ or $I = 0$.

STEP 3: *Strong Comparison Result.*

For the proofs of the **(SCR)**, we still consider $(I_0)^A$, $(I_0)_A$ but the **(SCR)** for either H_T, $H_T^{\rm reg}$ or the Kirchhoff condition. In the case of $H_T^{\rm reg}$, for example, we obtain

$$u \le (I^A)^+ \quad \text{and} \quad (I_A)^+ \le v \quad \text{in } \mathbb{R}^N \times (0, +\infty) \, .$$

To conclude in this case, we have to use Proposition 9.5.1 to pass to the limit by extracting a sequence of trajectories which converges to a regular trajectorie. The case of H_T is simpler.

STEP 4: *Passing to the limit to treat the case $A = \infty$.*

In the case where $A = +\infty$, we first notice that all solutions associated with initial data like $I_0(x) = A\mathbb{1}_{G_0}$, and slightly enlarging or slightly reducing the set G_0 are uniformly locally bounded with respect to A (this can be obtained by choosing appropriate trajectories such as straight lines). And the limiting function are

$$I^-(x,t) = \inf_{\mathcal{T}_{x,t},\theta} \left\{ \int_0^{t\wedge\theta} l(X(s), \dot{X}(s))ds; \; X(t) \in G_0 \right\} \, ,$$

$$I^+(x,t) = \inf_{\mathcal{T}^{\rm reg}_{x,t},\theta} \left\{ \int_0^{t\wedge\theta} l(X(s), \dot{X}(s))ds; \; X(t) \in G_0 \right\} \, .$$

Now, if u is a subsolution then, for all A and $C = \max_i(||c_i||_\infty)$, $\min(u, A - Ct)$ is also a subsolution associated to the initial data $A\mathbb{1}_{G_0}$. Indeed, sonce the Hamiltonians are convex, the infimum of two subsolutions remains a subsolution. We then use the first result to conclude. We can use a similar argument for the supersolution, using this time a comparison with $(I_A)^\pm$, depending of the result we want.

Q.E.D.

30.5 KPP-type problems with discontinuities (II): remarks on more general discontinuities

In the proof of Theorem 30.3.1, even if we hide it carefully inside the proof of Theorem 30.4.2, we use in an essential way the various notions of solutions which are described in Part II, namely flux- limited solutions and junction solutions together with results concerning their links.

This heavy sophisticated machinery is a weakness if we want to address the case of more general discontinuities for which we are not able to provide such a precise analysis. Therefore, it is natural to investigate what can be done in those more general cases.

30.5.1 In which cases can we conclude by using only the standard notion of Ishii viscosity solution?

In the framework of Chapter 30, i.e. with only a discontinuity of codimension 1 on an hyperplan, the answer is straightforward and this can be seen from two slightly different points of view: on one hand, in order to conclude, it is enough that the functions I^+ and I^- appearing in Theorem 30.4.1 and Theorem 30.4.2 are equal, and so are J^+ and J^-. Lemma 10.3.1 gives conditions under which this happens.

On the other hand, and this is a more general point of view, we can also look for conditions under which Ishii viscosity subsolutions are stratified subsolutions (since the supersolutions are the same). The conclusion then follows from the comparison result for stratified solutions. Since it is easy to see that, on the hyperplane, the $\mathbb{F}^N$-inequality on $\mathcal{H} \times (0, T_f)$ is the H_T-one, Lemma 10.3.1 still gives the answer.

In order to exploit this result, we recall that we have

$$H_i(x,p) := \frac{1}{2} a^{(i)}(x) p \cdot p - b^{(i)}(x) \cdot p + c^{(i)}(x) \ ,$$

and the computation of $m_1(x,p'), m_2(x,p')$ is easy:

$$m_i(x,p') = -\frac{1}{a^{(i)}(x) e_N \cdot e_N} \left(a^{(i)}(x) p' \cdot e_N - b^{(i)}(x) \cdot e_N \right) \ .$$

The condition $m_2(x,p') \geq m_1(x,p')$ for any (x,p') which is required in Lemma 10.3.1 in order to have $H_T = H_T^{\mathrm{reg}}$ leads to two properties by using the affine dependence in $p' \in \mathcal{H}$:

$$\frac{a^{(2)}(x) e_N}{a^{(2)}(x) e_N \cdot e_N} - \frac{a^{(1)}(x) e_N}{a^{(1)}(x) e_N \cdot e_N} = 0 \ , \tag{30.11}$$

and

$$\frac{b^{(2)}(x) \cdot e_N}{a^{(2)}(x) e_N \cdot e_N} \geq \frac{b^{(1)}(x) \cdot e_N}{a^{(1)}(x) e_N \cdot e_N} \ . \tag{30.12}$$

Indeed, the inequality $m_2(x,p') \geq m_1(x,p')$ for any (x,p') implies that the left-hand side of (30.11) is colinear to e_N while its scalar product with e_N is 0. Notice that in this computation, we have implicitly assumed that $N \geq 2$ but, if $N = 1$ the result remains true with only (30.12).

Under this condition, Theorem 30.3.1 can be proved using only the basic notion of viscosity solutions.

Remark 30.5.1 *Recalling that the costs for the associated control problems are*

$$\frac{1}{2}[a^{(i)}(x)]^{-1}(v - b^{(i)}(x)) \cdot (v - b^{(i)}(x)) - c^{(i)}(x)\,,$$

the stronger condition

$$\forall x \in \mathcal{H}\,, \quad b^{(2)}(x) \cdot e_N \geq 0 \geq b^{(1)}(x) \cdot e_N$$

is quite natural. Indeed with $b^{(1)}, b^{(2)}$ pointing towards $\mathcal{H}$, it is clear that a priori regular controls give better costs than singular ones. Condition (30.12) generalizes this simple case.

30.5.2 Can we go further?

The answer is yes following Section 21.2 and in particular, Proposition 21.2.3, even if it leads to very restrictive assumptions.

We consider the following example in the "cross case": consider Equations (30.2) which holds in $Q_i \subset \mathbb{R}^2$ where the Q_i's are the four quadrants in $\mathbb{R}^2$, namely

$$Q_1 = \{x_1 > 0, x_2 > 0\}\,, Q_2 = \{x_1 < 0, x_2 > 0\}\,,\ Q_3 = -Q_1\,,\ Q_4 = -Q_2\,.$$

To be able to apply Proposition 21.2.3, we assume that, for $i = 1, 2, 3, 4$, the $b^{(i)}$ are equal to 0 and that $a^{(i)}(x) = \lambda^{(i)}(x)Id$ in Q_i for some bounded, Lipschitz continuous function $\lambda^{(i)}$. We assume also the existence of some constant $\nu > 0$ such that $\lambda^{(i)}(x) \geq \nu$ in Q_i for any i.

Then, under natural assumptions on the regularity of the coefficients, the asymptotics of u_ε can easily be obtained in this framework: indeed

(i) $\underline{I}$ is an Ishii viscosity supersolution of the variational inequality in $\mathbb{R}^2 \times (0, +\infty)$,

(ii) $\overline{I}$ turns out to be a "stratified subsolution" of the variational inequality in $\mathbb{R}^2 \times (0, +\infty)$. Indeed, on the axes (except 0), i.e. on $\mathbf{M}^2$, the above analysis shows that $H_T = H_T^{\rm reg}$ inequality holds for $\overline{I}$ and therefore the $\mathbb{F}^2$-one holds too. At $x = 0$ for $t > 0$, i.e. on $\mathbf{M}^1$, we clearly have

$$\min(\overline{I}_t + \max_i(c^{(i)}(x), \overline{I}) \leq 0\,,$$

because all the inequalities $\min(\overline{I}_t + c^{(i)}(x), \overline{I}) \leq 0$ hold by passage to the limit (stability) from the Q_i domain. This is a case where Proposition 21.2.3 applies in a very simple way.

Hence, $\underline{I}$ and $\overline{I}$ are respectively stratified super and subsolution of the variational inequality and we can conclude since the comparison result for stratified solutions easily extend to this framework.

Proposition 21.2.3 allows to treat the following kind of KPP problems: we assume that $\mathbb{M} = (\mathbf{M}^k)_{k=0..N}$ is a stratification of $\mathbb{R}^N$ and that in the framework of Chapter 30, the u_ε are solutions of Equation 30.2 where

1. $a^{(i)}(x) = \lambda^{(i)}(x) Id$ in Ω_i where the Ω_i are the connected components of $\mathbf{M}^N$. We assume that the functions $\lambda^{(i)}$ are uniformly bounded and Lipschitz continuous functions and there exists a constant $\nu > 0$ such that $\lambda^{(i)}(x) \geq \nu$ in Ω_i for any i.
2. $b = 0$ in $\mathbb{R}^N$.
3. $f = f^{(i)}$ in Ω_i where the $f^{(i)}$ are KPP-nonlinearities, the $c^{(i)}$ being uniformly bounded and Lipschitz continuous on $\overline{\Omega_i}$.

Under these conditions, and if the initial data g is as in Chapter 30, the result is the

Proposition 30.5.2 *As $\varepsilon \to 0$, we have*

$$-\varepsilon \log(u_\varepsilon) \to I \quad \textit{locally uniformly in } \mathbb{R}^N \times (0, +\infty) \ ,$$

where I is the unique stratified solution of the equation with

$$\left\{ \begin{array}{ll} \min(I_t + H_i(x, DI), I) = 0 & \textit{in } \Omega_i \times (0, +\infty) \ , \\ I(x,0) = \left\{ \begin{array}{ll} 0 & \textit{if } x \in G_0, \\ +\infty & \textit{otherwise} \ . \end{array} \right. \end{array} \right. \tag{30.13}$$

oreover we have

$$u_\varepsilon(x,t) \to \left\{ \begin{array}{ll} 0 & \textit{in } \{I > 0\}, \\ 1 & \textit{in the interior of the set } \{I = 0\}. \end{array} \right.$$

Finally if Freidlin's condition holds then $I = \max(J, 0)$ where J is the unique stratified solution of

$$\left\{ \begin{array}{ll} J_t + H_i(x, DJ) = 0 & \textit{in } \Omega_i \times (0, +\infty) \ , \\ J(x,0) = \left\{ \begin{array}{ll} 0 & \textit{if } x \in G_0, \\ +\infty & \textit{otherwise} \ . \end{array} \right. \end{array} \right. \tag{30.14}$$

Function J is given by the following representation formula

$$J(x,t) = \inf\left\{\int_0^t l(y(s),\dot{y}(s))ds;\ y(0)=x,\ y(t)\in G_0,\ y\in H^1(0,t)\right\}\ ,$$

where $l(y(s),\dot{y}(s)) = \dfrac{1}{2}[\lambda^{(i)}(y(s))]^{-1}|\dot{y}(s)|^2 - c^{(i)}(y(s))$ *if* $y(s)\in\Omega_i$.

Several remarks on this results
(i) We have left this result with a slightly imprecise statement, giving the equations only in $\mathbf{M}^N\times(0,T_f)$ and defining l only in $\mathbf{M}^N\times(0,T_f)$. The next section will (at least partially) show why this is enough.
(ii) As above in the "cross case", the proof that $\overline{I}$ is a stratified subsolution comes from the arguments given the next section.
(iii) The first part of this result holds for example in the counter-example in dimension 1 given in Chapter 30, the only point is that Freidlin's condition is not satisfied.

Chapter 31

And What About Jumps?

31.1 The jumps we already take into account

The question we want to address in this section: which kind of control problems with jumps the formalism of Chapter 4 can take into account? And perhaps a first question could be: does it even allow to treat some of these problems?

In the formalism of Chapter 4, the dynamic-discount-cost is defined by

$$(\dot{X}, \dot{T}, \dot{D}, \dot{L})(s) = (b, c, l)(s) \in \mathbf{BCL}(X(s), T(s)) .$$

In this differential inclusion, s can be seen as an "artificial time": the state of the system is actually $x = X(s)$ at the "real time" $t = T(s)$. In the equation for T, namely $\dot{T}(s) = b^t(s)$, we can have $b^t(s) = 0$ on some interval $[s_1, s_2]$. If this is the case, the behavior of X can be interpreted as a jump in the "real time": indeed it is clear that the trajectory $s \mapsto X(s)$ remains continuous but, at time $t = T(s_1) = T(s_2)$, we observe a jump for the trajectory X from $X(s_1)$ (value of X when we arrive at time t) to $X(s_2)$ (value of X when we leave t). Hence the answer to the second question above seems to be yes. At least some of them.

To further analyze the situation, we examine a "pure jump" situation where $b^x = v \in \overline{B(0,R)} \subset \mathbb{R}^N$, $b^t \equiv 0$, $c \equiv 0$ and $l = l(v)$. More precisely, $\mathbf{BCL}(x,t)$ is independent of x and t and

$$\mathbf{BCL} = \{((v,0), 0, l(v))\}$$

We assume that $l(0) = 0$ and the first consequence of the convexity of **BCL** is that $l(\alpha v) = \alpha l(v)$ for any $v \in \overline{B(0,R)}$. An other assumption which avoids oscillating trajectories is that if $v = v_1 + v_2$ then $l(v) \leq l(v_1) + l(v_2)$. As a consequence of these two properties of l, l is a convex function of v.

Now we turn to the resolution of a very simple problem which looks like a $\mathbb{F}_{init}$ problem: if u_0 is a continuous function, we want to solve

$$\max(h(D_x u), u - u_0) = 0 \quad \text{in } \mathbb{R}^N \ ,$$

where $h(p_x) = \max_{|v|\le R}(-v \cdot p_x - l(v))$. The control interpretation suggests the solution

$$u(x) := \inf\left\{\int_0^\theta l(\dot X(s))ds + u_0(X(\theta)) : \ X(0) = x, \ |\dot X(s)| \le R, \ \theta > 0\right\} \ .$$

But by Jensen's inequality and the homogeneity of l

$$\int_0^\theta l(\dot X(s))ds \ge \theta l\left(\frac{1}{\theta}\int_0^\theta \dot X(s)ds\right) = \theta l\left(\frac{1}{\theta}(X(\theta) - x)\right) = l\left(X(\theta) - x\right) \ ;$$

and therefore

$$u(x) = \inf\left\{u_0(X(\theta)) + l\left(X(\theta) - x\right) : \ X(0) = x, \ |\dot X(s)| \le R, \ \theta > 0\right\} \ ,$$

or equivalently

$$u(x) = \inf_{y\in\mathbb{R}^N}\left\{u_0(y) + l\left(y - x\right)\right\} \ ,$$

which can be interpreted as the minimal value which can be obtained by making a jump from x to $y = X(\theta)$ with a cost $l(y - x) = l\left(X(\theta) - x\right)$ for this jump.

This very simple example gives an idea of the type of jumps which can be taken into account by the framework of Chapter 4. The next section examines cases which may not enter into this framework but which can be handled.

31.2 The jumps we could take into account

In the control literature, jumps arise in particular in inventory management and lead to quasi-variational inequality (QVI in short). We refer the reader to Bensoussan and Lions [44] for a study of such QVI in the framework of stochastic control/elliptic-parabolic pdes, which was the first situation where they were studied.

In their book, the jumps play a role via an operator $\mathcal{M}$ which is typically of the form

$$\mathcal{M}u(x) := \min_{\xi\in\Xi}\left(u(x + \xi) + k + C(\xi)\right) \ ,$$

where Ξ is a bounded or unbounded subset of $\mathbb{R}^N$, $k \ge 0$ is fixed cost and C is a cost depending on the size of the jump. In general one assume that C is a continuous

function such that $C(\xi) \geq 0$ and $C(0) = 0$ if $0 \in \Xi$. If Ξ is unbounded, it is generally assumed that C is coercive. In [44], a typical case is when $\Xi = [0, +\infty)^N$.

In the QVI, the complete Hamiltonian $\mathbb{F}$ is of the form $\max(\cdots, u - \mathcal{M}u)$ and as a by-product of this form or of the control problem, one has the inequality $u \leq \mathcal{M}u$ in $\mathbb{R}^N$ or $\mathbb{R}^N \times [0, T_f]$.

A favorable situation is when $k > 0$, both for second-order HJB equations as in [44] but also for first-order HJ-Equations: we refer for example to [14, 15] for simple ideas to treat such QVI in the continuous framework, both from the control and pde points-of-view. We point out that the classical comparison results of the continuous case extend without any change of assumptions to the QVI-case. And, besides of the proof via the localization procedure which has to be adapted, there are also no major changes in the discontinuous framework.

The case when $k = 0$ is more complicated and even desesperate if $\min_{\xi \in \Xi} (C(\xi)) = 0$, in particular if $0 \in \Xi$: indeed, in this case, no comparison can hold since, for any constant $\bar{c}$, we have

$$\bar{c} - \mathcal{M}\bar{c} = \bar{c} - \min_{\xi \in \Xi} (\bar{c} + C(\xi)) = -\min_{\xi \in \Xi} (C(\xi)) = 0 \, .$$

Hence all the constants are supersolutions.

Therefore either we are more or less back to the case when $k > 0$ if $0 \notin \Xi$ or we have to reinterpret the QVI in terms of gradient constraints as we did it in the previous section in order that these cases enters into the theory. This means that we should have the classical sublinearity assumption

$$C(\xi_1 + \xi_2) \leq C(\xi_1) + C(\xi_2) \quad \text{for any } \xi_1, \xi_2 \, ,$$

to gether with the homogeneity of degree 1.

Clearly we are not going to study these cases in details but we want to point out that jumps can help by ensuring the regularity of subsolutions on $\mathbf{M}^k$ if $0 \in \Xi$: indeed subsolutions of QVI satisfy

$$u(x) \leq u(x + \xi) + C(\xi) \quad \text{for all } \xi \in \Xi \, ,$$

and if there exists a sequence $(\xi_\varepsilon)_\varepsilon$ converging to 0 such that $x + \xi_\varepsilon \notin \mathbf{M}^k$, this provides the regularity on $\mathbf{M}^k$. And, for $\mathbf{M}^N$, it suffices to adapt this assumption to have the regularity from both side, and this is in particular true if $B(0, \eta) \subset \Xi$ for some $\eta > 0$.

In the next section, we present an example which is almost entering in the stratified framework and for which an additional information allows to prove the comparison result.

31.3 The jumps we cannot take into account but...

The aim of this section is to examine an HJ-problem which appears in Bouin, Calvez, Grenier and Nadin [50]; its formulation is highly non-standard and seems rather far from what we are doing in this book but we show how to analyze the different difficulties in light of the stratified approach.

The problem consists in looking for a function $u : [0, T_f) \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ which is solution of

$$\max(u_t(t,x,v)+v\cdot D_x u(t,x,v)-1, u(t,x,v)-m(t,x)-|v|^2) = 0 \quad \text{in } (0,T_f)\times\mathbb{R}^N\times\mathbb{R}^N,$$

$$m_t(t,x) \le 0 \quad \text{and} \quad m_t(t,x) = 0 \quad \text{if } \mathcal{S}(t,x) = \{0\}\ ,$$

where $m(t,x) = \min_{v'} u(t,x,v')$ and $\mathcal{S}(t,x)$ is the set of all v' where this min is achieved. These equations are complemented by an initial data

$$u(0,x,v) = u_0(x,v) \quad \text{in } \mathbb{R}^N\ ,$$

where u_0 is a continuous function such that $u_0(x,v) - |v|^2$ is bounded.

In order to analyze this problem, it is more convenient to consider $w(t,x,v) = u(t,x,v) - |v|^2$ which is expected to be a bounded continuous function and which first solves

$$\max(w_t(t,x,v)+v\cdot D_x w(t,x,v)-1, w(t,x,v)-\mathcal{M}w(t,x)) = 0 \quad \text{in } (0,T_f)\times\mathbb{R}^N\times\mathbb{R}^N,$$

where

$$\mathcal{M}w(t,x) := \min_{v'} \left(w(t,x,v') + |v'|^2\right)\ .$$

This equation for w generates several remarks: of course, this equation looks like a quasi-variational inequality which we presented in the previous section, i.e.

$$\mathcal{M}w(t,x) := \min_{v'} \left(w(t,x,v') + C(v')\right)\ ,$$

with $C(v') = |v'|^2$ but here the function C has all the disadvantages: it is not sublinear, nor homogeneous of degre 1 and $\min_{v'} C(v') = 0$. This causes a problem for the initial data, i.e. for the $\mathbb{F}_{init}$-equation which is

$$\max(w(0,x,v) - w_0(x,v), w(0,x,v) - \mathcal{M}w(0,x)) = 0 \quad \text{in } \mathbb{R}^N\times\mathbb{R}^N\ , \tag{31.1}$$

where $w_0(x,v) = u_0(x,v) - |v|^2$ since any constant is a supersolution and therefore the $\mathbb{F}_{init}$-equation does not determine uniquely $w(0,x,v)$.

Hence we are in the worst case because there is no way that the above equation could fit into the control framework we have described in Chapter 4.

We are going to try, anyway, to push the arguments as far as possible to show that we can also take advantage of some features of the QVI along the lines of the remarks we did at the end of the previous section. And, of course, we are going to forget the problem with the initial data, by assuming that it is achieved in the classical way: solving formally (31.1), which consists here as taking the maximal subsolution, the "natural" initial data should be $w(0,x,v) = \min(w_0(x,v), \mathcal{M}w_0(x))$.

To start our analysis, we remark that, by the equation, we also have

$$w(t,x,0) \leq \mathcal{M}w(t,x) \leq w(t,x,v') + |v'|^2 \quad \text{for any } v' \,, \tag{31.2}$$

and therefore the min is always achieved for $v' = 0$. Since $w(t,x,v') + |v'|^2 = u(t,x,v')$, we have $w(t,x,0) = m(t,x)$ and

$$\mathcal{S}(t,x) = \{v';\ w(t,x,v') + |v'|^2 = \mathcal{M}w(t,x)\} \,.$$

We deduce two properties from this remark: on one hand, on $\mathbf{M}^N := \{(t,x,v);\ v = 0\}$, we have the stratified inequality

$$w_t(t,x,0) \leq 0 \quad \text{in } (0,T_f) \times \mathbb{R}^N \,,$$

and, on the other hand, we have the unusual supersolution inequality

$$w_t(t,x,0) \geq 0 \quad \text{if } \mathcal{S}(t,x) = \{0\} \,,$$

where $\mathcal{S}(t,x)$ is defined above in terms of w and $\mathcal{M}$.

At this level of the analysis, we have a problem which cannot be formulated as a standard control problem satisfying the $(\mathbf{H_{BCL}})$ assumptions but we have apparently all the correct stratified inequalities on $\mathbf{M}^N$ which seems to be a discontinuity for the cost since the term $w_t(t,x,v) + v \cdot D_x w(t,x,v) - 1$ in the equation is associated to a cost 1 while the $w_t(t,x,0) \leq 0$ suggests a cost 0 on $\mathbf{M}^N$.

Now we turn to the standard assumptions in the stratified framework, namely **(TC)** and **(NC)**. For **(TC)**, the Hamiltonian $p_t + v \cdot p_x - 1$ satisfies **(TC-s)** and it can easily be seen that the term $w(t,x,v) - \mathcal{M}w(t,x)$ does not cause any problem for tangential regularization and we can even remark that a regularization in (t,x) can be performed even far from $\mathbf{M}^N$, allowing to assume that the subsolution are smooth in t and x. For **(NC)**, (31.2) gives more than needed.

Hence we can almost perform the proof of Theorem 20.4.1 except two additional difficulties: on one hand, since we are not in a standard control framework, we cannot

use Lemma 5.4.1. And on the other hand, the localization arguments are more tricky to apply here.

Complete failure? Not yet!

We sketch the main step to prove a comparison result. If w_1 and w_2 are respectively sub and supersolution of the above problem with $w_1(0,x,v) \le w_2(0,x,v)$ in $\mathbb{R}^N \times \mathbb{R}^N$, we consider, for some parameters $0 < \mu < 1$ close to 1, $\delta, \alpha, \eta > 0$ small enough

$$M = M(\mu,\delta,\eta) := \max_{[0,T_f]\times\mathbb{R}^N\times\mathbb{R}^N} (\mu w_1(t,x,v) - w_2(t,x,v) - \delta|v|^2 - \alpha(|x|^2+1)^{1/2} - \eta t) \ .$$

If $M > 0$, then it cannot be achieved for $t = 0$ [(1)] and if (t,x,v) is a maximum point, we have several cases

1. If $w_2(t,x,v) < \mathcal{M}w_2(t,x)$, then the conclusion follows easily since (i) because $t > 0$, we can assume without loss of generality that w_1 is smooth in t and x, (ii) for α small enough compared to η, $\mu w_1(t,x,v) - \alpha(|x|^2+1)^{1/2} - \eta t$ is a strict, smooth subsolution of $w_t(t,x,v) + v \cdot D_x w(t,x,v) - 1 = 0$ and (iii) this smooth subsolution is a test-function for w_2.

2. Therefore $w_2(t,x,v) \ge \mathcal{M}w_2(t,x) = w_2(t,x,v') + |v'|^2$ and, using that $w_1(t,x,v) \le w_1(t,x,v') + |v'|^2$, we have

$$M \le \mu(w_1(t,x,v') + |v'|^2) - (w_2(t,x,v') + |v'|^2) - \delta|v|^2 - \alpha(|x|^2+1)^{1/2} - \eta t \ ,$$

and, if $\delta < (1-\mu)$

$$\begin{aligned} M &\le \mu w_1(t,x,v') - w_2(t,x,v') - (1-\mu)|v'|^2 - \delta|v|^2 - \alpha(|x|^2+1)^{1/2} - \eta t \\ &\le \mu w_1(t,x,v') - w_2(t,x,v') - \delta|v'|^2 - \delta|v|^2 - \alpha(|x|^2+1)^{1/2} - \eta t \\ &\le M - \delta|v|^2 \ . \end{aligned}$$

Hence, necessarely $v = 0$ but examining more carefully the above inequalities and using $\delta < (1-\mu)$, we can also deduce that necessarely $v' = 0$. Hence $\mathcal{S}_2(t,x) = \{v';\ w_2(t,x,v') + |v'|^2\} = \mathcal{M}w_2(t,x)\} = \{0\}$ and we have $(w_2)_t(t,x,v) \ge 0$ and we obtain the contradiction since $\mu w_1(t,x,v) - \alpha(|x|^2+1)^{1/2} - \eta t$ is a strict, smooth subsolution of $w_t(t,x,0) = 0$. And the sketch of the proof is complete.

The reader may think—and he is right—that the above proof works because of a succession of miracles: it is clear that the "μ-trick", rather classical in this QVI-framework, allows to overcome in a perfect way the difficulty due to the non-standard

[(1)] By adding some large positive constant to w_1 and w_2, we may assume w.l.o.g that $w_1, w_2 \ge 0$ in $[0,T_f] \times \mathbb{R}^N \times \mathbb{R}^N$.

features of the QVI by leading us to the exact situation where we can use the rather exotic supersolution property on $v = 0$.

But the above example is a rare case where (at least in the spirit) some $b^t = 0$-controls play a key role and the above analysis shows, as we already mentioned it above, that these jumps can easily be taken into account (in particular in **(NC)**).

Chapter 32

And What About Networks?

32.1 Networks and stratified networks

We recall that a simple network in $\mathbb{R}^N$ is a set containing both points (called nodes) and segments (called edges), the edges connecting some of the nodes. To be concrete, we can give the example in $\mathbb{R}^2$ of a map with roads (or highways) connecting cities (or just cross-roads). In this most simple framework, edges are one-dimensional objects but, of course, more complicated situations can be considered.

One can define control problems and Hamilton-Jacobi Equations on these networks and Part III is strongly inspired by the theoretical works of Imbert and Monneau [113, 114, 116]) and Lions and Souganidis [137, 138] for treating various junctions conditions at nodes. This may give an idea of what can be done in this direction. Several works have also been devoted to consider applications, and in particular to traffic problems. We refer the reader to Imbert, Monneau and Zidani [115], Forcadel and Salazar [88],Forcadel, Salazar, Wilfredo and Zaydan [91], but our list is far from being complete and up-to-date.

32.1.1 About stratified networks

We want to outline in this section that the framework of stratified problems in Whitney stratification may lead to a rather general point of view for networks but we are going to do it in a simplified situation.

To do so, we consider the value function $U^\varepsilon : \mathbb{R}^N \times [0, T_f]$ defined in the framework of Chapter 21, with a stratification $\mathbb{M} = (\mathbf{M}^k)_k$ of the form

$$\mathbf{M}^k = \tilde{\mathbf{M}}^{k-1} \times [0, T_f] \quad \text{for } k \geq 1 \ ,$$

where $(\tilde{\mathbf{M}}^k)_k$ is a (AFS) of $\mathbb{R}^N$.On the other hand, $(b,c,l) \in \mathbf{BCL}(x,t)$ is replaced by $(b,c,l+\varepsilon^{-1}d(x))$ where d denotes the distance to $\mathbf{M}^0 \cup \cdots \cup \mathbf{M}^N$. Obviously this change of cost has the objective to make more and more expensive an excursion in $\mathbf{M}^{N+1}$ and therefore to force the dynamics to stay on the "network" $\bigcup_{k=1}^N \mathbf{M}^k$.

We recall that, by Chapter 21, the value-function U^ε is continuous and the unique solution of $\mathbb{F} = \varepsilon^{-1}d(x)$ in $\mathbb{R}^N \times [0,T_f]$ with $\mathbb{F}^k(x,t,U^\varepsilon,DU^\varepsilon) = 0$ on $\mathbf{M}^k$. Our main aim is to provide the asymptotic of U^ε and to do so, we first need to establish a key property of U^ε for which have to introduce new Hamiltonians.

To do so, we consider a connected component of $\tilde{\mathbf{M}}^k$, $\tilde{\mathbf{M}}^k_i$. By Lemma 3.3.2, $\tilde{\mathbf{M}}^k_i = \mathcal{O} \cap (x + V_k)$ for some k-dimensional vector space V_k independent of x. If $(y,t) \in \partial\tilde{\mathbf{M}}^k_i \times (0,T_f)$, we define the "inner dynamics" as

$$\mathbf{B}^{k,i}_{in}(y,t) := \{b = (b^x,b^t) \in \mathbf{B}(y,t) : \ y + sb^x \in \tilde{\mathbf{M}}^k_i \text{ for } s>0 \text{ small enough}\} \ ,$$

and $\mathbf{BCL}^{k,i}_{in}(y,t)$ is the subset of $(b,c,l) \in \mathbf{BCL}(y,t)$ such that $b \in \mathbf{B}^{k,i}_{in}(y,t)$. Of course, by the **(NC-BCL)** assumption, these sets are non empty. Finally we set

$$\mathbb{F}^{k,i}_{in}(y,t,r,p) = \max_{(b,c,l)\in\mathbf{BCL}^{k,i}_{in}(y,t)} (-b\cdot p + cr - l) \ .$$

Our result is the

Theorem 32.1.1 *Under Assumptions* $(\mathbf{H_{BCL}})$, **(TC-BCL)** *and* **(NC-BCL)**, *the value-functions* U^ε *converge locally uniformly on* $\bigcup_{k=1}^N \mathbf{M}^k$ *to a continuous function* $U : \bigcup_{k=1}^N \mathbf{M}^k \to \mathbb{R}$ *which is the unique solution of: for any* $k = 1,..,N$

$$\mathbb{F}^k(x,t,U,DU) = 0 \quad \text{on } \mathbf{M}^k \ ,$$

$$\mathbb{F}^{k,i}_{in}(x,t,U,DU) = 0 \quad \text{on } \partial\mathbf{M}^k_i \quad \text{for any } \ i \ .$$

Proof — We just sketch it, the details being tedious but straightforward at this point of the book.
1. The U^ε's are uniformly bounded on $\bigcup_{k=1}^N \mathbf{M}^k$: on one hand, $U^\varepsilon \geq U^\infty$ where U^∞ is the value function obtained by dropping the term $\varepsilon^{-1}d(x)$ in the cost and, on the other hand, the normal controllability implies that once we start from a point in $\bigcup_{k=1}^N \mathbf{M}^k$, we can stay on $\bigcup_{k=1}^N \mathbf{M}^k$.
2. We have $\underline{U} = \liminf_* U^\varepsilon = +\infty$ in $\mathbf{M}^{N+1}$ but, by the classical stability result, we have

$$\mathbb{F}(x,t,\underline{U},D\underline{U}) \geq 0 \quad \text{on } \bigcup_{k=1}^N \mathbf{M}^k \ .$$

This inequality reduces to

$$\mathbb{F}^k(x,t,\underline{U},D\underline{U})=0 \quad \text{on } \mathbf{M}^k\,,$$

for any k, using that, on each (flat) connected component of $\mathbf{M}^k$, if $\underline{U}-\phi$ has a minimum point at $(x,t)\in\mathbf{M}^k$, where ϕ is a smooth function, then $\underline{U}-\phi-p\cdot x$ has also a minimum point for any p which is orthogonal to $\tilde{\mathbf{M}}^k$ at x. Then the choice of p as a minimum point of $\mathbb{F}(x,t,\underline{U}(x,t),D\underline{\phi}(x,t)+p)$ gives the answer.
3. For the $\limsup^*$, we just take it on $\bigcup_{k=1}^N\mathbf{M}^k$ and using (of course) only the points of $\bigcup_{k=1}^N\mathbf{M}^k$. Denoting by $\overline{U}$ this $\limsup^*$, we have,

$$\mathbb{F}^k(x,t,\overline{U},D\overline{U})=0 \quad \text{on } \mathbf{M}^k\,.$$

And we can also pass to the limit in the $\mathbb{F}^{k,i}_{in}$-inequality which gives

$$\min(\mathbb{F}^k(x,t,\overline{U},D\overline{U}),\mathbb{F}^{k,i}_{in}(x,t,\overline{U},D\overline{U}))=0 \quad \text{on } \partial\mathbf{M}^k_i \quad \text{for any } k \text{ and } i\,.$$

4. Using the arguments of Lemma 26.4.1 with this last inequality allows to show that $\overline{U}$ is $\mathbf{M}^k_i$-regular at each point of $\partial\mathbf{M}^k_i$.
5. This last point allows to recopy exactly the stratified proof which provides the key inequality $\overline{U}\le\underline{U}$ on $\bigcup_{k=1}^N\mathbf{M}^k$ and the continuity/uniqueness of $U:=\overline{U}\le\underline{U}$.
Q.E.D.

32.1.2 Few examples

We begin with a very easy but relevant example where, in $\mathbb{R}^2$ (but this can easily be generalized to $\mathbb{R}^N$)

$$\tilde{\mathbf{M}}^0=\mathbb{Z}^N\ ,\ \tilde{\mathbf{M}}^1=(\mathbb{R}\times\mathbb{Z}\cup\mathbb{Z}\times\mathbb{R})\setminus\mathbb{Z}\times\mathbb{Z}\,,$$

and $\tilde{\mathbf{M}}^N=\mathbb{R}^2\setminus(\tilde{\mathbf{M}}^1\cup\tilde{\mathbf{M}}^0)$.

On this stratification, one can imagine a lot of control problem by, for instance, imposing a certain limitation of speed on each edge and a certain cost $E^-_{i,j}=((i,j),(i+1,j))$ or $E^+_{i,j}=((i,j),(i,j+1))$. For instance, if $x\in E^-_{i,j}$ or $x\in E^+_{i,j}$

$$\mathbf{BCL}_1(x,t):=\begin{cases}\{((b^x,-1),0,|b^x|/2),|b^x|\le 2\} & \text{if } i \text{ or } j \text{ is a prime}\\ \{((b^x,-1),0,2|b^x|),|b^x|\le 1\} & \text{otherwise}\,,\end{cases}$$

the "1" in $\mathbf{BCL}_1$ refereeing to $\tilde{\mathbf{M}}^1$. And if one insists to define $\mathbf{BCL}(x,t)$ on $\tilde{\mathbf{M}}^N\times(0,+\infty)$, we can define it as $\{((b^x,-1),0,\varepsilon^{-1}),b^x\in\overline{B(0,0.1)}\}$ and, on $\tilde{\mathbf{M}}^0\times(0,+\infty)$ (but also on $\tilde{\mathbf{M}}^1\times(0,+\infty)$), we just use the extension by upper semicontinuity.

Such example is very simple because each connected component of $\tilde{\mathbf{M}}^1$ (or $\mathbf{M}^2$) is extremely simple and we have no problem to obtain the regularity by using the complete controllability.

The second example *"ad augusta, per angusta"* in $\mathbb{R}^3$ is the case where, if (x_1, x_2, x_3) are the coordinates of $x \in \mathbb{R}^3$

$$\tilde{\mathbf{M}}^2 = \{\ x_1 = -1,\ (x_2, x_3) \neq (0,0)\} \cup \{\ x_1 = +1,\ (x_2, x_3) \neq (0,0)\}\ ,$$

$$\tilde{\mathbf{M}}^1 = (-1,1) \times \{(0,0)\}\ ,\ \tilde{\mathbf{M}}^0 = \{(-1,0,0),(1,0,0)\}\ ,$$

and $\tilde{\mathbf{M}}^3 = \mathbb{R}^3 \setminus (\tilde{\mathbf{M}}^2 \cup \tilde{\mathbf{M}}^1 \cup \tilde{\mathbf{M}}^0)$.

Here we just define the specific dynamic and cost $\mathbf{BCL}_i$ on $\tilde{\mathbf{M}}^i \times (0, +\infty)$

$$\mathbf{BCL}_2(x,t) := \{((b^x, -1), 0, |b^x|/2), |b^x| \leq 2\}\ ,$$

$$\mathbf{BCL}_1(x,t) := \{((b^x, -1), 0, 2|b^x|), |b^x| \leq 1\}\ ,$$

and we can see $\mathbf{BCL}_3(x,t)$ as being $\{((b^x, -1), 0, \varepsilon^{-1}), |b^x| \leq 2\}$. On $\tilde{\mathbf{M}}^0 \times (0, +\infty)$, we do not impose any particular cost, the **BCL** at such points (but also elsewhere) being computed using the upper semi-continuity of **BCL**.

What could be interesting in such example, at the "network level", is to force the dynamic X to go through $\tilde{\mathbf{M}}^1$ and this can be done by a suitable choice of the initial data. Choose for example

$$u_0(x_1, x_2, x_3) = -10x_1 + (1 - x_2^2)^+\ .$$

If we look at the control problem on $(\tilde{\mathbf{M}}^2 \cup \tilde{\mathbf{M}}^1 \cup \tilde{\mathbf{M}}^0) \times (0, +\infty)$, it is clear that, if we start from a point like $(-1, x_2, x_3)$ and if we stay on $\{x_1 = -1\}$, we are going to pay (at least) a final cost which is 10.

But if we decide to go directly to $(-1, 0, 0)$, to use the "channel" $\tilde{\mathbf{M}}^1$ and then to go to the point $(1, 1, 0)$ the total cost on the long run (i.e. for t large enough) will be

$$2^{-1}(x_2^2 + x_3^2)^{1/2} + 4 + 2^{-1} - 10\ .$$

The four terms represent successively the cost for joining $(-1, 0, 0)$, crossing the channel, going to the minimum point $(1, 1, 0)$ and finally the terminal cost.

Of course, if $(x_2^2 + x_3^2)^{1/2}$ is not too large, this strategy is far better than the other one.

Chapter 33

Further Discussions and Open Problems

Part VII

Appendices

Appendix A

Notations and Terminology

$\mathbb{F}, \mathbb{G}, H$	Generic Hamiltonians
$\mathcal{A}, \mathcal{E}$	Generic subsets of $\mathbb{R}^N$
$\mathcal{O}, \mathcal{F}$	Generic open and closed subsets of $\mathbb{R}^N$
$\mathcal{K}$	Generic compact subset of $\mathbb{R}^N$
$B(y,r)$	Open ball of center $y \in \mathbb{R}^k$ and of radius $r > 0$ for the Euclidian norm.
A	a compact, convex subset of $\mathbb{R}^p$
$\mathbb{A}$	the space of controls, $\mathbb{A} = L^\infty(0,T;A)$
$\mathbf{BCL}(\cdot,\cdot)$	set-valued map combining all the dynamics, costs, discount factors, p.99
(X,T,D,L)	a generic trajectory of the differential inclusion, p.100
$\mathcal{T}(x,t)$	space of controled trajectories such that $(X,T,D,L)(0) = (x,t,0,0)$, p.103
$\mathcal{T}^{\text{reg}}(x,t)$	space of regular controlled trajectories such that $(X,T,D,L)(0) = (x,t,0,0)$, p.177
z_*, z^*	lower and upper semi-continuous enveloppes, p.35
$u.s.c., l.s.c.$	upper/lower semi-continuous function, p.35
V_k	k-dimensional vectorial subspace, typically $V_k = \mathbb{R}^k \times \{0\}^{N-k}$, p.518
$Q_{r,h}^{x,t}$	the open cylinder $B(x,r) \times (t-h,t)$.
$Q_{r,h}^{x,t}[\mathcal{F}]$	the open cylinder $\big(B(x,r) \cap \mathcal{F}\big) \times (t-h,t)$, p.50.
USC-Sub($\mathcal{F}$)	set of u.s.c. subsolutions on $\mathcal{F}$, p.44
LSC-Sup($\mathcal{F}$)	set of l.s.c. supersolutions on $\mathcal{F}$, p.44
$\text{PC}^1(\mathbb{R}^N \times [0,T_f])$	piecewise C^1-smooth test functions, p.230

$\mathbb{M}$	a general regular stratification of $\mathbb{R}^N$, p.66
(AFS)	Admissible Flat Stratification, p.63
(RS)	Regular Stratification, p.66
(HJB-SD)	Hamilton-Jacobi-Bellman in Stratified Domains
(AHG)	Assumptions on the Hamiltonian in the General case
(LAHF)	Local Assumptions on the Hamiltonians in the Flat case
(TC)	Tangential Continuity (pde version), p.79
(TC-BCL)	Tangential Continuity (control version)
(NC$_w$)	Weak Normal Controllability, p.76
(NC)	Normal Controllability (pde version), p.80
(NC-BCL)	Normal Controllability (control version)
(Mon)	Monotonicity Assumption, p.80
(**SCR**)	Strong Comparison Result, p.41,43
(**LCR**)	Local Comparison Result, p.44
(**GCR**)	Global Comparison Result, p.44

Notions of solutions (see also appendix B for quick reference)

(CVS)	Ishii solutions / Classical viscosity solutions, p.160 (see also Section 3.1.1)
(FLS)	Flux-Limited Solution, p.238
(JVS)	Junction Viscosity solution, p.265
	Stratified solutions, p.339

NB: The "good framework for HJ Equations with discontinuities" is defined p.136.

Appendix B

Assumptions, Hypotheses, Notions of Solutions

The page number refers to the page where the assumption is stated for the first time in the book.

Basic or fundamental assumptions

- ($\mathbf{H}^{\text{class.}}_{\mathbf{BA-CP}}$) *Basic Assumptions on the Control Problem – Classical case*: p. 26

  (i) The function $u_0 : \mathbb{R}^N \to \mathbb{R}$ is a bounded, uniformly continuous function.

  (ii) The functions b, c, l are bounded, uniformly continuous on $\mathbb{R}^N \times [0, T_f] \times A$.

  (iii) There exists a constant $C_1 > 0$ such that, for any $x, y \in \mathbb{R}^N$, $t \in [0, T_f]$, $\alpha \in A$, we have

$$|b(x,t,\alpha) - b(y,t,\alpha)| \ \leq C_1|x-y| \ .$$

- ($\mathbf{H}_{\mathbf{BA-CP}}$) *Basic Assumptions on the Control Problem*: p. 30

  (i) The function $u_0 : \mathbb{R}^N \to \mathbb{R}$ is a bounded, continuous function.

  (ii) The functions b, c, l are bounded, continuous functions on $\mathbb{R}^N \times [0, T_f] \times A$ and the sets $(b, c, l)(x, t, A)$ are convex compact subsets of $\mathbb{R}^{N+2}$ for any $x \in \mathbb{R}^N$, $t \in [0, T_f]$ [(1)].

[(1)] The last part of this assumption which is not a loss of generality will be used for the connections with the approach by differential inclusions.

(*iii*) For any ball $B \subset \mathbb{R}^N$, there exists a constant $C_1(B) > 0$ such that, for any $x, y \in \mathbb{R}^N$, $t \in [0, T_f]$, $\alpha \in A$, we have

$$|b(x,t,\alpha) - b(y,s,\alpha)| \ \leq C_1(B)\left(|x-y| + |t-s|\right) \ .$$

- ($\mathbf{H_{BA-HJ}}$) *Basic Assumptions on the Hamilton-Jacobi equation*: p. 30

  There exists a constant $C_2 > 0$ and, for any ball $B \subset \mathbb{R}^N \times [0, T_f]$, for any $R > 0$, there exists constants $C_1(B,R) > 0, \gamma(R) \in \mathbb{R}$ and a modulus of continuity $m(B,R) : [0,+\infty) \to [0,+\infty)$ such that, for any $x, y \in B$, $t, s \in [0, T_f]$, $-R \leq r_1 \leq r_2 \leq R$ and $p, q \in \mathbb{R}^N$

$$|H(x,t,r_1,p) - H(y,s,r_1,p)| \leq C_1(B,R)[|x-y|+|t-s|]|p| + m(B,R)(|x-y|+|t-s|) \ ,$$

$$|H(x,t,r_1,p) - H(x,t,r_1,q)| \leq C_2|p-q| \ ,$$

$$H(x,t,r_2,p) - H(x,t,r_1,p) \geq \gamma(R)(r_2 - r_1) \ .$$

- ($\mathbf{H_{BA}}$-p_t) *Basic Assumption on the p_t-dependence,* p. 52

  For any $(x,t,r,p_x,p_t) \in \mathcal{F} \times (0,T_f] \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}$, the function $p_t \mapsto \mathbb{G}\big(x,t,r,(p_x,p_t)\big)$ is increasing and $\mathbb{G}\big(x,t,r,(p_x,p_t)\big) \to +\infty$ as $p_t \to +\infty$, uniformly for bounded x, t, r, p_x.

- ($\mathbf{H_{BA-Conv}}$) *Basic Assumption in the convex case*, p.58

  $H(x,t,r,p)$ is a locally Lipschitz function which is convex in (r,p). Moreover, for any ball $B \subset \mathbb{R}^N \times [0,T_f]$, for any $R > 0$, there exists constants $L = L(B,R), K = K(B,R) > 0$ and a function $G = G(B,R) : \mathbb{R}^N \to [1,+\infty[$ such that, for any $x, y \in B$, $t, s \in [0,T_f]$, $-R \leq u \leq v \leq R$ and $p \in \mathbb{R}^N$

$$D_p H(x,t,r,p) \cdot p - H(x,t,u,p) \geq G(p) - L \ ,$$

$$|D_x H(x,t,r,p)|, |D_t H(x,t,r,p)| \leq KG(p)(1+|p|) \ ,$$

$$D_r H(x,t,r,p) \geq 0 \ .$$

Stratification assumptions

- ($\mathbf{H_{ST}}$)$_{flat}$ *Structure of an admissible flat stratification*, p.63

  The family $\mathbb{M} = (\mathbf{M}^k)_{k=0..N}$ of disjoint submanifolds of $\mathbb{R}^N$ is said to be an *Admissible Flat Stratification* of $\mathbb{R}^N$ if $\mathbb{R}^N = \mathbf{M}^0 \cup \mathbf{M}^1 \cup \cdots \cup \mathbf{M}^N$ and

  (*i*) If $x \in \mathbf{M}^k$ for some $k = 0..N$, there exists $r = r_x > 0$ such that

(*a*) $B(x,r)\cap \mathbf{M}^k = B(x,r)\cap (x+V_k)$ for some $(x+V_k)\in V^{(k)}(x)$;
(*b*) For any $l<k$, $B(x,r)\cap \mathbf{M}^l=\emptyset$;
(*c*) For any $l>k$, $B(x,r)\cap \mathbf{M}^l$ is either empty or has at most a finite number of connected components;
(*d*) For any $l>k$, $B(x,r)\cap \mathbf{M}^l_j\neq\emptyset$ if and only if $x\in\partial\mathbf{M}^l_j$.

(*ii*) If $\mathbf{M}^k_i\cap\overline{\mathbf{M}^l_j}\neq\emptyset$ for some $l>k$ then $\mathbf{M}^k_i\subset\overline{\mathbf{M}^l_j}$.

(*iii*) The following inclusion holds: $\overline{\mathbf{M}^k}\subset\mathbf{M}^0\cup\mathbf{M}^1\cup\cdots\cup\mathbf{M}^k$.

- **N.B.** *Convergence in the sense of stratification is defined* pp. 364 and 366.

Assumptions for the differential inclusion and the value-function

- $(\mathbf{H_{BCL}})_{fund}$ *Fundamental assumptions on the set-valued map* **BCL**, p.99

  The set-valued map $\mathbf{BCL}:\mathbb{R}^N\times[0,T_f]\to\mathcal{P}(\mathbb{R}^{N+3})$ satisfies

  (*i*) The map $(x,t)\mapsto\mathbf{BCL}(x,t)$ has compact, convex images and is upper semi-continuous;

  (*ii*) There exists $M>0$, such that for any $x\in\mathbb{R}^N$ and $t>0$,

  $$\mathbf{BCL}(x,t)\subset\left\{(b,c,l)\in\mathbb{R}^{N+1}\times\mathbb{R}\times\mathbb{R}:|b|\leq M;|c|\leq M;|l|\leq M\right\},$$

- $(\mathbf{H_{BCL}})_{struct}$ *Structure assumptions on the set-valued map* **BCL**, p.101

  There exists $\underline{c},K>0$ such that

  (*i*) For all $x\in\mathbb{R}^N$, $t\in[0,T_f]$ and $b=(b^x,b^t)\in\mathbf{B}(x,t)$, $-1\leq b^t\leq 0$. Moreover, there exists $b=(b^x,b^t)\in\mathbf{B}(x,t)$ such that $b^t=-1$.

  (*ii*) For all $x\in\mathbb{R}^N$, $t\in[0,T_f]$, if $((b^x,b^t),c,l)\in\mathbf{BCL}(x,t)$, then $-Kb^t+c\geq 0$.

  (*iii*) For any $x\in\mathbb{R}^N$, there exists an element in $\mathbf{BCL}(x,0)$ of the form $((0,0),c,l)$ with $c\geq\underline{c}$.

  (*iv*) For all $x\in\mathbb{R}^N$, $t\in[0,T_f]$, if $(b,c,l)\in\mathbf{BCL}(x,t)$ then $\max(-b^t,c,l)\geq\underline{c}$.

- $(\mathbf{H_{BCL}})$ is just the conjunction of $(\mathbf{H_{BCL}})_{fund}$ and $(\mathbf{H_{BCL}})_{struct}$.

- $(\mathbf{H_U})$: the value-function U is locally bounded on $\overline{\Omega}\times[0,T_f]$, p.420.

Normal controllability, tangential continuity, Monotonicity

- **(TC)** *Tangential Continuity, HJ version*, p.79

  for any $x_1 = (y_1, z), x_2 = (y_2, z) \in B_\infty(\bar{x}, r)$, $|u| \leq R$, $p \in \mathbb{R}^N$, then

  $$|G(x_1, u, p) - G(x_2, u, p)| \leq C_1^R |y_1 - y_2|.|p| + m^R\big(|y_1 - y_2|\big)\,.$$

- **(TC-s)** *Strong Tangential Continuity*, p. 88

  For any $x_1 = (y_1, z), x_2 = (y_2, z) \in B_\infty(\bar{X}, r)$, $|u| \leq R$, $p = (p_y, p_z) \in \mathbb{R}^N$, then

  $$|G(X_1, u, p) - G(X_2, u, p)| \leq C_1^R |y_1 - y_2|.|p_y| + m^R\big(|y_1 - y_2|\big)\,.$$

- **(TC-BCL)** *Tangential Continuity, control version, in the case when* $\mathbb{M}$ *is an* (AFS) *with* $\mathbf{M}^k = (x,t) + V_k$, p.139

  For any $0 \leq k \leq N+1$ and for any $(x,t) \in \mathbf{M}^k$, there exists a constant $C_1 > 0$ and a modulus $m : [0, +\infty) \to \mathbb{R}^+$ such that, for any $j \geq k$, if $(y_1, t_1), (y_2, t_2) \in \mathbf{M}^j \cap B((x,t), r)$ with $(y_1, t_1) - (y_2, t_2) \in V_k$, then for any $(b_1, c_1, l_1) \in \mathbf{BCL}(y_1, t_1)$, there exists $(b_2, c_2, l_2) \in \mathbf{BCL}(y_2, t_2)$ such that

  $$|b_1 - b_2| \leq C_1(|y_1 - y_2| + |t_1 - t_2|) \quad , \quad |c_1 - c_2| + |l_1 - l_2| \leq m\big(|y_1 - y_2| + |t_1 - t_2|\big)\,.$$

- **($\mathbf{NC}_w$)** *Weak Normal Controllability*, p.76

  (i) If $N - k > 1$, there exists $e \in \mathbb{R}^{N-k}$ such that, for any $R > 0$, we have

  $$\mathbb{G}(X, u, (p_Y, Ce)) \to +\infty \quad \text{when } C \to +\infty\,,$$

  uniformly for $X = (Y, Z) \in B_\infty(\bar{X}, r)$, $|u| \leq R$, $|p_Y| \leq R$.

  (ii) If $N - k = 1$, this property holds for $e = +1$.

  (iii) If $N - k = 1$, this property holds for $e = -1$.

- **(NC)** *Normal Controllability, HJ version*, p. 80

  for any $x = (y, z) \in B_\infty(\bar{x}, r)$, $|u| \leq R$, $p = (p_y, p_z) \in \mathbb{R}^N$, then

  $$G(x, u, p) \geq C_2^R |p_z| - C_3^R |p_y| - C_4^R\,.$$

- (**NC**$_{\mathcal{H}}$) *Normal Controllability, codimension* 1 *case*, p. 163

  For any $(x,t) \in \mathcal{H} \times [0,T_f]$, there exists $\delta = \delta(x,t)$ and a neighborhood $\mathcal{V} = \mathcal{V}(x,t)$ such that, for any $(y,s) \in \mathcal{V}$

$$[-\delta,\delta] \subset \{b_1(y,s,\alpha_1)\cdot e_N,\ \alpha_1 \in A_1\} \quad \text{if } (y,s) \in \overline{\Omega_1}\ ,$$

$$[-\delta,\delta] \subset \{b_2(y,s,\alpha_2)\cdot e_N,\ \alpha_2 \in A_2\} \quad \text{if } (y,s) \in \overline{\Omega_2}\ ,$$

  where $e_N = (0,0\cdots,0,1) \in \mathbb{R}^N$.

- **(NC-BCL)** *Normal Controllability, multi-D case control version*, p.139

  There exists $\delta = \delta(x,t) > 0$, such that, for any $(y,s) \in B((x,t),r) \setminus \mathbf{M}^k$, one has

$$B(0,\delta) \cap V_k^{\perp} \subset P^{\perp}\left(\mathbf{B}(y,t)\right)\ .$$

- **(Mon)** *Monotonicity property*, p. 80

  For any $R > 0$, there exists $\lambda_R, \mu_R \in \mathbb{R}$, such that we have either $\Lambda_R > 0$ and for any $x \in B_\infty(\bar{x},r)$, $p = (p_y,p_z) \in \mathbb{R}^N$,

$$G(x,u_2,p) - G(x,u_1,p) \geq \lambda^R(u_2-u_1) \tag{B.1}$$

  for any $-R \leq u_1 \leq u_2 \leq R$, or (B.1) holds with $\lambda_R = 0$, we have $\mu_R > 0$ and

$$G(x,u_1,q) - G(x,u_1,p) \geq \mu^R(q_{y_1} - p_{y_1})\ , \tag{B.2}$$

  for any $q = (q_y,p_z)$ with $p_{y_1} < q_{y_1}$ and $p_{y_i} = q_{y_i}$ for $i = 2,...,p$.

  We say that (**Mon**-u) is satisfied if (B.1) holds and (**Mon**-p) is satisfied if (B.2) holds.

Localisation, convexity, subsolutions

- (**LOC1**) *localization hypothesis 1*, p.45

  If $\mathcal{F}$ is unbounded, for any $u \in$ USC-Sub($\mathcal{F}$), for any $v \in$ LSC-Sup($\mathcal{F}$), there exists a sequence $(u_\alpha)_{\alpha>0}$ of u.s.c. subsolutions of (3.3) such that $u_\alpha(x) - v(x) \to -\infty$ when $|x| \to +\infty$, $x \in \mathcal{F}$. Moreover, for any $x \in \mathcal{F}$, $u_\alpha(x) \to u(x)$ when $\alpha \to 0$.

- (**LOC2**) *localization hypothesis 2*, p.45

  For any $x \in \mathcal{F}$, $r > 0$, if $u \in$ USC-Sub$(\mathcal{F}^{x,r})$, there exists a sequence $(u^\delta)_{\delta>0}$ of functions in USC-Sub$(\mathcal{F}^{x,r})$ such that $\lfloor u^\delta - u \rfloor^x_{\partial\mathcal{F}^{x,r}} \geq \eta(\delta) > 0$ for any δ. Moreover, for any $y \in \mathcal{F}^{x,r}$, $u^\delta(y) \to u(y)$ when $\delta \to 0$.[(2)]

- (**LOC1**)-*evol localization hypothesis 1, evolution version*, p.50

  If $\mathcal{F}$ is unbounded, for any $u \in$ USC-Sub$(\mathcal{F} \times [0,T_f])$, for any $v \in$ LSC-Sup$(\mathcal{F} \times [0,T_f])$, there exists a sequence $(u_\alpha)_{\alpha>0}$ of u.s.c. subsolutions of (3.3) such that $u_\alpha(x,t) - v(x,t) \to -\infty$ when $|x| \to +\infty$, $x \in \mathcal{F}$, uniformly for $t \in [0,T_f]$. Moreover, for any $x \in \mathcal{F}$, $u_\alpha(x,t) \to u(x,t)$ when $\alpha \to 0$.

- (**LOC2**)-*evol localization hypothesis 2, evolution version*, p.50

  For any $x \in \mathcal{F}$, if $u \in$ USC-Sub$(Q^{x,t}_{r,\bar{h}}[\mathcal{F}])$ for some $0 < \bar{h} < t$, there exists $0 < h \leq \bar{h}$ and a sequence $(u^\delta)_{\delta>0}$ of functions in USC-Sub$(Q^{x,t}_{r,h}[\mathcal{F}])$ such that $\lfloor u^\delta - u \rfloor^{(x,t)}_{\partial_{\rm lat}Q} \geq \tilde{\eta}(\delta) > 0$ with $\tilde{\eta}(\delta) \to 0$ as $\delta \to 0$. Moreover, for any $y \in \mathcal{F}$, $u^\delta(y) \to u(y)$ when $\delta \to 0$.

- ($\mathbf{H_{Sub-HJ}}$) *Existence of a subsolution*, p.58

  There exists a C^1-function $\psi : \mathbb{R}^N \times [0,T_f] \to \mathbb{R}$ which is a subsolution of (3.10) and which satisfies $\psi(x,t) \to -\infty$ as $|x| \to +\infty$, uniformly for $t \in [0,T_f]$ and $\psi(x,0) \leq u_0(x)$ in $\mathbb{R}^N$.

- ($\mathbf{H_{Conv}}$) *convexity for a general Hamiltonian*, p. 84

  For any $x \in B_\infty(\bar{X}, r)$, the function $(u,p) \mapsto G(X,u,p)$ is convex.

- ($\mathbf{H_{QC-\mathbb{R}}}$) *Quasiconvex Hamiltonians in* $\mathbb{R}$, p. 145

  $f : \mathbb{R} \to \mathbb{R}$ is continuous, coercive and quasi-convex, i.e for any $a \in \mathbb{R}$, the lower level set $\{x : f(x) \leq a\}$ is convex or equivalently, for any $x, y \in \mathbb{R}$ and $\lambda \in (0,1)$,

  $$f(\lambda x + (1-\lambda)y) \leq \max\{f(x), f(y)\}\,.$$

- ($\mathbf{H_{QC}}$) *Quasi-convex Hamiltonians in the* e_N*-direction in* $\mathbb{R}^N$, p. 148

  For any fixed (x,t,r,p'), the function $h : s \mapsto H(x,t,r,p' + se_N)$ satisfies ($\mathbf{H_{QC-\mathbb{R}}}$). As a consequence, $H = \max(H^+, H^-)$ where $s \mapsto H^+(x,t,r,p' + p_N e_N)$ is decreasing and $s \mapsto H^-(x,t,r,p' + p_N e_N)$ is increasing.

[(2)] We recall that

$$\lfloor f \rfloor^x_{\partial\mathcal{F}^{x,r}} := f(x) - \max_{y \in \partial\mathcal{F}^{x,r}} f(y)\,.$$

Comparison results

- $(\mathbf{GCR})^{\mathcal{F}}$ *Global Comparison Result in* $\mathcal{F}$, p.44

  For any $u \in$ USC-Sub$(\mathcal{F})$, for any $v \in$ LSC-Sup$(\mathcal{F})$, we have $u \leq v$ on $\mathcal{F}$.

- $(\mathbf{LCR})^{\mathcal{F}}$ *Local Comparison Result in* $\mathcal{F}$, p.44

  For any $x \in \mathcal{F}$, there exists $r > 0$ such that, if $u \in$ USC-Sub$(\mathcal{F}^{x,r})$, $v \in$ LSC-Sup$(\mathcal{F}^{x,r})$ and $\max_{\overline{\mathcal{F}^{x,r}}}(u-v) > 0$, then

$$\max_{\overline{\mathcal{F}^{x,r}}}(u-v) \leq \max_{\partial\mathcal{F}^{x,r}}(u-v)\,.$$

- **(LCR)**-evol *Local Comparison Result, evolution case*, p.50

  For any $(x,t) \in \mathcal{F} \times (0,T_f]$, there exists $r > 0$, $0 < h < t$ such that, if $u \in$ USC-Sub$(Q^{x,t}_{r,h}[\mathcal{F}])$, $v \in$ LSC-Sup$(Q^{x,t}_{r,h}[\mathcal{F}])$ and $\max\limits_{\overline{Q^{x,t}_{r,h}[\mathcal{F}]}}(u-v) > 0$, then

$$\max_{\overline{Q^{x,t}_{r,h}[\mathcal{F}]}}(u-v) \leq \max_{\partial_p Q^{x,t}_{r,h}[\mathcal{F}]}(u-v)\,.$$

  N.B. here, $\partial_p Q^{x,t}_{r,h}[\mathcal{F}]$ stands for the parabolic boundary:
  $(\partial B(x,r) \cap \mathcal{F}) \times [t-h,t] \cup (\overline{B(x,r)} \cap \mathcal{F}) \times \{t-h\}$.

- $\mathbf{LCR}^{\psi}(\bar{x},\bar{t})$ *Local Comparison Result around* (x,t) *in the stratified case,* p.346

  There exists $r = r(\bar{x},\bar{t}) > 0$ and $h = h(\bar{x},\bar{t}) \in (0,\bar{t})$ such that, if u and v are respectively a strict stratified subsolution and a stratified supersolution of some ψ–Equation in $Q^{\bar{x},\bar{t}}_{r,h}$ and if $\max\limits_{\overline{Q^{\bar{x},\bar{t}}_{r,h}}}(u-v) > 0$, then

$$\max_{\overline{Q^{\bar{x},\bar{t}}_{r,h}}}(u-v) \leq \max_{\partial_p Q^{\bar{x},\bar{t}}_{r,h}}(u-v)\,.$$

  N.B. here, ψ-equation means an equation with obstacle ψ, a continuous function: $\max(\mathbb{F}(x,t,w,Dw), w-\psi) = 0$.

Notions of solutions

N.B. The following definitions are just gathered here as a quick reminder, the reader will find more details and the precise definition on the page given in reference.

- (CVS) *Ishii Solution for the hyperplane case*, p.160

  This is the "classical" notion of viscosity solution (hence the acronym (CVS)) where on the hyperplane the relaxed condition reads (in the viscosity sense)

$$\begin{cases} \max\Big(u_t + H_1(x,t,u,Du), u_t + H_2(x,t,u,Du)\Big) \geq 0\,, \\ \min\Big(u_t + H_1(x,t,u,Du), u_t + H_2(x,t,u,Du)\Big) \leq 0\,. \end{cases}$$

  The notion is "classical" in the sense that testing is done with test-functions in $C^1(\mathbb{R}^N \times [0,T_f])$ contrary to (FLS) and (JVS) below.

- (FLS) *Flux-Limited Solution*, p.238

  We are given a flux-limiter G on $\mathcal{H}$ (codim-1 discontinuity). Here, we use the extended $\mathrm{PC}^1(\mathbb{R}^N \times [0,T_f])$-test-functions.
  A locally bounded function $u : \mathbb{R}^N \times (0,T_f) \to \mathbb{R}$ is a (FLSub) of (HJ-Gen)-(FL) if it is a classical viscosity subsolution of (HJ-Gen) and if, for any test-function $\psi \in \mathrm{PC}^1(\mathbb{R}^N \times [0,T_f])$ and any local maximum point $(x,t) \in \mathcal{H} \times (0,T_f)$ of $u^* - \psi$ in $\mathbb{R}^N \times (0,T_f)$, at (x,t) the following inequality holds

$$\max\Big(\psi_t + G(x,t,u^*,D_{\mathcal{H}}\psi), \psi_t + H_1^+(x,t,u^*,D\psi_1), \psi_t + H_2^-(x,t,u^*,D\psi_2)\Big) \leq 0\,,$$

  where $u^* = u^*(x,t)$.
  A locally bounded function $v : \mathbb{R}^N \times (0,T_f) \to \mathbb{R}$ is a (FLSuper) of (HJ-Gen)-(FL) if it is a classical viscosity supersolution of (HJ-Gen) and if, for any test-function $\psi \in \mathrm{PC}^1(\mathbb{R}^N \times [0,T_f])$ and any local minimum point $(x,t) \in \mathcal{H} \times (0,T_f)$ of $v_* - \psi$ in $\mathbb{R}^N \times (0,T_f)$, at (x,t) the following inequality holds

$$\max\Big(\psi_t + G(x,t,v_*,D_{\mathcal{H}}\psi), \psi_t + H_1^+(x,t,v_*,D\psi_1), \psi_t + H_2^-(x,t,v_*,D\psi_2)\Big) \geq 0\,,$$

  where $v_* = v_*(x,t)$.
  A locally bounded function is a flux-limited solution if it is both a (FLSub) and a (FLSuper).

- (JVS) *Junction Viscosity solution*, p.265

  This notion is somehow "consistent" with the usual viscosity definitions. Given a flux-limiter G we use test-functions in $\mathrm{PC}^1(\mathbb{R}^N \times [0,T_f])$ for which we test as usual.
  A locally bounded function $u : \mathbb{R}^N \times (0,T_f) \to \mathbb{R}$ is a (JVSub) of (HJ-Gen)-(GJC) if it is a classical viscosity subsolution of (HJ-Gen) and if, for any test-function $\psi = (\psi_1,\psi_2) \in \mathrm{PC}^1(\mathbb{R}^N \times [0,T_f])$ and any local maximum point $(x,t) \in$

$\mathcal{H} \times (0,T_f)$ of $u^* - \psi$ in $\mathbb{R}^N \times (0,T_f)$,

$$\min\Big(G(x,t,\psi_t,D_{\mathcal{H}}\psi,\frac{\partial\psi_1}{\partial n_1},\frac{\partial\psi_2}{\partial n_2}),\psi_t+H_1(x,t,u^*,D\psi_1),\psi_t+H_2(x,t,u^*,D\psi_2)\Big)\le 0,$$

where u^* and the derivatives of ψ,ψ_1,ψ_2 are taken at (x,t).

A locally bounded function $v:\mathbb{R}^N\times(0,T_f)\to\mathbb{R}$ is a (JVSuper) of (HJ-Gen)-(GJC) if it is a classical viscosity supersolution of (HJ-Gen) and if, for any test-function $\psi=(\psi_1,\psi_2)\in \mathrm{PC}^1(\mathbb{R}^N\times[0,T_f])$ and any local minimum point $(x,t)\in\mathcal{H}\times(0,T_f)$ of $v_*-\psi$ in $\mathbb{R}^N\times(0,T_f)$,

$$\max\Big(G(x,t,\psi_t,D_{\mathcal{H}}\psi,\frac{\partial\psi_1}{\partial n_1},\frac{\partial\psi_2}{\partial n_2}),\psi_t+H_1(x,t,v_*,D\psi_1),\psi_t+H_2(x,t,v_*,D\psi_2)\Big)\ge 0,$$

where v_* and the derivatives of ψ,ψ_1,ψ_2 are taken at (x,t).
A (JVS) (i.e. a junction viscosity solution) is a locally bounded function which is both (JVSub) and (JVSuper).

- *Stratified solutions*, p.339

1. — (S-Super): A locally bounded function $v:\mathbb{R}^N\times[0,T_f[\to\mathbb{R}$ is a stratified supersolution of (HJB-S) if it is an Ishii supersolution of (20.1).

2. — (w-S-Sub): A locally bounded function $u:\mathbb{R}^N\times[0,T_f[\to\mathbb{R}$ is a *weak* stratified subsolution of (HJB-S) if

(*a*) for any $k=0,...,(N+1)$, u^* is an Ishii subsolution of

$$\mathbb{F}^k\big(x,t,u^*,Du^*\big)\le 0 \quad \text{on } \mathbf{M}^k,$$

(*b*) similarly, for $t=0$, and $k=0..N$, $u^*(x,0)$ is an Ishii subsolution of

$$\mathbb{F}^k_{init}(x,u^*(x,0),D_xu^*(x,0))\le 0 \quad \text{on } \mathbf{M}^k_0\,.$$

3. — (s-S-Sub): A locally bounded function $u:\mathbb{R}^N\times[0,T_f[\to\mathbb{R}$ is a *strong* stratified subsolution of (HJB-S) if it is a (w-S-Sub) and satisfies additionally

(*a*) $\mathbb{F}_*\big(x,t,u^*,Du^*\big)\le 0 \quad \text{in } \mathbb{R}^N\times(0,T_f)\,,$

(*b*) $(\mathbb{F}_{init})_*(x,u^*(x,0),D_xu^*(x,0))\le 0 \quad \text{in } \mathbb{R}^N\,.$

4. — A weak or strong *stratified solution* is a function which is both a (S-Super) and either a (w-S-Sub) or a (s-S-Sub).

"Good Assumptions" for the Network Approach

- $(\mathbf{H_{BA-HJ}})^+$ [p. 233]: assumption $(\mathbf{H_{BA-HJ}})$ in which we assume $\gamma(R) \geq 0$ for any R.

- $(\mathbf{H_{BA-CP}})^+$ [p. 233]: assumption $(\mathbf{H_{BA-CP}})$ in which we assume $c(x,t,\alpha) \geq 0$ for any x, t, α.

- **(NC-HJ)** *Normal controllability for general Hamiltonians*, p. 234

  For any $R > 0$, there exists constants $C_2^R, C_3^R, C_4^R > 0$ such that, for any $(x,t) \in \mathcal{H} \times (0,T_f)$ with $|x| \leq R$, $|u| \leq R$ and $p = (p', p_N)$ with $p' \in \mathbb{R}^{N-1}$, $p_N \in \mathbb{R}$,

  $$H(x,t,u,p) \geq C_2^R |p_N| - C_3^R |p'| - C_4^R \,.$$

- **(TC-HJ)** *Tangential Continuity for general Hamiltonians*, p. 234

  for any $R > 0$, there exists $C_1^R > 0$ and a modulus of continuity $m^R : [0,+\infty[\to [0,+\infty[$ such that for any $x = (x', x_N), y = (y', x_N)$ with $|x|, |y| \leq R$, $|x_N| \leq R^{-1}$, $t, s \in [0,T_f]$, $|u| \leq R$, $p = (p', p_N) \in \mathbb{R}^N$,

  $$|H(x,t,u,p) - H(y,s,u,p)| \leq C_1^R(|x'-y'| + |t-s|)|p'| + m^R\big(|x'-y'| + |t-s|\big) \,.$$

  With these assumptions we can formulate several "good assumptions" on H_1, H_2 depending on the context

- **(GA-Gen)** *General case*, p. 234: H_1, H_2 satisfy $(\mathbf{H_{BA-HJ}})^+$ and **(NC-HJ)**.

- **(GA-Conv)** *Convex case*, p. 234: H_1, H_2 satisfy **(GA-Gen)** and are convex in p.

- **(GA-QC)** *Quasi-convex case*, p. 234: H_1, H_2 satisfy **(GA-Gen)** and $(\mathbf{H_{QC}})$.

- **(GA-CC)** *Control case*, p. 234: $(\mathbf{H_{BA-CP}})^+$ and $(\mathbf{NC}_{\mathcal{H}})$ are satisfied.

  Assumptions on the junction condition G

- **(GA-ContG)** *A general assumption*, p. 235
  For any $R > 0$, there exist constants C_5^R, C_6^R such that, for any $x, y \in \mathcal{H}$, $t, s \in [0, T_f]$, $|r| \leq R$, $p_1', p_2' \in \mathbb{R}^{N-1}$, $a, b, c, a', b', c' \in \mathbb{R}$

  $$|G(x,t,a,p_1',b,c) - G(y,s,a,p_1',b,c)| \leq C_5^R(|x-y|+|t-s|)\big(1+|p_1'|+\varepsilon_0(|a|+|b|+|c|)\big) .$$

  $$|G(x,t,a',p_2',b',c') - G(x,t,a,p_1',b,c)| \leq C_6^R(|p_2'-p_1'|+(|a'-a|+|b'-b|+|c'-c|)) .$$

  With this assumption we can formulate several "good assumptions" on G depending on the context

- **(GA-G-FL)** *Flux-Limited*, p. 235: G is independent of a, b, c and **(GA-ContG)** holds with $\varepsilon_0 = 0$.
- **(GA-G-GKT)** *Kirchhoff type*, p. 235: **(GA-ContG)** holds with $\varepsilon_0 = 0$ and (14.2) holds with $\alpha \geq 0$, $\beta > 0$.
- **(GA-G-FLT)** *Flux-limited type*, p. 235: $G(x,t,a,p',b,c) = G_1(a,p',b,c) + G_2(x,t,a,p')$ where G_1 is a Lipschitz continuous function which satisfies (14.2) with $\alpha > 0$, $\beta = 0$ while G_2 satisfies **(GA-G-FL)**.

"Good Assumptions" for Stratified Problems

($\mathbf{H_{BA-SF}}$) *Basic Assumptions on the Stratified Framework*

(i) There exists a stratification $\mathbb{M} = (\mathbf{M}^k)_{k=0\ldots(N+1)}$ of $\mathbb{R}^N \times (0, T_f)$ such that $(x,t) \mapsto \mathbb{F}(x,t,r,p)$ is continuous on $\mathbf{M}^{N+1}$ and may be discontinuous on $\mathbf{M}^0 \cup \mathbf{M}^1 \cup \cdots \cup \mathbf{M}^N$. Moreover $(0_{\mathbb{R}^N}, 1) \notin (T_{(x,t)}\mathbf{M}^k)^\perp$ for any $(x,t) \in \mathbf{M}^k$ and for any $k = 1 \ldots N$ [(3)]. In the same way, there exists a stratification $\mathbb{M}_0 = (\mathbf{M}_0^k)_{k=0\ldots N}$ of $\mathbb{R}^N$ such that the Hamiltonian $x \mapsto \mathbb{F}_{init}(x, r, p_x)$ is continuous on $\mathbf{M}_0^N$ and may be discontinuous on $\mathbf{M}_0^0 \cup \mathbf{M}_0^1 \cup \cdots \cup \mathbf{M}_0^{N-1}$.

(ii) The "good framework for HJB Equations with discontinuities" holds for Equation (20.1) in $\mathcal{O} = \mathbb{R}^N \times (0, T_f)$ associated to the stratification $\mathbb{M}$.

(iii) The "good framework for HJB Equations with discontinuities" holds for the equation $\mathbb{F}_{init} = 0$ in $\mathcal{O} = \mathbb{R}^N$, associated to the stratification $\mathbb{M}_0$.

[(3)] This assumption, whose aim is to avoid "flat part" of $\mathbf{M}^k$ in time, will be redundant to the normal controllability assumption in $\mathbb{R}^N \times (0, T_f)$.

We recall that the assumptions for a "Good Framework for HJ Equations with Discontinuities" are that ($\mathbf{H_{BCL}}$), **(TC-BCL)** (p. 139) and **(NC-BCL)** (p. 139) hold. We refer to Section 5.5 where the connections with Hamiltonian assumptions **(Mon)**, **(TC)**, **(NC)** are described.

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