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THE FATOU COORDINATE FOR PARABOLIC DULAC GERMS

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Abstract. We study the class of parabolic Dulac germs of hyperbolic polycycles. For such germs we give a constructive proof of the existence of a *unique* Fatou coordinate, admitting an asymptotic expansion in the power-iterated logarithm monomials.

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1. Introduction

A classical problem for germs of diffeomorphisms on the real or complex line at a fixed point is the problem of embedding in the flow of vector fields or, equivalently, the construction of a Fatou coordinate in which the diffeomorphisms is the translation by one.

The problem is solved in most cases in the analytic setting. Denote by f the analytic germ of a diffeomorphism, and by λ the multiplier of the fixed point. The problem is fully solved in the following two cases: in the hyperbolic case ($|\lambda| \neq 1$) by König's and Bötcher's linearization theorem; in the parabolic, i.e. tangent-to-identity, case ($\lambda = 1$, to which also the case of λ being a root of unity is reduced) by the Leau-Fatou theorem, see e.g. [10].

We solve the problem of existence and uniqueness of Fatou coordinates in a non-analytic case: for parabolic Dulac germs. Dulac germs are germs f on $(\mathbb{R}^+, 0)$ analytic outside of zero, having the well-known *Dulac* power-log asymptotic expansion $\hat{f}$ and which are moreover *quasi-analytic*, i.e. which admit an extension to a complex domain ensuring the injectivity of the mapping $f \mapsto \hat{f}$ (see Definition 2.3). In fact, we build a Fatou coordinate which admits a *transasymptotic expansion* in power-logarithm monomials (see Section 3.1). This guarantees its uniqueness.

Our work on Dulac germs is motivated by the study of cyclicity of polycycles of analytic vector fields. Indeed, it is proved by Ilyashenko [7] that the first-return maps of hyperbolic polycycles are Dulac germs in the sense of Definition 2.3. The class of parabolic Dulac germs is the most interesting case in the study of the *cyclicity* of polycycles. Recall that the cyclicity is the maximal number of limit cycles which can appear in a neighborhood of the polycycle in an analytic deformation.

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Cycles are given by zeros of the *displacement germ* $\Delta(x)$, that is the Dulac germ minus the identity. In the case of a non-parabolic Dulac germ, the displacement germ is of the form $\Delta(x) = ax + o(x)$, with $a \neq 0$ [1]. In the parabolic case the displacement function is flatter than a function having a linear term, so one should expect higher cyclicity. This is indeed a theorem in the case of *hyperbolic loops* (*i.e.* polycycles with one vertex): there exist parabolic Dulac germs corresponding to homoclinic loops of arbitrary high cyclicity [16], whereas in the non-parabolic case the cyclicity is one [1]. In [13], universal bounds for cyclicity of hyperbolic polycycles of 2, 3 and 4 vertices are given under some generic conditions. They imply that the corresponding Dulac germ is non-parabolic. The cyclicity problem in the parabolic case, even for polycycles with such low number of vertices, is widely open.

The problem of embedding of diffeomorphisms in a flow has been studied previously in various non-analytic contexts. For a tangent-to-identity $\mathcal{C}^\infty$ germ f with a *non-flat contact* with the identity, the embedding of f in the flow of a smooth vector field is proved by Takens [20, Theorem 4]. Recall that a germ f has a *flat contact with the identity* if $f(x) - x$ tends to 0 faster than any power of x as $x \to 0$. The case of a parabolic $\mathcal{C}^\infty$ germ with a *flat contact* with the identity and 0 as an isolated fixed point, and an extra non-oscillation condition, is considered by Sergeraert [18, Theorem 3.1]. This result can be applied, for example, in the study of conjugation classes of some $\mathcal{C}^\infty$ codimension one foliations of the cylinder $\mathbb{S}^1 \times \mathbb{R}$. Finally, if f is a germ of $\mathcal{C}^r$ diffeomorphism of $\mathbb{R}^+$, $r \geq 2$, with 0 as an isolated fixed point, then a classical result of Szekeres says that f embeds in the flow of a $\mathcal{C}^1$ vector field, which is $\mathcal{C}^{r-1}$ outside 0 [19].

Note that none of these results applies as such in our framework. First, a parabolic Dulac germ f, while having analytic representatives on open intervals $(0, d)$, is not in general $\mathcal{C}^\infty$ at 0. Moreover, thanks to the *quasi-analyticity* result of [7], if f is not equal to the identity, then its Dulac asymptotic expansion is different from x. Hence, unlike the situation considered by Sergeraert, f does not have a flat contact with the identity. Finally, Szekeres' result could be applied to Dulac germs of class $\mathcal{C}^2$ at 0. However, Szekeres' methods do not lead to an asymptotic expansion of the Fatou coordinate in the power-iterated log scale.

In our study, the main difficulty consists in giving a meaning to the notion of transasymptotic expansion. In the classical case of Dulac germs, the problem does not occur as, in the expansions, each power of x is multiplied by a *polynomial* function in $\log(x)$. However, this is not the case anymore for the Fatou coordinate [11]. In order to build a Fatou coordinate, we perform a transfinite version of the classical Poincaré algorithm. In each step, we solve the same Abel equation on the formal and on the germ level. To this end, each time a power of x is multiplied in the expansion by an infinite series, we choose an appropriate representative of this series, consistently with the Abel equation. We call *integral section* this suitable choice of a representative (see Definitions 3.2 and 3.8).

The problem of the choice of a germ represented by an infinite series is a key problem in the study of analytic dynamical systems. The problem appears in Ilyashenko's solution of Dulac's problem. The solution is given by imposing the existence of an analytic extension of the germ to a sufficiently big complex domain

(quasi-analyticity), [7]. In our construction of the Fatou coordinate, the successive choices are done by imposing to the germs that appear in the process to be solutions of Abel equations.

Perspectives. Analytic classification of parabolic analytic germs was given by Ecalle-Voronin moduli (see [5, 21] or [8] for an overview). It was given by comparison of Fatou coordinates on corresponding sectors in the complex plane. We want to extend Ecalle-Voronin moduli to Dulac germs. We hence need Fatou coordinates defined on sufficently large complex sectors. Our main result is formulated on the real line (i.e. on $(0, d),\ d > 0$). However, by Ilyashenko [6], the Dulac germ extends to a *standard quadratic domain* in the complex domain. In this paper, we construct the Fatou coordinate in a small sector in the complex domain containing the interval $(0, d)$. Extension of the construction to maximal f-invariant domains should permit the description of the dynamics of the complex Dulac germ on a standard quadratic domain. By comparison of Fatou coordinates on different sectors, we expect to obtain in the future the definition of Ecalle-Voronin moduli for Dulac germs.

An additional motivation for our study is to answer the following question: *Can we* recognize *a Dulac germ by looking at the size of ε-neighborhoods of its orbits?* A similar question, but for analytic germs, was discussed in [14] and [15]. The embedding in a flow of a Dulac germ f will be used in [12] to define an appropriate generalization of the length of the ε-neighborhood of an orbit of f. The idea is to read the formal class and the Ecalle-Voronin moduli of a Dulac germ in the *parameter ε-space*, rather than in the phase space.

Organization of the paper. In Section 2 we formulate our main theorem about the existence and the uniqueness of the Fatou coordinate for a Dulac germ f and its Dulac expansion $\widehat{f}$ respectively.

In Section 3, we show the non-uniqueness of transasymptotic expansions in the power-iterated log scale in general. In order to remedy this flaw, we introduce the notion of *sectional asymptotic expansions* and define a particular type of such expansions adapted to Dulac germs and their corresponding Fatou coordinates which will ensure their uniqueness. The proof of the results of this section is given in Section 7.

We recall in Section 4 the classical notion of *embedding as the time-one map of a flow* and state the equivalence between the existence of a Fatou coordinate and an embedding in a flow, for analytic germs on open intervals and for parabolic transseries. The proof of these facts is also postponed to Section 7.

In Section 5 we give some examples of sectional asymptotic expansions.

Finally, Section 6 is dedicated to the precise description of the Fatou coordinate of a Dulac germ and to the proof of the Theorem. The existence of a Fatou coordinate is established simultaneously for a Dulac germ and its formal expansion. This allows in particular to prove that the Fatou coordinate of a Dulac germ admits a sectional asymptotic expansion in a power-iterated log scale, in the sense of Section 3. It is worth noticing that the proofs in Section 6 rely on the particular form of Dulac transseries. Part of the proof of the existence of the Fatou coordinate for Dulac germs in Section 6 is inspired by a similar classical result for parabolic analytic germs which is explained, for example, in [9].

2. Main definitions and results

We define here the classes of *power-iterated log transseries*, and recall the notions of *Dulac germs* and *Fatou coordinate*, which are needed to state the Theorem about the Fatou coordinate of a Dulac germ.

We first introduce several classes of transseries. We put $\boldsymbol{\ell}_0 := x$, $\boldsymbol{\ell} := \boldsymbol{\ell}_1 := \frac{1}{-\log x}$, and define inductively $\boldsymbol{\ell}_{j+1} = \boldsymbol{\ell} \circ \boldsymbol{\ell}_j$, $j \in \mathbb{N}$, as symbols for iterated logarithms.

Definition 2.1 (The classes $\widehat{\mathcal{L}}_j^\infty$ and $\widehat{\mathfrak{L}}$)**.** *Denote by $\widehat{\mathcal{L}}_j^\infty$, $j \in \mathbb{N}_0$, the set of all transseries of the type:*

$$\widehat{f}(x) = \sum_{i_0=0}^{\infty} \sum_{i_1=0}^{\infty} \cdots \sum_{i_j=0}^{\infty} a_{i_0,\dots,i_j} x^{\alpha_{i_0}} \boldsymbol{\ell}^{\alpha_{i_0,i_1}} \cdots \boldsymbol{\ell}_j^{\alpha_{i_0,\dots,i_j}}, \ a_{i_0,\dots,i_j} \in \mathbb{R}, \ x > 0, \tag{2.1}$$

where $(\alpha_{i_0,\dots,i_k})_{i_k \in \mathbb{N}}$ is a strictly increasing sequence of real numbers tending to $+\infty$ (or finite), for every $k = 0, \dots, j$. If, moreover, $\alpha_0 > 0$ (the infinitesimal cases), we denote the class by $\widehat{\mathcal{L}}_j$. The subset of $\widehat{\mathcal{L}}_1$ resp. $\widehat{\mathcal{L}}_1^\infty$ of transseries with only integer powers of $\boldsymbol{\ell}$ will be denoted by $\widehat{\mathcal{L}}$ resp. $\widehat{\mathcal{L}}^\infty$.

A monomial *in $\widehat{\mathcal{L}}_j^\infty$, $j \in \mathbb{N}_0$, is any term of the form $a x^{\gamma_0} \boldsymbol{\ell}^{\gamma_1} \cdots \boldsymbol{\ell}_j^{\gamma_j}$, $\gamma_i \in \mathbb{R}$, $i \in \{0, \dots, j\}$, and $a \in \mathbb{R} \setminus \{0\}$.*

A transseries $x^{\alpha_{i_0}} \big(\sum_{i_1=0}^{\infty} \cdots \sum_{i_j=0}^{\infty} a_{i_0,\dots,i_j} \boldsymbol{\ell}^{\alpha_{i_0,i_1}} \cdots \boldsymbol{\ell}_j^{\alpha_{i_0,\dots,i_j}} \big)$, $i_0 \in \mathbb{N}_0$, from (2.1) *is called a* block *of $\widehat{f} \in \widehat{\mathfrak{L}}$. That is, a block of $\widehat{f} \in \widehat{\mathfrak{L}}$ is a transseries containing all monomials from $\widehat{f}$ sharing the same power of x.*

For $\widehat{f} \in \widehat{\mathcal{L}}_j^\infty$, the support of *$\widehat{f}$, denoted by* $\mathrm{Supp}(\widehat{f})$*, is defined as the set of exponents of all monomials in $\widehat{f}$ with non-zero coefficients:*

$$\mathrm{Supp}(\widehat{f}) := \big\{ (\alpha_{i_0}, \alpha_{i_0,i_1}, \dots, \alpha_{i_0,i_1,\dots,i_j}) : \ a_{i_0,\dots,i_j} \neq 0 \big\}.$$

Put

$$\widehat{\mathfrak{L}} := \bigcup_{j \in \mathbb{N}_0} \widehat{\mathcal{L}}_j^\infty$$

for the class of all power-iterated logarithm transseries of finite depth in iterated logarithms.

Note that the classes $\widehat{\mathcal{L}}_j^\infty$, $j \in \mathbb{N}_0$, are the sub-classes of power-iterated logarithm transseries, whose support is any well-ordered subset of $\mathbb{R}^{j+1}$ (for the lexicographic order). We restrict only to the subclass with exponents forming a strictly increasing sequence tending to $+\infty$. In this paper, we work with Dulac germs and their expansions, for which this condition is verified.

Notice that $x = \boldsymbol{\ell}_0$. The classes $\widehat{\mathcal{L}}_0$ or $\widehat{\mathcal{L}}_0^\infty$ are made of formal power series:

$$\widehat{f}(x) = \sum_{i \in \mathbb{N}} a_i x^{\alpha_i}, \ a_i \in \mathbb{R}, \ x > 0,$$

such that $(\alpha_i)_i$ is a strictly increasing real sequence tending to $+\infty$.

For $\widehat{f} \in \widehat{\mathcal{L}}_j^\infty$, we denote by $\mathrm{Lt}(\widehat{f})$ its *leading term*, which is defined as the smallest term $a_{\gamma_0,\gamma_1,\dots,\gamma_j} \, x^{\gamma_0} \boldsymbol{\ell}^{\gamma_1} \boldsymbol{\ell}_2^{\gamma_2} \cdots \boldsymbol{\ell}_j^{\gamma_j}$ in $\widehat{f}$ (by the lexicographic order on the monomials)

with a non-zero coefficient $a_{\gamma_0,\gamma_1,\dots,\gamma_j} \neq 0$. The tuple $(\gamma_0, \gamma_1, \dots, \gamma_j)$ is called *the order* of $\widehat{f}$, and is denoted by $\mathrm{ord}(\widehat{f}) = (\gamma_0, \gamma_1, \dots, \gamma_j)$. The transseries $\widehat{f} \in \widehat{\mathcal{L}}_j$, $j \in \mathbb{N}_0$, is called *parabolic* if $\mathrm{ord}(\widehat{f}) = (1, 0, \dots, 0)$.

We denote by $\mathcal{G}$ the set of all germs at 0 of real functions defined on some open interval $(0, \varepsilon)$, $\varepsilon > 0$ (meaning that two functions f and g define the same germ if there exists an interval $(0, \varepsilon)$, $\varepsilon > 0$, where they coincide). Furthermore, by $\mathcal{G}_{AN} \subset \mathcal{G}$ we denote the set of all germs at 0 of real functions defined and *analytic* on some interval $(0, \varepsilon)$, $\varepsilon > 0$.

In the paper, we will use notation $o(\cdot)$ in two different contexts: for germs from $\mathcal{G}$ and for transseries from $\widehat{\mathfrak{L}}$. We always mean that $x \to 0$. In the case of germs, $f(x) = o(x^{\gamma_0}\boldsymbol{\ell}^{\gamma_1} \cdots \boldsymbol{\ell}_k^{\gamma_k})$ means that $\lim_{x\to 0} \frac{f(x)}{x^{\gamma_0}\boldsymbol{\ell}^{\gamma_1}\cdots\boldsymbol{\ell}_k^{\gamma_k}} = 0$. In the case of transseries, $\widehat{f}(x) = o(x^{\gamma_0}\boldsymbol{\ell}^{\gamma_1} \cdots \boldsymbol{\ell}_k^{\gamma_k})$ means that the leading monomial of $\widehat{f}$ is lexicographically of strictly bigger order than the monomial $x^{\gamma_0}\boldsymbol{\ell}^{\gamma_1} \cdots \boldsymbol{\ell}_k^{\gamma_k}$.

Definition 2.2. *Let $(\alpha_i)_i$ be a sequence of strictly positive real numbers. We say that $(\alpha_i)_i$ is an* increasing sequence of finite type *if it is of one of the following types:*

(1) finite and strictly increasing with i, or
(2) infinite, strictly increasing as $i \to \infty$ and finitely generated *(there exist strictly positive real generators $\beta_1, \dots, \beta_n$, such that for every α_i there exist $a_1^i, \dots, a_n^i \in \mathbb{N}$ such that $\alpha_i = \sum_{j=1}^n a_j^i \beta_j$).*

Note that in the case (2) *it necessarily follows that $\alpha_i \to \infty$, as $i \to \infty$.*

We now recall the definition of a *Dulac series* from [4], [6] or [17], and define what we mean by a *Dulac germ*.

Definition 2.3 (Dulac germs)**.**

1. We say that $\widehat{f} \in \widehat{\mathcal{L}}$ is a Dulac series ([4], [6], [17]) *if it is of the form:*

$$\widehat{f} = \sum_{i=1}^{\infty} P_i(\boldsymbol{\ell}) x^{\alpha_i}, \tag{2.2}$$

where $(P_i)_i$ is a sequence of polynomials and $(\alpha_i)_i$ an increasing sequence of finite type (see Definition 2.2).

2. We say that $f \in \mathcal{G}_{AN}$ is a Dulac germ *if:*

- *there exists a sequence (P_i) of* polynomials *and an increasing sequence of finite type $(\alpha_i)_i$ (see Definition 2.2), such that*

$$f - \sum_{i=1}^{n} P_i(\boldsymbol{\ell}) x^{\alpha_i} = o(x^{\alpha_n}), \ n \in \mathbb{N},$$

- *f is* quasi-analytic*: it can be extended to an analytic function to a standard quadratic domain in $\mathbb{C}$ with the same expansion* (2.2)*, as precisely defined by Ilyashenko, see* [6], [17].

If moreover $P_1 \equiv 1$, $\alpha_1 = 1$, and at least one of the polynomials P_i, $i > 1$, is not zero, then f is called a parabolic Dulac germ.

The quasi-analyticity property ensures that a Dulac germ f is uniquely determined by its Dulac asymptotic series $\widehat{f}$, see [6].

Note that the germs of first return maps of hyperbolic polycycles of planar analytic vector fields are Dulac germs in the sense of Definition 2.3, see e.g. [4, 6].

We recall finally what is the Fatou coordinate of a real germ:

Definition 2.4 (Fatou coordinate)**.**
1. Let f be an analytic germ on $(0,d)$, $d>0$. Let $(0,d)$ be invariant for f. We say that a strictly monotonic analytic germ Ψ on $(0,d)$ is a Fatou coordinate *for f if*

$$\Psi(f(x))-\Psi(x)=1,\ x\in(0,d). \tag{2.3}$$

2. Let $\widehat{f}\in\widehat{\mathcal{L}}$ be parabolic. We say that $\widehat{\Psi}\in\widehat{\mathfrak{L}}$ is a formal Fatou coordinate *for $\widehat{f}$ if the following equation is satisfied formally in $\widehat{\mathfrak{L}}$:*

$$\widehat{\Psi}(\widehat{f})-\widehat{\Psi}=1. \tag{2.4}$$

Classically, equation (2.3) is called the *Abel equation* for f.

We now formulate the main result of this paper. The notion of *sectional asymptotic expansions* is introduced in Section 3 to ensure the uniqueness of transfinite asymptotic expansions in $\widehat{\mathfrak{L}}$. In particular, the notion of *integral sections* is adapted to the Fatou coordinate.

The constructive proof of the Theorem and a more precise description of the Fatou coordinate for a Dulac germ is given in Section 6.

Theorem. *Let $f\in\mathcal{G}_{AN}$ be a parabolic Dulac germ and let $\widehat{f}\in\widehat{\mathcal{L}}$ be its Dulac expansion.*

1. There exists a unique (up to an additive constant) formal Fatou coordinate $\widehat{\Psi}$ for $\widehat{f}$ in $\widehat{\mathfrak{L}}$. It belongs to $\widehat{\mathcal{L}}_2^\infty$.

2. There exists a unique (up to an additive constant) Fatou coordinate $\Psi\in\mathcal{G}_{AN}$ for the germ f which admits a sectional asymptotic expansion with respect to an integral section in the class $\widehat{\mathfrak{L}}$ (in the sense of Definitions 3.3 and 3.13).

3. Let $\mathbf{s}$ be a fixed integral section and $\widehat{\Psi}\in\widehat{\mathcal{L}}_2^\infty$ the formal Fatou coordinate (with a fixed choice of the additive constant). Then there exists a choice of the additive constant in $\Psi\in\mathcal{G}_{AN}$ from 2. such that the formal Fatou coordinate $\widehat{\Psi}$ is the (unique) sectional asymptotic expansion of $\Psi\in\mathcal{G}_{AN}$ with respect to $\mathbf{s}$. The Fatou coordinate Ψ is of the form:

$$\Psi=\Psi_\infty+R,$$

where $\Psi_\infty\to\infty$ and $R=o(1)$, as $x\to 0$.

4. $\widehat{\Psi}=\widehat{\Psi}_\infty+\widehat{R}$, where $\widehat{\Psi}_\infty\in\widehat{\mathcal{L}}_2^\infty$ is the sectional asymptotic expansion of Ψ_∞ with respect to $\mathbf{s}$ and $\widehat{R}\in\widehat{\mathcal{L}}$ is the sectional asymptotic expansion of R with respect to $\mathbf{s}$.

Note that different choices of integral sections $\mathbf{s}$ in *3.* lead to change Ψ only by an additive constant $C\in\mathbb{R}$.

Note that $\Psi_\infty\to\infty$, $x\to 0$, is the *infinite* part, and $R=o(1)$, $x\to 0$, is the *infinitesimal* part. We call Ψ_∞ the *principal part* of Ψ.

Remark 2.5. Let $f \in \mathcal{G}_{AN}$ be a parabolic Dulac germ. Let $\mathrm{ord}(\mathrm{id} - f) = (\alpha_1, m)$, $m \in \mathbb{N}_0^-$, $\alpha_1 > 1$. The function R from the Theorem satisfies the *modified Abel* difference equation:

$$R(f(x)) - R(x) = \delta(x). \tag{2.5}$$

Here, δ is analytic on an open interval $(0, d)$, $d > 0$, and *small*: $\delta(x) = O(x^\gamma)$, with $\gamma > \alpha_1 - 1$.

Remark 2.6 (*Non-uniqueness* of the Fatou coordinate in $\mathcal{G}_{AN}$, if one does not require the existence of its asymptotic expansion in $\widehat{\mathfrak{L}}$)**.**

Note that any strictly monotonic $\Psi \in \mathcal{G}_{AN}$ whose inverse Ψ^{-1}, as a germ at infinity, satisfies $\Psi^{-1}(w+1) = f(\Psi^{-1}(w))$, is a Fatou coordinate for f. This gives us freedom of choice of Ψ^{-1} on the fundamental domain $[0, 1)$ and the rule for its extension at the neighborhood of ∞, thus, non-unicity of a Fatou coordinate for f.

In particular, let $\Psi_1 \in \mathcal{G}_{AN}$ be the Fatou coordinate constructed in the proof of the Theorem that admits a sectional expansion $\widehat{\Psi}_1 \in \widehat{\mathcal{L}}_2^\infty$. Let $\Psi_2 \in \mathcal{G}_{AN}$ be defined by $\Psi_2 := \Psi_1 + T_1 \circ \Psi_1$, where T_1 is any periodic function on $\mathbb{R}$ of period 1 whose derivative T_1' is bounded in $(-1, 1)$ (e.g. $T_1(x) = \frac{1}{4\pi}\sin(2\pi x),\ x \in \mathbb{R}$). It can be easily checked that Ψ_2 is also a Fatou coordinate for f (by Definition 2.4). It does not admit an expansion in $\widehat{\mathfrak{L}}$, due to periodicity of T_1.

Remark 2.7. Note that the Fatou coordinate $\Psi \in \mathcal{G}_{AN}$ for Dulac germ f in the Theorem admitting only a *sectional* asymptotic expansion in $\widehat{\mathfrak{L}}$ is *not* unique, as we will show in Example 6 in Subsection 6.1.5. On the other hand, we prove in Subsection 6.1.5 that a Fatou coordinate $\Psi \in \mathcal{G}_{AN}$ for f admitting a sectional asymptotic expansion with respect to an *integral sectional* in $\widehat{\mathfrak{L}}$ is unique.

3. Sectional asymptotic expansions

This paper is motivated by the study of Dulac germs (see Definition 2.3) and their Fatou coordinates. The Dulac asymptotic expansions involve monomials of the form $x^\alpha \log^p x$, $\alpha \in \mathbb{R}$, $p \in \mathbb{N}$, where each power of x is multiplied by a *polynomial* in $\log x$. The Dulac expansion of a germ is therefore uniquely given by the classical Poincaré algorithm. On the contrary, it turns out that the *asymptotic expansions* of Fatou coordinates of Dulac germs involve also *powers of x multiplied by possibly divergent series in iterated logarithms.* Hence, the classical Poincaré algorithm does not suffice to produce them. In this work we give a generalization of the algorithm to explain what it means for a germ from $\mathcal{G}$ to have a transfinite asymptotic expansion. In particular, our construction applies to Fatou coordinates.

Let us first illustrate our definitions on some examples.

Example 1. 1) Suppose that we want to express that the asymptotic expansion of the germ $f \in \mathcal{G}$ is the series $\widehat{f}(x) = x\left(1 + \boldsymbol{\ell} + \boldsymbol{\ell}^2 + \cdots\right) + x^2$. Obviously, we first require that, for every $p \in \mathbb{N}$, $f(x) - \sum_{n=0}^{p} x\boldsymbol{\ell}^n = o(x\boldsymbol{\ell}^p)$. This corresponds to the first steps of the Poincaré algorithm, which are indexed by integers. Now an extra step is needed, which may be thought of as indexed by the ordinal number ω. To perform this step, we can take advantage of the convergence of the series $\sum_{n\geq 0} \boldsymbol{\ell}^n = \frac{1}{1-\boldsymbol{\ell}}$ to require that $f(x) - x\frac{1}{1-\boldsymbol{\ell}}$ is equivalent to x^2 at the origin.

2) The above process does not work if the first power of x in the series $\widehat{f}$ is multiplied by a *divergent series* in $\boldsymbol{\ell}$. For example, how to give a meaning to the

statement that the series $\widehat{f}(x) = x\sum_{n\geq 0} n!\boldsymbol{\ell}^n + x^2$ is an asymptotic expansion of the germ $f \in \mathcal{G}$? Our answer consists in choosing a germ $g \in \mathcal{G}$ to which the series $x\sum_{n\geq 0} n!\boldsymbol{\ell}^n$ is asymptotic *in the classical sense*. The germ g can be seen as a *sum* of this series. Then we perform an extra step by requiring that $f(x) - g(x)$ is equivalent to x^2 at the origin. That is, we first follow the usual Poincaré algorithm along steps indexed by integers. Once we have reached the step indexed by the first *limit ordinal* ω, we associate a *sum* g to (in general divergent) series $x\sum_{n\geq 0} n!\boldsymbol{\ell}^n$ in order to proceed further.

3) In the same way, we say that a germ $f \in \mathcal{G}$ admits an asymptotic expansion $\widehat{f}(x) = x\sum_{n\geq 0} n!\boldsymbol{\ell}^n + x^2\sum_{n\geq 0}(n!)^2\boldsymbol{\ell}^n + x^3$ if there exist two germs $g_1, g_2 \in \mathcal{G}$ (two *sums*) such that:

- $x\sum_{n\geq 0} n!\boldsymbol{\ell}^n$ is the (classical Poincaré) asymptotic expansion of g_1 at 0,
- $x^2\sum_{n\geq 0}(n!)^2\boldsymbol{\ell}^n$ is the (classical) asymptotic expansion of g_2 at 0, and
- $f - g_1 - g_2$ is equivalent to x^3 at the origin.

In this case, we will say that $x\sum_{n\geq 0} n!\boldsymbol{\ell}^n + x^2\sum_{n\geq 0}(n!)^2\boldsymbol{\ell}^n$ is a *transfinite* asymptotic expansion of $g_1 + g_2$ and that $\widehat{f}$ is a *transfinite* asymptotic expansion of f.

Remark 3.1 (Non-uniqueness of asymptotic expansions in $\widehat{\mathfrak{L}}$ of germs from $\mathcal{G}$)**.** In Subsection 3.1 below, we give a general version of the method illustrated by the above examples. The usual algorithm due to Poincaré, which associates an asymptotic expansion to a germ, proceeds term by term along steps indexed by integers, that is, by ordinals less than ω. For germs of functions considered in this work, *transfinite* asymptotic expansions will be produced in $\widehat{\mathfrak{L}}$ by a *transfinite version* of this algorithm. The algorithm continues along steps indexed by ordinals (bigger than ω), as it is the case in the examples above. We see that, when we reach a step indexed by a *limit ordinal*, we have to provide a *sum* in order to continue.

This feature leads us to stress an important fact. Whereas the classical algorithm associates to a germ a well-defined, unique asymptotic expansion, the expansions produced in $\widehat{\mathfrak{L}}$ by the generalized method are in general not unique: in the examples above, different choices of *sums* g, g_1 and g_2 may lead to different asymptotic expansions. This non-uniqueness is illustrated by Example 2 below.

3.1. **Transfinite Poincaré algorithm.** Let the classes $\widehat{\mathcal{L}}_j^{\infty}$, $j \in \mathbb{N}$, and $\widehat{\mathfrak{L}}$ be as in Definition 2.1. In this section we define in full generality what it means for a series $\widehat{f} \in \widehat{\mathfrak{L}}$ to be an asymptotic expansion of a germ $f \in \mathcal{G}$. This definition can be seen as *transfinite* version of the usual definition of Poincaré.

Consider a germ $f \in \mathcal{G}$, a transseries $\widehat{f} \in \widehat{\mathfrak{L}}$ and an ordinal $\theta \geq 1$ (where 0 denotes the smallest ordinal). We say that $\widehat{f}$ is a *truncated asymptotic expansion of length θ of f*, and we write $f \underset{\theta}{\sim} \widehat{f}$, if there exist a sequence $(h_\alpha)_{0\leq\alpha<\theta}$ of elements of $\mathcal{G}$, and a sequence $(\widehat{f}_\alpha)_{0\leq\alpha<\theta}$ of elements of $\widehat{\mathfrak{L}}$, such that:

1. $\widehat{f} = \lim_{\nu<\theta}\widehat{f}_\nu$ (for the product topology with respect to the discrete topology introduced in [11]);
2. $h_0 = f$, $\widehat{f}_0 = 0$;
3. for all $0 < \alpha < \theta$, we have:

(a) if $\alpha := \nu + 1$ is a successor ordinal, then
  (i) either $h_\nu \to 0$ faster than *any* monomial from $\widehat{\mathfrak{L}}$ and $\theta = \alpha + 1$, $\widehat{f}_\alpha = \widehat{f}_\nu$ and $h_\alpha = h_\nu$,
  (ii) or there exists a monomial in $\widehat{\mathfrak{L}}$ (see Definition 2.1), denoted by $\mathrm{Lt}\,(h_\nu) \in \widehat{\mathfrak{L}}$, such that

$$\lim_{x\to 0} \frac{h_\nu\,(x)}{\mathrm{Lt}\,(h_\nu)} = 1,\ \widehat{f}_\alpha = \widehat{f}_\nu + \mathrm{Lt}\,(h_\nu)\,,\ h_\alpha = h_\nu - \mathrm{Lt}\,(h_\nu)\,,$$

(b) if $\alpha < \theta$ is a limit ordinal, then $\widehat{f}_\alpha = \lim_{\nu<\alpha} \widehat{f}_\nu$ and
  (i) either $\theta = \alpha + 1$ and, for every $\beta \in \mathbb{R}$, there exists a block of $\widehat{f}_\alpha$ whose monomials are smaller than x^β,
  (ii) or there exists a germ $g_\alpha \in \mathcal{G}$ with

$$g_\alpha \underset{\alpha}{\sim} \widehat{f}_\alpha \text{ and } h_\alpha = f - g_\alpha.$$

Notice that in case (b), the sequence $(\widehat{f}_\nu)_{\nu<\alpha}$ necessarily converges in the product topology with respect to the discrete topology introduced in [11], due to the fact that the orders of the monomials in the expansion strictly increase in each step.

Moreover, the existence of such a germ g_α is trivially guaranteed. Indeed, without any additional restrictions imposed on the choice, we always have a trivial choice of the germ: if $f \underset{\theta}{\sim} \widehat{f}$, then f itself admits an asymptotic expansion $\widehat{f}_\alpha$ of length α, $f \underset{\alpha}{\sim} \widehat{f}_\alpha$, for any $\alpha < \theta$. Of course, we may (and usually do) choose some other germ that admits the same expansion. In particular, a good choice of a germ will be given by an integral section (see Section 3.3).

Finally, in the cases 3.(a)(i) and 3.(b)(i) of the above definition, the series $\widehat{f}_\alpha$ is considered as a *total expansion* of f, rather than a *truncated expansion*. We call $\widehat{f}_\alpha$ simply an *asymptotic expansion of* f, and we write $f \sim \widehat{f}_\alpha$, without any reference to its length.

Actually, the germs g_α above can be seen as *sums* of the transseries $\widehat{f}_\alpha$. Since $\boldsymbol{\ell}_k$ are exponentially small with respect to $\boldsymbol{\ell}_{k+1}$, $k \in \mathbb{N}_0$, as $x \to 0$, because of the relation

$$e^{-\frac{1}{\boldsymbol{\ell}_{k+1}}} = \boldsymbol{\ell}_k,$$

different choices of sums g_α lead in general to different transfinite asymptotic expansions. This is illustrated in Example 2 below.

Example 2.

(1) $f(x) = \frac{x}{1-\boldsymbol{\ell}} + x^2$. We can also write it as $f(x) = x(\frac{1}{1-\boldsymbol{\ell}} + e^{-\frac{1}{\boldsymbol{\ell}}})$. The algorithm described above produces the intermediate series $\widehat{f}_\omega = x\left(1 + \boldsymbol{\ell} + \boldsymbol{\ell}^2 + \cdots\right)$. Now we can associate two sums to $\widehat{f}_\omega$, namely:

$$g_{\omega,1}\,(x) = \frac{x}{1-\boldsymbol{\ell}} \text{ and } g_{\omega,2}\,(x) = \frac{x}{1-\boldsymbol{\ell}} + x^2 = f\,(x)\,.$$

Note that the second sum corresponds to the ending remark in the algorithm, which says that f itself can always be chosen as a possible sum for any of the intermediate series $\widehat{f}_\alpha$, for $\alpha < \theta$ limit ordinal. Hence, depending

on the choice made at stage ω, the series:

$$\widehat{f}(x) = x(1+\boldsymbol{\ell}+\boldsymbol{\ell}^2+\boldsymbol{\ell}^3+\cdots), \text{ and } \widehat{f}(x) = x(1+\boldsymbol{\ell}+\boldsymbol{\ell}^2+\cdots)+x^2,$$

are two (equally good) asymptotic expansions of f in $\widehat{\mathcal{L}} \subset \widehat{\mathfrak{L}}$.

(2) $f(x) = \frac{x}{1-\boldsymbol{\ell}}$. Obviously, f admits an asymptotic expansion $\widehat{f} \in \widehat{\mathcal{L}}_1$, $\widehat{f} = x(1+\boldsymbol{\ell}+\boldsymbol{\ell}^2+\cdots)$. However, equally legitimate asymptotic expansions by the transfinite Poincaré algorithm from Remark 3.1 are, for example:

$$f(x) = x\big(\frac{1}{1-\boldsymbol{\ell}} + \boldsymbol{\ell}_k e^{-\frac{1}{\boldsymbol{\ell}}}\big) - x^2\boldsymbol{\ell}_k$$
$$\Rightarrow \widehat{f}(x) = x(1+\boldsymbol{\ell}+\boldsymbol{\ell}^2+\cdots) - x^2\boldsymbol{\ell}_k \ \in \widehat{\mathcal{L}}_k, \ k \in \mathbb{N}.$$

In this manner, we can easily generate non-unique asymptotic expansions of $f \in \mathcal{G}$ in $\widehat{\mathfrak{L}}$. Moreover, they can belong to any $\widehat{\mathcal{L}}_k$, $k \in \mathbb{N}$. Note additionally that there does not exist a minimal $j \in \mathbb{N}$ such that the expansion of f in $\widehat{\mathcal{L}}_j$ is unique. Indeed, if we put $\boldsymbol{\ell}$ instead of $\boldsymbol{\ell}_k$ in the example above, we get non-uniqueness of the expansion already in the class $\widehat{\mathcal{L}}_1$.

On the contrary, just note that in the class $\widehat{\mathcal{L}}_0^\infty$, the asymptotic expansion of a germ, if it exists, is unique (by the Poincaré algorithm).

3.2. **Sections. Sectional asymptotic expansions.** Based on the content of the previous section, we can now define the notion of *asymptotic transseries*:

Definition 3.2 (The set $\widehat{\mathcal{S}}$ of asymptotic transseries in $\widehat{\mathfrak{L}}$)**.** *A series $\widehat{f} \in \widehat{\mathfrak{L}}$ such that there exists an element $f \in \mathcal{G}$ with $f \sim \widehat{f}$ (as described in Section 3.1) is called an* asymptotic transseries*. We denote by $\widehat{\mathcal{S}} \subseteq \widehat{\mathfrak{L}}$ the set of all asymptotic transseries. By $\mathcal{S} \subseteq \mathcal{G}$, we denote the set of all germs which admit an asymptotic expansion in $\widehat{\mathcal{S}}$ (in the sense of Subsection 3.1).*

We have stressed in the previous section the non-uniqueness of asymptotic expansions of germs in $\mathcal{S}$. This comes from the multiple "sums" which can be associated to the intermediate series at each limit ordinal step of the transfinite Poincaré algorithm. Therefore, in Definition 3.2, asymptotic expansions of germs in $\widehat{\mathfrak{L}}$ (as well as in any $\widehat{\mathcal{L}}_j^\infty$, $j \geq 1$) are not unique. Note that the power asymptotic expansion of a germ (that is, the asymptotic expansion in $\widehat{\mathcal{L}}_0^\infty$), if it exists, is by classical Poincaré algorithm always unique. The uniqueness of the transfinite asymptotic expansion can be obtained by setting a "summation mapping" which associates to every series $\widehat{f}_\alpha$ indexed by limit ordinal α a well defined sum $g_\alpha \in \mathcal{G}$ (using the notations of Section 3.1).

Actually, picking a particular "sum" of a series among infinitely many possible ones is an operation quite similar to picking an element in the fiber over a point of the base space, in the situation of a fiber bundle. Hence in the sequel we call such a summation mapping a *section* (see the precise Definition 3.3), and the associated expansions *sectional asymptotic expansions*. In particular, we define later a particular choice of a section adapted to our problem that will guarantee the uniqueness of the asymptotic expansion of a germ with respect to this section.

It is convenient to impose upon our sections to be linear maps defined on appropriate vector subspaces of $\widehat{\mathcal{S}}$. Notice however that, while $\widehat{\mathcal{S}}$ is obviously a vector space, it is not the case for the set $\mathcal{S}$, as the following example shows.

Example 3 ($\mathcal{S}$ is not a vector space)**.** Take

$$f_1(x) = x\frac{1}{1-\boldsymbol{\ell}} \in \mathcal{G},\ f_2(x) = x\frac{1}{1-\boldsymbol{\ell}} + x^2\sin(x) \in \mathcal{G}$$

Note that f_2 can be written as $f_2(x) = x\big(\frac{1}{1-\boldsymbol{\ell}} + e^{-1/\boldsymbol{\ell}}\sin(e^{-1/\boldsymbol{\ell}})\big)$. Since both germs $\frac{1}{1-\boldsymbol{\ell}}$ and $\frac{1}{1-\boldsymbol{\ell}} + e^{-1/\boldsymbol{\ell}}\sin(e^{-1/\boldsymbol{\ell}})$ admit $1+\boldsymbol{\ell}+\boldsymbol{\ell}^2+\ldots \in \widehat{\mathcal{L}}_0$ as a power asymptotic expansion, both f_1 and f_2 admit $\widehat{f} = x(1+\boldsymbol{\ell}+\boldsymbol{\ell}^2+\ldots) \in \widehat{\mathcal{L}}_1$ as their asymptotic expansion in $\widehat{\mathfrak{L}}$ (as in Definition 3.2). On the other hand, $f_1(x) - f_2(x) = x^2\sin x$ obviously does not admit an asymptotic expansion in $\widehat{\mathfrak{L}}$.

Definition 3.3.

(*i*) *A* section $\mathbf{s}$ *is any linear mapping defined on some vector space* $\widehat{\mathcal{I}}_{\mathbf{s}} \subseteq \widehat{\mathcal{S}}$,

$$\mathbf{s} : \widehat{\mathcal{I}}_{\mathbf{s}} \to \mathcal{G}_{AN},$$

where $\widehat{g} \in \widehat{\mathcal{S}}$ *is an asymptotic expansion of* $\mathbf{s}(\widehat{g}) \in \mathcal{G}_{AN}$ (*obtained by the transfinite Poincaré algorithm from Section 3.1*).

We say that the germ $g = \mathbf{s}(\widehat{g})$ *is the* sectional sum of $\widehat{g}$ with respect to the section $\mathbf{s}$.

(*ii*) *We say that a germ* $f \in \mathcal{G}_{AN}$ *admits a* sectional asymptotic expansion $\widehat{f} \in \widehat{\mathcal{S}}$ with respect to the section $\mathbf{s}$ *if:*

- $f \sim \widehat{f}$ (*see Section 3.1*),
- *at each limit ordinal step* $0 < \alpha < \theta$ *it holds that* $\widehat{f}_\alpha \in \widehat{\mathcal{I}}_{\mathbf{s}}$ *and* $g_\alpha = \mathbf{s}(\widehat{f}_\alpha)$, *where* θ *is the length of the expansion* $\widehat{f}$ *of* f.

That is, the choice of germs $g_\alpha \in \mathcal{G}_{AN}$ *at limit ordinal steps of the transfinite Poincaré algorithm is uniquely determined by the section* $\mathbf{s}$.

Notice that there exist germs from $\mathcal{G}$ that admit sectional asymptotic expansions in $\widehat{\mathfrak{L}}$ with respect to some section, but which do not admit sectional asymptotic expansion in $\widehat{\mathfrak{L}}$ with respect to some other section. Take, for example, the germ

$$f(x) = x \cdot \Big(\frac{1}{1-\boldsymbol{\ell}} + x \cdot \sin\frac{1}{\boldsymbol{\ell}}\Big)$$

and any section $\mathbf{s}$ such that $\mathbf{s}\big(x\sum_{k=0}^{\infty}\boldsymbol{\ell}^k\big) = \frac{x}{1-\boldsymbol{\ell}}$. Obviously, f does not admit asymptotic expansion in $\widehat{\mathfrak{L}}$ with respect to $\mathbf{s}$. On the contrary, take any section $\mathbf{s}$ such that $\mathbf{s}\big(x\sum_{k=0}^{\infty}\boldsymbol{\ell}^k\big) = \frac{x}{1-\boldsymbol{\ell}} + e^{-\frac{1}{\boldsymbol{\ell}}}\sin\frac{1}{\boldsymbol{\ell}}$. Then the sectional asymptotic expansion of f with respect to the section $\mathbf{s}$ is then equal to $\widehat{f}(x) = x\sum_{k=0}^{\infty}\boldsymbol{\ell}^k$.

Proposition 3.4 (Uniqueness of the $\mathbf{s}$-sectional asymptotic expansion)**.** *The* $\mathbf{s}$-*sectional asymptotic expansion of a germ* $f \in \mathcal{G}$, *if it exists, is unique.*

The proof follows by the definition of $\mathbf{s}$-sectional asymptotic expansions and the transfinite Poincaré algorithm from Subsection 3.1.

On the other hand, note that by Definition 3.3, the injectivity of the mapping $f \mapsto \widehat{f}$, where $\widehat{f}$ is the $\mathbf{s}$-sectional asymptotic expansion of f, is *not* implied.

It is clear that all the sectional asymptotic expansions of a germ $f \in \mathcal{S}$ have the same leading term (see Definition 2.1). Hence we can put:

Definition 3.5. *Let* $f \in \mathcal{S}$. *The* leading term of f, *denoted by* $\mathrm{Lt}(f)$, *is the leading term* $\mathrm{Lt}(\widehat{f})$ *of its any sectional asymptotic expansion* $\widehat{f} \in \widehat{\mathcal{S}}$.

We define here what we mean by *convergent transseries* in $\widehat{\mathfrak{L}}$, which are canonically summable in some sense. We introduce a desired property of sections: we call it *coherence*. It means that a section respects convergence, that is, to a convergent transseries in $\widehat{\mathfrak{L}}$ assigns its sum.

Definition 3.6 (Convergent transseries in $\widehat{\mathfrak{L}}$)**.** *Let $\widehat{f} \in \widehat{\mathcal{L}}_j^\infty$, $j \in \mathbb{N}_0$. We will say that $\widehat{f}$ given by* (2.1) *is a* convergent transseries *if* (2.1) *is a summable family of monomials* (*summable pointwise on some open interval* $(0,d)$, $d>0$)*. That is, if there exists $d>0$ such that the multiple sum converges absolutely on* $(0,d)$*:*

$$\sum_{i_0=0}^{\infty}\sum_{i_1=0}^{\infty}\cdots\sum_{i_j=0}^{\infty}\left|a_{i_0\dots i_j}\right| x^{\alpha_{i_0}}\boldsymbol{\ell}^{\alpha_{i_0 i_1}}\cdots\boldsymbol{\ell}_j^{\alpha_{i_0\cdots i_j}}<\infty,\quad x\in(0,d)\,. \tag{3.1}$$

In that case, by $f\in\mathcal{G}$ we denote the sum of $\widehat{f}$ on $(0,d)$ *in the sense of summable families:*

$$\begin{aligned} f(x) := &\sum_{i_0=0}^{\infty}\sum_{i_1=0}^{\infty}\cdots\sum_{i_j=0}^{\infty} a^+_{i_0\dots i_j} x^{\alpha_{i_0}}\boldsymbol{\ell}^{\alpha_{i_0 i_1}}\cdots\boldsymbol{\ell}_j^{\alpha_{i_0\cdots i_j}} \\ &-\sum_{i_0=0}^{\infty}\sum_{i_1=0}^{\infty}\cdots\sum_{i_j=0}^{\infty} a^-_{i_0\dots i_j} x^{\alpha_{i_0}}\boldsymbol{\ell}^{\alpha_{i_0 i_1}}\cdots\boldsymbol{\ell}_j^{\alpha_{i_0\cdots i_j}},\quad x\in(0,d) \end{aligned} \tag{3.2}$$

where $a^+_{i_0\dots i_j} := \max\{a_{i_0\dots i_j},0\}>0$, $a^-_{i_0\dots i_j} := \max\{-a_{i_0\dots i_j},0\}>0$.

For exact definition and properties of summable families, see e.g. [3]. Note that, due to positivity, the order of summation in (3.1) and (3.2) is not important.

Definition 3.7 (Coherent sections)**.** *We say that a section* $\mathbf{s}$ *is* coherent *if it respects* convergence*. That is, if for every convergent $\widehat{f}\in\widehat{\mathcal{I}}_s\subseteq\widehat{\mathcal{S}}$ it holds that $\mathbf{s}(\widehat{f})=f$, where f is the sum of $\widehat{f}$ in the sense of Definition 3.6.*

Let $\mathbf{s}$ be a coherent section. Let $\widehat{f}$ convergent, with sum $f\in\mathcal{G}$, belong to $\widehat{\mathcal{I}}_{\mathbf{s}}$. By Fubini's theorem, all partial expansions of f, $\widehat{f}_\alpha$, with respect to $\mathbf{s}$, are also convergent. By Definition 3.3 of sections, it implies that also $\widehat{f}_\alpha\in\widehat{\mathcal{I}}_{\mathbf{s}}$. Therefore, the above definition of coherence is consistent.

3.3. **Integral sections.** We define here a particular family of coherent sections which we call the *integral sections*. The definition of integral section is based on Definition 3.8, which provides a formula for assigning a unique sum to a special type of divergent series in $\widehat{\mathcal{L}}_0^\infty$, which we call *integrally summable*. The formula is adapted to the Fatou coordinate constructed in Section 6. It uses the fact that the Fatou coordinate, as a formal transseries or as a germ at 0, is a solution of the Abel equation. See Example 4 (2) in Section 5 or Example 5 in Section 6 for a better understanding.

Definition 3.8 (Integrally summable series in $\widehat{\mathcal{L}}_0^\infty$)**.**

(1) *By $\widehat{\mathcal{L}}_0^I\subset\widehat{\mathcal{L}}_0^\infty$ we denote the set of all formal series*

$$\widehat{f}(y)=\sum_{n=N}^{\infty} a_n y^n\in\widehat{\mathcal{L}}_0^\infty,\ N\in\mathbb{Z},\ a_n\in\mathbb{R},$$

which are either:

(i) convergent on an open interval $(0,\delta)$*,* $\delta>0$*, or*
(ii) divergent and such that there exists $\alpha\in\mathbb{R}$*,* $\alpha\neq 0$*, for which*

$$\frac{d}{dx}\Big(x^{\alpha}\widehat{f}(\boldsymbol{\ell})\Big)=x^{\alpha-1}R(\boldsymbol{\ell}), \tag{3.3}$$

formally in $\widehat{\mathcal{L}}^{\infty}$*, where* R *is a convergent Laurent series.*

We call $\widehat{\mathcal{L}}_0^I$ *the set of* integrally summable series of $\widehat{\mathcal{L}}_0^{\infty}$*.*

(2)

(i) If $\widehat{f}$ *is convergent, we define its* integral sum *as the usual sum (on an interval* $(0,\delta)$*).*
(ii) If $\widehat{f}\in\widehat{\mathcal{L}}_0^I$ *is divergent, we define its* integral sum $f\in\mathcal{G}_{AN}$ *by:*

$$f(y):=\frac{\int_d^{e^{-1/y}}s^{\alpha-1}R\big(\boldsymbol{\ell}(s)\big)\,ds}{e^{-\frac{\alpha}{y}}}, \tag{3.4}$$

where $\boldsymbol{\ell}(s)=-\frac{1}{\log s}$*. We put* $d=0$ *if* $s^{\alpha-1}R\big(\boldsymbol{\ell}(s)\big)$ *is integrable at* 0 *(i.e.* $\alpha>0$*) and* $d>0$ *otherwise (i.e.* $\alpha<0$*).*

Remark 3.9. Note that the two parts of the definition of the integral sum are consistent. Indeed, in the case when $\widehat{f}$ is convergent, let R be a convergent series defined by (3.3) for $\alpha=0$. Now, the integral expression (3.4) for $f(y)$, with $\alpha=0$, $d>0$, coincides with the sum of $\widehat{f}$ up to a constant.

Remark 3.10. The idea behind the definition of an integral sum f of $\widehat{f}$ is that f is defined as a germ from $\mathcal{G}_{AN}$ such that the same equation as (3.3) is satisfied, but in the sense of germs:

$$\frac{d}{dx}\big(x^{\alpha}f(\boldsymbol{\ell})\big)=x^{\alpha-1}R(\boldsymbol{\ell}),\ x\in(0,d),\ d>0.$$

Note that the solution of (3.3) in the sense of germs is given by (3.4).

Proposition 3.11. *The exponent* $\alpha\neq 0$ *in Definition 3.8* (1) *(ii) is unique. We call such* α *the* exponent of integration of $\widehat{f}$*.*

The proof is in the Appendix. Note that if (3.3) were true for $\alpha=0$, it would imply that $\widehat{f}$ is convergent, so we can suppose $\alpha\neq 0$ for divergent series.

Remark 3.12.

(1) Note that the integral sum f of $\widehat{f}\in\widehat{\mathcal{L}}_0^I$, as defined by (3.4), is unique for $\alpha>0$. It is *unique only up to* $Ce^{\frac{\alpha}{y}}$, $C\in\mathbb{R}$, for $\alpha<0$, due to the possible choice of $d>0$. For $\alpha=0$ (that is, if $\widehat{f}$ is convergent), the integral sum is unique.

(2) Proposition 7.3 in the Appendix shows that $\widehat{f}\in\widehat{\mathcal{L}}_0^I$ is the power asymptotic expansion of its integral sums. In the case $\alpha<0$, the term $Ce^{\frac{\alpha}{x}}$ in the integral sum f is exponentially small with respect to monomials in $\widehat{\mathcal{L}}_0$. Therefore, adding such a term to f does not change its asymptotic expansion $\widehat{f}\in\widehat{\mathcal{L}}_0$.

Definition 3.13 (Integral sections)**.** *The vector subspace* $\widehat{\mathcal{I}}$ *of* $\widehat{\mathcal{S}}$ *generated by the transseries* $\widehat{g}(x)=x^{\alpha}\widehat{f}(\boldsymbol{\ell})$*, where* $\widehat{f}\in\widehat{\mathcal{L}}_0^I$ *and* $\alpha\in\mathbb{R}$ *is the exponent of integration of* $\widehat{f}$ *(or an arbitrary real number if* $\widehat{f}$ *convergent), is called the space of* integrally asymptotic transseries.

An integral section *is any coherent section* $\mathbf{s} : \widehat{\mathcal{I}} \to \mathcal{G}_{AN}$ *which, to every generator* $\widehat{g}(x) = x^\alpha \widehat{f}(\boldsymbol{\ell})$ *of* $\widehat{\mathcal{I}}$*, associates* $\mathbf{s}(\widehat{g}) = x^\alpha f(\boldsymbol{\ell})$*, where* $f \in \mathcal{G}_{AN}$ *is the integral sum of* $\widehat{f}$ *given by* (3.4).

Note that the germ f is unique only up to an additive constant (Remark 3.12). Likewise, the integral sections $\mathbf{s}$ are *linear* mappings *only up to an additive constant.*

4. Embedding in a one-parameter flow

In this section (Propositions 4.3 and 4.4) we discuss the close relationship between the existence of a (formal) Fatou coordinate for a germ $f \in \mathcal{G}_{AN}$ (resp. $\widehat{f}$) and an embedding of $f \in \mathcal{G}_{AN}$ (resp. $\widehat{f}$) in a (formal) one-parameter flow. For the explicit relation, consult also the constructive proofs of propositions in the Appendix.

Definition 4.1 (One-parameter local flow, standard definition)**.** *Let a and b be two continuous functions defined on $(0,d)$ such that $a(x) < 0 < b(x)$. A family $\{f^t\}_{t\in(a(x),b(x))}$ of functions defined on some open interval $(0,d)$, $d > 0$, is called a* one-parameter flow on $(0,d)$ *if:*

(1) $f^0(x) = x$, $x \in (0,d)$,
(2) If $s \in (a(x), b(x))$ *and* $t \in (a(f^s(x)), b(f^s(x)))$ *then* $t + s \in (a(x), b(x))$ *and* $f^{t+s}(x) = f^t(f^s(x))$, $x \in (0,d)$.

*We say that the one-parameter flow $\{f^t\}$ is a C^1-*flow *if the mapping $t \mapsto f^t(x)$ is of class $C^1\big((a(x), b(x))\big)$, for every $x \in (0,d)$.*

Let $f \in \mathcal{G}_{AN}$. We say that f embeds as the time-one map in a flow *$\{f^t\}$ if $f^1 = f$ (that is, if f^1 exists and is equal to f on some interval $(0,d)$).*

Definition 4.2 (Formal one-parameter flow, see [11])**.** *We say that a one-parameter family $\{\widehat{f}^t\}_{t\in\mathbb{R}}$, $\widehat{f}^t \in \widehat{\mathcal{L}}$, is a C^1-*formal flow *if:*

(1) $\widehat{f}^0 = \mathrm{id}$, $\widehat{f}^{t+s} = \widehat{f}^t \circ \widehat{f}^s$, $s, t \in \mathbb{R}$,
(2) $S := \cup_{t\in\mathbb{R}} \mathrm{Supp}(\widehat{f}^t)$ *is a well-ordered subset of* $\mathbb{R}_{>0} \times \mathbb{Z}$,
(3) $t \mapsto [\widehat{f}^t]_{(\alpha,m)}$ *is of the class* $C^1(\mathbb{R})$*, for every* $(\alpha, m) \in S$.

Here, $[\widehat{f}^t]_{(\alpha,m)}$ denotes the coefficient (which is a function of t) of the monomial $x^\alpha \boldsymbol{\ell}^m$ in $\widehat{f}^t$. Again, we say that $\widehat{f} \in \widehat{\mathcal{L}}$ embeds in the flow $\{\widehat{f}^t\}_{t\in\mathbb{R}}$ as the time-one map if $\widehat{f} = \widehat{f}^1$.

In [11], we have proved that any *parabolic* transseries $\widehat{f} \in \widehat{\mathcal{L}}$ is the time-one map of a unique C^1-flow in $\widehat{\mathcal{L}}$. The next proposition states that, accordingly, the formal Fatou coordinate exists and is unique.

Proposition 4.3 (Existence and uniqueness of the formal Fatou coordinate)**.** *Let $\widehat{f} \in \widehat{\mathcal{L}}$ be parabolic. The formal Fatou coordinate $\widehat{\Psi}$ exists and is unique in $\widehat{\mathfrak{L}}$, up to an additive constant. Moreover, $\widehat{\Psi} \in \widehat{\mathcal{L}}_2^\infty$. Let $\{\widehat{f}^t\}$ be the (unique) C^1-flow in $\widehat{\mathcal{L}}$ in which $\widehat{f}$ embeds as the time-one map. Then:*

$$\widehat{\Psi}(\widehat{f}^t(x)) - \widehat{\Psi}(x) = t, \ t \in \mathbb{R}. \tag{4.1}$$

For a more precise description of the formal Fatou coordinate of a parabolic $\widehat{f} \in \widehat{\mathcal{L}}$, see Remark 7.1 in the Appendix. If $\widehat{f}$ is moreover parabolic and Dulac, the

description is provided in Proposition 6.4.

The next proposition establishes, for an analytic germ in $\mathcal{G}_{AN}$, the equivalence between the existence of an analytic Fatou coordinate and the embedding of the germ in a C^1-flow:

Proposition 4.4. *Let f be an analytic function defined on some open interval $(0,d)$, $d>0$, such that $f(0)=0$ (in the limit sense) and $0<f(x)<x$, $x\in(0,d)$.*[1] *The following statements are equivalent:*

1. there exists a Fatou coordinate Ψ for f defined and analytic on $(0,d)$,

2. there exists a C^1-flow $\{f^t\}$, f_t defined and analytic on $(0,d)$, in which f can be embedded as the time one-map, and such that the function ξ defined on $(0,d)$ by

$$\xi := \frac{d}{dt} f^t\Big|_{t=0}$$

is non-oscillatory[2]*.*

In this case, for the Fatou coordinate Ψ and for the corresponding flow $\{f^t\}$, the following holds:

$$\Psi(f^t(x)) - \Psi(x) = t,\ x\in(0,d),\ t\in(\mathbb{R}_+,0). \tag{4.2}$$

The proofs of Propositions 4.3 and 4.4, are given in the Appendix. Also, see Remark 7.4 in the Appendix for the explanation of the importance of the *non-oscillatority assumption* in Proposition 4.4.

In the particular case of a *parabolic Dulac germ* f, the existence, uniqueness and the description of the Fatou coordinate for f, as well as of the formal Fatou coordinate for its Dulac expansion $\widehat{f}$, are given in Section 6 (in the proof of the Theorem) by means of an explicit construction.

5. Examples

Example 4 (Examples of sectional asymptotic expansions in $\widehat{\mathfrak{L}}$)**.**

(1) The asymptotic expansion of a Dulac germ $f\in\mathcal{G}_{AN}$ is unique and equal to its Dulac expansion:

$$\widehat{f} = \sum_{i=1}^{\infty} x^{\alpha_i} P_i(\boldsymbol{\ell}) \in \widehat{\mathcal{L}},$$

where $(\alpha_i)_i$ is an increasing sequence of finite type (Definition 2.2) and P_i is a sequence of polynomials. Indeed, since P_i are polynomials (finite sums of powers of $\boldsymbol{\ell}$), the asymptotic expansion $\widehat{f}$ of a Dulac germ f is unambigously defined by the standard Poincaré algorithm.

(2) The time-one map $f\in\mathcal{G}_{AN}$ of the vector field $X=\xi(x)\frac{d}{dx}$, $\xi(x)=x^2\boldsymbol{\ell}(x)^{-1}$, admits a unique Poincaré asymptotic expansion $\widehat{f}\in\widehat{\mathcal{L}}$ given by the formal exponential $\widehat{f}(x)=\mathrm{Exp}\big(x^2\boldsymbol{\ell}^{-1}\frac{d}{dx}\big).\mathrm{id}$. It is not transfinite, since every power of x in the expansion is multiplied by only finitely many powers of $\boldsymbol{\ell}$.

A Fatou coordinate $\Psi\in\mathcal{G}_{AN}$ of f is computed by

$$\Psi(x) = \int_d^x \frac{1}{\xi(x)}\,dx = \int_d^x x^{-2}\boldsymbol{\ell}(x)\,dx,\ d>0.$$

[1]That is, 0 is the only fixed point in $(0,d)$. Moreover, it is attracting.

[2]non-oscillatority means that there is no accumulation of zero points at 0

On the other hand, a formal Fatou coordinate of its expansion $\widehat{f}$ can be computed as a formal antiderivative in $\widehat{\mathfrak{L}}$ of $x^{-2}\boldsymbol{\ell}$ without constant term[3]:

$$\widehat{\Psi}(x) = \int \frac{dx}{\xi(x)} = \int x^{-2}\boldsymbol{\ell}\, dx,$$

see the proof of Proposition 4.3. By integration by parts,

$$\widehat{\Psi}(x) = -x^{-1}\sum_{n=1}^{\infty} n!\boldsymbol{\ell}^n.$$

The above series $\widehat{T}(\boldsymbol{\ell}) := \sum_{n=1}^{\infty} n!\boldsymbol{\ell}^n$ is a divergent series in $\widehat{\mathcal{L}}_0$. There are many ways to find a germ $T \in \mathcal{G}_{AN}$ which admits $\widehat{T}$ as its power asymptotic expansion. We choose one particular sum, the *integral sum* from Definition 3.8, which is uniquely defined up to a term Cx. That is, we choose an *integral* section $\mathbf{s}$. By Definition 3.13,

$$T := \frac{\int_d^x x^{-2}\boldsymbol{\ell}(x)\, dx}{-x^{-1}} \in \mathcal{G}_{AN},\ d > 0.$$

Due to the choice of $d > 0$, we get integral sums differing by a constant. Up to a constant, Ψ is the sectional sum of $\widehat{\Psi}$ with respect to the integral section $\mathbf{s}$ and $\widehat{\Psi}$ is the Poincaré asymptotic expansion of $\Psi(x) = -x^{-1}T(x) \in \mathcal{G}_{AN}$.

(3) Let $\mathbf{s}$ be a coherent section as defined in Definition 3.7. The following examples of germs in $\mathcal{G}_{AN}$ admit sectional asymptotic expansions with respect to $\mathbf{s}$ in $\widehat{\mathfrak{L}}$:

$$f(x) := F(x, \boldsymbol{\ell}, \boldsymbol{\ell}_2),\ g(x) := G(x, \frac{x}{\boldsymbol{\ell}}),$$

where F is a germ at the origin of a function of three variables analytic at $(0,0,0)$ and G a germ at the origin of a function of two variables analytic at $(0,0)$.

Let us show the statement for f (for g analogously). For y, z small enough, the germ $x \mapsto F(x,y,z)$ is analytic at 0. Therefore, for every $n \in \mathbb{N}$, we have:

$$F(x,y,z) = \sum_{k=0}^{n} g_k(y,z)x^k + R_n(x,y,z),$$

where $x \mapsto R_n(x,y,z)$ is analytic, for *fixed* small y, z. Take the *Lagrange form of the remainder* $R_n(x,y,z)$, for y, z fixed:

$$R_n(x,y,z) = \int_0^x \partial_t^{n+1}F(t,y,z)\frac{(x-t)^n}{n!}dt$$

Since F is analytic in y, z, we get that R_n is analytic in the three variables, and, moreover, we get the following *uniform* bound in y, z:

$$|R_n(x,y,z)| \leq C\frac{x^{n+1}}{(n+1)!},$$

on some small ball around $(0,0,0)$. Since R_n is analytic in y, z, $n \in \mathbb{N}$, we conclude that the functions $g_k(y,z)$ are also analytic in y, z, for all $k = 0, \ldots, n$, and we can repeat the same procedure for their expansions at the next level. Finally, f admits the unique sectional asymptotic expansion with respect to $\mathbf{s}$ equal to $\widehat{F}(x, \boldsymbol{\ell}, \boldsymbol{\ell}_2) \in \widehat{\mathcal{L}}_2$, where $\widehat{F}$ is the Taylor expansion of F at $(0,0,0)$.

[3]By *formal antiderivative in* $\widehat{\mathfrak{L}}$ of a transseries $\widehat{f} \in \widehat{\mathfrak{L}}$ *without constant term* we mean a transseries from $\widehat{\mathfrak{L}}$ obtained by integrating $\widehat{f}$ monomial by monomial, with constant of integration equal to 0.

As a simple example of this type of germ, take e.g.

$$f(x) = \frac{1}{1 - \frac{x}{1 - \frac{\boldsymbol{\ell}}{1-\boldsymbol{\ell}_2}}}.$$

Its sectional asymptotic expansion in $\widehat{\mathcal{L}}_2$ with respect to $\mathbf{s}$ is given by the transseries:

$$\widehat{f} = \sum_{k=0}^{\infty} x^k \Big(1 - \frac{\boldsymbol{\ell}}{1-\boldsymbol{\ell}_2}\Big)^{-k} = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{r=0}^{\infty} \binom{-k}{l}\binom{-l}{r}(-1)^{l+r} x^k \boldsymbol{\ell}^l \boldsymbol{\ell}_2^r.$$

6. Proof of the Theorem

The proof of the Theorem is constructive. Before proving the Theorem, we give an example illustrating our construction of the Fatou coordinate.

Example 5. Take a Dulac germ $f \in \mathcal{G}_{AN}$ with the expansion:

$$\widehat{f}(x) = x - x^2\boldsymbol{\ell}^{-1} + o(x^3)$$

The algorithm which will be described in this section is a block-by-block construction of the formal Fatou coordinate $\widehat{\Psi} \in \widehat{\mathfrak{L}}$ satisfying formally in $\widehat{\mathfrak{L}}$ the Fatou equation:

$$\widehat{\Psi}\big(x - x^2\boldsymbol{\ell}^{-1} + o(x^3)\big) - \widehat{\Psi}(x) = 1.$$

Blocks of a transseries from $\widehat{\mathfrak{L}}$ are defined in Definition 2.1. Let $\widehat{\Psi}_1$ be the leading block of $\widehat{\Psi}$, that is, $\widehat{\Psi} = \widehat{\Psi}_1 + h.o.b.$ Here, *h.o.b.* stands for *blocks of strictly higher order* (*in* x). Applying the formal Taylor expansion, the lowest-order block on the left-hand side is $-\widehat{\Psi}_1'(x) \cdot x^2\boldsymbol{\ell}^{-1}$, so it should equal 1 on the right-hand side. Therefore, $\widehat{\Psi}_1$ is given as the formal antiderivative of $-x^2\boldsymbol{\ell}$ in $\widehat{\mathfrak{L}}$ without constant term (see footnote on p. 15):

$$\widehat{\Psi}_1(x) = \int \frac{dx}{x^2 \log x}. \tag{6.1}$$

To continue, put $\widehat{\Psi} = \widehat{\Psi}_1 + \widehat{R}$ in the Fatou equation and repeat the procedure for the following blocks. By formal integration by parts, we see that

$$\widehat{\Psi}_1(x) = x^{-1}\sum_{n=1}^{\infty} n!\boldsymbol{\ell}^n. \tag{6.2}$$

For every $\boldsymbol{\ell} \in (0, d)$, the numeric series $\sum_{n=1}^{\infty} n!\boldsymbol{\ell}^n$ is divergent (and the formal series $\widehat{f}(\boldsymbol{\ell}) = \sum_{n=1}^{\infty} n!\boldsymbol{\ell}^n$ is Borel summable). However, following our construction, this series is *uniquely real summable*. That is, we also perform a parallel block by block construction on the level of germs. Let

$$\Psi_1(x) := \int_d^x \frac{dt}{t^2 \log t},\ d > 0,$$

be an analytic analogue of $\widehat{\Psi}_1$ from (6.1). Note that different choices of $d > 0$ lead to germs Ψ_1 differing by a constant. Obviously, $\Psi_1 \in \mathcal{G}_{AN}$ and therefore the integral sum $f(\boldsymbol{\ell}) := e^{-1/\boldsymbol{\ell}}\Psi_1(e^{-1/\boldsymbol{\ell}})$ of $\widehat{f}$ belongs to $\mathcal{G}_{AN}$. By Proposition 7.3, the power asymptotic expansion of $f \in \mathcal{G}_{AN}$ is equal to $\widehat{f}$. This procedure is formalized in Definition 3.8. Consequently, the Poincaré asymptotic expansions of Ψ_1 are, up to additive constants, equal to the formal series $\widehat{\Psi}_1$.

6.1. **The proof of the Theorem.**

Point 1. has already been proven in Proposition 4.3. We now give a constructive proof of the existence of $\widehat{\Psi} \in \widehat{\mathcal{L}}_2^{\infty}$ from 1. As illustrated in Example 5, simultaneously with formal Fatou coordinate $\widehat{\Psi}$, we construct a Fatou coordinate $\Psi \in \mathcal{G}_{AN}$ from point 2. and prove the relation between $\widehat{\Psi}$ and Ψ from 3. and 4. We follow in large part the construction of the Fatou coordinate for parabolic diffeomorphisms as explained, for example, in [8].

We proceed in four steps:

Step 1. In Subsection 6.1.1, we construct the formal Fatou coordinate $\widehat{\Psi}$, by solving *block by block* the formal Abel equation. By *block*, we mean the (formal) sum of all the monomials of $\widehat{f}$ which share a common power of x. We also get the precise form (6.21) of $\widehat{\Psi}$. Recall that the formal Fatou coordinate is unique by Proposition 4.3.

Simultaneously, we provide the "block by block" construction of a Fatou coordinate $\Psi \in \mathcal{G}_{AN}$, where the germ for each block (from $\mathcal{G}_{AN}$) is represented by an integral. We prove that each formal block is the Poincaré asymptotic expansion of the corresponding integrally defined germ from $\mathcal{G}_{AN}$, at least up to a constant term (Remark 3.12).

Additionally, we control the support of the formal Fatou coordinate, and prove that the powers of x it contains belong to a *finitely generated*[4] *lattice* .

Step 2. (see Subsection 6.1.2). The control of the support obtained in the previous step allows to conclude that the principal part $\widehat{\Psi}_{\infty}$ of the Fatou coordinate is obtained after finitely many steps of the "block by block" algorithm. We prove also that the principal part $\widehat{\Psi}_{\infty}$ is the sectional asymptotic expansion in $\widehat{\mathfrak{L}}$ with respect to any integral section $\mathbf{s}$ of the principal part Ψ_{∞}, up to a constant term depending on the choice of the integral section. That is, depending on the choice of constants of integration in the integrals for the infinite part.

Step 3. In Subsection 6.1.3, we solve the modified Abel equation (2.5) for the remaining infinitesimal part of the Fatou coordinate $\widehat{R}$. The infinitesimal part R is obtained directly from the equation in the form of a *convergent* series, following the method explained in [8]. We prove in Section 6.1.4 that the formal infinitesimal part $\widehat{R}$ of $\widehat{\Psi}$ obtained blockwise (in countably many steps) is indeed the sectional asymptotic expansion in $\widehat{\mathcal{L}}$ with respect to any integral section $\mathbf{s}$ of the infinitesimal part R of Ψ.

Step 4. Finally, in Subsection 6.1.5, we prove the uniqueness of the formal Fatou coordinate $\widehat{\Psi} \in \widehat{\mathfrak{L}}$ up to an additive constant. Furthermore, we prove the uniqueness, up to an additive constant, of the Fatou coordinate $\Psi \in \mathcal{G}_{AN}$ admitting an integral sectional asymptotic expansion in $\widehat{\mathfrak{L}}$. Moreover, for a fixed constant term in $\widehat{\Psi} \in \widehat{\mathcal{L}}_2^{\infty}$ and for a fixed integral section $\mathbf{s}$, any such Fatou coordinate $\Psi \in \mathcal{G}_{AN}$ constructed in the proof admits $\widehat{\Psi}$ as its sectional asymptotic expansion with respect to $\mathbf{s}$, up to an additive constant.

If we change the integral section $\mathbf{s}$, the sectional asymptotic expansion with respect to $\mathbf{s}$ of a fixed Fatou coordinate $\Psi \in \mathcal{G}_{AN}$ changes only by an additive constant.

[4]Every element is a linear combination over positive integers of finitely many generators of the lattice.

6.1.1. **The Fatou coordinate and the control of the support.**

Let

$$f(x) = x - x^{\alpha_1} P_1(\boldsymbol{\ell}^{-1}) - x^{\alpha_2} P_2(\boldsymbol{\ell}^{-1}) + o(x^{\alpha_2})$$

be a parabolic Dulac germ. Here, $1 < \alpha_1 < \alpha_2 < \dots$ is an increasing sequence of finite typeor a strictly increasing sequence tending to $+\infty$, whose elements belong to a finitely generated (see the footnote[5]) sub-semigroup of $(\mathbb{R}_{>0}, +)$ and P_i are *polynomials*. Let $\widehat{f} \in \widehat{\mathcal{L}}$ be its Dulac expansion.

We construct the formal Fatou coordinate $\widehat{\Psi}$ satisfying the formal Abel equation

$$\widehat{\Psi}(\widehat{f}) - \widehat{\Psi} = 1 \tag{6.3}$$

block by block and we control its support. By one *block*, we mean the sum of all monomials sharing a common power of x, as defined in Definition 2.1. We call the power of x of a block the *order of the block.* In each step, we consider the construction from two sides:

1. the side of formal transseries, and
2. the side of analytic germs in $\mathcal{G}_{AN}$.

Let us now describe the induction step. Put

$$\widehat{\Psi} = \widehat{\Psi}_1 + \widehat{R}_1,$$

where $\widehat{\Psi}_1$ represents the lowest-order block of $\widehat{\Psi}$. Since we search for a solution $\widehat{\Psi}$ in $\widehat{\mathfrak{L}}$, and since $\widehat{f}$ is parabolic, we can expand the Abel equation (6.3) using Taylor expansions:

$$\widehat{\Psi}' \cdot \widehat{g} + \frac{1}{2!}\widehat{\Psi}'' \cdot \widehat{g}^2 + \cdots = 1,$$

where $\widehat{g} = \mathrm{id} - \widehat{f} = x^{\alpha_1} P_1(\boldsymbol{\ell}^{-1}) + x^{\alpha_2} P_2(\boldsymbol{\ell}^{-1}) + \cdots$. The term $(\widehat{\Psi}_1)' \cdot x^{\alpha_1} P_1(\boldsymbol{\ell}^{-1})$ is the block of the strictly lowest order on the left-hand side, so it should equal 1 on the right-hand side. Therefore,

$$(\widehat{\Psi}_1)' = \frac{x^{-\alpha_1}}{P_1(\boldsymbol{\ell}^{-1})} = x^{-\alpha_1} Q_1(\boldsymbol{\ell}).$$

Here, Q_1 is a rational function. We obtain the formal antiderivative $\widehat{\Psi}_1^{\infty} \in \widehat{\mathcal{L}}_2^{\infty}$ by expanding the integral *formally, using integration by parts*, without adding constant term, as explained in the footnote on p. 15 (see Proposition 7.3 in the Appendix).

On the other hand, we define:

$$\Psi_1(x) := \int_d^x t^{-\alpha_1} Q_1(\boldsymbol{\ell})\, dt, \ d > 0.$$

Obviously, $\Psi_1 \in \mathcal{G}_{AN}$. Note that $\alpha_1 > 1$ or ($\alpha_1 = 1$, [5] $\mathrm{ord}(Q_1) \leq 1$), so $\Psi_1(x) \to \infty$, as $x \to 0$. Note that Ψ_1 is unique only up to an additive constant (free choice of $d > 0$). In the sequel, this will be the case for infinite blocks (i.e. which tend to ∞, as $x \to 0$). On the other hand, infinitesimal blocks (which tend to 0, as $x \to \infty$) will give unique germs.

By Proposition 7.3 in the Appendix, $\widehat{\Psi}_1$ is the sectional asymptotic expansion in $\widehat{\mathcal{L}}_2^{\infty}$ of $\Psi_1 \in \mathcal{G}_{AN}$ with respect to any integral section, up to an additive constant.

[5]Here and in the sequel, by $\mathrm{ord}(Q_1)$ we mean the order (as defined after Definition 2.1) of the power series $\widehat{Q}_1(\boldsymbol{\ell})$ in $\boldsymbol{\ell}$, which is the power asymptotic expansion of the rational function Q_1 at $\boldsymbol{\ell} = 0$.

The Abel equation for R_1 becomes:

$$R_1(f(x)) - R_1(x) = 1 - \big(\Psi_1(f(x)) - \Psi_1(x)\big) = 1 - \int_x^{f(x)} t^{-\alpha_1} Q_1(\boldsymbol{\ell})\, dt. \tag{6.4}$$

Let us denote by $\delta_1(x) := 1 - \int_x^{f(x)} t^{-\alpha_1} Q_1(\boldsymbol{\ell})\, dt$ the new right-hand side of the equation. Obviously, $\delta_1 \in \mathcal{G}_{AN}$, as a difference of analytic germs.

On the other hand, applying Taylor expansion to $\widehat{\Psi}_1$, the *formal* Abel equation becomes:

$$\begin{aligned} \widehat{R}_1(\widehat{f}(x)) - \widehat{R}_1(x) &= 1 - \big(\widehat{\Psi}_1(\widehat{f}(x)) - \widehat{\Psi}_1(x)\big), \\ &= 1 - (\widehat{\Psi}_1)'\widehat{g} - \frac{1}{2!}(\widehat{\Psi}_1)''\widehat{g}^2 + \cdots, \\ &= 1 - \frac{x^{-\alpha_1}}{P_1(\boldsymbol{\ell}^{-1})}\widehat{g} - \frac{1}{2!}\Big(\frac{x^{-\alpha_1}}{P_1(\boldsymbol{\ell}^{-1})}\Big)'\widehat{g}^2 + \cdots =: \widehat{\delta}_1(x). \end{aligned} \tag{6.5}$$

Note that, due to the fact that the power of x in the leading term of $\widehat{f}$ is strictly bigger than 1, the Taylor formula above converges formally and is true in the formal setting. This is a classical result that can be checked in e.g. [2]. We denote the right-hand side of (6.5) by $\widehat{\delta}_1 \in \widehat{\mathcal{L}}_1^\infty$. It can be checked from the above computation that the leading block in $\widehat{\delta}_1$ is of order $\min\{\alpha_1 - 1, \alpha_2 - \alpha_1\}$. Similarly, we have $\delta_1 = O(x^{\min\{\alpha_2-\alpha_1,\alpha_1-1\}-\varepsilon})$, for every $\varepsilon > 0$. By parallel constructions, $\widehat{\delta}_1$ is the sectional asymptotic expansion of δ_1 with respect to a coherent section (series in $\boldsymbol{\ell}$ are convergent).

Furthermore, it can be seen that $\widehat{\delta}_1$ consists of blocks of the type $x^\beta R(\boldsymbol{\ell}^{-1})$, where R is a rational function whose denominator can only be a *positive* integer power of the polynomial $P_1(\boldsymbol{\ell}^{-1})$, and β belongs to the set:

$$\mathcal{R}_1 := \{(\alpha_{n_1} + \cdots + \alpha_{n_k} - k) - (\alpha_1 - 1),\ k \in \mathbb{N}\},$$

where $\alpha_{n_i} > 1$ are powers of x in $\widehat{f}$. Since the sequence $(\alpha_i)_i$ is finitely generated, the set $\mathcal{R}_1$ belongs to a finitely generated lattice. In particular, it is well-ordered by the standard order $<$ on $\mathbb{R}$.

Now, we repeat the same procedure of elimination for germs (*resp.* for formal series), with right-hand side δ_1 (*resp.* $\widehat{\delta}_1$) instead of 1. We put $\widehat{R}_1 = \widehat{\Psi}_2^\infty + \widehat{R}_2$. To eliminate the first block from $\widehat{\delta}_1$, say $x^\beta R(\boldsymbol{\ell}^{-1})$, we take:

$$(\widehat{\Psi}_2)' = \frac{x^\beta R(\boldsymbol{\ell}^{-1})}{x^{\alpha_1} P_1(\boldsymbol{\ell}^{-1})} = x^{\beta-\alpha_1} Q_2(\boldsymbol{\ell}), \tag{6.6}$$

where Q_2 is a rational function.

In the same step, we define the germ $\Psi_2 \in \mathcal{G}_{AN}$:

$$\Psi_2(x) := \int_d^x t^{-(\alpha_1-\beta)} Q_2(\boldsymbol{\ell}(t))\, dt, \tag{6.7}$$

where $d > 0$ if $\alpha_1 - \beta > 1$ or $(\alpha_1 = \beta + 1,\ \mathrm{ord}(Q_2) \le 1)$, and $d = 0$ if $\alpha_1 - \beta > 1$ or $(\alpha_1 = \beta+1,\ \mathrm{ord}(Q_2) \le 1)$. Obviously, $\Psi_2^\infty \in \mathcal{G}_{AN}$. Note that there are no new singularities created in $Q_2(\boldsymbol{\ell})$, since the denominator of $\frac{R(\boldsymbol{\ell}^{-1})}{P_1(\boldsymbol{\ell}^{-1})}$ is just a positive integer power of $P_1(\boldsymbol{\ell}^{-1})$.

Note here that, depending on the order of the right-hand side in (6.6) (that is, depending on the step of the algorithm), in (6.7) we get either $\Psi_2(x) \to \infty$ (infinite block) or $\Psi_2(x) = o(1)$ (infinitesimal block), as $x \to 0$. In the former case, we may choose the constant in Ψ_2 arbitrarily (any small $d > 0$), while in the latter we put $d = 0$ in order to have $\Psi_2(0) = 0$ and thus to avoid constant terms in the infinitesimal part of the Fatou coordinate. We will show below that there are only finitely many infinite steps, but at most countably many infinitesimal steps.

Repeating the same procedure, we conclude that the blocks in $\widehat{\delta}_2$ are of the form $x^\beta R(\boldsymbol{\ell}^{-1})$, where R is again a rational function whose denominator can only be a *positive* integer power of the polynomial $P_1(\boldsymbol{\ell}^{-1})$, and and

$$\beta \in \mathcal{R}_2 := \{(\alpha_{n_1} + \cdots + \alpha_{n_k} - k) - 2(\alpha_1 - 1),\ k \in \mathbb{N},\ k \geq 2\} \cup \mathcal{R}_1.$$

At the r-th step of this procedure, the powers of x in $\widehat{\delta}_r$ are:

$$\beta \in \mathcal{R}_r := \cup_{p\in\mathbb{N},\, p\leq r}\{(\alpha_{n_1} + \cdots + \alpha_{n_k} - k) - p(\alpha_1 - 1),\ k \in \mathbb{N},\ k \geq p\}.$$

Therefore, the monomials appearing in the algorithm on the right-hand side of the modified Abel equation are always of the form $x^\beta R(\boldsymbol{\ell}^{-1})$, where

$$\beta \in \mathcal{R} := \bigcup_{r\in\mathbb{N}} \{(\alpha_{n_1} + \cdots + \alpha_{n_k} - k) - r(\alpha_1 - 1) |\ k \in \mathbb{N},\ k \geq r\} =$$
$$= \{(\alpha_{n_1} - \alpha_1) + \cdots + (\alpha_{n_r} - \alpha_1) + (\alpha_{n_{r+1}} - 1) + \cdots + (\alpha_{n_k} - 1),\ r \leq k,\ r, k \in \mathbb{N}\}.$$

That is, the set $\mathcal{R}$ is the set of *all* finite sums of nonegative elements of the form $(\alpha_i - \alpha_1)$ and $(\alpha_i - 1)$, where $\alpha_i \geq 1$, $i \in \mathbb{N}$, is the sequence of powers of x in the Dulac expansion $\widehat{f}(x)$. Since $(\alpha_i)_i$ belong to a finitely generated lattice, it is the same for the elements of $\mathcal{R}$. In particular, $\mathcal{R}$ is well-ordered. Its order type is ω, and its elements form a sequence tending to $+\infty$. Since all the powers of x in the common support of all the right-hand sides δ in the course of the algorithm belong to $\mathcal{R}$, they can either be ordered in an *infinite strictly increasing sequence tending to* $+\infty$ or there are *only finitely many* of them. In the latter case, the block by block algorithm terminates in finitely many steps. Otherwise, it needs ω steps to terminate. In any case, the construction by blocks of the formal Fatou coordinate is not transfinite, as it terminates in at most ω steps.

Furthermore, thanks to the direct relation of $\widehat{\Psi}_r$ and the leading block of $\widehat{\delta}_r$ described in (6.6), and by Proposition 7.3, we also see that the support of $\widehat{\Psi}$ is well-ordered.

6.1.2. **The principal (*infinite*) part of the Fatou coordinate.** Let $\alpha_1 > 1$ be such that $\mathrm{ord}(\mathrm{id} - \widehat{f}) = (\alpha_1, m)$, $m \in \mathbb{Z}^-$, as above. We have proved in Section 6.1.1 that the orders of the blocks on the right-hand sides of the Fatou equation in the course of eliminations belong to a finitely generated lattice. The order of the leading block on the right-hand side in every step strictly increases. Therefore, it follows that after *finitely many steps* of block by block eliminations, the order of the right-hand side $\widehat{\delta}$ becomes strictly bigger than $\alpha_1 - 1$.

We denote by $r_0 \in \mathbb{N}$ the smallest number such that, after r_0 steps, the Abel equation becomes:

$$R_{r_0}(f(x)) - R_{r_0}(x) = o(x^{\alpha_1 - 1}).$$

The r_0-th step is the *critical step* between the infinite and the infinitesimal part of the Fatou coordinate. That is:

$$R_{r_0-1}(x) \to \infty,\ R_{r_0}(x) \to 0 \text{ as } x \to 0.$$

This is a direct consequence of the following Proposition 6.1:

Proposition 6.1 (Order of the blocks in the algorithm)**.** *Let $\beta \in \mathcal{R}$ be the order of the leading block of the right-hand side $\widehat{\delta}_k \in \widehat{\mathcal{L}}$ of the equation*

$$\widehat{R}(\widehat{f}) - \widehat{R} = \widehat{\delta}_k.$$

Then the leading block $\widehat{\Psi}_k \in \widehat{\mathcal{L}}_2^\infty$ of $\widehat{R}$ is a block of order $\beta - (\alpha_1 - 1)$. Moreover,

a) if $\beta > \alpha_1 - 1$, then $\widehat{\Psi}_k \in \widehat{\mathcal{L}}$,
b) if $\beta < \alpha_1 - 1$, then $\widehat{\Psi}_k \in \widehat{\mathcal{L}}^\infty$,
c) if $\beta = \alpha_1 - 1$, then $\widehat{\Psi}_k \in \widehat{\mathcal{L}}_2^\infty$.

Proof. By Taylor expansion, as described in the algorithm in Section 6.1.1,

$$\widehat{\Psi}_k = \int x^{-(\alpha_1-\beta)}\widehat{Q}(\boldsymbol{\ell})\, dx,$$

where $\widehat{Q}$ is an asymptotic expansion of a rational function, and $\int$ denotes the formal integral. The result now follows by Proposition 7.3. □

It follows that there exists an index $r_0 \in \mathbb{N}$, called the *critical index*, such that all the infinite blocks of Ψ or $\widehat{\Psi}$ are exactly those (finitely many) indexed by $r \le r_0$. Hence we define *the principal part* of $\Psi \in \mathcal{G}_{AN}$ or $\widehat{\Psi} \in \widehat{\mathcal{L}}_2^\infty$ by:

$$\Psi^\infty = \Psi_1 + \cdots + \Psi_{r_0},$$
$$\widehat{\Psi}^\infty = \widehat{\Psi}_1 + \cdots + \widehat{\Psi}_{r_0}.$$

Note that, by the integral definition (6.7) (1) of infinite blocks Ψ_i, $i = 1 \dots r_0$, due to arbitrary choices of $d > 0$, Ψ^∞ defined above is unique only up to an additive constant. Therefore, by Proposition 7.3, parallel construction and by the definition of integral sections and sectional asymptotic expansions in Section 3, the formal principal part $\widehat{\Psi}^\infty \in \widehat{\mathcal{L}}_2^\infty$ obtained by the blockwise integration by parts is the sectional asymptotic expansion of Ψ^∞ with respect to any integral section $\mathbf{s}$, up to appropriate choices of constant terms in both Ψ^∞ and $\widehat{\Psi}^\infty$. Also, the change of the integral section $\mathbf{s}$ leads to the change in constant terms in Ψ^∞ or in $\widehat{\Psi}^\infty$.

6.1.3. **The infinitesimal part of the Fatou coordinate.** Let r_0 be the *critical index* defined at the end of Section 6.1.2. The germ $R = \Psi - \Psi^\infty$, $R \in \mathcal{G}_{AN}$, satisfies the difference equation:

$$R(f(x)) - R(x) = \delta(x), \tag{6.8}$$

where $\delta(x) = O(x^\gamma)$, with $\gamma > \alpha_1 - 1$, $\delta \in \mathcal{G}_{AN}$. Note that this is the first step of the block by block algorithm for which we obtain an *infinitesimal* solution. That is, $R = o(1)$, as $x \to 0$.

On the one hand, we continue solving formally block by block (expanding integrals by integration by parts). We have already proved at the end of Section 6.1.1 that we terminate the formal block by block algorithm in countably many steps:

$$\widehat{R} = \sum_{i\in\mathbb{N}} \widehat{R}_{r_0+i}. \tag{6.9}$$

By Proposition 6.1, $\widehat{R}_{r_0+i} \in \widehat{\mathfrak{L}}$, $i \in \mathbb{N}$, are blocks of strictly increasing orders, so $\widehat{R} \in \widehat{\mathcal{L}}$.

On the other hand, in each step we get a germ $R_{r_0+i} \in \mathcal{G}_{AN}$, $i \in \mathbb{N}$, which is defined by an appropriate integral in (6.7), with $d = 0$. Note that $R_{r_0+i} = o(1)$, as $x \to 0$, as a consequence of the fact that we have put $d = 0$.

Therefore, by Proposition 7.3, $\widehat{R}_{r_0+i}$ is exactly the sectional asymptotic expansion of R_{r_0+i} in $\widehat{\mathcal{L}}$ with respect to any integral section.

Instead of proving that the infinite series of analytic germs from $\mathcal{G}_{AN}$ is an analytic germ from $\mathcal{G}_{AN}$, once that we have reached the equation for the infinitesimal part (6.8), we directly construct a germ $R \in \mathcal{G}_{AN}$, $R(x) = o(1)$, satisfying (6.8), by following the classical construction explained in [8]. We define:

$$R(x) := -\sum_{k=0}^{\infty} \delta\big(f^{\circ k}(x)\big). \tag{6.10}$$

We prove in Proposition 6.2 that the above sum with $\delta \in \mathcal{G}_{AN}$ and $\delta(x) = O(x^\gamma)$, $\gamma > \alpha_1 - 1$, converges for sufficiently small $d > 0$ to an analytic function defined on $(0, d)$. That is, that R defined by (6.10) is well-defined and $R \in \mathcal{G}_{AN}$. Directly by Proposition 6.2 below, we get that $R(x) = o(1)$, as $x \to 0$.

Now, $\Psi = \Psi^\infty + R$, $\Psi \in \mathcal{G}_{AN}$, is a Fatou coordinate for f.

Proposition 6.2. *Let $f \in \mathcal{G}_{AN}$ be the Dulac germ as above. Let $\alpha_1 > 1$ be the order of the leading block in the Dulac expansion of* $\mathrm{id} - f$. *Let $\delta \in \mathcal{G}_{AN}$ be an analytic germ on some open interval $(0, d)$, $d > 0$, satisfying $\delta(x) = O(x^\gamma)$, $\gamma > \alpha_1 - 1$, obtained in the block-by-block construction described above. Then h defined by the series*

$$h(x) := -\sum_{k=0}^{\infty} \delta\big(f^{\circ k}(x)\big) \tag{6.11}$$

belongs to $\mathcal{G}_{AN}$. Moreover, for every $\varepsilon > 0$, $h(x) = o(x^{\gamma-(\alpha_1-1+\varepsilon)})$, as $x \to 0$.

Proof. The idea is to follow the procedure described in e.g. Loray [8] for regular germs at 0, and to apply Weierstrass theorem. To be able to apply Weierstrass theorem, we need to extend f and δ from $(0, d) \subset \mathbb{R}^+$ analytically to an open, f-invariant set $U \subset \mathbb{C}$ in the complex plane, chosen such that the sum

$$h(z) := -\sum_{k=0}^{\infty} \delta\big(f^{\circ k}(z)\big) \tag{6.12}$$

converges normally on U. Then, in particular, the sum $-\sum_{k=0}^{\infty} \delta\big(f^{\circ k}(x)\big)$ converges for sufficiently small $x \in (0, d)$ to $h \in \mathcal{G}_{AN}$.

The *quasi-analyticity* of the Dulac germ $f \in \mathcal{G}_{AN}$ (see [7, Definition 2 and Corollary 3]) means that f can be extended from $(0, d)$ to an analytic function $f(z)$ defined on a standard quadratic domain Q, as defined in Ilyashenko [7].

Beware that here we work in the original chart and not in the logarithmic chart as Ilyashenko does. More precisely, we restrict to an open sectorial domain V_P at the origin, centered at the positive real line.

We construct now an open set $U \subset V_P$ containing $(0, d)$, $d > 0$, which is *invariant under f*. Moreover we prove the existence on U of bounds which guarantee normal convergence of (6.12). We adapt the construction given in Loray [8].

Moreover, the function δ can be analytically extended to $U \subset V_P$. Indeed, the function δ is the sum of finitely many integrals, see (6.4). Each of these integrals is well-defined and analytic on V_P. Moreover, by [7, Corollary 3], the same Dulac asymptotic expansion at 0 of f holds on the whole domain V_P. Therefore, the asymptotic behavior at 0 of δ on V_P coincides with its behavior on the real line.

Let us now return to the construction of the f-invariant open set $U \subset V_P$ where we show the normal convergence of the series.

Let $\widehat{f}(z) = z - az^{\alpha_1}\boldsymbol{\ell}^m + h.o.t.$, $a \in \mathbb{R}$, be the Dulac expansion of f. We make a change of coordinates

$$w = \Psi_0(z) = \frac{1}{a(\alpha_1 - 1)} z^{-\alpha_1+1}\boldsymbol{\ell}^{-m}. \tag{6.13}$$

Note that Ψ_0 is the leading term in the expansion of the Fatou coordinate of f. The idea is to transform f to *almost* a translation by 1. It can be shown that this change of variables is a well-defined, univalued analytic homeomorphism $\Psi_0 : V_P \to \Psi_0(V_P)$, for an appropriate choice of a small sector V_P centered at $(0, d)$. Indeed, up to a scalar multiplication, we can write Ψ_0 as the composition:

$$\Psi_0 = f_3 \circ f_2 \circ f_1,\ f_1(z) = z^{-\frac{\alpha_1-1}{m}},\ f_2(z) = z \cdot (-\log z),\ f_3(z) = z^m. \tag{6.14}$$

All three functions are analytic homeomorphisms on corresponding domains. To see that f_2 is an analytic homeomorphism, we can consider the function $h(z) = -\log f_2(e^{-z}) = z - \log z$, which is a small perturbation of identity at infinity.

We construct now an f-invariant domain inside V_P. We work in the w-plane at $\infty = \Psi_0(0)$ (see (6.13)). We show that there exists an open subset of the image $\Psi_0(V_P)$ which is $\tilde{f}$-invariant, where $\tilde{f}$ is the analogue of f in the w-plane, defined in (6.15) below. Its pre-image by Ψ_0 is then an open subset U of V_P in the z-plane, which contains $(0, d)$ and which is f-invariant.

Let us analyze the *shape* of $\Psi_0(V_P)$. Since Ψ_0 is a homeomorphism from V_P to $\Psi_0(V_P)$, $\Psi_0(V_P)$ is open and connected, since V_P is. In fact, by decomposition (6.14) of Ψ_0, we see that Ψ_0 maps sector V_P to *almost* a sector, that is, the image $\Psi_0(V_P)$ contains a sector at ∞ (of sufficiently big radius and small opening). Denote this sector by $W_{R_1,\theta_1} \subset \Psi_0(V_P)$.

Let f_0 be the germ from $\mathcal{G}_{AN}$ with the Fatou coordinate Ψ_0, that is, defined by

$$f_0(z) := \Psi_0^{-1}\big(\Psi_0(z) + 1\big),\ z \in V_P.$$

The map Ψ_0 is analytic on V_P and $\Psi_0(z) + 1$ remains in the image $\Psi_0(V_P)$ for every $z \in V_P$, as we show below when analyzing the image $\Psi_0(V_P)$, after possibly reducing the radius and the opening of V_P. Hence, f_0 is well-defined and analytic on V_P. By the Taylor expansion and explicit formula for Ψ_0, we get that $f_0(z) = z - az^{\alpha_1}\boldsymbol{\ell}^m + o(z^{\alpha_1}\boldsymbol{\ell}^m)$. Note that the leading term of the expansion of $f_0 - \mathrm{id}$ is the same as the leading term of $f - \mathrm{id}$.

We define $\tilde{f}(w) := \Psi_0 \circ f \circ \Psi_0^{-1}(w)$ (an analogue of f in the w-plane, defined on the image by Ψ_0 of an f-invariant subset of V_P). In this new coordinate $w = \Psi_0(z)$, it can be checked that there exists a germ $h(w)$ such that:

$$\begin{aligned} \tilde{f}(w) &= \Psi_0(f(z)) = \Psi_0(f_0(z) + R(z)) \\ &= \Psi_0(f_0(z)) + \Psi_0'(f_0(z)) \cdot R(z) + h.o.t. \\ &= \Psi_0(z) + 1 + g(z) = w + 1 + h(w), \end{aligned} \tag{6.15}$$

and $h(w) = o(1)$ as $w \to +\infty$. The function R above is defined by $f(z) = f_0(z) + R(z)$, so R is analytic on V_P with $R(z) = cz^{\alpha_2}\boldsymbol{\ell}^k + o(z^{\alpha_2}\boldsymbol{\ell}^k)$, $c \in \mathbb{R}$, $(\alpha_2, k) \succ (\alpha_1, m)$, $z \to 0$, $z \in V_P$. Thus,

(6.16)
$$h(w) = \begin{cases} C(\log w)^{-p} + o((\log w)^{-p}),\ p \in \mathbb{N}, & \text{if } \alpha_2 = \alpha_1, \\ Cw^{-\beta}\log^r w + o(w^{-\beta}\log^r w),\ \beta > 0,\ r \in \mathbb{R}, & \text{if } \alpha_2 > \alpha_1,\quad C \in \mathbb{R}. \end{cases}$$

It is easier to get estimates and $\tilde{f}$-invariance in the w-plane. Let $R > 0$. If $|w| > R$ then from (6.16) we get:

$$|h(w)| \le \begin{cases} c(\log R)^{-p}, & \text{if } \alpha_2 = \alpha_1, \\ cR^{-(\beta-\varepsilon)},\ \varepsilon > 0,\ \beta - \varepsilon > 0, & \text{if } \alpha_2 > \alpha_1,\quad c > 0. \end{cases} \tag{6.17}$$

We now construct an $\tilde{f}$-invariant sector $W_{R,\alpha_R} \subset \Psi_0(V_P)$. To this end we define $\alpha_R > 0$ such that

$$\sin\alpha_R := \begin{cases} c(\log R)^{-p}, & \text{if } \alpha_2 = \alpha_1, \\ cR^{-(\beta-\varepsilon)},\ \varepsilon > 0, & \text{if } \alpha_2 > \alpha_1. \end{cases} \tag{6.18}$$

Take R big enough so that the open sector W_{R,α_R} in the w-plane of radius $|w| > R$ and of opening $2\alpha_R$ is contained in W_{R_1,θ_1} ($R > R_1$ and $\alpha_R < \theta_1$). This is possible, since $\alpha_R \to 0$, as $R \to \infty$.

By construction, on W_{R,α_R} it holds that, for every $w \in W_{R,\alpha_R}$, the sector centered at w with opening $2\alpha_R$ is completely contained in W_{R,α_R}. Using (6.15), (6.17) and (6.18), we obtain, for every w such that $|w| \ge R$, that the whole orbit $\{\tilde{f}^{\circ n}(w) :\ n \in \mathbb{N}_0\}$ remains in the open sector centered at w with opening $2\alpha_R$. Therefore, the open sector W_{R,α_R} is invariant for $\widetilde{f}(w)$.

Moreover, for $w \in W_{R,\alpha_R}$ we get, by (6.15), (6.17), the following estimates:

$$|\tilde{f}^{\circ n}(w) - w - n| \le \begin{cases} cnR^{-(\beta-\varepsilon)}, \\ cn(\log R)^{-p}. \end{cases}$$

It follows that:

$$|\tilde{f}^{\circ n}(w)| \ge C_R \cdot n,\ C_R > 0,\ w \in W_{R,\alpha_R}. \tag{6.19}$$

The sector W_{R,α_R} is open and contains $(R, +\infty) \subset \mathbb{R}_+$. Denote by $U \subset \mathbb{C}$ its pre-image in the z-plane by the homeomorphism Ψ_0^{-1}:

$$U := \Psi_0^{-1}(W_{R,\alpha_R}).$$

Obviously, U is an open set, containing $(0, d)$, and invariant under f. From (6.19), returning to the variable z we get:

$$|f^{\circ n}(z)| \le D_R \cdot n^{-(\alpha_1 - 1 - \varepsilon)},\ n \in \mathbb{N},\ D_R > 0,\ z \in U,$$

for any small $\varepsilon > 0$ such that $\alpha_1 - 1 - \varepsilon > 0$. Using this estimate, f-invariance of U and the fact that $\delta(z) = O(z^\gamma)$, $\gamma > \alpha_1 - 1$, $z \in U$, it holds that the series (6.11) is well-defined and converges normally on U. By the Weierstrass theorem, the function h defined by the series (6.11) is analytic on U. In particular, $h\big|_{(0,d)} \in \mathcal{G}_{AN}$.

Having shown the convergence of the series (6.11) towards an analytic germ $h \in \mathcal{G}_{AN}$, the rest of the proof can be carried through on $(0,d)$. Let $\varepsilon > 0$ such that $\gamma > \alpha_1 - 1 + \varepsilon$. By Proposition 6.3, we get that there exists $d > 0$ such that

$$f^{\circ k}(x) \leq \big(\frac{k}{2}\big)^{-\frac{1}{\alpha_1 - 1 + \varepsilon}},\ x \in (0,d).$$

The last point of the proposition follows from the following inequalities:

$$|h(x)| = \Big| -\sum_{k=0}^{\infty} \delta(f^{\circ k}(x)) \Big| \leq C \sum_{k \geq 0} x^{\gamma} (1 + \frac{k}{2} x^{\alpha_1 - 1 + \varepsilon})^{-\frac{\gamma}{\alpha_1 - 1 + \varepsilon}}$$
$$\leq C_1 x^{\gamma} \int_0^{\infty} (1 + \frac{t}{2} x^{\alpha_1 + \varepsilon - 1})^{-\frac{\gamma}{\alpha_1 - 1 + \varepsilon}}\, dt \leq O(x^{\gamma - (\alpha_1 - 1 + \varepsilon)}),\ x \to 0+,$$

where $C > 0$ and $C_1 > 0$ are constants. □

Proposition 6.3. *Consider an analytic function f on $(0,d)$ satisfying $0 < f(x) < x$, $x \in (0,d)$. Let $\alpha > 0$ such that, for every $\varepsilon > 0$:*

$$\lim_{x \to 0} \frac{x^{\alpha + \varepsilon}}{x - f(x)} = 0.$$

Then for every $\varepsilon > 0$ there exist $n_0 \in \mathbb{N}$ and $d_1 > 0$, such that

$$0 < f^{\circ n}(x) < x\Big(1 + \frac{n}{2} x^{\alpha + \varepsilon - 1}\Big)^{-\frac{1}{\alpha + \varepsilon - 1}},\ x \in (0, d_1),\ n \geq n_0. \tag{6.20}$$

Proof. Take $\varepsilon > 0$ and $f_1(x) = x - x^{\alpha + \varepsilon}$. We prove the proposition for f_1, and the statement for f follows. To prove the estimate, we use the change of variables $w = \frac{1}{(\alpha + \varepsilon - 1) x^{\alpha + \varepsilon}}$, $w \in (M, \infty)$, by which f_1 becomes:

$$F_1(w) = w + 1 + O(w^{-1}).$$

It is now easy to end the proof working with F_1. □

6.1.4. **The sectional asymptotic expansions of the Fatou coordinate with respect to integral sections.** We prove here that $R \in \mathcal{G}_{AN}$ defined by (6.10) admits the formal infinitesimal part of the Fatou coordinate $\widehat{R} \in \widehat{\mathcal{L}}$ constructed in (6.9) as its sectional asymptotic expansion with respect to *any* integral section $\mathbf{s}$. We have already proven in Subsection 6.1.2 that $\Psi^{\infty} \in \mathcal{G}_{AN}$ admits $\widehat{\Psi}^{\infty} \in \widehat{\mathcal{L}}_2^{\infty}$ as its sectional asymptotic expansion with respect to any integral section for appropriate choices of constant terms in Ψ^{∞}, $\widehat{\Psi}^{\infty}$. Consequently, the Fatou coordinate $\Psi = \Psi^{\infty} + R \in \mathcal{G}_{AN}$ will admit the formal Fatou coordinate $\widehat{\Psi} = \widehat{\Psi}^{\infty} + \widehat{R} \in \widehat{\mathcal{L}}_2^{\infty}$ as its sectional asymptotic expansion with respect to any integral section, with appropriate choice of constant terms in Ψ^{∞}, $\widehat{\Psi}^{\infty}$. Finally, different choices of integral sections lead to different choices of the constants in Ψ or in $\widehat{\Psi}$, if $\widehat{\Psi}$ is to be the sectional asymptotic expansion of Ψ with respect to the new integral section.

Put $h_n := R - \sum_{i=1}^{n} R_{r_0 + i}$, $n \in \mathbb{N}$. Obviously, $h_n \in \mathcal{G}_{AN}$ and $h_n = o(1)$ (since $R = o(1)$ and $R_{r_0+i} = o(1)$, $i \in \mathbb{N}$). It can easily be checked that h_n satisfies the difference equation with the right-hand side $\delta_n(x) = O(x^{\gamma_n})$, where $\gamma_n \to \infty$ and $\delta_n \in \mathcal{G}_{AN}$. Iterating the equation for h_n and passing to limit as for $R(x)$ before, we get that h_n is necessarily given by the formula:

$$h_n(x) = -\sum_{k=0}^{\infty} \delta_n(f^{\circ k}(x)).$$

By Proposition 6.2, we get that $h_n = O(x^{\beta_n})$, where $\beta_n \to \infty$. Together with the fact that $\widehat{R}_{r_0+i} \in \widehat{\mathfrak{L}}$ is the sectional asymptotic expansion of $R_{r_0+i} \in \mathcal{G}_{AN}$, with respect to any integral section (see Proposition 7.3), $i \in \mathbb{N}$, this proves that $R \in \mathcal{G}_{AN}$, $R = o(1)$, admits $\widehat{R} \in \widehat{\mathfrak{L}}$ as the sectional asymptotic expansion with respect to any integral section.

6.1.5. **Uniqueness of the Fatou coordinate.**

By Proposition 4.3, the formal Fatou coordinate $\widehat{\Psi} \in \widehat{\mathcal{L}}_2^\infty$ is unique in $\widehat{\mathfrak{L}}$, up to an additive constant.

Example 6 (Non-uniqueness of a Fatou coordinate with only sectional asymptotic expansion in $\mathfrak{L}$)**.** Let $\Psi \in \mathcal{G}_{AN}$ be the Fatou coordinate for f Dulac constructed in the algorithm. It admits a sectional asymptotic expansion $\widehat{\Psi} \in \widehat{\mathcal{L}}_2^\infty$ with respect to an integral section:

$$\widehat{\Psi} = \sum_{i=1}^{\infty} x^{\alpha_i} \widehat{f}_i(\boldsymbol{\ell}),$$

where $\alpha_1 < 0$. The integral section attributes to partial expansion $\sum_{i=1}^n x^{\alpha_i} \widehat{f}_i(\boldsymbol{\ell})$ at each limit ordinal stage $n \in \mathbb{N}$ the *sum* $\sum_{i=1}^n x^{\alpha_i} f_i(\boldsymbol{\ell}) \in \mathcal{G}_{AN}$, where $f_i \in \mathcal{G}_{AN}$ are integral sums of $\widehat{f}_i \in \widehat{\mathcal{L}}_0^I$.

Take now another Fatou coordinate for f:

$$\Psi_1 = \Psi + \sin(2\pi\Psi).$$

Obviously, $\Psi_1 \in \mathcal{G}_{AN}$ and satisfies the Abel equation. We prove that Ψ_1 admits the same $\widehat{\Psi}$ as its sectional asymptotic expansion if we take e.g. the section $\mathbf{s}$ that maps:

$$\begin{aligned} &x^{\alpha_1} \widehat{f}_1 \mapsto x^{\alpha_1} \big(f_1(\boldsymbol{\ell}) + e^{\frac{\alpha_1}{\boldsymbol{\ell}}} \sin(2\pi\Psi(e^{-1/\boldsymbol{\ell}})) \big), \\ &x^{\alpha_i} \widehat{f}_i \mapsto x^{\alpha_i} f_i, \ i \geq 2. \end{aligned}$$

Note that the real sine function is bounded, so $e^{\frac{\alpha_1}{\boldsymbol{\ell}}} \sin(2\pi\Psi(e^{-1/\boldsymbol{\ell}}))$, $\alpha_1 < 0$, is exponentially small with respect to $\boldsymbol{\ell}$. Therefore $f_1(\boldsymbol{\ell}) + e^{\frac{\alpha_1}{\boldsymbol{\ell}}} \sin(2\pi\Psi(e^{-1/\boldsymbol{\ell}}))$ admits Poincaré asymptotic expansion $\widehat{f}_1(\boldsymbol{\ell})$. To conclude, $\Psi_1(x)$ admits the sectional asymptotic expansion $\widehat{\Psi}$ with respect to section $\mathbf{s}$ (which is not integral sectional).

We prove now that the Fatou coordinate for a Dulac germ f admitting a sectional asymptotic expansion in $\widehat{\mathfrak{L}}$ with respect to an integral section *is* unique (up to an additive constant). Suppose the contrary, that is, that $\Psi_1 \in \mathcal{G}_{AN}$ and $\Psi_2 \in \mathcal{G}_{AN}$ are two Fatou coordinates for f that admit integral sectional asymptotic expansions in $\widehat{\mathfrak{L}}$. Note that if Ψ_1 and Ψ_2 admit integral sectional asymptotic expansions in $\widehat{\mathfrak{L}}$, then $\Psi_1 - \Psi_2$ admits an integral sectional asymptotic expansion in $\widehat{\mathfrak{L}}$ (this can be easily seen by definition of integral sectional asymptotic expansions as sum of blocks $\sum_{i\in\mathbb{N}} \int_*^x x^{\gamma_i} R_i(\boldsymbol{\ell}) dx$, where R_i is a rational function, and $\gamma_i \in \mathbb{R}$ strictly increase as $i \to \infty$). This is not the case for only sectional asymptotic expansions in $\widehat{\mathfrak{L}}$. Take, for example, Ψ and Ψ_1 from Example 6 which both admit sectional asymptotic expansions in $\widehat{\mathfrak{L}}$, but their difference, $\sin(2\pi\Psi)$, obviously does not admit an asymptotic expansion in $\widehat{\mathfrak{L}}$.

The difference $\Psi_1 - \Psi_2 \in \mathcal{G}_{AN}$ satisfies the equation:

$$(\Psi_1 - \Psi_2)(f(x)) = (\Psi_1 - \Psi_2)(x).$$

Since $\Psi_1 - \Psi_2$ admits an integral sectional asymptotic expansion, we have that (by blocks) there exists $\beta \in \mathbb{R}$, a rational function R not identically equal to 0 and $\delta > 0$ such that

$$\Psi_1(x) - \Psi_2(x) = \int_*^x R(\boldsymbol{\ell})x^\beta + o(x^{\beta+1+\delta}),\ x \to 0.$$

Since $f(x) = x + x^{\alpha_1}P_1(\boldsymbol{\ell}) + o(x^{\alpha_1+\delta})$, $x \to 0$, by Taylor formula we get:

$$(\Psi_1 - \Psi_2)'(\xi_x) \cdot \left(x^{\alpha_1}P_1(\boldsymbol{\ell}) + o(x^{\alpha_1+\delta})\right) = 0,$$

where $\xi_x = x + o(x)$, $x \to 0$. We conclude $(\Psi_1 - \Psi_2)'(\xi_x) = 0$ for small x, but this is not possible since $(\Psi_1 - \Psi_2)'(\xi_x) \sim x^\beta R(\boldsymbol{\ell}) \sim x^\beta \boldsymbol{\ell}^M$, $M \in \mathbb{Z}$, as $x \to 0$. The last approximation follows since $R(\boldsymbol{\ell})$ is a rational function not identically equal to 0.

6.2. **A precise form of the formal Fatou coordinate.** In the course of the proof of the Theorem (in Subsection 6.1.1), we have also proved a more precise form of the formal Fatou coordinate $\widehat{\Psi}$ as stated below in Proposition 6.4. In particular, we have proved that there is only one monomial in $\widehat{\Psi}$ which contains the double logarithm.

Proposition 6.4 (The formal Fatou coordinate of a parabolic Dulac germ)**.** *Let $\widehat{\Psi} \in \widehat{\mathcal{L}}_2^\infty$ be the (unique in $\widehat{\mathfrak{L}}$ up to an additive constant) formal Fatou coordinate for a parabolic Dulac germ $f \in \mathcal{G}_{AN}$. Then there exists $\rho \in \mathbb{R}$ such that $\widehat{\Psi} - \rho\boldsymbol{\ell}_2^{-1} \in \widehat{\mathfrak{L}}^\infty$, and*

$$\widehat{\Psi} - \rho\boldsymbol{\ell}_2^{-1} = \sum_{i=1}^\infty x^{\alpha_i}\widehat{f}_i(\boldsymbol{\ell}). \tag{6.21}$$

Here,

1. *$\alpha_1 < 0$,*
2. *α_i is a strictly increasing real sequence tending to $+\infty$ (finitely generated),*
3. *$\widehat{f}_i \in \widehat{\mathcal{L}}_0^\infty$ is a formal Laurent series given by:*

$$\widehat{f}_i(\boldsymbol{\ell}) = \frac{\int x^{\alpha_i - 1}\widehat{R}_i(\boldsymbol{\ell})\,dx}{x^{\alpha_i}}, \tag{6.22}$$

where $\widehat{R}_i$ is a convergent series (*more precisely, the power asymptotic expansion of a rational function*).

Note that, in general, $\widehat{f}_i(y)$ is a *divergent power series* for every positive value $y > 0$. Nevertheless, it is what we call *integrally summable* in Definition 3.8.

7. Appendix

The following Remark is used in Section 4.

Remark 7.1 (Description of the formal Fatou coordinate for parabolic transseries)**.** We explain here another way of deducing that the formal Fatou coordinate $\widehat{\Psi} \in \widehat{\mathfrak{L}}$ of a parabolic $\widehat{f} \in \widehat{\mathcal{L}}$ belongs to the class $\widehat{\mathcal{L}}_2^\infty$ and its precise form.

Recall the formal normal form $\widehat{f}_0$ of $\widehat{f}$ in $\widehat{\mathcal{L}}$ deduced in [11], given as the formal time-one map of a simple vector field:

$$\widehat{f} = x + ax^\alpha \boldsymbol{\ell}^m + \text{h.o.t.}, \text{ with } \alpha > 1, \text{ or } \alpha = 1 \text{ and } m \in \mathbb{N},$$

$$\widehat{f}_0(x) = \operatorname{Exp}\big(\widehat{\xi}_0(x)\frac{\mathrm{d}}{\mathrm{d}x}\big)\cdot \mathrm{id},$$

$$\widehat{\xi}_0(x) = \frac{ax^\alpha \boldsymbol{\ell}^m}{1 + \frac{a\alpha}{2}x^{\alpha-1}\boldsymbol{\ell}^m - \big(\frac{am}{2} + \frac{b}{a}\big)x^{\alpha-1}\boldsymbol{\ell}^{m+1}}\frac{\mathrm{d}}{\mathrm{d}x},\ b \in \mathbb{R}.$$

By the existence part of Proposition 4.3, we look for a formal Fatou coordinate of $\widehat{f}_0$ as the formal antiderivative of $\frac{1}{\widehat{\xi}_0}$:

$$\begin{aligned}\widehat{\Psi}_0 =& \frac{1}{a}\int x^{-\alpha}\boldsymbol{\ell}^{-m}\,dt + \frac{\alpha}{2}\log x + (\frac{m}{2} + \frac{b}{a^2})\int \frac{dx}{x\log x}\\ =& \frac{1}{a}\widehat{h}(x) - \frac{\alpha}{2}\boldsymbol{\ell}^{-1} + (\frac{m}{2} + \frac{b}{a^2})\boldsymbol{\ell}_2^{-1} + C,\ C \in \mathbb{R}.\end{aligned}$$

Here, $\widehat{h} \in \widehat{\mathcal{L}}^\infty$ is obtained by repeated formal *integration by parts*:

$$\int x^{-\alpha}\boldsymbol{\ell}^{-m}\,dx = \begin{cases}\frac{1}{1-\alpha}x^{-\alpha+1}\boldsymbol{\ell}^{-m} + \frac{m}{1-\alpha}\int x^{-\alpha}\boldsymbol{\ell}^{-m+1}\,dx, & \alpha \neq 1,\\ -\frac{1}{m+1}\boldsymbol{\ell}^{-m-1} + C,\ C \in \mathbb{R}, & \alpha = 1,\ m \in \mathbb{N}.\end{cases} \tag{7.1}$$

In particular, in the case when $m \in \mathbb{N}$, $\widehat{h}$ contains only finitely many monomials.

Now put $\widehat{\Psi} := \widehat{\Psi}_0 \circ \widehat{\varphi}$, where $\widehat{\varphi} \in \widehat{\mathcal{L}}$ parabolic is the formal change of variables reducing $\widehat{f}$ to $\widehat{f}_0$. It is easy to check that the composition $\widehat{\Psi}$ belongs to $\widehat{\mathcal{L}}_2^\infty$. It is obviously a Fatou coordinate for $\widehat{f}$, since $\widehat{\Psi}_0$ satisfies the Abel equation for $\widehat{f}_0$. Moreover, by Proposition 4.3, the formal Fatou coordinate of $\widehat{f}$ is unique in $\widehat{\mathfrak{L}}$ (up to a constant term).

Therefore, at most one term with double logarithm $\boldsymbol{\ell}_2^{-1}$ appears in the formal Fatou coordinate $\widehat{\Psi}$ of $\widehat{f}$. It corresponds to the residual term $bx^{2\alpha-1}\boldsymbol{\ell}^{2m+1}$ in the normal form $\widehat{f}_0(x) = x + ax^\alpha\boldsymbol{\ell}^m + bx^{2\alpha-1}\boldsymbol{\ell}^{2m+1}$.

7.1. **Propositions for Section 3.**

Proof of Proposition 3.11. Suppose that $\widehat{f}$ is divergent and that there are two exponents of integration $\alpha \neq \beta$ for $\widehat{f}$. Then

$$\frac{d}{dx}\big(x^\alpha \widehat{f}(\boldsymbol{\ell})\big) = x^{\alpha-1}R_1(\boldsymbol{\ell}),\ \frac{d}{dx}\big(x^\beta \widehat{f}(\boldsymbol{\ell})\big) = x^{\beta-1}R_2(\boldsymbol{\ell}).$$

Therefore,

$$\begin{aligned}\frac{d}{dx}\big(x^\alpha \widehat{f}(\boldsymbol{\ell})\big) &= \frac{d}{dx}\big(x^{\alpha-\beta}\cdot x^\beta \widehat{f}(\boldsymbol{\ell})\big) = (\alpha-\beta)x^{\alpha-1}\widehat{f}(\boldsymbol{\ell}) + x^{\alpha-\beta}\frac{d}{dx}\big(x^\beta \widehat{f}(\boldsymbol{\ell})\big) =\\ &= (\alpha-\beta)x^{\alpha-1}\widehat{f}(\boldsymbol{\ell}) + x^{\alpha-1}R_2(\boldsymbol{\ell})\\ &\Rightarrow x^{\alpha-1}R_1(\boldsymbol{\ell}) = (\alpha-\beta)x^{\alpha-1}\widehat{f}(\boldsymbol{\ell}) + x^{\alpha-1}R_2(\boldsymbol{\ell}).\end{aligned}$$

Since R_1 and R_2 are both convergent Laurent series and $\alpha - \beta \neq 0$, this is a contradiction with divergence of $\widehat{f}$. □

Proposition 7.2. *Let $\alpha \in \mathbb{R}$, $m \in \mathbb{Z}$. Let*

$$a(x) := \int_d^x t^{-\alpha}\boldsymbol{\ell}^m\,dt,$$

with $d > 0$ if $\alpha > 1$ or $(\alpha = 1,\ m \leq 1)$, and $d = 0$ if $\alpha < 1$ or $(\alpha = 1,\ m > 1)$. The integral is divergent at 0 in the first case and convergent in the second. Then $a \in \mathcal{G}_{AN}$ and

$$a(x) = \begin{cases} O(x^{-\alpha+1}\boldsymbol{\ell}^m),\ x \to 0+, & \alpha \neq 1, \\ \frac{\boldsymbol{\ell}^{m-1}}{m-1} + C,\ C \in \mathbb{R}, & \alpha = 1,\ m < 1, \\ \boldsymbol{\ell}_2^{-1} + C,\ C \in \mathbb{R}, & \alpha = 1,\ m = 1, \\ \frac{\boldsymbol{\ell}^{m-1}}{1-m}, & \alpha = 1,\ m > 1. \end{cases}$$

The proof is based on elementary calculus.

The following proposition is an easy consequence of Proposition 7.2 and integration by parts:

Proposition 7.3.
(1) *Let $\widehat{R} \in \widehat{\mathcal{L}}_0^\infty$ and let $n_0 := \mathrm{ord}(\widehat{\mathrm{R}})$, $n_0 \in \mathbb{Z}$. For $\alpha \in \mathbb{R}$ consider the transseries $\widehat{F}$ and $\widehat{f}$ respectively defined by:*

$$\widehat{F}(x) = \frac{\int x^{\alpha-1}\widehat{R}(\boldsymbol{\ell}(x))\,dx}{x^\alpha} \text{ and } \widehat{f}(y) = \widehat{F}\left(e^{-1/y}\right), \tag{7.2}$$

where the numerator of $\widehat{F}(x)$ is the formal antiderivative in $\widehat{\mathfrak{L}}$ of $x^{\alpha-1}\widehat{R}(\boldsymbol{\ell}(x))$ without constant term (as explained in the footnote on p. 15). Then $\widehat{F} \in \widehat{\mathcal{L}}_2^\infty$ and $\widehat{f} \in \widehat{\mathcal{L}}_1^\infty$. Moreover:

a) if $\alpha \neq 0$ then $\widehat{f} \in \widehat{\mathcal{L}}_0^\infty$ and

$$\widehat{f}(y) = \sum_{n=0}^{\infty} a_n y^{n_0+n},\ a_n \in \mathbb{R};$$

b) if $\alpha = 0$ then

$$\widehat{f}(y) = \begin{cases} \sum_{n=0}^{-n_0} a_n y^{n_0-1+n} + b\log y + \sum_{n=0}^{\infty} b_n y^n, & n_0 \leq 1, \\ \sum_{n=0}^{\infty} a_n y^{n_0-1+n}, & n_0 > 1,\ a_n, b_n, b \in \mathbb{R}. \end{cases}$$

(2) *Let $R \in \mathcal{G}_{AN}$ admit the integer power asymptotic expansion $\widehat{R} \in \widehat{\mathcal{L}}_0^\infty$. Let $f \in \mathcal{G}_{AN}$ be given by:*

$$f(y) := \frac{\int_d^{e^{-1/y}} s^{\alpha-1}R\big(\boldsymbol{\ell}(s)\big)\,ds}{e^{-\frac{\alpha}{y}}}, \tag{7.3}$$

where $d > 0$ if $\alpha < 0$ or $(\alpha = 0$ and $n_0 \leq 1)$, and $d = 0$ if $\alpha > 0$ or $(\alpha = 0$ and $n_0 > 1)$.

Then $\widehat{f} \in \widehat{\mathcal{L}}_0^\infty$ defined in (7.2) is the power asymptotic expansion of $f \in \mathcal{G}_{AN}$ if $\alpha \neq 0$. If $\alpha = 0$, $\widehat{f}$ is the Poincaré asymptotic expansion of f up to a constant.

7.2. **Proofs of propositions from Section 4.**

Proof of Proposition 4.3.

The chain rule. As a prerequisite for the proof, we prove that the chain rule is valid in our formal setting. That is, if $\widehat{\Psi} \in \widehat{\mathcal{L}}_2^\infty$ and $\{\widehat{f}^t\},\ \widehat{f}^t \in \widehat{\mathcal{L}}$ is a C^1-flow as defined in [11, Def.1.2], then:

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(\widehat{\Psi}(\widehat{f}^t(x))\big) = \widehat{\Psi}'\big(\widehat{f}^t(x)\big) \cdot \frac{\mathrm{d}}{\mathrm{d}t}\widehat{f}^t(x)$$

holds formally in $\widehat{\mathcal{L}}_2^\infty$. Here, $\frac{\mathrm{d}}{\mathrm{d}t}$ applied to a transseries means the derivation monomial by monomial. Since $\widehat{f}^t(x) = x + \text{h.o.t.}$ with coefficients in $C^1(\mathbb{R})$, it stems from *Neuman's Lemma* (see [2]) that the coefficients of $\widehat{\Psi}(\widehat{f}^t(x))$ also belong to $C^1(\mathbb{R})$.

It is sufficient to prove the equality in $\widehat{\mathcal{L}}_2^\infty$ for a single monomial $m(x)$ from the support $\mathrm{Supp}(\widehat{\Psi})$:

$$\frac{\mathrm{d}}{\mathrm{d}t}(m(\widehat{f}^t(x))) = m'(\widehat{f}^t(x)) \cdot \frac{\mathrm{d}}{\mathrm{d}t}\widehat{f}^t(x).$$

Both sides share a common well-ordered support. Take any monomial from this support. By *Neumann's lemma*, on both sides only finitely many monomials from $\widehat{f}^t$ contribute to it. Now the equality holds if we replace $\widehat{f}^t$ by the finite sum of its terms corresponding to these first monomials. Therefore, the coefficients of every monomial on both sides coincide.

The existence. Take $\widehat{\Psi}$ to be the formal antiderivative in $\widehat{\mathfrak{L}}$ without constant term (as explained in the footnote on p. 15) of $1/\widehat{\xi}$, where

$$\widehat{\xi} := \frac{\mathrm{d}}{\mathrm{d}t}\widehat{f}^t\Big|_{t=0}.$$

We prove that $\widehat{\Psi}$ satisfies the equation (2.4) for the formal Fatou coordinate. Integrating formally $1/\widehat{\xi}$ (every monomial is formally integrated by parts), we conclude that $\widehat{\Psi} \in \widehat{\mathcal{L}}_2^\infty$. Indeed, since $\widehat{\xi} \in \widehat{\mathcal{L}}$, we get that $1/\widehat{\xi} \in \widehat{\mathcal{L}}^\infty$. In the integration process, the double logarithm $\boldsymbol{\ell}_2^{-1}$ is generated when integrating the monomial $x^{-1}\boldsymbol{\ell}$.

Using $\frac{\mathrm{d}}{\mathrm{d}t}\widehat{f}^t = \widehat{\xi}(\widehat{f}^t)$, by the chain rule proved above, we get that

$$\frac{d}{dt}\big(\widehat{\Psi}(\widehat{f}_t(x))\big) = \widehat{\Psi}'(\widehat{f}^t(x)) \cdot \frac{\mathrm{d}}{\mathrm{d}t}\widehat{f}^t(x) = \frac{1}{\widehat{\xi}(\widehat{f}^t(x))} \cdot \frac{\mathrm{d}}{\mathrm{d}t}\widehat{f}^t(x) = 1.$$

Integrating this equality from 0 to t gives the equality (4.1). In particular, $\widehat{\Psi}(\widehat{f}(x)) - \widehat{\Psi}(x) = 1$.

The uniqueness. Suppose that there exist two formal Fatou coordinates $\widehat{\Psi}_1,\ \widehat{\Psi}_2 \in \widehat{\mathfrak{L}}$. Let $\widehat{\Psi} := \widehat{\Psi}_1 - \widehat{\Psi}_2$. Then $\widehat{\Psi} \in \widehat{\mathfrak{L}}$, that is, $\widehat{\Psi} \in \widehat{\mathcal{L}}_j^\infty$ for some $j \in \mathbb{N}_0$, and it satisfies:

$$\widehat{\Psi}(\widehat{f}(x)) - \widehat{\Psi}(x) = 0.$$

Since $\widehat{f} \in \widehat{\mathcal{L}}$, by Taylor expansion in $\widehat{\mathcal{L}}_j^\infty$, we get:

$$\widehat{\Psi}' \cdot \widehat{g} + \frac{1}{2!}\widehat{\Psi}'' \cdot \widehat{g}^2 + \cdots = 0.$$

If $\widehat{\Psi}' \neq 0$ in $\widehat{\mathcal{L}}_j^\infty$, since $\mathrm{ord}(\widehat{g}) \succ (1,0)$, the leading term of the left-hand side is the non-zero leading term of $\widehat{\Psi}' \cdot \widehat{g}$, which is a contradiction. Therefore, $\widehat{\Psi}' = 0$, so $\widehat{\Psi} = C,\ C \in \mathbb{R}$ (which can be easily checked by analysing leading terms or looked up in [2]). $\square$

Proof of Proposition 4.4. We prove both directions.

1. Suppose that an analytic Fatou coordinate Ψ for f exists on $(0,d)$. By definition it is strictly monotonic on $(0,d)$. Since $f(x) < x$, Ψ is strictly decreasing on $(0,d)$. Therefore its image is the interval $(\Psi(d), +\infty)$. Then the family $\{f^t\}$ of analytic functions on $(0,d)$, $d > 0$, defined by:

$$f^t(x) := \Psi^{-1}\big(\Psi(x) + t\big), \tag{7.4}$$

is well-defined and analytic for $t \in (\Psi(d) - \Psi(x), +\infty)$, $x \in (0,d)$. Note that, for every $x \in (0,d)$, due to monotonicity of Ψ, $\Psi(d) - \Psi(x) < 0$, and t can be extended to $+\infty$ in the positive direction. We conclude that f^1 exists for all $x \in (0,d)$ and $\{f^t\}$ gives a C^1-flow in which f embeds as the time-one map.

Furthermore,

$$\xi := \frac{\mathrm{d}}{\mathrm{d}t} f^t\Big|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t}\Psi^{-1}\big(\Psi(x) + t\big)\Big|_{t=0} = \frac{1}{\Psi'}.$$

Since Ψ is a strictly monotonic on $(0,d)$, either $\Psi' > 0$ or $\Psi' < 0$ in $(0,d)$, so ξ is non-oscillatory in $(0,d)$.

2. The vector field whose flow is given by the C^1-family $\{f^t\}$, $t \in \mathbb{R}$, of analytic functions on $(0,d)$, is given by the formula:

$$X = \xi(x)\frac{\mathrm{d}}{\mathrm{d}x}, \text{ where } \xi := \frac{\mathrm{d}}{\mathrm{d}t} f^t\Big|_{t=0}.$$

Obviously, ξ is also analytic on $(0,d)$. Take Ψ to be the antiderivative of $1/\xi$. That is, $\Psi' = \frac{1}{\xi}$. We prove that Ψ is a Fatou coordinate for f, that is, satisfies (2.3). We solve the differential equation for the flow:

$$\begin{aligned} \dot{x} &= \xi(x) \\ \frac{\mathrm{d}x}{\xi(x)} &= \mathrm{d}t, \\ t &= \int_x^{f^t(x)} \frac{ds}{\xi(s)}. \end{aligned}$$

We get that

$$\Psi(f^t(x)) - \Psi(x) = t,\ x \in (0,d),\ t \in (\mathbb{R}^+, 0) \text{ (the exact interval depending on } x).$$

In particular, $\Psi(f(x)) - \Psi(x) = 1$ (note that by assumption f embeds in $\{f^t\}$ on $(0,d)$ as time-1 map, so f is defined for every $x \in (0,d)$.

Moreover, since $\Psi' = \frac{1}{\xi}$, and ξ does not change sign in some interval $(0,d)$, Ψ is strictly monotonic in the same interval. □

Remark 7.4 (The importance of *non-oscillatority* in Proposition 4.4)**.** Consider the flow $\{f^t\}_t$ of an analytic vector field $X = \xi\frac{\mathrm{d}}{\mathrm{dx}}$ on $(0,d)$. Take a non-singular point $x_0 > 0$ of the vector field ($\xi(x_0) \neq 0$). Then, by (4.2), the Fatou coordinate Ψ_{x_0} is defined at a point x as the time $t \in \mathbb{R}$ such that $f^t(x_0) = x$. In particular, $\Psi_{x_0}(x_0) = 0$. Obviously, Ψ_{x_0} cannot be defined at any singular (equilibrium) point of vector field ξ.

For example, the flow $\{f^t\}_{t\in\mathbb{R}}$ of the analytic vector field $X = x^2\sin(1/x)\frac{\mathrm{d}}{\mathrm{d}x}$ on $(0,d)$ consists of analytic maps on $(0,d)$. But, as the vector field X admits infinitely many singular points which accumulate at the origin, we cannot define a Fatou coordinate on any interval $(0,d_1)$.

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