

## Prove optimality with a heuristic? Do you want to bet? Case study on Graph Coloring Problem

Find the chromatic number  $\chi(G)$ : the smallest number of colors needed to color a graph  $G$

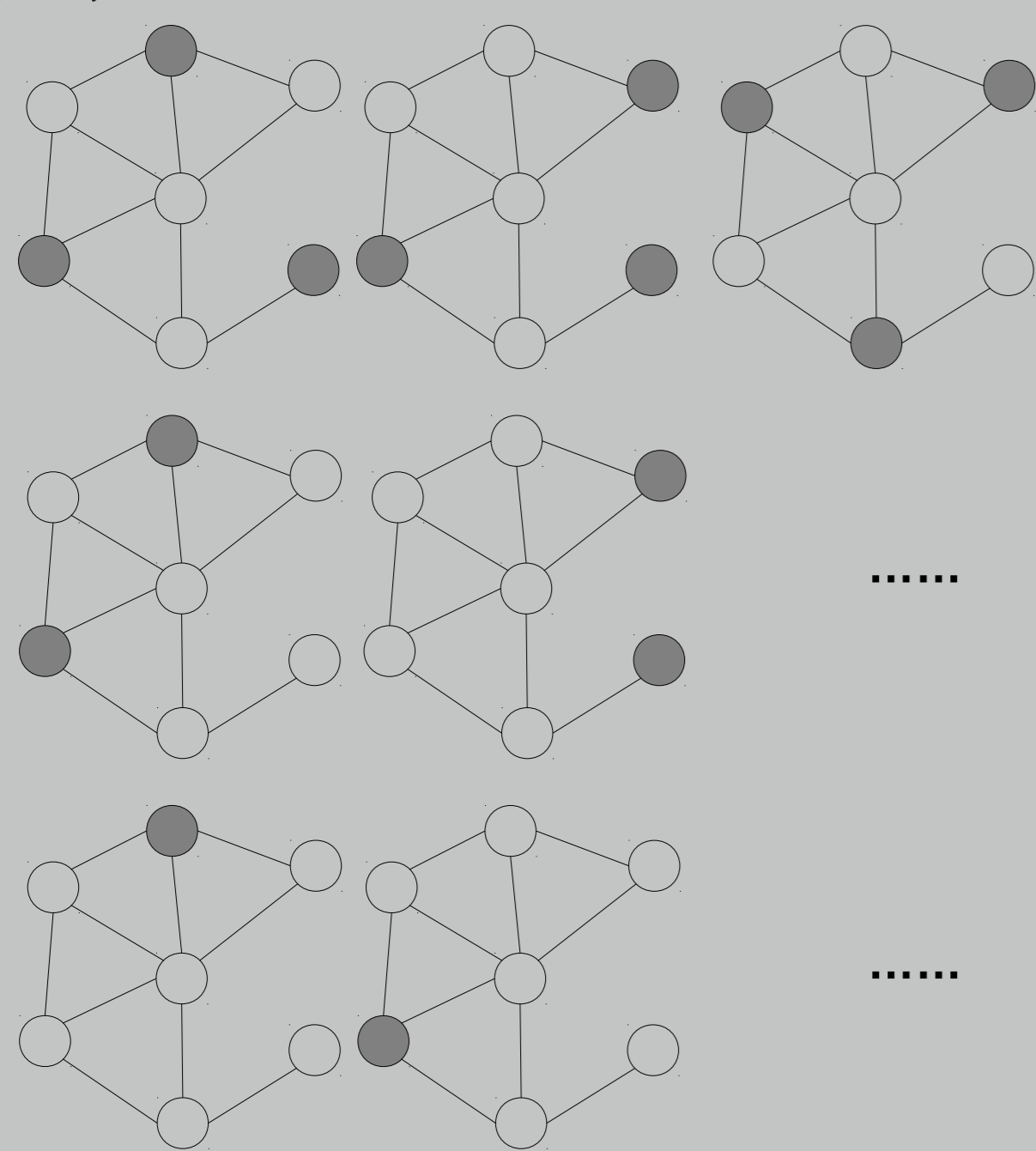
### Theorem: optimality by counting solutions

Let  $k > 0$  and  $G$  a graph,  $\mathcal{N}(G, k)$ : the number of different  $k$ -colorings of  $G$ ,  $i(G)$ : the number of independent sets (or stables) of  $G$ ,  
if  $i(G) > \mathcal{N}(G, k) > 0$ , then  $\chi(G) = k$ .

### Demonstration / Application

#### Counting all Independent Sets (IS) of a graph

▶  $i(G)$  = number of Independent Sets (IS) of a graph



3 IS of size 3.

11 IS of size 2.

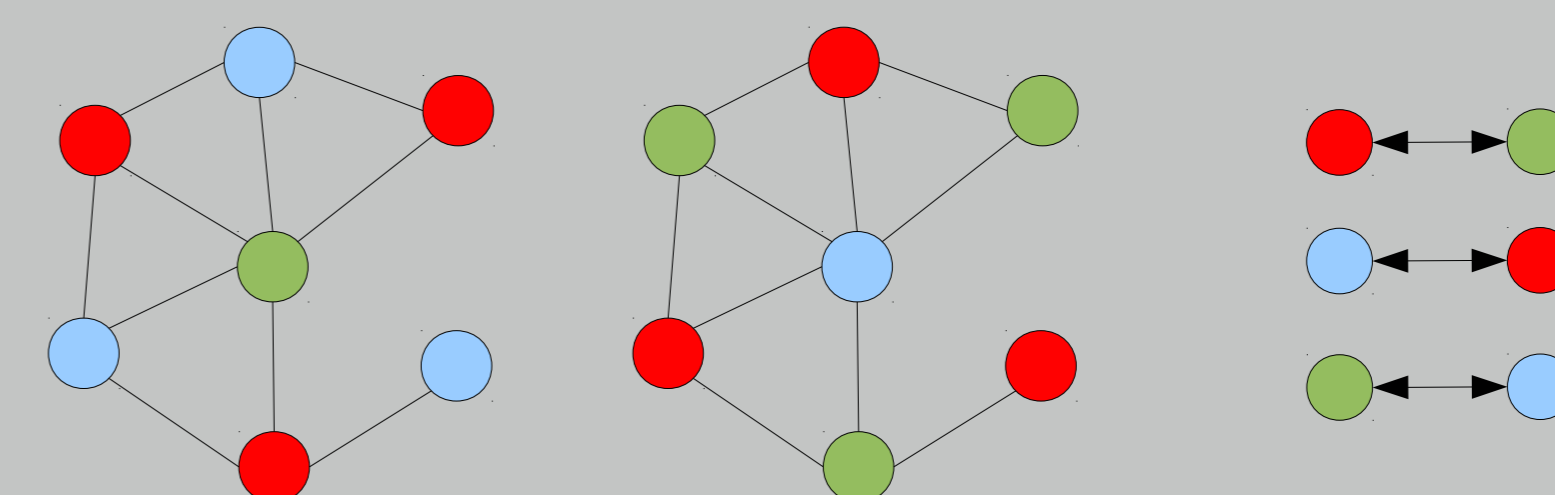
7 IS of size 1.

$$i(G) = 21 \text{ IS}$$

- ▶ NP-hard problem
- ▶ Need only a lower bound (simple problem)
- ▶ Counting all maximal IS
  - ▶ Efficient exact method CLIQUER on complementary graph

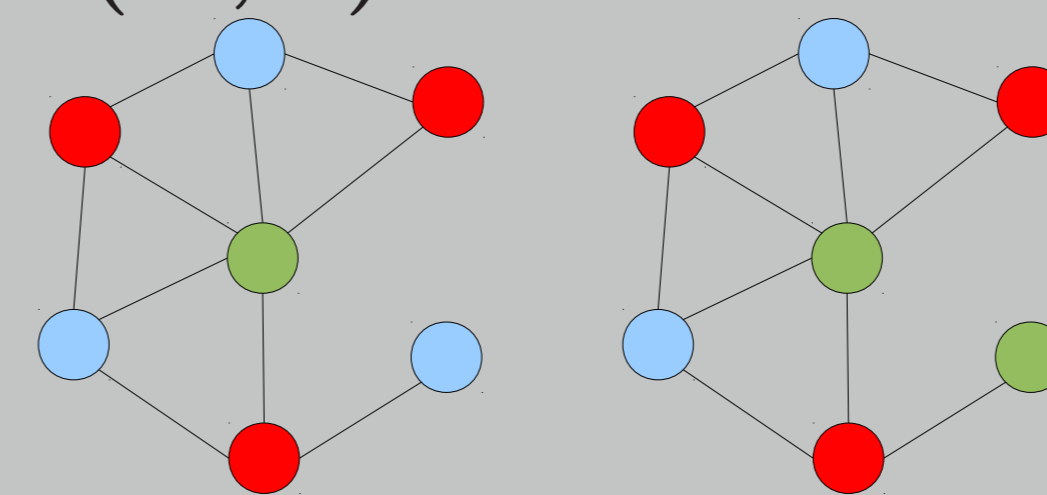
#### Counting number of different $k$ -colorings

▶ Same  $k$ -colorings if permutation of colors:



These two 3-colorings are identical

▶  $\mathcal{N}(G, k)$ : the number of different  $k$ -colorings of  $G$



Two 3-colorings,  $\mathcal{N}(G, 3) = 2$

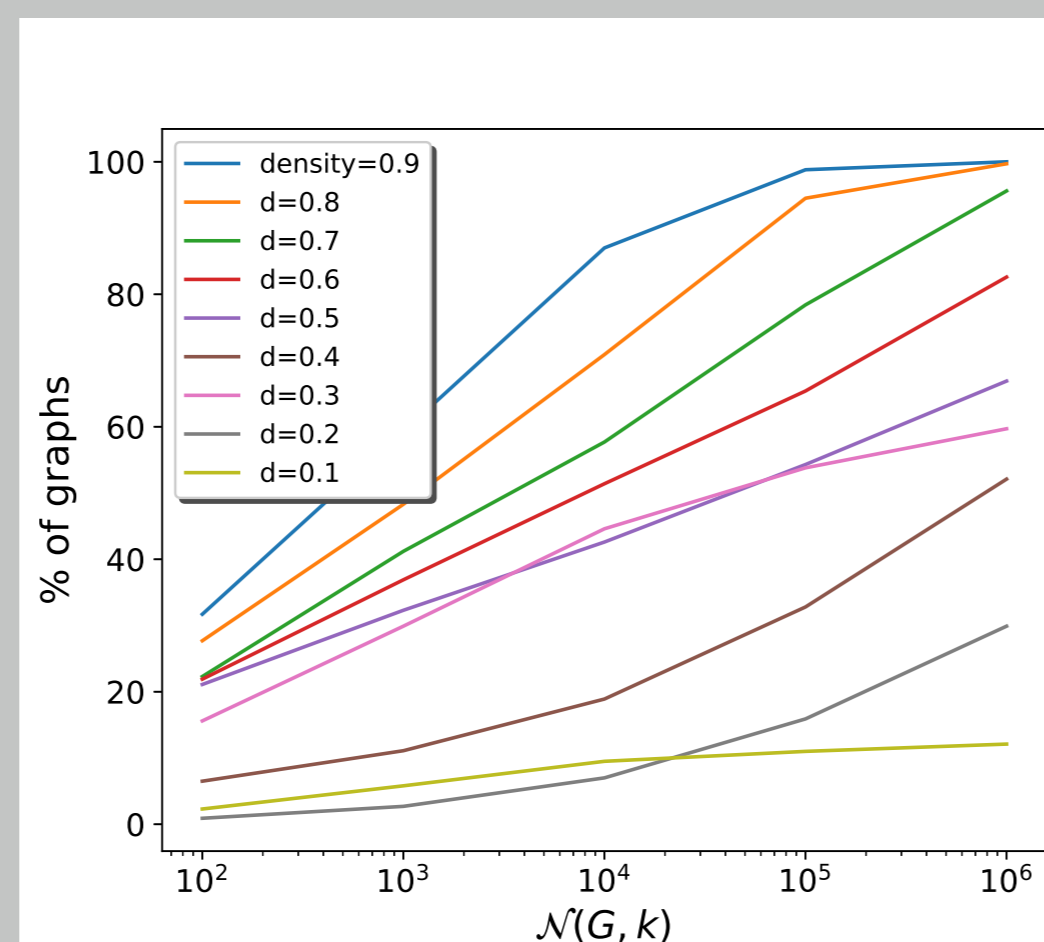
- ▶ #P-complete problem (#SAT / #CSP problems)
- ▶ Need only an upper bound  $\Rightarrow$  difficult problem
  - ▶ Use a modified version of DSATUR for small graphs
  - ▶ Use an estimation based on multi-run of HEAD (one of the best  $k$ -coloring algorithm: memetic algorithm=evolutionary algorithm+tabu search) [Journal of Heuristics, 2018]

### Conclusion

▶ If it exists at least one 2-coloring  $\Rightarrow$  it exists at least 21 3-colorings. As it is not the case:  $i(G) > \mathcal{N}(G, 3) > 0 \Rightarrow \chi(G) = 3$

### Theorem applicable on graphs with low $\mathcal{N}(G, k)$

- ▶ Generate 1000 random graphs with 40/50/60/70 vertices and density 0.1-0.9
- ▶ Count the number of optimal colorings  $\mathcal{N}(G, k)$
- ▶ 50% of graphs (50 vertices, density  $> 0.3$ ) have less than  $10^5$  optimal colorings



### Optimality Clue: let $G$ a graph and $k > 0$

1. Build a sample of  $t = 1000$   $k$ -colorings of  $G$ :
  - ▶ multi-run of HEAD [github.com/graphcoloring/HEAD]
2. Count the number of different  $k$ -colorings inside the sample =  $p$
3. Estimate an upper bound of the  $\mathcal{N}(G, k)$  as  $UB(p, t)$
4. Compute  $i(G)$ , the number of IS, or at least a lower bound
5. If  $i(G) > UB(p, t)$ , then conclude to **Optimality Clue**:  
Chances are that  $k$  is equal to  $\chi(G)$

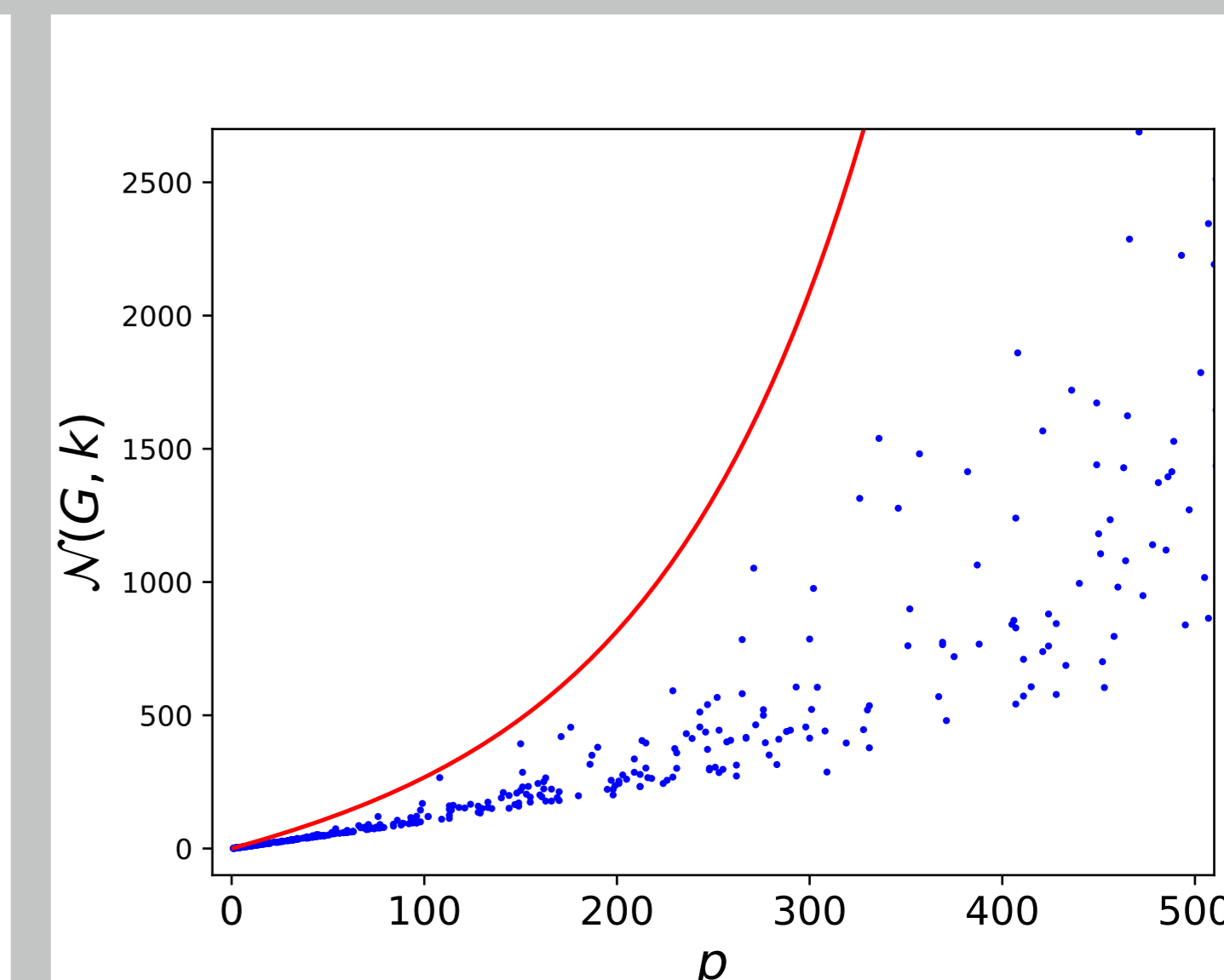
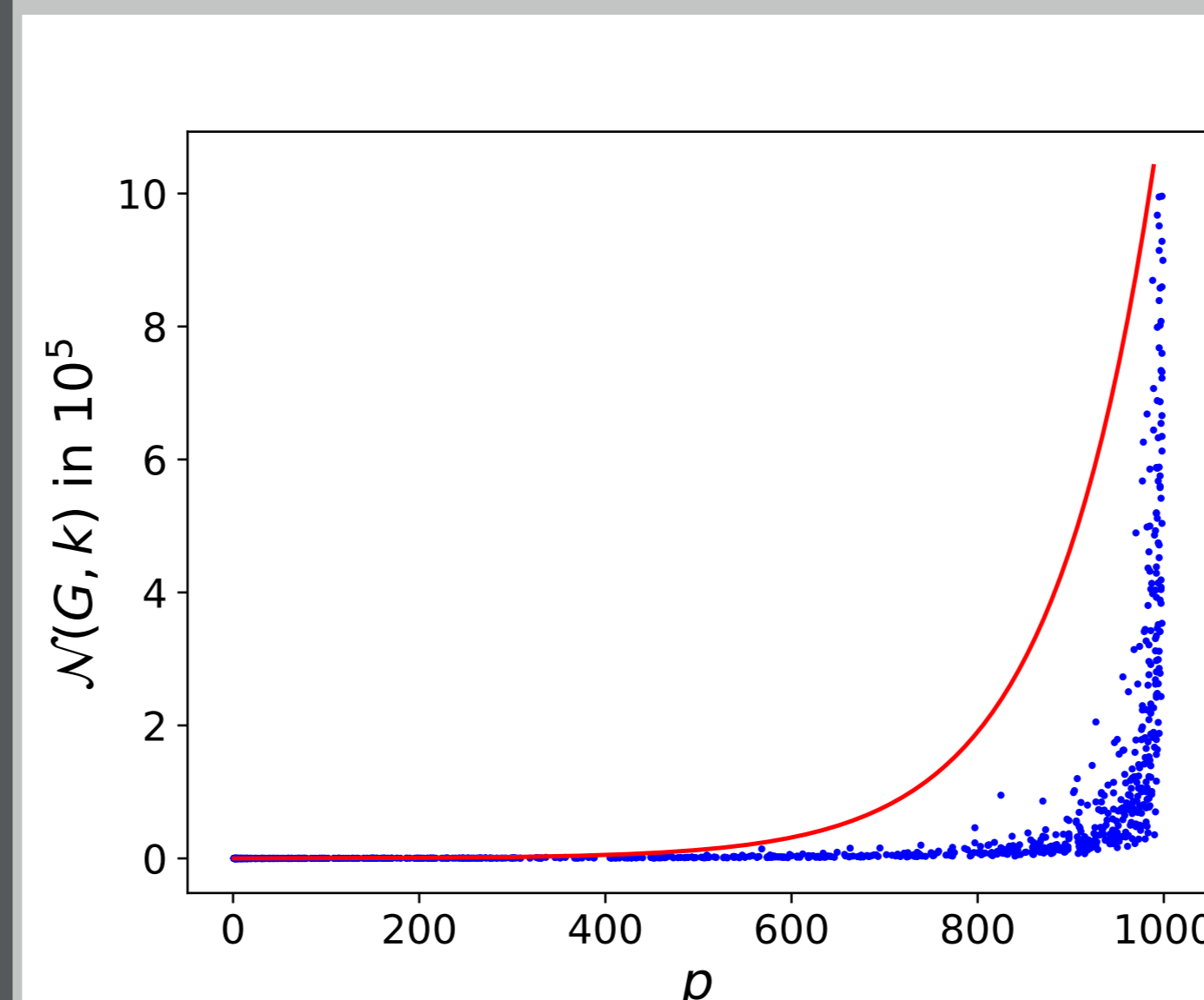
### Some Results on challenging DIMACS graphs

Instances	$ V $	$d$	$\chi(G)$	$k$	$i(G)$	$\mathcal{N}(G, k)$	$t$	$p$	$UB(p, t)$	Opt. Clue
DSJC125.5	125	0.5	17	17	537 508	?	1000	784	164 879	True
DSJC500.5	500	0.5	?	47	$>10000000$	?	341	281	32731	True
DSJC500.9	500	0.9	?	126	35 165	?	1000	949	726 585	False
flat1000_50_0	1000	0.49	?	50	$>10000000$	?	1000	1	2	True
flat1000_60_0	1000	0.49	?	60	$>10000000$	?	1000	1	2	True
le450_15c	450	0.17	15	15	$>10000000$	?	1000	919	554 866	True
le450_15d	450	0.17	15	15	$>10000000$	?	1000	579	26 041	True
queen7_7	49	0.4	7	7	1 862	4	1000	4	8	True

### Find, $UB(p, t)$ , an upper bound of $\mathcal{N}(G, k)$

- ▶ Experimental tests on more than 2000 graphs (RCBII / DIMACS)
- ▶ Random graphs / Size: 40 to 100 vertices / Density: 0.1 to 0.9
- ▶ For each graph, generate a sample of  $k$ -colorings with  $k = \chi(G)$ 
  - ▶  $t$ : size of the sample of  $k$ -colorings ( $t = 1000$ )
  - ▶  $p$ : number of different  $k$ -colorings in the sample
- ▶ Estimate an upper bound of the  $\mathcal{N}(G, k)$  as  $UB(p, t)$  (red line):

$$UB(p, t) = \begin{cases} p + p^{\frac{t+p}{t}} & \text{if } p < t \times 0.99 \\ +\infty & \text{otherwise} \end{cases}$$



### Who wants to win her weight in bottles of wine?

- ▶ The first one to find a counterexample of the **Optimality Clue**
  - ▶ Find a graph  $G$  that has the **Optimality Clue** with  $k$
  - ▶ Find a legal coloring of  $G$  with less than  $k$  colors