Find the chromatic number by counting k-colorings with a randomized heuristic

Alexandre Gondran, Laurent Moalic

Find the chromatic number by counting k-colorings with a randomized heuristic

Alexandre Gondran[1], Laurent Moalic[2]

[1] ENAC, Toulouse, France, `alexandre.gondran@enac.fr`
[2] UHA, Mulhouse, France, `laurent.moalic@uha.fr`

1 Introduction

The main drawback of a heuristic is not to guarantee the optimality of the solutions found, nor even the distance to the optimal value (except for the approximation algorithms). Should we be satisfied to find only *good solutions* for large NP-hard problems ? We propose in this paper a new kind of optimality based on the counting of solutions. The idea is that, for a minimization problem, the more the value of the objective function decreases the less there are admissible solutions with this value of the objective function. Therefore, when the number of solutions with a given value of the objective function is small enough, we have the optimality. It is particularly true for graph coloring problem. This new kind of optimality is named *weak optimality* or *experimental optimality* in that it relies on an estimator of the number of legal k-colorings (admissible solutions for a given value of the objective function). The problem of counting solutions of NP-complete problems has been widely studied for boolean satisfiability problem (called #SAT) or constraint satisfaction problem (called #CSP; k-coloring problem is a special case of CSP). These problems are known as #P-complete [6].

2 Graph coloring problem

Coloring the graph $G(V, E)$ consists in assigning a color to each vertex $v \in V$ in such a way that two vertices u, v connected by an edge $e = (u, v) \in E$ have different colors. Let k be a positive integer, we call a legal k-coloring of G, a coloring using k colors. $\chi(G)$, the chromatic number of G, is the minimum number of colors necessary to color G. Find $\chi(G)$ is an NP-hard problem. While exact methods have been developed to solve small problems, only heuristic approaches currently solve the graph coloring problem with a greater number of vertices, but without guarantee of optimality. In particular, hybridization of heuristics [5], such as memetic approaches [2, 7], and in particular with reduced population [3, 4], perform very good results, both in terms of solution quality and computational time. We denote $\mathcal{N}(G, k)$ the number of different k-colorings of G and $i(G)$ the number of stable (or independant set, i.e. a set of non-adjacent two-to-two vertices) of G. Our work is based on the following theorem:

Theorem 1. *Let $k > 0$ and G a graph, if $i(G) > \mathcal{N}(G, k) > 0$, then $\chi(G) = k$.*

It is therefore sufficient to find an upper bound of $\mathcal{N}(G,k)$ which is smaller than $i(G)$ to prove optimality. Our method is the following: we construct an estimator of an upper bound of $\mathcal{N}(G,k)$ from a sample of k-colorings found using our memetic algorithm HEAD [4]. This estimator does not have an absolute guaranty of being an upper bound, which is why we call this optimality *weak* or *experimental*.

3 Tests

To evaluate our approach, we perform tests on random graphs from RCBII benchmark [1] and on graphs of DIMACS benchmark. We process as follows :

1. For a given integer $k > 0$, we run the memetic algorithm *HEAD* on a graph instance as many times as needed to obtain $t = 1000$ legal k-colorings. Those solutions are *the solutions sample*.
2. We count, p, the number of different solutions inside the sample.
3. We estimate an upper bound of the $\mathcal{N}(G,k)$ as $UB(p,t)$:

$$UB(p,t) = \begin{cases} p + p^{\alpha^{\frac{t+p}{t}}} & \text{if } p < t \times 0.99 \\ +\infty & \text{otherwise} \end{cases} \tag{1}$$

   with $\alpha = 1.01$. This upper bound was built from a *reference data set* consisting of around 1000 random graphs from RCBII benchmark.
4. We compute the number of independent set of G, $i(G)$, or at least a lower bound if $i(G) > 10^6$, with an exact algorithm.
5. If $i(G) > UB(p,t)$, then the solutions are *experimental optimum* or *weak optimum* for the upper bound $UB : \chi_{weak}(G) = k$. Otherwise, there are not *weak optimum* for the upper bound UB.

We obtain many results as the *weak optimality* of the challenging DSJC500.5 graph with 47 colors.

References

1. Fabio Furini, Virginie Gabrel, and Ian-Christopher Ternier. An improved dsatur-based branch-and-bound algorithm for the vertex coloring problem. *Networks*, 69(1):124–141, 2017.
2. Philippe Galinier and Jin-Kao Hao. Hybrid evolutionary algorithms for graph coloring. *Journal of Combinatorial Optimization*, 3(4):379–397, 1999.
3. Laurent Moalic and Alexandre Gondran. Heuristic rope team: A parallel algorithm for graph coloring. In *Proceedings of the Genetic and Evolutionary Computation Conference*, GECCO '17, pages 314–320, New York, NY, USA, 2017. ACM.
4. Laurent Moalic and Alexandre Gondran. Variations on memetic algorithms for graph coloring problems. *Journal of Heuristics*, 24(1):1–24, Feb 2018.
5. Olawale Titiloye and Alan Crispin. Parameter tuning patterns for random graph coloring with quantum annealing. *PLoS ONE*, 7(11):e50060, 11 2012.
6. Leslie G. Valiant. The complexity of enumeration and reliability problems. *SIAM J. Comput.*, 8(3):410–421, 1979.
7. Qinghua Wu and Jin-Kao Hao. Coloring large graphs based on independent set extraction. *Computers & Operations Research*, 39(2):283–290, 2012.