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ON ALGEBRAS AND GROUPS OF FORMAL SERIES OVER A SMALL CATEGORY, NON-ABELIAN STOCHASTIC COSURFACES AND TOPOLOGICAL QUANTUM FIELD THEORIES

JEAN-PIERRE MAGNOT

Abstract. Based on the example of stochastic cosurfaces, we develop the notion of groups of series indexed by a graded small category, with coefficients in a Frölicher algebra. The presence of the notion of small category is motivated by the cobordism-like relations of stochastic cosurfaces, while the use of Frölicher algebra is a framework adapted to convolution of probabilities. Basic description of these groups, which generalize classical groups of series, is given, and the example raised by non-abelian stochatic cosurfaces of any dimension is studied.

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Introduction

In [24, 28], an algebra and a group of formal series of operators is described in order to rewrite the integration of the KP hierarchy in a non formal way. One of the main advances of this work is to get a (non formal) principal bundle where the concept of holonomy makes sense rigorously. The geometric objects under consideration are diffeological or Frölicher groups, which are regular in the sense that the exponential map exists and is smooth.

Diffeological spaces, first described in the 80's by Souriau and his coworkers (see e.g. [11, 15, 19, 32]) are generalizations of manifolds that enables differential geometry without charts. Independently, Frölicher spaces give a more rigid framework, that also generalize the notion of manifolds [10, 12, 16]. The comparison of the two frameworks has been made independently in [22] and in [34], see e.g. [24].

The present work is concerned with series indexed by manifolds with boundary, carrying so called "cut-and paste" properties of composition in the spirit of cobordism composition. Our main motivation is a quite old serie of works on so-called stochastic cosurfaces [1, 2, 3, 4, 5] where, in a heuristic way, time is considered as taking values in a non-linear manifold of dimension $d \geq 2$, andwhere time slices are given by parts of $(d-1)-$dimensional skeletons with particular properties. Such models have applications in fields of mathematical physics such as lattice models and Higgs fields among others, and which recovers 2D Yang-Mills theories as particular examples, in the fully developped work [20]. Analyzing the conditions that are technically necessary to develop cobordism-like series that fit with these works, it appear that the key elements are very weak: one only needs associativity of the composition law (which needs not totality as in the case of cobordism examples) and a $\mathbb{N}^*-$ grading on the required set of indexes. This explains our choice for small categories, $\mathbb{N}-$graded, where neutral elements need to be of order 0. By the need of probability spaces, and the use of their classical topologies (and related differential structures) we need to consider Frölicher algebras in order to consider topologies such as the vague topology on measures, along the lines of the remarks given in [27].

From these preliminary considerations, we present here a setting for a safe development of the group of series where stochastic cosurfaces offer a wide class of examples, reaching some topological quantum field theory-like developments. As and easy example, we show how one enlarges straightway the example of [24]: in $q-$deformation, indexes are monomials q^n, and in the example that we develop the base algebra is $A = Cl(M, E)$, the algebra of classical pseudo-differential operators. It is not an enlargeable Lie algebra, but the formal series

$$G = 1 + \sum_{n>0} q^n A$$

is a regular Frölicher Lie group (section 1.7). For the sake of completeness we quote two works [7, 8] on the Butcher group and some of its generalizations, that are very similar to what we develop in this article. Initially, the author was not

aware of these papers and intended to make also remarks on the Butcher group. But these two works are more complete, and the section on the Butcher group has been deleted from the text.

As announced we want to consider indexes that are obtained replacing $\mathbb{N}$ by a small category: this is the case of path-like or cobordism-like formal series (section 3 and after). In this example, the strategy is based on defining a $\mathbb{N}-$grading on the indexes, such that the neutral elements have to be of order 0. On such a setting, no element is invertible except possibily the ones of order 0, which shows that the setting of groupoid is still too restrictive. This is the case for the cobordism composition which is not considered up to homotopy, when we work on well-chosen families of manifolds M, viewed as morphisms of cobordism, that are embedded in a fixed target space N.

Our next feature in this work is to consider a grading induced by the volume in the case of embeddings, that we call length and which carries the order of the indexes. We also have to choose an algebra A. By our motivating examples, this setting is not void, we start from works [1, 2, 3, 4, 5] that introduced so-called stochastic cosurface. We enlarge the settings of the previous references, and adapt them to build families of measures indexed by cobordism, such that, if γ and γ' are two morphisms of cobordism that can be composed into $\gamma\gamma'$, then for the corresponding measures, we get $\mu(\gamma\gamma') = \mu(\gamma)\mu(\gamma')$ (convolution product). By the way, the mapping $\gamma \mapsto \mu(\gamma)$ can be understood as a formal serie over a family Γ of morphisms for cobordisms.

These results show that the initial investigations of [1, 2, 3, 4, 5], carried out for:

- $d = 2$ stochastic cosurfaces with arbitrary Lie group
- abelian cosurfaces of any dimension

carry, with mild considerations, some possible generalizations for non necessarily abelian stochastic cosurfaces at any dimension. Even if cut-and-paste formulas require more attention, see section 5 and section 6, where effects due to the presence of non-abelian groups required the introduction of an order in the slice of the non-linear time, the series of measures that we produce offer more usual expressions, in terms of series of measures, of the complex effects of non-abelian theories. Heuristically speaking, series of measures furnish classical treatise and expression of non-abelian (non-linear?) effects that seem to have been ignored in previous works because of their complete novelty in the their properties, to our knowledge. After these last remarks, further investigations are required to elucidate which are the key properties for cosurfaces dynamics, from a deterministic of stochastic viewpoint.

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1. Regular Frölicher Lie groups of series of unbounded operators

1.1. Diffeological spaces and Frölicher spaces.

Definition 1.1. [32], see e.g. [15]. Let X be a set.

• A **p-parametrization** of dimension p (or p-plot) on X is a map from an open subset O of $\mathbb{R}^p$ to X.
• A **diffeology** on X is a set $\mathcal{P}$ of parametrizations on X, called plots of the diffeology, such that, for all $p \in \mathbb{N}$,

- any constant map $\mathbb{R}^p \to X$ is in $\mathcal{P}$;
- Let I be an arbitrary set of indexes; let $\{f_i : O_i \to X\}_{i\in I}$ be a family of compatible maps that extend to a map $f : \bigcup_{i\in I} O_i \to X$. If $\{f_i : O_i \to X\}_{i\in I} \subset \mathcal{P}$, then $f \in \mathcal{P}$.
- Let $f \in \mathcal{P}$, defined on $O \subset \mathbb{R}^p$. Let $q \in \mathbb{N}$, O' an open subset of $\mathbb{R}^q$ and g a smooth map (in the usual sense) from O' to O. Then, $f \circ g \in \mathcal{P}$.

• If $\mathcal{P}$ is a diffeology on X, then $(X, \mathcal{P})$ is called a **diffeological space**.
Let $(X, \mathcal{P})$ and $(X', \mathcal{P}')$ be two diffeological spaces; a map $f : X \to X'$ is **differentiable** (=smooth) if and only if $f \circ \mathcal{P} \subset \mathcal{P}'$.

Remark 1.2. Any diffeological space $(X, \mathcal{P})$ can be endowed with the weakest topology such that all the maps that belong to $\mathcal{P}$ are continuous. We do not dwell deeper on this fact in this work because it is not closely related to the main themes of this paper.

We now introduce Frölicher spaces, see [12], using the terminology defined in [16].

Definition 1.3. • A **Frölicher** space is a triple $(X, \mathcal{F}, \mathcal{C})$ such that

- $\mathcal{C}$ is a set of paths $\mathbb{R} \to X$,
- $\mathcal{F}$ is the set of functions from X to $\mathbb{R}$, such that a function $f : X \to \mathbb{R}$ is in $\mathcal{F}$ if and only if for any $c \in \mathcal{C}$, $f \circ c \in C^\infty(\mathbb{R}, \mathbb{R})$;
- A path $c : \mathbb{R} \to X$ is in $\mathcal{C}$ (i.e. is a **contour**) if and only if for any $f \in \mathcal{F}$, $f \circ c \in C^\infty(\mathbb{R}, \mathbb{R})$.

• Let $(X, \mathcal{F}, \mathcal{C})$ and $(X', \mathcal{F}', \mathcal{C}')$ be two Frölicher spaces; a map $f : X \to X'$ is **differentiable** (=smooth) if and only if $\mathcal{F}' \circ f \circ \mathcal{C} \subset C^\infty(\mathbb{R}, \mathbb{R})$.

Any family of maps $\mathcal{F}_g$ from X to $\mathbb{R}$ generates a Frölicher structure $(X, \mathcal{F}, \mathcal{C})$ by setting, after [16]:

- $\mathcal{C} = \{c : \mathbb{R} \to X$ such that $\mathcal{F}_g \circ c \subset C^\infty(\mathbb{R}, \mathbb{R})\}$
- $\mathcal{F} = \{f : X \to \mathbb{R}$ such that $f \circ \mathcal{C} \subset C^\infty(\mathbb{R}, \mathbb{R})\}$.

In this case we call $\mathcal{F}_g$ a **generating set of functions** for the Frölicher structure $(X, \mathcal{F}, \mathcal{C})$. One easily see that $\mathcal{F}_g \subset \mathcal{F}$. This notion is useful for this paper since it allows us to describe a Frölicher structure in a simple way. A Frölicher space $(X, \mathcal{F}, \mathcal{C})$ carries a natural topology, which is the pull-back topology of $\mathbb{R}$ via $\mathcal{F}$. We note that in the case of a finite dimensional differentiable manifold X we can take $\mathcal{F}$ the set of all smooth maps from X to $\mathbb{R}$, and $\mathcal{C}$ the set of all smooth paths from $\mathbb{R}$ to X. In this case the underlying topology of the Frölicher structure is the same as the manifold topology [16]. In the infinite dimensional case, there is to our knowledge no complete study of the relation between the Frölicher topology and the manifold topology; our intuition is that these two topologies can differ.

We also remark that if $(X, \mathcal{F}, \mathcal{C})$ is a Frölicher space, we can define a natural diffeology on X by using the following family of maps f defined on open domains

$D(f)$ of Euclidean spaces (see [22]):

$$\mathcal{P}_\infty(\mathcal{F}) = \coprod_{p\in\mathbb{N}} \{ f : D(f) \to X;\ \mathcal{F}\circ f \in C^\infty(D(f),\mathbb{R}) \quad \text{(in the usual sense)}\} . \tag{1.1}$$

If X is a differentiable manifold, this diffeology has been called the **nébuleuse diffeology** by P. Iglesias-Zemmour, see [15]. We can easily show the following:

Proposition 1.4. [22] *Let$(X,\mathcal{F},\mathcal{C})$ and $(X',\mathcal{F}',\mathcal{C}')$ be two Frölicher spaces. A map $f : X \to X'$ is smooth in the Frölicher sense if and only if it is smooth for the underlying diffeologies $\mathcal{P}_\infty(\mathcal{F})$ and $\mathcal{P}_\infty(\mathcal{F}')$.*

Thus, we can also state:

Smooth manifold $\Rightarrow$ Frölicher space $\Rightarrow$ Diffeological space

A deeper analysis of these implications has been given in [34]. The next remark is inspired on this work and on [22]; it is based on [16, p.26, Boman's theorem].

Remark 1.5. The set of contours $\mathcal{C}$ of a Frölicher space $(X,\mathcal{F},\mathcal{C})$ **does not** give us a diffeology, because a diffeology needs to be stable under restriction of domains. In the case of paths in $\mathcal{C}$ the domain is always $\mathbb{R}$. However, $\mathcal{C}$ defines a "minimal diffeology" $\mathcal{P}_1(\mathcal{F})$ whose plots are smooth parametrizations which are locally of the type $c\circ g$, where $g \in \mathcal{P}_\infty(\mathbb{R})$ and $c \in \mathcal{C}$. Within this setting, we can replace $\mathcal{P}_\infty$ by $\mathcal{P}_1$ in Proposition 1.4.

We also remark that given an algebraic structure, we can define a corresponding compatible diffeological structure. For example, a $\mathbb{R}-$vector space equipped with a diffeology is called a diffeological vector space if addition and scalar multiplication are smooth (with respect to the canonical diffeology on $\mathbb{R}$). An analogous definition holds for Frölicher vector spaces. We will also consider diffeological groups.

Remark 1.6. Frölicher, c^∞ (the "smooth convenient setting" of [16]) and Gâteaux smoothness are the same notion if we restrict to a Fréchet context, see [16, Theorem 4.11]. Indeed, for a smooth map $f : (F,\mathcal{P}_1(F)) \to \mathbb{R}$ defined on a Fréchet space with its 1-dimensional diffeology, we have that $\forall (x,h) \in F^2$, the map $t \mapsto f(x+th)$ is smooth as a classical map in $\mathcal{C}^\infty(\mathbb{R},\mathbb{R})$. And hence, it is Gâteaux smooth. The converse is obvious.

1.2. **Frölicher completion of a diffeological space.** We now finish the comparison of the notions of diffeological and Frölicher space following mostly [34]:

Theorem 1.7. *Let $(X,\mathcal{P})$ be a diffeological space. There exists a unique Frölicher structure $(X,\mathcal{F}_\mathcal{P},\mathcal{C}_\mathcal{P})$ on X such that for any Frölicher structure $(X,\mathcal{F},\mathcal{C})$ on X, these two equivalent conditions are fulfilled:*

(i) the canonical inclusion is smooth in the sense of Frölicher $(X,\mathcal{F}_\mathcal{P},\mathcal{C}_\mathcal{P}) \to (X,\mathcal{F},\mathcal{C})$

(ii) the canonical inclusion is smooth in the sense of diffeologies $(X,\mathcal{P}) \to (X,\mathcal{P}(\mathcal{F}))$.

Moreover, $\mathcal{F}_\mathcal{P}$ is generated by the family

$$\mathcal{F}_0 = \{f : X \to \mathbb{R} \text{ smooth for the usual diffeology of } \mathbb{R}\}.$$

Proof. Let $(X,\mathcal{F},\mathcal{C})$ be a Frölicher structure satisfying *(ii)*. Let $p \in P$ of domain O. $\mathcal{F}\circ p \in C^\infty(O,\mathbb{R})$ in the usual sense. Hence, if $(X,\mathcal{F}_\mathcal{P},\mathcal{C}_\mathcal{P})$is the Frölicher

structure on X generated by the set of smooth maps $(X, \mathcal{P}) \to \mathbb{R}$, we have two smooth inclusions

$$(X, \mathcal{P}) \to (X, \mathcal{P}(\mathcal{F}_{\mathcal{P}})) \text{ in the sense of diffeologies}$$

and then

$$(X, \mathcal{F}_{\mathcal{P}}, \mathcal{C}_{\mathcal{P}}) \to (X, \mathcal{F}, \mathcal{C}) \text{ in the sense of Frölicher.}$$

□

Definition 1.8. [34] A **reflexive** diffeological space is a diffeological space $(X, \mathcal{P})$ such that $\mathcal{P} = \mathcal{P}(\mathcal{F}_{\mathcal{P}})$.

Theorem 1.9. [34] *The category of Frölicher spaces is exactly the category of reflexive diffeological spaces.*

This last theorem allows us to make no difference between Frölicher spaces and reflexive diffeological spaces. We shall call them Frölicher spaces, even when working with their underlying diffeologies.

1.3. **Push-forward, quotient and trace.** We give here only the results that will be used in the sequel. For an overview on diffeologies, see [32] or more recently [15]

Proposition 1.10. [22] *Let $(X, \mathcal{P})$ be a diffeological space, and let X' be a set. Let $f : X \to X'$ be a surjective map. Then, the set*

$$f(\mathcal{P}) = \{u \text{ such that } u \text{ restricts to some maps of the type } f \circ p; p \in \mathcal{P}\}$$

*is a diffeology on X', called the **push-forward diffeology** on X' by f.*

We have now the tools needed to describe the diffeology on a quotient:

Proposition 1.11. *let $(X, \mathcal{P})$ b a diffeological space and $\mathcal{R}$ an equivalence relation on X. Then, there is a natural diffeology on $X/\mathcal{R}$, noted by $\mathcal{P}/\mathcal{R}$, defined as the push-forward diffeology on $X/\mathcal{R}$ by the quotient projection $X \to X/\mathcal{R}$.*

Given a subset $X_0 \subset X$, where X is a Frölicher space or a diffeological space, we can define on trace structure on X_0, induced by X.

• If X is equipped with a diffeology $\mathcal{P}$, we can define a diffeology $\mathcal{P}_0$ on X_0 setting

$$\mathcal{P}_0 = \{p \in \mathcal{P} \text{such that the image of } p \text{ is a subset of } X_0\}.$$

• If $(X, \mathcal{F}, \mathcal{C})$ is a Frölicher space, we take as a generating set of maps $\mathcal{F}_g$ on X_0 the restrictions of the maps $f \in \mathcal{F}$. In that case, the contours (resp. the induced diffeology) on X_0 are the contours (resp. the plots) on X which image is a subset of X_0.

1.4. **Cartesian products and projective limits.** The category of Sikorski differential spaces is not cartesianly closed, see e.g. [10]. This is why we prefer to avoid the questions related to cartesian products on differential spaces in this text, and focuse on Frölicher and diffeological spaces, since the cartesian product is a tool essential for the definition of configuration spaces.

In the case of diffeological spaces, we have the following [32]:

Proposition 1.12. *Let $(X, \mathcal{P})$ and $(X', \mathcal{P}')$ be two diffeological spaces. We call **product diffeology** on $X \times X'$ the diffeology $\mathcal{P} \times \mathcal{P}'$ made of plots $g : O \to X \times X'$ that decompose as $g = f \times f'$, where $f : O \to X \in \mathcal{P}$ and $f' : O \to X' \in \mathcal{P}'$.*

Then, in the case of a Frölicher space, we derive very easily, compare with e.g. [16]:

Proposition 1.13. *Let $(X, \mathcal{F}, \mathcal{C})$ and $(X', \mathcal{F}', \mathcal{C}')$ be two Frölicher spaces, with natural diffeologies $\mathcal{P}$ and $\mathcal{P}'$. There is a natural structure of Frölicher space on $X \times X'$ which contours $\mathcal{C} \times \mathcal{C}'$ are the 1-plots of $\mathcal{P} \times \mathcal{P}'$.*

We can even state the same results in the case of infinite products, in a very trivial way by taking the cartesian products of the plots or of the contours. Let us now give the description of what happens for projective limits of Frölicher and diffeological spaces.

1.5. **Fully regular Frölicher Lie groups.**

Definition 1.14. Let G be a group, equiped with a diffeology $\mathcal{P}$. We call G a **diffeological group** if both multiplication and inversion are smooth.

An analogous definition holds for Frölicher groups. Following Iglesias-Zemmour and Leslie's pioneering work, [15, 19], see e.g. [17] for a quite unprecise approach on Frölicher spaces which ignores [6], we give conditions for the existence of such a tangent space at the identity element e (which is precisely the identity mapping when the group G is a group of transformations).

Definition 1.15. The diffeological group G is a **diffeological Lie group** if and only if ${}^iT_eG$ is a vector space and if the derivative of the Adjoint action of G on ${}^iT_eG$ defines a Lie bracket. In this case, we call ${}^iT_eG$ the Lie algebra of G, and we denote generically by $\mathfrak{g}$.

Let us precise the algebraic, diffeological and Frölicher structures of $\mathfrak{g}$. This section is mostly inspired on [24].

Remark 1.16. Let G be a diffeological Lie group with Lie algebra $\mathfrak{g} = \{\partial_t c(0) : c \in C_e \text{ and } c(0) = e\}$. We note that this definition coincides with the classical definition of the Lie algebra of a finite dimensional Lie group via germs of paths at e. We have:

- Let $(X, Y) \in \mathfrak{g}^2$, $X + Y = \partial_t(c.d)(0)$ where $c, d \in C_e^2$, $c(0) = d(0) = e$, $X = \partial_t c(0)$ and $Y = \partial_t d(0)$.
- Let $(X, g) \in \mathfrak{g} \times G$, $Ad_g(X) = \partial_t(gcg^{-1})(0)$ where $c \in C_e$, $c(0) = e$, and $X = \partial_t c(0)$.
- Let $(X, Y) \in \mathfrak{g}^2$, $[X, Y] = \partial_t(Ad_{c(t)}Y)$ where $c \in C_e$, $c(0) = e$, and $X = \partial_t c(0)$.

All these operations are smooth and thus well-defined as operations on Frölicher spaces as well.

Let us now precise the statement and proof of [19, Proposition 1.6.].

Proposition 1.17. *Let G be a diffeological group. Then the tangent cone at the identity element, ${}^iT_eG$, is a diffeological vector space.*

Now we go back to the problem of equipping ${}^iT_eG$ with a Lie algebra structure. We remark that, actually, the condition on the existence of a Lie bracket obtained from differentiation of the adjoint action of the group G on ${}^iT_eG$ cannot be relaxed: it has to be assumed in order to give a structure of Lie algebra to ${}^iT_eG$. However, if both G and ${}^iT_eG$ are smoothly embedded into a diffeological algebra A, and if

group multiplication and ${}^iT_eG-$addition are the pull-back of the operations on A, then the Lie bracket exists and is defined by the standard relation

$$[u,v] = uv - vu \tag{1.2}$$

as long as ${}^iT_eG$ is stable under (1.2). We can ask the following natural question, perhaps reminiscent of Hilbert's classical problem on the relation between topological groups and Lie groups:

Does there exist a diffeological (or Frölicher) group
which is not a diffeological (or Frölicher) Lie group ?

Let us now concentrate on diffeological and Frölicher Lie groups. We write $\mathfrak{g} = {}^iT_eG$. The basic properties of adjoint and coadjoint actions, and of Lie brackets, remain globally the same as in the case of finite-dimensional Lie groups, and the proofs are similar: we only need to replace charts by plots of the underlying diffeologies (see e.g. [19] for further details, and [6] for the case of Frölicher Lie groups), as soon as we have checked that the Lie algebra $\mathfrak{g}$ is a diffeological Lie algebra, i.e. a diffeological vector space equipped with a smooth Lie bracket.

Definition 1.18. A Frölicher Lie group G with Lie algebra $\mathfrak{g}$ is called **regular** if and only if there is a smooth map

$$Exp : C^\infty([0,1],\mathfrak{g}) \to C^\infty([0,1],G)$$

such that $g(t) = Exp(v(t))$ is the unique solution of the differential equation

$$\left\{ \begin{array}{rcl} g(0) & = & e \\ \dfrac{dg(t)}{dt}\, g(t)^{-1} & = & v(t)\ , \qquad v(t) \in C^\infty([0,1],\mathfrak{g})\ . \end{array} \right. \tag{1.3}$$

We define the exponential function as follows:

$$\begin{array}{rcl} exp : \mathfrak{g} & \to & G \\ v & \mapsto & exp(v) = g(1)\ , \end{array}$$

where g is the image by Exp of the constant path v.

We can also define the Riemann integral of smooth $\mathfrak{g}-$valued functions.

Definition 1.19. Let $(V,\mathcal{F},\mathcal{C})$ be a Frölicher vector space, i.e. a vector space V equipped with a Frölicher structure compatible with vector space addition and scalar multiplication. The space $(V,\mathcal{F},\mathcal{C})$ is **regular** if there is a smooth map

$$\int_0^{(.)} : C^\infty([0;1];V) \to C^\infty([0;1],V)$$

such that $\int_0^{(.)} v = u$ if and only if u is the unique solution of the differential equation

$$\left\{ \begin{array}{l} u(0) = 0 \\ u'(t) = v(t)\ . \end{array} \right.$$

This definition applies, for instance, if V is a complete locally convex topological vector space, equipped with its natural Frölicher structure given by the Frölicher completion of its nébuleuse diffeology, see [15, 22, 24]. We finish with a natural definition, with the terminology due to E. Reyes given in [28]:

Definition 1.20. Let G be a Frölicher Lie group with Lie algebra $\mathfrak{g}$. Then, G is **fully regular** if both G and $\mathfrak{g}$ are regular in the sense of definitions 1.18 and 1.19 respectively.

For completeness, let us mention that —following terminology used in the early investigations on infinite dimensional Lie theory ([30]; see also [29])— a regular Lie algebra $\mathfrak{g}$ is said to be **enlargeable** if there exists a (not necessarily regular) Frölicher Lie group G with Lie algebra $\mathfrak{g}$.

Theorem 1.21. [24] *Let G be a regular Frölicher Lie group with Lie algebra $\mathfrak{g}$. Let $\mathfrak{g}_1$ be a Lie subalgebra of $\mathfrak{g}$, and set $G_1 = Exp(C^\infty([0;1];\mathfrak{g}_1))(1)$. If $Ad_{G_1 \cup G_1^{-1}}(\mathfrak{g}_1) = \mathfrak{g}_1$,*

$$i.e. \quad \forall g \in Exp(C^\infty([0;1];\mathfrak{g}_1))(1), \quad \forall v \in \mathfrak{g}_1, \quad Ad_g v \in \mathfrak{g}_1 \text{ and } Ad_{g^{-1}} v \in \mathfrak{g}_1 \ ,$$

then G_1 is a Frölicher subgroup of G.

The first known example is the following [24]:

Proposition 1.22. *Let $(G_n)_{n\in\mathbb{N}}$ be a sequence of Banach Lie groups, increasing for $\supset$, and such that the inclusions are Lie group morphisms. Let $G = \bigcap_{n\in\mathbb{N}} G_n$. Then, G is a Frölicher regular Lie group with regular Lie algebra $\mathfrak{g} = \bigcap_{n\in\mathbb{N}} \mathfrak{g}_n$.*

Let us notice that there exists non regular Frölicher Lie groups, see [25], where as there is no example of Fréchet Lie group that has been proved to be non regular [16]. We now turn to key results from [24]:

Theorem 1.23. *Let $(A_n)_{n\in\mathbb{N}^*}$ be a sequence of (Frölicher) vector spaces which are regular, equipped with a graded smooth multiplication operation on $\bigoplus_{n\in\mathbb{N}^*} A_n$, i.e. a multiplication such that for each $n, m \in \mathbb{N}^*$, $A_n.A_m \subset A_{n+m}$ is smooth with respect to the corresponding Frölicher structures. Let us define the (non unital) algebra of formal series*

$$\mathcal{A} = \left\{ \sum_{n\in\mathbb{N}^*} a_n \,|\, \forall n \in \mathbb{N}^*, a_n \in A_n \right\},$$

equipped with the Frölicher structure of an infinite product. Then, the set

$$1 + \mathcal{A} = \left\{ 1 + \sum_{n\in\mathbb{N}^*} a_n | \forall n \in \mathbb{N}^*, a_n \in A_n \right\}$$

is a Frölicher Lie group with regular Frölicher Lie algebra $\mathcal{A}$. Moreover, the exponential map defines a smooth bijection $\mathcal{A} \to 1 + \mathcal{A}$.

Notation: for each $u \in \mathcal{A}$, we note by $[u]_n$ the A_n-component of u.

Theorem 1.24. *Let*

$$1 \longrightarrow K \xrightarrow{i} G \xrightarrow{p} H \longrightarrow 1$$

be an exact sequence of Frölicher Lie groups, such that there is a smooth section $s : H \to G$, and such that the subset diffeology from G on $i(K)$ coincides with the push-forward diffeology from K to $i(K)$. We let

$$0 \longrightarrow \mathfrak{k} \xrightarrow{i'} \mathfrak{g} \xrightarrow{p} \mathfrak{h} \longrightarrow 0$$

be the corresponding sequence of Lie algebras. Then,

- *The Lie algebras $\mathfrak{k}$ and $\mathfrak{h}$ are regular if and only if the Lie algebra $\mathfrak{g}$ is regular;*
- *The Frölicher Lie groups K and H are regular if and only if the Frölicher Lie group G is regular.*

1.6. **Groups of series that are regular Fréchet Lie groups.** We now asume that the algebras $\mathcal{A}_n$ of Theorem 1.23 are Fréchet vector spaces, and that the bilinear multiplication

$$\mathcal{A}_n \times \mathcal{A}_m \to \mathcal{A}_{n+m}$$

is smooth. Then we get the following theorem:

Theorem 1.25. *The group $1+\mathcal{A}$ is a regular Fréchet Lie group with Lie algebra $\mathcal{A}$.*

Proof. The exponential $\mathcal{A} \to 1+\mathcal{A}$ is already shown to be bijective, and $\mathcal{A}$, seen as a vector space, is endowed with the semi-norms on each $\mathcal{A}_n$. Let us now show that the exponential of paths $C^\infty([0;1],\mathcal{A}) \to C^\infty([0;1];1+\mathcal{A})$ is smooth, which will complete the proof. Let $v \in C^\infty([0,1];\mathcal{A})$. Let $s \in [0;1]$ and let $j = \lfloor ns \rfloor$. We define

$$u_n(s) = \left(1 + \left(s - \frac{j}{n}\right) v\left(\frac{j}{n}\right)\right) \prod_{i=1}^{j} \left(1 + \frac{1}{n} v\left(\frac{j-i}{n}\right)\right).$$

We have that

$$\lim_{n\to+\infty} \partial_s u_n(s).u_n^{-1}(s) = \lim_{n\to+\infty} v\left(\frac{j}{n}\right)\left(1 + \left(s - \frac{j}{n}\right) v\left(\frac{j}{n}\right)\right)^{-1} = v(s).$$

Moreover, the A_m component of the product converges to a sum of integrals of the type

$$\int_{1\geq s_1 \geq \ldots \geq s_k \geq 0} \left[\prod_{i=1}^{k} v(s_i)\right]_m (ds)^k$$

for $k \leq m$, which shows the convergence to a path $u \in C^\infty([0;1];1+\mathcal{A})$ satisfying

$$\partial_s u(s).u^{-1}(s) = v(s)$$

which smoothly depends on the path $v \in C^\infty([0;1],\mathcal{A})$ in the Fréchet sense. □

1.7. **Examples of $q-$deformed pseudo-differential operators.** In our work of Lax-type equations, we use the following group from [24]:Let E be a smooth vector bundle over a compact manifold without boundary M. We denote by $Cl(M,E)$ (resp. $Cl^k(M,E)$) the space of classical pseudo-differential operators (resp. classical pseudo-differential operators of order k) acting on smooth sections of E. We denote by $Cl^*(M,\mathbb{C}^n)$, $Cl^{0,*}(M,\mathbb{C}^n)$ the groups of the units of the algebras $Cl(M,\mathbb{C}^n)$ and $Cl^0(M,\mathbb{C}^n)$. Notice that $Cl^{0,*}(M,\mathbb{C}^n)$ is a CBH Lie group, and belong to a wider class of such groups that is studied in [14].

.

Definition 1.26. Let q be a formal parameter. We define the algebra of formal series

$$Cl_q(M,E) = \left\{ \sum_{t\in\mathbb{N}^*} q^k a_k | \forall k \in \mathbb{N}^*, a_k \in Cl(M,E) \right\}.$$

This is obviously an algebra, graded by the order (the valuation) into the variable q. Thus, setting

$$\mathcal{A}_n = \{q^n a_n | a_n \in Cl(M,E)\},$$

we can set $\mathcal{A} = Cl_q(M,E)$ and state the following consequence of Theorem 1.25:

Corollary 1.27. *The group $1 + Cl_q(M,E)$ is a regular Fréchet Lie group with Lie algebra $Cl_q(M,E)$.*

Let $Cl^{0,*}(M,E)$ be the Lie group of invertible pseudo-differential operators of order 0. This group is known to be a regular Lie group since Omori, but the most efficient proof is actually in [14], to our knowledge. We remark a short exact sequence of Frölicher Lie groups:

$$0 \to 1 + Cl_q(M,E) \to Cl^{0,*}(M,E) + Cl_t(M,E) \to Cl^{0,*}(M,E) \to 0,$$

which satisfies the conditions of Theorem 2.3, in its version for regular Lie groups that can be found in [16]. Thus, we have the following:

Theorem 1.28. *$Cl^{0,*}(M,E) + Cl_q(M,E)$ is a regular Lie group with Lie algebra $Cl^0(M,E) + Cl_q(M,E)$.*

Remark 1.29. One could also develop some similar examples, which could stand as a generalized version, with other algebras of non classical operators, as desired. These "generalized" versions depend on the contexts and the necessities of the studied models. These examples are not developed here in order to avoid some too long lists of examples constructed in the same spirit.

2. Algebras and groups of series over graded small categories

We mimick and extend the procedure used in [24] Let $(I,*)$ be a small category with neutral elements e. By small category, we require that $*$ is associative, with neutral element(s) and that it is not necessarily total. Let A_i be a family of regular Frölicher vector spaces indexed by I. The family $\{\mathcal{A}_i; i \in I\}$ is equipped with a multiplication, associative and distributive with respect to addition in the vector spaces $\mathcal{A}_i$, such that

$$A_i.A_j \begin{cases} \subset A_{i*j} & \text{if } i*j \text{ exists} \\ = 0 & \text{otherwise.} \end{cases}$$

and smooth for the Frölicher structures. Let $\mathcal{A}$ be the vector space of formal series of the type

$$a = \sum_{i \in I} a_i \quad ; \quad a_i \in \mathcal{A}_i$$

such that, for each $k \in I$, there is a finite number of indexes $(i,j) \in I^2$ such that $i * j = k$. Notice that, with such a definition, isf $E \subset ord^{-1}(0)$, $\mathcal{A}_E$ is an algebra. From now, we assume $\mathcal{A}$ unital, and we note its unit element 1. We are not sure that, with this kind of definition, the exponential exists. In order to make the previous theorems valid, we have to define an adequate $\mathbb{N}-$grading.

Definition 2.1. Let I as above, such that, there is a $\mathbb{N}-$grading, that is, a morphism of small categories $ord : I \to \mathbb{N}$, such that the order of any unital element is 0. Let A_i be family of regular Frölicher vector spaces indexed by I. Let

$$\mathcal{A}_0 \subset \mathcal{A} = \left\{ \sum_{i \in I - ord^{-1}(0)} a_i | a_i \in \mathcal{A}_i \right\}$$

be an algebra. The Frölicher vector space $\mathcal{A}_0$ is called Frölicher $I-$graded regular algebra if and only if it is equipped with a multiplication, associative and distributive with respect addition, smooth for the induced Frölicher structure.

Proposition 2.2. *Let $\mathcal{A}$ be a Frölicher $I-$graded regular algebra. It is the Lie algebra of the Frölicher regular Lie group $1+\mathcal{A}$.*

Proof. This is a straight application of theorem 1.23 □

Theorem 2.3. *Let $\mathcal{A}=\bigoplus_{i\in I}\mathcal{A}_i$ be a Frölicher $I-$graded regular algebra. Let G be a regular Frölicher Lie group, acting on $\mathcal{A}$ componentwise. Then,*

$$G\oplus A$$

is a regular Frölicher Lie group.

Proof. Considering the exact sequence

$$0\rightarrow 1+\mathcal{A}\rightarrow G\oplus\mathcal{A}\rightarrow G\rightarrow 0$$

there is a (global) slice $G\rightarrow G\oplus\{0_{\mathcal{A}}\}$ so that Theorem 1.24 applies . □

Remark 2.4. Notice that this small section is written in a heuristic way, in order to describe another kind of example that will be described rigorously in future works. Let I be a family of manifolds, stable and finitely generated under cartesian product. Cartesian product is the composition law. We remark that it is graded with respect to the dimension of the manifold. A standard singleton can be added to I as a neutral element of dimension 0. Let J be a family of finite rank vector bundles over the family of manifolds I, stable under tensor product. The scalar field can be understood as a neutral element of dimension 0. By the way, Proposition 2.2 applies to the following algebras :

- algebras of smooth sections of the finite rank vector bundles of J
- algebras of operators acting on these sections.

When $I=\left\{(S^1)^n|n\in\mathbb{N}^*\right\}$, $J=\left\{(S^1)^n\times\mathbb{C}|n\in\mathbb{N}^*\right\}$, and when the algebras under consideration are $Cl(S^1,\mathbb{C})^{\otimes n}$, we recognize a framework in the vincinity of the example given at the end of [23].

3. Path-like and Cobordism-like deformations

We describe here a setting where the indexes which live an a small category. This example recovers, passing to homotopy classes, the cobordism setting. We do not wish to consider homotopy invariant properties, and describe some kind of "pseudo-cobordism". We now consider the set

$$Gr=\coprod_{m\in\mathbb{N}^*}Gr_m$$

where Gr_m is the set of m-dimensional connected oriented manifolds M, possibly with boundary, where the boundaries ∂M are separated into two disconnected parts: the initial part $\alpha(M)$ and the final part $\beta(M)$. Then, we have a composition law $*$, called cobordism composition in the rest of the text, defined by the following relation:

Definition 3.1. Let $m\in\mathbb{N}^*$. Let $M,M'\in Gr_m$. Then $M''=M*M'\in Gr_m$ exists if

(1) $\alpha(M)=\beta(M')\neq\emptyset$, up to diffeomorphism

(2) $\alpha(M'') = \alpha(M')$
(3) $\beta(M'') = \beta(M)$
(4) M'' cuts into two pieces $M'' = M \cup M'$ with $M \cap M' = \alpha(M) = \beta(M')$.

This composition, that we call **cobordism composition**, extends naturally to embedded manifolds:

Definition 3.2. Let N be a smooth (finite dimensinal) manifold.

$$Gr(N) = \coprod_{m \in \mathbb{N}^*} \coprod_{M \in Gr_m} Emb(M, N).$$

where the notation $Emb(M, N)$ denotes the smooth manifold of smooth embeddings of M into N.

Notice that since $dim(N) < \infty$, we have $m \leq dim(N)$. We recall that that $Gr(N)$ is naturally a smooth manifold, since $Emb(M, N)$ is a smooth manifold [16], and that $*$ is obviously smooth because it is smooth in the sense of the underlying diffeologies. When we only consider manifolds without boundary (in this case, cobordism composition is not defined), these spaces are called non linear grassmanians in the litterature, which explains the notations.

Definition 3.3.
- Let $I = (Gr \times \mathbb{N}^*) \coprod (\emptyset, 0)$, graded by the second component. Assuming $\emptyset$ as a neutral element for $*$, we extend the cobordism composition into a composition, also noted $*$, defined as:
$$(M, p) * (M', p') = (M * M', p + p')$$
when $M * M'$ is defined. WE call **length** of (M, p) the number $len(M, p) = p$.
- Let $I(N) = (Gr(N) \times \mathbb{N}^*) \coprod (\emptyset, 0)$, graded by the second component. Assuming $\emptyset$ as a neutral element for $*$, we extend the cobordism composition into a composition, also noted $*$, defined as:
$$(M, p) * (M', p') = (M * M', p + p')$$
when $M * M'$ is defined.
- Let $m \in \mathbb{N}^*$. We note by I_m and $I_m(N)$ the set of indexes based on Gr_m and on $Gr_m(N)$ respectively

Let us now turn to $q-$deformed groups and algebras. For these definitions, the length number carries the natural grading for series. Let A be a regular Frölicher algebra. Let $m \in \mathbb{N}^*$. Let

$$\mathcal{A}_{I_m} = \left\{ \sum_{(M,n) \in I_m} q^n a_{M,n} | a_{M;n} \in A \right\}$$

and let

$$\mathcal{A}_{I_m}(N) = \left\{ \sum_{(\phi,n) \in I_m(N)} q^n a_{\phi,n} | a_{M;n} \in A \right\}.$$

Theorem 3.4. *Let $\Gamma \subset \coprod_{m \in \mathbb{N}^*} I_m$, resp. $\Gamma(N) \subset \coprod_{m \in \mathbb{N}^*} I_m(N)$, be a family of indexes, stable under $*$, such that $\forall m \in \mathbb{N}^*$,*

(1) *$\forall m \in \mathbb{N}^*$, $\Gamma \cap I_m$ is finite or, more generally;*
(2) *$\forall \gamma \in \Gamma$, the set of pairs $(\gamma', \gamma'') \in \Gamma^2$ such that $\gamma = \gamma' * `\gamma''$ is finite.*

(3)

$$\mathcal{A}_\Gamma = \left\{ \sum_{(M,n)\in\Gamma} q^n a_{M,n} | a_{M;n} \in A \right\},$$

resp.

$$\mathcal{A}_\Gamma(N) = \left\{ \sum_{(\phi,n)\in\Gamma(N)} q^n a_{\phi,n} | a_{\phi;n} \in A \right\},$$

is a regular algebra (in the sense of regular vector spaces).

Then

$$1_A + \mathcal{A}_{\Gamma-\{(\emptyset,0)\}}$$

is a Lie group with Lie algebra A_Γ. Moreover, for each Frölicher Lie group G with Lie algebra $\mathfrak{g}$ such that $G \subset A^*$ smoothly,

$$G \oplus \mathcal{A}_{\Gamma-\{(\emptyset,0)\}}$$

is a regular Frölicher Lie group with Lie algebra $\mathfrak{g}\oplus\mathcal{A}_{\Gamma-\{(\emptyset,0)\}}$. Moreover, the results are the same replacing Γ by $\Gamma(N)$.

Proof. Following condition (1) and (2), in the (possibly infinite sum) $\sum_{(M,n)\in\Gamma} q^n a_{M,n}$, each power q^n has only a finite number of $A-$coefficients since there is only a finite number of possible indexes for each q^n. So that, Proposition 2.2 and Theorem 2.3 apply. The same arguments are also valid when replacing Γ by $\Gamma(N)$. □

4. Markov Cosurfaces in codimension 1

4.1. **Settings.** Let M be a $d-$dimensional connected oriented Riemannian manifold. Let $\mathcal{H}_\vee$ be the set of embedded, oriented, smooth, closed, connected hypersurfaces (codimension 1 submanifolds) of M with piecewise smooth border. What we call hypersurface is mostly smooth hypersurfaces on the mnifold M, but since we need piecewise smooth oriented hypersurfaces, we need to build them by induction, gluing together the smooth components. What we get at the end is a space of oriented piecewise smooth hypersurfaces, with piecewise smooth border.

Definition 4.1. We set $\mathcal{H}_\vee^{(1)} = \mathcal{H}_\vee$. For $d \geq 2$, we define by induction:

- Let $(s_1, s_2) \in \mathcal{H}_\vee \times \mathcal{H}_\vee$. If
  (1) $s_1 \cap s_2 \subset \partial s_1 \cap \partial s_2$ is a $(d-2)$ piecewise smooth manifold and
  (2) the orientations induced on $s_1 \cap s_2$ by s_1 and s_2 are opposite,

then we define $s_1 \vee s_2$ to be the oriented piecewise smooth hypersurface of M obtained by gluing s_1 and s_2 along their common border. The orientation of $s_1 \vee s_2$ is the one induced by s_1 and s_2. The set of all such hypersurfaces is denoted by $\mathcal{H}_\vee^{(2)}$.

- Let $(s_1, s_2) \in \mathcal{H}_\vee^{(n-1)} \times \mathcal{H}_\vee$. If
  (1) $s_1 \cap s_2 \subset \partial s_1 \cap \partial s_2$ is a $(d-2)$ piecewise smooth manifold and
  (2) the orientations induced on $s_1 \cap s_2$ by s_1 and s_2 are opposite,

then we define in the same way $s_1 \vee s_2$. The set of such hypersurfaces is denoted by $\mathcal{H}_\vee^{(n)}$.

- We set $\Sigma_\vee = \bigcup_{n\in\mathbb{N}^*} \mathcal{H}_\vee^{(n)}$.

In all the article, we shall assume also that the connected components of ∂s are in $\Sigma_{\vee,n-1}$ if $s \in \Sigma_{\vee,n}$, for $n \in \mathbb{N}^*$.

Remark 4.2. If M is $2-$dimensional, it might seem that definition 4.1 generalizes the composition of unparametrized piecewise smooth paths, setting $\mathcal{H}$ to be the set of smooth paths and $\vee$ the groupoïd composition law of oriented piecewise smooth paths. In fact, we need to reformulate the definition for $d = 2$ in order to fit with the usual composition of paths. Let us look at the following example. Let $M = \mathbb{R}^2$, and let s_1 and s_2 the paths parametrized by $s_1(t) = (\cos(\pi t), \sin(\pi t))$ and $s_2(t) = (-\cos(\pi t), -\sin(\pi t))$ for $t \in [0, 1]$. We have $\partial s_1 = \partial s_2 = \{(-1; 0); (1; 0)\}$, with "opposite orientations" (i.e. the endpoint of s_1 (resp. s_2) is the initial point of s_2 (resp. s_1)) so that paths s_1 and s_2 can be composed. But in order to have a loop, one has to determine which point among $\{(-1; 0); (1; 0)\}$ will be the initial point. The choice comes with the order in the composition of paths: $s_1 * s_2$ or $s_2 * s_1$. Such a choice cannot be done with definition 4.1 because the law $\vee$ is obviously commutative in $\mathcal{H}_\vee \times \mathcal{H}_\vee$.

Now, we define $\mathcal{H}_*$ the set of (unparametrized, but oriented) smooth hypersurfaces s on the oriented manifold M, equipped in addition with a prescribed orientation of smooth components of ∂s. **Initial parts** of ∂s, noted $\alpha(s)$, are those for which the prescribed orientation is opposite to the one induced by s, and the **final parts**, noted $\beta(s)$, are the ones for which they coïncide. For $d = 2$, the orientation of paths can prescribe naturally initial and final points. This is the (apparently natural) choice that has been made in [2] but we remark here that this choice is not necessary. The picture of the following definition will be merely the same as the one of definition 4.1, but each smooth component of the border of the hypersurface is assigned to be either initial or final. In order to keep the coherence with the loop composition,

- we can glue together a final part with an initial part,
- and the final parts and the initial parts can be the same set-theorically, just as in the case of a loop starting and finishing at the same point.

Here is the construction:

Definition 4.3. We set $\mathcal{H}_*^{(1)} = \mathcal{H}_*$. we define by induction:

• Let $(s_1, s_2) \in \mathcal{H}_* \times \mathcal{H}_*$. Let $a = \alpha(s_1) \cap \beta(s_2)$. We define $s_1 * s_2$ as the oriented piecewise smooth hypersurface of M obtained by gluing s_1 and s_2 on a and denoted by $s_1 \cup_a s_2$. The orientation of $s_1 * s_2$ is the one induced by s_1 and s_2 on $s_1 \cup_a s_2$ By the way, we have $\alpha(s_1 * s_2) = \alpha(s_2) \cup (\alpha(s_1) - a)$, $\beta(s_1 * s_2) = \beta(s_1) \cup (\beta(s_2) - a)$, and $\partial(s_1 * s_2) = \alpha(s_1 * s_2) \coprod \beta(s_1 * s_2)$. The set of such hypersurfaces is denoted by $\mathcal{H}_*^{(2)}$.

• Let $(s_1, s_2) \in \mathcal{H}_*^{(n-1)} \times \mathcal{H}_*$. then we define in the same way $s_1 * s_2$. The set of such hypersurfaces is denoted by $\mathcal{H}_*^{(n)}$.

• We set $\Sigma_* = \bigcup_{n \in \mathbb{N}^*} \mathcal{H}_*^{(n)}$.

Notice that there is a forgetful map $\Sigma_* \to \Sigma_\vee$ only for the hypersurfaces $s \in \Sigma^*$ that have no self-intersection. The following example, based on the Möbius band, shows that this restriction is needed. **Example.** Let us fix $M = \mathbb{R}^3$ and let

$$s_1 = \left\{ \left(\cos(\pi t), \sin(\pi t), s - \frac{1}{2} \right) | (t, s) \in [0; 1]^2 \right\}$$

such that $\alpha(s_1) = \left\{(\cos(\pi t), \sin(\pi t), s)|(t,s) \in (\partial[0;1])^2, t=0\right\}$, $\beta(s_1) = \partial s_1 - \alpha(s_1)$ and let

$$s_2 = \left\{(-\cos(\pi t), -\sin(\pi t)(1+cos(\pi t)), s - cos(\pi t))|(t,s) \in [0;1]^2\right\}$$

such that

$$\alpha(s_2) = \left\{\left(-\cos(\pi t), -\sin(\pi t)\left(1+\frac{sin(\pi t)}{2}\right), s - cos(\pi t) - \frac{1}{2}\right)|(t,s) \in (\partial[0;1])^2, t=0\right\},$$

$\beta(s_2) = \partial s_2 - \alpha(s_2)$. If one glues topologically s_1 and s_2, we get the Möbius band which is non orientable. So that $s_1 \vee s_2$ is not defined in this case. By our choices of initial ad final parts, $s_1 * s_2$ and $s_2 * s_1$ exist both, because they can be represented by the "cut" Möbius band, with

$$\alpha(s_1 * s_2) = \alpha(s_2) = \left\{(1;0;s)| -\frac{1}{2} \leq s \leq \frac{1}{2}\right\},$$

$$\beta(s_1 * s_2) = \beta(s_1) = \left\{(1;0;s)| -\frac{1}{2} \leq s \leq \frac{1}{2}\right\},$$

and with

$$\alpha(s_2 * s_1) = \alpha(s_1) = \left\{(-1;0;s)| -\frac{1}{2} \leq s \leq \frac{1}{2}\right\},$$

$$\beta(s_2 * s_1) = \beta(s_2) = \left\{(-1;0;s)| -\frac{1}{2} \leq s \leq \frac{1}{2}\right\}.$$

One can say that this is not natural since e.g. $\alpha(s_1 * s_2)$ is not in the border of the underlying C^0-manifold, this is one of the reasons why we discuss this example in details. Moreover, this fits with the natural composition of paths: ignoring the third coordinate, we get back the classical composition of paths, for which loops are topologically without border but have a starting point and an endpoint.

In what follow, Σ_M represents either $\Sigma_\vee$ or Σ_* with an adequate choice of Lie group G (we choose G to be abelian for $\Sigma_\vee$). Anyway, we denote the group law of G by the operation of multiplication, and we note $s_1 s_2$ for $s_1 \vee s_2$ or $s_1 * s_2$.

Definition 4.4. A $G-$valued **cosurface** is a map

$$c : \Sigma_M \to G$$

such that

(1) $\forall (s_1, s_2) \in \Sigma_M \times \mathcal{H}, c(s_1 \vee s_2) = c(s_1)c(s_2)$ and
(2) We denote by $\tilde{s}$ the same cosurface as $s \in \Sigma_M$ with opposite orientation. Then $\forall s \in \Sigma_M, c(\tilde{s}) = c(s)^{-1}$.

Let $\tau_s(c) = c(s)$. Let $\Gamma_{M,G}$ the set of $G-$valued cosurfaces of M equipped with the the smallest $\sigma-$ algebra making measurable the collection of maps

$$\{\tau_s : \Gamma_{M,G} \to G | s \in \Sigma_M\}.$$

Let $(\Omega, \mathcal{B}, p)$ be any probability space.

Definition 4.5. A **stochastic cosurface** is a map

$$C : \Omega \times \Sigma_M \to G$$

such that:

(1) $\forall \omega \in \Omega$, $C(\omega, .) \in \Gamma_{M,G}$.
(2) the map $\omega \in \Omega \mapsto C(\omega, .)$ is a $\Gamma_{M,G}-$valued measurable map.

For a subset $\Lambda \subset M$ we consider the $\sigma-$algebra $\mathfrak{T}(\Lambda)$ generated by stochastic cosurfaces $C(s)$ where $s \subset \Lambda$. In other words,

$$\mathfrak{T}(\Lambda) = \sigma\left\{\{C \in \Gamma_{M,G} | C(s) \in B\} \,|s \subset \Lambda; B \text{ Borel subset of } G\right\}.$$

Now, we have to define finite sequences of hypersurfaces that we denote by **complex**.

Definition 4.6. Let $n \in \mathbb{N}^*$. An $n-$**complex** on M is a $n-$uple $K = (s_1, ...s_n) \in (\Sigma_M)^n$ such that $s_i \neq s_j$ for $i \neq j$. We define

$$C(K) = (C_1(s_1), ..., C_n(s_n))$$

where each $C_i \in \Gamma(M; G)$. We note by $\mathcal{K}$ the set of complexes of any length n.

Notice that a complex is an **ordered** sequence, and related with this order there is a natural notion of **subcomplex** of a complex K. If $K = (s_1, ..., s_n)$, a subcomplex L is a subsequence of K, that is

$$\exists l < m \leq n, L = (s_l, ..., s_m) = (s_i)_{l \leq i \leq m}.$$

Now, we need to recognize the complexes that define skeletons of a partition of the manifold M.

Definition 4.7. A $n-$complex $K = (s_1, ..., s_n)$ is **regular** if $\forall (i, j) \in {\mathbb{N}_n}^2$,

$$i \neq j \Rightarrow s_i \cap s_j \subset \partial s_i \cap \partial s_j.$$

Notice that the definition does not consider initial and final parts of the borders. In the sequel, since many ways to understand complexes can be useful (set-theory, topological spaces, sequences, oriented manifolds) we shall use the standard notations in these various fields and we shall precise in what sense we use them only if the notations carry any ambiguity.

Definition 4.8. Let K be a regular $n-$complex. K is called **saturated** if and only if $\bigcup_{i \in \mathbb{N}_n} s_i$ defines the borders of a covering of M by connected and simply connected closed subsets. In other words, there is a familly $(A_k)_k$ of closed connected and simply connected subsets of M such that

(1) $\bigcup_k A_k = M$
(2) for two any indexes k and k', if $k \neq k'$,

$$A_k \cap A_{k'} \subset \partial A_k \cap \partial A_{k'} \subset \bigcup_{i \in \mathbb{N}_n} s_i.$$

We say that a regular $n-$complex K **splits** M through the subcomplex $L = (s_i)_{l \leq i \leq m}$ if

(1) $\bigcup_{s \in L} s$ splits M into two connected components M^+ and M^- and
(2) $\bigcup_{i<l} s_i \subset Adh(M^-)$; we note $K^- = (s_i)_{i<l}$" $= K \cap M^-$",
(3) $\bigcup_{i>m} s_i \subset Adh(M^+)$; we note $K^+ = (s_i)_{i>m}$" $= K \cap M^+$".

(here, Adh means topological closure)

Examples around the 2-cube $ABCDEFGH \subset \mathbb{R}^3$:

$ABCDEFGH$ is the 2-cube, and we assume that each coordinate of A, B, C, D, E, F, G and H is equal to ± 1.

- Let us consider the (empty) $2-$cube $ABCDEFGH$ as a piecewise smooth hypersurface of $\mathbb{R}^3$. By the orientation of $\mathbb{R}^3$, and since the cube divides $\mathbb{R}^3$ into an inside part and an outside part, each face is oriented so that $ABCDEFGH \in \Sigma_\vee$. The 2-cube divides $\mathbb{R}^3$ into two parts, that we recognize as interior and exterior, which are connected an simply connected (but one is non contractible). So that, it splits $\mathbb{R}^3$. We can also say that we have a regular $6-$complex made of the faces of $ABCDEFGH$ (where we have to choose an order which is non canonical).

-Let us now project $ABCDEFGH$ into S^2 radially. Then the segments of $ABCDEFGH$ define a class of regular complexes on S^2. The complex of the segments is not uniquely defined because of the order that we have to choose, and also because of the orientations of the segments that we have to choose. This will yield different possible splittings. For example,if we consider the the complex $K = (s_1, ...s_{12})$ defined by

$$\begin{aligned} K &= \big((A;B),(B;E),(B;C),(C;D),(C;G),(A;E),(E;F), \\ &\quad (F;G),(G;H),(H;D),(D;A),(E;H)\big) \end{aligned}$$

we have the subcomplex

$$L = ((A;E),(E;F),(F;G),(G;H),(H;D),(D;A))$$

that splits K. We have $(S^2)^+ = AEHD \cup EFGH$ and $(S^2)^- = ABCD \cup ABFE \cup BCGF \cup CDGH$ (which are both contractible) and finally $K^+ = (E;H)$ and $K^- = ((A;B),(B;E),(B;C),(C;D),(C;G))$.

Definition 4.9. Let C be a stochastic cosurface. C is said to be a **Markov cosurface** if for each $n \in \mathbb{N}^*$ and for each regular $n-$complex K which splits through a subcomplex $L = (s_i)_{l \leq i \leq m}$, for each couple of map (f^+, f^-) for which the above expectations exists and f^+ (resp. f^-) is $\mathfrak{T}(M^+ \cup \bigcup_{l \leq i \leq m} s_i)-$measurable (resp. $\mathfrak{T}(M^- \cup \bigcup_{l \leq i \leq m} s_i)-$measurable), we have:

$$\mathbb{E}(f^+ f^- | \mathcal{T}(\bigcup_{l \leq i \leq m} s_i)) = \mathbb{E}(f^+ | \mathfrak{T}(\bigcup_{l \leq i \leq m} s_i))\mathbb{E}(f^- | \mathfrak{T}(\bigcup_{l \leq i \leq m} s_i)).$$

4.2. **Markov cosurfaces and Markov semigroups.** Let λ be the Haar measure on G which is now assumed unimodular. We introduce a projective system of probability measures on $\{G^K; K \in \mathcal{K}\}$. For this, we use a partial order on $\mathcal{K}$.

Proposition 4.10. *Let $(K, K') \in \mathcal{K}^2$ such that $K = K'$ in the set-theoric sense. We write $K \prec K'$ if $\forall s \in K$, there exists a subcomplex L' of K' such that s is the composition of the elements of L', ordered by indexes. $\prec$ is an order on $\mathcal{K}$.*

Proof. Comparing this proposition with [5], we already have that $\prec$ is only a preorder. So that, we need only to check reflexivity. Let $s \in K$. Taking $L' = \{s\}$, we get $K \prec K$. Moreover, let $(K, K') \in \mathcal{K}^2$, if $K \prec K'$ and $K' \prec K$, $\forall s \in K, s \in K'$ and $\forall s' \in K', s \in K$ and hence K and K' have the same hypersurfaces, indexed with respect to the same order. □

We now recall the standard definition of filters for the order $\prec$.

Definition 4.11. A filter $P \subset \mathcal{K}$ is such that:

(1) $\forall (K, K') \in P^2, \exists K'' \in P, (K \prec K'' \wedge K' \prec K'')$.
(2) $(\forall K \in \mathcal{K}, \exists K' \in P, K \prec K') \Rightarrow K \in P$.

Let Q_t be a convolution semigroup of probability measures on G with densities, i.e. $Q_t = q_t.\lambda$ satisfies

(1) $Q_0 = \delta_e$ (Dirac measure at the unit element)
(2) $\forall s, t \in (\mathbb{R}_+^*)^2$, $\forall x \in G$, $(q_t q_s)(x) = \int_G q_s(xy^{-1}) q_t(y) d\lambda(y) = q_{t+s}(x)$
(3) $\lim_{t\to 0} Q_t = \delta_e$ weakly
(4) $\forall (x, y) \in G^2, q_{(.)}(xy) = q_{(.)}(yx)$

Now, we need to separate the exposition among the two approaches of cosurfaces, one on $\Sigma_\vee$ and the other on Σ_*. In both cases, we fix $K \in P$ a regular saturated complex with associated domains $D = \{A_1, ...A_m\}$. Each $A \in D$ is oriented through the orientation of M.

• On $\Sigma_\vee$.
We define

$$\varphi_A(s) = \begin{cases} s & \text{if } s \subset \partial A \text{ has the same orientation as } \partial A \\ \tilde{s} & \text{if } s \subset \partial A \text{ has the opposite orientation from } \partial A \\ \emptyset & \text{if } s \not\subset \partial A \end{cases}$$

We set

$$\phi_A(C(K)) = \prod_{s\in K} C \circ \varphi_A(s).$$

(this product is a convolution product of measures)

• On Σ_*. Here, $K = (s_1, ..., s_m)$ is ordered by indexation. Then, we have to work by induction to define φ_A.

- Let $s_j \in K$ be the first element in K such that $s_j \subset \partial A$. For $i < j$, we set $\varphi_A(s_i) = \emptyset$. Then we compare the orientation of ∂A with the one of s_j as in the case of $\Sigma_M^\vee$ to define $\varphi_A(s_j)$.
- Assume now that we have determined φ_A till the index j. Take l to be the first index after j such that $s_l \subset \partial A$. As before, for $j < i < l$, we set $\varphi_A(s_i) = \emptyset$. First, compare the orientation of s_l with the one of ∂A and change s_l into $\tilde{s}_l$ if necessary as before. Notice that final parts of s_j and initial parts of s_l are not considered here. This enables anyway to define

$$\phi_A(C(K)) = C \circ \varphi_A(s_1)...C \circ \varphi_A(s_n) \tag{4.1}$$

Both in the case of $\Sigma_\vee$ and in the case of Σ_*, we set

Definition 4.12.

$$\mu_K^Q(C) = \prod_{i=1}^{k} q_{|A_i|} \left(\phi_{A_i} \circ C(K)\right).$$

Remark 4.13. When $d = 2$, changing the orientation of the path $s \in \mathcal{H}$ is the same as permuting its initial and its final points. Then, the procedure described for Σ_* makes also final parts and initial parts coïnciding.

4.3. **Action of the symmetric group.** Looking at the definition 4.12 of μ_K, we easily see that the value of μ_K is independent of the order of the sequences $A = (A_1, ..., A_k)$ since the group G is unimodular. Unlikely, there is no invariance under reordering K in the non abelian case (see the definition of ϕ_A in equation 4.1). Here, ϕ_A depends on the order of the saturated complex $K = (s_1, ..., s_n)$. So that, the action of the n-symmetric group $\mathfrak{G}_n$ on indexes of n-saturated complexes

$$(\sigma, K = (s_1, ...s_n)) \longmapsto \sigma.K = \left(s_{\sigma(1)}, ..., s_{\sigma(n)}\right)$$

generates an action $\sigma.\mu_K = \mu_{\sigma.K}$. Setting $\mathcal{KS}$ to be the set of saturated complexes on M, noting by $\mathfrak{G}_\infty$ the group of bijections on $\mathbb{N}^*$, we get an action $\mathfrak{G}_\infty \times \mathcal{KS} \to \mathcal{KS}$ in the following way: completing $K = (s_1, ..., s_n) \in \mathcal{KS}$ into $\hat{K} = (s_1, ..., s_n, \emptyset, ...) \in \Sigma_*^{\mathbb{N}^*}$, a bijection $\sigma \in \mathfrak{G}_\infty$ on indexes gives a sequence $\sigma.\hat{K}$ with only n elements different from $\emptyset$. We define $\sigma.K$ to be the n-saturated complex (indexed by $\mathbb{N}_n = \{1, 2, ..., n\}$) as the collection $\{s_1, ..., s_n\}$, ordered $\sigma.\hat{K}$ indexwise.

4.4. **Examples.** This selection of examples is based on earlier works [1, 2, 3, 4, 5] where only $d = 2$ examples on Σ_* or examples on $\Sigma_\vee$ were considered. An example on Σ_* with $d = 3$ will be given later because the tools needed have to be much clarified.

4.4.1. *The $d = 2$ holonomy cosurface.* ([2], compare with the settings described in [20]) Let M be a $2-$manifold. Let $\mathcal{P}(M)$be the space of piecewise oriented smooth paths, with canonical initial and final points ("canonical" means induced by the path orientation). By the way, open paths in $\mathcal{P}(M)$ can be identified with $\Sigma_\vee$ and there is a map $\Sigma_* \to \mathcal{P}(M)$ which coïncides with the forgetful map $\Sigma_* \to \Sigma_\vee$ on open paths and with a map on loops that is changing the initial and final parts if necessary. Let G be a Lie group and let $P = M \times G$ the trivial principal bundle over M with structure group G. Let θ be a connection on P and we note by Hol_θ the holonomy mapping $\mathcal{P}(M) \to G$ for which the horizontal lift starts at $(\alpha(p), e_G)$. Let $s \in \Sigma_*$ that we identify with the corresponding path that we note also s. We define the holonomy cosurface c by

$$c(s) = Hol_\theta(\tilde{s}).$$

We need here to invert the orientation of the path because of the right action of the holonomy group on the principal bundle P.

Remark 4.14. Since the definition of (non stochastic) cosurfaces does not require any measure, one can take for G any Lie group on which the notion of horizontal lift of a path with respect to a connection is well defined. At this level, the construction is valid for any (finite dimensional) Lie group, but for any Banach Lie group, or regular Fréchet Lie group, regular c_∞-Lie group [16], as well as for regular frölicher Lie group [24].

Remark 4.15. This approach is quite similar to the approach of gauge theories via quantum loop gravity approach, see e.g. [26] for an up-to date paper. However, much open questions remain when one wish to work along the lines of this viewpoint.

Then choosing Q as the heat semi-group on G, we get a stochastic cosurface picture of $2d$-Yang-Mills fields (see e.g. [20], [31] and references therein for an extensive work when M is a 2-dimensional manifold and the topology is non trivial).

4.4.2. *Markov cosurfaces and lattice models.* [2, 5] Let $L_\epsilon = \mathbb{Z}^d$. Let U be an invariant function on a compact group G and a "coupling constant" $\beta > 0$. Let Λ be a bounded subset of L_ϵ and let us define the (normalized) probability measure

$$\mu_\epsilon^\Lambda = \frac{1}{Z_{\Lambda,\epsilon}} \exp\left(-\beta \sum_{\gamma \subset \Lambda} U(C(\partial\gamma))\right) \prod_{\gamma \subset \Lambda} dC(\partial\gamma)$$

where γ is an elementary cell, $\partial\gamma$ the boundary $C(\partial\gamma)$ a variable associated to $\partial\gamma$ with values in G. In the sens of projective limits of measures, the limit $\Lambda \to L_\epsilon$

exists and defines a Gibbs-like lattice cosurface. In the cases $G = U(1), SU(2)$, the continuum limit $\epsilon \to 0$ for μ_ϵ has been shown to exist for appropriate U and by a suitable choice of $\beta(\epsilon)$ such that $\lim_{\epsilon\to 0} \beta(\epsilon) = +\infty$.

4.4.3. *Markov cosurfaces and Higgs fields in 2-dimensional space time.* ([5], see also [1]) Let Λ be a bounded non empty subset of $\mathbb{Z}^2$. Cosurfaces C are defined along the edges of $\mathbb{Z}^2$. Let G be a compact Lie group, equipped with a representation ρ on an Euclide space V with scalar product $< .; . >$ and norm $|.|$. Let $(\lambda, \mu) \in \left(\mathbb{R}_+^*\right)^2$ and let φ be a $V-$ valued random field over $\mathbb{Z}^2$. We define a probability measure$\mu_{\lambda,\Lambda}$ on $\mathbb{Z}^2$ with support in Λ by

$$\mu_{\lambda,\Lambda}(d\phi) = \frac{1}{Z_\Lambda} e^{-\frac{\lambda}{2}\sum_{x\in\Lambda}\left(B+\frac{\mu^2}{\lambda}\right)|\varphi(x)|^2} e^{-\frac{\lambda}{2}\sum_{x,y\in\Lambda}<\varphi(x),\rho\circ C(xy)\varphi(y)>} \prod_{x\in\Lambda} d\varphi(x).$$

where $C(xy)$ is the evaluation of the Markov cosurface C on the path xy. Replaing the lattice $\mathbb{Z}^2$ by $\epsilon\mathbb{Z}^2$, with suitable choice of $\lambda(\epsilon, \mu(\epsilon)$, yields continuum limit Higgs models as $\epsilon \to 0$. In the cases $G = U(1), SU(2)$, with Q_t the heat semi-group, our Higgs fields coincide with the Higgs fields in the physics litterature. For a description of the mathematical construction of the continuum limit, keeping C fixed, see [5].

5. Cosurfaces without underlying manifolds

Let us now consider a Hilbert space H. Mimicking section 4, we show how the notions can be extended without the codimension 1 assumption.

5.1. **Settings.** Let M be a $d-$dimensional connected oriented Riemannian manifold. Let $\mathcal{H}_{\vee,n}$ (resp. $\mathcal{H}_{*,n}$ be the set of embedded, oriented, smooth, closed, connected n-submanifolds of H with piecewise smooth border (resp. the set of embedded, oriented, smooth, closed, connected n-submanifolds of H with piecewise smooth border and with initial and final parts). Let us recall the assumptions for the $\vee$ composition:

- Let $(s_1, s_2) \in \mathcal{H}_{\vee,n} \times \mathcal{H}_{\vee,n}$. If
  (1) $s_1 \cap s_2 \subset \partial s_1 \cap \partial s_2$ is a non empty $(d-2)$ piecewise smooth manifold and
  (2) the orientations induced on $s_1 \cap s_2$ by s_1 and s_2 are opposite.

then we define $s_1 \vee s_2$ by gluing s_1 and s_2 along their common border. The orientation of $s_1 \vee s_2$ is the one induced by s_1 and s_2, and by the gluing conditions, if $s_1 \vee s_2$ is itself a (non piecewise) smooth submanifold of H, then there is an orientation on $s_1 \vee s_2$ that generates the orientation of s_1 and of s_2.

We denote by $\Sigma_{\vee,n}$ the corresponding space of piecewise smooth n-submanifolds of H, and set

$$\Sigma_{\vee,H} = \coprod_{n\in\mathbb{N}} \Sigma_{\vee,n}.$$

Notice that we can extend $\vee$ componentwise to $\Sigma_{\vee,H}$.

Let us do the same for $*$. $\mathcal{H}_{*,n}$ is the set of (unparametrized, but oriented) n-submanifolds s of H equipped in addition with a prescribed orientation of smooth components of ∂s. Initial parts of ∂s, noted $\alpha(s)$, are those for which the prescribed orientation is opposite to the one induced by s, and the final parts, noted $\beta(s)$, are the ones for which they coïncide. These are exactly the definitions given in section

4, we can define $\Sigma_{*,n}$ the corresponding space of piecewise smooth submanifolds and we set

$$\Sigma_{*,H} = \coprod_{n\in\mathbb{N}} \Sigma_{*,n}$$

with extension of $*$ componentwise. In what follows, when it carries no ambiguity, we omit the notations $\vee$ or $*$ to denote the composition rule by multiplication, with an adequate choice of Lie group G (we choose G to be abelian for $\Sigma_{\vee,H}$). Anyway, we note the group law of G by multiplication too. The following notions can be extended straightaway; the only conceptual difference is that the orientation of a surface s cannot be compared with the orientation of an underlying manifold M. The manifold M is replaced by the CW-complex obtained by gluing the domains A_k along the borders.

Definition 5.1. • A G-valued cosurface is a map

$$c : \Sigma_H \to G$$

such that

(1) $\forall (s_1, s_2) \in \Sigma_H \times \mathcal{H}, c(s_1 s_2) = c(s_1)c(s_2)$ and
(2) We denote by $\tilde{s}$ the same n-submanifold as $s \in \Sigma_M$ with opposite orientation on s and ∂s. Then $\forall s \in \Sigma_M, c(\tilde{s}) = c(s)^{-1}$.

We denote by $\Gamma(G)$ the set of all G-valued cosurfaces.

• Let $m \in \mathbb{N}^*$. An $(m,n)-$**complex** on H is an $m-$uple $K = (s_1, ...s_m) \in (\Sigma_n)^m$ such that $s_i \neq s_j$ for $i \neq j$. We define

$$C(K) = (C(s_1), ..., C(s_n))$$

where each $C_i \in \Gamma(M; G)$. We denote by $\mathcal{K}_n$ the set of complexes of dimension n and of any length m.

• An $n-$complex $K = (s_1, ..., s_n)$ is **regular** if $\forall (i,j) \in {\mathbb{N}_n}^2$,

$$i \neq j \Rightarrow s_i \cap s_j \subset \partial s_i \cap \partial s_j.$$

• There is a natural notion of subcomplex of a complex K. If $K = (s_1, ..., s_n)$, a subcomplex L is a subsequence of K, that is

$$\exists l < m \leq n, L = (s_l, ..., s_m) = (s_i)_{l\leq i\leq m}.$$

• Let K be a regular $n-$complex. K is called **saturated** if and only if there is a $(n+1)-$ complex A such that $\bigcup_{i\in\mathbb{N}_n} s_i$ defines the borders of the $(n+1)-$surfaces of A by connected and simply connected closed subsets. In other words, the family $(A_k)_k$ of closed connected and simply connected subsets of H defines, by gluing along K, a CW-complex also noted M such that

(1) $\bigcup_k A_k = M$
(2) for any two indexes k and k', if $k \neq k'$,

$$A_k \cap A_{k'} \subset \partial A_k \cap \partial A_{k'} \subset \bigcup_{i\in\mathbb{N}_n} s_i.$$

• We say that a complex K **splits** through the subcomplex $L = (s_i)_{l\leq i\leq l'}$ if $\bigcup_{s\in L} s$ splits M (as a subset of H) into two topologically connected components M^+ and M^- such that $K \cap M^+ = (s_i)_{i>m}$ and $K \cap M^- = (s_i)_{i>l'}$

5.2. **Dimension extension.** We use here the idea of the previous defintion, gluing together simply connected $(n+1)-$surfaces A_i along a $n-$complex K in order to get, by induction on the dimension N of the surfaces, a construction of $n-$cosurfaces from lower dimensions.

Definition 5.2. A complex $K \in \mathcal{K}_n$ is called **saturated** if it can be embedded into a $(n+1)$-submanifold M of H, $M \in VCob_{m+1}$, for which it is a complex for cobordism. M is called a **saturation** of K.

In this definition, we then obtain M by gluing along K a family of elements of Σ_{n+1} which are connected and simply connected.

Definition 5.3. A complex $K \in \mathcal{K}_{\vee,n}$ is called **weakly saturated** if there is a complex $A = (A_1, ..., A_k) \in \mathcal{K}_{*,n+1}$ obtained

- topologically by gluing each set A_i on $L \subset K$ with respect to the borders ∂A_i such that there is a bijective map $\partial A_i \to L$.
- the orientations of the borders on ∂A_i correspond to the orientations of the submanifolds in the sequence L.
- any $s \in K$ is at least glued once.

A is called a **weak saturation** of K.

Notice that with this definition, the orientation of each A_i is left free of choice. Moreover, the definition can be extended straightaway by replacing $\vee$ by $*$ since the initial and final parts of $s \in K$ do not interfer with the gluing.

Definition 5.4. Let c_K be defined on the surfaces $A \in \Sigma_{*,n+1}$ that are smooth with piecewise smooth border $\partial A \subset K$. Then we define

$$c_K(A) = \prod_{s \in \partial A \subset K} c \circ \varphi_A(s)$$

(the product is with respect to the order in K).

Proposition 5.5. *If G is abelian, c_K is the restriction of a cosurface c' on $\Sigma_{*,n+1}$ that coïncide with c_K on any A where it is defined.*

Proof. First, for a complex K_1 such that $K \prec K_1$, we can get the values of c on K_1. Then, all we have to show is that, given A an $(n+1)$-submanifold of H with piecewise smooth border along K and taking $A' = (A_1, ...A_l)$ such that $A \prec A'$, in other words $A = A_1 * ... * A_l$, taking K' the (unordered) skeleton of the gluing $A_1 * ... * A_l$, one has

$$c_K(A) = c_{K'}(A_1)...c_{K'}(A_l).$$

There is of course an ambiguity on the order of K' but since G is abelian, for each $s \in K'$, we have only to count the number of indexes $j \in \mathbb{N}_l$ such that $\varphi_{A_j}(s) = s$ and compare it to the number of indexes such that $\varphi_{A_j}(s) = \tilde{s}$. Since we are in the $*$-composition, we have at each edge $s \not\subset K$ only one index of each type, which are the A_j's for which s is in the initial part and the final part respectively. So that the contributions that are "interior" compensate. □

Definition 5.6. Let $K \in \mathcal{K}_n$. An **overcomplex** K' of K is a weakly saturated complex such that there exists a complex K_1 with $K \prec K_1$ and $K_1 \subset K'$ with preserved order.

Proposition 5.7. *If G is non abelian, c_K is the restriction of a cosurface c' on $\Sigma_{*,n+1}$ such that: for any overcomplex K' of K, for any $s \in K' \backslash K$, there exists c'' a cosurface on $\Sigma_{*,n+1}$,*

$$c'(s) = c''_K(s) \in Z(G).$$

Proof. Let A be a complex as in definition 5.4. The main points are to know

- what happens for K_1 such that $K \prec K_1$? Here again, we can make use of the complex c (we recall that if $K \prec K_1$, the order is preserved).

- What happens for an overcomplex K' that the skeleton of a complex A' such that A? what to do with the "interiors"? Let K_1 be the complex made of elements $s \in K'$ Let is assign arbitrarily a value $c''(s) = g \in Z(G)$ (e.g. take $g = e$) if $s \in K' - K_1$. We have

$$\begin{aligned}\prod_{i=1}^{l} c_K(A_i) &= \prod_{i=1}^{l} \prod_{s \in \partial A_i \subset K} c \circ \varphi_{A_i}(s) \\ &= \prod_{i=1}^{l} \prod_{s \in \partial A \subset K_1} c \circ \varphi_{A_i}(s)\end{aligned}$$

on one hand. And on the other hand,

$$\begin{aligned}\prod_{j=1}^{l'} c_K(A'_j) &= \prod_{i=1}^{l'} \prod_{s \in \partial A'_j \subset K} c \circ \varphi_{A_j}(s) \\ &= \left(\prod_{j=1}^{l'} \prod_{s \in \partial A \subset K_1} c \circ \varphi_{A_i}(s)\right)\left(\prod_{j=1}^{l'} \prod_{s \in \partial A \subset K - K_1} c \circ \varphi_{A_i}(s)\right) \\ &= \left(\prod_{j=1}^{l'} \prod_{s \in \partial A \subset K_1} c \circ \varphi_{A_i}(s)\right).e\end{aligned}$$

for the same reasons as in the last proof. □

Example: Dimension extension of the holonomy cosurface

We have given a way to extend cosurfaces to higher dimensions. Let us now use it to show that the notion of non abelian cosurface is not void on $\Sigma_{*,n}$ for $n > 1$. For this, let us consider the infinite lattice

$$\mathbb{Z}^\infty = \left\{(u_n)_{n \in \mathbb{N}} \in \mathbb{Z}^{\mathbb{N}} | u_n \neq 0 \text{for a finite number of indexes } \right\}$$

Let G be a unimudular Lie group, and let θ be a connection on $H \times G$. Along the edges of this lattice in particular and more generally on any path, we can define the holonomy cosurface as in section 4.4.1. Let K be a weakly saturated $1-$ complex along the edges of $\mathbb{Z}^\infty$. Let us now consider the squares that are described by this lattice, and more especially those that are gluing along K. They are $2-$submanifolds on H and we can define a cosurface $c_{2,K}$ on $\Sigma_{2,*}$ that reads as in theorem 5.7. Then, for K "large enough", we can find a weakly saturated complex of squares $A = (A_1, ..., A_n)$ to reproduce the procedure to get a non abelian cosurface on cubes, and this until we reach the dimension d for which there no longer exists any weakly saturated complex "based" on K.

For example, take the cube ABCDEFGH and take a complex made of its 12 (oriented) segments and θ a $SU(N)$-connection. With this choice, $Z(G) = e$ so that $c_{2,K}$ is uniquely determined. Then $c_{2,K}$ is non abelian on the 6 faces of the cube, and by choosing an order on the faces, i.e. by choosing complex A made of its 6 faces, $c_{2,K}$ extends to a cosurface $c_{3,A,K}$ which is trivial except on the cube viewed as an element of $\Sigma_{3,*}$. If one wants to get into another dimension (e.g. on the hypercube), the cosurfaces obtained are trivial except for the chosen manifolds along the cube.

6. Cosurfaces and Cobordisms

By 'manifold' M we always mean a smooth finite dimensional manifold, possibly with boundary; if the boundary, denoted ∂M, is void, the manifold is said to be *closed*. If N is a an oriented manifold we denote by N^- the manifold N with the opposite orientation.

Let X_1 and X_2 be oriented closed submanifolds, both of dimension $d-1$, where d is a positive integer. By a **pre-cobordism** $(Y, \phi_1, \phi_2) : X_1 \to X_2$ we mean an oriented manifold Y along with an orientation preserving diffeomorphism

$$\phi : X_1^- \sqcup X_2 \to \partial Y.$$

An isomorphism from a pre-cobordism $Y : X_1 \to X_2$ to a pre-cobordism $(Y', \phi_1', \phi_2') : X_1 \to X_2$ is an orientation-preserving diffeomorphism $f : Y \to Y'$ such that $f \circ \phi_1 = \phi_1'$ and $f \circ \phi_2 = \phi_2'$. A **cobordism** is a pre-cobordism up to isomorphisms.

Next, if $Y : X_1 \to X_2$ is a cobordism, and Y is equipped with a top-dimensional volume form (a measure of volume) we say that Y is a *volume* pre-cobordism. A **volume cobordism** is then an equivalence class of such pre-cobordisms, where the equivalence relation is obtained by using only orientation preserving and volume-preserving diffeomorphisms. A theorem of Morse guarantees that any two diffeomorphic compact oriented manifolds of equal volume are diffeomorphic by means of a volume-preserving diffeomorphism, and so to restrict the considerations to volume cobordism is not a huge restriction).

To keep notation under control, we will simply think of a cobordism from X_1 to X_2 as an oriented manifold Y, of dimension d, running "from" X_1 "to" X_2. Composition of cobordisms is defined in the natural way. The "identity" cobordism $X \to X$ is given by the oriented manifold $X \times [0,1]$ along with the mapping

$$X^- \sqcup X \to X \times [0,1]$$

which takes $p \in X^-$ to $(p, 0)$ and $p \in X$ to $(p, 1)$. Let $\mathbf{VCob}_d$ be the category whose objects are $d-1$ dimensional closed oriented manifolds and whose morphisms are volume cobordisms.

We may also work within a fixed oriented d-dimensional manifold M, equipped with a volume form, and operate only with cobordisms which are (full dimensional) submanifolds of M. Let VCob_M be the set of all such cobordisms.

6.1. **Adapted saturated complexes.** Consider Y and Y' two morphisms in the category $VCob_d$, seen as two d-dimensional manifolds equipped with their borders and volume form. Assume also that $\alpha(Y) = \beta(Y')$ so that $Y \circ Y'$ exists in $VCob_d$. Fix now a regular saturated complex K'' in $Y \circ Y'$ and for the $*$-construction so

that the set $K'' \cap \alpha(Y)$ is made of complexes on each connected component of $\alpha(Y')$. What we want to construct is a composition rule for saturated complexes adapted to the composition of morphisms in $VCob_d$. Namely, we want to build two complexes $K \subset Y$ and $K' \subset Y'$ and a "composition rule" based on the composition $*$ for which $K * K' = K''$.

Definition 6.1. Let $Y \in Mor(VCob_d)$ and let K b a regular saturated complex of Y. Then K is **adapted** if for each $x \in \partial Y \cap K$ and for each$s \in K$ such that $x \in s$,

$$x \in \partial s$$

and

$$x \in \alpha(s) \Leftrightarrow x \in \alpha(Y).$$

Intuitively, the complex K is adapted if it satisfies a property of transversality on the border of Y, and if the initial and final prts of Y coincide with the corresponding initial and final parts of the surfaces of K reaching the border. Now, let

$$Y'' = Y \circ Y'.$$

We want to build a regular saturated complex of Y'' that splits into Y and Y'. For this, we need 3 parts:

$$\begin{cases} K_1 & = & \{\sigma \in K'' | \sigma \subset \alpha(Y) = \beta(Y')\} \\ K_0 & = & \{\sigma \in K'' | \sigma \subset Y \text{ and } \sigma \notin K_1\}, \\ K_0' & = & \{\sigma \in K'' | \sigma \subset Y' \text{ and } \sigma \notin K_1\} \end{cases}$$

such that K_1 is a covering of $\alpha(Y)$, and K_0 and K_0' are adapted saturated complexes of Y and Y' respectively. Notice that under this condition, $\alpha(Y)$ is a $m-1$-manifold, and the borders ∂s, with $s \in K_1$, can define a complex on $\alpha(Y)$ by their smooth components up to re-ordering. This is what we precise first, and then we give a precise construction from cutting and pasting.

6.2. **Border reduction.** Let K b an adapted complex on $Y \in Mor(VCob_d)$. Let A be the covering of Y with respect to K and let us consider $A_k \in A$ such that $\partial A_k \cap \alpha(Y)$ has a non empty interior in ∂Y (one can replace here $\alpha(Y)$ by $\beta(Y)$). Let $\tilde{A}_k$ be the closure in ∂Y of the interior of a connected component of $\partial A_k \cap \partial Y$. This is a connected subset of ∂Y, not necessarily simply connected.

• Orientation $\tilde{A}_k$ is a $(d-1)$ manifold with boundary, with the orientation induced by the orientation of the border of Y.

• Initial and final parts Now, let us consider $\partial \tilde{A}_k$. This is a $(d-2)$ piecewise smooth manifold, since it is a subset of $\bigcup_{s \in K} \partial s$. Let $s \in K \cap \partial A_k$ such that $s \cap \tilde{A}_k \subset \partial \tilde{A}_k$. Then the orientation on $s \cap \partial \tilde{A}_k$ is the one induced by the orientation of s, which defines whether it is an initial or a final part.

Notice that we have here no induced order from the adapted complex K to the border reduction. The **border reduction** is a non ordered regular complex on ∂Y, which is non necessarily saturated because it defines a partition of ∂Y into subsets which are non necessarily simply connected, with orientations induced by Y and K.

6.3. **Complexes for cobordism, cosurfaces and measures part I: cutting.** We now give a more restricted class of complexes.

Definition 6.2. We say that K'' is an n-**complex for cobordism** if $K'' = K''_a \cup K''_\alpha \cup K''_\beta$ if

- K''_α is a covering of $\alpha(Y'') = \alpha(Y')$
- K''_β is a covering of $\beta(Y) = \beta(Y'')$
- K''_a is a saturated complex of Y'',

We now need to say how we "cut" $Y'' \in Mor(VCob_d)$. Let $Y, Y' \in Mor(VCob_d)$ such that $Y'' = Y \circ Y'$ exists. We say that we can cut (Y'', K'') if there exists $\theta \in \mathcal{P}(\mathbb{N}_n)$ such that

$$K''_a = \{\sigma_i \in K'' | i \in \theta\}$$

is an adapted complex in Y' that splits into $K''_a = K_a \cup K'_a \cup K''_b$, where K_a and K'_a are adapted complexes of Y and Y' respectively, and

$$K''_b = \{\sigma_i \in K'' | i \notin \theta\}$$

which defines a covering of $\alpha(Y)$

The sets $K_a, K'_a, K''_\alpha, K''_\beta$ and K_b are equipped with the order induced by K'', and gathering the correxponding parts, we get two complexes for cobordism

- $K = K_a \cup K_b \cup K''_\beta$ on Y
- $K' = K'_a \cup K_\alpha \cup K_b$ on Y'.

Remark 6.3. K_b, as a subcomplex of K'', splits M.

Now let us turn to measures. For this, we now take a stochastic cosurface C''_N on Y'' adapted to the cobordism, that is one that can be divided into two stochastic cosurfaces C_N and C'_N on Y and Y' which coincide on

$$\alpha(Y) \cap \Sigma_* = \{\sigma \in \Sigma_* | \sigma \subset \alpha(Y)\}.$$

Since the order on K'' determines the orders on subcomplexes, for each domain A''_i we define $\phi_{A_i} \circ C''$ that equals to $\phi_{A_i} \circ C$ or $\phi_{A_i} \circ C'$ (we recall that we have $A_i \subset Y$ or $A_i \subset Y'$ since K'' is a complex for cobordism) and each domain is connected and simply connected. If G is non abelian, we assume that the indexation of the family $(A_k)_k$ such that the indexes of the domains in Y' are in the beginning of the list, and that the indexes of the domains in Y are in the end of the list. If G is abelian, this assumption is not necessary.

Theorem 6.4.

$$\mu_K . \mu_{K'} = \mu_{K''}.$$

Proof. Let us build two groups in the formula of definition 4.12, namely, with the notations of Theorem 6.4,

$$\mu^Q_{K''}(c'') = \prod_{i=1}^{k} q_{|A_i|}\left(\phi_{A_i} \circ c''(K)\right).$$

This formula does not depend on the order among the indexes $\mathbb{N}_k$, so that we can define a twofold partition I, J of $\mathbb{N}_k$ defined as follows: I (resp. J) is the set of

indexes i such that $A_i \subset Y$ (resp. $A_i \subset Y'$). Then

$$\begin{aligned}
\mu^Q_{K''}(c'') &= \left(\prod_{i\in I} q_{|A_i|}\left(\phi_{A_i}\circ c''(K)\right)\right).\left(\prod_{j\in J} q_{|A_j|}\left(\phi_{A_j}\circ c''(K)\right)\right)\\
&= \left(\prod_{i\in I} q_{|A_i|}\left(\phi_{A_i}\circ c(K)\right)\right).\left(\prod_{j\in J} q_{|A_j|}\left(\phi_{A_j}\circ c'(K)\right)\right)\\
&= \mu^Q_K(c).\mu^Q_{K'}(c') \quad \square
\end{aligned}$$

Remark 6.5. In order to get saturated complexes we had to add a complex on the border of the manifold Y. This assumption was not explicitely present in the papers [1, 2, 4, 5] where open manifolds were also considered. For volume cobordism, only compact manifolds with boundary are considered. The link with finite volume open manifolds can be done with the particular case of cosurfaces c such that, for any complex for cobordism K, we have the property $c|_{K_b} = e$.

6.4. **Complexes for cobordism, cosurfaces and measures part II: pasting.** Now let us consider the inverse problem, and let us only point out extra facts that give "anomalies" to the pasting procedure. Let us consider $Y, Y' \in VCob_d$, equipped with two complexes for cobordisms K and K' and two cosurfaces c and c'. Here are the conditions to be able to build up a complex for cobordism K''on $Y'' = Y \circ Y'$:

(A) $K_b \cap \alpha(Y) = K'_b \cap \beta(Y')$ with corresponding orientations, initial and final parts on each hypersurface and on each border.

(B) $c|_{K_b\cap\alpha(Y)} = c'|_{K'_b\cap\beta(Y')}$.

With this, one can build up c'', but one cannot build up K'' in an unified way. This depends on a choice of reindexation, compatible with the orders of K_a, K_b, $K'b$ and $K'a$ that we have recovered by "extraction" from K''.

Proposition 6.6. *There exists such a cobordism K'' with the properties (A) and (B).*

Proof. Let us start with

$$K''_0 = K_b \cap \alpha(Y) = K'_b \cap \beta(Y') = (s_1, ..., s_l).$$

We build up by induction a complex K'' which satisfies (A) and (B). In a complex K_e, for $s \in K_e$, we note by $bef_{K_e}(s)$ the subcomplex of elements of K_e before s in the list, and by $aft_{K_e}(s)$ the subcomplex of elements that are after s in the list.

• First step

Let

$$K''_1 = bef_K(s_1) \cup bef_{K'}(s_1) \cup$$

(this union is an ordered union, made first of the ordered set $bef_K(s_1)$, secondly of $bef_{K'}(s_1)$ and finally of K_0).

• Intermediate steps

Let $i \in \mathbb{N}_{l-1}$. Assume that we know K''_i. Set

$$\begin{aligned}
K''_{i+1} &= bef_{K''_i}(s_{i+1}) \cup (aft_K(s_i) \cap bef_K(s_{i+1})) \cup\\
&\quad (aft_{K'}(s_i) \cap bef_{K'}(s_{i+1})) \cup aft_{K''_i}(s_i).
\end{aligned}$$

(with ordered union)

- Final step We have obtained K''_l which last element is s_l. Then
$$K'' = K''_l \cup aft_K(s_l) \cup aft_{K'}(s_l).$$

Then one can extract K and K' from K'' with the desired order. □

Since the corresponding coverings $A = (A_i)_{i\in I}$ of Y and $A' = (A'_j)_{j\in J}$ of Y' are well defined and since all the quantities depend only on the indexation of the hypersurfaces on the borders of each domain, with an order already defined by K and K' and that *will not be changed* while passing to K'', the quantity

$$\left(\prod_{i\in I} q_{|A_i|}\left(\phi_{A_i}\circ c(K)\right)\right)\cdot\left(\prod_{j\in J} q_{|A_j|}\left(\phi_{A_j}\circ c'(K)\right)\right) = \mu^Q_K(c)\mu^Q_{K'}(c')$$

corresponds to the (classical) definition of $\mu^Q_{K''}(c'')$ for any possible choice of indexation for K''.

7. Algebras and groups of series: applications to cobordism and complexes

Let us now gather the framework of cobordism and stochastic cosurfaces into series. First, fix $m > 1$ the dimension of the theory of cobordism. The set of indexes is the one described in Theorem 3.4, namely, $\Gamma \subset \coprod_{p\in\mathbb{N}^*} I_p$, resp. $\Gamma(N) \subset \coprod_{p\in\mathbb{N}^*} I_p(N)$, be a family of indexes, stable under $*$, such that $\forall p \in \mathbb{N}^*$,

(1) $\forall p \in \mathbb{N}^*$, $\Gamma \cap I_p$ is finite or, more generally;
(2) $\forall \gamma \in \Gamma$, the set of pairs $(\gamma', \gamma'') \in \Gamma^2$ such that $\gamma = \gamma' * `\gamma''$ is finite.

We fix a family
$$\mathcal{A}_\Gamma = \prod_{\gamma\in\Gamma} \mathcal{A}_\gamma$$
of diffeological vector spaces $\mathcal{A}_\gamma$ that are regular (with respect to classical integration of paths) which is equipped with a multiplication $*$ which is defined componentwise
$$\mathcal{A}_\gamma \times \mathcal{A}_{\gamma'} \to \mathcal{A}_{\gamma*\gamma'}$$
diffeologically smooth, and such that $(\mathcal{A}_\Gamma, +, *)$ is an algebra.

Let us now turn to our motivating example. Let Γ be a family of piecewise smooth manifolds, made along the infinite lattice $\mathbb{Z}^\infty$, of fixed dimension m. Let us also normalize the volume of a $m-$cube to 1. The family Γ is $\mathbb{N}-$graded by the volume, and assume that we have a stochastic cosurface on o $(m-1)-$cubes, either defined directly, or by dimension extension. Assume now that the complex K, supporting the family Γ, is a (maybe infinite) complex for cobordism. Then, following Theorem 6.4, we have the map
$$\mu_K : M \in \Gamma \mapsto \mu_K(M) \in \mathcal{A} \subset M(\Omega),$$
where $M(\Omega)$ is the space of measures on Ω, and $\mathcal{A}$ is a complete vector space of measures such that convolution is associative. Then, we are in the context of application of the main theorems of this paper, and we have:

Theorem 7.1. *For a fixed choice of stochastic cosurface, the mapping $\gamma \in \Gamma \mapsto \mu_K(Gamma)$ defines an element of $\mathcal{A}_\Gamma$*

The proof is obvious.

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