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ON THE DIFFERENTIAL GEOMETRY OF NUMERICAL SCHEMES AND WEAK SOLUTIONS OF FUNCTIONAL EQUATIONS.

JEAN-PIERRE MAGNOT

ABSTRACT. We exhibit differential geometric structures that arise in numerical methods, based on the construction of Cauchy sequences, that are currently used to prove explicitly the existence of weak solutions to functional equations. We describe the geometric framework, highlight several examples and describe how two well-known proofs fit with our setting. The first one is a re-interpretation of the classical proof of an implicit functions theorem in an ILB setting, for which our setting enables us to state an implicit functions theorem without additional norm estimates, and the second one is the finite element method of the Dirichlet problem where the set of triangulations appear as a smooth set of parameters. In both case, smooth dependence on the set of parameters is established. Before that, we develop the necessary theoretical tools, namely the notion of Cauchy diffeology on spaces of Cauchy sequences and a new generalization of the notion of tangent space to a diffeological space.

Keywords: Diffeology, Cauchy sequences, implicit function theorem, ILB spaces, triangulations.

MSC (2010): 46T20; 58C15; 58B10; 47J25

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INTRODUCTION

The aim of this paper is to describe some differential geometric properties of the analysis of weak solutions of functional equations, especially partial differential equations. We highlight a generalized differential geometric structure, called diffeology (we call it generalized differential geometry because this setting does not involve atlases), as well as a definition of (abstract) symmetries, based on the notion of numerical schemes that are commonly used in constructing explicit solutions to functional equations of the type

$$F(u, q) = 0$$

where u is a function and q is a parameter. We refine for this goal two theoretical aspects of diffeologies, that carry a language which is quite user-friendly. First we define a new tangent space on a diffeological space, and secondly we define a diffeology on Cauchy sequences, that we call Cauchy diffeology, that seems adapted to this setting. We illustrate the results and the settings of this paper by several examples. We finish with two worked-out examples adapted from well-known frameworks. First we analyze the problem of implicit functions under the light of weak solutions. This enables us to state it in the ILB setting [36], for degree 0 map which do not carry additional norm estimates as in classical statements [17, 19, 36]. The difference of our result with these classical approaches is discussed in details, as well as the correspondence with the geometry of weak solutions. Secondly we complete the study of the degree 1 finite elements method for the Dirichlet problem. Here existence and uniqueness of solutions is well-known but we concentrate our efforts on the forgotten aspect of the diffeology of the space of triangulations of the domain, and we show how triangulations can be considered as a smooth space of parameters for numerical schemes built up from the finite elements method.

Let us describe with more details the contents of this paper. We recall the necessary material on diffeologies in section 1.

In section 2, we describe a diffeology, new to our knowledge, which appears as a refined diffeology of a diffeological space. This refined diffeology is inherited from the group of diffeomorphisms, and gives rise to a new (fourth) definition of the tangent space of a diffeological space. These definitions are useful for the generalization of the notion of symmetries proposed in section 3.2.

In section 3.1, we describe the so-called Cauchy diffeology on Cauchy sequences, for which

- (1) the limit map is smooth
- (2) the index maps $ev_k : (u_n)_{n \in \mathbb{N}} \mapsto u_k$ are smooth ($k \in \mathbb{N}$).

We show that the two conditions are necessary, by well-chosen examples. The combination of the two conditions then appears to give the good conditions.

In section 3.2, we find another motivation for the diffeology: when solving numerically a PDE, we build a sequence (u_n) which converges to the solution u , but when u_n is "close enough" to u , computer representations of u uses the approximate solution u_n . Thus, other u and u_n need to be smooth under the parameters and the initial conditions. After describing what can be a general setting for numerical methods, we address an open question on paradoxal solutions of Euler equations for perfect fluids. Cauchy sequences appear mostly where convergence of sequences is needed. This tool is basically topological, as well as the notion of convergence. But especially in analysis of ordinary or partial differential equations, once Cauchy

sequences enabled to construct solutions, a classical question is the smooth dependence on the initial conditions and/or the parameters. Out of a well-established manifold structure on the parameters or the initial conditions, and even if these structures are given, diffeologies appear as an easy way to formalize smoothness, taking rid of the technical requirements of the more rigid framework of manifolds.

In section 4, we show that part of the classical hypothesis of classical (smooth) implicit functions theorems can be relaxed focusing on the diffeology of Cauchy sequences instead of starting with strong estimates on the considered functions. We concentrate on an implicit function theorem on ILB sequences of Banach spaces (E_i) and (F_i) and smooth maps $f_i : O_i \subset E_i \times F_i \rightarrow F_i$. Uniform estimates on the family f_i are not necessary to define a diffeological domain $D \subset \bigcap_{i \in \mathbb{N}} O_i$ and a smooth map $u : D \rightarrow \bigcap_{i \in \mathbb{N}} F_i$ such that $\forall i \in \mathbb{N}, f_i(x, u(x)) = 0$. Then, we discuss on the interest of Nash-Moser estimates, or bornological estimates, in order to give some “control” on the domain D , and the link with the notion of well-posedness is addressed. We finish with a corresponding “free of estimates” Fröbenius theorem, again with the only help of rewritten classical proofs.

In section 5, we show how one of the most classical numerical methods, namely the finite elements method for the Dirichlet problem, fits with our setting. For the equation $\Delta u = f$, with Dirichlet conditions at the border, we define the set of parameters as composed by the possible functions f and the set of triangulations \mathcal{T} . Smoothness of the solution u on f is already known, but we show here that the sequence (u_n) of approximations of u through the finite element method is smooth for the Cauchy diffeology, with respect to the chosen triangulation and the function f . For this purpose, the adapted differential geometry of the space of triangulation is described in terms of diffeologies.

1. PRELIMINARIES ON DIFFEOLOGY

This section provides background on diffeology and related topics necessary for the rest of this paper. The main reference is [21], and the reader should consult this for proofs. A complementary non exhaustive bibliography is [2, 3, 4, 9, 7, 8, 10, 11, 12, 16, 20, 24, 29, 30, 39, 48, 51].

1.1. Basics of Diffeology. In this subsection we review the basics of the theory of diffeological spaces; in particular, their definition, categorical properties, as well as their induced topology.

Definition 1.1 (Diffeology). *Let X be a set. A **parametrisation** of X is a map of sets $p : U \rightarrow X$ where U is an open subset of Euclidean space (no fixed dimension). A **diffeology** \mathcal{P} on X is a set of parametrisations satisfying the following three conditions:*

- (1) (*Covering*) For every $x \in X$ and every non-negative integer n , the constant function $p : \mathbb{R}^n \rightarrow \{x\} \subseteq X$ is in \mathcal{P} .
- (2) (*Locality*) Let $p : U \rightarrow X$ be a parametrisation such that for every $u \in U$ there exists an open neighbourhood $V \subseteq U$ of u satisfying $p|_V \in \mathcal{P}$. Then $p \in \mathcal{P}$.
- (3) (*Smooth Compatibility*) Let $(p : U \rightarrow X) \in \mathcal{P}$. Then for every n , every open subset $V \subseteq \mathbb{R}^n$, and every smooth map $F : V \rightarrow U$, we have $p \circ F \in \mathcal{P}$.

A set X equipped with a diffeology \mathcal{P} is called a **diffeological space**, and is denoted by (X, \mathcal{P}) . When the diffeology is understood, we will drop the symbol \mathcal{P} . The parametrisations $p \in \mathcal{P}$ are called **plots**.

Definition 1.2 (Diffeologically Smooth Map). *Let (X, \mathcal{P}_X) and (Y, \mathcal{P}_Y) be two diffeological spaces, and let $F: X \rightarrow Y$ be a map. Then we say that F is **diffeologically smooth** if for any plot $p \in \mathcal{P}_X$,*

$$F \circ p \in \mathcal{P}_Y.$$

Diffeological spaces with diffeologically smooth maps form a category. This category is complete and co-complete, and forms a quasi-topos (see [1]).

Proposition 1.3. [48, 21] *Let (X', \mathcal{P}) be a diffeological space, and let X be a set. Let $f: X \rightarrow X'$ be a map. We define $f^*(\mathcal{P})$ the **pull-back diffeology** as*

$$f^*(\mathcal{P}) = \{p: D(p) \rightarrow X \mid \exists n \in \mathbb{N}^*, D(p) \text{ is open in } \mathbb{R}^n \text{ and } f \circ p \in \mathcal{P}\}.$$

Proposition 1.4. [48, 21] *Let (X, \mathcal{P}) be a diffeological space, and let X' be a set. Let $f: X \rightarrow X'$ be a map. We define $f_*(\mathcal{P})$ the **push-forward diffeology** as the coarsest (i.e. the smallest for inclusion) among the diffeologies on X' , which contains $f \circ \mathcal{P}$.*

Definition 1.5. *Let (X, \mathcal{P}) and (X', \mathcal{P}') be two diffeological spaces. A map $f: X \rightarrow X'$ is called a **subduction** if $\mathcal{P}' = f_*(\mathcal{P})$.*

. In particular, we have the following constructions.

Definition 1.6 (Product Diffeology). *Let $\{(X_i, \mathcal{P}_i)\}_{i \in I}$ be a family of diffeological spaces. Then the **product diffeology** \mathcal{P} on $X = \prod_{i \in I} X_i$ contains a parametrisation $p: U \rightarrow X$ as a plot if for every $i \in I$, the map $\pi_i \circ p$ is in \mathcal{P}_i . Here, π_i is the canonical projection map $X \rightarrow X_i$.*

In other words, in last definition, $\mathcal{P} = \bigcap_{i \in I} \pi_i^*(\mathcal{P}_i)$ and each π_i is a subduction.

Definition 1.7 (Subset Diffeology). *Let (X, \mathcal{P}) be a diffeological space, and let $Y \subseteq X$. Then Y comes equipped with the **subset diffeology**, which is the set of all plots in \mathcal{P} with image in Y .*

If X is a smooth manifold, finite or infinite dimensional, modelled on a complete locally convex topological vector space, we define the **nebulae diffeology**

$$\mathcal{P}(X) = \{p \in C^\infty(O, X) \text{ (in the usual sense) } \mid O \text{ is open in } \mathbb{R}^d, d \in \mathbb{N}^*\}.$$

1.2. Frölicher spaces.

Definition 1.8. • A **Frölicher space** is a triple $(X, \mathcal{F}, \mathcal{C})$ such that

- \mathcal{C} is a set of paths $\mathbb{R} \rightarrow X$,
- A function $f: X \rightarrow \mathbb{R}$ is in \mathcal{F} if and only if for any $c \in \mathcal{C}$, $f \circ c \in C^\infty(\mathbb{R}, \mathbb{R})$;
- A path $c: \mathbb{R} \rightarrow X$ is in \mathcal{C} (i.e. is a **contour**) if and only if for any $f \in \mathcal{F}$, $f \circ c \in C^\infty(\mathbb{R}, \mathbb{R})$.

• Let $(X, \mathcal{F}, \mathcal{C})$ et $(X', \mathcal{F}', \mathcal{C}')$ be two Frölicher spaces, a map $f: X \rightarrow X'$ is **differentiable** (=smooth) if and only if one of the following equivalent conditions is fulfilled:

- $\mathcal{F}' \circ f \circ \mathcal{C} \subset C^\infty(\mathbb{R}, \mathbb{R})$
- $f \circ \mathcal{C} \subset \mathcal{C}'$

- $\mathcal{F}' \circ f \subset \mathcal{F}$

Any family of maps \mathcal{F}_g from X to \mathbb{R} generate a Frölicher structure $(X, \mathcal{F}, \mathcal{C})$, setting [23]:

- $\mathcal{C} = \{c : \mathbb{R} \rightarrow X \text{ such that } \mathcal{F}_g \circ c \subset C^\infty(\mathbb{R}, \mathbb{R})\}$
- $\mathcal{F} = \{f : X \rightarrow \mathbb{R} \text{ such that } f \circ \mathcal{C} \subset C^\infty(\mathbb{R}, \mathbb{R})\}$.

One easily see that $\mathcal{F}_g \subset \mathcal{F}$. This notion will be useful in the sequel to describe in a simple way a Frölicher structure. A Frölicher space carries a natural topology, which is the pull-back topology of \mathbb{R} via \mathcal{F} . In the case of a finite dimensional differentiable manifold, the underlying topology of the Frölicher structure is the same as the manifold topology. In the infinite dimensional case, these two topologies differ very often.

Let us now compare Frölicher spaces with diffeological spaces, with the following diffeology called "nebulae":

$$\mathcal{P}_\infty(\mathcal{F}) = \coprod_{p \in \mathbb{N}} \{f \text{ p-parametrization on } X; \mathcal{F} \circ f \in C^\infty(O, \mathbb{R}) \text{ (in the usual sense)}\}.$$

With this construction, we get a natural diffeology when X is a Frölicher space. In this case, one can easily show the following:

Proposition 1.9. [29] *Let $(X, \mathcal{F}, \mathcal{C})$ and $(X', \mathcal{F}', \mathcal{C}')$ be two Frölicher spaces. A map $f : X \rightarrow X'$ is smooth in the sense of Frölicher if and only if it is smooth for the underlying nebulae diffeologies.*

Thus, we can also state intuitively:

$$\text{smooth manifold} \Rightarrow \text{Frölicher space} \Rightarrow \text{Diffeological space}$$

With this construction, any complete locally convex topological vector space is a diffeological vector space, that is, a vector space for which addition and scalar multiplication is smooth. The same way, any finite or infinite dimensional manifold X has a nebulae diffeology, which fully determines smooth functions from or with values in X . We now finish the comparison of the notions of diffeological and Frölicher space following mostly [29, 51]:

Theorem 1.10. *Let (X, \mathcal{P}) be a diffeological space. There exists a unique Frölicher structure $(X, \mathcal{F}_\mathcal{P}, \mathcal{C}_\mathcal{P})$ on X such that for any Frölicher structure $(X, \mathcal{F}, \mathcal{C})$ on X , these two equivalent conditions are fulfilled:*

(i) *the canonical inclusion is smooth in the sense of Frölicher $(X, \mathcal{F}_\mathcal{P}, \mathcal{C}_\mathcal{P}) \rightarrow (X, \mathcal{F}, \mathcal{C})$*

(ii) *the canonical inclusion is smooth in the sense of diffeologies $(X, \mathcal{P}) \rightarrow (X, \mathcal{P}_\infty(\mathcal{F}))$.*

Moreover, $\mathcal{F}_\mathcal{P}$ is generated by the family

$$\mathcal{F}_0 = \{f : X \rightarrow \mathbb{R} \text{ smooth for the usual diffeology of } \mathbb{R}\}.$$

Proof. Let $(X, \mathcal{F}, \mathcal{C})$ be a Frölicher structure satisfying (ii). Let $p \in P$ of domain O . $\mathcal{F} \circ p \in C^\infty(O, \mathbb{R})$ in the usual sense. Hence, if $(X, \mathcal{F}_\mathcal{P}, \mathcal{C}_\mathcal{P})$ is the Frölicher structure on X generated by the set of smooth maps $(X, \mathcal{P}) \rightarrow \mathbb{R}$, we have two smooth inclusions

$$(X, \mathcal{P}) \rightarrow (X, \mathcal{P}_\infty(\mathcal{F}_\mathcal{P})) \text{ in the sense of diffeologies}$$

and

$$(X, \mathcal{F}_{\mathcal{P}}, \mathcal{C}_{\mathcal{P}}) \rightarrow (X, \mathcal{F}, \mathcal{C}) \text{ in the sense of Frölicher.}$$

Proposition 1.9 ends the proof. \square

Definition 1.11. [51] A **reflexive** diffeological space is a diffeological space (X, \mathcal{P}) such that $\mathcal{P} = \mathcal{P}_{\infty}(\mathcal{F}_{\mathcal{P}})$.

Theorem 1.12. [51] The category of Frölicher spaces is exactly the category of reflexive diffeological spaces.

This last theorem allows us to make no difference between Frölicher spaces and reflexive diffeological spaces. We shall call them Frölicher spaces, even when working with their underlying diffeologies.

A deeper analysis of these implications has been given in [51]. The next remark is inspired on this work and on [29]; it is based on [23, p.26, Boman's theorem].

Remark 1.13. We notice that the set of contours \mathcal{C} of the Frölicher space $(X, \mathcal{F}, \mathcal{C})$ **does not** give us a diffeology, because a diffeology needs to be stable under restriction of domains. In the case of paths in \mathcal{C} the domain is always \mathbb{R} . However, \mathcal{C} defines a “minimal diffeology” $\mathcal{P}_1(\mathcal{F})$ whose plots are smooth parameterizations which are locally of the type $c \circ g$, where $g \in \mathcal{P}_{\infty}(\mathbb{R})$ and $c \in \mathcal{C}$. Within this setting, a map $f : (X, \mathcal{F}, \mathcal{C}) \rightarrow (X', \mathcal{F}', \mathcal{C}')$ is smooth if and only if it is smooth $(X, \mathcal{P}_{\infty}(\mathcal{F})) \rightarrow (X', \mathcal{P}_{\infty}(\mathcal{F}'))$ or equivalently smooth $(X, \mathcal{P}_1(\mathcal{F})) \rightarrow (X', \mathcal{P}_1(\mathcal{F}'))$

We apply the results on product diffeologies to the case of Frölicher spaces and we derive very easily, (compare with e.g. [23]) the following:

Proposition 1.14. Let $(X, \mathcal{F}, \mathcal{C})$ and $(X', \mathcal{F}', \mathcal{C}')$ be two Frölicher spaces equipped with their natural diffeologies \mathcal{P} and \mathcal{P}' . There is a natural structure of Frölicher space on $X \times X'$ which contours $\mathcal{C} \times \mathcal{C}'$ are the 1-plots of $\mathcal{P} \times \mathcal{P}'$.

We can even state the result above for the case of infinite products; we simply take cartesian products of the plots or of the contours. We also remark that given an algebraic structure, we can define a corresponding compatible diffeological structure. For example, a \mathbb{R} -vector space equipped with a diffeology is called a diffeological vector space if addition and scalar multiplication are smooth (with respect to the canonical diffeology on \mathbb{R}), see [21, 39, 41]. An analogous definition holds for Frölicher vector spaces. Other examples will arise in the rest of the text.

Remark 1.15. Frölicher, c^{∞} and Gateaux smoothness are the same notion if we restrict to a Fréchet context, see [23, Theorem 4.11]. Indeed, for a smooth map $f : (F, \mathcal{P}_1(F)) \rightarrow \mathbb{R}$ defined on a Fréchet space with its 1-dimensional diffeology, we have that $\forall (x, h) \in F^2$, the map $t \mapsto f(x + th)$ is smooth as a classical map in $C^{\infty}(\mathbb{R}, \mathbb{R})$. And hence, it is Gateaux smooth. The converse is obvious.

1.3. Quotient and subsets. We give here only the results that will be used in the sequel.

We have now the tools needed to describe the diffeology on a quotient:

Proposition 1.16. let (X, \mathcal{P}) be a diffeological space and \mathcal{R} an equivalence relation on X . Then, there is a natural diffeology on X/\mathcal{R} , noted by \mathcal{P}/\mathcal{R} , defined as the push-forward diffeology on X/\mathcal{R} by the quotient projection $X \rightarrow X/\mathcal{R}$.

Given a subset $X_0 \subset X$, where X is a Frölicher space or a diffeological space, we can define on subset structure on X_0 , induced by X .

• If X is equipped with a diffeology \mathcal{P} , we can define a diffeology \mathcal{P}_0 on X_0 , called **subset diffeology** [48, 21] setting

$$\mathcal{P}_0 = \{p \in \mathcal{P} \text{ such that the image of } p \text{ is a subset of } X_0\}.$$

Example 1.17. *Let X be a diffeological space. Let us note by*

$$X^\infty = \{(x_n)_{n \in \mathbb{N}} \in X^\mathbb{N} \mid \{n \mid x_n \neq 0\} \text{ is a finite set}\}$$

Then this is a diffeological space, as a subset of $X^\mathbb{N}$.

• If $(X, \mathcal{F}, \mathcal{C})$ is a Frölicher space, we take as a generating set of maps \mathcal{F}_g on X_0 the restrictions of the maps $f \in \mathcal{F}$. In that case, the contours (resp. the induced diffeology) on X_0 are the contours (resp. the plots) on X which image is a subset of X_0 .

1.4. Projective limits and vector pseudo-bundles. Let us now give the description of what happens for projective limits of Frölicher and diffeological spaces.

Proposition 1.18. *Let Λ be an infinite set of indexes which can even be uncountable.*

• *Let $\{(X_\alpha, \mathcal{P}_\alpha)\}_{\alpha \in \Lambda}$ be a family of diffeological spaces indexed by Λ totally ordered for inclusion, with $(i_{\beta, \alpha} : X_\alpha \rightarrow X_\beta)_{(\alpha, \beta) \in \Lambda^2}$ the family of inclusion maps which are assumed smooth maps. If $X = \bigcap_{\alpha \in \Lambda} X_\alpha$, then X carries the **projective diffeology** \mathcal{P} which is the pull-back of the diffeologies \mathcal{P}_α of each X_α via the family of inclusion maps $(f_\alpha : X \rightarrow X_\alpha)_{\alpha \in \Lambda}$. The diffeology \mathcal{P} is made of plots $g : O \rightarrow X$ such that for each $\alpha \in \Lambda$,*

$$f_\alpha \circ g \in \mathcal{P}_\alpha.$$

This is the biggest diffeology for which the maps f_α are smooth.

• *We have the same kind of property for Frölicher spaces: let $\{(X_\alpha, \mathcal{F}_\alpha, \mathcal{C}_\alpha)\}_{\alpha \in \Lambda}$ be a family of Frölicher spaces indexed by Λ , a non-empty set totally ordered for inclusion. With the sae notations, there is a natural structure of Frölicher space on $X = \bigcap_{\alpha \in \Lambda} X_\alpha$, for which the corresponding contours*

$$\mathcal{C} = \bigcap_{\alpha \in \Lambda} \mathcal{C}_\alpha$$

are some 1-plots of $\mathcal{P} = \bigcap_{\alpha \in \Lambda} \mathcal{P}_\alpha$. A generating set of functions for this Frölicher space is the set of maps of the type:

$$\bigcup_{\alpha \in \Lambda} \mathcal{F}_\alpha \circ f_\alpha.$$

Now, following [39], in which the ideas from [48, last section] have been developed to vector spaces. The notion of quantum structure has been introduced in [48] as a generalization of principal bundles, and the notion of vector pseudo-bundle in [39]. The common idea consist in the description of fibered objects made of a total (diffeological) space E , over a diffeological space X and with a canonical smooth projection $\pi : E \rightarrow X$ such as, $\forall x \in X$, $\pi^{-1}(x)$ is endowed with a (smooth) algebraic structure, but for which we do not assume the existence of a system of local trivialization.

- (1) For a diffeological vector pseudo-bundle, the fibers $\pi^{-1}(x)$ are assumed diffeological vector spaces, i.e. vector spaces where addition and multiplication over a diffeological field of scalars (e.g. \mathbb{R} or \mathbb{C}) is smooth. We notice that [39] only deals with finite dimensional vector spaces.
- (2) For a so-called “structure quantique” (i.e. “quantum structure”) following the terminology of [48], a diffeological group G is acting on the right, smoothly and freely on a diffeological space E . The space of orbits $X = E/G$ defines the base of the quantum structure $\pi : E \rightarrow X$, which generalize the notion of principal bundle by not assuming the existence of local trivialization. In this picture, each fiber $\pi^{-1}(x)$ is isomorphic to G .

From these two examples, we can generalize the picture:

Definition 1.19. *Let E and X be two diffeological spaces and let $\pi : E \rightarrow X$ be a smooth surjective map. Then $\pi : E \rightarrow X$ is a **diffeological fiber pseudo-bundle** if and only if π is a subduction.*

By the way, we can give the following definitions:

Definition 1.20. *Let $\pi : E \rightarrow X$ be a diffeological fiber pseudo-bundle. Then:*

- (1) *Let \mathbb{K} be a diffeological field. $\pi : E \rightarrow X$ is a diffeological \mathbb{K} -vector pseudo-bundle if there exists:*

- a smooth fiberwise map $\cdot : \mathbb{K} \times E \rightarrow E$,
- a smooth fiberwise map $+$: $E^{(2)} \rightarrow E$ where

$$E^{(2)} = \coprod_{x \in X} \{(u, v) \in E^2 \mid (u, v) \in \pi^{-1}(x)\}$$

equipped by the pull-back diffeology of the canonical map $E^{(2)} \rightarrow E^2$, such that $\forall x \in X$, $(\pi^{-1}(x), +, \cdot)$ is a diffeological \mathbb{K} -vector bundle.

- (2) $\pi : E \rightarrow X$ is a **diffeological gauge pseudo-bundle** if there exists
 - a smooth fiberwise involutive map $(\cdot)^{-1} : E \rightarrow E$,
 - a smooth fiberwise map $\cdot : E^{(2)} \rightarrow E$*such that $\forall x \in X$, $(\pi^{-1}(x), \cdot)$ is a diffeological group with inverse map $(\cdot)^{-1}$.*
- (3) $\pi : E \rightarrow X$ is a **diffeological principal pseudo-bundle** if there exists a diffeological gauge pseudo-bundle $\pi' : E' \rightarrow X$ such that, considering

$$E \times_X E' = \coprod_{x \in X} \{(u, v) \mid (u, v) \in \pi^{-1}(x) \times \pi'^{-1}(x)\}$$

equipped by the pull-back diffeology of the canonical map $E \times_X E' \rightarrow E \times E'$, there exists a smooth map $E \times_X E' \rightarrow E$ which restricts fiberwise to a smooth free and transitive right-action

$$\pi^{-1}(x) \times \pi'^{-1}(x) \rightarrow \pi^{-1}(x).$$

- (4) $\pi : E \rightarrow X$ is a **Souriau quantum structure** if it is a diffeological principal pseudo-bundle with diffeological gauge (pseudo-)bundle $X \times G \rightarrow X$.

2. TANGENT SPACES, DIFFEOLOGY AND GROUP OF DIFFEOMORPHISMS

There are actually two main definitions, (2) and (3) below, of the tangent space of a diffeological space (X, \mathcal{P}) , while definition (1) of the tangent cone will be used

in the sequel. These definitions are very similar to the definitions of the two tangent spaces in the c^∞ -setting given in [23].

- (1) the **internal tangent cone**. This construction is extended to the category of diffeologies from the definitions in [11] on Frölicher spaces, and very similar to the kinematic tangent space defined in [23]. For each $x \in X$, we consider

$$C_x = \{c \in C^\infty(\mathbb{R}, X) \mid c(0) = x\}$$

and take the equivalence relation \mathcal{R} given by

$$c\mathcal{R}c' \Leftrightarrow \forall f \in C^\infty(X, \mathbb{R}), \partial_t(f \circ c)|_{t=0} = \partial_t(f \circ c')|_{t=0}.$$

The internal tangent cone at x is the quotient

$${}^i T_x X = C_x / \mathcal{R}.$$

If $X = \partial_t c(t)|_{t=0} \in {}^i T_X$, we define the simplified notation

$$Df(X) = \partial_t(f \circ c)|_{t=0}.$$

Under these constructions, we can define the total space of internal tangent cones

$${}^i T X = \coprod_{x \in X} {}^i T_x X$$

with canonical projection $\pi : u \in {}^i T_x X \mapsto x$, and equipped with the diffeology defined by the plots $p : D(p) \subset \mathbb{R}^n \rightarrow {}^i T X$ defined through the plots $p' : (t, z) \in \mathbb{R} \times D(p) \rightarrow X \in \mathcal{P}$ by

$$p(z) = \partial_t p'(t, z)|_{t=0}.$$

With this construction, ${}^i T X$ is a diffeological fiber pseudo-bundle. This construction can be criticized because, depending on the base diffeology \mathcal{P} on X , there can exist “too few” maps $f \in C^\infty(X, \mathbb{R})$ and hence some germs of paths c cannot be separated via evaluations by smooth functions f . However, this approach seems sufficient when dealing with locally convex topological spaces, from a private comment by P. Iglesias-Zemmour based on [23].

- (2) The **internal tangent space** is defined in [18, 8], based on germs of paths defined via colimits in categories.
- (3) the **external** tangent space ${}^e T X$, defined as the set of derivations on $C^\infty(X, \mathbb{R})$. [23, 21].

It is shown in [11] that the internal tangent cone at a point x is not a vector space in many examples. This motivates [8], where a subtle construction intends to get rid of the potential lack of smooth test functions $X \rightarrow \mathbb{R}$. For finite dimensional manifold, definitions (1), (2) and (3) coincide.

However, to our best knowledge, there is still one aspect that is not investigated yet among all possible ways to generalize tangent spaces. For this purpose, we need to recall the following definitions from [21]:

Definition 2.1. *Let (X, \mathcal{P}) and (X', \mathcal{P}') be two diffeological spaces. Let $S \subset C^\infty(X, X')$ be a set of smooth maps. The **functional diffeology** on S is the diffeology \mathcal{P}_S made of plots*

$$\rho : D(\rho) \subset \mathbb{R}^k \rightarrow S$$

such that, for each $p \in \mathcal{P}$, the maps $\Phi_{\rho,p} : (x, y) \in D(p) \times D(\rho) \mapsto \rho(y)(x) \in X'$ are plots of \mathcal{P}' .

With this definition, we have the classical fundamental property for calculus of variations and for composition:

Proposition 2.2. [21] *Let X, Y, Z be diffeological spaces*

(1)

$$C^\infty(X \times Y, Z) = C^\infty(X, C^\infty(Y, Z)) = C^\infty(Y, C^\infty(X, Z))$$

as diffeological spaces equipped with functional diffeologies.

(2) *The composition map*

$$C^\infty(X, Y) \times C^\infty(Y, Z) \rightarrow C^\infty(X, Z)$$

is smooth.

Let us now investigate tangent spaces from the viewpoint of diffeomorphisms. On a finite dimensional manifold M , the Lie algebra of the ILH Lie group of diffeomorphisms [36], defined as the tangent cone at the identity map, is the space of smooth vector fields, i.e. smooth sections of TM . On a non-compact, locally compact manifold, the situation is quite similar [32] while the group of diffeomorphisms is no longer a Fréchet manifold but a Frölicher Lie group. On these groups, the underlying diffeology is the functional diffeology, as well as for the more general definition of $Diff(X)$ when X is a diffeological space [21].

We now get the necessary material to give the following definition. In the rest of this section, $(X, \mathcal{F}, \mathcal{C})$ is a Frölicher space.

Definition 2.3. *We use here the notations that we used before for the definition of the internal tangent cone. Let ${}^d T_x X$ be the subset of ${}^i T_x X$ defined by*

$${}^d T_x X = {}^d C_x / \mathcal{R}$$

with

$${}^d C_x = \{c \in C_x \mid \exists \gamma \in C^\infty(\mathbb{R}, Diff(X)), c(\cdot) = \gamma(\cdot)(x) \text{ and } \gamma(0) = Id_X\}$$

Through this definition, ${}^d T_x X$ is intrinsically linked with the tangent space at the identity ${}^i T_{Id_X} Diff(X)$ described in [28] for any diffeological group (i.e. group equipped with a diffeology which makes composition and inversion smooth), see e.g. [33], as smooth vector space.

Remark 2.4. *Let $\gamma \in C^\infty(\mathbb{R}, Diff(X))$ such that $\gamma(0)(x) = x$. Then $\lambda(x) = (\gamma(0))^{-1} \circ \gamma(\cdot)(x)$ defines a smooth path $\lambda \in {}^d C_x$. By the way,*

$${}^d C_x = \{c \in C_x \mid \exists \gamma \in C^\infty(\mathbb{R}, Diff(X)), c(\cdot) = \gamma(\cdot)(x) \text{ and } \gamma(0) = Id_X\}$$

Definition 2.5. *Let X be a Frölicher space. we define, by*

$${}^d T X = \coprod_{x \in X} {}^d T_x X$$

the diff-tangent bundle of X .

By the way, we can get easily the following observations:

Proposition 2.6. *Let (X, \mathcal{P}) be a reflexive diffeological space, and let \mathcal{P}_{Diff} be the functional diffeology on $Diff(X)$.*

- (1) *There exists a diffeology $\mathcal{P}(Diff) \subset \mathcal{P}$ generated by the family of push-forward diffeologies :*

$$\{(ev_x)_*(\mathcal{P}_{Diff}) \mid x \in X\}.$$

- (2) $\forall x \in X, {}^d T_x X$ *is the internal tangent cone of $(X, \mathcal{P}(Diff))$ at x .*
 (3) $\forall x \in X, {}^d T_x X$ *is a diffeological vector space*
 (4) *The total diff-tangent space*

$${}^d T X = \coprod_{x \in X} {}^d T X \subset {}^i T X$$

is a vector pseudo-bundle for the subset diffeology inherited from ${}^i T X$ and also for the diffeology of internal tangent space of (X, \mathcal{P}_{Diff}) .

Proof. (1) is a consequence of the definition of push-forward diffeologies the following way: the family

$$\{\mathcal{P} \text{ diffeology on } X \mid \forall x \in X, (ev_x)_*(\mathcal{P}_{Diff}) \subset \mathcal{P}\}$$

has a minimal element by Zorn Lemma.

(2) follows from remark 2.4.

(3): The diffeology $\mathcal{P}(Diff)$ coincides with the diffeology made of plots which are locally of the form $ev_x \circ p$, where $x \in X$ and p is a plot of the diffeology of $Diff(X)$. We have that ${}^i T_{Id} Diff(X)$ is a diffeological vector space, following [28]. This relation follows from the differentiation of the multiplication of the group: given two paths γ_1, γ_2 in $C^\infty(\mathbb{R}, Diff(X))$, with $\gamma_1(0) = \gamma_2(0) = Id$, if $X_i = \partial_t \gamma_i(0)$ for $i \in \{1; 2\}$, then

$$X_1 + X_2 = \partial_t(\gamma_1 \cdot \gamma_2)(0).$$

Reading locally plots in $\mathcal{P}(Diff)$, we can consider only plots of the form $ev_x \circ p$, where p is a plot in $Diff(X)$ such that $p(0) = Id_X$. By the way the vector space structure on ${}^d T_x X$ is inherited from ${}^i T_{Id} Diff(X)$ via evaluation maps.

In order to finish to check (3), we prove directly (4) by describing its diffeology.

For this, we consider

$$C_0^\infty(\mathbb{R}, Diff(X)) = \{\gamma \in C_0^\infty(\mathbb{R}, Diff(X)) \mid \gamma(0) = Id_X\}.$$

Let ${}^d C = \coprod_{x \in X} C_x$. The total evaluation map

$$\begin{aligned} ev : X \times C_0^\infty(\mathbb{R}, Diff(X)) &\rightarrow {}^d C \\ (x, \gamma) &\mapsto ev_x \circ \gamma \end{aligned}$$

is fiberwise (over X), and onto. By the way we get a diffeology on ${}^d C$ which is the push-forward diffeology of $X \times C_0^\infty(\mathbb{R}, Diff(X))$ by ev . Passing to the quotient, we get a diffeology on ${}^d T X$ which makes each fiber ${}^d T_x X$ a diffeological vector space trivially. □

Example 2.7. *Let us consider $X \subset \mathbb{R}^2$ defined by*

$$X = \{(x, y) \in \mathbb{R}^2 \mid xy = 0\}.$$

\mathbb{R}^2 *is a Frölicher space (equipped with its nebulae diffeology $\mathcal{P}_\infty(\mathbb{R}^2)$) and X has a subset diffeology made of plots of three types:*

- *plots of the subset diffeology of $X_1 = \{(x, y) \in \mathbb{R}^2 \mid y = 0\}$ which is (diffeologically) isomorphic to $(\mathbb{R}, \mathcal{P}_\infty(\mathbb{R}))$*

- plots of the subset diffeology of $X_2 = \{(x, y) \in \mathbb{R}^2 \mid x = 0\}$ which is (diffeologically) isomorphic to $(\mathbb{R}, \mathcal{P}_\infty(\mathbb{R}))$
- plots which are locally in $\mathcal{P}_\infty(X_1)$ or $\mathcal{P}_\infty(X_2)$, obtained by gluing along $X_1 \cap X_2 = \{(0; 0)\}$ where plots have to be stationary.

Let

$$\mathcal{F}_1 = \{f \in C^\infty(\mathbb{R}^2, \mathbb{R}) \mid \forall(x, y, y') \in \mathbb{R}^3, f(x, y) = f(x, y')\}$$

and let

$$\mathcal{F}_2 = \{f \in C^\infty(\mathbb{R}^2, \mathbb{R}) \mid \forall(x, y, x') \in \mathbb{R}^3, f(x, y) = f(x', y)\}$$

The subset diffeology of X is generated by $\mathcal{F}_1 \cup \mathcal{F}_2$, i.e. p is a plot of this diffeology if and only if $(\mathcal{F}_1 \cup \mathcal{F}_2) \circ p \subset C^\infty(D(p), \mathbb{R})$. So that it is reflexive.

Let us now highlight the internal tangent cone.

- $\forall x \in \mathbb{R}^*$, ${}^i T_{(x,0)} X$ and ${}^i T_{(0,x)} X$ are both diffeologically isomorphic to \mathbb{R}
- The internal tangent cone at the origin ${}^i T_{(0;0)} X \sim \mathbb{R} \cup \mathbb{R} (\subset \mathbb{R}^2 = T_{(0;0)} \mathbb{R}^2)$ is a cone, and which completes to \mathbb{R}^2 along the lines of the work by Christensen and Wu [8].

Let us now consider $g \in \text{Diff}(X)$. G is continuous for the D -topology of X . Let us consider $z = g(0; 0)$. The set $X - \{(0; 0)\}$ has four connected components for the D -topology, so that $g(X - \{(0; 0)\}) = g(X) - \{z\}$ has also four connected components. By the way, $z = (0; 0)$ and hence $(0; 0)$ is a fixed point for any diffeomorphism g of X . This shows that

$${}^d T_{(0;0)} X = \{0\} \neq {}^i T_{(0;0)} X$$

and hence

$${}^d T X \neq {}^i T X$$

while ${}^d T X \subset {}^i T X$.

3. CAUCHY DIFFEOLOGY AND SMOOTH NUMERICAL SCHEMES

3.1. The Cauchy diffeology. Let $X \subset Y$, let $j : X \rightarrow Y$ the canonical inclusion. In what follows, we assume that:

- X is a diffeological space with diffeology \mathcal{P} .
- Y is a (T_2) sequentially complete uniform space [5, Topologie générale, Chapter II].
- $\forall p \in \mathcal{P}$, $j \circ p$ is a continuous map.
- Y is equipped with a diffeology \mathcal{P}_0 .

The diffeology \mathcal{P}_0 is made of plots which can be not continuous for the uniform space topology. This is why we consider the following diffeology:

- Let $\mathcal{F}_0 = \{f \in C^0(Y, \mathbb{R}) \mid \forall p \in \mathcal{P}_0, f \circ p \text{ is smooth}\}$
- Let $\mathcal{C} = \mathcal{C}(\mathcal{F}_0)$ and $\mathcal{F} = \mathcal{F}(\mathcal{C})$.
- Let $\mathcal{P}' = \{p \in \mathcal{P}_\infty(\mathcal{F}) \mid p \text{ is continuous}\}$.

Notice that if $C^0(Y, \mathbb{R})$ determines the topology of Y by pull-back of the topology of \mathbb{R} , the last point is not necessary, and $\mathcal{P}' = \mathcal{P}(\mathcal{F})$.

Definition 3.1. We note by $\mathcal{C}(X, Y)$ the subspace of $X^\mathbb{N}$ made of sequence $(x_n)_{n \in \mathbb{N}}$ such that $(j(x_n))_{n \in \mathbb{N}}$ is Cauchy in Y .

When $(X, \mathcal{P}) = (Y, \mathcal{P}_0)$ as diffeological spaces, we shall use the notation $\mathcal{C}(X)$ instead of $\mathcal{C}(X, X)$. There exists two naive "natural" diffeologies on $\mathcal{C}(X)$:

- As an infinite product, $X^{\mathbb{N}}$ carries a natural diffeology $\mathcal{P}_{\mathbb{N}}$, and by the inclusion $\mathcal{C}(X) \subset X^{\mathbb{N}}$, the set $\mathcal{C}(X)$ can be endowed with the subset diffeology.
- If X is complete, $\mathcal{C}(X)$ can be equipped with the pull-back diffeology

$$\left(\lim_{n \rightarrow +\infty} \right)^* (\mathcal{P}').$$

The first one is natural from the view point of approximation theory: an approximation scheme of $x \in X$ generates a sequence $(x_n)_{n \in \mathbb{N}}$ which converges to x and x_n is considered as a “reasonable” approximation of x when the index n is “large enough”. The second one is natural for the viewpoint of the limit: Cauchy sequences are usually constructed in order to get convergence in complete spaces. So that, it is natural also to assume that the limit of a smooth plot on $\mathcal{C}(X)$ generates a smooth plot on X . If X is not complete for the considered uniform space, we assume that \bar{X} , the completion of X , is equipped with a metric and is a diffeological space, with D -topology stronger than the metric topology. In order to define the most useful diffeology on $\mathcal{C}(X)$ we need to gather these two crucial aspects: smoothness of the limit and smoothness of evaluation map $ev_{=} : (x_n) \mapsto x_n$ (for $n \in \mathbb{N}$). The comparison of the two diffeologies is not very fruitful straightway, as it is shown in next proposition:

Proposition 3.2. *If there exists two distinct points x and y in X which are connected by a smooth path $\gamma : [0; 1] \rightarrow X$, then*

$$\lim_{n \rightarrow +\infty} : (\mathcal{C}(X), \mathcal{P}_{\mathbb{N}}) \rightarrow (X, \mathcal{P})$$

is not smooth.

Proof. Let \mathbf{x} be the constant Cauchy sequence which converges to x , and let \mathbf{y} be the constant Cauchy sequence which converges to y . Let us define a path $\Gamma : [0; 1] \rightarrow \mathcal{C}(X)$ by the relations, for $n \geq 0$:

$$(1) \quad \Gamma(t)_n = \begin{cases} x & \text{if } t < \frac{1}{n+1} \\ \gamma\left((n+2)\left(t - 1 + \frac{1}{n+1}\right)\right) & \text{if } t \in \left[1 - \frac{1}{n+1}; 1 - \frac{1}{n+2}\right] \\ y & \text{if } t > 1 - \frac{1}{n+2} \end{cases}$$

This path is smooth for $\mathcal{P}_{\mathbb{N}}$, and

$$\begin{cases} \lim_{n \rightarrow +\infty} \Gamma(0) & = x \\ \forall t, \lim_{n \rightarrow +\infty} \Gamma(t) & = y \end{cases}$$

On this path, the map $\lim_{n \rightarrow +\infty}$ is not smooth. \square

We illustrate also that the two diffeologies are very different by the following example, already given in [14]:

Example 3.3. *Let us equip $\mathbb{Q} \subset \mathbb{R}$ with the subset diffeology of \mathbb{R} . This is the discrete diffeology. Thus the diffeology $\mathcal{P}_{\mathbb{N}}$ on $\mathcal{C}(\mathbb{Q})$ is the discrete diffeology, and hence the map $\lim_{n \rightarrow +\infty} : \mathcal{C}(\mathbb{Q}) \rightarrow \mathbb{R}$ is smooth. However, $\mathcal{P}_{\mathbb{N}} \neq (\lim_{n \rightarrow +\infty})^* \mathcal{P}_{\infty}(\mathbb{R})$.*

Since we have motivated before the use of the two diffeologies, and since both diffeologies cannot be compared in an efficient way, the following definition becomes natural:

Definition 3.4. Let X and Y as above. We define the **Cauchy diffeology** on $\mathcal{C}(X, Y)$, noted \mathcal{P}_C , as

$$\mathcal{P}_C = \mathcal{P}_{\mathbb{N}} \cap \left(\lim_{n \rightarrow +\infty} \right)^* (\mathcal{P}),$$

where \lim is the Y -limit and $\mathcal{P}_{\mathbb{N}}$ is the subset diffeology in $\mathcal{C}(X, Y)$ inherited from $X^{\mathbb{N}}$.

From example 3.3 we see that the ambient space X needs to have “enough” plots to make the Cauchy diffeology interesting. There exist already a non-trivial example of Cauchy diffeology in the literature, in the setting of an ultrametric completion:

Example 3.5. Diffeologies compatible with a valuation and ultrametric completion. [14] Let (X, val) be a \mathbb{K} -algebra with valuation. assuming that \mathbb{K} is equipped with the discrete diffeology, let \mathcal{P} be a diffeology on X such that

- (1) addition and multiplication $A \times A \rightarrow A$ are smooth,
- (2) scalar multiplication $\mathbb{K} \times A \rightarrow A$ is smooth
- (3) Let $p \in \mathbb{Z}$ and let $X_p = \{x \in X \mid \text{val}(x) \geq p\}$. Let us equip X/X_p by the quotient diffeology and let us equip the ultrametric completion \hat{X} with the pull-back diffeology of the projection maps $\pi_p : \hat{X} \rightarrow X/X_p$ for $p \in \mathbb{Z}$. Then the diffeology on X is the pull-back of diffeology on \hat{X} through the canonical inclusion $X \rightarrow \hat{X}$.

Then, the map $\lim_{n \rightarrow +\infty} : \mathcal{C}(X) \rightarrow \hat{X}$ is smooth [14, Theorem 2.1], in other words, the Cauchy diffeology coincides with the subset diffeology of $\mathcal{C}(X) \subset X^{\mathbb{N}}$, i.e. $\mathcal{P}_C = \mathcal{P}_{\mathbb{N}}$.

Let us give another example, where the Cauchy diffeology is more complex.

Example 3.6. Cauchy diffeology on $\mathbb{P}(X)$. Let X be a compact metric space, equipped with its Borel σ -algebra. The space of probabilities on X , $\mathbb{P}(X)$ is a convex subspace of the space of measures on X and the set of extremals (the “border”) of $\mathbb{P}(X)$ is the set of Dirac measures $\delta(X) = \{\delta_x; x \in X\}$. The Monte Carlo method is based on isobarycentres of Dirac measures, and any probability measure $\mu \in \mathbb{P}(X)$ is a limit of isobarycentres of Dirac measures for the vague topology, in other words,

$$\forall \mu \in \mathbb{P}(X), \exists (x_n) \in X^{\mathbb{N}}, \forall f \in C^0(X, \mathbb{R}), \int_X f d\mu = \lim_{n \rightarrow +\infty} \frac{1}{n+1} \sum_{k=0}^n f(x_k).$$

Conversely, each sequence of Dirac measures generate this way a probability measure. Equipped with the Prokhorov metric, $\mathbb{P}(X)$ is a compact metric space [43]. If X is a diffeological space, the family $C^\infty(X, \mathbb{R})$ generates a structure of Frölicher space, and hence we can define the diffeology \mathcal{P}' on $\mathbb{P}(X)$ following the notations of the section. Thus, we get a Cauchy diffeology on $\mathcal{C}(\delta(X), \mathbb{P}(X))$. This diffeology contains smooth paths for the diffeology $\mathcal{P}_{\mathbb{N}}$

$$\gamma : [0; 1] \rightarrow (x_0(t), \dots, x_n(t), \dots)$$

with values in sequences that are stationnary after a fixed rank N . Thus the Cauchy diffeology is not the discrete diffeology. Moreover, this diffeology does contain, also, paths that are not targeted in sequences stationnary after finite rank, such as, when $U = [0; 1]$, $t \mapsto (x_n(t))$ with

$$x_n(t) = \frac{t}{n+1}.$$

This example gives a first motivation for the following theorem:

Proposition 3.7. *Let (X, \mathcal{P}_X) and (Y, \mathcal{P}_Y) be a diffeological vector spaces, such that $X \subset Y$ with smooth inclusion, and such that Y is equipped with a translation-invariant metric which generates a topology τ which is weaker than its D -topology. Assume also that X is dense in Y for τ . Then $(\mathcal{C}(X), \mathcal{P}_{\mathcal{C}})$ is a diffeological vector space.*

Proof. Since $\mathcal{C}(X)$ is a vector space, it is a diffeological vector space for $\mathcal{P}_{\mathbb{N}}$. Since $\lim_{n \rightarrow +\infty}$ is a linear map, $(\lim_{n \rightarrow +\infty})^*(\mathcal{P}_Y)$ is also a diffeology of diffeological vector space on $\mathcal{C}(X)$. Thus, so is $\mathcal{P}_{\mathcal{C}}$. \square

3.2. A theory of smooth numerical schemes. We omit in this section to precise the diffeologies under consideration when these are obvious ones: nebulae (reflexive) diffeologies for Fréchet spaces of smooth manifolds, of arbitrary, fixed diffeologies for diffeological spaces.

Let X, Y be Fréchet spaces, let Z be a locally convex topological vector space, and let Q be a diffeological space of parameters.

- Assume that $X \subset Y$ with smooth and dense inclusion
- Let us consider the space of Cauchy sequences $\mathcal{C}(X, Y)$ that are Cauchy sequences in X with respect to the uniform structure on Y .

Definition 3.8. *A **smooth functional equation** is defined by a smooth map $F : X \times Q \rightarrow Z$ and by the condition*

$$(2) \quad F(u, q) = 0$$

A Y -**smooth numerical scheme** is a smooth map

$$Num_F(Y) : Q \rightarrow \mathcal{C}(X, Y)$$

such that, if $(x_n)_{n \in \mathbb{N}} \in Num_F(Y)(q)$ for $q \in Q$,

$$\lim_{n \rightarrow +\infty} F(x_n) = 0.$$

We call the image space

$$\mathcal{S}_Y(F) = \left\{ \lim_{n \rightarrow +\infty} x_n \in C^\infty(Q, Y) \mid (x_n)_{n \in \mathbb{N}} \in Num_F(Y) \right\}$$

the space of solutions of (2) with respect to $Num_F(Y)$.

If the solution is unique in Y , so called well-posedness of the solution with respect to the set of parameters Q is ensured by the obtention of smooth numerical schemes. The difference that we formalize here between the base space X which serves as a domain for the functions $F(\cdot, q)$, with $q \in Q$, and the space Y where we will take place what we can call weak solutions in Y , is a classical feature in solving functional equations. One may think that this differentiation is due to methods that are not efficient enough to solve the equation. This can be true in some cases. In some other approaches one can get a solution which is a priori in Y , but which is proved to be in X by a refined analysis. Let us suggest by the following example (which could be considered as a “toy example”, regardless to [6] and references there in) that the necessity of Y relies directly on the lack of relatively compact neighborhoods in the topology of X .

Example 3.9. Let $X = C^\infty(S^1, \mathbb{R})$, let $Z = \mathbb{R}$ and let $\Delta = -\frac{d^2}{dx^2}$ be the (positive) Laplacian on S^1 . Let us consider the heat operator $e^{-t\Delta}$ which is a smoothing operator for $t > 0$ and which converges weakly to Id_{L^2} when $t \rightarrow 0^+$. Let $w \in L^2(S^1, \mathbb{R})$. We consider the following equation:

$$F(u, t, w) = 0$$

with $u \in X$, $Q = \mathbb{R}_+ \times L^2(S^1, \mathbb{R})$ and

$$(3) \quad F : X \times Q \rightarrow \mathbb{R}$$

$$(4) \quad (u, (t, w)) \mapsto \frac{1}{2\pi} \int_{S^1} ((e^{-t\Delta}u) - w)^2$$

$w \in L^2(S^1, \mathbb{R})$ This equation has an unique solution $u = e^{t\Delta}w$ only when $w \in \{e^{-t\Delta}v \mid v \in L^2(S^1, \mathbb{R})\}$ but one can implement a numerical scheme even when w has not enough regularity. For example, since F is a convex functional, one may consider a gradient method as a priori adapted.

$$D_u F(v) = \frac{1}{2\pi} \left(\int_{S^1} (e^{-2t\Delta}u)v - 2 \int_{S^1} (e^{-t\Delta}w)v \right),$$

and then taking the L^2 -gradient

$$\nabla F(u) = e^{-2t\Delta}u - 2e^{-t\Delta}w$$

which leads to define a sequence (u_n) , fixing $u_0 \in X$, setting by induction, for $n \in \mathbb{N}$ such that u_n has been constructed, the term $u_{n+1} = u_n - \gamma \nabla F(u_n)$ is obtained by the constant γ such that $F(u_n) - \gamma \|\nabla F(u_n)\|_{L^2} = 0$. For “bad” choices of u_0 , of the parameter w and for $t > 0$, convergence of the sequence (u_n) is a priori accomplished neither for convergence in X nor for weak L^2 -convergence, while the sequence remains in the set defined by:

$$U = \{u \in X \mid F(u) \leq F(u_0)\},$$

and requires an adapted space Y for which U is bounded and with compact closure.

Since smooth dependence on the parameters is ensured, a notion of (smooth) symmetries can be derived by extension of the classical notions of symmetries.

One can equip $\mathcal{S}_Y(F)$ with one of the following diffeologies which appear natural to us:

- The push-forward diffeology $\mathcal{P}^{(1)} = \lim_* \mathcal{P}_C$ which can differ from the subset diffeology inherited from Y .
- $\mathcal{P}^{(2)}$, the Frölicher completion of $\mathcal{P}^{(1)}$
- $\mathcal{P}^{(3)} = \mathcal{P}^{(2)}(Diff)$, the diff-diffeology of $\mathcal{P}^{(2)}$.

Definition 3.10. The set of Y -symmetries, noted by $Sym(F, X, Y)$ is defined as the set of smooth maps

$$\Phi : \mathcal{S}_Y(F) \rightarrow \mathcal{S}_Y(F)$$

which have a smooth inverse.

In other words, $Sym(F, X, Y) = Diff(\mathcal{S}_Y(F))$. We can apply the settings described in section 2.

Proposition 3.11. $Sym(F, X, Y)$ is a diffeological group.

Definition 3.12. *The set of (X, Y) -weak symmetries, noted by $WSym(F, X, Y)$, is defined as the set of smooth maps*

$$\Phi : Num_F(Y) \rightarrow Num_F(Y)$$

which have a smooth inverse.

In other words, $WSym(F, X, Y) = Diff(Num_Y(F))$.

Proposition 3.13. *$WSym(F, X, Y)$ is a diffeological group.*

One can define the same way infinitesimal symmetries and weak infinitesimal symmetries taking the tangent space at identity of the groups $Sym(F, X, Y)$ and $WSym(F, X, Y)$. These spaces can be viewed as sets of smooth sections of ${}^dTS_Y(F)$ and ${}^dTWSym(F, X, Y)$.

Example 3.14. Open problem on an example: paradoxal solutions of perfect fluid dynamics

Let $n \geq 2$. Let us consider the perfect fluid equations on a compact interval $I =]0; T[$ (at finite time), with variables $u \in C^\infty(I \times \mathbb{R}^n, \mathbb{R}^n)$ (the velocity) and $p \in C^\infty(I \times \mathbb{R}^n, \mathbb{R})$, and with (fixed) external force f :

$$(5) \quad \begin{cases} \frac{\partial u}{\partial t} - \nabla_v v + \nabla p & = f \\ \operatorname{div}(u) & = 0 \end{cases}$$

Since Scheffer and Shnirelman [45, 46, 47] it is known that these equations have surprising weak solutions, compactly supported in space and time. Advances in this problem have been recently performed in series of papers initiated by [25, 26, 27], see also [49] for an overview of the two first references. Let us focus with the following central theorem, see e.g. [49, Theorem 1.4]:

Theorem 3.15. *Let Ω be an open subset of \mathbb{R}^n , let e be an uniformly continuous function $]0; T[\times \Omega \rightarrow \mathbb{R}_+^*$ with $e \in L^\infty(]0; T[; L^1(\Omega))$. $\forall \eta > 0$, there exists a weak solution (u, p) of (5) with $f = 0$ such that:*

- (1) $u \in C^0(]0; T[, L_w^2(\mathbb{R}))^n$ (the subscript “w” refers to the weak topology)
- (2) $u(t, x) = 0$ if $(t, x) \in]0; T[\times \Omega$, in particular, $\forall x \in \mathbb{R}^n$, $u(0, x) = u(T, x) = 0$.
- (3) $\frac{|u(t, x)|^2}{2} = \frac{n}{2} p(t, x) = e(t, x)$ almost everywhere on Ω and for all $t \in]0; T[$
- (4) $\sup_{t \in I} \|u(t, \cdot)\|_{H^{-1}(\Omega)} < \eta$.
- (5) there exists a sequence (u_n, p_n) such that (u_n, p_n) converges to (u, p) for strong $L^2(dt, dx)$ -convergence, where each (u_m, p_m) is a (classical) solution of (5) with force f_m , and such that (f_n) converges to 0 in the sense of distributions.

Actually, the sequence f_m can be obtained from convex integration, the sequence (u_m, p_m) is a Cauchy sequence of (smooth) classical solutions of (5), which is L^2 -Cauchy, and a natural question is the following:

Problem *The control function e and the sequence (f_m) appear as parameters to define the Cauchy sequence $(u_m, p_m)_{m \in \mathbb{N}}$. One can wonder if there is a diffeology on the parameter set $Q = \{(e, (f_m))\} \subset L^\infty(]0; T[; L^1(\Omega)) \times \mathcal{CD}(]0; T[\times \Omega]$ (different from the discrete diffeology!) such that the map $(e, (f_m)) \mapsto (u_m, p_m) \in$*

$\mathcal{C}(\mathcal{D}([0; T[\times\Omega)^{n+1})$ is smooth. under these conditions, the description of (projectable) symmetries of the set of weak solutions becomes an important open question, while the infinitesimal symmetries [35] and the projectable symmetries [44] of the set of (non-weak) solutions are well-known.

This opens the question of the diffeologies involved in convex integration and in the h-principle.

4. IMPLICIT FUNCTIONS THEOREM FROM THE VIEWPOINT OF NUMERICAL SCHEMES

We set the following notations, following [36]: Let $\mathbf{E} = (E_i)_{i \in \mathbb{N}}$ and $\mathbf{F} = (F_i)_{i \in \mathbb{N}}$ be two ILB vector spaces, i.e. $\forall i > j, E_i \subset E_j$ and $F_i \subset F_j$, with smooth inclusion and density, and set $E_\infty = \bigcap_{i \in \mathbb{N}} E_i$, $F_\infty = \bigcap_{i \in \mathbb{N}} F_i$ with projective limit topology. Let O_0 be an open neighborhood of $(0; 0)$ in $E_0 \times F_0$, let $\mathbf{O} = (O_i)_{i \in I}$ with $O_i = O_0 \cap (E_i \times F_i)$, for $i \in \mathbb{N} \cup \{\infty\}$. Let us now have a flash-back on classical methods in getting an implicit functions theorem in this setting, under the light of numerical schemes and trying to omit additional norm estimates which are essential in topological approaches of this classical proof. Each of these aspects will be discussed in detailed comments and remarks after the statement and the proof of the following implicit functions theorem.

Theorem 4.1. *Let*

$$f_i : O_i \rightarrow F_i, \quad i \in \mathbb{N} \cup \{\infty\}$$

be a collection of smooth maps satisfying the following condition:

$$i > j \Rightarrow f_j|_{O_i} = f_i$$

and such that,

$$\forall i, f_i(0; 0) = 0; \quad D_2 f_i(0; 0) = Id_{F_i}.$$

Then, there exists a domain $D \subset O_\infty$ and an unique function $u : D \rightarrow F_\infty$ such that:

- $\forall x \in D, f_\infty(x, u(x)) = 0$
- *For the subset diffeology of $D \subset E_\infty, u : D \rightarrow F_\infty$ is smooth .*

Proof. Let $g_i = Id_{F_i} - f_i$, for $i \in \mathbb{N} \cup \{\infty\}$. With this condition,

$$D_2 g_i(0; 0) = 0.$$

We note by $\phi_{i,x} = g_i(x, \cdot)$. Let $D'_i \subset E_i$ be the set defined by the following:

$$x \in D'_i \Leftrightarrow \begin{cases} \forall n \in \mathbb{N}, \phi_{i,x}^n(0) \in O_i \\ (\phi_{i,x}^n(0))_{n \in \mathbb{N}} \in \mathcal{C}(O_i, O_i) \end{cases}$$

We define, for $x \in D'_i$,

$$u_i(x) = \lim_{n \rightarrow +\infty} \phi_{i,x}^n(0) \in \bar{O}_i.$$

For each $x \in D'_i$, if $u_i(x) \in O_i$, we get $g_i(x, u_i(x)) = 0$ and hence $f_i(x, u_i(x)) = 0$. The family of maps $\{g_i; i \in \mathbb{N}\}$ restricts to a map g on O_∞ , and hence each map $\phi_{i,x}$ restricts the same way to a map ϕ_x . We set

$$D' = \bigcap_{i \in \mathbb{N}} (D'_i \cap u_i^{-1}(O_i))$$

and define $D \subset D'$ the maximal domain which contains 0 such that the map

$$x \mapsto (\phi_x^n(0))_{n \in \mathbb{N}} \in \bigcap_{i \in \mathbb{N}} \mathcal{C}(O_i, O_i)$$

is smooth for the projective limit diffeology. Then, the map u which is the restriction of and map u_i to D is a smooth map and,

$$\forall x \in D, f(x, u(x)) = 0.$$

Uniqueness follows from the classical implicit function theorem applied to f_0 . \square

After this proof, adapted from the classical proof of the implicit functions theorem in Banach spaces, see e.g. [38], we must notice several points:

- (1) Here, the uniqueness of the function u is not stated. Moreover, the domain D is not precised. Each of these two points, for each index $i \in \mathbb{N}$ in [38] are treated assuming a contractibility property on g_i , which enables to show both that the domain D'_i contains an open neighborhood U_i of 0, and that there exists V_i a neighborhood of 0 in F_i such that $(x, u_i(x))$ is the only value in $\{x\} \times V_i$ such that $f_i(x, u_i(x)) = 0$, which is the uniqueness of the map. Another way to give the same argument would state that the implicit function theorem on Banach spaces proves uniqueness on O_0 , so that it proves the uniqueness for and $O_i, i \in \mathbb{N} \cup \{\infty\}$. But the domain for uniqueness of the solution on O_∞ is a subset of D .
- (2) Passing to the projective limit, the question of the nature of the domain D can be adressed. As an infinite intersection of open sets, it cannot be stated that D is an open subset of E_∞ . We could add the remark that, skeptically, D can be restricted to $\{(0; 0)\}$. Thinking with an intuition based on finite dimensional problems, one could think that a domain D which is not a neighborhood of 0 is not interesting, because all norms are equivalent. This is the motivation for Nash-Moser estimates [17], with which one manages to show that the domain D is an open neighborhood of 0. This comes from the natural desire to control the "size" of the domain D . But noticing that $E_1 \subset E_0$ is not an open subset of E_0 unless $E_1 = E_0$, one can see that D is not open is not disqualifying. We can here consider D as the maximal domain on which such a proof applies. We already know that the function u will be unique and smooth in the spirit of [23] who consider smoothness in non open domains. Only the nature of D is unknown, which is the object of additional estimates.
- (3) Since Nash-Moser estimates appears rather artificial even if very useful, and only designed to "pass to the projective limit" some open domain, and noticing that the condition of contraction enables to bound the difference $\|\phi_{i,x}^{n+1}(0) - \phi_{i,x}^n(0)\|_{F_i}$ by a converging geometric sequence, one can try to generalize this procedure and apply the notion of bornology. This gives the theorem stated in [19]. But Theorem 4.1 deals with a wider class of functions. Here again, in [19] the domain is controlled, non as an open subset but as a bounded set of a bornology (which can be very small).
- (4) Summarizing these two facts and the detailed proof given before, one gets smoothness of the solution u of the equation $f(x, u(x)) = 0$ and its uniqueness on $D \cap O_0$. This relates implicit functions theorem to well-posedness,

i.e. the problem of unique existence of a unique solution smoothly dependent on the set of parameters $Q = D \cap O_0$. The same proof, applied to the functions f_i on the domain O_∞ , produce weak solutions $u_i(x) \in F_i$, depending smoothly on the parameter $x \in D$, to the equation $f_\infty(x, u(x)) = 0$.

- (5) Pushing forward this last point, again applying the classical implicit functions theorem on Banach spaces [38], one gets implicit functions $u_i : D_i \rightarrow F_i$ where D_i is a neighborhood of 0 in O_i . Restricting to $Q_i = D_i \cap O_\infty$, one can say that Q_i is a set of parameters for solving the equation $F(u, q) = 0$ with $F(u, x) = f_\infty(x, u(x))$. Each function u_i appears as the unique element of $\mathcal{S}_{F_i}(F)$, with admissible set of parameters Q_i . The set D appears as the set of admissible parameters at each index $i \in \mathbb{N}$, i.e. $D = \bigcap_{i \in \mathbb{N}} Q_i$.

The same way, we can state the corresponding Frobenius theorem:

Theorem 4.2. *Let*

$$f_i : O_i \rightarrow L(E_i, F_i), \quad i \in -\mathbb{N}$$

be a collection of smooth maps satisfying the following condition:

$$i > j \Rightarrow f_j|_{O_i} = f_i$$

and such that,

$$\begin{aligned} \forall (x, y) \in O_i, \forall a, b \in E_i \\ (D_1 f_i(x, y)(a)(b) + (D_2 f_i(x, y))(f_i(x, y)(a))(b) = \\ (D_1 f_i(x, y)(b)(a) + (D_2 f_i(x, y))(f_i(x, y)(b))(a). \end{aligned}$$

Then, $\forall (x_0, y_0) \in O_\infty$, there exists diffeological subspace space D of O_∞ that contains (x_0, y_0) and a smooth map $J : D \rightarrow F_\infty$ such that

$$\forall (x, y) \in D, \quad D_1 J(x, y) = f_i(x, J(x, y))$$

and, if $D_{x_0} = \{(x_0, y) \in D\}$,

$$J_i(x_0, \cdot) = Id_{D_{x_0}}.$$

Proof. We consider

$$G_i = C_0^1([0, 1], F_i) = \{\gamma \in C^1([0, 1], F_i) | \gamma(0) = 0\}$$

and

$$H_i = C^0([0, 1], F_i),$$

endowed with their usual topologies, and nebulae underlying diffeology. Obviously, if $i < j$, the injections $G_j \subset G_i$ and $H_j \subset H_i$ are smooth.

Let us consider open subsets $B_0 \subset E_0$ and $B'_0 \subset F_0$, such that

$$(x_0, y_0) \subset B_0 \times B'_0 \subset O_0.$$

Let us consider the open set

$$B''_0 = \{\gamma \in G_0 | \forall t \in [0, 1], \gamma(t) \in B'_0\}.$$

We set $B_i = B_0 \cap E_i$, $B'_i = B'_0 \cap F_i$ and $B''_i = B''_0 \cap G_i$. Then, we define, for $i \in \mathbb{N} \cup \{\infty\}$,

$$\begin{aligned} g_i : B_i \times B'_i \times B''_i \rightarrow H_i \\ g(x, y, \gamma)(t) = \frac{d\gamma}{dt}(t) - f_i(t(x - x_0) + x_0, y + \gamma(t)) \cdot (x - x_0). \end{aligned}$$

In order to apply the last implicit function theorem, we must calculate, avoiding the subscripts i for easier reading:

$$D_3g(x_0, y_0, 0)(\delta) = \frac{d\delta}{dt}.$$

Thus, we can apply the implicit function theorem to

$$f = \left(\int_0^{\cdot} \right) \circ g.$$

We can define the function α as the unique function on the domain $D \subset B_\infty \times B'_\infty$ such that

$$\begin{cases} \alpha(x_0, y_0) = 0 \\ g_\infty(x, y, \alpha(x, y)) = 0, \quad \forall (x, y) \in D \end{cases}.$$

We set $J(x, y) = y + \alpha(x, y)(1)$. \square

5. THE FRÖLICHER SPACE OF TRIANGULATIONS AND THE FINITE ELEMENTS METHOD

Let us consider the 2-dimensional Dirichlet problem. Let Ω be a bounded connected open subset of \mathbb{R}^2 , and assume that the border $\partial\Omega = \bar{\Omega} - \Omega$ is a polygonal curve. The Dirichlet problem is given by the following PDE:

$$\begin{cases} \Delta u = f \\ u|_{\partial\Omega} = 0 \end{cases}$$

where $f \in \mathcal{D}(\Omega, \mathbb{R})$, Δ is the Laplacian, and u is the solution of the Dirichlet problem.

Let us analyze the set of triangulation as a smooth set of parameters for the finite element method, with as an example the Dirichlet problem.

5.1. Outlook of the finite elements method (degree 1) for the Dirichlet problem. One classical way to solve the problem is to approximate u by a sequence $(u_n)_{n \in \mathbb{N}}$ in the Sobolev space $H_0^1(\Omega, \mathbb{R})$ which converges to u for the H_0^1 -convergence. For this, based on a triangulation τ_0 with 0-vertices $(s_k^0)_{k \in K_0}$, where K_0 is an adequate set of indexes, and we consider the H_0^1 -orthogonal family $(\delta_{s_k^{(0)}})_{k \in K_0}$ of continuous, piecewise affine maps on each interior domain of triangulation, defined by

$$\delta_{s_k^{(0)}}(s_j^{(0)}) = \delta_{j,k} \text{ (Kronecker symbol).}$$

With this setting, u_0 is a linear combination of $(\delta_{s_k^{(0)}})_{k \in K_0}$ such that

$$\forall k \in K_0, \left(\Delta u_0, \delta_{s_k^{(0)}} \right)_{H^{-1} \times H_0^1} = \left(f, \delta_{s_k^{(0)}} \right)_{L_2}.$$

This is a finite dimensional linear equation, which can be solved by inversion of a $|K_0|$ -dimensional matrix. Then we refine the triangulation τ_0 adding the centers

of 1-vertices to make the triangulation τ_1 , and by induction, we get a sequence of triangulation $(\tau_n)_{n \in \mathbb{N}}$, and by the way a sequence of families

$$\left(\left(\delta_{s_k}^{(n)} \right)_{k \in K_n} \right)_{n \in \mathbb{N}},$$

which determines the sequence $(u_n)_{n \in \mathbb{N}}$ which converges to u . Then through operator analysis, we know that u is smooth and that there exists a smooth inverse to the Lapacian Δ^{-1} adapted to the Dirichlet problem such that $u = \Delta^{-1}f$. For the rest of the section, we equip $H_0^1(\Omega, \mathbb{R})$ by the Frölicher structure generated by

$$\left\{ (\cdot, f)_{H_0^1} \mid f \in C_c^\infty(\Omega, \mathbb{R}) \right\}.$$

5.2. Differential geometry of the space of triangulations. Let us now fully develop an approach based on the remarks given in [31]. For this, the space of triangulations of Ω is considered itself as a Frölicher space, and the mesh of triangulations which makes the finite element method converge will take place, as the function f , among the set of parameters Q . We describe here step by step the Frölicher structure on the space of triangulations. By the way, we begin with a lemma which is adapted from so-called gluing results present in [34, 40, 42] to the context which is of interest for us.

Lemma 5.1. *Let us assume that X is a topological space, and that there is a collection $\{(X_i, \mathcal{F}_i, \mathcal{C}_i)\}_{i \in I}$ of Frölicher spaces, together with continuous maps $\phi_i : X_i \rightarrow X$. Then we can define a Frölicher structures on X setting*

$$\mathcal{F}_{I,0} = \{f \in C^0(X, \mathbb{R}) \mid \forall i \in I, f \circ \phi_i \circ \mathcal{C}_i \subset C^\infty(\mathbb{R}, \mathbb{R})\},$$

we define \mathcal{C}_I the contours generated by the family $\mathcal{F}_{I,0}$, and $\mathcal{F}_I = \mathcal{F}(\mathcal{C}_I)$.

Let M be a smooth manifold for dimension n . Let

$$(6) \quad \Delta_n = \{(x_0, \dots, x_n) \in \mathbb{R}_+^{n+1} \mid x_0 + \dots + x_n = 1\}$$

be the standard n -simplex, equipped with its subset diffeology. It is easy to show that this diffeology is reflexive through Boman's theorem already mentioned, and hence we can call it Frölicher space $(\Delta_n, \mathcal{F}_{\Delta_n}, \mathcal{C}_{\Delta_n})$, and we note its associated reflexive diffeology by $\mathcal{P}(\Delta_n)$.

Definition 5.2. *A **smooth triangulation** of M is a family $\tau = (\tau_i)_{i \in I}$ where $I \subset \mathbb{N}$ is a set of indexes, finite or infinite, each τ_i is a smooth map $\Delta_n \rightarrow M$, and such that:*

- (1) $\forall i \in I, \tau_i$ is a (smooth) embedding, i.e. a smooth injective map such that $(\tau_i)_*(\mathcal{P}(F_{\Delta_n}))$ is also the subset diffeology of $\tau_i(\Delta_n)$ as a subset of M .
- (2) $\bigcup_{i \in I} \tau_i(\Delta_n) = M$. (covering)
- (3) $\forall (i, j) \in I^2, \tau_i(\Delta_n) \cap \tau_j(\Delta_n) \subset \tau_i(\partial \Delta_n) \cap \tau_j(\partial \Delta_n)$. (intersection along the borders)
- (4) $\forall (i, j) \in I^2$ such that $D_{i,j} = \tau_i(\Delta_n) \cap \tau_j(\Delta_n) \neq \emptyset$, for each $(n-1)$ -face F of $D_{i,j}$, the "transition maps" " $\tau_j^{-1} \circ \tau_i' : \tau_i^{-1}(F) \rightarrow \tau_j^{-1}(F)$ are affine maps.

Under these conditions, we equip the triangulated manifold (M, τ) with a Frölicher structure $(\mathcal{F}_I, \mathcal{C}_I)$, generated by the smooth maps τ_i applying Lemma 5.1. The following result is obtained from the construction of \mathcal{F} and \mathcal{C} :

Theorem 5.3. *The inclusion $(M, \mathcal{F}, \mathcal{C}) \rightarrow M$ is smooth.*

Proof. Here the manifold M is considered as a reflexive diffeological space equipped with its reflexive diffeology $\mathcal{P}_\infty(M)$ and with its associated Frölicher structure $(\mathcal{F}(\mathcal{P}_\infty(M)), \mathcal{C}(\mathcal{P}_\infty(M)))$. Let $f \in C^\infty((M, \mathcal{P}_\infty(M)), \mathbb{R})$, i.e. a (classical) smooth map $f \in C^\infty(M, \mathbb{R})$. Since each τ_i is a smooth map $\Delta_n \rightarrow M$,

$$f \circ \tau_i \circ C \in C^\infty(\mathbb{R}, \mathbb{R})$$

and hence

$$C^\infty((M, \mathcal{P}_\infty(M)), \mathbb{R}) \subset \mathcal{F}_{I,0} \subset \mathcal{F}_I.$$

□

Remark 5.4. Maps in \mathcal{F}_I can be intuitively identified as some piecewise smooth maps $M \rightarrow \mathbb{R}$, which are of class C^0 along the 1-skeleton of the triangulation. We have proved also that $\mathcal{C}_I \subset \mathcal{P}_\infty(M)$. Some characteristic elements of \mathcal{C}_I can be understood as paths which are smooth (in the classical sense) on the interiors of the domains of the simplexes of the triangulation, and that fulfill some more restrictive conditions while crossing the 1-skeleton of the triangulation. For example, paths that are (locally) stationnary at the 1-skeleton are in \mathcal{C}_I .

Remark 5.5. While trying to define a Frölicher structure from a triangulation, one could also consider

$$\mathcal{C}_{I,0} = \{ \gamma \in C^0(\mathbb{R}, M) \mid \forall i \in I, \forall f \in C_c^\infty(\phi_i(\Delta_n), \mathbb{R}), f \circ \gamma \in C^\infty(\mathbb{R}, \mathbb{R}) \}$$

where $C_c^\infty(\phi_i(\Delta_n), \mathbb{R})$ stands for compactly supported smooth functions $M \rightarrow \mathbb{R}$ with support in $\phi_i(\Delta_n)$. Then define

$$\mathcal{F}'_I = \{ f : M \rightarrow \mathbb{R} \mid f \circ \mathcal{C}_{I,0} \in C^\infty(\mathbb{R}, \mathbb{R}) \}$$

and

$$\mathcal{C}'_I = \{ C : \mathbb{R} \rightarrow M \mid \mathcal{F}'_I \circ C \in C^\infty(\mathbb{R}, \mathbb{R}) \}.$$

We get here another construction, but which does not understand as smooth maps $M \rightarrow \mathbb{R}$ the maps δ_k already mentionned.

Now, let us fix the set of indexes I and fix a so-called **model triangulation** τ . We note by \mathcal{T}_τ the set of triangulations τ' of M such that the corresponding 1-skeletons are diffeomorphic to the 1-skeleton of τ (in the Frölicher category).

Definition 5.6. Since $\mathcal{T}_\tau \subset C^\infty(\Delta_n, M)^I$, we can equip \mathcal{T}_τ with the subset Frölicher structure, in other words, the Frölicher structure on \mathcal{T}_τ whose generating family of contours \mathcal{C} are the contours in $C^\infty(\Delta_n, M)^I$ which lie in \mathcal{T}_τ .

We define the full space of triangulations \mathcal{T} as the disjoint union of the spaces of the type \mathcal{T}_τ , with disjoint union Frölicher structure. With this notation, in the sequel and when it carries no ambiguity, the triangulations in \mathcal{T}_τ is equipped with a fixed set of indexes I (which is impossible to fix for \mathcal{T}). We need now to describe the procedure which intends to refine the triangulation and define a sequence of triangulations $(\tau_n)_{n \in \mathbb{N}}$. We can now consider the refinement operator, which is the operator which divides a simplex Δ_n into a triangulation.

Definition 5.7. Let $m \in \mathbb{N}$, with $m \geq 3$. Let

$$\mu = \{ \mu_i : \Delta_n \rightarrow \Delta_n \mid i \in \mathbb{N}_m \}$$

be a smooth triangulation of Δ_n such that the 0-vertices of μ in $\partial\Delta_n$ are stable under re-indexation of the coordinates of Δ_n . Let $\tau \in \mathcal{T}$. Then we define

$$\mu(\tau) = \{ f \circ \mu_i \mid i \in \mathbb{N}_m \text{ and } f \in \tau \}.$$

With this definition, $\mu(\tau)$ is trivially a triangulation of M if τ is a triangulation of M . The conditions imposed in the definition ensures that the refinement map maps an triangulation to another triangulation, that is, if τ is a triangulation, $\mu(\tau)$ is also a triangulation. The delicate needed condition is that the new 0-vertices added to τ in $\mu(\tau)$ are matching.

Theorem 5.8. *The map $\mu : \mathcal{T} \rightarrow \mathcal{T}$ is smooth.*

Proof. Composition map

$$C^\infty(\Delta_n, \Delta_n) \times C^\infty(\Delta_n, M) \rightarrow C^\infty(\Delta_n, M)$$

is smooth, so that it extends canonically (coefficientwise) to a smooth map

$$\Phi : C^\infty(\Delta_n, \Delta_n)^{\mathbb{N}_n} \times C^\infty(\Delta_n, M)^I \rightarrow C^\infty(\Delta_n, M)^{\mathbb{N}_m \times I}.$$

Let us fix $\mu \in C^\infty(\Delta_n, \Delta_n)^{\mathbb{N}_n}$ a smooth triangulation of Δ_n , for a fixed model triangulation $\tau = \{\tau_i\}_{i \in I}$ the map μ is a restriction $\mathcal{T}_\tau \rightarrow \mathcal{T}_{\mu(\tau)}$ of the map $\Phi(\mu, \cdot)$. So that, $\mu \in C^\infty(\mathcal{T}_\tau, \mathcal{T}_{\mu(\tau)})$ and extending this result to \mathcal{T} as a disjoint union, we get $\mu \in C^\infty(\mathcal{T}, \mathcal{T})$. \square

Definition 5.9. *Let $\tau \in \mathcal{T}$. We define the μ -refined sequence of triangulations $\mu^{\mathbb{N}}(\tau) = (\tau_n)_{n \in \mathbb{N}}$ by*

$$\begin{cases} \tau_0 & = \tau \\ \tau_{n+1} & = \mu(\tau_n) \end{cases}$$

Proposition 5.10. *The map*

$$\mu^{\mathbb{N}} : \mathcal{T} \rightarrow \mathcal{T}^{\mathbb{N}}$$

is smooth (with $\mathcal{T}^{\mathbb{N}}$ equipped with the infinite product Frölicher structure).

In the case of the Dirichlet problem, we consider a subspace of \mathcal{T}_τ .

Proof. It follows from smoothness of $\mu : \mathcal{T} \rightarrow \mathcal{T}$. \square

Lemma 5.11. *Let us fix an indexation of the 0-vertices of Δ_n . Let $\tau = (\tau_i)_{i \in I} \in \text{mathcal{T}}$ and let $(i, j) \in I \times \mathbb{N}_{n+1}$. Let $x_j(\tau_i) \in M$ be the image by τ_i of the j -th 0-vertex of Δ_n . Then for fixed indexes i and j , the map $\tau \in \mathcal{T} \mapsto x_j(\tau_i) \in M$ is smooth.*

Proof. It follows from the smoothness of evaluation maps. \square

Let Ω be a bounded connected open subset of \mathbb{R}^n , and assume that the border $\partial\Omega = \bar{\Omega} - \Omega$ is a polygonal curve. Since \mathbb{R}^n is a vector space, we can consider the space of affine triangulations:

$$\text{Aff}\mathcal{T}_\tau = \{\tau' \in \mathcal{T}_\tau \mid \forall i, \tau'_i \text{ is (the restriction to } \Delta_n \text{ of) an affine map}\}.$$

We define $\text{Aff}\mathcal{T}$ from $\text{Aff}\mathcal{T}_\tau$ the same way we defined \mathcal{T} from \mathcal{T}_τ , via disjoint union. We equip $\text{Aff}(\mathcal{T}_\tau)$ and $\text{Aff}(\mathcal{T})$ with their subset diffeology. We use here the notations of last Lemma.

Theorem 5.12. *Let*

$$c : \mathbb{R} \rightarrow \text{Aff}(\mathcal{T}_\tau)$$

be a path on $\text{Aff}(\mathcal{T}_\tau)$. Then

$$c \text{ is smooth} \Leftrightarrow \forall (i, j) \in I \times \mathbb{N}_{n+1}, t \mapsto x_j(c(t)_i) \text{ is smooth.}$$

Proof. Let $x \in \Delta_n$. We consider the normalized barycentric coordinates

$$(\alpha_1(x), \dots, \alpha_{n+1}(x))$$

which correspond to the coordinates of $x \in R^{n+1}$. when one use the identification (6). The map

$$x \in \Delta_n \mapsto (\alpha_1(x), \dots, \alpha_{n+1}(x))$$

is smooth. Let $c : \mathbb{R} \rightarrow \mathcal{T}$ such that, $\forall (i, j) \in I \times \mathbb{N}_n$, the maps $t \mapsto x_j(c(t)_i)$ are smooth. We fix $i \in I$ and define a smooth plot $p \in \mathcal{P}(\Delta_n)$. Then the map $(\alpha_1 \circ p, \dots, \alpha_{n+1} \circ p)$ is smooth and since τ_i is affine,

$$\tau_i \circ p = \sum_{j=0}^{n+1} (\alpha_j \circ p) \cdot x_j.$$

We replace τ_i by $c(t)_i$ in this formula, made of smooth operations, which shows that the maps $t \mapsto c(t)_i$ are smooth for the diffeology defined in Definition 5.6 applying Proposition 2.2. \square

Proposition 5.13. *Let μ be a fixed affine triangulation of Δ_n . The map $\mu^{\mathbb{N}}$ restricts to a smooth map from the set of affine triangulations of Ω to se set of sequences of affine triangulations of Ω .*

Proof. Follows from Proposition 5.10. \square

5.3. Back to the Dirichlet problem. With a sequence of affine triangulations $(\tau_n)_{n \in \mathbb{N}}$ defined as before on a suitable domain Ω of \mathbb{R}^n , we wish to establish smoothness of the family of maps δ defined before with respect to the underlying triangulation. For this, we extend first the family of H_0^1 -functions δ to \mathcal{T} .

Definition 5.14. *Let $\tau \in \mathcal{T}$, indexed by the set I . Let a be a 0-vertex of τ We note by $St(a)$ the domain described as*

$$\cup \{Im(\tau_i) \mid i \in I \text{ and } a \in Im(\tau_i)\}.$$

Let us define the following maps:

- for $(i, j) \in I \times \mathbb{N}_{n+1}$, let $\delta_{i,j}^\tau : \Omega \rightarrow \mathbb{R}$ be the map defined by

$$\delta_{i,j}^\tau(x) = \begin{cases} 0 & \text{if } x \notin Im(\tau_i) \\ \alpha_j(\tau_i^{-1}(x)) & \text{if } x \in Im(\tau_i) \end{cases}$$

- Let $\{x_k\}_{k \in K}$ be the set of 0-vertices in Ω of the triangulation τ , indexed by K . Let δ_{x_k} be the map defined by

$$\delta_{x_k}^\tau(x) = \begin{cases} 0 & \text{if } x \notin St(x_k) \\ \delta_{i,j}^\tau(x) & \text{if } x \in Im(\tau_i) \cap St(x_k) \text{ and } x_k = x_j(\tau_i) \end{cases}$$

We remark that this definition is consistent by condition (4) of Definition 5.2, which ensures that “gluing along the borders” is possible, that is, $\forall ((i, j), (k, l)) \in (I \times \mathbb{N}_{n+1})^2$, if $x_k = x_j(\tau_i) = x_l(\tau_k)$, for $x \in Im(\tau_i) \cap Im(\tau_k)$,

$$\delta_{i,j}^\tau(x) = \delta_{k,l}^\tau(x).$$

With the previous notations, we have:

Lemma 5.15. $\forall \tau \in \mathcal{T}, \forall k \in K, \delta_{x_k}^\tau \in H_0^1 \cap C^0(\Omega)$.

Proof. The map $\delta_{x_k}^\tau$ is:

- smooth on each interior of domain $Im(\tau_i)$

- C^0 in Ω

So that, it is a continuous map, piecewise smooth. \square

By the way we define a map

$$\delta : \mathcal{T}_\tau \rightarrow (H_0^1 \cap C^0(\Omega))^I$$

which extends, if I is finite and if (τ_n) is a μ -refined sequence, to a map

$$\mu^{\mathbb{N}}(\mathcal{T}_\tau) \rightarrow \left((H_0^1 \cap C^0(\Omega))^\infty \right)^{\mathbb{N}}$$

or, if $I = \mathbb{N}$, to a map

$$\mu^{\mathbb{N}}(\mathcal{T}_\tau) \rightarrow \left((H_0^1 \cap C^0(\Omega))^{\mathbb{N}} \right)^{\mathbb{N}}.$$

Theorem 5.16. *Let $\tau \in \mathcal{T}$. The map*

$$\delta : \mathcal{T}_\tau \rightarrow (H_0^1 \cap C^0(\Omega))^I$$

is smooth.

Proof. Let us fix $k \in K$ and $f \in C_c^\infty(\Omega, \mathbb{R})$. Let $h = \Delta f \in C_c^\infty(\Omega, \mathbb{R})$. Let p be a plot in the nebulae diffeology of Frölicher structure on \mathcal{T}_τ . Let $\beta : D(p) \rightarrow H_0^1(\Omega, \mathbb{R})$ be the map defined by,

$$\forall x \in D(p), \quad \gamma(x) = \delta_{x_k}^{p(x)} \in H_0^1(\Omega, \mathbb{R}).$$

Let $i \in I$. We define $h_{i,x} : \Delta_n \rightarrow \mathbb{R}$ by

$$h_{i,x} = h \circ \gamma(x)_i.$$

Then

$$\begin{aligned} (\delta_{x_k}^{p(x)}, f)_{H_0^1} &= (\delta_{x_k}^{p(x)}, h)_{L^2} \\ &= \sum \int_{\Delta_n} \alpha_j(y) h_{i,x}(y) |J(\tau_i(y))| dy \end{aligned}$$

where, in this last equation, the sum Σ is among the indexed in i which correspond to $St(x_k)$, y is such that $\tau_i(y) = x$ and $J(\tau_i(y))$ is the Jacobian determinant. Let $c : \mathbb{R} \rightarrow D(p)$ be a smooth path. In order to prove the theorem, via Boman theorem already cited, it is sufficient to prove that $t \mapsto (\delta_{x_k}^{p \circ c(t)}, f)_{H_0^1}$ is smooth for each smooth path c . We have that

$$t \mapsto h_{i,c(t)}$$

is smooth in $C^\infty(\Delta_n, \mathbb{R})$ and by the way,

$$t \mapsto \int_{\Delta_n} \alpha_j(y) h_{i,x}(y) |J(\tau_i(y))| dy$$

is smooth. \square

Now, let us fix μ a triangulation of Δ_n , consider the μ -refinement scheme in \mathcal{T} which introduces, for each $\tau \in \mathcal{T}$, a sequence τ_n , and a family of functions δ^{τ_n} . For fixed index n , we solve the problem

$$(\Delta u, \delta_{x_k}^{\tau_n})_{H^{-1} \times H_0^1} = (f, \delta_{x_k}^{\tau_n})_{H^{-1} \times H_0^1}$$

in the vector space spanned by the family of functions $\delta_{x_k}^{\tau_n}$, where each x_k is a 0-vertex of τ_n . If K is the set of indexes k , we get $A^{\tau_n} \in M_{K \times K}(\mathcal{C})$ which is invertible, defined by

$$A_{k,l}^{\tau_n} = (\Delta \delta_{x_k}^{\tau_n}, \delta_{x_l}^{\tau_n})_{H^{-1} \times H_0^1},$$

$v \in \mathbb{C}^K$ defined by

$$(f, \delta_{x_k}^{\tau_n})_{H^{-1} \times H_0^1}$$

and define

$$u_n = (A^{\tau_n})^{-1} v.$$

Theorem 5.17. *The map $(\tau'_0, f) \in \mathcal{T} \times C^\infty(\Omega, \mathbb{R}) \mapsto (u_n)_{n \in \mathbb{N}}$ is a smooth H_0^1 -numerical scheme for the Dirichlet problem.*

Proof. This follows from the construction of v , A^{τ_n} and from the smoothness of inversion of matrices. Smoothness of the limit is already known through classical analysis of the Dirichlet problem. \square

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