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On the Thomas–Fermi approximation of the ground state in a \mathcal{PT} -symmetric confining potential

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Abstract

For the stationary Gross–Pitaevskii equation with harmonic real and linear imaginary potentials in the space of one dimension, we study the ground state in the limit of large densities (large chemical potentials), where the solution degenerates into a compact Thomas–Fermi approximation. We prove that the Thomas–Fermi approximation can be constructed by using the unstable manifold theorem for a planar dynamical system. To justify the Thomas–Fermi approximation, the existence problem can be reduced to the Painlevé-II equation, which admits a unique global Hastings–McLeod solution. We illustrate numerically that an iterative approach to solving the existence problem converges but give no analytical proof of this result. Generalizations are discussed for the stationary Gross–Pitaevskii equation with harmonic real and localized imaginary potentials.

1 Introduction

Ground states of the repulsive Bose–Einstein condensates placed in a harmonic (magnetic) confinement are global minimizers of the Gross–Pitaevskii energy [2]. For the large–density atomic gas, these ground states are well approximated by a compact function, which is referred to as the Thomas–Fermi approximation. The Thomas–Fermi approximation was rigorously justified using calculus of variations [8].

It was also discovered in several independent studies [3, 4, 12] that the nearly compact Thomas–Fermi approximation of the ground state has a super-exponential spatial decay outside a transitional layer, where the ground state satisfies the Painlevé-II equation [7]. A particular solution of the Painlevé-II equation referred to as the Hastings–McLeod solution [9, 10] reconstructs the Thomas–Fermi approximation by means of a change of dependent and independent variables [2, Section 1.2.3].

Justification of the Painlevé-II equation in the context of the radially symmetrical ground states was developed in our previous work [6], where it was shown that the Painlevé-II equation remains valid uniformly on the spatial scale if the confining potential is purely harmonic. These results were further used for several purposes. Expansions of energy for the Thomas–Fermi approximation were studied in [5]. Excited states of the stationary Gross–Pitaevskii equation were constructed with the method of Lyapunov–Schmidt reductions in [14]. More general non-radial trapping potentials were included by Karali and Sourdis [11], where the Painlevé-II equation does not hold uniformly on the spatial scale but is valid nevertheless in a transitional layer.

Very recently, ground states of the repulsive Bose–Einstein condensates were considered under the presence of a complex-valued potential, which expresses a certain balance between losses and gains occurring in the atomic gases. This potential is symmetric with respect to the simultaneous parity (\mathcal{P}) and time-reversal (\mathcal{T}) transformations, hence it is referred to as the \mathcal{PT} -symmetric potential. Thomas–Fermi approximations in a localized \mathcal{PT} -symmetric potential added to the harmonic potential were numerically considered in [1]. Ground and excited states in a linear \mathcal{PT} -symmetric potential added to the harmonic potential were numerically constructed in [16]. In both works, it was discovered that the existence of the ground state may be fragile in the presence of the \mathcal{PT} -symmetric potential, because the branch of the ground state may disappear due to the coalescence with the branch of the first excited state. Precise predictions on where this saddle-node bifurcation occurs and whether the ground state can be extended to the Thomas–Fermi (large-density) limit were not detailed in these works.

In the present work, we study existence of the ground state in the Thomas–Fermi limit for the Gross–Pitaevskii equation with a \mathcal{PT} -symmetric potential. Because no variational principle can be formulated for the \mathcal{PT} -symmetric potential, existence of ground states can not be established using calculus of variations. We use again the transformation of the stationary Gross–Pitaevskii equation to the Painlevé-II equation with the Hastings-McLeod solution. Persistence of the Hastings-McLeod solution is analyzed analytically and numerically with an iterative approach.

Our starting point is the stationary Gross–Pitaevskii equation with the \mathcal{PT} -symmetric harmonic potential

$$\mu U(X) = (-\partial_X^2 + X^2 + 2i\alpha X + |U(X)|^2) U(X), \quad X \in \mathbb{R}, \quad (1.1)$$

where $\mu \in \mathbb{R}$ is the chemical potential, $\alpha \in \mathbb{R}$ is the gain–loss coefficient, and $U : \mathbb{R} \rightarrow \mathbb{C}$ is the wave function for the steady state. The spectrum of the linearized operator

$$L_0 := -\partial_X^2 + X^2 + 2i\alpha X = -\partial_Z^2 + (Z + i\alpha)^2 + \alpha^2$$

coincides with the spectrum of the operator $\tilde{L}_0 = -\partial_Z^2 + Z^2 + \alpha^2$. Therefore, the spectrum of L_0 is purely discrete, real, and bounded from below. The ground state of the stationary Gross–Pitaevskii equation (1.1) bifurcates from the smallest eigenvalue $\mu_0 = 1 + \alpha^2$ of the operator L_0 and exists for $\mu \geq \mu_0$. This local bifurcation of the ground state (as well as excited states) was formally considered by Zezyulin and Konotop [16].

As the Thomas–Fermi approximation is derived in the large-density limit $\mu \rightarrow \infty$, we introduce the change of variables $\mu = \varepsilon^{-1}$ and $U(X) = \varepsilon^{-1/2} u(\varepsilon^{1/2} X)$ with small positive ε . The stationary Gross–Pitaevskii equation (1.1) is now written in the form

$$\left(1 - x^2 - 2i\alpha\varepsilon^{1/2}x - |u(x)|^2\right) u(x) = -\varepsilon^2 u''(x), \quad x \in \mathbb{R}, \quad (1.2)$$

where $x = \varepsilon^{1/2} X$. To incorporate the complex phase of u produced by the gain-loss term, we use the polar form $u = \varphi e^{i\theta}$ for the ground state solutions with $|u(x)| > 0$ for all $x \in \mathbb{R}$. Splitting the scalar equation (1.2) for real and imaginary parts, we obtain the system

$$\begin{cases} (1 - x^2 - \varphi^2(x) - \varepsilon^2(\theta')^2(x)) \varphi(x) = -\varepsilon^2 \varphi''(x), \\ (\varphi^2 \theta')'(x) = 2\alpha \varepsilon^{-3/2} x \varphi^2(x), \end{cases} \quad x \in \mathbb{R}. \quad (1.3)$$

As we are looking for spatially decaying solutions with $\varphi^2 \theta'(x) \rightarrow 0$ as $|x| \rightarrow \infty$, it is clear that the following constraint must be satisfied:

$$\int_{\mathbb{R}} x \varphi^2(x) dx = 0. \quad (1.4)$$

In particular, if φ^2 and θ' are even in x , the constraint is satisfied, and the parity requirement implies that the stationary solution is \mathcal{PT} -symmetric with $u(-x) = \bar{u}(x)e^{i\theta_0}$, where $\theta_0 \in \mathbb{R}$. Note in passing that the question whether \mathcal{PT} -symmetric equations may admit non- \mathcal{PT} -symmetric spatially decaying solutions is open, recent results in this direction were obtained by Yang [15] with perturbation techniques.

Using the scaled variable $\xi(x) := \varepsilon\theta'(x)$ and scaled parameter $\alpha = \varepsilon^{1/2}\eta$, we obtain the final form of the existence problem:

$$\begin{cases} (1 - x^2 - \varphi^2(x) - \xi^2(x)) \varphi(x) = -\varepsilon^2 \varphi''(x), \\ (\varphi^2 \xi)'(x) = 2\eta x \varphi^2(x), \end{cases} \quad x \in \mathbb{R}. \quad (1.5)$$

The existence problem has two parameters η and ε and we are looking for the ground state with even and strictly positive φ in the limit of small ε . The other parameter η can be either ε -independent or ε -dependent and we shall later specify the conditions on this parameter to ensure that the ground state exists in the limit of small ε .

When $\eta = 0$, we can choose $\xi \equiv 0$ and the existence problem (1.5) reduces to the stationary Gross–Pitaevskii equation, studied in our previous work [6]. In the most general case, we can solve the second equation of system (1.5) uniquely from the boundary condition $\lim_{x \rightarrow \pm\infty} \varphi^2(x)\xi(x) = 0$. In this way, we obtain the integral representation

$$\xi(x) = \frac{2\eta}{\varphi^2(x)} \int_{-\infty}^x s \varphi^2(s) ds, \quad (1.6)$$

which allows us to close the first equation of system (1.5) as an integro–differential equation.

The formal Thomas–Fermi limit corresponds to the solution of the truncated problem

$$\begin{cases} 1 - x^2 - \varphi^2(x) - \xi^2(x) = 0, \\ (\varphi^2 \xi)'(x) = 2\eta x \varphi^2(x), \end{cases} \quad x \in [-1, 1], \quad (1.7)$$

subject to the boundary conditions $\varphi(\pm 1) = \xi(\pm 1) = 0$. In the following theorem, we state the existence of suitable solutions to the limiting problem (1.7) for a sufficiently small but ε -independent η . Since the component ξ is uniquely determined by (1.6), we set

$$\xi(x) = \frac{2\eta}{\varphi^2(x)} \int_{-1}^x s \varphi^2(s) ds, \quad x \in (-1, 1), \quad (1.8)$$

and state the result in terms of φ only.

Theorem 1.1 *There exists $\eta_0 > 0$ such that for any $|\eta| < \eta_0$, the truncated existence problem (1.7)–(1.8) admits a unique solution $\varphi_{\text{TF}} \in C^\infty(-1, 1)$ such that $\varphi_{\text{TF}}(x) > 0$ for all $x \in (-1, 1)$ and*

$$\varphi_{\text{TF}}^2(x) = 1 - x^2 + \mathcal{O}((1 - x^2)^2) \quad \text{as } |x| \rightarrow 1. \quad (1.9)$$

Figure 1 illustrates components φ (left) and ξ (right) of the Thomas–Fermi solution in Theorem 1.1 for three different values of η . The numerical solution is obtained with the fourth-order Runge–Kutta method applied to the closed first-order differential equation for variable ξ , after the variable φ^2 is eliminated from the system (1.7). The solution terminates at $\eta_0 \approx 0.93$ because the derivative of ξ diverges near $x = 0$. The numerical approximations illustrate the statement of Theorem 1.1 that the Thomas–Fermi approximation exists only for $|\eta| < \eta_0$, where the value of η_0 is finite.

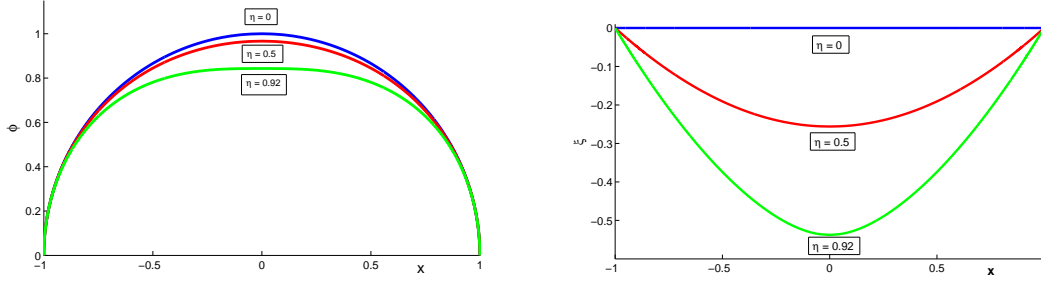


Figure 1: Components φ (left) and ξ (right) for the numerical solution to the limiting problem (1.7) for three different values of η .

Remark 1.2 *From Theorem 1.1 and numerical illustrations, we can see that the Thomas–Fermi radius ($|x| = 1$ in this particular case) is independent of the gain-loss parameter η and that the \mathcal{PT} -symmetric linear potential leads to the decrease of the ground state amplitude φ near the center of the harmonic potential ($x = 0$). These two facts appear to be universal for spatially decaying \mathcal{PT} -symmetric potentials.*

The limit leading to the Painlevé-II equation appears after the formal change of dependent and independent variables near the Thomas–Fermi radius ($x = 1$):

$$\varphi(x) = \varepsilon^{1/3}\nu(y), \quad \xi(x) = \varepsilon^{2/3}\chi(y), \quad y = \frac{1-x^2}{\varepsilon^{2/3}}. \quad (1.10)$$

The new variables satisfy the modified existence problem:

$$\begin{cases} 4\nu''(y) + y\nu(y) - \nu^3(y) = \varepsilon^{2/3} (4y\nu''(y) + 2\nu'(y) + \chi^2(y)\nu(y)), \\ (\nu^2\chi)'(y) = -\eta\nu^2(y), \end{cases} \quad y \in (-\infty, \varepsilon^{-2/3}), \quad (1.11)$$

subject to the decay condition $\nu(y) \rightarrow 0$ as $y \rightarrow -\infty$. The truncated version of the first equation in system (1.11) is the Painlevé-II equation

$$4\nu''(y) + y\nu(y) - \nu^3(y) = 0, \quad y \in \mathbb{R}, \quad (1.12)$$

which admits a unique solution ν_0 [9] satisfying the following asymptotic behavior [7]

$$\nu_0(y) = \begin{cases} y^{1/2} - \frac{1}{2}y^{-5/2} + \mathcal{O}(y^{-11/2}) & \text{as } y \rightarrow +\infty, \\ \pi^{-1/2}|y|^{-1/4}e^{-\frac{1}{3}|y|^{3/2}}(1 + \mathcal{O}(|y|^{-3/4})) & \text{as } y \rightarrow -\infty. \end{cases} \quad (1.13)$$

This function ν_0 is referred to as the Hastings–McLeod solution of the Painlevé-II equation (1.12). Moreover, the asymptotic expansion as $y \rightarrow +\infty$ in (1.13) can be differentiated term by term.

The persistence of the Hastings–McLeod solution ν_0 with respect to small perturbation terms in ε needs to be considered within the modified existence problem (1.11). For technical reasons, it is easier to control the perturbation terms if η is ε -dependent. Even with this simplification,

we obtain a partial progress towards the proof of persistence of ν_0 . Since the component χ is uniquely determined by integrating the second equation of system (1.11), we set again

$$\chi(y) = -\frac{\eta}{\nu^2(y)} \int_{-\infty}^y \nu^2(s) ds \quad (1.14)$$

and state the desired result in terms of ν only.

Conjecture 1.3 *Let ν_0 be the Hastings–McLeod solution of the Painlevé-II equation, defined by (1.12) and (1.13). For any $q > \frac{5}{6}$, there exist $\varepsilon_q > 0$, $\eta_q > 0$, and $C_q > 0$ such that for every $\varepsilon \in (0, \varepsilon_q)$ and $|\eta| < \eta_q \varepsilon^q$, the coupled system (1.11) admits a unique solution $\nu_P \in C^\infty(-\infty, \varepsilon^{-2/3})$ such that $\nu_P(y) > 0$ for all $y \in (-\infty, \varepsilon^{-2/3})$, $\nu_P(y) \rightarrow 0$ as $y \rightarrow -\infty$, and*

$$\sup_{y \in (-\infty, \varepsilon^{-2/3})} |\nu_P(y) - \nu_0(y)| \leq C_q \begin{cases} \varepsilon^{2q-4/3} |\log(\varepsilon)|^{1/2}, & q \leq 1, \\ \varepsilon^{2/3}, & q > 1. \end{cases} \quad (1.15)$$

Remark 1.4 *It follows from the bound (1.15) that for every $x \in (-1, 1)$, we have*

$$\varepsilon^{2/3} \nu_P^2(y) \rightarrow 1 - x^2 \quad \text{as } \varepsilon \rightarrow 0, \quad (1.16)$$

where $y = \frac{1-x^2}{\varepsilon^{2/3}}$.

The bound (1.15) in Conjecture 1.3 is found from the rigorous analysis of the solution of the persistence problem if a suitable bounded function χ is substituted in the first equation of the system (1.11). For this reduced problem, we can prove existence of the solution for the component ν satisfying the bound (1.15) (see Theorem 4.1 below). When this solution for the component ν is used in the integral equation (1.14), we can also fully characterize properties of the component χ (see Lemmas 5.1 and 5.2 below). By alternating solutions of these two uncoupled problems, we can develop a simple iterative method, which approximates numerically solutions of the coupled system (1.11). Although this numerical method is found to converge extremely fast, we still lack nice Lipschitz properties of the integral equation (1.14) in order to achieve a rigorous proof of the statement in Conjecture 1.3.

Figure 2 illustrates components ν (left) and χ (right) of the numerical solution of the coupled system (1.11) for $\varepsilon = 0.0067$ and three different values of η . The numerical solution is obtained with the iterative method described above. Because the component ν is close to the Hastings–McLeod solution ν_0 , the difference between the three cases of η is not visible on the left panel of the figure. The convergence of the numerical method is lost for $\eta \approx \varepsilon^{0.15}$, which may signal that no solution of the coupled system (1.11) exists for such large values of η .

This paper is organized as follows. Section 2 gives the proof of Theorem 1.1 on the existence of the compact Thomas–Fermi approximation. Section 3 describes properties of decaying solutions of the system (1.5) closed with the integral equation (1.6). Section 4 gives the details of how a solution of the persistence problem can be obtained if a suitable bounded function χ is substituted in the first equation of the system (1.11) without computing it from the integral equation (1.14). Section 5 is devoted to the study of the integral equation (1.14). Section 6 illustrates numerically the persistence of the Hastings–McLeod solution beyond the Painlevé-II equation in Conjecture 1.3. Section 7 discusses generalizations of our results to spatially decaying \mathcal{PT} -symmetric potentials superposed with the harmonic confining potential.

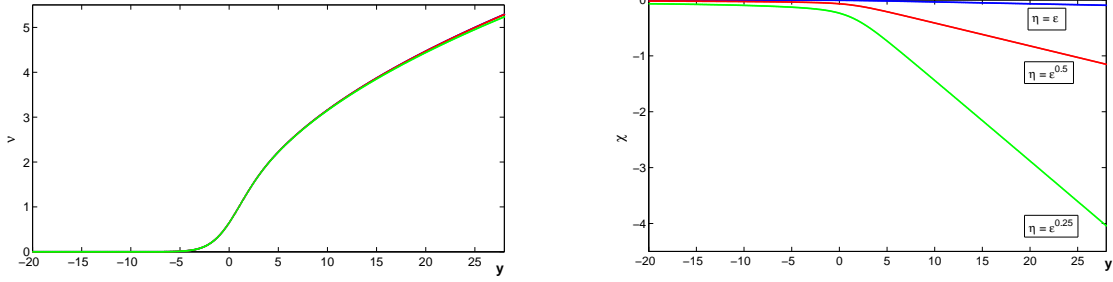


Figure 2: Components ν (left) and χ (right) for the numerical solution to the coupled system (1.11) with $\varepsilon = 0.0067$ and three different values of η .

2 Proof of Theorem 1.1

First, we rewrite the truncated problem (1.7) in terms of the variable $z := 1 - x^2$ and introduce a new function $\omega(z) := \varphi^2(x)$. Hence the truncated problem (1.7) is rewritten as

$$\begin{cases} \omega(z) = z - \xi^2(z), \\ (\omega\xi)'(z) = -\eta\omega(z), \end{cases} \quad z \in [0, 1], \quad (2.1)$$

subject to the boundary conditions $\omega(0) = \xi(0) = 0$. Here and in what follows, we use the same notation ξ for the function of variables x and z . The system of two equations (2.1) can be closed as a first-order non-autonomous differential equation for ξ as a function of z :

$$\frac{d}{dz}(z\xi - \xi^3) = -\eta(z - \xi^2), \quad z \in [0, 1], \quad (2.2)$$

starting with $\xi(0) = 0$. Formally, equation (2.2) is solved by the power series expansion given by

$$\xi(z) = -\frac{1}{2}\eta z \left[1 + \frac{1}{12}\eta^2 z + \frac{1}{24}\eta^4 z^2 + \mathcal{O}(\eta^6 z^3) \right]. \quad (2.3)$$

This expansion suggests us to write ξ under the form

$$\xi(z) = -\frac{1}{2}\eta z \psi(\zeta), \quad \zeta := \eta^2 z. \quad (2.4)$$

Note that if ξ satisfies the power series expansion (2.3), then ψ satisfies the boundary condition $\psi(0) = 1$. Moreover, straightforward substitution imply that ψ solves the first-order differential equation

$$\frac{d\psi}{d\zeta} = \frac{2(1 - \psi) - \frac{1}{2}\zeta\psi^2 + \frac{3}{4}\zeta\psi^3}{\zeta(1 - \frac{3}{4}\zeta\psi^2)}, \quad \zeta \in [0, \eta^2], \quad (2.5)$$

starting with $\psi(0) = 1$.

In order to prove Theorem 1.1, we first prove existence and uniqueness of a suitable solution for $\psi(\zeta)$. For this purpose, we shall transform the first-order non-autonomous equation (2.5) into a planar dynamical system, where the point $(\zeta, \psi) = (0, 1)$ is an equilibrium state with a unique unstable manifold extending to the domain $\zeta > 0$. To do this, we set $\tau := \log(\zeta)$ as an

evolutionary variable of the planar dynamical system and rewrite equation (2.5) in the dynamical system form

$$\begin{cases} \dot{\zeta} = \zeta, \\ \dot{\psi} = \frac{2(1-\psi) - \frac{1}{2}\zeta\psi^2 + \frac{3}{4}\zeta\psi^3}{1 - \frac{3}{4}\zeta\psi^2}, \end{cases} \quad (2.6)$$

where the dot stands for the derivative in τ . We can see that $(\zeta, \psi) = (0, 1)$ is a saddle point of the dynamical system (2.6) and that the dynamical system is analytic near this point.

The stable manifold of the linearized system at the critical point $(0, 1)$ corresponds to the eigenvalue -2 and is the line $\zeta = 0$. The unstable manifold of the linearized system at the critical point $(0, 1)$ corresponds to the eigenvalue $\lambda = 1$ and is the line $\psi - 1 = \frac{1}{12}\zeta$, which also follows from the power series expansion (2.3). There exists a unique solution of the linearized system for $\zeta > 0$ such that $\psi(\tau) \rightarrow 1$ and $\zeta(\tau) \rightarrow 0$ as $\tau \rightarrow -\infty$. By the Unstable Manifold Theorem, there exists a unique solution of the full nonlinear system (2.6) with the same properties and this solution is tangent to the unstable manifold of the linearized system at $(0, 1)$ in the sense that

$$\lim_{\zeta \rightarrow 0} \frac{\psi - 1 - \frac{1}{12}\zeta}{\zeta} = 0,$$

again in agreement with the power series expansion (2.3). This solution exists at least locally, e.g. for $(-\infty, \tau_0)$ for some $\tau_0 \in \mathbb{R}$. It is not clear if it exists globally or not, because the unstable manifold on the plane (ζ, ψ) may intersect the curve $3\zeta\psi^2 = 4$, where the dynamical system (2.6) is singular.

We transfer now the result of the Unstable Manifold Theorem back to the solutions of the truncated problem (2.1). The solution to the dynamical system (2.6) we have constructed for $\tau \in (-\infty, \tau_0)$ provides a solution $\psi(\zeta)$ to the first-order equation (2.5) for $\zeta \in [0, \zeta_0)$, with $\zeta_0 := e^{\tau_0}$. Using the scaling transformation (2.4), we obtain the existence of a solution $\xi(z)$ to the first-order equation (2.2) for $z \in [0, \zeta_0\eta^{-2})$. For $|\eta| < \eta_0 = \zeta_0^{1/2}$, this interval includes the interval $[0, 1]$. Then, $\omega(z) = z - \xi^2(z)$ is a smooth solution of the truncated problem (2.1) for all $z \in [0, 1]$. It follows from (2.4) and (2.5) that there is a positive constant C such that

$$|\xi(z)| + |\xi'(z)| \leq C|\eta| \quad \text{for all } z \in [0, 1]. \quad (2.7)$$

Therefore, for sufficiently small values of η , we have

$$\omega'(z) = 1 - 2\xi(z)\xi'(z) > 0, \quad \text{for all } z \in [0, 1],$$

hence, the solution satisfies $\omega(z) > 0$ for all $z \in (0, 1]$ and $\omega(z) = z + \mathcal{O}(z^2)$ as $z \rightarrow 0$. Defining $\varphi_{\text{TF}}(x) = \sqrt{\omega(1-x^2)}$ for $x \in [-1, 1]$, we complete the proof of Theorem 1.1.

3 Properties of decaying solutions

Here we assume the existence of an even, \mathcal{C}^1 , spatially decaying solution of the system of differential equations (1.5). We prove that such a solution has a fast decay at infinity with a specific rate and remains positive at least outside the Thomas–Fermi interval $[-1, 1]$. Both parameters ε and η are considered to be positive and fixed.

Lemma 3.1 Assume that φ is an even C^1 solution of system (1.5) such that $\varphi(x) \rightarrow 0$ as $|x| \rightarrow \infty$ and satisfies for large values of $|x|$,

$$\varphi(x) > 0, \quad \left| \int_{-\infty}^x s\varphi^2(s)ds \right| < \infty. \quad (3.1)$$

Assume that $1 - \varepsilon - \varepsilon^2\eta^2 > 0$. Then, there is $\gamma > 0$ such that

$$\varphi(x) \underset{|x| \rightarrow \infty}{\sim} \gamma|x|^{\frac{1-\varepsilon-\varepsilon^2\eta^2}{2\varepsilon}} e^{-\frac{x^2}{2\varepsilon}}, \quad (3.2)$$

and $\varphi(x) > 0$ for all $|x| \geq 1$.

Proof. We justify the decay (3.2) from the Unstable/Stable Manifold Theorem and the WKB theory. Let us consider decaying solutions of the linear second-order differential equation

$$\varepsilon^2 \varphi_{\infty}''(x) + (1 - x^2 - \xi(x)^2) \varphi_{\infty}(x) = 0, \quad (3.3)$$

where ξ is defined by the integral formula (1.6) computed at $\varphi = \varphi_{\infty}$. By the WKB method without turning points [13, Chapter 7.2], for a fixed positive ε , decaying solutions of (3.3) satisfying (3.1) exist and are all proportional to the particular solution φ_{∞} given by

$$\varphi_{\infty}(x) = \frac{e^{\varepsilon^{-1} \int B(x) dx}}{\sqrt{B(x)}}, \quad (3.4)$$

where $B(x)$ satisfies

$$B(x) = \sqrt{x^2 + \xi(x)^2 - 1 + \frac{\varepsilon^2(2BB'' - 3(B')^2)}{4B^2}}. \quad (3.5)$$

Since $B(x) \rightarrow +\infty$ as $x \rightarrow -\infty$, integration by parts yields

$$\int_{-\infty}^x s\varphi_{\infty}^2(s)ds = -\frac{\varepsilon|x|}{2B(x)^2} e^{2\varepsilon^{-1} \int B(x) dx} \left[1 + \mathcal{O}\left(\frac{\varepsilon}{|x|B(x)}\right) + \mathcal{O}\left(\frac{\varepsilon B'(x)}{B^2(x)}\right) \right] \quad \text{as } x \rightarrow -\infty.$$

It follows from the integral formula (1.6) that

$$\xi(x) = -\frac{\varepsilon\eta|x|}{B(x)} \left[1 + \mathcal{O}\left(\frac{\varepsilon}{|x|B(x)}\right) + \mathcal{O}\left(\frac{\varepsilon B'(x)}{B^2(x)}\right) \right] \quad \text{as } x \rightarrow -\infty. \quad (3.6)$$

From (3.5) and (3.6), we deduce that $\xi(x) = o(|x|)$ and $B(x) = \mathcal{O}(x)$ as $x \rightarrow -\infty$. By Taylor expansions, this further specifies the asymptotic expansions

$$\xi(x) = -\varepsilon\eta \left[1 + \mathcal{O}\left(\frac{1}{x^2}\right) \right] \quad \text{as } x \rightarrow -\infty$$

and

$$B(x) = |x| + \frac{\varepsilon^2\eta^2 - 1}{2|x|} + \mathcal{O}\left(\frac{1}{|x|^3}\right) \quad \text{as } x \rightarrow -\infty.$$

By using (3.4), we obtain

$$\varphi_\infty(x) = |x|^{\frac{1-\varepsilon-\varepsilon^2\eta^2}{2\varepsilon}} e^{-\frac{x^2}{2\varepsilon}} \left[1 + \mathcal{O}\left(\frac{1}{x^2}\right) \right] \quad \text{as } x \rightarrow -\infty.$$

This asymptotic decay recovers (3.2) by the Unstable Manifold Theorem, which states that decaying solutions φ of system (1.5) are all proportional to the decaying solution φ_∞ of the linear equation (3.3) as $x \rightarrow -\infty$.

To justify positivity of φ , we represent the first equation in (1.5) as follows:

$$\varepsilon^2 \frac{d^2\varphi}{dx^2} = (x^2 - 1 + \xi^2 + \varphi^2)\varphi. \quad (3.7)$$

By the decay (3.2), we have $\varphi(x) > 0$ and $\varphi'(x) > 0$ for large negative values of x . Then $\varphi''(x) > 0$ for all $x \in (-\infty, -1)$, so that $\varphi(x) > 0$ for all $x \in (-\infty, -1]$. ■

4 Mapping $\chi \rightarrow \nu$

Here we consider system (1.11) for a family of functions $\chi \in L^\infty(-\infty, \varepsilon^{-2/3})$, which depend on ε and η . We assume that there are constants $C_+ \geq 1$ and $C_- > 0$ such that for every $\varepsilon > 0$ small enough and every $\eta \in \mathbb{R}$, the function χ satisfies

$$\frac{1}{C_+}|\eta|y \leq |\chi(y)| \leq C_+|\eta|(1+y), \quad y \in (0, \varepsilon^{-2/3}) \quad (4.1)$$

and

$$|\chi(y)| \leq C_-|\eta|, \quad y \in (-\infty, 0). \quad (4.2)$$

Additionally, we assume the asymptotic behavior

$$\chi(y) \underset{y \rightarrow -\infty}{\sim} -\varepsilon^{1/3}\eta. \quad (4.3)$$

Under these assumptions on χ , we consider the scalar equation

$$4\nu''(y) + y\nu(y) - \nu^3(y) = \varepsilon^{2/3} (4y\nu''(y) + 2\nu'(y) + \chi^2(y)\nu(y)), \quad y \in (-\infty, \varepsilon^{-2/3}). \quad (4.4)$$

The Hastings–McLeod solution ν_0 solves (4.4) for $\varepsilon = 0$. For $\varepsilon > 0$ small, we are looking for a solution ν to the scalar equation (4.4) near ν_0 . Thus, using the decomposition $\nu = \nu_0 + R$, we rewrite (4.4) as

$$M_\varepsilon R = H_\varepsilon + N_\varepsilon(R), \quad (4.5)$$

where the linear operator M_ε , the source term H_ε and the nonlinear function $N_\varepsilon(R)$ are given by

$$M_\varepsilon = -4(1 - \varepsilon^{2/3}y)^{1/2}\partial_y(1 - \varepsilon^{2/3}y)^{1/2}\partial_y + W_0(y), \quad (4.6)$$

$$H_\varepsilon := -\varepsilon^{2/3} (4y\nu_0''(y) + 2\nu_0'(y) + \chi^2(y)\nu_0(y)) \quad (4.7)$$

and

$$N_\varepsilon(R) := -3\nu_0 R^2 - R^3 - \varepsilon^{2/3}\chi^2 R, \quad (4.8)$$

with $W_0(y) := 3\nu_0^2(y) - y$. By Lemma 2.2 in [6], there is a positive constant W_{\min} such that

$$W_0(y) \geq W_{\min} \quad \text{for all } y \in \mathbb{R}. \quad (4.9)$$

Let us define the Hilbert spaces L_ε^2 and H_ε^1 as the sets of functions in $L_{\text{loc}}^1(-\infty, \varepsilon^{2/3})$ with finite squared norms

$$\begin{aligned} \|u\|_{L_\varepsilon^2}^2 &:= \int_{-\infty}^{\varepsilon^{-2/3}} (1 - \varepsilon^{2/3}y)^{-1/2} |u(y)|^2 dy, \\ \|u\|_{H_\varepsilon^1}^2 &:= \int_{-\infty}^{\varepsilon^{-2/3}} (1 - \varepsilon^{2/3}y)^{-1/2} \left[4(1 - \varepsilon^{2/3}y) |u'(y)|^2 + W_0(y) |u(y)|^2 \right] dy. \end{aligned}$$

By Lemma 2.3 in [6], M_ε is defined as a self-adjoint unbounded invertible operator on L_ε^2 and for $\varepsilon > 0$ small enough, the inverse operator satisfies the ε -independent bound

$$\forall f \in L_\varepsilon^2, \quad \|M_\varepsilon^{-1}f\|_{H_\varepsilon^1} \leq W_{\min}^{-1/2} \|f\|_{L_\varepsilon^2}. \quad (4.10)$$

By the implicit function theorem arguments, we obtain the following result.

Theorem 4.1 *Let ν_0 be the Hastings–McLeod solution of the Painlevé-II equation, defined by (1.12) and (1.13). Let $\chi \in L^\infty(-\infty, \varepsilon^{-2/3})$ satisfy (4.1)–(4.3). For any $q > \frac{5}{6}$, there exist $\varepsilon_q > 0$, $\eta_q > 0$, and $C_q > 0$ such that for every $\varepsilon \in (0, \varepsilon_q)$ and $|\eta| < \eta_q \varepsilon^q$, there exists a unique solution $R \in H_\varepsilon^1$ of equation (4.5) such that*

$$\|R\|_{H_\varepsilon^1} \leq C_q \begin{cases} \varepsilon^{2q-4/3} |\log(\varepsilon)|^{1/2}, & \text{if } q \leq 1, \\ \varepsilon^{2/3}, & \text{if } q > 1. \end{cases} \quad (4.11)$$

If $\nu := \nu_0 + R$, then $\nu(y) > 0$ for all $y \in (-\infty, \varepsilon^{-2/3})$ and there is $\gamma > 0$ such that

$$\nu(y) \underset{y \rightarrow -\infty}{\sim} \gamma |y|^{\frac{1-\varepsilon-\varepsilon^2\eta^2}{4\varepsilon}} e^{-\frac{|y|}{2\varepsilon^{1/3}}}. \quad (4.12)$$

Furthermore, if $\nu_{1,2}$ correspond to $\chi_{1,2}$, then there exists an ε -independent positive constant C such that

$$\|\nu_1 - \nu_2\|_{H_\varepsilon^1} \leq C \varepsilon^{2/3} \|(\chi_1^2 - \chi_2^2) \nu_1\|_{L_\varepsilon^2}. \quad (4.13)$$

The proof of this theorem is divided into three subsections.

4.1 Nonlinear and residual terms $N_\varepsilon(R)$ and H_ε

First, we note the following embedding property.

Lemma 4.2 *There exists $C > 0$ such that if $\varepsilon > 0$ is small enough and if $u \in H_\varepsilon^1$, then $u \in C^0(-\infty, \varepsilon^{-2/3})$ satisfies*

$$\|u\|_{L^\infty(-\infty, \varepsilon^{-2/3})} \leq C \|u\|_{H_\varepsilon^1}. \quad (4.14)$$

Proof. We introduce the map T_ε defined for $u \in L_\varepsilon^2$ by

$$(T_\varepsilon u)(z) = u(\varepsilon^{-2/3} - \varepsilon^{2/3} z^2), \quad z \in \mathbb{R}.$$

In [6], we showed that T_ε is an isometry between L_ε^2 and the space $L_{\text{even}}^2(\mathbb{R})$ of even squared-integrable functions on \mathbb{R} . Also, T_ε induces an isometry between H_ε^1 and

$$H_w^1 = \left\{ f \in L_{\text{even}}^2(\mathbb{R}) : \|f\|_{H_w^1}^2 := \int_{\mathbb{R}} (|f'(z)|^2 + W_0(\varepsilon^{-2/3} - \varepsilon^{2/3} z^2)|f(z)|^2) dz < \infty \right\}.$$

As a result, Sobolev embedding implies for every $u \in H_\varepsilon^1$ that

$$\|u\|_{L^\infty(-\infty, \varepsilon^{2/3})} = \|T_\varepsilon u\|_{L^\infty(\mathbb{R})} \lesssim \|T_\varepsilon u\|_{H^1(\mathbb{R})} \lesssim \|T_\varepsilon u\|_{H_w^1} = \|u\|_{H_\varepsilon^1},$$

which yields (4.14). ■

Next, we write $N_\varepsilon(R) = N_0(R) + \Delta N_\varepsilon(R)$, where

$$N_0(R) = -3\nu_0 R^2 - R^3, \quad \Delta N_\varepsilon(R) = -\varepsilon^{2/3} \chi^2 R,$$

with a given $\chi \in L^\infty(-\infty, \varepsilon^{-2/3})$. We estimate the nonlinear terms in the following lemma.

Lemma 4.3 *There exists $C > 0$ such that for every $R \in H_\varepsilon^1$, we have $N_0(R) \in L_\varepsilon^2$ with*

$$\|N_0(R)\|_{L_\varepsilon^2} \leq C \left(\|\nu_0\|_{L^\infty(-\infty, \varepsilon^{-2/3})} \|R\|_{H_\varepsilon^1}^2 + \|R\|_{H_\varepsilon^1}^3 \right). \quad (4.15)$$

There exists $C > 0$ such that for $\varepsilon > 0$ small enough, for $\chi \in L^\infty(-\infty, \varepsilon^{-2/3})$ satisfying (4.1) and (4.2), and for every $R \in H_\varepsilon^1$, we have

$$\|\Delta N_\varepsilon(R)\|_{L_\varepsilon^2} \leq C \varepsilon^{-2/3} \eta^2 \|R\|_{H_\varepsilon^1}. \quad (4.16)$$

Proof. By Sobolev embedding of $H^1(\mathbb{R})$ into $L^p(\mathbb{R})$ for any $p \geq 2$, for every $R \in H_\varepsilon^1$ and $p = 2, 3$, we have

$$\|R^p\|_{L_\varepsilon^2} = \|T_\varepsilon(R^p)\|_{L^2} = \|T_\varepsilon R\|_{L^{2p}}^p \lesssim \|T_\varepsilon R\|_{H^1}^p \lesssim \|T_\varepsilon R\|_{H_w^1}^p = \|R\|_{H_\varepsilon^1}^p,$$

which yields bound (4.15).

Similarly, from (4.1) and (4.2), we have

$$\|\Delta N_\varepsilon(R)\|_{L_\varepsilon^2} \leq \varepsilon^{2/3} \|\chi\|_{L^\infty(-\infty, \varepsilon^{-2/3})}^2 \|R\|_{L_\varepsilon^2} \leq C \varepsilon^{-2/3} \eta^2 \|R\|_{L_\varepsilon^2},$$

which yields bound (4.16). ■

Remark 4.4 *Since $\|\nu_0\|_{L^\infty(-\infty, \varepsilon^{-2/3})} = \mathcal{O}(\varepsilon^{-1/3})$, bound (4.15) implies that, for fixed $C_0 > 0$ and $\alpha > 1/3$, N_0 maps the ball of radius $C_0 \varepsilon^\alpha$ centered at the origin in H_ε^1 into itself, provided ε is small enough. Moreover, estimating $N_0(R_1) - N_0(R_2)$ similarly, one can show that if ε is small enough, N_0 induces a contraction on these balls.*

Remark 4.5 *The term $\varepsilon^{2/3} \chi^2$ in $\Delta N_\varepsilon(R)$ is a small bounded perturbation to the linear operator M_ε if $\eta = \mathcal{O}(\varepsilon^q)$ with $q > \frac{1}{3}$, which is satisfied if $q > \frac{5}{6}$.*

Finally, we write $H_\varepsilon = H_0 + \Delta H_\varepsilon$, where

$$H_0 = -\varepsilon^{2/3} (4y\nu_0''(y) + 2\nu_0'(y)), \quad \Delta H_\varepsilon = -\varepsilon^{2/3} \chi^2 \nu_0,$$

with a given $\chi \in L^\infty(-\infty, \varepsilon^{-2/3})$. We estimate the residual terms in the following lemma.

Lemma 4.6 *There exists $C > 0$ such that*

$$\|H_0\|_{L_\varepsilon^2} \leq C\varepsilon^{2/3}, \quad (4.17)$$

There exists $C > 0$ such that for $\varepsilon > 0$ small enough and for $\chi \in L^\infty(-\infty, \varepsilon^{-2/3})$ satisfying (4.1) and (4.2), we have

$$\|\Delta H_\varepsilon\|_{L_\varepsilon^2} \leq C\varepsilon^{-4/3} \eta^2 |\log(\varepsilon)|^{1/2}. \quad (4.18)$$

Proof. The first term H_0 was analyzed in [6]. The bound (4.17) holds because $4y\nu_0''(y) + 2\nu_0'(y) \underset{y \rightarrow +\infty}{=} \mathcal{O}(y^{-7/2})$, whereas this function decays even faster as $y \rightarrow -\infty$.

The second term is analyzed with the following auxiliary result,

$$\begin{aligned} \int_0^{\varepsilon^{-2/3}} \frac{dy}{(1 - \varepsilon^{2/3}y)^{1/2}(1+y)} &= \int_0^{\varepsilon^{-2/3}} \frac{dy}{1+y} + \int_0^{\varepsilon^{-2/3}} \frac{1 - (1 - \varepsilon^{2/3}y)^{1/2}}{(1 - \varepsilon^{2/3}y)^{1/2}(1+y)} dy \\ &= \log(1 + \varepsilon^{-2/3}) + \int_0^1 \frac{1 - (1-t)^{1/2}}{(1-t)^{1/2}(\varepsilon^{2/3} + t)} dt \\ &= -\frac{2}{3} \log(\varepsilon) + \mathcal{O}(1), \end{aligned} \quad (4.19)$$

where for the last equality, we have used Lebesgue's theorem, which is possible since for every $t \in (0, 1)$ and $\varepsilon > 0$,

$$\frac{1 - (1-t)^{1/2}}{(1-t)^{1/2}(\varepsilon^{2/3} + t)} \leq g(t) := \begin{cases} \frac{t}{(1-t)^{1/2}t} \sup_{s \in [0,t]} \frac{1}{2\sqrt{1-s}} = \frac{1}{2(1-t)} & \text{if } t \in (0, 1/2) \\ \frac{2}{(1-t)^{1/2}} & \text{if } t \in (1/2, 1) \end{cases}$$

and $g \in L^1(0, 1)$. Hence, from (4.1) and (4.2), we have

$$\|\Delta H_\varepsilon\|_{L_\varepsilon^2} \leq \varepsilon^{2/3} \|\chi\|_{L^\infty(-\infty, \varepsilon^{-2/3})}^2 \|\nu_0\|_{L_\varepsilon^2} \leq C\varepsilon^{-2/3} \eta^2 \|\nu_0\|_{L_\varepsilon^2} \leq C\varepsilon^{-4/3} \eta^2 \|(1 + |y|)^{-1} \nu_0\|_{L_\varepsilon^2},$$

where $(1 + |y|)^{-1} \nu_0 = \mathcal{O}(y^{-1/2})$ as $y \rightarrow \infty$ and decays fast as $y \rightarrow -\infty$. By using (4.19), this bound yields (4.18). \blacksquare

4.2 Existence and properties of $R \in H_\varepsilon^1$

For $\varepsilon > 0$ small enough, let $\chi \in L^\infty(-\infty, \varepsilon^{-2/3})$ satisfy (4.1), (4.2), and (4.3). Then, we prove the existence of a unique solution $R \in H_\varepsilon^1$ of equation (4.5) satisfying (4.11) provided that $\eta = \mathcal{O}(\varepsilon^q)$ as $\varepsilon \rightarrow 0$ for any $q > \frac{5}{6}$.

The existence of R follows from a fixed-point argument in \mathcal{B}_ε , where \mathcal{B}_ε denotes the ball of H_ε^1 centered at the origin, with radius

$$\rho_\varepsilon := C_q \begin{cases} \varepsilon^{2q-4/3} |\log(\varepsilon)|^{1/2}, & \text{if } q \leq 1, \\ \varepsilon^{2/3}, & \text{if } q > 1, \end{cases} \quad (4.20)$$

for some $C_q > 0$. Indeed, inverting M_ε , we rewrite (4.5) as the fixed point equation

$$R = \Phi(R), \quad \Phi(R) := M_\varepsilon^{-1}(H_0 + \Delta H_\varepsilon + N_0(R) + \Delta N_\varepsilon(R)), \quad (4.21)$$

By bounds (4.10), (4.15), (4.16), (4.17), and (4.18), we obtain

$$\|\Phi(R)\|_{H_\varepsilon^1} \leq C \left(\varepsilon^{2/3} + \varepsilon^{2q-4/3} |\log(\varepsilon)|^{1/2} + \varepsilon^{-1/3} \rho_\varepsilon^2 + \rho_\varepsilon^3 + \varepsilon^{2q-2/3} \rho_\varepsilon \right).$$

If $q > 1$, then $2q - \frac{4}{3} > \frac{2}{3}$ and for $\varepsilon > 0$ small enough, $\varepsilon^{2/3} \gg \varepsilon^{2q-4/3} |\log(\varepsilon)|^{1/2}$, thus the operator Φ maps the ball \mathcal{B}_ε to itself. If $q \leq 1$ but $q > \frac{5}{6}$, then $\varepsilon^{2/3} \ll \varepsilon^{2q-4/3} |\log(\varepsilon)|^{1/2}$ and the operator Φ maps the ball \mathcal{B}_ε to itself. Similarly, one can show that Φ is a contraction on the ball \mathcal{B}_ε , see Remarks 4.4 and 4.5.

Next, we set $\nu := \nu_0 + R$ and prove positivity of $\nu(y)$ for all $y \in (-\infty, \varepsilon^{-2/3})$ and decay of $\nu(y)$ as $y \rightarrow -\infty$, according to the asymptotic behavior (4.12).

By Sobolev embedding (4.14), we have $\|R\|_{L^\infty(-\infty, \varepsilon^{-2/3})} \leq C\rho_\varepsilon$, where $\rho_\varepsilon \rightarrow 0$ as $\varepsilon \rightarrow 0$. Since ν_0 is increasing and $\nu_0(0) > 0$, then $\nu(y) > 0$ for all $y \in (0, \varepsilon^{-2/3})$ and $\varepsilon > 0$ small enough. Additionally, we know that $R \in \mathcal{C}^0(-\infty, \varepsilon^{-2/3})$ and $R(y) \rightarrow 0$ as $y \rightarrow -\infty$. By bootstrapping arguments, we obtain a higher regularity of $\nu \in \mathcal{C}^2(-\infty, \varepsilon^{-2/3})$, and hence $\nu \in \mathcal{C}^\infty(-\infty, \varepsilon^{-2/3})$.

Next, coming back to the variable x in the transformation (1.10), we can see that $\varphi(x) = \varepsilon^{1/3}\nu(y)$ satisfies the first equation of system (1.5), whereas since χ satisfies (4.3), then

$$\xi \underset{|x| \rightarrow \infty}{\sim} -\varepsilon\eta.$$

By the same method as in the proof of Lemma 3.1, we obtain positivity of $\nu(y)$ for $y \in (-\infty, 0)$ and the decay of $\nu(y) \rightarrow 0$ as $y \rightarrow -\infty$. The decay behavior (4.12) follows from the decay behavior (3.2) by the change of variables (1.10).

4.3 Lipschitz continuity of the map $\chi \mapsto \nu$

We prove the bound (4.13) and hence complete the proof of Theorem 4.1. First, we write equation (4.4) for ν_1 and ν_2 related to χ_1 and χ_2 . Taking the difference and denoting $\delta\nu := \nu_1 - \nu_2$, we obtain

$$(M_\varepsilon + \Delta W_1 + \Delta W_2) \delta\nu = -\varepsilon^{2/3}(\chi_1^2 - \chi_2^2)\nu_1, \quad (4.22)$$

where $\Delta W_1 := \varepsilon^{2/3}\chi_2^2$ and $\Delta W_2 := \nu_1^2 + \nu_1\nu_2 + \nu_2^2 - 3\nu_0^2$. By the assumptions (4.1) and (4.2), there is an ε -independent positive constant C such that

$$\|\Delta W_1\|_{L^\infty(-\infty, \varepsilon^{-2/3})} \leq C\eta^2\varepsilon^{-2/3},$$

which shows that, if $\eta = \mathcal{O}(\varepsilon^q)$ with $q > \frac{1}{3}$ as $\varepsilon \rightarrow 0$, then ΔW_1 is a small bounded perturbation to the positive potential W_0 in M_ε . On the other hand, denoting $\nu_{1,2} = \nu_0 + R_{1,2}$, we have

$$\Delta W_2 = 3\nu_0(R_1 + R_2) + R_1^2 + R_1R_2 + R_2^2.$$

Since both R_1 and R_2 belongs to \mathcal{B}_ε with radius (4.20), there is another ε -independent positive constant C such that

$$\|\Delta W_2\|_{L^\infty(-\infty, \varepsilon^{-2/3})} \leq C\rho_\varepsilon\varepsilon^{-1/3}.$$

If $q > \frac{5}{6}$, then ΔW_2 is another small bounded perturbation to the positive potential W_0 in M_ε . Hence $(M_\varepsilon + \Delta W_1 + \Delta W_2)$ is an invertible operator with an ε -independent bound on its inverse from L_ε^2 to H_ε^1 . Therefore, we obtain from equation (4.22) that

$$\|\delta\nu\|_{H_\varepsilon^1} = \varepsilon^{2/3} \|(M_\varepsilon + \Delta W_1 + \Delta W_2)^{-1} (\chi_1^2 - \chi_2^2)\nu_1\|_{H_\varepsilon^1} \leq C\varepsilon^{2/3} \|(\chi_1^2 - \chi_2^2)\nu_1\|_{L_\varepsilon^2},$$

which yields the bound (4.13).

5 Mapping $\nu \rightarrow \chi$

Here we consider the integral formula (1.14) for a family of positive functions $\nu \in L^2(-\infty, \varepsilon^{-2/3}) \cap \mathcal{C}^0(-\infty, \varepsilon^{-2/3})$, which depends on ε . This integral formula defines a solution of the second equation in system (1.11). We assume that there are constants $C_+ \geq 1$ and $C_- > 0$ such that for every $\varepsilon > 0$ small enough, the function ν satisfies

$$\frac{1}{C_+}y \leq \nu^2(y) \leq C_+(1+y), \quad y \in (0, \varepsilon^{-2/3}), \quad (5.1)$$

and

$$\frac{\int_{-\infty}^y \nu^2(s) ds}{\nu^2(y)} \leq C_-, \quad y \in (-\infty, 0). \quad (5.2)$$

In addition, we assume that there is $\gamma > 0$ such that ν satisfies the asymptotic decay

$$\nu(y) \underset{y \rightarrow -\infty}{\sim} \gamma |y|^{\frac{1-\varepsilon-\varepsilon^2\eta^2}{4\varepsilon}} e^{-\frac{|y|}{2\varepsilon^{1/3}}}. \quad (5.3)$$

We shall study the mapping $\nu \rightarrow \chi$, defined on some neighborhood of ν_0 in a suitable space such that (1.14) provides a bounded function χ . First, we obtain the following elementary result.

Lemma 5.1 *Let $\nu \in L^2(-\infty, \varepsilon^{-2/3}) \cap \mathcal{C}^0(-\infty, \varepsilon^{-2/3})$ satisfy (5.1)–(5.3). Then, $\chi \in L^\infty(-\infty, \varepsilon^{-2/3})$ is well-defined by the integral formula (1.14) and satisfies properties (4.1)–(4.3).*

Proof. Thanks to assumptions (5.1) and (5.2), we obtain

$$\frac{(1+y)^2 - 1}{2C_+} \leq \|\nu\|_{L^2(-\infty, y)}^2 \leq C_+ \left(C_- + \frac{(1+y)^2 - 1}{2} \right), \quad y \in [0, \varepsilon^{-2/3}].$$

Hence, the lower and upper bounds on $|\chi(y)|$ in (4.1) follow from the lower and upper bounds on $\nu^2(y)$ in (5.1). Bound (4.2) follows from the definition (1.14) and bound (5.2). Finally, the asymptotic decay (5.3) gives

$$\frac{\int_{-\infty}^y \nu^2(s) ds}{\nu^2(y)} \underset{y \rightarrow -\infty}{\sim} \varepsilon^{1/3},$$

which is equivalent to the property (4.3). ■

The following result gives a Lipschitz continuity property of the mapping $\nu \rightarrow \chi$ in a neighborhood of ν_0 .

Lemma 5.2 Let $\chi_{1,2}$ be defined by (1.14) for $\nu_{1,2} \in L^2(-\infty, \varepsilon^{-2/3}) \cap C^0(-\infty, \varepsilon^{-2/3})$, where $\nu_{1,2}$ satisfy (5.1)–(5.2) and are close to ν_0 so that for a positive δ , they satisfy the bound

$$\|\nu_{1,2} - \nu_0\|_{L^2(-\infty, \varepsilon^{-2/3})} + \|\nu_{1,2} - \nu_0\|_{L^\infty(-\infty, \varepsilon^{-2/3})} \leq \delta.$$

For $\varepsilon > 0$ sufficiently small, there is an ε -independent positive constant C such that

$$\|\chi_1 - \chi_2\|_{L^\infty(0, \varepsilon^{-2/3})} \leq C|\eta| \left(\|\nu_1 - \nu_2\|_{L^2(-\infty, \varepsilon^{-2/3})} + \varepsilon^{-1/3} \|\nu_1 - \nu_2\|_{L^\infty(0, \varepsilon^{-2/3})} \right). \quad (5.4)$$

Furthermore, for any fixed $y_0 \in (-\infty, 0)$ and $\varepsilon > 0$ sufficiently small, there is another ε -independent positive constant $C(y_0)$ such that

$$\|\chi_1 - \chi_2\|_{L^\infty(y_0, 0)} \leq C(y_0)|\eta| \left(\|\nu_1 - \nu_2\|_{L^2(-\infty, \varepsilon^{-2/3})} + \|\nu_1 - \nu_2\|_{L^\infty(-\infty, \varepsilon^{-2/3})} \right). \quad (5.5)$$

Proof. We write

$$\chi_1(y) - \chi_2(y) = -\frac{\eta}{\nu_1^2(y)} \int_{-\infty}^y (\nu_1^2(s) - \nu_2^2(s)) ds - \frac{\chi_2(y)}{\nu_1^2(y)} (\nu_1^2(y) - \nu_2^2(y)). \quad (5.6)$$

By using the Cauchy–Schwarz and triangle inequalities, we obtain

$$\begin{aligned} |\chi_1(y) - \chi_2(y)| &\leq \frac{|\chi_1(y)|}{\|\nu_1\|_{L^2(-\infty, y)}} \left(1 + \frac{\|\nu_2\|_{L^2(-\infty, y)}}{\|\nu_1\|_{L^2(-\infty, y)}} \right) \|\nu_1 - \nu_2\|_{L^2(-\infty, y)} \\ &\quad + \frac{|\chi_2(y)|}{\nu_1^2(y)} (\nu_1(y) + \nu_2(y)) |\nu_1(y) - \nu_2(y)|. \end{aligned} \quad (5.7)$$

Since ν_0 is increasing, we have $\nu_0(0) > 0$ and $\|\nu_0\|_{L^2(-\infty, 0)} > 0$. As a result, for $\delta > 0$ sufficiently small, we have

$$\inf_{y \in [0, \varepsilon^{-2/3}]} \nu_{1,2}(y) > \frac{1}{2} \nu_0(0)$$

and

$$\|\nu_{1,2}\|_{L^2(-\infty, 0)} > \frac{1}{2} \|\nu_0\|_{L^2(-\infty, 0)}.$$

Using moreover (4.1) and (5.1) in the bound (5.7) for $y \in (0, \varepsilon^{-2/3})$, we obtain (5.4).

Using (5.7) for $y \in (-\infty, 0)$ and (4.2), we obtain for $y \in (y_0, 0)$

$$\begin{aligned} \|\chi_1 - \chi_2\|_{L^\infty(y_0, 0)} &\leq C_- |\eta| \left(1 + \frac{\|\nu_2\|_{L^2(-\infty, 0)}}{\|\nu_1\|_{L^2(-\infty, y_0)}} \right) \frac{\|\nu_1 - \nu_2\|_{L^2(-\infty, 0)}}{\|\nu_1\|_{L^2(-\infty, y_0)}} \\ &\quad + C_- |\eta| \left(1 + \frac{\sup_{y \in [y_0, 0]} |\nu_2(y)|}{\inf_{y \in [y_0, 0]} |\nu_1(y)|} \right) \frac{\|\nu_1 - \nu_2\|_{L^\infty(y_0, 0)}}{\inf_{y \in [y_0, 0]} |\nu_1(y)|}, \end{aligned}$$

Provided

$$\delta < \frac{1}{2} \min [\|\nu_0\|_{L^2(-\infty, 0)}, \nu_0(y_0)],$$

the triangle inequality $|\nu_1| \geq |\nu_0| - |\nu_1 - \nu_0|$ yields

$$\begin{aligned} \|\chi_1 - \chi_2\|_{L^\infty(y_0, 0)} &\leq C_- |\eta| \left(1 + \frac{\|\nu_0\|_{L^2(-\infty, 0)} + \delta}{\|\nu_0\|_{L^2(-\infty, y_0)} - \delta} \right) \frac{\|\nu_1 - \nu_2\|_{L^2(-\infty, \varepsilon^{-2/3})}}{\|\nu_0\|_{L^2(-\infty, y_0)} - \delta} \\ &\quad + C_- |\eta| \left(1 + \frac{\|\nu_0\|_{L^\infty(-\infty, 0)} + \delta}{\nu_0(y_0) - \delta} \right) \frac{\|\nu_1 - \nu_2\|_{L^\infty(-\infty, \varepsilon^{-2/3})}}{\nu_0(y_0) - \delta}, \end{aligned} \quad (5.8)$$

which yields (5.5). ■

There are two main reasons why we are unable to prove convergence of an iterative method for obtaining solutions of the system (1.11) by coupling the maps $\chi \rightarrow \nu$ and $\nu \rightarrow \chi$ together. These two reasons are related to the decay of the component ν to zero as $y \rightarrow -\infty$, after which finding the component χ from the second equation of system (1.11) becomes a degenerate problem.

First, it is hard to justify the bound (5.2) for the solution ν in Theorem 4.1 because the correction term R decays much slower than the Hastings–McLeod solution ν_0 . Although the bound (5.2) holds for sufficiently large negative y , thanks to the asymptotic decay (5.3), the asymptotic formula (5.3) blows up if we fix the value of y and send ε to zero. As a result, the asymptotic decay formula (5.3) cannot be used to justify bound (5.2) for all $y \in (-\infty, 0)$.

Second, the Lipschitz continuity of Lemma 5.2 is established for a fixed value of y_0 . However, to match the solution ν with the asymptotic decay formula (5.3), we need to consider the value of y_0 to be ε -dependent such that $y_0 \rightarrow -\infty$ as $\varepsilon \rightarrow 0$. So far, we have not obtained good bounds on the Lipschitz constant $C(y_0)$ in the limit of $y_0 \rightarrow -\infty$.

6 Solution in Conjecture 1.3 via a numerical iterative method

We shall develop an iterative numerical scheme to illustrate the validity of the existence result stated in Conjecture 1.3. The leading-order solution for the component ν of the problem (1.11) is the Hastings–McLeod solution ν_0 of the Painlevé-II equation defined by (1.12)–(1.13). Let us define the zero iteration for the component χ by

$$\chi_0(y) = -\frac{\eta}{\nu_0^2(y)} \int_{-\infty}^y \nu_0^2(s) ds. \quad (6.1)$$

Properties of this function are described by the following proposition.

Proposition 6.1 $\chi_0 \in \mathcal{C}^\infty(\mathbb{R})$ and satisfies the asymptotic behavior

$$\chi_0(y) = -\eta \begin{cases} \frac{1}{2}y + \frac{3}{2}y^{-2} + \mathcal{O}(y^{-5}) & \text{as } y \rightarrow +\infty \\ |y|^{-1/2} + \mathcal{O}(|y|^{-5/4}) & \text{as } y \rightarrow -\infty \end{cases} \quad (6.2)$$

Proof. First, since $\nu_0(y) > 0$ for all $y \in \mathbb{R}$ [9, 10] and $\nu_0^2(y)$ decays fast as $y \rightarrow -\infty$, the integral formula (6.1) defines $\chi_0(y)$ for every $y \in \mathbb{R}$. Moreover, since $\nu_0 \in \mathcal{C}^\infty(\mathbb{R})$, then $\chi_0 \in \mathcal{C}^\infty(\mathbb{R})$. We shall now consider the asymptotic behavior of χ_0 as $y \rightarrow \pm\infty$.

For $y \rightarrow -\infty$, we use the asymptotic behavior of ν_0 given by (1.13) and integration by parts to obtain

$$\begin{aligned} \int_{-\infty}^y \nu_0^2(s) ds &= \frac{1}{\pi} \int_{-\infty}^y |s|^{-1/2} e^{-\frac{2}{3}|s|^{3/2}} \left(1 + \mathcal{O}(|s|^{-3/4})\right) ds \\ &\underset{y \rightarrow -\infty}{=} \frac{1}{\pi|y|} e^{-\frac{2}{3}|y|^{3/2}} \left(1 + \mathcal{O}(|y|^{-3/4})\right). \end{aligned}$$

Dividing this expression by $\nu_0^2(y)$ and using the asymptotic behavior (1.13) as $y \rightarrow -\infty$, we obtain the second line of (6.2).

For $y \rightarrow +\infty$, we use the asymptotic behavior (1.13) and write

$$\int_{-\infty}^y \nu_0^2(s) ds \underset{y \rightarrow +\infty}{=} \frac{1}{2}y^2 + y^{-1} + \mathcal{O}(y^{-4}). \quad (6.3)$$

Dividing this expression by $\nu_0^2(y)$, we obtain the first line of (6.2). ■

Replacing χ by χ_0 in the scalar equation (4.4), we obtain the first iteration ν_1 from Theorem 4.1. Because the asymptotic behavior (4.3) is replaced by the asymptotic behavior given in the second line of (6.2), the asymptotic decay (4.12) is modified as follows:

$$\nu_1(y) \underset{y \rightarrow -\infty}{\sim} \gamma_1 |y|^{\frac{1-\varepsilon}{4\varepsilon}} e^{-\frac{|y|}{2\varepsilon^{1/3}}}. \quad (6.4)$$

Nevertheless, the integral formula (1.14) can be used with ν_1 to obtain the first iterate χ_1 , which satisfies the same properties (4.1)–(4.3) in spite of the modified asymptotic decay (6.4). Then, we compute the second iterates ν_2 from Theorem 4.1 and χ_2 from the integral formula (1.14), and continue on this computational algorithm.

Next, we implement this iterative scheme numerically to show that the sequence $\{(\nu_n, \chi_n)\}_{n \in \mathbb{N}}$ converges to a solution of the coupled system (1.11).

We first approximate numerically the Hastings–McLeod solution ν_0 of the Painlevé-II equation (1.12). We use the second-order Heun’s method supplemented with a shooting algorithm. The solution is shown on the left panel of Figure 3. The dashed lines showing asymptotical expansions (1.13) for $y \geq 1$ and $y \leq -1$ are not distinguished from the numerical approximations (dots). We truncate the solution at $y_0 = 28$ and choose $\varepsilon = y_0^{-3/2} = 0.0067$. Then, we use the composite trapezoidal rule and approximate the component χ_0 from the integral equation (6.1) for $\eta = \varepsilon$. The solution is shown on the right panel of Figure 3. Again, the dashed lines show asymptotical expansions (6.2) for $y \geq 1$ and $y \leq -1$.

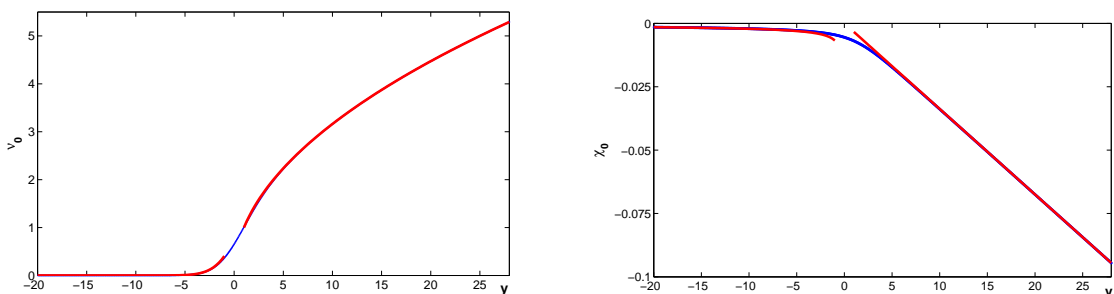


Figure 3: Components ν_0 (left) and χ_0 (right) for the numerical approximation of the Hastings–McLeod solution of the Painlevé-II equation (1.12) and the integral equation (6.1) for $\eta = \varepsilon$ with $\varepsilon = 0.0067$.

Finally, we use the iterative method to obtain the sequence $\{(\nu_n, \chi_n)\}_{n \in \mathbb{N}}$ numerically. At each step, the numerical solution for ν_n is obtained from the scalar equation (4.4) with $\chi = \chi_{n-1}$ by the result of Theorem 4.1. Implemented numerically with a second-order difference method, it takes just very few iterations to obtain a suitable approximation for ν_n . Then, the numerical solution for χ_n is obtained from the integral equation (1.14) with $\nu = \nu_n$ by applying the composite trapezoidal rule. The iterations are terminated when the difference between two subsequent approximations becomes smaller than 10^{-15} . For the same value of $\varepsilon = 0.0067$, the numerical method converges in 3 iterations for $\eta = \varepsilon$, in 6 iterations for $\eta = \varepsilon^{0.5}$, and in 11 iterations for $\eta = \varepsilon^{0.25}$. No convergence of this method was found for $\eta = \varepsilon^{0.15}$.

Figure 4 shows details of the numerical solution for $\eta = \varepsilon$. The top left panel shows the component $R := \nu - \nu_0$ of the final iterate of the numerical solution. The top right panel shows

the component χ , where the dashed line indicates the asymptotic value (4.3) for large negative y . The bottom left panel shows the component ν (dots) in comparison with the asymptotic decay behavior (1.13) of the Hastings–McLeod solution ν_0 (dashed line). It is clear from the semi-logarithmic scale that the component ν decays slower, which agrees with the asymptotic behavior (4.12). The bottom right panel shows the component ν (dots) in comparison with the growth condition (1.13) of the Hastings–McLeod solution ν_0 (dashed line). Because the values of R are small for $y = \mathcal{O}(\varepsilon^{-2/3})$, the components ν and ν_0 have similar growth rate. The situation changes when the value of η is larger, e.g. for $\eta = \varepsilon^{0.25}$, when the values of R become large near the end $y = y_0$ of the computational interval. For such large values of η , the bound (4.11) cannot be justified.

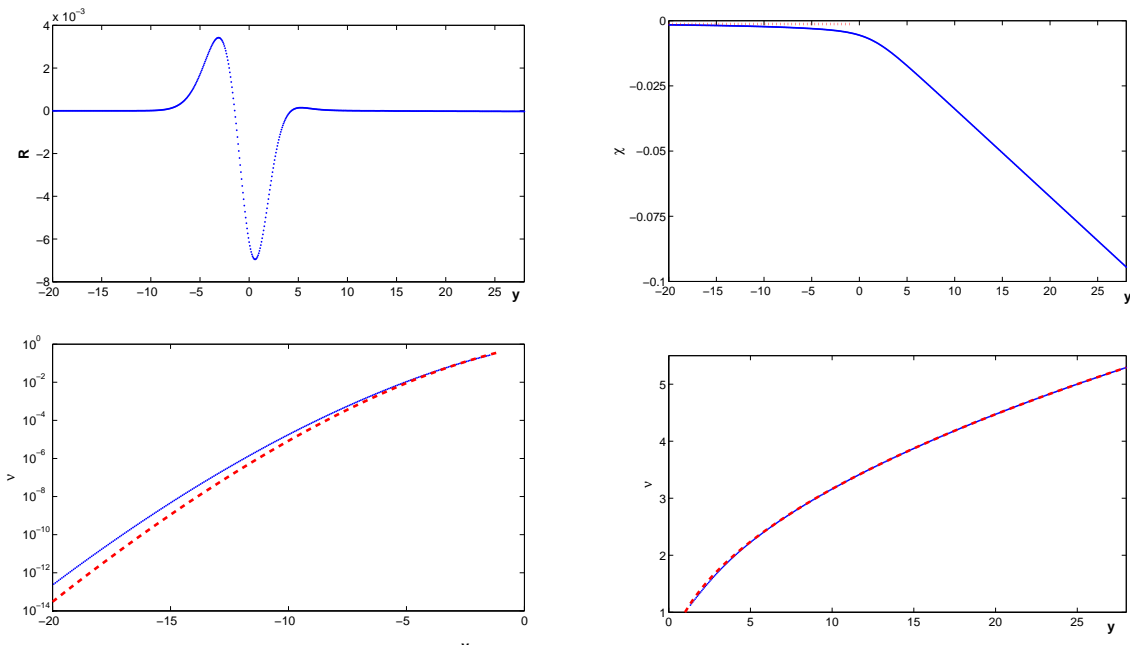


Figure 4: Details of the numerical approximation of the solution of the coupled system (1.11) for $\eta = \varepsilon$ with $\varepsilon = 0.0067$: component R (top left panel), component χ (top right panel), component ν (bottom panels) in comparison with various asymptotic values shown by dashed lines.

7 Discussion

Here we discuss the Thomas–Fermi limit of the ground state in the stationary Gross–Pitaevskii equation with a more general \mathcal{PT} -symmetric potential:

$$\mu U(X) = (-\partial_X^2 + X^2 + 2i\alpha XW(X) + |U(X)|^2) U(X), \quad X \in \mathbb{R}, \quad (7.1)$$

where W is an even, bounded, and decaying potential. Changing the variables $\mu = \varepsilon^{-1}$, $x = \varepsilon^{1/2}X$, and $U(X) = \varepsilon^{-1/2}\varphi(x)e^{i\theta(x)}$, and using the scaled variable $\xi(x) := \varepsilon\theta'(x)$ and scaled

parameter $\alpha = \varepsilon^{1/2}\eta$, we obtain the existence problem in the form,

$$\begin{cases} (1 - x^2 - \varphi^2(x) - \xi^2(x)) \varphi(x) = -\varepsilon^2 \varphi''(x), \\ (\varphi^2 \xi)'(x) = 2\eta x W(\varepsilon^{-1/2}x) \varphi^2(x), \end{cases} \quad x \in \mathbb{R}. \quad (7.2)$$

The existence problem has now two scales x and $X = \varepsilon^{-1/2}x$ thanks to the bounded and decaying potential W . As a result, the analysis of this existence problem at least for finite and even large values of η can be performed by a straightforward asymptotic method.

Solving the second equation of system (7.2) uniquely from the condition $\lim_{x \rightarrow \pm\infty} \varphi^2(x)\xi(x) = 0$, we obtain an integral representation

$$\xi(x) = \frac{2\eta\varepsilon}{\varphi^2(x)} \int_{-\infty}^{\varepsilon^{-1/2}x} sW(s)\varphi^2(\varepsilon^{1/2}s)ds. \quad (7.3)$$

Assuming now that $\varphi \in \mathcal{C}^2$ near $x = 0$ and $X^3W(X) \in L^1(\mathbb{R})$, we expand (7.3) into the asymptotic approximation,

$$\xi(x) = 2\eta\varepsilon \left(\int_{-\infty}^{\varepsilon^{-1/2}x} sW(s)ds \right) \frac{\varphi^2(0) + \mathcal{O}(\varepsilon)}{\varphi^2(x)}. \quad (7.4)$$

This asymptotic approximation shows that the phase-related component ξ gives a contribution to the \mathcal{PT} -symmetric ground state only if η is as large as $\mathcal{O}(\varepsilon^{-1})$ as $\varepsilon \rightarrow 0$ and that this contribution is only affecting the ground state in the tiny region $|x| = \mathcal{O}(\varepsilon^{1/2})$ around the origin as $\varepsilon \rightarrow 0$. Therefore, the solution φ of the existence problem (7.2) for $\eta = \mathcal{O}(\varepsilon^{-1})$ is close to the solution of the existence problem (7.2) with $\eta = 0$ (which was justified in our previous work [6]), except for the values $|x| = \mathcal{O}(\varepsilon^{1/2})$, where the solution φ is close to the modified Thomas–Fermi approximation

$$\varphi_{\text{TF}}^2(x) = 1 - x^2 - 4\eta^2\varepsilon^2 \left(\int_{-\infty}^{\varepsilon^{-1/2}x} sW(s)ds \right)^2, \quad |x| \leq C\varepsilon^{1/2}, \quad (7.5)$$

where C is ε -independent. From the requirement $\varphi_{\text{TF}}^2(0) > 0$, we find the existence interval $\eta \in (-\eta_0, \eta_0)$ of the \mathcal{PT} -symmetric ground state at the Thomas–Fermi limit, where

$$\eta_0 := \frac{1}{2\varepsilon \left| \int_{-\infty}^0 sW(s)ds \right|}.$$

Note that the breakdown of the ground state occurs at the origin $x = 0$, because the absolute value of the integral $\int_{-\infty}^{\varepsilon^{-1/2}x} sW(s)ds$ quickly drops when x deviates from the origin. Therefore, we reiterate the two facts mentioned in Remark 1.2: the Thomas–Fermi radius $|x| = 1$ is independent of the gain-loss parameter η and the \mathcal{PT} -symmetric potential leads to the decrease of the ground state amplitude φ near the center $x = 0$ of the harmonic potential.

Justification of the asymptotic approximations above for the ground state of the existence problem (7.2) appears to be a simple analytical problem if W is bounded and decaying, while $\eta = \mathcal{O}(\varepsilon^{-1})$ as $\varepsilon \rightarrow 0$. We do not include this justification analysis in the present work.

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