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Clément Gallo
Institut Montpellierain Alexander Grothendieck, Université de Montpellier,
CC051, Place Eugène Bataillon, 34095 Montpellier Cedex, France
clement.gallo@math.univ-montp2.fr

Abstract

We prove existence and uniqueness of a positive solution to a system of two coupled Gross-Pitaevskii equations. We give a full asymptotic expansion of this solution into powers of the semi classical parameter ε in the Thomas–Fermi limit $\varepsilon \to 0$.

Résumé

On montre l'existence et l'unicité d'une solution positive pour un système de deux équations de Gross-Pitaevskii couplées. On donne un développement asymptotique complet de cette solution en terme de puissances du paramètre semi-classique ε, dans la limite de Thomas-Fermi $\varepsilon \to 0$.

Keywords. Gross-Pitaevskii equation, Ground State, Bose-Einstein, Thomas-Fermi limit, asymptotic expansion.

MSC codes. 35Q55, 34E05, 35E05.

1 Introduction

Recent experiments with Bose–Einstein condensates [PS] have stimulated new interest in the Gross–Pitaevskii equation with a harmonic potential. This equation can be written as

$$i\varepsilon u_t + \varepsilon^2 \Delta u + (1 - |x|^2)u - |u|^2 u = 0, \quad x \in \mathbb{R}^d, \quad t \in \mathbb{R}_+, \tag{1.1}$$

where $u(t, x)$ denotes the complex valued wave function of the Bose gas, and ε is a small parameter. The limit $\varepsilon \to 0$ corresponds to the Thomas–Fermi approximation of a nearly compact atomic cloud [Fer], [T]. At equilibrium and in the absence of rotation, the condensate is described by the ground–state, which is a positive, time independent solution $u(t, x) = \eta_\varepsilon(x)$ to (1.1). The ground state minimizes the Gross–Pitaevskii energy

$$E_\varepsilon(u) = \int_{\mathbb{R}^d} \left(\varepsilon^2 |\nabla u|^2 + (|x|^2 - 1)|u|^2 + \frac{1}{2}|u|^4 \right) dx \tag{1.2}$$

among functions with finite energy. The understanding of the profile of the ground state is particularly important [A]. It is well known (see for instance [IM]) that in the Thomas–Fermi limit $\varepsilon \to 0$, the ground state η_ε converges to the Thomas–Fermi's compactly supported function

$$\eta_0(x) = \begin{cases} (1 - |x|^2)^{1/2} & \text{for } |x| < 1, \\ 0 & \text{for } |x| > 1. \end{cases} \tag{1.3}$$

The function η_0 has a singularity at $|x| = 1$, whereas for $\varepsilon > 0$, η_ε is regular. The question of the description of the behaviour of η_ε close to the turning point $|x| = 1$ as $\varepsilon \to 0$ has been adressed by Dalfovo, Pitaevskii and Stringari [DPS] and by Fetter and Feder [FF] on a formal level. Among other

reasons, this question is relevant because an important part of the kinetic energy is concentrated in the region $|x| \approx 1$ (see also [G]). In particular, it is shown in [DPS] and [FF] that it is possible to describe η_ε close to $|x| = 1$ as $\varepsilon \to 0$ thanks to solutions of the Painlevé II equation. This analysis has been made rigorous in [GP], where a full asymptotic expansion of η_ε in terms of powers of $\varepsilon^{2/3}$ is calculated. The proof consists in introducing a new variable $y = (1 - |x|^2)/\varepsilon^{2/3}$ that blows up the solution close to the turning point $|x| = 1$, writing $\eta_\varepsilon(x) = \varepsilon^{1/3}\nu_\varepsilon(y)$ and solving the equation satisfied by ν_ε in terms of the variable y. It turns out that the variable y makes it possible to describe the behaviour of η_ε as $\varepsilon \to 0$ not only close to the turning point, but also globally for all $x \in \mathbb{R}^d$. In [KS], Karali and Sourdis have extended this result to more general potentials.

The purpose of this paper is to adapt the result obtained in [GP] to the case of a two–component Bose–Einstein condensate. As we shall see, one of the new difficulties we are facing to is that the ground state has now two turning points instead of one in the case of a scalar Gross-Pitaevskii equation. As a matter of fact, it will be necessary to use three different variables to describe the ground state, instead of one for the scalar equation. Then, denoting by η_1 and η_2 the wave functions of the two components, η_1 and η_2 solve the following system of two coupled Gross-Pitaevskii equations with quadratic potentials,

$$\begin{cases} \varepsilon^2\Delta\eta_1 + \left(\mu_1 - |x|^2\right)\eta_1 - 2\alpha_1\eta_1^3 - 2\alpha_0\eta_2^2\eta_1 = 0 \\ \varepsilon^2\Delta\eta_2 + \left(\mu_2 - |x|^2\right)\eta_2 - 2\alpha_2\eta_2^3 - 2\alpha_0\eta_1^2\eta_2 = 0, \end{cases} \tag{1.4}$$

where $\alpha_0, \alpha_1, \alpha_2 > 0$, $\mu_1, \mu_2 > 0$ are chemical potentials, ε is a small parameter and $x \in \mathbb{R}^d$ where the dimension d is 1,2 or 3. Ground states of this system have also been studied in the case $d = 2$ and with different methods by Aftalion, Noris and Sourdis [ANS]. They prove various estimates on the difference between the Ground state and the Thomas-Fermi limit, which can be recovered by using the full asymptotic expansion of the ground state we prove here.

For convenience, we define

$$\Gamma_1 = 1 - \frac{\alpha_0}{\alpha_1}, \quad \Gamma_2 = 1 - \frac{\alpha_0}{\alpha_2}, \quad \Gamma_{12} = 1 - \frac{\alpha_0^2}{\alpha_1\alpha_2}.$$

We will consider here only values of the parameters such that the two components of the Thomas–Fermi limit (η_{10}, η_{20}) are supported and do not vanish on disks centered at $x = 0$, in opposition with other cases where one component is supported in an annulus and the other one in a disk. More specific conditions are given below. One of the differences between this case and the one component case is that, as we shall see in the next section, the Thomas–Fermi limit (η_{10}, η_{20}) has now two turning points. Thus, we have to introduce two different new variables. We will still be able to give a full asymptotic expansion of (η_1, η_2) into powers of ε in the limit $\varepsilon \to 0$, but functions of each of these two new variables will appear in every term of the expansion.

1.1 Calculation of the Thomas-Fermi limit

We are interested in solutions of (1.4) which converge in the Thomas-Fermi limit $\varepsilon \to 0$ to functions η_{10} and η_{20} which are both supported in a disk, with respective radii R_1 and R_2 (for $j = 1, 2$, $R_j = \inf\{R > 0, \mathrm{Supp}\eta_{j0} \subset B(0, R)\}$), and such that (η_{10}, η_{20}) solves (1.4) with $\varepsilon = 0$. Let us recall the arguments leading to the expression of the Thomas–Fermi profile (η_{10}, η_{20}) of the ground state, as it has been done in [AMW]. Up to a change of the indices, we assume (see Remark 1.2 below for the case $R_1 = R_2$)

$$R_1 < R_2.$$

From our definition of R_1 and R_2, we have $\eta_{10}(x) = \eta_{20}(x) = 0$ for $|x| \geqslant R_2$. For $R_1 < |x| < R_2$, $\eta_{10}(x) = 0$, and the second equation in (1.4) implies

$$\eta_{20}(x)^2 = \frac{\mu_2 - |x|^2}{2\alpha_2}.$$

Thus,

$$\mu_2 = R_2^2, \tag{1.5}$$

and $\eta_{20}(x) > 0$ for $|x| = R_1$, which implies that $\eta_{10}(x) \neq 0$ and $\eta_{20}(x) \neq 0$ for $|x| \approx R_1$, $|x| < R_1$. If $\varepsilon = 0$, $\eta_1 \neq 0$ and $\eta_2 \neq 0$, then (1.4) can be rewritten into a non-homogeneous linear system in the variables η_1^2, η_2^2. Solving this system (for the peculiar case $\Gamma_{12} = 0$, see Remark 1.3 below), we get, for $|x| \approx R_1$ and $|x| < R_1$,

$$\eta_{10}(x)^2 = \frac{1}{2\alpha_1\Gamma_{12}}\left(\mu_1 - \frac{\alpha_0}{\alpha_2}\mu_2 - \Gamma_2|x|^2\right), \tag{1.6}$$

$$\eta_{20}(x)^2 = \frac{1}{2\alpha_2\Gamma_{12}}\left(\mu_2 - \frac{\alpha_0}{\alpha_1}\mu_1 - \Gamma_1|x|^2\right). \tag{1.7}$$

In particular, since η_{10} vanishes on the sphere $|x| = R_1$ (or equivalently, using the continuity of η_{20} on the same sphere), we deduce

$$\mu_1 = \frac{\alpha_0}{\alpha_2}\mu_2 + \Gamma_2 R_1^2 = \frac{\alpha_0}{\alpha_2}R_2^2 + \Gamma_2 R_1^2. \tag{1.8}$$

Moreover, we also infer from the positiveness of η_{10}^2 and (1.6) that the condition

$$\Gamma_2/\Gamma_{12} > 0 \tag{1.9}$$

has to be satisfied. Finally, (1.7) and the assumption of positiveness of η_2 on the disk with radius R_2 (and not on an annulus) yields

$$\frac{1}{\Gamma_{12}}\left(\mu_2 - \frac{\alpha_0}{\alpha_1}\mu_1\right) > 0,$$

which can be rewritten in terms of R_1 and R_2 as

$$R_2^2 > \frac{\alpha_0}{\alpha_1}\frac{\Gamma_2}{\Gamma_{12}}R_1^2. \tag{1.10}$$

As a result, provided that the parameters satisfy conditions (1.9) and (1.10),

$$\eta_{10}(x) = \begin{cases} \left(\frac{\Gamma_2}{2\alpha_1\Gamma_{12}}\right)^{1/2}(R_1^2 - |x|^2)^{1/2} & \text{if} \quad |x| \leqslant R_1 \\ 0 & \text{if} \quad |x| \geqslant R_1 \end{cases} \tag{1.11}$$

and

$$\eta_{20}(x) = \begin{cases} \left(\frac{R_2^2 - R_1^2}{2\alpha_2} + \frac{\Gamma_1}{2\alpha_2\Gamma_{12}}(R_1^2 - |x|^2)\right)^{1/2} & \text{if} \quad |x| \leqslant R_1 \\ \left(\frac{R_2^2 - |x|^2}{2\alpha_2}\right)^{1/2} & \text{if} \quad R_1 \leqslant |x| \leqslant R_2 \\ 0 & \text{if} \quad |x| \geqslant R_2 \end{cases} \tag{1.12}$$

define a solution for $\varepsilon = 0$ to the system (1.4), which, taking into account (1.5) and (1.8), can be rewritten as

$$\begin{cases} \varepsilon^2\Delta\eta_1 + \left(\frac{\alpha_0}{\alpha_2}(R_2^2 - R_1^2) + R_1^2 - |x|^2\right)\eta_1 - 2\alpha_1\eta_1^3 - 2\alpha_0\eta_2^2\eta_1 = 0 \\ \varepsilon^2\Delta\eta_2 + \left(R_2^2 - |x|^2\right)\eta_2 - 2\alpha_2\eta_2^3 - 2\alpha_0\eta_1^2\eta_2 = 0. \end{cases} \tag{1.13}$$

Remark 1.1 *From (1.5) and (1.8), $\mu_2 - \mu_1 = \Gamma_2(R_2^2 - R_1^2)$. Thus, under the extra assumption $\Gamma_2 > 0$ (an assumption which will be made later), the assumption $R_1 < R_2$ implies $\mu_1 < \mu_2$.*

Remark 1.2 *If $R_1 = R_2 = R$ and $\Gamma_{12} \neq 0$, $\eta_{10}(x)$ and $\eta_{20}(x)$ are given by (1.6) and (1.7) for $|x| \leqslant R$, and they both vanish at $|x| = R$. We infer $\Gamma_2/\Gamma_{12} > 0$, $\Gamma_1/\Gamma_{12} > 0$ (if $\Gamma_1 = 0$ or $\Gamma_2 = 0$, then one of the two components is identically equal to 0, and therefore we are brought back to the study of one simple equation, like the one which was studuied in [GP]) and*

$$\frac{1}{\Gamma_2}\left(\mu_1 - \frac{\alpha_0}{\alpha_2}\mu_2\right) = R^2 = \frac{1}{\Gamma_1}\left(\mu_2 - \frac{\alpha_0}{\alpha_1}\mu_1\right),$$

which implies $\mu_1 = \mu_2 = \mu$. Then, for $\varepsilon > 0$, if η denotes the ground state of

$$\varepsilon^2\Delta\eta + (\mu - |x|^2)\eta - 2|\Gamma_{12}|\eta^3 = 0$$

(which, up to a rescaling, is the one which is described in [GP]), then

$$(\eta_1, \eta_2) = \Big((|\Gamma_2|/\alpha_1)^{1/2} \eta, (|\Gamma_1|/\alpha_2)^{1/2} \eta \Big)$$

solves (1.4).

Remark 1.3 *If $\Gamma_{12} = 0$, then an analysis similar to the one which is done above implies $\alpha_1 = \alpha_2 = \alpha_0 = \alpha$ and $\mu_1 = \mu_2 = \mu$. Then,*

$$(\eta_1, \eta_2) = (\eta, \eta)$$

solves (1.4), where η is the ground state solution of

$$\varepsilon^2 \Delta \eta + (\mu - |x|^2)\eta - 4\alpha\eta^3 = 0,$$

which is described in [GP] (up to a rescaling).

1.2 Goal and strategy

Our goal is to construct a solution (η_1, η_2) of (1.13) for $\varepsilon > 0$ sufficiently small, and to describe its convergence to (η_{10}, η_{20}) as $\varepsilon \to 0$. The first step consists in constructing aproximate solutions of (1.13). Because of the singularities of η_{10} and η_{20} at $|x| = R_1$ and $|x| = R_2$, $(\eta_1(x), \eta_2(x))$ will be described by functions of different variables, depending on the region of $\mathbb{R}^d$ x belongs to. We write $\mathbb{R}^d = D_0 \cup D_1 \cup D_2$, where

$$D_0 \quad = \quad \Big\{ x \in \mathbb{R}^d \big| |x|^2 \leqslant R_1^2 - \varepsilon^\beta \Big\},$$

$$D_1 \quad = \quad \Big\{ x \in \mathbb{R}^d \big| R_1^2 - 2\varepsilon^\beta \leqslant |x|^2 \leqslant R_1^2 + 2\varepsilon^\beta \Big\}$$

and

$$D_2 = \Big\{ x \in \mathbb{R}^d \big| |x|^2 \geqslant R_1^2 + \varepsilon^\beta \Big\},$$

where $\beta \in (0, 2/3)$ is some number that will be fixed later (note that $D_0 \cap D_1$ and $D_1 \cap D_2$ are not empty). Then, for $x \in D_0$, $(\eta_1(x), \eta_2(x))$ will be described as a function of the variable $z = R_1^2 - |x|^2$, whereas for $j = 1, 2$ and $x \in D_j$, it will be described as a function of the real variables y_j given by

$$y_j = \frac{R_j^2 - |x|^2}{\varepsilon^{2/3}}. \tag{1.14}$$

In order to be more specific, let us introduce the following truncation functions. Let φ be a $\mathcal{C}^\infty$ function on $\mathbb{R}$ wich is identically equal to 0 on $\mathbb{R}_-$ and identically equal to 1 on $[1, +\infty)$. Then, let us define

$$\Phi_\varepsilon(z) = \varphi\left(\frac{z - \varepsilon^\beta}{2\varepsilon^\beta - \varepsilon^\beta} \right),$$

such that $\Phi_\varepsilon(z) \equiv 0$ for $z \leqslant \varepsilon^\beta$ and $\Phi_\varepsilon(z) \equiv 1$ for $z \geqslant 2\varepsilon^\beta$, which means (if $\Phi_\varepsilon(z) = \Phi_\varepsilon(R_1^2 - |x|^2)$ is considered as a function of the variable x, also denoted Φ_ε for convenience) that $\mathrm{Supp}\Phi_\varepsilon \subset D_0$ and $\Phi_\varepsilon \equiv 1$ for $x \in D_0 \backslash D_1$. Similarly, we set

$$\chi_\varepsilon(y_1) = \left(1 - \varphi\left(\frac{\varepsilon^{2/3} y_1 - \varepsilon^\beta}{2\varepsilon^\beta - \varepsilon^\beta} \right) \right) \varphi\left(\frac{\varepsilon^{2/3} y_1 + 2\varepsilon^\beta}{-\varepsilon^\beta + 2\varepsilon^\beta} \right),$$

such that $\chi_\varepsilon(y_1) \equiv 0$ for $y_1 \geqslant 2\varepsilon^{\beta - 2/3}$ and $y_1 \leqslant -2\varepsilon^{\beta-2/3}$, whereas $\chi_\varepsilon(y_1) \equiv 1$ for $-\varepsilon^{\beta-2/3} \leqslant y_1 \leqslant \varepsilon^{\beta-2/3}$, which means (if $\chi_\varepsilon(y_1) = \chi_\varepsilon((R_1^2 - |x|^2)/\varepsilon^{2/3})$ is considered as a function of the variable x, also denoted χ_ε) that $\mathrm{Supp}\chi_\varepsilon \subset D_1$ and $\chi_\varepsilon \equiv 1$ for $x \in D_1 \backslash (D_0 \cup D_2)$. We also define

$$\Psi_\varepsilon(y_2) = 1 - \varphi\Big(\frac{z}{\varepsilon^\beta} + 2 \Big) = 1 - \varphi\left(-\frac{R_2^2 - R_1^2}{\varepsilon^\beta} + \varepsilon^{2/3-\beta} y_2 + 2 \right),$$

such that $\Psi_\varepsilon(y_2) \equiv 0$ for $y_2 \geqslant \frac{R_2^2-R_1^2}{\varepsilon^{2/3}} - \varepsilon^{\beta-2/3}$ and $\Psi_\varepsilon(y_2) \equiv 1$ for $y_2 \leqslant \frac{R_2^2-R_1^2}{\varepsilon^{2/3}} - 2\varepsilon^{\beta-2/3}$, which means (if $\Psi_\varepsilon(y_2) = \Psi_\varepsilon((R_2^2-|x|^2)/\varepsilon^{2/3})$ is considered as a function of x, also denoted Ψ_ε) that $\mathrm{Supp}\Psi_\varepsilon \subset D_2$ and $\Psi_\varepsilon \equiv 1$ for $x \in D_2 \backslash D_1$. Formally, we look for (η_1, η_2) under the form

$$\left\{ \begin{array}{l} \eta_1(x) = \Phi_\varepsilon \omega(z) + \varepsilon^{1/3} \chi_\varepsilon \nu(y_1) \\ \eta_2(x) = \Phi_\varepsilon \tau(z) + \varepsilon^{1/3} \chi_\varepsilon \lambda(y_1)^{1/2} + \varepsilon^{1/3} \Psi_\varepsilon \mu(y_2), \end{array} \right. \tag{1.15}$$

in such a way that

$$\text{for } x \in D_0, \quad (\eta_1, \eta_2)(x) \approx (\omega(z), \tau(z)), \tag{1.16}$$

$$\text{for } x \in D_1, \quad (\eta_1, \eta_2)(x) \approx \varepsilon^{1/3} \left(\nu(y_1), \lambda(y_1)^{1/2} \right) \tag{1.17}$$

and

$$\text{for } x \in D_2, \quad (\eta_1, \eta_2)(x) \approx \left(0, \varepsilon^{1/3} \mu(y_2) \right). \tag{1.18}$$

We look for approximate values of the functions ω, τ, ν, λ and μ by using a multi-scale analysis. Namely, we write

$$\begin{array}{l} \omega = \omega_0 + \varepsilon^2 \omega_1 + \varepsilon^4 \omega_2 + \cdots \\ \tau = \tau_0 + \varepsilon^2 \tau_1 + \varepsilon^4 \tau_2 + \cdots \\ \nu = \nu_0 + \varepsilon^{2/3} \nu_1 + \varepsilon^{4/3} \nu_2 + \cdots \\ \lambda = \lambda_{-1} \varepsilon^{-2/3} + \lambda_0 + \varepsilon^{2/3} \lambda_1 + \varepsilon^{4/3} \lambda_2 + \cdots \\ \mu = \mu_0 + \varepsilon^{2/3} \mu_1 + \varepsilon^{4/3} \mu_2 + \cdots \end{array} \tag{1.19}$$

The ω_j's, τ_j's, ν_j's, λ_j's and μ_j's in this expansions are ε-independent functions, which are chosen in such a way that (1.16), (1.17) and (1.18) provide at least formally solutions to (1.13) at any order. Then, we prove rigorously that the truncation of the formal asymptotic expansions we have obtained are indeed approximations of positive solutions to (1.13) which converge to (η_{10}, η_{20}) as $\varepsilon \to 0$. For this purpose, we use the ansatz

$$\left\{ \begin{array}{l} \eta_1(x) = \Phi_\varepsilon \omega(z) + \varepsilon^{1/3} \left(\chi_\varepsilon \nu(y_1) + \varepsilon^{2(N+1)/3} P(x) \right) \\ \eta_2(x) = \Phi_\varepsilon \tau(z) + \varepsilon^{1/3} \left(\chi_\varepsilon \lambda(y_1)^{1/2} + \Psi_\varepsilon \mu(y_2) + \varepsilon^{2(N+1)/3} Q(x) \right). \end{array} \right. \tag{1.20}$$

where ω, τ, ν, λ and μ are now truncations up to some finite order ($N \in \mathbb{N}$ for ν, λ, $M = M(N)$ for ω, τ and $L = L(N)$ for μ) of the formal series (1.19), and P, Q are remainder terms. A fixed point theorem provides the existence of P, Q as well as estimates which ensure that the remainder terms in (1.20) are indeed small. The better ω, τ, ν, λ and μ are chosen (that is, the larger is N), the smaller is $\varepsilon^{2(N+1)/3}(P, Q)$. The functional space in which (P, Q) is obtained is $H_w^1(\mathbb{R}^d)^2$, where

$$H_w^1(\mathbb{R}^d) = \left\{ f \in H^1(\mathbb{R}^d) \mid \min(|y_1|, |y_2|)^{1/2} f \in L^2(\mathbb{R}^d) \right\}.$$

$H_w^1(\mathbb{R}^d)^2$ is endowed with the norm

$$\|(P, Q)\|_{H_w^1(\mathbb{R}^d)^2} = \left(\int_{\mathbb{R}^d} \left(|\nabla P|^2 + |\nabla Q|^2 \right) dx + \int_{\mathbb{R}^d} \max(1, \min(|y_1|, |y_2|))(|P|^2 + |Q|^2) dx \right)^{1/2}.$$

Remark 1.4 *Note that the set $H_w^1(\mathbb{R}^d)^2$ does not depend on ε, even though it's norm does. However, this norm has been chosen in such a way that the norm of the continuous embedding of $H_w^1(\mathbb{R}^d)^2$ into $H^1(\mathbb{R}^d)^2$ is uniformly bounded in ε.*

Once we have constructed (η_1, η_2), we would like to estimate in different norms the difference between the exact solution (η_1, η_2) and its approximation

$$\left\{ \begin{array}{l} \eta_{1app}(x) = \Phi_\varepsilon \omega(z) + \varepsilon^{1/3} \chi_\varepsilon \nu(y_1) \\ \eta_{2app}(x) = \Phi_\varepsilon \tau(z) + \varepsilon^{1/3} \chi_\varepsilon \lambda(y_1)^{1/2} + \varepsilon^{1/3} \Psi_\varepsilon \mu(y_2), \end{array} \right. \tag{1.21}$$

where ω, τ, ν, λ and μ are truncations of the formal series (1.19) up to some fixed orders M_0 for ω and τ, N_0 for ν and λ and L_0 for μ. However, the estimates on P and Q provided by the fixed point argument are not very good. So in order to get better estimates on $\eta_j - \eta_{japp}$ $(j = 1, 2)$, we proceed as follows. We choose three large integers $M > M_0$, $N > N_0$ and $L > L_0$ and write η_j as in (1.20) (with truncations of the formal power series at orders M, N and L instead of M_0, N_0 and L_0). Then, the estimate on $\eta_j - \eta_{japp}$ is obtained thanks to estimates on $\varepsilon^{2m}\omega_m$ and $\varepsilon^{2m}\tau_m$ for $M_0 + 1 \leqslant m \leqslant M$, on $\varepsilon^{2n/3}\nu_n$ and $\varepsilon^{2n/3}\lambda_n$ for $N_0 + 1 \leqslant n \leqslant N$ and on $\varepsilon^{2n/3}\mu_n$ for $L_0 + 1 \leqslant n \leqslant L$. The estimates on P and Q provided by the fixed point argument are good enough to ensure that $\varepsilon^{2N/3+1}P$ and $\varepsilon^{2N/3+1}Q$ are negligible in comparison with the other terms in the expression of $\eta_j - \eta_{japp}$.

In our main result below, we give estimates on the $L^p(\mathbb{R}^d)$ and $H^1(\mathbb{R}^d)$ norms of $\eta_j - \eta_{japp}$ for $j = 1, 2$. Note however that depending on the need of the reader, our strategy can give many other informations on $\eta_j - \eta_{japp}$ (see Remark 1.6 below).

Theorem 1.5 *Let $d \in \{1, 2, 3\}$, and $\alpha_0, \alpha_1, \alpha_2 > 0$, $R_2 > R_1 > 0$ such that $\Gamma_2, \Gamma_{12} > 0$ and such that (1.10) is satisfied. Then, for $\varepsilon > 0$ sufficiently small, (1.13) has a unique solution $(\eta_1, \eta_2) \in \mathcal{C}_0(\mathbb{R}^d)^2$ such that the two components η_1 and η_2 are both positive. Moreover, if $M_0, N_0, L_0 \in \mathbb{N}$, if $\beta \in (0, \varepsilon^{2/3})\backslash\mathbb{Q}$, if $\omega_m = \omega_m(z), \tau_m = \tau_m(z), \nu_n = \nu_n(y_1), \lambda_n = \lambda_n(y_1), \mu_n = \mu_n(y_2)$ are the functions given by (2.6), (2.7), (2.10), (2.19), (2.30), (2.39), (2.34), (2.44) and (2.42), then*

$$\|\eta_1 - \eta_{1app}\|_E = \mathcal{O}\left(\varepsilon^{\gamma_1(E)}\right) \quad \text{and} \quad \|\eta_2 - \eta_{2app}\|_E = \mathcal{O}\left(\varepsilon^{\gamma_2(E)}\right), \tag{1.22}$$

where E can be either $L^p(\mathbb{R}^d)$ for any $p \in [2, +\infty]$ or $H^1(\mathbb{R}^d)$,

$$\eta_{1app} = \Phi_\varepsilon \sum_{m=0}^{M_0} \varepsilon^{2m}\omega_m + \varepsilon^{1/3}\chi_\varepsilon \sum_{n=0}^{N_0} \varepsilon^{2n/3}\nu_n,$$

$$\eta_{2app} = \Phi_\varepsilon \sum_{m=0}^{M_0} \varepsilon^{2m}\tau_m + \varepsilon^{1/3}\chi_\varepsilon \left(\sum_{n=-1}^{N_0} \varepsilon^{2n/3}\lambda_n\right)^{1/2} + \varepsilon^{1/3}\Psi_\varepsilon \sum_{n=0}^{L_0} \varepsilon^{2n/3}\mu_n$$

and for $p \in [2, +\infty]$,

$$\gamma_1(L^p(\mathbb{R}^d)) = \begin{cases} \min\big((2-3\beta)M_0 + 2 - 5\beta/2 + \beta/p\ ,\ 1 + 2/(3p)\big) & \text{if } N_0 = 0 \\ \min\big((2-3\beta)M_0 + 2 - 5\beta/2 + \beta/p\ ,\ 5/3 + 2/(3p)\big) & \text{if } N_0 = 1 \text{ and } p > 2 \\ \min\big((2-3\beta)M_0 + 2 - 5\beta/2 + \beta/p\ ,\ 2 - \delta\big) & \text{if } N_0 = 1 \text{ and } p = 2 \\ \min\big((2-3\beta)M_0 + 2 - 5\beta/2 + \beta/p\ ,\ \beta N_0 + 2 - 3\beta/2 + \beta/p\big) & \text{if } N_0 \geqslant 2 \end{cases}$$

where $\delta > 0$ is arbitrarily small, and

$$\gamma_2(L^p(\mathbb{R}^d)) = \begin{cases} \min\big((2-3\beta)M_0 + 2 - 2\beta + \beta/p\ ,\ 4/3 + 2/(3p)\ ,\ 2L_0/3 + 1 + 2/(3p)\big) & \text{if } N_0 = 0 \\ \min\big((2-3\beta)M_0 + 2 - 2\beta + \beta/p\ ,\ \beta N_0 + 2 - \beta + \beta/p\ ,\ 2L_0/3 + 1 + 2/(3p)\big) & \text{if } N_0 \geqslant 1, \end{cases}$$

whereas

$$\gamma_1(H^1(\mathbb{R}^d)) = \begin{cases} \min\big((2-3\beta)(M_0+1)\ ,\ 2/3\big) & \text{if } N_0 = 0 \\ \min\big((2-3\beta)(M_0+1)\ ,\ 4/3\big) & \text{if } N_0 = 1 \\ \min\big((2-3\beta)(M_0+1)\ ,\ 2 - \delta\big) & \text{if } N_0 = 2 \\ \min\big((2-3\beta)(M_0+1)\ ,\ \beta(N_0-2) + 2\big) & \text{if } N_0 \geqslant 3 \end{cases}$$

and

$$\gamma_2(H^1(\mathbb{R}^d)) = \begin{cases} \min\big((2-3\beta)(M_0+1) + \beta/2\ ,\ 1\ ,\ 2L_0/3 + 2/3\big) & \text{if } N_0 = 0 \\ \min\big((2-3\beta)(M_0+1) + \beta/2\ ,\ 5/3\ ,\ 2L_0/3 + 2/3\big) & \text{if } N_0 = 1 \\ \min\big((2-3\beta)(M_0+1) + \beta/2\ ,\ \beta(N_0 - 3/2) + 2\ ,\ 2L_0/3 + 2/3\big) & \text{if } N_0 \geqslant 2. \end{cases}$$

Remark 1.6 *Depending on the value of M_0, N_0, L_0 and p, the value of the parameter $\beta \in (0, 2/3)$ can be adjusted in such a way that the values of γ_1 and γ_2 are as large as possible. If we are only interested in the approximation of one of the two components η_j, one can even choose $\beta \in (0, 2/3)$ to optimize γ_j without considering the other component. In some cases, one can be interested in estimations on the norms of $\eta_1 - \eta_{1app}$ and $\eta_2 - \eta_{2app}$, not on $\mathbb{R}^d$ as a whole, but only on a subdomain like D_0, D_1 or D_2. In each minimum in the expressions of γ_1 and γ_2 in the statement of the theorem, the first argument corresponds to the rate of convergence of the norm in D_0, the second one to the rate of convergence of the norm in D_1 and the third one (for η_2) to the rate of convergence of the norm in D_2. The L^p and H^1 norms of the restriction of $\eta_1 - \eta_{1app}$ to D_2 converge to 0 faster than any power of ε as $\varepsilon \to 0$.*

In the following corollary, we write more expicitely upper bounds on the rates of convergence of $\eta_1 - \eta_{1app}$ and $\eta_2 - \eta_{2app}$ to 0 in the particular and important case where $M_0 = N_0 = L_0 = 0$ and $E = L^2, L^\infty$ or H^1.

Corollary 1.7 *If $\beta \in (0, 1/3)$, we have*

$$\eta_1 = \Phi_\varepsilon \omega_0 + \varepsilon^{1/3} \chi_\varepsilon \nu_0 + \left\{ \begin{array}{l} \mathcal{O}_{L^2(\mathbb{R}^d)}(\varepsilon^{4/3}) \\ \mathcal{O}_{L^\infty(\mathbb{R}^d)}(\varepsilon) \\ \mathcal{O}_{H^1(\mathbb{R}^d)}(\varepsilon^{2/3}) \end{array} \right.$$

and

$$\eta_2 = \Phi_\varepsilon \tau_0 + \varepsilon^{1/3} \chi_\varepsilon \left(\frac{\lambda_{-1}}{\varepsilon^{2/3}} + \lambda_0 \right) + \varepsilon^{1/3} \Psi_\varepsilon \mu_0 + \left\{ \begin{array}{l} \mathcal{O}_{L^2(\mathbb{R}^d)}(\varepsilon^{4/3}) \\ \mathcal{O}_{L^\infty(\mathbb{R}^d)}(\varepsilon) \\ \mathcal{O}_{H^1(\mathbb{R}^d)}(\varepsilon^{2/3}). \end{array} \right.$$

1.3 Organization of the paper

In Section 2, we calculate formally all the functions ω_j's, τ_j's, ν_j's, λ_j's and μ_j's appearing in the formal series (1.19), in such a way that truncations of these series provide at least formally, through the ansatz (1.16), (1.17) and (1.18), solutions to (1.13) at any order. We also study asymptotic behaviours of these functions. In Section 3, we study the functions obtained by truncations of the formal series. In particular, if ω, τ, ν, λ, μ denote these truncations, we estimate the order at which (1.16), (1.17) and (1.18) solve (1.13), respectively on D_0, D_1 and D_2. We also check that (1.16) and (1.17) are close one from another on $D_0 \cap D_1$ and that (1.17) and (1.18) are close one from another on $D_1 \cap D_2$. Section 4 is devoted to the proof of the main result.

Notations.

- If A and B are two quantities depending on a parameter x belonging to some set D, the claim "for $x \in D$, $A(x) \lesssim B(x)$" means "there exists $C > 0$ such that for every $x \in D$, $A(x) \leqslant CB(x)$".
- Let $F(x)$ be a function defined in a neighborhood of ∞. Given $\alpha \in \mathbb{R}$, $\{f_m\}_{m\in\mathbb{N}} \in \mathbb{R}$, and $\gamma > 0$, the notation

$$F(x) \underset{x\to\infty}{\approx} x^\alpha \sum_{m=0}^{\infty} f_m x^{-\gamma m}$$

  means that for every $M \in \mathbb{N}$,

$$F(x) - x^\alpha \sum_{m=0}^{M} f_m x^{-\gamma m} = \mathcal{O}(x^{\alpha-\gamma(M+1)}) \quad \text{as } x \to \infty,$$

  and, moreover, that the asymptotic series can be differentiated term by term. We use the same notation if $\gamma < 0$ and if F is defined in a neighborhood of 0.
- $\mathcal{C}_0(\mathbb{R}^d)$ denotes the space of continuous functions on $\mathbb{R}^d$ that converge to 0 at infinity.
- If $(f_\varepsilon)_{0<\varepsilon<\varepsilon_0}$ is a sequence of functions such that for every ε, f_ε belongs to some Banach space E_ε that may depend on ε, if $\alpha \in \mathbb{R}$, $f_\varepsilon = \mathcal{O}_{E_\varepsilon}(\varepsilon^\alpha)$ (respectively $f_\varepsilon = o_{E_\varepsilon}(\varepsilon^\alpha)$) means that $\|f_\varepsilon\|_{E_\varepsilon}/\varepsilon^\alpha$ remains bounded (respectively converges to 0) as $\varepsilon \to 0$.

2 Formal asymptotic expansions

2.1 Asymptotic behaviour of $\nu_0, \mu_0, \lambda_{-1}, \lambda_0$.

We are looking for a solution (η_1, η_2) to (1.13) which converges to the Thomas-Fermi approximation as $\varepsilon \to 0$. Namely, for every $x \in \mathbb{R}^d$,

$$\eta_1(x) \underset{\varepsilon\to 0}{\longrightarrow} \eta_{10}(x), \quad \eta_2(x) \underset{\varepsilon\to 0}{\longrightarrow} \eta_{20}(x). \tag{2.1}$$

The convergence of (η_1, η_2) (expressed using the ansatz (1.20)) to the Thomas-Fermi limit determines the asymptotic behaviour of $\nu(y_1), \mu(y_2), \lambda(y_1)$ as $y_1, y_2 \to \pm\infty$. We will construct the functions $\nu_0, \mu_0, \lambda_{-1}$ and λ_0 in such a way that they capture entirely this asymptotic behaviour. More precisely,

$$
\begin{array}{llll}
\text{for } |x| > R_2, & \varepsilon^{1/3}\mu_0(y_2) \underset{\varepsilon\to 0}{\longrightarrow} 0 & \text{yields} & \mu_0(y_2) \underset{y_2\to-\infty}{\longrightarrow} 0, \\
\text{for } R_1 < |x| < R_2, & \varepsilon^{1/3}\mu_0(y_2) \underset{\varepsilon\to 0}{\longrightarrow} \left(\frac{R_2^2-|x|^2}{2\alpha_2}\right)^{1/2} & \text{yields} & \mu_0(y_2) \underset{y_2\to+\infty}{\sim} \left(\frac{y_2}{2\alpha_2}\right)^{1/2}, \\
\text{for } R_1 < |x| < R_2, & \varepsilon^{1/3}\nu_0(y_1) \underset{\varepsilon\to 0}{\longrightarrow} 0 & \text{yields} & \nu_0(y_1) \underset{y_1\to-\infty}{\longrightarrow} 0, \\
\text{for } |x| < R_1, & \varepsilon^{1/3}\nu_0(y_1) \underset{\varepsilon\to 0}{\longrightarrow} \left(\frac{\Gamma_2}{2\alpha_1\Gamma_{12}}(R_1^2-|x|^2)\right)^{1/2} & \text{yields} & \nu_0(y_1) \underset{y_1\to+\infty}{\sim} \left(\frac{\Gamma_2 y_1}{2\alpha_1\Gamma_{12}}\right)^{1/2} \\
\text{for } R_1 < |x| < R_2, & \varepsilon^{1/3}\left(\frac{\lambda_{-1}(y_1)}{\varepsilon^{2/3}}+\lambda_0(y_1)\right)^{1/2} \underset{\varepsilon\to 0}{\longrightarrow} \left(\frac{R_2^2-|x|^2}{2\alpha_2}\right)^{1/2} = \left(\frac{R_2^2-R_1^2}{2\alpha_2}+\frac{R_1^2-|x|^2}{2\alpha_2}\right)^{1/2} & & \\
 & & \text{yields} & \lambda_{-1}(y_1) \underset{y_1\to-\infty}{\longrightarrow} \frac{R_2^2-R_1^2}{2\alpha_2}, \\
 & & & \lambda_0(y_1) \underset{y_1\to-\infty}{\sim} \frac{y_1}{2\alpha_2}, \\
\text{for } |x| < R_1, & \varepsilon^{1/3}\left(\frac{\lambda_{-1}(y_1)}{\varepsilon^{2/3}}+\lambda_0(y_1)\right)^{1/2} \underset{\varepsilon\to 0}{\longrightarrow} \left(\frac{R_2^2-R_1^2}{2\alpha_2}+\frac{\Gamma_1}{2\alpha_2\Gamma_{12}}(R_1^2-|x|^2)\right)^{1/2} & & \\
 & & \text{yields} & \lambda_{-1}(y_1) \underset{y_1\to+\infty}{\longrightarrow} \frac{R_2^2-R_1^2}{2\alpha_2}, \\
 & & & \lambda_0(y_1) \underset{y_1\to+\infty}{\sim} \frac{\Gamma_1 y_1}{2\alpha_2\Gamma_{12}}.
\end{array}
$$

2.2 Expansions of ω and τ in D_0

In the domain D_0, we look for (η_1, η_2) solution of (1.13) under the form (1.16). It follows that $\omega(z)$ and $\tau(z)$ have to solve for $z \in (0, R_1^2)$ the following system of differential equations

$$
\begin{aligned}
-2d\varepsilon^2\omega' + 4(R_1^2-z)\varepsilon^2\omega'' + \left(\frac{\alpha_0}{\alpha_2}(R_2^2-R_1^2)+z\right)\omega - 2\alpha_1\omega^3 - 2\alpha_0\tau^2\omega &= 0 \qquad (2.2)\\
-2d\varepsilon^2\tau' + 4(R_1^2-z)\varepsilon^2\tau'' + \left(R_2^2-R_1^2+z\right)\tau - 2\alpha_2\tau^3 - 2\alpha_0\omega^2\tau &= 0. \qquad (2.3)
\end{aligned}
$$

Then, we look for ω and τ under the form of formal power series in the parameter ε^2:

$$
\omega = \sum_{m=0}^{\infty} \varepsilon^{2m}\omega_m, \quad \tau = \sum_{m=0}^{\infty} \varepsilon^{2m}\tau_m.
$$

Plugging these expansions into (2.2), we get

$$
\begin{aligned}
&-2d\sum_{m=1}^{\infty}\varepsilon^{2m}\omega'_{m-1} + 4(R_1^2-z)\sum_{m=1}^{\infty}\varepsilon^{2m}\omega''_{m-1} + \left(\frac{\alpha_0}{\alpha_2}(R_2^2-R_1^2)+z\right)\sum_{m=0}^{\infty}\varepsilon^{2m}\omega_m \\
&-2\alpha_1\sum_{m=0}^{\infty}\varepsilon^{2m}\sum_{m_1+m_2+m_3=m}\omega_{m_1}\omega_{m_2}\omega_{m_3} - 2\alpha_0\sum_{m=0}^{\infty}\varepsilon^{2m}\sum_{m_1+m_2+m_3=m}\omega_{m_1}\tau_{m_2}\tau_{m_3} = 0, \quad (2.4)
\end{aligned}
$$

whereas (2.3) yields

$$
\begin{aligned}
&-2d\sum_{m=1}^{\infty}\varepsilon^{2m}\tau'_{m-1} + 4(R_1^2-z)\sum_{m=1}^{\infty}\varepsilon^{2m}\tau''_{m-1} + \left(R_2^2-R_1^2+z\right)\sum_{m=0}^{\infty}\varepsilon^{2m}\tau_m \\
&-2\alpha_2\sum_{m=0}^{\infty}\varepsilon^{2m}\sum_{m_1+m_2+m_3=m}\tau_{m_1}\tau_{m_2}\tau_{m_3} - 2\alpha_0\sum_{m=0}^{\infty}\varepsilon^{2m}\sum_{m_1+m_2+m_3=m}\omega_{m_1}\omega_{m_2}\tau_{m_3} = 0. \qquad (2.5)
\end{aligned}
$$

At order $m = 0$, we deduce that ω_0^2, τ_0^2 have to solve in the domain $z \in (0, R_1^2)$ (a range of values of z for which they are expected not to vanish) the linear system

$$
\begin{aligned}
\alpha_1\omega_0^2 + \alpha_0\tau_0^2 &= \frac{1}{2}\left(\frac{\alpha_0}{\alpha_2}(R_2^2-R_1^2)+z\right) \\
\alpha_0\omega_0^2 + \alpha_2\tau_0^2 &= \frac{1}{2}\left(R_2^2-R_1^2+z\right).
\end{aligned}
$$

As already mentioned in (1.11) and (1.12), it follows that

$$\omega_0^2 \quad = \quad \frac{\Gamma_2}{2\alpha_1\Gamma_{12}} z \tag{2.6}$$

and

$$\tau_0^2 = \frac{R_2^2 - R_1^2}{2\alpha_2} + \frac{\Gamma_1}{2\alpha_2\Gamma_{12}} z. \tag{2.7}$$

For $m \geqslant 1$, (2.4) and (2.5) imply that

$$M \begin{bmatrix} \omega_m \\ \tau_m \end{bmatrix} = \begin{bmatrix} 2d\omega'_{m-1} + 4(z-R_1^2)\omega''_{m-1} + 2\alpha_1 \sum\limits_{\substack{m_1+m_2+m_3=m \\ m_1,m_2,m_3<m}} \omega_{m_1}\omega_{m_2}\omega_{m_3} + 2\alpha_0 \sum\limits_{\substack{m_1+m_2+m_3=m \\ m_1,m_2,m_3<m}} \omega_{m_1}\tau_{m_2}\tau_{m_3} \\ 2d\tau'_{m-1} + 4(z-R_1^2)\tau''_{m-1} + 2\alpha_2 \sum\limits_{\substack{m_1+m_2+m_3=m \\ m_1,m_2,m_3<m}} \tau_{m_1}\tau_{m_2}\tau_{m_3} + 2\alpha_0 \sum\limits_{\substack{m_1+m_2+m_3=m \\ m_1,m_2,m_3<m}} \omega_{m_1}\omega_{m_2}\tau_{m_3} \end{bmatrix}, \tag{2.8}$$

where

$$M \quad = \quad -4 \begin{bmatrix} \alpha_1\omega_0^2 & \alpha_0\omega_0\tau_0 \\ \alpha_0\omega_0\tau_0 & \alpha_2\tau_0^2 \end{bmatrix}. \tag{2.9}$$

Thus, the functions ω_m, τ_m for $m \geqslant 1$ can be calculated thanks to the recursion relation

$$\begin{bmatrix} \omega_m \\ \tau_m \end{bmatrix} = \frac{1}{\alpha_1\alpha_2\Gamma_{12}\omega_0^2\tau_0^2} \begin{bmatrix} \alpha_2\tau_0^2 & -\alpha_0\omega_0\tau_0 \\ -\alpha_0\omega_0\tau_0 & \alpha_1\omega_0^2 \end{bmatrix} \times \tag{2.10}$$
$$\begin{bmatrix} -\frac{d}{2}\omega'_{m-1} - (z-R_1^2)\omega''_{m-1} - \frac{\alpha_1}{2} \sum\limits_{\substack{m_1+m_2+m_3=m \\ m_1,m_2,m_3<m}} \omega_{m_1}\omega_{m_2}\omega_{m_3} - \frac{\alpha_0}{2} \sum\limits_{\substack{m_1+m_2+m_3=m \\ m_1,m_2,m_3<m}} \omega_{m_1}\tau_{m_2}\tau_{m_3} \\ -\frac{d}{2}\tau'_{m-1} - (z-R_1^2)\tau''_{m-1} - \frac{\alpha_2}{2} \sum\limits_{\substack{m_1+m_2+m_3=m \\ m_1,m_2,m_3<m}} \tau_{m_1}\tau_{m_2}\tau_{m_3} - \frac{\alpha_0}{2} \sum\limits_{\substack{m_1+m_2+m_3=m \\ m_1,m_2,m_3<m}} \omega_{m_1}\omega_{m_2}\tau_{m_3} \end{bmatrix}.$$

From this relation, we deduce useful informations about the behaviour of ω_m and τ_m for $z \in (0, R_1^2]$.

Lemma 2.1 *For every $m \geqslant 1$, there exists $(w_{m,n})_{n\geqslant 0}, (t_{m,n})_{n\geqslant 0} \in \mathbb{R}^{\mathbb{N}}$ such that*

$$\omega_m(z) \underset{z\to 0}{\approx} z^{1/2-3m} \sum_{n=0}^{\infty} w_{m,n} z^n \tag{2.11}$$

and

$$\tau_m(z) \underset{z\to 0}{\approx} z^{1-3m} \sum_{n=0}^{\infty} t_{m,n} z^n. \tag{2.12}$$

In particular, there is a constant $c_m > 0$ such that

$$\forall z \in (0, R_1^2], \quad |\omega_m(z)| \leqslant c_m z^{1/2-3m} \quad \text{and} \quad |\tau_m(z)| \leqslant c_m z^{1-3m}.$$

Remark 2.2 *Note that for $m = 0$, (2.11) is also true (with $w_{0,n} = 0$ for $n \geqslant 1$), whereas (2.12) has to be replaced by the Taylor expansion of τ_0 at $z = 0$, which can be written as*

$$\tau_0(z) \underset{z\to 0}{\approx} \left(\frac{R_2^2 - R_1^2}{2\alpha_2}\right)^{1/2} + \sum_{n=0}^{\infty} t_{0,n} z^{1+n} \tag{2.13}$$

for some $(t_{0,n})_{n\geqslant 0} \in \mathbb{R}^{\mathbb{N}}$.

Proof. From (2.10), ω_1 and τ_1 can be explicitely expressed by

$$\begin{bmatrix} \omega_1 \\ \tau_1 \end{bmatrix} = \begin{bmatrix} \frac{-d\omega'_0/2 - (z-R_1^2)\omega''_0}{\alpha_1\Gamma_{12}\omega_0^2} + \frac{\alpha_0(d\tau'_0/2 + (z-R_1^2)\tau''_0)}{\alpha_1\alpha_2\Gamma_{12}\omega_0\tau_0} \\ \frac{\alpha_0(d\omega'_0/2 + (z-R_1^2)\omega''_0)}{\alpha_1\alpha_2\Gamma_{12}\omega_0\tau_0} - \frac{d\tau'_0/2 + (z-R_1^2)\tau''_0}{\alpha_2\Gamma_{12}\tau_0^2} \end{bmatrix}. \tag{2.14}$$

Then, it follows from (2.6), (2.7) and the expansions as $\varepsilon \to 0$ of τ'_0, τ''_0, $1/\tau_0$ and $1/\tau_0^2$ that (2.11)-(2.12) hold for $m = 1$. Let $m \geqslant 2$ and assume that (2.11) and (2.12) are true for m replaced by any integer between 1 and $m-1$. Then (2.11)-(2.12) also hold at order m thanks to (2.10), the recursion assumption, (2.6) and (2.13). ∎

Remark 2.3 *A consequence of Lemma 2.1 is that for every $x \in D_0$ (which in terms of the variable z, means $R_1^2 \geqslant z \geqslant \varepsilon^\beta$), for every $m \geqslant 1$,*

$$|\varepsilon^{2m}\omega_m(z)| \leqslant c_m \varepsilon^{\beta/2+m(2-3\beta)}, \quad |\varepsilon^{2m}\tau_m(z)| \leqslant c_m \varepsilon^{\beta+m(2-3\beta)}.$$

In particular, since we have chosen $\beta \in (0, 2/3)$, for every $M \geqslant 1$,

$$\left\| \sum_{m=0}^{M} \varepsilon^{2m}\omega_m - \omega_0 \right\|_{L^\infty(D_0)} \underset{\varepsilon\to 0}{\longrightarrow} 0 \quad \text{and} \quad \left\| \sum_{m=0}^{M} \varepsilon^{2m}\tau_m - \tau_0 \right\|_{L^\infty(D_0)} \underset{\varepsilon\to 0}{\longrightarrow} 0,$$

and for a fixed value of M, the larger is $m \in \{0, \cdots, M\}$, the smaller are the $L^\infty(D_0)$ norms of $\varepsilon^{2m}\omega_m$ and $\varepsilon^{2m}\tau_m$ in the limit $\varepsilon \to 0$.

2.3 Expansion of μ in D_2

For $x \in D_2$, we look for a solution (η_1, η_2) to (1.13) under the form (1.18). Thus, μ is constructed in such a way that $\eta_2(x) = \varepsilon^{1/3}\mu(y_2)$ solves, for $|x| > R_1$,

$$\varepsilon^2 \Delta\eta_2 + \left(R_2^2 - |x|^2\right)\eta_2 - 2\alpha_2\eta_2^3 = 0, \tag{2.15}$$

which means that for $y_2 < (R_2^2 - R_1^2)/\varepsilon^{2/3}$,

$$4|x|^2\mu''(y_2) - 2d\varepsilon^{2/3}\mu'(y_2) + y_2\mu(y_2) - 2\alpha_2\mu(y_2)^3 = 0. \tag{2.16}$$

Moreover, we are looking for a solution η_2 that converges to η_{20} for $|x| > R_1$. Thus, as already discussed in Section 2.1, μ has to satisfy the following asymptotics:

$$\mu(y_2) \underset{y_2\to-\infty}{\longrightarrow} 0, \qquad \mu(y_2) \underset{y_2\to+\infty}{\sim} \left(\frac{y_2}{2\alpha_2}\right)^{1/2}.$$

We rescale to change the unknown function μ into γ, defined by

$$\mu(y_2) = \frac{R_2^{1/3}}{(2\alpha_2)^{1/2}}\gamma\left(\frac{y_2}{R_2^{2/3}}\right).$$

Then, it turns out that μ solves (2.16) if and only if γ solves the differential equation

$$4(1 - \tilde{\varepsilon}^{2/3}y)\gamma''(y) - 2d\tilde{\varepsilon}^{2/3}\gamma'(y) + y\gamma(y) - \gamma(y)^3 = 0, \quad -\infty < y \leqslant \frac{R_2^2 - R_1^2}{R_2^2}\tilde{\varepsilon}^{-2/3}, \tag{2.17}$$

where $\tilde{\varepsilon} = \varepsilon/R_2^2$. In [GP], we have constructed a solution γ of this equation for $y \in (-\infty, \tilde{\varepsilon}^{-2/3}]$ (γ was denoted $\nu_{\tilde{\varepsilon}}$ in that paper). Moreover, this solution, for any $N \in \mathbb{N}$, can be expressed under the form (see below for an explanation of the notations)

$$\gamma(y) = \sum_{n=0}^{N} \tilde{\varepsilon}^{2n/3}\gamma_n(y) + \tilde{\varepsilon}^{2(N+1)/3}R_{N,\tilde{\varepsilon}}(y).$$

Thus,

$$\mu(y_2) = \frac{R_2^{1/3}}{(2\alpha_2)^{1/2}} \sum_{n=0}^{N} \tilde{\varepsilon}^{2n/3}\gamma_n\left(\frac{y_2}{R_2^{2/3}}\right) + \tilde{\varepsilon}^{2(N+1)/3}\frac{R_2^{1/3}}{(2\alpha_2)^{1/2}}R_{N,\tilde{\varepsilon}}\left(\frac{y_2}{R_2^{2/3}}\right). \tag{2.18}$$

In particular, the functions μ_n introduced in (1.19) are given for every $n \geqslant 0$ by

$$\mu_n(y_2) = \frac{R_2^{1/3}}{(2\alpha_2)^{1/2}}R_2^{-4n/3}\gamma_n\left(\frac{y_2}{R_2^{2/3}}\right). \tag{2.19}$$

The functions γ_n and $R_{N,\varepsilon}$ mentioned above have been defined as follows in [GP].

- γ_0 is the Hastings-McLeod solution of the Painlevé-II equation, that is the unique solution of

$$4\gamma_0''(y) + y\gamma_0(y) - \gamma_0(y)^3 = 0, \quad y \in \mathbb{R}, \tag{2.20}$$

with the asymptotic behaviour

$$\gamma_0(y) \underset{y\to+\infty}{\sim} y^{1/2}, \quad \gamma_0(y) \underset{y\to-\infty}{\longrightarrow} 0.$$

- for $1 \leqslant n \leqslant N$, γ_n is the unique solution of

$$-4\gamma_n''(y) + W_0(y)\gamma_n(y) = F_n(y), \quad y \in \mathbb{R}, \tag{2.21}$$

which goes to 0 as $y \to \pm\infty$, where

$$W_0(y) = 3\gamma_0^2(y) - y \tag{2.22}$$

and

$$F_n(y) = -\sum_{\substack{n_1,\, n_2,\, n_3 \,<\, n \\ n_1+n_2+n_3 \,=\, n}} \gamma_{n_1}(y)\gamma_{n_2}(y)\gamma_{n_3}(y) - 2d\gamma_{n-1}'(y) - 4y\gamma_{n-1}''(y),$$

- $R_{N,\tilde\varepsilon}$ solves

$$-4(1-\tilde\varepsilon^{2/3}y)R_{N,\tilde\varepsilon}'' + 2\tilde\varepsilon^{2/3}dR_{N,\tilde\varepsilon}' + W_0R_{N,\tilde\varepsilon} = F_{N,\tilde\varepsilon}(y, R_{N,\tilde\varepsilon}),\ y \in (-\infty, \tilde\varepsilon^{-2/3}], \tag{2.23}$$

where

$$\begin{aligned} F_{N,\tilde\varepsilon}(y,R) \;=\; & -(4y\nu_N'' + 2d\nu_N') - \sum_{n=0}^{2N-1} \tilde\varepsilon^{2n/3} \sum_{\substack{n_1+n_2+n_3 \,=\, n+N+1 \\ 0\,\leqslant\, n_1,\, n_2,\, n_3 \,\leqslant\, N}} \gamma_{n_1}\gamma_{n_2}\gamma_{n_3} \\ & - \left(3\sum_{n=1}^{2N} \tilde\varepsilon^{2n/3} \sum_{\substack{n_1+n_2 \,=\, n \\ 0\,\leqslant\, n_1,\, n_2 \,\leqslant\, N}} \gamma_{n_1}\gamma_{n_2} \right) R - \left(3\sum_{n=N+1}^{2N+1} \tilde\varepsilon^{2n/3}\gamma_{n-(N+1)} \right) R^2 - \tilde\varepsilon^{4(N+1)/3}R^3. \end{aligned}$$

The analysis below requires the precise knowledge of the behaviour of $\gamma_n(y)$ as $y \to \pm\infty$. This behaviour was already described in [GP], and it is summarized in the next two propositions:

Proposition 2.4 *The behaviour of γ_0 as $y \to -\infty$ is described by*

$$\gamma_0(y) = \frac{1}{\sqrt{\pi}(-y)^{1/4}} \exp\left(-\frac{1}{3}(-y)^{3/2}\right)\left(1 + \mathcal{O}(|y|^{-3/4})\right) \underset{y\to-\infty}{\approx} 0, \tag{2.24}$$

whereas as $y \to +\infty$,

$$\gamma_0(y) \underset{y\to+\infty}{\approx} y^{1/2}\sum_{n=0}^{\infty} a_n y^{-3n}, \tag{2.25}$$

where $a_0 = 1$, and for $n \geqslant 0$,

$$a_{n+1} = 2\left(9n^2 - \frac{1}{4}\right)a_n - \frac{1}{2}\sum_{\substack{n_1+n_2+n_3=n+1 \\ n_1,n_2,n_3\leqslant n}} a_{n_1}a_{n_2}a_{n_3}.$$

Remark 2.5 *The calculation of the first terms in (2.25) gives*

$$\gamma_0(y) = y^{1/2} - \frac{1}{2}y^{-5/2} - \frac{73}{8}y^{-11/2} + O(y^{-17/2}). \tag{2.26}$$

Proposition 2.6 *For every $n \geqslant 1$,*

$$\gamma_n(y) \underset{y\to+\infty}{\approx} y^{1/2-2n}\sum_{m=0}^{\infty} g_{n,m}y^{-3m} \text{ for some } \{g_{n,m}\}_{m\in\mathbb{N}},$$

and $$\gamma_n(y) \underset{y\to-\infty}{\approx} 0.$$

Moreover, if $d = 1$, for every $n \geqslant 1$, $g_{n,0} = 0$.
For instance, $\gamma_1(y) \underset{y\to+\infty}{\sim} \frac{5(7-d)}{4}y^{-9/2}$ if $d = 1$, whereas $\gamma_1(y) \underset{y\to+\infty}{\sim} = \frac{1-d}{2}y^{-3/2}$ if $d = 2, 3$.

2.4 Expansions of ν and λ in D_1

For $x \in D_1$, we formally look for a solution (η_1, η_2) to (1.13) under the form given in (1.17). Then, it turns out that ν and λ have to solve

$$\begin{aligned} -2d\varepsilon^{2/3}\nu' + 4R_1^2\nu'' - 4\varepsilon^{2/3}y_1\nu'' + \frac{\alpha_0}{\alpha_2}\frac{R_2^2 - R_1^2}{\varepsilon^{2/3}}\nu + y_1\nu - 2\alpha_1\nu^3 - 2\alpha_0\lambda\nu &= 0 \quad (2.27) \\ -d\varepsilon^{2/3}\lambda\lambda' - (R_1^2 - \varepsilon^{2/3}y_1)\lambda'^2 + 2(R_1^2 - \varepsilon^{2/3}y_1)\lambda\lambda'' + y_2\lambda^2 - 2\alpha_2\lambda^3 - 2\alpha_0\nu^2\lambda^2 &= 0 \quad (2.28) \end{aligned}$$

Moreover, we are looking for solutions (η_1, η_2) that converge to the Thomas-Fermi limit (η_{10}, η_{20}) as $\varepsilon \to 0$. As a result, according to Section 2.1, ν and λ have to satisfy the following asymptotics. On the one side, if $R_1 < |x| < R_2$ is fixed, $\varepsilon \to 0$ if and only if $y_1 \to -\infty$, and

$$\nu(y_1) \underset{y_1 \to -\infty}{\longrightarrow} 0, \qquad \lambda(y_1) \underset{y_1 \to -\infty}{\sim} \left(\frac{R_2^2 - R_1^2}{2\alpha_2\varepsilon^{2/3}} + \frac{y_1}{2\alpha_2} \right).$$

On the other side, if $|x| < R_1$ is fixed, $\varepsilon \to 0$ if and only if $y_1 \to +\infty$, and

$$\nu(y_1) \underset{y_1 \to +\infty}{\sim} \left(\frac{\Gamma_2 y_1}{2\alpha_1\Gamma_{12}} \right)^{1/2}, \qquad \lambda(y_1) \underset{y_1 \to +\infty}{\sim} \frac{R_2^2 - R_1^2}{2\alpha_2\varepsilon^{2/3}} + \frac{\Gamma_1 y_1}{2\alpha_2\Gamma_{12}}.$$

We formally develop ν and λ into powers of $\varepsilon^{2/3}$:

$$\nu(y_1) = \sum_{n=0}^{\infty} \varepsilon^{2n/3}\nu_n(y_1), \quad \lambda(y_1) = \sum_{n=-1}^{\infty} \varepsilon^{2n/3}\lambda_n(y_1), \tag{2.29}$$

and we plug these expansions of ν and λ into (2.27). We obtain

$$\begin{aligned} -2d\sum_{n=1}^{\infty}\varepsilon^{2n/3}\nu'_{n-1} + 4R_1^2\sum_{n=0}^{\infty}\varepsilon^{2n/3}\nu''_n - 4y_1\sum_{n=1}^{\infty}\varepsilon^{2n/3}\nu''_{n-1} + \frac{\alpha_0}{\alpha_2}(R_2^2 - R_1^2)\sum_{n=-1}^{\infty}\varepsilon^{2n/3}\nu_{n+1} + y_1\sum_{n=0}^{\infty}\varepsilon^{2n/3}\nu_n \\ -2\alpha_1\sum_{n=0}^{\infty}\varepsilon^{2n/3}\sum_{n_1+n_2+n_3=n}\nu_{n_1}\nu_{n_2}\nu_{n_3} - 2\alpha_0\sum_{n=-1}^{\infty}\varepsilon^{2n/3}\sum_{\substack{n_1+n_2=n,\\ n_1 \geqslant -1,\, n_2 \geqslant 0}}\lambda_{n_1}\nu_{n_2} = 0. \end{aligned}$$

At order $n = -1$, we get, in agreement with the asymptotics of λ_{-1} given in Section 2.1,

$$\lambda_{-1}(y_1) = \frac{R_2^2 - R_1^2}{2\alpha_2}, \tag{2.30}$$

and therefore the equation can be simplified into

$$\begin{aligned} -2d\sum_{n=1}^{\infty}\varepsilon^{2n/3}\nu'_{n-1} + 4R_1^2\sum_{n=0}^{\infty}\varepsilon^{2n/3}\nu''_n - 4y_1\sum_{n=1}^{\infty}\varepsilon^{2n/3}\nu''_{n-1} + y_1\sum_{n=0}^{\infty}\varepsilon^{2n/3}\nu_n \\ -2\alpha_1\sum_{n=0}^{\infty}\varepsilon^{2n/3}\sum_{n_1+n_2+n_3=n}\nu_{n_1}\nu_{n_2}\nu_{n_3} - 2\alpha_0\sum_{n=0}^{\infty}\varepsilon^{2n/3}\sum_{\substack{n_1+n_2=n,\\ n_1 \geqslant 0,\, n_2 \geqslant 0}}\lambda_{n_1}\nu_{n_2} = 0 \end{aligned} \tag{2.31}$$

At order $n = 0$, we obtain

$$4R_1^2\nu''_0 + y_1\nu_0 - 2\alpha_1\nu_0^3 - 2\alpha_0\lambda_0\nu_0 = 0. \tag{2.32}$$

Similarly, plugging (2.29) into (2.28) and multiplying by $\varepsilon^{4/3}$, we get

$$
\begin{aligned}
&-d\sum_{n=1}^{\infty}\varepsilon^{2n/3}\sum_{\substack{n_1+n_2=n-3\\ n_1,\,n_2\geqslant -1}}\lambda'_{n_1}\lambda_{n_2}+2R_1^2\sum_{n=0}^{\infty}\varepsilon^{2n/3}\sum_{\substack{n_1+n_2=n-2\\ n_1,\,n_2\geqslant -1}}\lambda''_{n_1}\lambda_{n_2}\\
&-2y_1\sum_{n=1}^{\infty}\varepsilon^{2n/3}\sum_{\substack{n_1+n_2=n-3\\ n_1,\,n_2\geqslant -1}}\lambda''_{n_1}\lambda_{n_2}-R_1^2\sum_{n=0}^{\infty}\varepsilon^{2n/3}\sum_{\substack{n_1+n_2=n-2\\ n_1,\,n_2\geqslant -1}}\lambda'_{n_1}\lambda'_{n_2}\\
&+y_1\sum_{n=1}^{\infty}\varepsilon^{2n/3}\sum_{\substack{n_1+n_2=n-3\\ n_1,\,n_2\geqslant -1}}\lambda'_{n_1}\lambda'_{n_2}+(R_2^2-R_1^2)\sum_{n=-1}^{\infty}\varepsilon^{2n/3}\sum_{\substack{n_1+n_2=n-1\\ n_1,\,n_2\geqslant -1}}\lambda_{n_1}\lambda_{n_2}\\
&+y_1\sum_{n=0}^{\infty}\varepsilon^{2n/3}\sum_{\substack{n_1+n_2=n-2\\ n_1,\,n_2\geqslant -1}}\lambda_{n_1}\lambda_{n_2}-2\alpha_2\sum_{n=-1}^{\infty}\varepsilon^{2n/3}\sum_{\substack{n_1+n_2+n_3=n-2\\ n_1,\,n_2,\,n_3\geqslant -1}}\lambda_{n_1}\lambda_{n_2}\lambda_{n_3}\\
&-2\alpha_0\sum_{n=0}^{\infty}\varepsilon^{2n/3}\sum_{\substack{n_1+n_2+n_3+n_4=n-2\\ n_1,\,n_2\geqslant -1,\ n_3,\,n_4\geqslant 0}}\lambda_{n_1}\lambda_{n_2}\nu_{n_3}\nu_{n_4}=0
\end{aligned}
\tag{2.33}
$$

This equation at order $n=-1$ is satisfied thanks to (2.30). At order $n=0$, we obtain

$$
\lambda_0(y_1)=\frac{y_1}{2\alpha_2}-\frac{\alpha_0}{\alpha_2}\nu_0(y_1)^2. \tag{2.34}
$$

From (2.32) and (2.34), we infer the equation satisfied by ν_0:

$$
4R_1^2\nu_0''+\Gamma_2 y_1\nu_0-2\alpha_1\Gamma_{12}\nu_0^3=0. \tag{2.35}
$$

Moreover, according to Section 2.1, the asymptotic behaviour we need for ν_0 is

$$
\nu_0(y_1)\underset{y_1\to+\infty}{\sim}\left(\frac{\Gamma_2 y_1}{2\alpha_1\Gamma_{12}}\right)^{1/2},\quad \nu_0(y_1)\underset{y_1\to-\infty}{\to}0.
$$

Looking for ν_0 under the form

$$
\nu_0(y_1)=\frac{R_1^{1/3}|\Gamma_2|^{1/3}}{(2\alpha_1)^{1/2}|\Gamma_{12}|^{1/2}}\gamma\left(\frac{|\Gamma_2|^{1/3}y_1}{R_1^{2/3}}\right),
$$

ν_0 solves (2.35) if and only if γ solves

$$
4\mathrm{sign}(\Gamma_2)\gamma''(y)+y\gamma(y)-\gamma(y)^3=0,\quad y\in\mathbb{R}, \tag{2.36}
$$

with the boundary conditions

$$
\gamma(y)\underset{y\to+\infty}{\sim}\sqrt{y},\quad \gamma(y)\underset{y\to-\infty}{\to}0. \tag{2.37}
$$

If the sign of Γ_2 (which is the same as the sign of Γ_{12} according to (1.9)) is negative, it can be easily seen that (2.36) has no non-trivial solution with fast decay to 0 as $y\to-\infty$. Indeed, if γ solves (2.36) with $\gamma'(y)\to 0$ and $y\gamma(y)^2\to 0$ as $y\to-\infty$, then by integration between $-\infty$ and y, we get

$$
2\mathrm{sign}(\Gamma_2)\gamma'(y)^2=-y\frac{\gamma(y)^2}{2}+\int_{-\infty}^{y}\frac{\gamma(t)^2}{2}dt+\frac{\gamma(y)^4}{4},
$$

which implies $\gamma\equiv 0$ if $\Gamma_2<0$ and $y<0$. Also, from now on, we assume

$$
\Gamma_2>0,\qquad \Gamma_{12}>0. \tag{2.38}
$$

Under this condition, γ has to be the Hastings-McLeod solution γ_0 of the Painlevé II equation (2.20), and

$$
\nu_0(y_1)=\frac{R_1^{1/3}\Gamma_2^{1/3}}{(2\alpha_1)^{1/2}\Gamma_{12}^{1/2}}\gamma_0\left(\frac{\Gamma_2^{1/3}y_1}{R_1^{2/3}}\right). \tag{2.39}
$$

Thanks to (2.30), equation (2.31) at order $n \geqslant 1$ gives

$$\begin{aligned} & 4R_1^2\nu_n'' + y_1\nu_n - 6\alpha_1\nu_0^2\nu_n - 2\alpha_0\lambda_0\nu_n - 2\alpha_0\lambda_n\nu_0 \\ = \quad & 2d\nu_{n-1}' + 4y_1\nu_{n-1}'' + 2\alpha_1 \sum_{\substack{n_1+n_2+n_3=n \\ 0\leqslant n_1,n_2,n_3\leqslant n-1}} \nu_{n_1}\nu_{n_2}\nu_{n_3} + 2\alpha_0 \sum_{\substack{n_1+n_2=n, \\ 1\leqslant n_1,n_2\leqslant n-1}} \lambda_{n_1}\nu_{n_2}. \end{aligned} \tag{2.40}$$

On the other side, equation (2.33) at order $n \geqslant 1$ yields

$$\begin{aligned} & -d \sum_{\substack{n_1+n_2=n-3 \\ n_1,n_2\geqslant -1}} \lambda_{n_1}'\lambda_{n_2} + 2R_1^2 \sum_{\substack{n_1+n_2=n-2 \\ n_1,n_2\geqslant -1}} \lambda_{n_1}''\lambda_{n_2} \\ & -2y_1 \sum_{\substack{n_1+n_2=n-3 \\ n_1,n_2\geqslant -1}} \lambda_{n_1}''\lambda_{n_2} - R_1^2 \sum_{\substack{n_1+n_2=n-2 \\ n_1,n_2\geqslant -1}} \lambda_{n_1}'\lambda_{n_2}' \\ & +y_1 \sum_{\substack{n_1+n_2=n-3 \\ n_1,n_2\geqslant -1}} \lambda_{n_1}'\lambda_{n_2}' + (R_2^2-R_1^2) \sum_{\substack{n_1+n_2=n-1 \\ n_1,n_2\geqslant 0}} \lambda_{n_1}\lambda_{n_2} + 2(R_2^2-R_1^2)\lambda_{-1}\lambda_n \\ & +y_1 \sum_{\substack{n_1+n_2=n-2 \\ n_1,n_2\geqslant -1}} \lambda_{n_1}\lambda_{n_2} - 2\alpha_2 \sum_{\substack{n_1+n_2+n_3=n-2 \\ n-1\geqslant n_1,n_2,n_3\geqslant -1}} \lambda_{n_1}\lambda_{n_2}\lambda_{n_3} - 6\alpha_2\lambda_{-1}^2\lambda_n \\ & -2\alpha_0 \sum_{\substack{n_1+n_2+n_3+n_4=n-2 \\ n_1,n_2\geqslant -1,\ n-1\geqslant n_3,n_4\geqslant 0}} \lambda_{n_1}\lambda_{n_2}\nu_{n_3}\nu_{n_4} - 4\alpha_0\lambda_{-1}^2\nu_0\nu_n = 0, \end{aligned} \tag{2.41}$$

therefore for $n \geqslant 1$,

$$\lambda_n = -2\frac{\alpha_0}{\alpha_2}\nu_0\nu_n + \frac{2\alpha_2}{(R_2^2-R_1^2)^2}\delta_n, \tag{2.42}$$

where

$$\begin{aligned} \delta_n \quad = \quad & \sum_{\substack{n_1+n_2=n-3 \\ n_1,n_2\geqslant -1}} \left(-d\lambda_{n_1}'\lambda_{n_2} - 2y_1\lambda_{n_1}''\lambda_{n_2} + y_1\lambda_{n_1}'\lambda_{n_2}'\right) \\ & + \sum_{\substack{n_1+n_2=n-2 \\ n_1,n_2\geqslant -1}} \left(2R_1^2\lambda_{n_1}''\lambda_{n_2} - R_1^2\lambda_{n_1}'\lambda_{n_2}' + y_1\lambda_{n_1}\lambda_{n_2}\right) + (R_2^2-R_1^2) \sum_{\substack{n_1+n_2=n-1 \\ n_1,n_2\geqslant 0}} \lambda_{n_1}\lambda_{n_2} \\ & -2\alpha_2 \sum_{\substack{n_1+n_2+n_3=n-2 \\ n-1\geqslant n_1,n_2,n_3\geqslant -1}} \lambda_{n_1}\lambda_{n_2}\lambda_{n_3} - 2\alpha_0 \sum_{\substack{n_1+n_2+n_3+n_4=n-2 \\ n_1,n_2\geqslant -1,\ n-1\geqslant n_3,n_4\geqslant 0}} \lambda_{n_1}\lambda_{n_2}\nu_{n_3}\nu_{n_4} \end{aligned} \tag{2.43}$$

At this stage, we have constructed λ_{-1}, ν_0 and λ_0, which are given respectively by (2.30), (2.39) and (2.34). For $n \geqslant 1$, the λ_n's and the ν_n's are constructed by induction as follows. Let $n \geqslant 1$, and assume that the λ_k's and the ν_k's are known for every $k \leqslant n-1$. Then, plugging (2.42) and (2.34) into (2.40), ν_n has to solve

$$T\nu_n = F_n, \tag{2.44}$$

where

$$T = -4R_1^2\partial_{y_1}^2 + W(y_1), \quad W(y_1) = 6\alpha_1\Gamma_{12}\nu_0^2 - \Gamma_2 y_1 \tag{2.45}$$

and

$$F_n = -\frac{4\alpha_0\alpha_2\nu_0}{(R_2^2-R_1^2)^2}\delta_n - 2d\nu_{n-1}' - 4y_1\nu_{n-1}'' - 2\alpha_1 \sum_{\substack{n_1+n_2+n_3=n \\ 0\leqslant n_1,n_2,n_3\leqslant n-1}} \nu_{n_1}\nu_{n_2}\nu_{n_3} - 2\alpha_0 \sum_{\substack{n_1+n_2=n, \\ 1\leqslant n_1,n_2\leqslant n-1}} \lambda_{n_1}\nu_{n_2} \tag{2.46}$$

Note that only λ_k's and ν_k's for $k \leqslant n-1$ appear in (2.46) and (2.43). Once (2.44) has been solved, λ_n is given by (2.42). In order to invert T in (2.44), one needs to understand the behaviour of $F_n(y_1)$

as $y_1 \to \pm\infty$. Thus, δ_n, F_n, ν_n and λ_n will be constructed recursively in such a way that for every $n \geqslant 1$,

$$\delta_n(y_1) \underset{y_1\to-\infty}{\approx} y_1^{n-2} \sum_{0\leqslant m\leqslant(n-2)/3} \widetilde{D}_{n,m} y_1^{-3m}, \quad \delta_n(y_1) \underset{y_1\to+\infty}{\approx} y_1^{n-2} \sum_{m=0}^{+\infty} D_{n,m} y_1^{-3m}, \tag{2.47}$$

$$F_n(y_1) \underset{y_1\to-\infty}{\approx} 0, \quad F_n(y_1) \underset{y_1\to+\infty}{\approx} y_1^{n-3/2} \sum_{m=0}^{+\infty} F_{n,m} y_1^{-3m}, \tag{2.48}$$

$$\nu_n(y_1) \underset{y_1\to-\infty}{\approx} 0, \quad \nu_n(y_1) \underset{y_1\to+\infty}{\approx} y_1^{n-5/2} \sum_{m=0}^{+\infty} N_{n,m} y_1^{-3m}, \tag{2.49}$$

$$\lambda_n(y_1) \underset{y_1\to-\infty}{\approx} y_1^{n-2} \sum_{0\leqslant m\leqslant(n-2)/3} \widetilde{L}_{n,m} y_1^{-3m}, \quad \lambda_n(y_1) \underset{y_1\to+\infty}{\approx} y_1^{n-2} \sum_{m=0}^{+\infty} L_{n,m} y_1^{-3m}, \tag{2.50}$$

where the $D_{n,m}$'s, $F_{n,m}$'s, $N_{n,m}$'s, $L_{n,m}$'s, $\widetilde{D}_{n,m}$'s and $\widetilde{L}_{n,m}$'s are some real coefficients. Note that thanks to (2.39), (2.34), (2.24) and (2.25), ν_0 and λ_0 admit similar expansions. However, the power of the leading term in the expansions they satisfy as $y_1 \to +\infty$ (and for λ_0, also as $y_1 \to -\infty$) is higher of three units to the one which would be given by (2.49) and (2.50) for $n = 0$. More precisely, we have

$$\nu_0(y_1) \underset{y_1\to-\infty}{\approx} 0, \quad \nu_0(y_1) \underset{y_1\to+\infty}{\approx} y_1^{1/2} \sum_{m=0}^{+\infty} N_{0,m} y_1^{-3m} \tag{2.51}$$

and

$$\lambda_0(y_1) \underset{y_1\to-\infty}{\approx} \frac{y_1}{2\alpha_2}, \quad \lambda_0(y_1) \underset{y_1\to+\infty}{\approx} y_1 \sum_{m=0}^{+\infty} L_{0,m} y_1^{-3m}, \tag{2.52}$$

where

$$N_{0m} = \left(\frac{\Gamma_2}{2\alpha_1\Gamma_{12}}\right)^{1/2} \left(\frac{R_1^2}{\Gamma_2}\right)^m a_m \tag{2.53}$$

and

$$L_{00} = \frac{\Gamma_1}{2\alpha_2\Gamma_{12}} \quad \text{and for } m \geqslant 1, \quad L_{0m} = -\frac{\alpha_0}{\alpha_2} \sum_{m_1+m_2=m} N_{0m_1} N_{0m_2}. \tag{2.54}$$

Next, let us explain why δ_1, F_1, ν_1 and λ_1 admit asymptotic expansions like the ones given in (2.47), (2.48), (2.49), (2.50) and let us calculate explicitely the first terms in these expansions. Thanks to (2.43) for $n = 1$ as well as (2.30), (2.34), (2.52), (2.53), (2.54),we have

$$\begin{aligned} \delta_1 &= 2y_1\lambda_{-1}\lambda_0 + 2R_1^2\lambda_0''\lambda_{-1} + (R_2^2 - R_1^2)\lambda_0^2 - 6\alpha_2\lambda_0^2\lambda_{-1} - 4\alpha_0\lambda_{-1}\lambda_0\nu_0^2 \\ &= 2R_1^2\lambda_0''\lambda_{-1} \underset{y_1\to+\infty}{=} \frac{3\alpha_0 R_1^4(R_2^2 - R_1^2)}{\alpha_1\alpha_2^2\Gamma_{12}} y_1^{-4} + O(y_1^{-7}). \end{aligned} \tag{2.55}$$

Thus, the asymptotics as $y_1 \to +\infty$ in (2.47) holds with

$$D_{10} = 0 \quad \text{and} \quad D_{11} = \frac{3\alpha_0 R_1^4(R_2^2 - R_1^2)}{\alpha_1\alpha_2^2\Gamma_{12}}. \tag{2.56}$$

From (2.52) and (2.55), we also infer that $\delta_1 \underset{y_1\to-\infty}{\approx} 0$, which is the asymptotics as $y_1 \to -\infty$ in (2.47).

Then, (2.46) yields

$$F_1 = -\frac{4\alpha_0\alpha_2\nu_0}{(R_2^2-R_1^2)^2}\delta_1 - 2d\nu_0' - 4y_1\nu_0'' \tag{2.57}$$
$$\underset{y_1\to+\infty}{=} \left(\frac{\Gamma_2}{2\alpha_1\Gamma_{12}}\right)^{1/2}(1-d)y_1^{-1/2} + \frac{R_1^2}{(2\alpha_1\Gamma_2\Gamma_{12})^{1/2}}\left(\frac{5}{2}(7-d) - \frac{12\alpha_0^2\Gamma_2R_1^2}{\alpha_1\alpha_2\Gamma_{12}(R_2^2-R_1^2)}\right)y_1^{-7/2} + O(y_1^{-13/2}).$$

Thus, (2.51) and (2.47) for $n=1$ imply that (2.48) for $n=1$ holds with

$$F_{10} = \left(\frac{\Gamma_2}{2\alpha_1\Gamma_{12}}\right)^{1/2}(1-d) \quad \text{and} \quad F_{11} = \frac{R_1^2}{(2\alpha_1\Gamma_2\Gamma_{12})^{1/2}}\left(\frac{5}{2}(7-d) - \frac{12\alpha_0^2\Gamma_2R_1^2}{\alpha_1\alpha_2\Gamma_{12}(R_2^2-R_1^2)}\right).$$

In order to calculate ν_1 from (2.44), let us first notice that the function W defined in (2.45) coincides, up to a rescaling, to the function $W_0(y) = 3\gamma_0(y)^2 - y$ which was studied in [GP]. On the other side, W can be expressed in terms of λ_0 thanks to (2.34). Namely,

$$W(y_1) = \Gamma_2^{2/3}R_1^{2/3}W_0\left(\frac{\Gamma_2^{1/3}y_1}{R_1^{2/3}}\right) = \left(\frac{3\alpha_1\Gamma_1}{\alpha_0} + 2\Gamma_2\right)y_1 - \frac{6\alpha_1\alpha_2\Gamma_{12}}{\alpha_0}\lambda_0(y_1). \tag{2.58}$$

In particular, there exists $C>0$ such that $W(y_1) \geqslant C$ for every $y_1\in\mathbb{R}$, and W admits the asymptotic expansions

$$W(y_1) \underset{y_1\to-\infty}{\approx} -\Gamma_2 y_1, \quad W(y_1) \underset{y_1\to+\infty}{\approx} y_1\left(2\Gamma_2 - \frac{6\alpha_1\alpha_2\Gamma_{12}}{\alpha_0}\sum_{m=1}^{+\infty}L_{0,m}y_1^{-3m}\right) \tag{2.59}$$

In the case $d=1$, since $F_{10}=0$, $F_1\in L^2(\mathbb{R})$, and ν_1 is obtained by inversion of T, which is a Schrödinger operator on $L^2(\mathbb{R})$. Moreover, thanks to (2.59) and the positiveness of W, Lemma 2.1 in [GP] implies that the solution ν_1 to (2.44) admits asymptotic expansions like the ones given in (2.49), with

$$N_{1,0}=0, \quad N_{1,1} = -\frac{6R_1^2}{(2\alpha_1\Gamma_{12}\Gamma_2)^{1/2}}\left(\frac{\alpha_0^2R_1^2}{\alpha_1\alpha_2(R_2^2-R_1^2)\Gamma_{12}} - \frac{5}{4\Gamma_2}\right).$$

In the cases $d=2,3$, $F_{10}\neq 0$, and therefore $F_1\notin L^2(\mathbb{R})$. We construct the solution ν_1 to (2.44) by using the same trick as in [GP]. Namely, we look for ν_1 under the form

$$\nu_1 = \frac{F_{1,0}y_1^{-1/2}}{W(y_1)}\Phi(y_1) + \widetilde{\nu}_1,$$

where $\Phi\in\mathcal{C}^\infty(\mathbb{R})$ is such that $\Phi(y_1)\equiv 0$ for $y_1\leqslant 1/2$ and $\Phi(y_1)\equiv 1$ for $y_1\geqslant 1$, in such a way that

$$(-4R_1^2\partial_{y_1}^2 + W(y_1))\widetilde{\nu}_1 = F_1 - F_{1,0}y_1^{-1/2}\Phi(y_1) + 4R_1^2\frac{d^2}{dy_1^2}\left[\frac{F_{1,0}y_1^{-1/2}}{W(y_1)}\Phi(y_1)\right]. \tag{2.60}$$

The right hand side of (2.60) behaves now like $O(y_1^{-7/2})$ as $y_1\to+\infty$, and its behaviour at $-\infty$ is the same as the one of F_1, therefore the right hand side in (2.60) belongs to $L^2(\mathbb{R})$, and (2.60) has a unique solution $\widetilde{\nu}_1$ in $L^2(\mathbb{R})$. Moreover, again thanks to Lemma 2.1 in [GP], we deduce the existence of asymptotic expansions for $\widetilde{\nu}_1$ as $y_1\to\pm\infty$, with $\widetilde{\nu}_1(y_1) \underset{y_1\to+\infty}{=} O(y_1^{-9/2})$. These expansions for $\widetilde{\nu}_1$ imply that ν_1 has expansions like in (2.49), with

$$N_{1,0} = \frac{F_{1,0}}{2\Gamma_2} = \frac{1-d}{2(2\alpha_1\Gamma_2\Gamma_{12})^{1/2}}.$$

Then, from (2.42) and (2.55),

$$\lambda_1 = -2\frac{\alpha_0}{\alpha_2}\nu_0\nu_1 + \frac{4\alpha_2R_1^2}{(R_2^2-R_1^2)^2}\lambda_0''\lambda_{-1} \tag{2.61}$$

and thanks to (2.51), (2.55) and (2.49) for $n = 1$, λ_1 as asymptotic expansions like in (2.50) (in particular, $\lambda_1(y_1) \underset{y_1 \to -\infty}{\approx} 0$), with

$$L_{1,0} = \frac{\alpha_0(d-1)}{2\alpha_1\alpha_2\Gamma_{12}}, \quad \text{and if } d = 1, \quad L_{1,1} = \frac{3\alpha_0 R_1^2}{2\alpha_1\alpha_2\Gamma_{12}}\left(\frac{4R_1^2}{(R_2^2-R_1^2)\Gamma_{12}} - \frac{5}{\Gamma_2}\right).$$

For $n = 2$, (2.43) and (2.46) give similarly (after simplifications involving also (2.30), (2.34) and (2.61))

$$\delta_2 \quad = \quad -d\lambda_0'\lambda_{-1} - 2y_1\lambda_0''\lambda_{-1} - 2R_1^2\lambda_0''\lambda_0 - R_1^2\lambda_0'^2 + 2R_1^2\lambda_1''\lambda_{-1} - 2\alpha_0\lambda_{-1}^2\nu_1^2 \tag{2.62}$$

and

$$F_2 \quad = \quad -\frac{4\alpha_0\alpha_2\nu_0}{(R_2^2-R_1^2)^2}\delta_2 - 2d\nu_1' - 4y_1\nu_1'' - 6\alpha_1\nu_0\nu_1^2 - 2\alpha_0\lambda_1\nu_1 \tag{2.63}$$

which implies, thanks to the expansions calculated previously for λ_0, ν_0, λ_1 and ν_1 that δ_2 and F_2 satisfy respectively (2.47) and (2.48) for $n = 2$, with

$$D_{2,0} = -\frac{\Gamma_1\left(d\Gamma_{12}(R_2^2-R_1^2) + \Gamma_1 R_1^2\right)}{4\alpha_2^2\Gamma_{12}^2}, \quad \widetilde{D}_{2,0} = -\frac{d(R_2^2-R_1^2)+R_1^2}{4\alpha_2^2} \tag{2.64}$$

and

$$F_{2,0} = \frac{\alpha_0}{\alpha_2}\left(\frac{\Gamma_2}{2\alpha_1\Gamma_{12}}\right)^{1/2}\frac{\Gamma_1}{\Gamma_{12}^2}\frac{(d\Gamma_{12}(R_2^2-R_1^2)+\Gamma_1R_1^2)}{(R_2^2-R_1^2)^2}.$$

In order to solve (2.44) for $n = 2$, we look for ν_2 under the form

$$\nu_2 \quad = \quad \frac{F_{2,0}}{W(y_1)}y_1^{1/2}\Phi(y_1) + \widetilde{\nu}_2 \tag{2.65}$$

Then ν_2 solves (2.44) for $n = 2$ if and only if $\widetilde{\nu}_2$ solves

$$(-4R_1^2\partial_{y_1}^2 + W(y_1))\widetilde{\nu}_2 \quad = \quad F_2 - F_{2,0}y_1^{1/2}\Phi(y_1) + 4R_1^2\frac{d^2}{dy_1^2}\left[\frac{F_{2,0}y_1^{1/2}}{W(y_1)}\Phi(y_1)\right] \underset{y_1\to+\infty}{=} O(y_1^{-5/2}).$$

In particular, the right hand side in this equation belongs to $L^2(\mathbb{R})$. Thus by inversion of T like for $n = 1$ and $d = 2, 3$, and coming back to (2.65), ν_2 satisfies (2.49) for $n = 2$, with

$$N_{2,0} = \frac{F_{2,0}}{2\Gamma_2} = \frac{\alpha_0}{\alpha_2}\frac{1}{(2\alpha_1\Gamma_2\Gamma_{12})^{1/2}}\frac{\Gamma_1}{\Gamma_{12}^2}\frac{(d\Gamma_{12}(R_2^2-R_1^2)+\Gamma_1R_1^2)}{2(R_2^2-R_1^2)^2}.$$

As a result, from (2.42) for $n = 2$, λ_2 satisfies (2.50) for $n = 2$, with

$$L_{2,0} = -\frac{\Gamma_1\left(d\Gamma_{12}(R_2^2-R_1^2)+\Gamma_1R_1^2\right)}{2(R_2^2-R_1^2)^2\Gamma_{12}^3\alpha_2} \quad \text{and} \quad \widetilde{L}_{2,0} = -\frac{d(R_2^2-R_1^2)+R_1^2}{2\alpha_2(R_2^2-R_1^2)}. \tag{2.66}$$

Next, let us fix $n \geqslant 3$. We assume that we have constructed the ν_k's for $k \in \{1 \cdots n-1\}$, and that asymptotic expansions (2.47), (2.48), (2.49), (2.50) with n replaced by each of these k's are satisfied. Then it is clear from (2.43) and (2.46) that $F_n \approx 0$ as $y_1 \to -\infty$ as indicated in (2.48). In order to study the asymptotic expansion of δ_n as $y_1 \to +\infty$, let us first focus on the first sum in the right hand side of (2.43). If $n_1, n_2 \geqslant 1$ and $n_1 + n_2 = n - 3$, then it follows from (2.50) that $h_{n_1n_2}(y_1) := -d\lambda_{n_1}'\lambda_{n_2} - 2y_1\lambda_{n_1}''\lambda_{n_2} + y_1\lambda_{n_1}'\lambda_{n_2}'$ admits an asymptotic expansions which can be written as

$$y_1^{n-l}\sum_{m=0}^{+\infty} c_m y_1^{-3m}, \tag{2.67}$$

for some coefficients $(c_m)_{m\in\mathbb{N}}$, with $l = 8$. Thus, we deduce that

$$\sum_{\substack{n_1+n_2=n-3\\ n_1,n_2\geqslant 1}} h_{n_1n_2}(y_1) \underset{y_1\to+\infty}{\approx} y_1^{n-8}\sum_{m=0}^{+\infty}\check{D}_{n,m+2}y_1^{-3m} = y_1^{n-2}\sum_{m=2}^{+\infty}\check{D}_{n,m}y_1^{-3m},$$

for some coefficients $(\check{D}_{n,m})_{m\geqslant 2}$. Similarly, $h_{n_1n_2}(y_1)$ has an asymptotic expansion which can be written as (2.67) with $l=5$ if $n_1\geqslant 1$ and $n_2\leqslant 0$ or $n_1\leqslant 0$ and $n_2\geqslant 1$, and with $l=2$ if $n_1,n_2\leqslant 0$. As a result, the first sum in the right hand side of (2.43) admits an asymptotic expansion as $y_1\to+\infty$ like (2.67) with $l=2$, and in order to calculate the leading term cy_1^{n-2} in this expansion, one has to consider only the terms of the sum corresponding to indices $n_1,n_2\in\{-1,0\}$. The same kind of arguments applied to the other terms in the right hand side of (2.43) yields the asymptotic expansion of δ_n as $y_1\to+\infty$ given in (2.47). Moreover, in order to express D_{n0}, the only terms of the right hand side of (2.43) which have to be considered are written in the calculation below, where we use (2.52), (2.30), (2.50), (2.51) and (2.49)

$$
\begin{aligned}
\delta_n \underset{y_1\to+\infty}{=} & \left(-d\lambda_0'\lambda_{n-3}-2y_1\lambda_0''\lambda_{n-3}+y_1\lambda_0'\lambda_{n-3}'\right)\mathbf{1}_{\{n=3\}}\\
&+2y_1\lambda_{-1}\lambda_{n-1}+2y_1\lambda_0\lambda_{n-2}+2(R_2^2-R_1^2)\lambda_0\lambda_{n-1}-12\alpha_2\lambda_{-1}\lambda_0\lambda_{n-1}-6\alpha_2\lambda_0^2\lambda_{n-2}\\
&-8\alpha_0\lambda_{-1}\lambda_0\nu_0\nu_{n-1}-4\alpha_0\lambda_{-1}\lambda_{n-1}\nu_0^2-4\alpha_0\lambda_0^2\nu_0\nu_{n-2}-4\alpha_0\lambda_0\lambda_{n-2}\nu_0^2+O(y_1^{n-5})\\
\underset{y_1\to+\infty}{=} & \left(-d\frac{\Gamma_1}{2\alpha_2\Gamma_{12}}\frac{\Gamma_1}{2\alpha_2\Gamma_{12}}y_1+y_1\frac{\Gamma_1}{2\alpha_2\Gamma_{12}}\frac{\Gamma_1}{2\alpha_2\Gamma_{12}}\right)\mathbf{1}_{\{n=3\}}\\
&+2y_1\frac{R_2^2-R_1^2}{2\alpha_2}y_1^{n-3}L_{n-1,0}+2y_1\left(\frac{\Gamma_1}{2\alpha_2\Gamma_{12}}y_1\right)y_1^{n-4}L_{n-2,0}\\
&+2(R_2^2-R_1^2)\left(\frac{\Gamma_1}{2\alpha_2\Gamma_{12}}y_1\right)y_1^{n-3}L_{n-1,0}\\
&-12\alpha_2\frac{R_2^2-R_1^2}{2\alpha_2}\left(\frac{\Gamma_1}{2\alpha_2\Gamma_{12}}y_1\right)y_1^{n-3}L_{n-1,0}-6\alpha_2\left(\frac{\Gamma_1}{2\alpha_2\Gamma_{12}}y_1\right)^2y_1^{n-4}L_{n-2,0}\\
&-8\alpha_0\frac{R_2^2-R_1^2}{2\alpha_2}\left(\frac{\Gamma_1}{2\alpha_2\Gamma_{12}}y_1\right)\left(\left(\frac{\Gamma_2}{2\alpha_1\Gamma_{12}}\right)^{1/2}y_1^{1/2}\right)y_1^{n-7/2}N_{n-1,0}\\
&-4\alpha_0\frac{R_2^2-R_1^2}{2\alpha_2}y_1^{n-3}L_{n-1,0}\left(\left(\frac{\Gamma_2}{2\alpha_1\Gamma_{12}}\right)^{1/2}y_1^{1/2}\right)^2\\
&-4\alpha_0\left(\frac{\Gamma_1}{2\alpha_2\Gamma_{12}}y_1\right)^2\left(\left(\frac{\Gamma_2}{2\alpha_1\Gamma_{12}}\right)^{1/2}y_1^{1/2}\right)y_1^{n-9/2}N_{n-2,0}\\
&-4\alpha_0\left(\frac{\Gamma_1}{2\alpha_2\Gamma_{12}}y_1\right)y_1^{n-4}L_{n-2,0}\left(\left(\frac{\Gamma_2}{2\alpha_1\Gamma_{12}}\right)^{1/2}y_1^{1/2}\right)^2+O(y_1^{n-5})\\
\underset{y_1\to+\infty}{=} & y_1\frac{(1-d)\Gamma_1^2}{4\alpha_2^2\Gamma_{12}^2}\mathbf{1}_{\{n=3\}}+y_1^{n-2}\left(\frac{R_2^2-R_1^2}{\alpha_2}L_{n-1,0}+\frac{\Gamma_1}{\alpha_2\Gamma_{12}}L_{n-2,0}\right)+(R_2^2-R_1^2)y_1^{n-2}\frac{\Gamma_1}{\alpha_2\Gamma_{12}}L_{n-1,0}\\
&-\frac{3(R_2^2-R_1^2)\Gamma_1}{\alpha_2\Gamma_{12}}y_1^{n-2}L_{n-1,0}-\frac{3\Gamma_1^2}{2\alpha_2\Gamma_{12}^2}y_1^{n-2}L_{n-2,0}-2\alpha_0\frac{(R_2^2-R_1^2)\Gamma_1\Gamma_2^{1/2}N_{n-1,0}}{\alpha_2^2(2\alpha_1)^{1/2}\Gamma_{12}^{3/2}}y_1^{n-2}\\
&-\alpha_0\frac{(R_2^2-R_1^2)\Gamma_2L_{n-1,0}}{\alpha_1\alpha_2\Gamma_{12}}y_1^{n-2}-\frac{\alpha_0\Gamma_1^2\Gamma_2^{1/2}N_{n-2,0}}{\alpha_2^2(2\alpha_1)^{1/2}\Gamma_{12}^{5/2}}y_1^{n-2}-\frac{\alpha_0\Gamma_1\Gamma_2L_{n-2,0}}{\alpha_1\alpha_2\Gamma_{12}^2}y_1^{n-2}+O(y_1^{n-5})\\
\underset{y_1\to+\infty}{=} & y_1^{n-2}D_{n,0}+O(y_1^{n-5}),
\end{aligned}
\tag{2.68}
$$

with

$$
\begin{aligned}
D_{n,0} = & \frac{(1-d)\Gamma_1^2}{4\alpha_2^2\Gamma_{12}^2}\mathbf{1}_{\{n=3\}}-\frac{(R_2^2-R_1^2)\Gamma_1}{\alpha_2\Gamma_{12}}L_{n-1,0}-\frac{\Gamma_1^2}{2\alpha_2\Gamma_{12}^2}L_{n-2,0}\\
&-2\frac{\alpha_0(R_2^2-R_1^2)\Gamma_1\Gamma_2^{1/2}}{\alpha_2^2(2\alpha_1)^{1/2}\Gamma_{12}^{3/2}}N_{n-1,0}-\frac{\alpha_0\Gamma_1^2\Gamma_2^{1/2}}{\alpha_2^2(2\alpha_1)^{1/2}\Gamma_{12}^{5/2}}N_{n-2,0}.
\end{aligned}
\tag{2.69}
$$

The existence of an asymptotic expansion of $\delta_n(y_1)$ as $y_1\to-\infty$ like the one given in (2.47) follows

from (2.43) similarly as for the expansion at $y_1 = +\infty$. Moreover, like in (2.68), we obtain

$$\begin{aligned}\delta_n &\underset{y_1\to-\infty}{=} \left(-d\lambda_0'\lambda_{n-3} + y_1\lambda_0'\lambda_{n-3}'\right)\mathbf{1}_{\{n=3\}} \\ &\quad +2y_1\lambda_{-1}\lambda_{n-1} + 2y_1\lambda_0\lambda_{n-2} + 2(R_2^2 - R_1^2)\lambda_0\lambda_{n-1} - 12\alpha_2\lambda_{-1}\lambda_0\lambda_{n-1} - 6\alpha_2\lambda_0^2\lambda_{n-2} + O(y_1^{n-5}) \\ &= \widetilde{D}_{n,0}y_1^{n-2} + O(y_1^{n-5}), \end{aligned} \tag{2.70}$$

with

$$\widetilde{D}_{n,0} = \frac{1-d}{4\alpha_2^2}\mathbf{1}_{\{n=3\}} - \left(\widetilde{L}_{n-1,0}\frac{R_2^2 - R_1^2}{\alpha_2} + \frac{\widetilde{L}_{n-2,0}}{2\alpha_2}\right). \tag{2.71}$$

Then, (2.48) follows from (2.46), (2.51), (2.47) and the recursion assumption. Moreover,

$$F_{n,0} = -\frac{4\alpha_0\alpha_2}{(R_2^2 - R_1^2)^2}\left(\frac{\Gamma_2}{2\alpha_1\Gamma_{12}}\right)^{1/2} D_{n,0}. \tag{2.72}$$

Then, using the same trick as for $n = 2$, we look for a solution ν_n of (2.44) under the form

$$\nu_n = \frac{F_{n,0}}{W(y_1)}y_1^{n-3/2}\Phi(y_1) + \widetilde{\nu_n}.$$

ν_n solves (2.44) if and only if $\widetilde{\nu_n}$ solves

$$T\widetilde{\nu_n} = \widetilde{F_n}, \quad \text{where} \quad \widetilde{F_n} = F_n - F_{n,0}y_1^{n-3/2}\Phi(y_1) + 4R_1^2\frac{d^2}{dy_1^2}\left[\frac{F_{n,0}y_1^{n-3/2}}{W(y_1)}\Phi(y_1)\right].$$

The function $\widetilde{F_n}$ defined just above admits expansions as $y_1 \to \pm\infty$ which are similar to those satisfied by F_n and given in (2.48), except that in the expansion of $\widetilde{F_n}$ as $y_1 \to +\infty$, the power of y_1 in the leading term is smaller from three units than the one of F_n. By iterating this process a finite number of times, we are brought back to solve an equation like (2.44), but with a right hand side which is in $L^2(\mathbb{R})$. Thanks to Lemma 2.1 in [GP] and (2.59), it turns out that ν_n satisfies (2.49), where the coefficient in the leading term as $y_1 \to +\infty$ is

$$N_{n,0} = \frac{F_{n,0}}{2\Gamma_2} = -\frac{2\alpha_0\alpha_2}{(R_2^2 - R_1^2)^2\,(2\alpha_1\Gamma_{12}\Gamma_2)^{1/2}}D_{n,0}. \tag{2.73}$$

Finally, from (2.42), (2.51), (2.49), (2.47) and (2.73), we deduce that λ_n satisfies (2.50), with

$$L_{n,0} = -2\frac{\alpha_0}{\alpha_2}\left(\frac{\Gamma_2}{2\alpha_1\Gamma_{12}}\right)^{1/2} N_{0,n} + \frac{2\alpha_2}{(R_2^2 - R_1^2)^2}D_{n0} = \frac{2\alpha_2}{(R_2^2 - R_1^2)^2\Gamma_{12}}D_{n0}, \tag{2.74}$$

and

$$\widetilde{L}_{n,0} = \frac{2\alpha_2\widetilde{D}_{n,0}}{(R_2^2 - R_1^2)^2}, \tag{2.75}$$

which completes the recursion and proves that (2.47), (2.48), (2.49) and (2.50) hold for every $n \geqslant 1$. In addition, one can compute explicitely the coefficients of the leading terms in the expansions of δ_n, F_n, ν_n and λ_n as $y_1 \to \pm\infty$. Indeed, as $y_1 \to +\infty$, according to (2.69), (2.73) and (2.74), we have, for

every $n \geqslant 3$,

$$\begin{aligned}
D_{n,0} &= \frac{(1-d)\Gamma_1^2}{4\alpha_2^2\Gamma_{12}^2}\mathbf{1}_{\{n=3\}} - \frac{(R_2^2-R_1^2)\Gamma_1}{\alpha_2\Gamma_{12}}\frac{2\alpha_2}{(R_2^2-R_1^2)^2\Gamma_{12}}D_{n-1,0} - \frac{\Gamma_1^2}{2\alpha_2\Gamma_{12}^2}\frac{2\alpha_2}{(R_2^2-R_1^2)^2\Gamma_{12}}D_{n-2,0} \\
&\quad + 2\frac{\alpha_0(R_2^2-R_1^2)\Gamma_1\Gamma_2^{1/2}}{\alpha_2^2(2\alpha_1)^{1/2}\Gamma_{12}^{3/2}}\frac{2\alpha_0\alpha_2}{(R_2^2-R_1^2)^2\left(2\alpha_1\Gamma_{12}\Gamma_2\right)^{1/2}}D_{n-1,0} \\
&\quad + \frac{\alpha_0\Gamma_1^2\Gamma_2^{1/2}}{\alpha_2^2(2\alpha_1)^{1/2}\Gamma_{12}^{5/2}}\frac{2\alpha_0\alpha_2}{(R_2^2-R_1^2)^2\left(2\alpha_1\Gamma_{12}\Gamma_2\right)^{1/2}}D_{n-2,0} \\
&= \frac{(1-d)\Gamma_1^2}{4\alpha_2^2\Gamma_{12}^2}\mathbf{1}_{\{n=3\}} - \frac{2\Gamma_1}{(R_2^2-R_1^2)\Gamma_{12}^2}D_{n-1,0} - \frac{\Gamma_1^2}{(R_2^2-R_1^2)^2\Gamma_{12}^3}D_{n-2,0} \\
&\quad + \frac{2\alpha_0^2\Gamma_1}{\alpha_1\alpha_2\Gamma_{12}^2(R_2^2-R_1^2)}D_{n-1,0} + \frac{\alpha_0^2\Gamma_1^2}{\alpha_1\alpha_2\Gamma_{12}^3(R_2^2-R_1^2)^2}D_{n-2,0} \\
&= \frac{(1-d)\Gamma_1^2}{4\alpha_2^2\Gamma_{12}^2}\mathbf{1}_{\{n=3\}} - \frac{2\Gamma_1}{(R_2^2-R_1^2)\Gamma_{12}}D_{n-1,0} - \frac{\Gamma_1^2}{(R_2^2-R_1^2)^2\Gamma_{12}^2}D_{n-2,0}.
\end{aligned}$$

Thus, defining for every $n \in \mathbb{N}$

$$d_n = \left(\frac{(R_2^2-R_1^2)\Gamma_{12}}{\Gamma_1}\right)^n D_{n0}, \tag{2.76}$$

we infer thanks to (2.56) and (2.64)

$$d_1 = 0, \quad d_2 = -(R_2^2-R_1^2)^2\frac{d\Gamma_{12}(R_2^2-R_1^2)+\Gamma_1R_1^2}{4\alpha_2^2\Gamma_1}, \quad d_3 = (R_2^2-R_1^2)^2\frac{(1+d)\Gamma_{12}(R_2^2-R_1^2)+2\Gamma_1R_1^2}{4\alpha_2^2\Gamma_1},$$

and

$$\forall n \geqslant 4,\ d_n = -2d_{n-1} - d_{n-2}.$$

It follows that for $n \geqslant 2$,

$$d_n = -\frac{(-1)^n(R_2^2-R_1^2)^2\left(\Gamma_{12}(R_2^2-R_1^2)(d+n-2)+(n-1)\Gamma_1R_1^2\right)}{4\alpha_2^2\Gamma_1},$$

and therefore

$$D_{n0} = -\left(\frac{-\Gamma_1}{\Gamma_{12}(R_2^2-R_1^2)}\right)^n \frac{(R_2^2-R_1^2)^2\left(\Gamma_{12}(R_2^2-R_1^2)(n+d-2)+(n-1)\Gamma_1R_1^2\right)}{4\alpha_2^2\Gamma_1}.$$

Coming back to (2.73) and (2.74), we get, for $n \geqslant 3$,

$$N_{n,0} = \left(\frac{-\Gamma_1}{\Gamma_{12}(R_2^2-R_1^2)}\right)^n \frac{\alpha_0(\Gamma_{12}(R_2^2-R_1^2)(n+d-2)+\Gamma_1R_1^2(n-1))}{2\alpha_2\Gamma_1\left(2\alpha_1\Gamma_{12}\Gamma_2\right)^{1/2}} \tag{2.77}$$

and

$$L_{n,0} = -\left(\frac{-\Gamma_1}{\Gamma_{12}(R_2^2-R_1^2)}\right)^n \frac{\Gamma_{12}(R_2^2-R_1^2)(n+d-2)+\Gamma_1R_1^2(n-1)}{2\alpha_2\Gamma_1\Gamma_{12}}. \tag{2.78}$$

Similarly, as $y_1 \to -\infty$, for $n \geqslant 3$, from (2.71) and (2.75) we get

$$\widetilde{D}_{n,0} = \frac{1-d}{4\alpha_2^2}\mathbf{1}_{\{n=3\}} - \frac{2}{R_2^2-R_1^2}\widetilde{D}_{n-1,0} - \frac{1}{(R_2^2-R_1^2)^2}\widetilde{D}_{n-2,0}.$$

Since from (2.47) for $n=1$ and (2.64), we have

$$\widetilde{D}_{1,0} = 0, \quad \widetilde{D}_{2,0} = -\frac{d(R_2^2-R_1^2)+R_1^2}{4\alpha_2^2},$$

we deduce

$$\widetilde{D}_{3,0} = \frac{d(R_2^2-R_1^2)+R_1^2+R_2^2}{4\alpha_2^2(R_2^2-R_1^2)}$$

and for $n \geqslant 4$,

$$\widetilde{D}_{n,0} = \frac{(-1)^{n+1}}{4\alpha_2^2(R_2^2-R_1^2)^{n-2}}((d-2+n)(R_2^2-R_1^2)+(n-1)R_1^2),$$

and therefore thanks to (2.75), we obtain

$$\widetilde{L}_{n,0} = \frac{(-1)^{n+1}}{2\alpha_2(R_2^2-R_1^2)^{n}}((d-2+n)(R_2^2-R_1^2)+(n-1)R_1^2).$$

The main results obtained in this section are summarized in the following proposition.

Proposition 2.7

$$\begin{aligned}
\lambda_{-1}(y_1) &= \frac{R_2^2-R_1^2}{2\alpha_2}\\
\nu_0(y_1) &= \frac{R_1^{1/3}\Gamma_2^{1/3}}{(2\alpha_1)^{1/2}\Gamma_{12}^{1/2}}\gamma_0\left(\frac{\Gamma_2^{1/3}y_1}{R_1^{2/3}}\right) \underset{y_1\to+\infty}{=} \left(\frac{\Gamma_2}{2\alpha_1\Gamma_{12}}\right)^{1/2} y_1^{1/2}+O(y_1^{-5/2})\\
\nu_0(y_1) &\underset{y_1\to-\infty}{\approx} 0\\
\lambda_0(y_1) &= \frac{y_1}{2\alpha_2}-\frac{\alpha_0}{\alpha_2}\nu_0(y_1)^2 \underset{y_1\to+\infty}{=} \frac{\Gamma_1}{2\alpha_2\Gamma_{12}}y_1+O(y_1^{-2})\\
\lambda_0(y_1) &\underset{y_1\to-\infty}{\approx} \frac{y_1}{2\alpha_2}\\
\nu_1(y_1) &= T^{-1}\left(-\frac{8\alpha_0\alpha_2R_1^2\lambda_0''\lambda_{-1}\nu_0}{(R_2^2-R_1^2)^2}-2d\nu_0'-4y_1\nu_0''\right)\\
&\underset{y_1\to+\infty}{=} \begin{cases} \frac{1-d}{2(2\alpha_1\Gamma_2\Gamma_{12})^{1/2}}y_1^{-3/2}+O(y_1^{-9/2}) & \text{if } d=2,3\\ -\frac{6R_1^2}{(2\alpha_1\Gamma_{12}\Gamma_2)^{1/2}}\left(\frac{\alpha_0^2R_1^2}{\alpha_1\alpha_2(R_2^2-R_1^2)\Gamma_{12}}-\frac{5}{4\Gamma_2}\right)y_1^{-9/2}+O(y_1^{-15/2}) & \text{if } d=1\end{cases}\\
\nu_1(y_1) &\underset{y_1\to-\infty}{\approx} 0\\
\lambda_1(y_1) &= -2\frac{\alpha_0}{\alpha_2}\nu_0\nu_1+\frac{4\alpha_2R_1^2}{(R_2^2-R_1^2)^2}\lambda_0''\lambda_{-1}\\
&\underset{y_1\to+\infty}{=} \begin{cases} \frac{\alpha_0(d-1)}{2\alpha_1\alpha_2\Gamma_{12}}y_1^{-1}+O(y_1^{-4}) & \text{if } d=2,3\\ \frac{3\alpha_0R_1^2}{2\alpha_1\alpha_2\Gamma_{12}}\left(\frac{4R_1^2}{(R_2^2-R_1^2)\Gamma_{12}}-\frac{5}{\Gamma_2}\right)y_1^{-4}+O(y_1^{-7}) & \text{if } d=1.\end{cases}\\
\lambda_1(y_1) &\underset{y_1\to-\infty}{\approx} 0
\end{aligned}$$

and for $n \geqslant 2$,

$$\begin{aligned}
\nu_n(y_1) &\underset{y_1\to+\infty}{=} \left(\frac{-\Gamma_1}{\Gamma_{12}(R_2^2-R_1^2)}\right)^n \frac{\alpha_0(\Gamma_{12}(R_2^2-R_1^2)(n+d-2)+\Gamma_1R_1^2(n-1))}{2\alpha_2\Gamma_1\left(2\alpha_1\Gamma_{12}\Gamma_2\right)^{1/2}}y_1^{n-5/2}+O(y_1^{n-11/2})\\
\nu_n(y_1) &\underset{y_1\to-\infty}{\approx} 0\\
\lambda_n(y_1) &\underset{y_1\to+\infty}{=} -\left(\frac{-\Gamma_1}{\Gamma_{12}(R_2^2-R_1^2)}\right)^n \frac{\Gamma_{12}(R_2^2-R_1^2)(n+d-2)+\Gamma_1R_1^2(n-1)}{2\alpha_2\Gamma_1\Gamma_{12}}y_1^{n-2}+O(y_1^{n-5})\\
\lambda_n(y_1) &\underset{y_1\to-\infty}{=} \frac{(-1)^{n+1}}{2\alpha_2(R_2^2-R_1^2)^n}((d-2+n)(R_2^2-R_1^2)+(n-1)R_1^2)y_1^{n-2}+O(y_1^{n-5}).
\end{aligned}$$

3 Truncation of the asymptotic expansions

In section 2, we have explained how to calculate asymptotic expansions into powers of ε of ω, τ, ν, λ and μ in such a way that (1.15), (1.16), (1.17), (1.18) and (1.19) provide formally solutions to (1.13) at any order. However, we have not said anything about the convergence of these formal series. In this section, we prove that the truncations of the formal series at a finite order provide approximate solutions to (1.13) at a arbitrarily high order in terms of powers of ε. More precisely, M, N and L are

three fixed positive integers, and we set in all the section

$$\begin{aligned}\omega(z)&=\sum_{m=0}^{M}\varepsilon^{2m}\omega_m(z), \qquad & \tau(z)&=\sum_{m=0}^{M}\varepsilon^{2m}\tau_m(z),\\ \nu(y_1)&=\sum_{n=0}^{N}\varepsilon^{2n/3}\nu_n(y_1), \qquad & \lambda(y_1)&=\sum_{n=-1}^{N}\varepsilon^{2n/3}\lambda_n(y_1),\\ & & \mu(y_2)&=\sum_{n=0}^{L}\varepsilon^{2n/3}\mu_n(y_2),\end{aligned} \tag{3.1}$$

where the ω_m's, τ_m's, ν_n's, λ_n's and μ_n's are the ones calculated in Section 2. The way integers M, N and L are chosen is explained in Sections 3.5 and 3.6 below.

3.1 Consistency of the ansatz

Ansatz (1.20) requires the calculation of $\lambda(y_1)^{1/2}$ for $x \in \mathrm{Supp}\chi_\varepsilon \subset D_1$. So it makes sense to combine (1.19) and (3.1) only if the function λ given by (3.1) satisfies $\lambda(y_1) \geqslant 0$ for $x \in \mathrm{Supp}\chi_\varepsilon$. We next show that the last inequality indeed holds for $x \in D_1$.

Lemma 3.1 *Let $N > 0$ and λ given by (3.1). There exists $C > 0$ (which might depend on N) such that for $\varepsilon \in (0,1]$ sufficiently small, for every $x \in D_1$,*

$$\lambda(y_1) \geqslant C\varepsilon^{-2/3}.$$

Proof. Let $x \in D_1$. Then $y_2 \geqslant (R_2^2 - R_1^2)/\varepsilon^{2/3} - 2\varepsilon^{\beta-2/3}$, $-2\varepsilon^{\beta-2/3} \leqslant y_1 \leqslant 2\varepsilon^{\beta-2/3}$, and since γ_0 is increasing and $\gamma_0(y) \underset{y\to+\infty}{\sim} \sqrt{y}$, we get on the one side

$$\varepsilon^{-2/3}\lambda_{-1}+\lambda_0(y_1) = \frac{y_2}{2\alpha_2} - \frac{\alpha_0}{\alpha_2}\nu_0(y_1)^2 \geqslant \frac{R_2^2-R_1^2}{2\alpha_2\varepsilon^{2/3}} - \frac{\varepsilon^{\beta-2/3}}{\alpha_2} - \frac{\alpha_0}{\alpha_2}\nu_0(2\varepsilon^{\beta-2/3})^2 = \frac{R_2^2-R_1^2}{2\alpha_2\varepsilon^{2/3}} + O(\varepsilon^{\beta-2/3}),$$

whereas for $n \geqslant 1$, thanks to (2.50)

$$\begin{aligned}|\varepsilon^{2n/3}\lambda_n(y_1)| &\leqslant c_n\varepsilon^{2n/3}\left(\mathbf{1}_{\{|y_1|\leqslant 1\}} + |y_1|^{n-2}\mathbf{1}_{\{|y_1|\geqslant 1\}}\right) \leqslant \tilde{c}_n\varepsilon^{2n/3}\max(1,\varepsilon^{(\beta-2/3)(n-2)})\\ &\leqslant \tilde{c}_n\max(\varepsilon^{2n/3},\varepsilon^{\beta(n-2)+4/3}) = O(\varepsilon^{2/3}),\end{aligned} \tag{3.2}$$

for some $c_n > 0$ and $\tilde{c}_n = 2^{n-2}c_n$. As a result, for ε sufficiently small, we have $\lambda(y_1) \geqslant \frac{R_2^2-R_1^2}{4\alpha_2\varepsilon^{2/3}}$ for every $x \in D_1$. . ∎

3.2 Truncation of (ω,τ) in D_0

In this section, we prove that (3.1) provides an approximate solution to (1.13) in D_0 at an arbitrarily high order. For convenience, we use the same notation ω for the functions $z \mapsto \omega(z)$ and $x \mapsto \omega(z) = \omega(R_1^2 - |x|^2)$.

Lemma 3.2 *Let $M \geqslant 1$ be an integer, $\beta \in (0,2/3)$ and ω,τ given by (3.1). Then*

$$\left\|\varepsilon^2\Delta\omega + \frac{\alpha_0}{\alpha_2}(R_2^2-R_1^2)\omega + z\omega - 2\alpha_1\omega^3 - 2\alpha_0\tau^2\omega\right\|_{L^\infty(D_0)} = O(\varepsilon^{(2-3\beta)M+2-3\beta/2})$$

and

$$\left\|\varepsilon^2\Delta\tau + (R_2^2-R_1^2+z)\tau - 2\alpha_2\tau^3 - 2\alpha_0\omega^2\tau\right\|_{L^\infty(D_0)} = O(\varepsilon^{(2-3\beta)M+2-2\beta}).$$

Proof. Thanks to (2.6), (2.7) and (2.8), we have

$$\begin{aligned}
&\varepsilon^2\Delta\omega+\frac{\alpha_0}{\alpha_2}(R_2^2-R_1^2)\omega+z\omega-2\alpha_1\omega^3-2\alpha_0\tau^2\omega\\
&=\sum_{m=1}^{M+1}\varepsilon^{2m}\Delta\omega_{m-1}+\frac{\alpha_0}{\alpha_2}(R_2^2-R_1^2)\sum_{m=0}^{M}\varepsilon^{2m}\omega_m+z\sum_{m=0}^{M}\varepsilon^{2m}\omega_m.\\
&\quad-2\alpha_1\sum_{m=0}^{3M}\varepsilon^{2m}\sum_{\substack{m_1+m_2+m_3=m\\0\leqslant m_1,m_2,m_3\leqslant M}}\omega_{m_1}\omega_{m_2}\omega_{m_3}-2\alpha_0\sum_{m=0}^{3M}\varepsilon^{2m}\sum_{\substack{m_1+m_2+m_3=m\\0\leqslant m_1,m_2,m_3\leqslant M}}\tau_{m_1}\tau_{m_2}\omega_{m_3}\\
&=\varepsilon^{2(M+1)}\Delta\omega_M-2\alpha_1\sum_{m=M+1}^{3M}\varepsilon^{2m}\sum_{\substack{m_1+m_2+m_3=m\\0\leqslant m_1,m_2,m_3\leqslant M}}\omega_{m_1}\omega_{m_2}\omega_{m_3}\\
&\quad-2\alpha_0\sum_{m=M+1}^{3M}\varepsilon^{2m}\sum_{\substack{m_1+m_2+m_3=m\\0\leqslant m_1,m_2,m_3\leqslant M}}\tau_{m_1}\tau_{m_2}\omega_{m_3}.
\end{aligned}\tag{3.3}$$

From Lemma 2.1, (2.6), (2.7) and Remark 2.2, we infer that for every $x\in D_0$,

$$\begin{aligned}
&|\varepsilon^2\Delta\omega+\frac{\alpha_0}{\alpha_2}(R_2^2-R_1^2)\omega+z\omega-2\alpha_1\omega^3-2\alpha_0\tau^2\omega|\\
&\lesssim\ \varepsilon^{2(M+1)}z^{-3/2-3M}+\sum_{m=M+1}^{3M}\varepsilon^{2m}z^{3/2-3m}+\sum_{m=M+1}^{3M}\varepsilon^{2m}z^{5/2-3m}\lesssim\varepsilon^{(2-3\beta)M+2-3\beta/2}.
\end{aligned}\tag{3.4}$$

Similarly,

$$\begin{aligned}
&\varepsilon^2\Delta\tau+(R_2^2-R_1^2+z)\tau-2\alpha_2\tau^3-2\alpha_0\omega^2\tau\\
&=\sum_{m=1}^{M+1}\varepsilon^{2m}\Delta\tau_{m-1}+(R_2^2-R_1^2+z)\sum_{m=0}^{M}\varepsilon^{2m}\tau_m\\
&\quad-2\alpha_2\sum_{m=0}^{3M}\varepsilon^{2m}\sum_{\substack{m_1+m_2+m_3=m\\0\leqslant m_1,m_2,m_3\leqslant M}}\tau_{m_1}\tau_{m_2}\tau_{m_3}-2\alpha_0\sum_{m=0}^{3M}\varepsilon^{2m}\sum_{\substack{m_1+m_2+m_3=m\\0\leqslant m_1,m_2,m_3\leqslant M}}\omega_{m_1}\omega_{m_2}\tau_{m_3}\\
&=\varepsilon^{2(M+1)}\Delta\tau_M-2\alpha_2\sum_{m=M+1}^{3M}\varepsilon^{2m}\sum_{\substack{m_1+m_2+m_3=m\\0\leqslant m_1,m_2,m_3\leqslant M}}\tau_{m_1}\tau_{m_2}\tau_{m_3}\\
&\quad-2\alpha_0\sum_{m=M+1}^{3M}\varepsilon^{2m}\sum_{\substack{m_1+m_2+m_3=m\\0\leqslant m_1,m_2,m_3\leqslant M}}\omega_{m_1}\omega_{m_2}\tau_{m_3},
\end{aligned}\tag{3.5}$$

thus for $x\in D_0$,

$$\begin{aligned}
&|\varepsilon^2\Delta\tau+(R_2^2-R_1^2+z)\tau-2\alpha_2\tau^3-2\alpha_0\omega^2\tau|\\
&\lesssim\ \varepsilon^{(2-3\beta)M+2-\beta}+\sum_{m=M+1}^{3M}\varepsilon^{2m}z^{1-3m}\lesssim\varepsilon^{(2-3\beta)M+2-2\beta}.
\end{aligned}\tag{3.6}$$

∎

3.3 Truncation of $(\varepsilon^{1/3}\nu,\varepsilon^{1/3}\lambda^{1/2})$ in D_1

Lemma 3.3 *Let $N\geqslant 4$ be an integer, and ν,λ given by (3.1). Then*

$$\left\|\varepsilon^2\Delta\left(\varepsilon^{1/3}\nu\right)+\left(\frac{\alpha_0}{\alpha_2}(R_2^2-R_1^2)+z\right)\varepsilon^{1/3}\nu-2\alpha_1\left(\varepsilon^{1/3}\nu\right)^3-2\alpha_0\left(\varepsilon^{1/3}\lambda^{1/2}\right)^2\varepsilon^{1/3}\nu\right\|_{L^\infty(D_1)}=O(\varepsilon^{\beta N+4-7\beta/2})$$

and

$$\left\|\varepsilon^2\Delta\left(\varepsilon^{1/3}\lambda^{1/2}\right)+(R_2^2-R_1^2+z)\varepsilon^{1/3}\lambda^{1/2}-2\alpha_2\left(\varepsilon^{1/3}\lambda^{1/2}\right)^3-2\alpha_0\left(\varepsilon^{1/3}\nu\right)^2\varepsilon^{1/3}\lambda^{1/2}\right\|_{L^\infty(D_1)}=O(\varepsilon^{\beta N+2-\beta}).$$

Proof. Using (3.1), (2.32) and (2.40) for $n\in\{1,\cdots,N\}$, we get

$$\begin{aligned}
&\varepsilon^{-1}\left(\varepsilon^2\Delta\left(\varepsilon^{1/3}\nu\right)+\left(\frac{\alpha_0}{\alpha_2}(R_2^2-R_1^2)+z\right)\varepsilon^{1/3}\nu-2\alpha_1\left(\varepsilon^{1/3}\nu\right)^3-2\alpha_0\left(\varepsilon^{1/3}\lambda^{1/2}\right)^2\varepsilon^{1/3}\nu\right)\\
&= \varepsilon^{4/3}\Delta\nu+\frac{\alpha_0}{\alpha_2}\frac{R_2^2-R_1^2}{\varepsilon^{2/3}}\nu+y_1\nu-2\alpha_1\nu^3-2\alpha_0\lambda\nu\\
&= -2d\varepsilon^{2/3}\nu'+4R_1^2\nu''-4\varepsilon^{2/3}y_1\nu''+\frac{\alpha_0}{\alpha_2}\frac{R_2^2-R_1^2}{\varepsilon^{2/3}}\nu+y_1\nu-2\alpha_1\nu^3-2\alpha_0\lambda\nu\\
&= -2d\sum_{n=1}^{N+1}\varepsilon^{2n/3}\nu'_{n-1}+4R_1^2\sum_{n=0}^{N}\varepsilon^{2n/3}\nu''_n-4y_1\sum_{n=1}^{N+1}\varepsilon^{2n/3}\nu''_{n-1}+\frac{\alpha_0}{\alpha_2}(R_2^2-R_1^2)\sum_{n=-1}^{N-1}\varepsilon^{2n/3}\nu_{n+1}\\
&\quad+y_1\sum_{n=0}^{N}\varepsilon^{2n/3}\nu_n-2\alpha_1\sum_{n=0}^{3N}\varepsilon^{2n/3}\sum_{\substack{n_1+n_2+n_3=n\\0\leqslant n_1,n_2,n_3\leqslant N}}\nu_{n_1}\nu_{n_2}\nu_{n_3}-2\alpha_0\sum_{n=-1}^{2N}\varepsilon^{2n/3}\sum_{\substack{n_1+n_2=n\\-1\leqslant n_1\leqslant N\\0\leqslant n_2\leqslant N}}\lambda_{n_1}\nu_{n_2}\\
&= -2d\sum_{n=1}^{N+1}\varepsilon^{2n/3}\nu'_{n-1}+4R_1^2\sum_{n=0}^{N}\varepsilon^{2n/3}\nu''_n-4y_1\sum_{n=1}^{N+1}\varepsilon^{2n/3}\nu''_{n-1}\\
&\quad+y_1\sum_{n=0}^{N}\varepsilon^{2n/3}\nu_n-2\alpha_1\sum_{n=0}^{3N}\varepsilon^{2n/3}\sum_{\substack{n_1+n_2+n_3=n\\0\leqslant n_1,n_2,n_3\leqslant N}}\nu_{n_1}\nu_{n_2}\nu_{n_3}-2\alpha_0\sum_{n=0}^{2N}\varepsilon^{2n/3}\sum_{\substack{n_1+n_2=n\\0\leqslant n_1,n_2\leqslant N}}\lambda_{n_1}\nu_{n_2}\\
&= -2d\varepsilon^{2(N+1)/3}\nu'_N-4y_1\varepsilon^{2(N+1)/3}\nu''_N\\
&\quad-2\alpha_1\sum_{n=N+1}^{3N}\varepsilon^{2n/3}\sum_{\substack{n_1+n_2+n_3=n\\0\leqslant n_1,n_2,n_3\leqslant N}}\nu_{n_1}\nu_{n_2}\nu_{n_3}-2\alpha_0\sum_{n=N+1}^{2N}\varepsilon^{2n/3}\sum_{\substack{n_1+n_2=n\\0\leqslant n_1,n_2\leqslant N}}\lambda_{n_1}\nu_{n_2} \qquad (3.7)\\
&= \varepsilon^{2(N+1)/3}\left[-2d\nu'_N-4y_1\nu''_N-2\alpha_1\sum_{n=0}^{2N-1}\varepsilon^{2n/3}\sum_{\substack{n_1+n_2+n_3=n+N+1\\0\leqslant n_1,n_2,n_3\leqslant N}}\nu_{n_1}\nu_{n_2}\nu_{n_3}\right.\\
&\quad\left.-2\alpha_0\sum_{n=0}^{N-1}\varepsilon^{2n/3}\sum_{\substack{n_1+n_2=n+N+1\\0\leqslant n_1,n_2\leqslant N}}\lambda_{n_1}\nu_{n_2}\right].
\end{aligned}$$

Thus, if we note that for $x\in D_1=\left\{x\in\mathbb{R}^d|\,-2\varepsilon^{\beta-2/3}\leqslant y_1\leqslant 2\varepsilon^{\beta-2/3}\right\}$, $|\varepsilon^{2/3}y_1|\lesssim\varepsilon^\beta\to 0$ as $\varepsilon\to 0$, we have thanks to (2.49), (2.50) and (2.51)

$$\begin{aligned}
&\varepsilon\left|\varepsilon^{4/3}\Delta\nu+\frac{\alpha_0}{\alpha_2}\frac{R_2^2-R_1^2}{\varepsilon^{2/3}}\nu+y_1\nu-2\alpha_1\nu^3-2\alpha_0\lambda\nu\right|\\
&\lesssim \varepsilon^{2N/3+5/3}\max(1,y_1)^{N-7/2}+\varepsilon^{2N/3+5/3}\sum_{n=0}^{2N-1}\varepsilon^{2n/3}\max(1,y_1)^{n+N+1-5/2-5/2+1/2}\\
&\quad+\varepsilon^{2N/3+5/3}\sum_{n=0}^{N-1}\varepsilon^{2n/3}\max(1,y_1)^{n+N+1-5/2-2}\\
&\lesssim \varepsilon^{2N/3+5/3}\max(1,y_1)^{N-7/2}\lesssim\varepsilon^{2N/3+5/3}\max(1,\varepsilon^{(\beta-2/3)(N-7/2)})=\varepsilon^{\beta N+4-7\beta/2}, \qquad (3.8)
\end{aligned}$$

where the last equality holds because $N \geqslant 4$. The first estimate of the lemma is proved. Similarly, from (3.1), (2.30), (2.34) and (2.41), we deduce

$$
\begin{aligned}
&\varepsilon^2\Delta\left(\varepsilon^{1/3}\lambda^{1/2}\right) + (R_2^2 - R_1^2 + z)\varepsilon^{1/3}\lambda^{1/2} - 2\alpha_2\left(\varepsilon^{1/3}\lambda^{1/2}\right)^3 - 2\alpha_0\left(\varepsilon^{1/3}\nu\right)^2\varepsilon^{1/3}\lambda^{1/2}\\
&= \varepsilon\lambda^{-3/2}\left[-d\varepsilon^{2/3}\lambda\lambda' - (R_1^2 - \varepsilon^{2/3}y_1)\lambda'^2 + 2(R_1^2 - \varepsilon^{2/3}y_1)\lambda\lambda'' + y_2\lambda^2 - 2\alpha_2\lambda^3 - 2\alpha_0\nu^2\lambda^2\right]\\
&= \varepsilon^{-1/3}\lambda^{-3/2}\left[-d\sum_{n=1}^{2N+3}\varepsilon^{2n/3}\sum_{\substack{n_1+n_2=n-3\\-1\leqslant n_1,n_2\leqslant N}}\lambda'_{n_1}\lambda_{n_2} + 2R_1^2\sum_{n=0}^{2N+2}\varepsilon^{2n/3}\sum_{\substack{n_1+n_2=n-2\\-1\leqslant n_1,n_2\leqslant N}}\lambda''_{n_1}\lambda_{n_2}\right.\\
&\quad -2y_1\sum_{n=1}^{2N+3}\varepsilon^{2n/3}\sum_{\substack{n_1+n_2=n-3\\-1\leqslant n_1,n_2\leqslant N}}\lambda''_{n_1}\lambda_{n_2} - R_1^2\sum_{n=0}^{2N+2}\varepsilon^{2n/3}\sum_{\substack{n_1+n_2=n-2\\-1\leqslant n_1,n_2\leqslant N}}\lambda'_{n_1}\lambda'_{n_2}\\
&\quad +y_1\sum_{n=1}^{2N+3}\varepsilon^{2n/3}\sum_{\substack{n_1+n_2=n-3\\-1\leqslant n_1,n_2\leqslant N}}\lambda'_{n_1}\lambda'_{n_2} + (R_2^2 - R_1^2)\sum_{n=-1}^{2N+1}\varepsilon^{2n/3}\sum_{\substack{n_1+n_2=n-1\\-1\leqslant n_1,n_2\leqslant N}}\lambda_{n_1}\lambda_{n_2}\\
&\quad +y_1\sum_{n=0}^{2N+2}\varepsilon^{2n/3}\sum_{\substack{n_1+n_2=n-2\\-1\leqslant n_1,n_2\leqslant N}}\lambda_{n_1}\lambda_{n_2} - 2\alpha_2\sum_{n=-1}^{3N+2}\varepsilon^{2n/3}\sum_{\substack{n_1+n_2+n_3=n-2\\-1\leqslant n_1,n_2,n_3\leqslant N}}\lambda_{n_1}\lambda_{n_2}\lambda_{n_3}\\
&\quad \left.-2\alpha_0\sum_{n=0}^{4N+2}\varepsilon^{2n/3}\sum_{\substack{n_1+n_2+n_3+n_4=n-2\\-1\leqslant n_1,n_2\leqslant N\\0\leqslant n_3,n_4\leqslant N}}\lambda_{n_1}\lambda_{n_2}\nu_{n_3}\nu_{n_4}\right]\\
&= \varepsilon^{-1/3}\lambda^{-3/2}\left[-d\sum_{n=N+1}^{2N+3}\varepsilon^{2n/3}\sum_{\substack{n_1+n_2=n-3\\-1\leqslant n_1,n_2\leqslant N}}\lambda'_{n_1}\lambda_{n_2} + 2R_1^2\sum_{n=N+1}^{2N+2}\varepsilon^{2n/3}\sum_{\substack{n_1+n_2=n-2\\-1\leqslant n_1,n_2\leqslant N}}\lambda''_{n_1}\lambda_{n_2}\right.\\
&\quad -2y_1\sum_{n=N+1}^{2N+3}\varepsilon^{2n/3}\sum_{\substack{n_1+n_2=n-3\\-1\leqslant n_1,n_2\leqslant N}}\lambda''_{n_1}\lambda_{n_2} - R_1^2\sum_{n=N+1}^{2N+2}\varepsilon^{2n/3}\sum_{\substack{n_1+n_2=n-2\\-1\leqslant n_1,n_2\leqslant N}}\lambda'_{n_1}\lambda'_{n_2}\\
&\quad +y_1\sum_{n=N+1}^{2N+3}\varepsilon^{2n/3}\sum_{\substack{n_1+n_2=n-3\\-1\leqslant n_1,n_2\leqslant N}}\lambda'_{n_1}\lambda'_{n_2} + (R_2^2 - R_1^2)\sum_{n=N+1}^{2N+1}\varepsilon^{2n/3}\sum_{\substack{n_1+n_2=n-1\\-1\leqslant n_1,n_2\leqslant N}}\lambda_{n_1}\lambda_{n_2}\\
&\quad +y_1\sum_{n=N+1}^{2N+2}\varepsilon^{2n/3}\sum_{\substack{n_1+n_2=n-2\\-1\leqslant n_1,n_2\leqslant N}}\lambda_{n_1}\lambda_{n_2} - 2\alpha_2\sum_{n=N+1}^{3N+2}\varepsilon^{2n/3}\sum_{\substack{n_1+n_2+n_3=n-2\\-1\leqslant n_1,n_2,n_3\leqslant N}}\lambda_{n_1}\lambda_{n_2}\lambda_{n_3}\\
&\quad \left.-2\alpha_0\sum_{n=N+1}^{4N+2}\varepsilon^{2n/3}\sum_{\substack{n_1+n_2+n_3+n_4=n-2\\-1\leqslant n_1,n_2\leqslant N\\0\leqslant n_3,n_4\leqslant N}}\lambda_{n_1}\lambda_{n_2}\nu_{n_3}\nu_{n_4}\right] \qquad (3.9)
\end{aligned}
$$

In order to estimate this quantity, we consider separately each sum appearing in the bracket in the right hand side of (3.9). Let us focus for instance on the first one. If $n \geqslant N+1$, $n_1+n_2 = n-3$ and $n_1, n_2 \geqslant 1$, then we infer from (2.50) that for $x \in D_1$ (which implies $|y_1| \lesssim \varepsilon^{\beta-2/3}$), we have

$$
\begin{aligned}
\varepsilon^{2n/3}|\lambda'_{n_1}\lambda_{n_2}| &\lesssim \varepsilon^{2n/3}\max(1,|y_1|)^{n-8} \lesssim \max(\varepsilon^{2n/3}, \varepsilon^{2n/3+(\beta-2/3)(n-8)}) = \max(\varepsilon^{2n/3}, \varepsilon^{\beta n+8(2/3-\beta)})\\
&\lesssim \max(\varepsilon^{2(N+1)/3}, \varepsilon^{\beta(N+1)+8(2/3-\beta)}).
\end{aligned}
$$

If one of the two indices n_1, n_2 belongs to $\{-1, 0\}$, whereas the other one is larger than or equal to 1, we infer similarly thanks to (2.30), (2.34) and (2.50) that

$$\varepsilon^{2n/3}|\lambda'_{n_1}\lambda_{n_2}| \lesssim \max(\varepsilon^{2(N+1)/3}, \varepsilon^{\beta(N+1)+5(2/3-\beta)}).$$

Finally, If $N \geqslant 3$, the conditions $n_1 + n_2 = n - 3 \geqslant N - 2$ excludes the case where both n_1 and n_2 belong to $\{-1, 0\}$. Using similar arguments as well as Lemma 3.1, we deduce that for $x \in D_1$ and N large enough,

$$\begin{aligned}&\left|\varepsilon^2\Delta\left(\varepsilon^{1/3}\lambda^{1/2}\right) + (R_2^2 - R_1^2 + z)\varepsilon^{1/3}\lambda^{1/2} - 2\alpha_2\left(\varepsilon^{1/3}\lambda^{1/2}\right)^3 - 2\alpha_0\left(\varepsilon^{1/3}\nu\right)^2\varepsilon^{1/3}\lambda^{1/2}\right| \\ &\quad\lesssim\quad \varepsilon^{-1/3}\varepsilon\max(\varepsilon^{2(N+1)/3}, \varepsilon^{\beta(N+1)+2(2/3-\beta)}) \lesssim \varepsilon^{\beta N+2-\beta}. \end{aligned} \tag{3.10}$$

∎

3.4 Truncation of $(0, \varepsilon^{1/3}\mu)$ in D_2

Lemma 3.4 *Let $L \geqslant 1$ be an integer and μ be given by (3.1). There exists $C > 0$ such that for $x \in \mathbb{R}^d$ and $\varepsilon \in]0, 1]$,*

$$\left|\varepsilon^2\Delta\left(\varepsilon^{1/3}\mu\right) + (R_2^2 - R_1^2 + z)\varepsilon^{1/3}\mu - 2\alpha_2\left(\varepsilon^{1/3}\mu\right)^3\right| \leqslant \frac{C\varepsilon^{2L/3+5/3}}{1 + |y_2|^{2L+1/2}}, \tag{3.11}$$

where $y_2 = (R_2^2 - |x|^2)/\varepsilon^{2/3}$.

Corollary 3.5 *Under the same assumptions, there is $h \in L^2 \cap L^\infty(\mathbb{R}^d)$ such that for every $x \in \mathbb{R}^d$ and $\varepsilon \in]0, 1]$,*

$$\left|\varepsilon^2\Delta\left(\varepsilon^{1/3}\mu\right) + (R_2^2 - R_1^2 + z)\varepsilon^{1/3}\mu - 2\alpha_2\left(\varepsilon^{1/3}\mu\right)^3\right| \leqslant \varepsilon^{2L/3+5/3}h(x). \tag{3.12}$$

Corollary 3.6 *Under the same assumptions, there is $C > 0$ such that for $x \in D_1 \cap D_2$ and $\varepsilon \in]0, 1]$,*

$$\left|\varepsilon^2\Delta\left(\varepsilon^{1/3}\mu\right) + (R_2^2 - R_1^2 + z)\varepsilon^{1/3}\mu - 2\alpha_2\left(\varepsilon^{1/3}\mu\right)^3\right| \leqslant C\varepsilon^{2L+2}. \tag{3.13}$$

Proof of Lemma 3.4. Taking into account the equations satisfied by the μ_n's, namely

$$4R_2^2\mu_0'' + y_2\mu_0 - 2\alpha_2\mu_0^3 = 0 \tag{3.14}$$

for $n = 0$ and

$$4R_2^2\mu_n'' = 2\alpha_2 \sum_{n_1+n_2+n_3=n} \mu_{n_1}\mu_{n_2}\mu_{n_3} + 2d\mu'_{n-1} + 4y_2\mu''_{n-1} - y_2\mu_n \tag{3.15}$$

for $n \geqslant 1$, we infer

$$\begin{aligned}&\varepsilon^2\Delta\left(\varepsilon^{1/3}\mu\right) + (R_2^2 - R_1^2 + z)\varepsilon^{1/3}\mu - 2\alpha_2\left(\varepsilon^{1/3}\mu\right)^3 \\ &\quad= \quad \varepsilon(\varepsilon^{4/3}\Delta\mu + y_2\mu - 2\alpha_2\mu^3) \\ &\quad= \quad \varepsilon\left(-2d\varepsilon^{2(L+1)/3}\mu'_L - 4y_2\varepsilon^{2(L+1)/3}\mu''_L - 2\alpha_2\sum_{n=L+1}^{3L}\varepsilon^{2n/3}\sum_{\substack{n_1+n_2+n_3=n\\0\leqslant n_1,n_2,n_3\leqslant L}}\mu_{n_1}\mu_{n_2}\mu_{n_3}\right) \\ &\quad= \quad \varepsilon^{2L/3+5/3}\left(-2d\mu'_L - 4y_2\mu''_L - 2\alpha_2\sum_{n=0}^{2L-1}\varepsilon^{2n/3}\sum_{\substack{n_1+n_2+n_3=n+L+1\\0\leqslant n_1,n_2,n_3\leqslant L}}\mu_{n_1}\mu_{n_2}\mu_{n_3}\right). \end{aligned} \tag{3.16}$$

Let us define for $y \in \mathbb{R}$

$$h_0(y) = (1+|y|^{2L+1/2}) \max \left(|\mu_L'(y)|, |y\mu_L''(y)|, \max_{\substack{0 \leqslant n \leqslant 2L-1 \\ n_1+n_2+n_3 = n+L+1 \\ 0 \leqslant n_1,n_2,n_3 \leqslant L}} |\mu_{n_1}(y)\mu_{n_2}(y)\mu_{n_3}(y)| \right).$$

Thanks to (2.19) and Propositions 2.4 and 2.6, h_0 is uniformly bounded on $\mathbb{R}$. The lemma follows. ∎

Proof of Corollary 3.5. For $x \in \mathbb{R}^d$ and $\varepsilon \leqslant 1$, one has

$$\frac{1}{1+|y_2|^{2L+1/2}} = \frac{1}{1+\left(\frac{R_2^2-|x|^2)}{\varepsilon^{2/3}}\right)^{2L+1/2}} \leqslant h(x) = \begin{cases} 1 & \text{if} \quad |x|^2 \leqslant 2R_2^2 \\ \frac{1}{1+\left(\frac{|x|^2}{2}\right)^{2L+1/2}} & \text{if} \quad |x|^2 \geqslant 2R_2^2 \end{cases}$$

The corollary follows, since $L \geqslant 1$ and $d \leqslant 3$ imply $h \in L^2(\mathbb{R}^d)$. ∎

Proof of Corollary 3.6. The corollary follows from Lemma 3.11 and from the inequality

$$\frac{1}{1+|y_2|^{2L+1/2}} \lesssim \varepsilon^{4L/3+1/3},$$

that holds for $x \in D_1 \cap D_2$. ∎

3.5 Comparison of (ω, τ) and $\varepsilon^{1/3}(\nu, \lambda^{1/2})$ in $D_0 \cap D_1$

Lemma 3.7 *Let $N \in \mathbb{N}^*$, $M \geqslant \frac{\beta}{2-3\beta} N$, and ω, τ given by (3.1). Then for every $l \geqslant 0$,*

$$\left\| \omega^{(l)} - \sum_{\substack{(m,n)\in\mathbb{N}^2 \\ (2-3\beta)m+\beta n \leqslant \beta N}} \varepsilon^{2m} w_{m,n} \frac{d^l}{dz^l}\left(z^{1/2-3m+n}\right) \right\|_{L^\infty(D_0\cap D_1)} \underset{\varepsilon\to 0}{=} o\left(\varepsilon^{\beta(N+1/2-l)}\right) \tag{3.17}$$

and

$$\left\| \tau^{(l)} - \lambda_{-1}^{1/2}\mathbf{1}_{l=0} - \sum_{\substack{(m,n)\in\mathbb{N}^2 \\ (2-3\beta)m+\beta n \leqslant \beta N}} \varepsilon^{2m} t_{m,n} \frac{d^l}{dz^l}\left(z^{1+n-3m}\right) \right\|_{L^\infty(D_0\cap D_1)} \underset{\varepsilon\to 0}{=} o(\varepsilon^{\beta(N+1-l)}). \tag{3.18}$$

where the $w_{m,n}$'s and the $t_{m,n}$'s are defined in Lemma 2.1 and (2.13).

Proof. From Lemma 2.1, for every $l \geqslant 0$,

$$\begin{aligned} \omega^{(l)}(z) = \sum_{m=0}^{M} \varepsilon^{2m} \omega_m^{(l)}(z) \quad &\underset{z\to 0}{\approx} \quad \sum_{m=0}^{M} \varepsilon^{2m} \sum_{n=0}^{\infty} w_{m,n} \frac{d^l}{dz^l}\left(z^{1/2-3m+n}\right) \\ &\underset{z\to 0}{\approx} \quad \sum_{k=-3M}^{\infty} \sum_{\substack{(m,n)\in\{0,\cdots,M\}\times\mathbb{N} \\ n-3m=k}} \varepsilon^{2m} w_{m,n} \frac{d^l}{dz^l}\left(z^{1/2+k}\right). \end{aligned} \tag{3.19}$$

Thus, since $x \in D_0 \cap D_1$ implies $\varepsilon^\beta \leqslant z \leqslant 2\varepsilon^\beta \to 0$ as $\varepsilon \to 0$,

$$\begin{aligned}
\omega^{(l)}(z) &\underset{z\to 0}{=} \sum_{k=-3M}^{N} \Big(\sum_{\substack{(m,n)\in\{0,\cdots,M\}\times\mathbb{N}\\ n-3m=k}} \varepsilon^{2m} w_{m,n} \Big) \frac{d^l}{dz^l}\left(z^{1/2+k}\right) + o(z^{1/2+N-l}) \\
&\underset{z\to 0}{=} \sum_{\substack{(m,n)\in\{0,\cdots,M\}\times\mathbb{N}\\ n-3m\leqslant N}} \varepsilon^{2m} w_{m,n} \frac{d^l}{dz^l}\left(z^{1/2+n-3m}\right) + o(z^{1/2+N-l}) \\
&\underset{z\to 0}{=} \sum_{\substack{(m,n)\in\{0,\cdots,M\}\times\mathbb{N}\\ n-3m\leqslant N\\ 2m+\beta(1/2+n-3m-l)\leqslant\beta(1/2+N-l)}} \varepsilon^{2m} w_{m,n} \frac{d^l}{dz^l}\left(z^{1/2+n-3m}\right) + o_{L^\infty(D_0\cap D_1)}(\varepsilon^{\beta(1/2+N-l)}) \\
&\underset{z\to 0}{=} \sum_{\substack{(m,n)\in\mathbb{N}^2\\ (2-3\beta)m+\beta n\leqslant\beta N}} \varepsilon^{2m} w_{m,n} \frac{d^l}{dz^l}\left(z^{1/2+n-3m}\right) + o_{L^\infty(D_0\cap D_1)}(\varepsilon^{\beta(1/2+N-l)}). \qquad (3.20)
\end{aligned}$$

Note that the assumption on M in the statement of the Lemma ensures that the set $\{(m,n) \in \mathbb{N}^2, (2-3\beta)m + \beta n \leqslant \beta N\}$ is a triangle included in the rectangle $\{0,\cdots,M\}\times\{0,\cdots,N\}$. Similarly, we infer from Lemma 2.1 and (2.13) that

$$\begin{aligned}
\tau^{(l)}(z) = \sum_{m=0}^{M} \varepsilon^{2m}\tau_m^{(l)}(z) &\underset{z\to 0}{\approx} \lambda_{-1}^{1/2}\mathbf{1}_{l=0} + \sum_{m=0}^{M}\varepsilon^{2m}\sum_{n=0}^{\infty} t_{m,n}\frac{d^l}{dz^l}\left(z^{1-3m+n}\right) \\
&\underset{z\to 0}{\approx} \lambda_{-1}^{1/2}\mathbf{1}_{l=0} + \sum_{k=-3M}^{\infty}\ \sum_{\substack{(m,n)\in\{0,\cdots,M\}\times\mathbb{N}\\ n-3m=k}} \varepsilon^{2m} t_{m,n}\ \frac{d^l}{dz^l}\left(z^{1+k}\right). \\
&\underset{z\to 0}{=} \lambda_{-1}^{1/2}\mathbf{1}_{l=0} + \sum_{k=-3M}^{N}\ \sum_{\substack{(m,n)\in\{0,\cdots,M\}\times\mathbb{N}\\ n-3m=k}} \varepsilon^{2m} t_{m,n}\ \frac{d^l}{dz^l}\left(z^{1+k}\right) + o(z^{N+1-l}).
\end{aligned}$$

Thus,

$$\begin{aligned}
\tau^{(l)}(z) &\underset{\varepsilon\to 0}{=} \lambda_{-1}^{1/2}\mathbf{1}_{l=0} + \sum_{\substack{(m,n)\in\{0,\cdots,M\}\times\mathbb{N}\\ 2m+(1+n-3m-l)\beta\leqslant(N+1-l)\beta}} \varepsilon^{2m} t_{m,n}\ \frac{d^l}{dz^l}\left(z^{1+n-3m}\right) + o_{L^\infty(D_0\cap D_1)}(\varepsilon^{\beta(N+1-l)}) \\
&\underset{\varepsilon\to 0}{=} \lambda_{-1}^{1/2}\mathbf{1}_{l=0} + \sum_{\substack{(m,n)\in\mathbb{N}^2\\ (2-3\beta)m+\beta n\leqslant\beta N}} \varepsilon^{2m} t_{m,n}\ \frac{d^l}{dz^l}\left(z^{1+n-3m}\right) + o_{L^\infty(D_0\cap D_1)}(\varepsilon^{\beta(N+1-l)}). \qquad (3.21)
\end{aligned}$$

∎

Lemma 3.8 *Let $N \geqslant 1$. We assume that $\beta \in (0,2/3)\backslash\mathbb{Q}$. There exist two families of numbers $(n_{m,n})_{m\geqslant 0,n\geqslant 0}$ and $(l_{m,n})_{m\geqslant 0,n\geqslant 0}$ which do not depend on N such that if ν and λ are given by (3.1), then for $l = 0,1,2$,*

$$\left\| \frac{d^l}{dz^l}\left(\varepsilon^{1/3}\nu(y_1)\right) - \sum_{\substack{(m,n)\in\mathbb{N}^2\\ (2-3\beta)m+\beta n\leqslant\beta N}} \varepsilon^{2m} n_{m,n}\frac{d^l}{dz^l}\left(z^{1/2-3m+n}\right)\right\|_{L^\infty(D_0\cap D_1)} \underset{\varepsilon\to 0}{=} o\left(\varepsilon^{\beta(N+1/2-l)}\right) \qquad (3.22)$$

and

$$\left\| \frac{d^l}{dz^l}\left(\varepsilon^{1/3}\lambda(y_1)^{1/2}\right) - \lambda_{-1}^{1/2}\mathbf{1}_{l=0} - \sum_{\substack{(m,n)\in\mathbb{N}^2\\ (2-3\beta)m+\beta n\leqslant\beta N}} \varepsilon^{2m} l_{m,n}\ \frac{d^l}{dz^l}\left(z^{1+n-3m}\right)\right\|_{L^\infty(D_0\cap D_1)} \underset{\varepsilon\to 0}{=} o(\varepsilon^{\beta(N+1-l)}) \qquad (3.23)$$

Proof. For $x \in D_0 \cap D_1$, we have $2\varepsilon^{\beta-2/3} \geqslant y_1 \geqslant \varepsilon^{\beta-2/3} \to +\infty$ as $\varepsilon \to 0$. Thus, we infer from (2.49) and (2.51) that for every $l \geqslant 0$,

$$\begin{aligned}
&\frac{d^l}{dy_1^l}\left(\varepsilon^{1/3}\nu(y_1)\right)\\
&\underset{y_1\to+\infty}{\approx} \varepsilon^{1/3}\sum_{m=0}^{\infty} N_{0,m}\frac{d^l}{dy_1^l}\left(y_1^{1/2-3m}\right) + \varepsilon^{1/3}\sum_{n=1}^{N}\varepsilon^{2n/3}\sum_{m=0}^{\infty} N_{n,m}\frac{d^l}{dy_1^l}\left(y_1^{n-5/2-3m}\right)\\
&\underset{y_1\to+\infty}{\approx} \sum_{m=0}^{\infty} N_{0,m}\varepsilon^{1/3}\frac{d^l}{dy_1^l}\left(y_1^{1/2-3m}\right) + \sum_{k=-N}^{\infty}\sum_{\substack{(n,m)\in\{1,\cdots,N\}\times\mathbb{N}\\ 3m-n=k}} N_{n,m}\varepsilon^{1/3+2n/3}\frac{d^l}{dy_1^l}\left(y_1^{-5/2-k}\right)\\
&\underset{y_1\to+\infty}{=} \sum_{0\leqslant m\leqslant\frac{\beta N}{2-3\beta}} N_{0,m}\varepsilon^{1/3}\frac{d^l}{dy_1^l}\left(y_1^{1/2-3m}\right) + \varepsilon^{1/3}o(y_1^{1/2-\frac{3\beta N}{2-3\beta}-l})\\
&\qquad + \sum_{-N\leqslant k\leqslant\frac{3\beta N-2}{2-3\beta}-3}\ \sum_{\substack{(n,m)\in\{1,\cdots,N\}\times\mathbb{N}\\ 3m-n=k}} N_{n,m}\varepsilon^{1/3+2n/3}\frac{d^l}{dy_1^l}\left(y_1^{-5/2-k}\right) + \varepsilon o(y_1^{-5/2-\frac{3\beta N-2}{2-3\beta}+3-l}).
\end{aligned}$$

Thus, for $x \in D_0 \cap D_1$, we have

$$\begin{aligned}
&\frac{d^l}{dy_1^l}\left(\varepsilon^{1/3}\nu(y_1)\right)\\
&= \varepsilon^{2l/3}\sum_{0\leqslant m\leqslant\frac{\beta}{2-3\beta}N}\varepsilon^{2m}N_{0,m}\frac{d^l}{dz^l}\left(z^{1/2-3m}\right) + \varepsilon^{2l/3}\sum_{\substack{(n,m)\in\{1,\cdots,N\}\times\mathbb{N}\\ 3m-n\leqslant\frac{3\beta N-2}{2-3\beta}-3}} N_{n,m}\varepsilon^{2+2m}\frac{d^l}{dz^l}\left(z^{-5/2+n-3m}\right)\\
&\quad + o_{L^\infty(D_0\cap D_1)}(\varepsilon^{\beta(N+1/2)+(2/3-\beta)l})\\
&= \varepsilon^{2l/3}\sum_{0\leqslant m\leqslant\frac{\beta}{2-3\beta}N}\varepsilon^{2m}N_{0,m}\frac{d^l}{dz^l}\left(z^{1/2-3m}\right) + \varepsilon^{2l/3}\sum_{\substack{(n,m)\in\{1,\cdots,N\}\times\mathbb{N}^*\\ 3m-n\leqslant\frac{3\beta N-2}{2-3\beta}}} N_{n,m-1}\varepsilon^{2m}\frac{d^l}{dz^l}\left(z^{1/2+n-3m}\right)\\
&\quad + o_{L^\infty(D_0\cap D_1)}(\varepsilon^{\beta(N+1/2)+(2/3-\beta)l})\\
&= \varepsilon^{2l/3}\sum_{0\leqslant m\leqslant\frac{\beta}{2-3\beta}N}\varepsilon^{2m}N_{0,m}\frac{d^l}{dz^l}\left(z^{1/2-3m}\right) + \varepsilon^{2l/3}\sum_{\substack{(n,m)\in\mathbb{N}^{*2}\\ (2-3\beta)m+\beta n\leqslant\beta N}} N_{n,m-1}\varepsilon^{2m}\frac{d^l}{dz^l}\left(z^{1/2+n-3m}\right)\\
&\quad + o_{L^\infty(D_0\cap D_1)}(\varepsilon^{\beta(N+1/2)+(2/3-\beta)l}),
\end{aligned} \tag{3.24}$$

where in the last equality, we have neglected all the terms in the sum over (n, m) which can be incorporated in the rest term, and we have used that the condition

$$(n, m) \in \mathbb{N}^{*2}, \quad (2 - 3\beta)m + \beta n \leqslant \beta N \tag{3.25}$$

clearly implies $n \leqslant N$ (even $n < N$, in fact), as well as

$$3m - n \leqslant \frac{3\beta N - 2}{2 - 3\beta}. \tag{3.26}$$

Indeed, (3.25) can be rewritten as

$$\frac{1}{3}(3m - n) + n\left(\frac{1}{3} + \frac{\beta}{2 - 3\beta}\right) \leqslant \frac{\beta N}{2 - 3\beta}, \tag{3.27}$$

which yields (3.26) if we take into account that $n \geqslant 1$. The result follows from the change of variable $z = \varepsilon^{2/3}y_1$, with

$$n_{m,n} = \begin{cases} N_{0,m} & \text{if} \quad n = 0\\ 0 & \text{if} \quad n \geqslant 1 \text{ and } m = 0\\ N_{n,m-1} & \text{if} \quad n \geqslant 1 \text{ and } m \geqslant 1. \end{cases}$$

Similarly as for $\varepsilon^{1/3}\nu$, we have

$$\begin{aligned}
&\frac{d^l}{dy_1^l}\left(\varepsilon^{2/3}\lambda(y_1)\right)\\
&\underset{y_1\to+\infty}{\approx} \quad \lambda_{-1}\mathbf{1}_{\{l=0\}}+\varepsilon^{2/3}\sum_{m=0}^{\infty}L_{0,m}\frac{d^l}{dy_1^l}\left(y_1^{1-3m}\right)+\varepsilon^{2/3}\sum_{n=1}^{N}\varepsilon^{2n/3}\sum_{m=0}^{\infty}L_{n,m}\frac{d^l}{dy_1^l}\left(y_1^{n-2-3m}\right)\\
&\underset{y_1\to+\infty}{\approx} \quad \lambda_{-1}\mathbf{1}_{\{l=0\}}+\varepsilon^{2/3}\sum_{m=0}^{\infty}L_{0,m}\frac{d^l}{dy_1^l}\left(y_1^{1-3m}\right)+\varepsilon^{2/3}\sum_{n=1}^{N}\varepsilon^{2n/3}\sum_{m=1}^{\infty}L_{n,m-1}\frac{d^l}{dy_1^l}\left(y_1^{1+n-3m}\right)\\
&\underset{y_1\to+\infty}{\approx} \quad \lambda_{-1}\mathbf{1}_{\{l=0\}}+\sum_{n=0}^{N}\varepsilon^{2(n+1)/3}\sum_{m=0}^{\infty}\check{L}_{n,m}\frac{d^l}{dy_1^l}\left(y_1^{1+n-3m}\right), \qquad (3.28)
\end{aligned}$$

with

$$\check{L}_{n,m}=\begin{cases} L_{0,m} & \text{if} \quad n=0\\ 0 & \text{if} \quad n\geqslant 1 \text{ and } m=0\\ L_{n,m-1} & \text{if} \quad n\geqslant 1 \text{ and } m\geqslant 1.\end{cases}$$

Thus,

$$\begin{aligned}
&\frac{d^l}{dy_1^l}\left(\varepsilon^{2/3}\lambda(y_1)\right)\\
&\underset{y_1\to+\infty}{\approx} \quad \lambda_{-1}\mathbf{1}_{\{l=0\}}+\sum_{k=-N}^{\infty}\sum_{\substack{(n,m)\in\{0,\cdots,N\}\times\mathbb{N}\\ 3m-n=k}}\check{L}_{n,m}\varepsilon^{2(n+1)/3}\frac{d^l}{dy_1^l}\left(y_1^{1-k}\right)\\
&\underset{y_1\to+\infty}{=} \quad \lambda_{-1}\mathbf{1}_{\{l=0\}}+\sum_{-N\leqslant k\leqslant\frac{3\beta N}{2-3\beta}}\sum_{\substack{(n,m)\in\{0,\cdots,N\}\times\mathbb{N}\\ 3m-n=k}}\check{L}_{n,m}\varepsilon^{2(n+1)/3}\frac{d^l}{dy_1^l}\left(y_1^{1-k}\right)+\varepsilon^{2/3}o(y_1^{1-\frac{3\beta N}{2-3\beta}-l})\\
&\underset{y_1\to+\infty}{=} \quad \lambda_{-1}\mathbf{1}_{\{l=0\}}+\sum_{\substack{(n,m)\in\{0,\cdots,N\}\times\mathbb{N}\\ 3m-n\leqslant\frac{3\beta N}{2-3\beta}}}\check{L}_{n,m}\varepsilon^{2(n+1)/3}\frac{d^l}{dy_1^l}\left(y_1^{1-3m+n}\right)+\varepsilon^{2/3}o(y_1^{1-\frac{3\beta N}{2-3\beta}-l}). \qquad (3.29)
\end{aligned}$$

Therefore for $x\in D_0\cap D_1$,

$$\begin{aligned}
&\frac{d^l}{dy_1^l}\left(\varepsilon^{2/3}\lambda(y_1)\right)\\
&\underset{y_1\to+\infty}{=} \quad \lambda_{-1}\mathbf{1}_{\{l=0\}}+\sum_{\substack{(n,m)\in\{0,\cdots,N\}\times\mathbb{N}\\ 3m-n\leqslant\frac{3\beta N}{2-3\beta}}}\check{L}_{n,m}\varepsilon^{2m}\varepsilon^{2l/3}\frac{d^l}{dz^l}\left(z^{1-3m+n}\right)+\varepsilon^{2l/3}o_{L^\infty(D_0\cap D_1)}(\varepsilon^{\beta(N+1-l)}).\\
&\underset{y_1\to+\infty}{=} \quad \lambda_{-1}\mathbf{1}_{\{l=0\}}+\varepsilon^{2l/3}\sum_{\substack{(n,m)\in\mathbb{N}^2\\ (2-3\beta)m+\beta n\leqslant\beta N}}\check{L}_{n,m}\varepsilon^{2m}\frac{d^l}{dz^l}\left(z^{1-3m+n}\right)+\varepsilon^{2l/3}o_{L^\infty(D_0\cap D_1)}(\varepsilon^{\beta(N+1-l)}). \qquad (3.30)
\end{aligned}$$

thanks to the same remark as in (3.27). In particular, for $l=0$, we get

$$\begin{aligned}
&\varepsilon^{1/3}\lambda(y_1)^{1/2} \qquad (3.31)\\
&= \left(\lambda_{-1} + \sum_{\substack{(n,m)\in\mathbb{N}^2\\(2-3\beta)m+\beta n\leqslant\beta N}} \check{L}_{n,m}\varepsilon^{2m}z^{1-3m+n} + o_{L^\infty(D_0\cap D_1)}(\varepsilon^{\beta(N+1)})\right)^{1/2}\\
&= \lambda_{-1}^{1/2} + \sum_{k=1}^{N+1} c_k \left(\sum_{\substack{(n,m)\in\mathbb{N}^2\\(2-3\beta)m+\beta n\leqslant\beta N}} \check{L}_{n,m}\varepsilon^{2m}z^{1-3m+n}\right)^k + o_{L^\infty(D_0\cap D_1)}(\varepsilon^{\beta(N+1)})\\
&= \lambda_{-1}^{1/2} + \sum_{k=1}^{N+1} c_k z^k \sum_{\substack{((n_1,m_1),\cdots,(n_k,m_k))\in(\mathbb{N}^2)^k\\ \forall j\in\{1,\cdots,k\},\ (2-3\beta)m_j+\beta n_j\leqslant\beta N}} \prod_{j=1}^k \check{L}_{n_j,m_j}(\varepsilon^2 z^{-3})^{m_1+\cdots+m_k} z^{n_1+\cdots+n_k} + o_{L^\infty(D_0\cap D_1)}(\varepsilon^{\beta(N+1)})\\
&= \lambda_{-1}^{1/2} + \sum_{k=1}^{N+1} c_k z^k \sum_{\substack{(n,m)\in\mathbb{N}^2\\(2-3\beta)m+\beta n\leqslant\beta(N+1-k)}} \varepsilon^{2m}z^{n-3m} \sum_{\substack{((n_1,m_1),\cdots,(n_k,m_k))\in(\mathbb{N}^2)^k\\ n_1+\cdots+n_k=n\\ m_1+\cdots+m_k=m}} \prod_{j=1}^k \check{L}_{n_j,m_j} + o_{L^\infty(D_0\cap D_1)}(\varepsilon^{\beta(N+1)})\\
&= \lambda_{-1}^{1/2} + \sum_{\substack{(n,m)\in\mathbb{N}^2\\(2-3\beta)m+\beta n\leqslant\beta N}} \varepsilon^{2m}z^{n-3m+1} \underbrace{\sum_{k=1}^{n+1} c_k \sum_{\substack{((n_1,m_1),\cdots,(n_k,m_k))\in(\mathbb{N}^2)^k\\ n_1+\cdots+n_k=n-k+1\\ m_1+\cdots+m_k=m}} \prod_{j=1}^k \check{L}_{n_j,m_j}}_{=:l_{m,n}} + o_{L^\infty(D_0\cap D_1)}(\varepsilon^{\beta(N+1)}),
\end{aligned}$$

where the c_k's are some real coefficients. So, we have proved (3.23) for $l=0$. In order to prove (3.23) for $l=1$, we first write

$$\frac{d}{dy_1}\left(\varepsilon^{1/3}\lambda(y_1)^{1/2}\right) = \frac{1}{2}\frac{d}{dy_1}\left(\varepsilon^{2/3}\lambda(y_1)\right)\left(\varepsilon^{2/3}\lambda(y_1)\right)^{-1/2}.$$

Then, note that $\left(\varepsilon^{2/3}\lambda(y_1)\right)^{-1/2}$ has the same kind of asymptotic expansion as the one that appears in the right hand side of (3.31). Indeed, the same calculation can be done with the power $1/2$ replaced by $-1/2$, which only changes the values of the c_k's. Thus, for some coefficients $(\alpha_{m,n})_{m,n\in\mathbb{N}^2}$, we have

$$\left(\varepsilon^{2/3}\lambda(y_1)\right)^{-1/2} = \lambda_{-1}^{-1/2} + \sum_{\substack{(n,m)\in\mathbb{N}^2\\(2-3\beta)m+\beta n\leqslant\beta N}} \varepsilon^{2m}z^{n-3m+1}\alpha_{m,n} + o_{L^\infty(D_0\cap D_1)}(\varepsilon^{\beta(N+1)}) \qquad (3.32)$$

From the product of this expansion with (3.30) for $l=1$, we infer that

$$\begin{aligned}
\frac{d}{dy_1}\left(\varepsilon^{1/3}\lambda(y_1)^{1/2}\right) &= \frac{1}{2}\left(\varepsilon^{2/3}\sum_{\substack{(n,m)\in\mathbb{N}^2\\(2-3\beta)m+\beta n\leqslant\beta N}}\check{L}_{n,m}\varepsilon^{2m}\frac{d}{dz}\left(z^{1-3m+n}\right)+\varepsilon^{2/3}o_{L^\infty(D_0\cap D_1)}(\varepsilon^{\beta N})\right)\\
&\quad\times\left(\lambda_{-1}^{-1/2}+\sum_{\substack{(n,m)\in\mathbb{N}^2\\(2-3\beta)m+\beta n\leqslant\beta N}}\varepsilon^{2m}z^{n-3m+1}\alpha_{m,n}+o_{L^\infty(D_0\cap D_1)}(\varepsilon^{\beta(N+1)})\right)\\
&= \frac{\lambda_{-1}^{-1/2}\varepsilon^{2/3}}{2}\sum_{\substack{(n,m)\in\mathbb{N}^2\\(2-3\beta)m+\beta n\leqslant\beta N}}\check{L}_{n,m}\varepsilon^{2m}(1-3m+n)z^{n-3m}\\
&\quad+\frac{\varepsilon^{2/3}}{2}\sum_{\substack{(n,m)\in\mathbb{N}^*\times\mathbb{N}\\(2-3\beta)m+\beta n\leqslant\beta N}}\varepsilon^{2m}z^{n-3m}\sum_{\substack{n_1,n_2,m_1,m_2\in\mathbb{N}\\n_1+n_2=n-1\\m_1+m_2=m}}\check{L}_{n_1,m_1}(1-3m_1+n_1)\alpha_{m_2,n_2}\\
&\quad+o_{L^\infty(D_0\cap D_1)}(\varepsilon^{\beta N+2/3})\\
&= \varepsilon^{2/3}\sum_{\substack{(m,n)\in\mathbb{N}^2\\(2-3\beta)m+\beta n\leqslant\beta N}}l'_{m,n}\varepsilon^{2m}z^{n-3m}+o_{L^\infty(D_0\cap D_1)}(\varepsilon^{\beta N+2/3}),
\end{aligned}\tag{3.33}$$

for some coefficients $l'_{m,n}\in\mathbb{R}$. In order to prove (3.23), it is now sufficient to establish that for every $m,n\geqslant 0$, the $l'_{m,n}$'s and the $l_{m,n}$'s, defined respectively in (3.33) and (3.31), are related by $l'_{m,n}=(1+n-3m)l_{m,n}$. For this purpose, we note, for $z\in[\varepsilon^\beta,2\varepsilon^\beta]$ $\theta(z)=\varepsilon^{1/3}\lambda(y_1)^{1/2}$, such that according to (3.31) and (3.33),

$$\theta(z)=\lambda_{-1}^{1/2}+\sum_{\substack{(m,n)\in\mathbb{N}^2\\(2-3\beta)m+\beta n\leqslant\beta N}}l_{m,n}\varepsilon^{2m}z^{n-3m+1}+o_{L^\infty(D_0\cap D_1)}(\varepsilon^{\beta(N+1)})\tag{3.34}$$

and

$$\theta'(z)=\sum_{\substack{(m,n)\in\mathbb{N}^2\\(2-3\beta)m+\beta n\leqslant\beta N}}l'_{m,n}\varepsilon^{2m}z^{n-3m}+o_{L^\infty(D_0\cap D_1)}(\varepsilon^{\beta N}).\tag{3.35}$$

Then, we have on the one side from (3.34)

$$\theta(2\varepsilon^\beta)-\theta(\varepsilon^\beta)=\sum_{\substack{(m,n)\in\mathbb{N}^2\\(2-3\beta)m+\beta n\leqslant\beta N}}l_{m,n}(2^{n-3m+1}-1)\varepsilon^{(2-3\beta)m+\beta(n+1)}+o_{L^\infty(D_0\cap D_1)}(\varepsilon^{\beta(N+1)}),\tag{3.36}$$

whereas on the other side, thanks to (3.35),

$$\begin{aligned}
\theta(2\varepsilon^\beta)-\theta(\varepsilon^\beta) &= \sum_{\substack{(m,n)\in\mathbb{N}^2\\(2-3\beta)m+\beta n\leqslant\beta N}}l'_{m,n}\varepsilon^{2m}\int_{\varepsilon^\beta}^{2\varepsilon^\beta}z^{n-3m}dz+o_{L^\infty(D_0\cap D_1)}(\varepsilon^{\beta(N+1)})\\
&= \sum_{\substack{(m,n)\in\mathbb{N}^2\\(2-3\beta)m+\beta n\leqslant\beta N\\n-3m\neq-1}}\frac{l'_{m,n}}{n-3m+1}(2^{n-3m+1}-1)\varepsilon^{(2-3\beta)m+\beta(n+1)}\\
&\quad+\ln(2)\sum_{\substack{(m,n)\in\mathbb{N}^2\\(2-3\beta)m+\beta n\leqslant\beta N\\n-3m=-1}}l'_{m,n}\varepsilon^{2m}+o_{L^\infty(D_0\cap D_1)}(\varepsilon^{\beta(N+1)}).
\end{aligned}\tag{3.37}$$

Since β is not rational, the family of functions of the variable ε, $(\varepsilon^{2m+\beta(n+1-3m)})_{(m,n)\in\mathbb{N}^2}$ is linearly independent, and we deduce by comparison of (3.36) and (3.37) that $l'_{m,n}=(n-3m+1)l_{m,n}$, in both cases $n-3m+1\neq 0$ and $n-3m+1=0$. (3.23) for $l=1$ follows. The proof for $l=2$ is similar. ∎

Lemma 3.9 *Let $N \geqslant 2$ be an integer, $\varepsilon_0 > 0$ and $\beta \in (0, 2/3)\backslash\mathbb{Q}$. Let $(\theta_1, \theta_2)_{0<\varepsilon\leqslant\varepsilon_0}$ be a sequence of pairs of regular functions defined for $z \in [\varepsilon^\beta, 2\varepsilon^\beta]$, such that*

$$\left\| \varepsilon^2 \Delta\theta_1 + \frac{\alpha_0}{\alpha_2}(R_2^2 - R_1^2)\theta_1 + z\theta_1 - 2\alpha_1\theta_1^3 - 2\alpha_0\theta_2^2\theta_1 \right\|_{L^\infty(D_0\cap D_1)} = o\left(\varepsilon^{\beta(N+1/2)}\right) \tag{3.38}$$

and

$$\left\| \varepsilon^2 \Delta\theta_2 + (R_2^2 - R_1^2)\theta_2 + z\theta_2 - 2\alpha_2\theta_2^3 - 2\alpha_0\theta_1^2\theta_2 \right\|_{L^\infty(D_0\cap D_1)} = o\left(\varepsilon^{\beta(N+1)}\right) \tag{3.39}$$

are satisfied, where $\Delta\theta_j$ refers to $\sum_{k=1}^d \frac{\partial^2}{\partial x_k^2}\left(\theta_j(R_1^2 - |x|^2)\right) = -2d\theta_j'(z) + 4(R_1^2 - z)\theta_j''(z)$ (with $z = R_1^2 - |x|^2$). We assume that there exists two families of real numbers $p_{m,n}$, $q_{m,n}$, defined for every $(m,n) \in \mathbb{N}^2$ such that $(2-3\beta)m + \beta n \leqslant \beta N$, such that

$$\forall l \in \{0,1,2\}, \quad \left\| \theta_1^{(l)} - \sum_{\substack{(m,n)\in\mathbb{N}^2 \\ (2-3\beta)m+\beta n\leqslant\beta N}} \varepsilon^{2m} p_{m,n} \frac{d^l}{dz^l}\left(z^{1/2-3m+n}\right) \right\|_{L^\infty(D_0\cap D_1)} \underset{\varepsilon\to 0}{=} o\left(\varepsilon^{\beta(N+1/2-l)}\right) \tag{3.40}$$

and

$$\forall l \in \{0,1,2\}, \quad \left\| \theta_2^{(l)} - \lambda_{-1}^{1/2}\mathbf{1}_{\{l=0\}} - \sum_{\substack{(m,n)\in\mathbb{N}^2 \\ (2-3\beta)m+\beta n\leqslant\beta N}} \varepsilon^{2m} q_{m,n} \frac{d^l}{dz^l}\left(z^{1+n-3m}\right) \right\|_{L^\infty(D_0\cap D_1)} \underset{\varepsilon\to 0}{=} o(\varepsilon^{\beta(N+1-l)}) \tag{3.41}$$

Then, equations (3.40), (3.41), (3.38) and (3.39) entirely determine the values of the $p_{m,n}$'s and the $q_{m,n}$'s for $(2-3\beta)m + \beta n \leqslant \beta(N-1)$. Moreover, these coefficients do not depend on N or β.

Proof. For convenience, for every $(m,n) \in \mathbb{N}^2$, we denote $p'_{m,n} = (1/2 - 3m + n)p_{m,n}$, $p''_{m,n} = (-1/2 - 3m + n)(1/2 - 3m + n)p_{m,n}$, $q'_{m,n} = (1 + n - 3m)q_{m,n}$ and $q''_{m,n} = (n - 3m)(1 + n - 3m)q_{m,n}$. For functions (θ_1, θ_2) that satisfy (3.40) and (3.41), let us calculate the function that appears in the

left hand side of (3.38), evaluated at $z=\varepsilon^\beta$. In the calculation below, implicitely, $\theta_j=\theta_j(\varepsilon^\beta)$.

$$
\begin{aligned}
&\varepsilon^2\Delta\theta_1+\frac{\alpha_0}{\alpha_2}(R_2^2-R_1^2)\theta_1+z\theta_1-2\alpha_1\theta_1^3-2\alpha_0\theta_2^2\theta_1\\
&= -2\sum_{\substack{m,n\geqslant 0\\(2-3\beta)m+\beta n\leqslant\beta N}}(dp'_{m,n}+2p''_{m,n})\varepsilon^{(2-3\beta)(m+1)+\beta(n+2)+\beta/2}\\
&\quad +4R_1^2\sum_{\substack{m,n\geqslant 0\\(2-3\beta)m+\beta n\leqslant\beta N}}p''_{m,n}\varepsilon^{(2-3\beta)(m+1)+\beta(n+1)+\beta/2}\\
&\quad +\frac{\alpha_0}{\alpha_2}(R_2^2-R_1^2)\sum_{\substack{m,n\geqslant 0\\(2-3\beta)m+\beta n\leqslant\beta N}}p_{m,n}\varepsilon^{(2-3\beta)m+\beta n+\beta/2}+\sum_{\substack{m,n\geqslant 0\\(2-3\beta)m+\beta n\leqslant\beta N}}p_{m,n}\varepsilon^{(2-3\beta)m+\beta(n+1)+\beta/2}\\
&\quad -2\alpha_1\sum_{\substack{m,n\geqslant 0\\(2-3\beta)m+\beta n\leqslant\beta N}}\Big(\sum_{\substack{m_1,m_2,m_3,n_1,n_2,n_3\geqslant 0\\m_1+m_2+m_3=m\\n_1+n_2+n_3=n}}p_{m_1,n_1}p_{m_2,n_2}p_{m_3,n_3}\Big)\varepsilon^{(2-3\beta)m+\beta n+3\beta/2}\\
&\quad -2\alpha_0\lambda_{-1}\sum_{\substack{m,n\geqslant 0\\(2-3\beta)m+\beta n\leqslant\beta N}}p_{m,n}\varepsilon^{(2-3\beta)m+\beta n+\beta/2}\\
&\quad -4\alpha_0\lambda_{-1}^{1/2}\sum_{\substack{m,n\geqslant 0\\(2-3\beta)m+\beta n\leqslant\beta N}}\Big(\sum_{\substack{m_1,m_2,n_1,n_2\geqslant 0\\n_1+n_2=n\\m_1+m_2=m}}q_{m_1,n_1}p_{m_2,n_2}\Big)\varepsilon^{(2-3\beta)m+\beta n+3\beta/2}\\
&\quad -2\alpha_0\sum_{\substack{m,n\geqslant 0\\(2-3\beta)m+\beta n\leqslant\beta N}}\Big(\sum_{\substack{m_1,m_2,m_3,n_1,n_2,n_3\geqslant 0,\\n_1+n_2+n_3=n\\m_1+m_2+m_3=m}}q_{m_1,n_1}q_{m_2,n_2}p_{m_3,n_3}\Big)\varepsilon^{(2-3\beta)m+\beta n+5\beta/2}+o(\varepsilon^{\beta N+\beta/2})\\
&= -2\sum_{\substack{m\geqslant 1,n\geqslant 2\\(2-3\beta)m+\beta n\leqslant\beta N}}(dp'_{m-1,n-2}+2p''_{m-1,n-2})\varepsilon^{(2-3\beta)m+\beta n+\beta/2}\\
&\quad +4R_1^2\sum_{\substack{m,n\geqslant 1\\(2-3\beta)m+\beta n\leqslant\beta N}}p''_{m-1,n-1}\varepsilon^{(2-3\beta)m+\beta n+\beta/2}+\sum_{\substack{m\geqslant 0,n\geqslant 1\\(2-3\beta)m+\beta n\leqslant\beta N}}p_{m,n-1}\varepsilon^{(2-3\beta)m+\beta n+\beta/2}\\
&\quad -2\alpha_1\sum_{\substack{m\geqslant 0,n\geqslant 1\\(2-3\beta)m+\beta n\leqslant\beta N}}\Big(\sum_{\substack{m_1,m_2,m_3,n_1,n_2,n_3\geqslant 0\\m_1+m_2+m_3=m\\n_1+n_2+n_3=n-1}}p_{m_1,n_1}p_{m_2,n_2}p_{m_3,n_3}\Big)\varepsilon^{(2-3\beta)m+\beta n+\beta/2}\\
&\quad -4\alpha_0\lambda_{-1}^{1/2}\sum_{\substack{m\geqslant 0,n\geqslant 1\\(2-3\beta)m+\beta n\leqslant\beta N}}\Big(\sum_{\substack{n_1,n_2,m_1,m_2\geqslant 0,\\n_1+n_2=n-1\\m_1+m_2=m}}q_{m_1,n_1}p_{m_2,n_2}\Big)\varepsilon^{(2-3\beta)m+\beta n+\beta/2}\qquad(3.42)\\
&\quad -2\alpha_0\sum_{\substack{m\geqslant 0,n\geqslant 2\\(2-3\beta)m+\beta n\leqslant\beta N}}\Big(\sum_{\substack{n_1,n_2,n_3,m_1,m_2,m_3\geqslant 0\\n_1+n_2+n_3=n-2\\m_1+m_2+m_3=m}}q_{m_1,n_1}q_{m_2,n_2}p_{m_3,n_3}\Big)\varepsilon^{(2-3\beta)m+\beta n+\beta/2}+o(\varepsilon^{\beta N+\beta/2}),
\end{aligned}
$$

where we have used (2.30). Since β is not rational, the functions $\Big((0,\varepsilon_0)\ni\varepsilon\mapsto\varepsilon^{(2-3\beta)m+\beta n}\Big)_{m,n\in\mathbb{N}^2}$ are two by two distinct, and therefore linearly independent. According to (3.38), we deduce from (3.42):

- for $m=0$ and $1\leqslant n\leqslant N$,

$$
\begin{aligned}
&p_{0,n-1}-2\alpha_1\sum_{\substack{n_1,n_2,n_3\geqslant 0\\n_1+n_2+n_3=n-1}}p_{0,n_1}p_{0,n_2}p_{0,n_3}\\
&\qquad -4\alpha_0\lambda_{-1}^{1/2}\sum_{\substack{n_1,n_2\geqslant 0\\n_1+n_2=n-1}}q_{0,n_1}p_{0,n_2}-2\alpha_0\sum_{\substack{n_1,n_2,n_3\geqslant 0\\n_1+n_2+n_3=n-2}}q_{0,n_1}q_{0,n_2}p_{0,n_3}\;=\;0,
\end{aligned}
$$

  which can be rewritten as

$$
p_{0,0}-2\alpha_1p_{0,0}^3-4\alpha_0\lambda_{-1}^{1/2}q_{0,0}p_{0,0}\;=\;0\qquad(3.43)
$$

for $n=1$, and

$$p_{0,n-1}\left(1-6\alpha_1 p_{0,0}^2-4\alpha_0\lambda_{-1}^{1/2}q_{0,0}\right)-4\alpha_0\lambda_{-1}^{1/2}p_{0,0}q_{0,n-1} \tag{3.44}$$
$$= 2\alpha_1\sum_{\substack{0\leqslant n_1,n_2,n_3<n-1\\ n_1+n_2+n_3=n-1}} p_{0,n_1}p_{0,n_2}p_{0,n_3}+4\alpha_0\lambda_{-1}^{1/2}\sum_{\substack{0\leqslant n_1,n_2<n-1\\ n_1+n_2=n-1}} q_{0,n_1}p_{0,n_2}+2\alpha_0\sum_{\substack{n_1,n_2,n_3\geqslant 0\\ n_1+n_2+n_3=n-2}} q_{0,n_1}q_{0,n_2}p_{0,n_3}$$

for $n\geqslant 2$.

- for $1\leqslant m\leqslant \beta(N-1)/(2-3\beta)$ and $n=1$,

$$4R_1^2p''_{m-1,0}+p_{m,0}-2\alpha_1\sum_{\substack{m_1,m_2,m_3\geqslant 0\\ m_1+m_2+m_3=m}} p_{m_1,0}p_{m_2,0}p_{m_3,0}-4\alpha_0\lambda_{-1}^{1/2}\sum_{\substack{m_1,m_2\geqslant 0\\ m_1+m_2=m}} q_{m_1,0}p_{m_2,0} = 0,$$

which can be rewritten as

$$p_{m,0}\left(1-6\alpha_1 p_{0,0}^2-4\alpha_0\lambda_{-1}^{1/2}q_{0,0}\right)-4\alpha_0\lambda_{-1}^{1/2}p_{0,0}q_{m,0}$$
$$= -4R_1^2p''_{m-1,0}+2\alpha_1\sum_{\substack{0\leqslant m_1,m_2,m_3<m\\ m_1+m_2+m_3=m}} p_{m_1,0}p_{m_2,0}p_{m_3,0}+4\alpha_0\lambda_{-1}^{1/2}\sum_{\substack{0\leqslant m_1,m_2<m\\ m_1+m_2=m}} q_{m_1,0}p_{m_2,0} \tag{3.45}$$

- for $m\geqslant 1$ and $n\geqslant 2$ such that $(2-3\beta)m+\beta n\leqslant \beta N$,

$$-2(dp'_{m-1,n-2}+2p''_{m-1,n-2})+4R_1^2p''_{m-1,n-1}+p_{m,n-1}$$
$$-2\alpha_1\sum_{\substack{m_1,m_2,m_3,n_1,n_2,n_3\geqslant 0\\ m_1+m_2+m_3=m\\ n_1+n_2+n_3=n-1}} p_{m_1,n_1}p_{m_2,n_2}p_{m_3,n_3}$$
$$-4\alpha_0\lambda_{-1}^{1/2}\sum_{\substack{m_1,m_2,n_1,n_2\geqslant 0\\ n_1+n_2=n-1\\ m_1+m_2=m}} q_{m_1,n_1}p_{m_2,n_2}-2\alpha_0\sum_{\substack{m_1,m_2,m_3,n_1,n_2,n_3\geqslant 0\\ n_1+n_2+n_3=n-2\\ m_1+m_2+m_3=m}} q_{m_1,n_1}q_{m_2,n_2}p_{m_3,n_3} = 0,$$

which can be rewritten as

$$\left(1-6\alpha_1 p_{0,0}^2-4\alpha_0\lambda_{-1}^{1/2}q_{0,0}\right)p_{m,n-1}-4\alpha_0\lambda_{-1}^{1/2}p_{0,0}q_{m,n-1} \tag{3.46}$$
$$= 2(dp'_{m-1,n-2}+2p''_{m-1,n-2})-4R_1^2p''_{m-1,n-1}+2\alpha_1\sum_{\substack{m_1,m_2,m_3,n_1,n_2,n_3\geqslant 0\\ m_1+m_2+m_3=m\\ n_1+n_2+n_3=n-1\\ \forall j\in\{1,2,3\},(m_j,n_j)\neq(m,n-1)}} p_{m_1,n_1}p_{m_2,n_2}p_{m_3,n_3}$$
$$+4\alpha_0\lambda_{-1}^{1/2}\sum_{\substack{m_1,m_2,n_1,n_2\geqslant 0,\\ n_1+n_2=n-1\\ m_1+m_2=m\\ \forall j\in\{1,2\},(m_j,n_j)\neq(m,n-1)}} q_{m_1,n_1}p_{m_2,n_2}+2\alpha_0\sum_{\substack{m_1,m_2,m_3,n_1,n_2,n_3,m_3\geqslant 0,\\ n_1+n_2+n_3=n-2\\ m_1+m_2+m_3=m}} q_{m_1,n_1}q_{m_2,n_2}p_{m_3,n_3}.$$

Next, we perform the same kind of calculations with the function that appears in the left hand side of (3.39).

$$
\begin{aligned}
&\varepsilon^2\Delta\theta_2 + (R_2^2 - R_1^2)\theta_2 + z\theta_2 - 2\alpha_2\theta_2^3 - 2\alpha_0\theta_1^2\theta_2 \\
&\quad = \quad -2 \sum_{\substack{m,n\geqslant 0\\ (2-3\beta)m+\beta n\leqslant \beta N}} (dq'_{m,n} + 2q''_{m,n})\varepsilon^{(2-3\beta)(m+1)+\beta(n+2)+\beta} \\
&\qquad +4R_1^2 \sum_{\substack{m,n\geqslant 0\\ (2-3\beta)m+\beta n\leqslant \beta N}} q''_{m,n}\varepsilon^{(2-3\beta)(m+1)+\beta(n+1)+\beta} \\
&\qquad +(R_2^2 - R_1^2)\lambda_{-1}^{1/2} + (R_2^2 - R_1^2) \sum_{\substack{m,n\geqslant 0\\ (2-3\beta)m+\beta n\leqslant \beta N}} q_{m,n}\varepsilon^{(2-3\beta)m+\beta n+\beta} \\
&\qquad +\varepsilon^{\beta}\lambda_{-1}^{1/2} + \sum_{\substack{m\geqslant 0, n\geqslant 0\\ (2-3\beta)m+\beta n\leqslant \beta N}} q_{m,n}\varepsilon^{(2-3\beta)m+\beta(n+1)+\beta} \\
&\qquad -2\alpha_2\lambda_{-1}^{3/2} - 6\alpha_2\lambda_{-1} \sum_{\substack{m\geqslant 0, n\geqslant 0\\ (2-3\beta)m+\beta n\leqslant \beta N}} q_{m,n}\varepsilon^{(2-3\beta)m+\beta n+\beta} \\
&\qquad -6\alpha_2\lambda_{-1}^{1/2} \sum_{\substack{m,n\geqslant 0\\ (2-3\beta)m+\beta n\leqslant \beta N}} \Big(\sum_{\substack{m_1,m_2,n_1,n_2\geqslant 0\\ m_1+m_2=m\\ n_1+n_2=n}} q_{m_1,n_1}q_{m_2,n_2}\Big)\varepsilon^{(2-3\beta)m+\beta(n+1)+\beta} \\
&\qquad -2\alpha_2 \sum_{\substack{m\geqslant 0, n\geqslant 0\\ (2-3\beta)m+\beta n\leqslant \beta N}} \Big(\sum_{\substack{m_1,m_2,m_3,n_1,n_2,n_3\geqslant 0,\\ n_1+n_2+n_3=n\\ m_1+m_2+m_3=m}} q_{m_1,n_1}q_{m_2,n_2}q_{m_3,n_3}\Big)\varepsilon^{(2-3\beta)m+\beta(n+2)+\beta} \\
&\qquad -2\alpha_0\lambda_{-1}^{1/2} \sum_{\substack{m,n\geqslant 0\\ (2-3\beta)m+\beta n\leqslant \beta N}} \Big(\sum_{\substack{m_1,m_2,n_1,n_2\geqslant 0\\ m_1+m_2=m\\ n_1+n_2=n}} p_{m_1,n_1}p_{m_2,n_2}\Big)\varepsilon^{(2-3\beta)m+\beta n+\beta} \\
&\qquad -2\alpha_0 \sum_{\substack{m,n\geqslant 0\\ (2-3\beta)m+\beta n\leqslant \beta N}} \Big(\sum_{\substack{m_1,m_2,m_3,n_1,n_2,n_3\geqslant 0\\ m_1+m_2+m_3=m\\ n_1+n_2+n_3=n}} p_{m_1,n_1}p_{m_2,n_2}q_{m_3,n_3}\Big)\varepsilon^{(2-3\beta)m+\beta(n+1)+\beta} + o(\varepsilon^{\beta(N+1)}).
\end{aligned}
$$

Thus, changing the indices and throwing away all the terms that can be incorporated in the rest,

$$
\begin{aligned}
&\varepsilon^2\Delta\theta_2+(R_2^2-R_1^2)\theta_2+z\theta_2-2\alpha_2\theta_2^3-2\alpha_0\theta_1^2\theta_2\\
&\quad=\quad -2\sum_{\substack{m\geqslant 1,n\geqslant 2\\(2-3\beta)m+\beta n\leqslant\beta N}}(dq'_{m-1,n-2}+2q''_{m-1,n-2})\varepsilon^{(2-3\beta)m+\beta n+\beta}\\
&\qquad +4R_1^2\sum_{\substack{m,n\geqslant 1\\(2-3\beta)m+\beta n\leqslant\beta N}}q''_{m-1,n-1}\varepsilon^{(2-3\beta)m+\beta n+\beta}\\
&\qquad +(R_2^2-R_1^2)\sum_{\substack{m,n\geqslant 0\\(2-3\beta)m+\beta n\leqslant\beta N}}q_{m,n}\varepsilon^{(2-3\beta)m+\beta n+\beta}\\
&\qquad +\varepsilon^\beta\lambda_{-1}^{1/2}+\sum_{\substack{m\geqslant 0,n\geqslant 1\\(2-3\beta)m+\beta n\leqslant\beta N}}q_{m,n-1}\varepsilon^{(2-3\beta)m+\beta n+\beta}\\
&\qquad -6\alpha_2\lambda_{-1}\sum_{\substack{m\geqslant 0,n\geqslant 0\\(2-3\beta)m+\beta n\leqslant\beta N}}q_{m,n}\varepsilon^{(2-3\beta)m+\beta n+\beta}\\
&\qquad -6\alpha_2\lambda_{-1}^{1/2}\sum_{\substack{m\geqslant 0,n\geqslant 1\\(2-3\beta)m+\beta n\leqslant\beta N}}\Big(\sum_{\substack{m_1,m_2,n_1,n_2\geqslant 0\\m_1+m_2=m\\n_1+n_2=n-1}}q_{m_1,n_1}q_{m_2,n_2}\Big)\varepsilon^{(2-3\beta)m+\beta n+\beta}\\
&\qquad -2\alpha_2\sum_{\substack{m\geqslant 0,n\geqslant 2\\(2-3\beta)m+\beta n\leqslant\beta N}}\Big(\sum_{\substack{m_1,m_2,m_3,n_1,n_2,n_3\geqslant 0,\\n_1+n_2+n_3=n-2\\m_1+m_2+m_3=m}}q_{m_1,n_1}q_{m_2,n_2}q_{m_3,n_3}\Big)\varepsilon^{(2-3\beta)m+\beta n+\beta}\\
&\qquad -2\alpha_0\lambda_{-1}^{1/2}\sum_{\substack{m,n\geqslant 0\\(2-3\beta)m+\beta n\leqslant\beta N}}\Big(\sum_{\substack{m_1,m_2,n_1,n_2\geqslant 0\\m_1+m_2=m\\n_1+n_2=n}}p_{m_1,n_1}p_{m_2,n_2}\Big)\varepsilon^{(2-3\beta)m+\beta n+\beta}\qquad(3.47)\\
&\qquad -2\alpha_0\sum_{\substack{m\geqslant 0,n\geqslant 1\\(2-3\beta)m+\beta n\leqslant\beta N}}\Big(\sum_{\substack{m_1,m_2,m_3,n_1,n_2,n_3\geqslant 0\\m_1+m_2+m_3=m\\n_1+n_2+n_3=n-1}}p_{m_1,n_1}p_{m_2,n_2}q_{m_3,n_3}\Big)\varepsilon^{(2-3\beta)m+\beta n+\beta}+o(\varepsilon^{\beta(N+1)}).
\end{aligned}
$$

According to (3.39), the right hand side of (3.47) is equal to 0, up to the rest term $o(\varepsilon^{\beta(N+1)})$. Thus, the linear independance of the family of functions of ε, $\Big(\varepsilon^{(2-3\beta)m+\beta n}\Big)_{m,n\geqslant 0}$ yields:

- for $m=0$, $n=0$, thanks to (2.30), we get

$$-2(R_2^2-R_1^2)q_{0,0}=\lambda_{-1}^{1/2}(2\alpha_0p_{0,0}^2-1),\qquad(3.48)$$

- for $m=0$ and $1\leqslant n\leqslant N$,

$$
\begin{aligned}
&(R_2^2-R_1^2)q_{0,n}+q_{0,n-1}-6\alpha_2\lambda_{-1}q_{0,n}-6\alpha_2\lambda_{-1}^{1/2}\sum_{\substack{n_1,n_2\geqslant 0,\\n_1+n_2=n-1}}q_{0,n_1}q_{0,n_2}\\
&\qquad -2\alpha_2\sum_{\substack{n_1,n_2,n_3\geqslant 0,\\n_1+n_2+n_3=n-2}}q_{0,n_1}q_{0,n_2}q_{0,n_3}-2\alpha_0\lambda_{-1}^{1/2}\sum_{\substack{n_1,n_2\geqslant 0\\n_1+n_2=n}}p_{0,n_1}p_{0,n_2}-2\alpha_0\sum_{\substack{n_1,n_2,n_3\geqslant 0\\n_1+n_2+n_3=n-1}}p_{0,n_1}p_{0,n_2}q_{0,n_3}=0,
\end{aligned}
$$

which, using (2.30), can be rewritten as

$$
\begin{aligned}
&-2(R_2^2-R_1^2)q_{0,n}-4\alpha_0p_{0,0}\lambda_{-1}^{1/2}p_{0,n}\qquad(3.49)\\
&\quad=\quad -q_{0,n-1}+6\alpha_2\lambda_{-1}^{1/2}\sum_{\substack{n_1,n_2\geqslant 0,\\n_1+n_2=n-1}}q_{0,n_1}q_{0,n_2}+2\alpha_2\sum_{\substack{n_1,n_2,n_3\geqslant 0,\\n_1+n_2+n_3=n-2}}q_{0,n_1}q_{0,n_2}q_{0,n_3}\\
&\qquad +2\alpha_0\lambda_{-1}^{1/2}\sum_{\substack{0\leqslant n_1,n_2<n\\n_1+n_2=n}}p_{0,n_1}p_{0,n_2}+2\alpha_0\sum_{\substack{n_1,n_2,n_3\geqslant 0\\n_1+n_2+n_3=n-1}}p_{0,n_1}p_{0,n_2}q_{0,n_3},
\end{aligned}
$$

- for $1 \leqslant m \leqslant \beta N/(2-3\beta)$ and $n = 0$,

$$(R_2^2 - R_1^2) q_{m,0} - 6\alpha_2 \lambda_{-1} q_{m,0} - 2\alpha_0 \lambda_{-1}^{1/2} \sum_{\substack{m_1,m_2 \geqslant 0, \\ m_1+m_2=m}} p_{m_1,0} p_{m_2,0} = 0,$$

that is

$$-2(R_2^2 - R_1^2) q_{m,0} - 4\alpha_0 p_{0,0} \lambda_{-1}^{1/2} p_{m,0} = 2\alpha_0 \lambda_{-1}^{1/2} \sum_{\substack{0 \leqslant m_1,m_2 < m, \\ m_1+m_2=m}} p_{m_1,0} p_{m_2,0}. \tag{3.50}$$

- for $m \geqslant 1$ and $n \geqslant 1$ such that $(2-3\beta)m + \beta n \leqslant \beta N$,

$$\begin{aligned}
&-2(dq'_{m-1,n-2} + 2q''_{m-1,n-2}) \mathbf{1}_{\{n \geqslant 2\}} + 4R_1^2 q''_{m-1,n-1} + (R_2^2 - R_1^2) q_{m,n} + q_{m,n-1} \\
&-6\alpha_2 \lambda_{-1} q_{m,n} - 6\alpha_2 \lambda_{-1}^{1/2} \sum_{\substack{m_1,m_2,n_1,n_2 \geqslant 0 \\ m_1+m_2=m, \\ n_1+n_2=n-1}} q_{m_1,n_1} q_{m_2,n_2} - 2\alpha_2 \sum_{\substack{m_1,m_2,m_3,n_1,n_2,n_3 \geqslant 0, \\ n_1+n_2+n_3=n-2 \\ m_1+m_2+m_3=m}} q_{m_1,n_1} q_{m_2,n_2} q_{m_3,n_3} \\
&-2\alpha_0 \lambda_{-1}^{1/2} \sum_{\substack{m_1,m_2,n_1,n_2 \geqslant 0 \\ m_1+m_2=m \\ n_1+n_2=n}} p_{m_1,n_1} p_{m_2,n_2} - 2\alpha_0 \sum_{\substack{m_1,m_2,m_3,n_1,n_2,n_3 \geqslant 0 \\ m_1+m_2+m_3=m \\ n_1+n_2+n_3=n-1}} p_{m_1,n_1} p_{m_2,n_2} q_{m_3,n_3} = 0
\end{aligned}$$

which can be rewritten as

$$\begin{aligned}
&-2(R_2^2 - R_1^2) q_{m,n} - 4\alpha_0 \lambda_{-1}^{1/2} p_{0,0} p_{m,n} \\
&= 2(dq'_{m-1,n-2} + 2q''_{m-1,n-2}) \mathbf{1}_{\{n \geqslant 2\}} - 4R_1^2 q''_{m-1,n-1} - q_{m,n-1} + 6\alpha_2 \lambda_{-1}^{1/2} \sum_{\substack{0 \leqslant m_1,m_2,n_1,n_2 \\ n_1+n_2=n-1 \\ m_1+m_2=m}} q_{m_1,n_1} q_{m_2,n_2} \\
&\quad + 2\alpha_2 \sum_{\substack{m_1,m_2,m_3,n_1,n_2,n_3 \geqslant 0 \\ n_1+n_2+n_3=n-2 \\ m_1+m_2+m_3=m}} q_{m_1,n_1} q_{m_2,n_2} q_{m_3,n_3} \\
&\quad + 2\alpha_0 \lambda_{-1}^{1/2} \sum_{\substack{m_1,m_2,n_1,n_2 \geqslant 0 \\ m_1+m_2=m \\ n_1+n_2=n \\ \forall j \in \{1,2\}, (m_j,n_j) \neq (m,n)}} p_{m_1,n_1} p_{m_2,n_2} + 2\alpha_0 \sum_{\substack{m_1,m_2,m_3,n_1,n_2,n_3 \geqslant 0 \\ m_1+m_2+m_3=m \\ n_1+n_2+n_3=n-1}} p_{m_1,n_1} p_{m_2,n_2} q_{m_3,n_3}
\end{aligned} \tag{3.51}$$

Next, we show that the system of equations satisfied by the $p_{m,n}$'s and $q_{m,n}$'s has a unique solution such that $p_{0,0} > 0$. First, plugging

$$q_{0,0} = \frac{1 - 2\alpha_0 p_{0,0}^2}{2(R_2^2 - R_1^2)} \lambda_{-1}^{1/2}, \tag{3.52}$$

(which comes from (3.48)) into (3.43) and using also (2.30), we get

$$p_{0,0} = \left(\frac{\Gamma_2}{2\alpha_1 \Gamma_{12}} \right)^{1/2}, \tag{3.53}$$

and

$$q_{0,0} = \frac{\Gamma_1}{2\Gamma_{12}(2\alpha_2(R_2^2 - R_1^2))^{1/2}}. \tag{3.54}$$

Next, for $1 \leqslant n \leqslant N-1$, the $q_{0,n}$'s and the $p_{0,n}$'s are constructed recursively thanks to (3.49) as well as (3.44) with n replaced by $n+1$. We solve the system obtained by combination of these two equations by inverting the matrix

$$M = \begin{bmatrix} -2(R_2^2 - R_1^2) & -4\alpha_0 p_{0,0} \lambda_{-1}^{1/2} \\ -4\alpha_0 p_{0,0} \lambda_{-1}^{1/2} & 1 - 6\alpha_1 p_{0,0}^2 - 4\alpha_0 \lambda_{-1}^{1/2} q_{0,0} \end{bmatrix} = \begin{bmatrix} -2(R_2^2 - R_1^2) & -2\alpha_0 \left(\frac{(R_2^2 - R_1^2)\Gamma_2}{\alpha_1 \alpha_2 \Gamma_{12}} \right)^{1/2} \\ -2\alpha_0 \left(\frac{(R_2^2 - R_1^2)\Gamma_2}{\alpha_1 \alpha_2 \Gamma_{12}} \right)^{1/2} & -2\frac{\Gamma_2}{\Gamma_{12}} \end{bmatrix},$$

where we have used (2.30), (3.53) and (3.54). The determinant of M is

$$\det M = 4(R_2^2 - R_1^2)\Gamma_2 > 0,$$

therefore M is invertible, and there is a unique possible choice for $(q_{0,n}, p_{0,n})$ for $1 \leqslant n \leqslant N-1$ such that the assumptions of the lemma are satisfied. Then, for $1 \leqslant m \leqslant \beta(N-1)/(2-3\beta)$, the $q_{m,0}$'s and the $p_{m,0}$'s are constructed recursively thanks to (3.45) and (3.50) by inverting the same matrix M. Finally, if $m \geqslant 1$, $n \geqslant 1$ and $(2-3\beta)m + \beta n \leqslant \beta(N-1)$ and if the $q_{k,l}$'s and the $p_{k,l}$'s are known for every $k \leqslant m$, $l \leqslant n$ and $(k,l) \neq (m,n)$, $(q_{m,n}, p_{m,n})$ is entirely determined because the system made of (3.46) for n replaced by $n+1$ and (3.51) has a unique solution thanks to the invertibility of M. This way, we prove recursively that the assumptions of the lemma determine completely the values of the coefficients $q_{m,n}$ and $p_{m,n}$, provided $(2-3\beta)m + \beta n \leqslant \beta(N-1)$. ∎

Lemma 3.10 *Let $N \geqslant 3$ be an integer, $M \geqslant \frac{\beta}{2-3\beta}N$, and ω, τ, ν, λ given by (3.1). Then for $l = 0, 1, 2$, we have*

$$\left\| \frac{d^l}{dz^l}\left(\omega(z) - \varepsilon^{1/3}\nu(y_1)\right)\right\|_{L^\infty(D_0\cap D_1)} = o(\varepsilon^{\beta(N-1/2-l)}) \tag{3.55}$$

and

$$\left\| \frac{d^l}{dz^l}\left(\tau(z) - \varepsilon^{1/3}\lambda(y_1)^{1/2}\right)\right\|_{L^\infty(D_0\cap D_1)} = o(\varepsilon^{\beta(N-l)}). \tag{3.56}$$

Proof. The assumptions (3.40) and (3.41) made on $(\theta_1(z), \theta_2(z))$ in Lemma 3.9 are satisfied by $(\omega(z), \tau(z))$ thanks to Lemma 3.7, and also by $(\varepsilon^{1/3}\nu(y_1), \varepsilon^{1/3}\lambda(y_1)^{1/2})$ thanks to Lemma 3.8. Assumptions (3.38) and (3.39) are satisfied by $(\omega(z), \tau(z))$ thanks to (3.4) and (3.6), and they are also satisfied by $(\varepsilon^{1/3}\nu(y_1), \varepsilon^{1/3}\lambda(y_1)^{1/2})$ thanks to Lemma 3.3. Therefore, Lemma 3.9 ensures that for every $(m,n) \in \mathbb{N}^2$ such that $(2-3\beta)m + \beta n \leqslant \beta(N-1)$, $w_{m,n} = n_{m,n}$ and $t_{m,n} = l_{m,n}$. In particular, (3.55) and (3.56) are satisfied. ∎

3.6 Comparison of $\varepsilon^{1/3}(\nu, \lambda^{1/2})$ and $(0, \varepsilon^{1/3}\mu)$ in $D_1 \cap D_2$

We first give an expansion of $\varepsilon^{1/3}(\nu, \lambda^{1/2})$ into powers of ε in $D_1 \cap D_2$, as $\varepsilon \to 0$.

Lemma 3.11 *Let $N \geqslant 1$ be an integer. There exist a family of numbers $(\widetilde{l}_{m,n})_{m\geqslant 0, n\geqslant 0}$ which does not depend on N such that if ν and λ are given by (3.1), then for every $\alpha > 0$,*

$$\left\| \frac{d^l}{dz^l}\left(\varepsilon^{1/3}\nu(y_1)\right)\right\|_{L^\infty(D_1\cap D_2)} \underset{\varepsilon\to 0}{=} o\left(\varepsilon^{\alpha}\right) \tag{3.57}$$

and

$$\left\| \frac{d^l}{dz^l}\left(\varepsilon^{1/3}\lambda(y_1)^{1/2}\right) - \lambda_{-1}^{1/2}\mathbf{1}_{l=0} - \sum_{\substack{(m,n)\in\mathbb{N}^2 \\ (2-3\beta)m+\beta n\leqslant \beta N \\ 1+n-3m\geqslant 0}} \varepsilon^{2m}\widetilde{l}_{m,n}\ \frac{d^l}{dz^l}\left(z^{1+n-3m}\right)\right\|_{L^\infty(D_1\cap D_2)} \underset{\varepsilon\to 0}{=} o(\varepsilon^{\beta(N+1-l)}) \tag{3.58}$$

Proof. For $x \in D_1 \cap D_2$, we have $-2\varepsilon^{\beta-2/3} \leqslant y_1 \leqslant -\varepsilon^{\beta-2/3} \to -\infty$ as $\varepsilon \to 0$. Thus, (3.57) follows from (2.49) and (2.51). As for (3.58), we proceed like in the proof of Lemma 3.8. First, from (2.50)

and (2.52), we have

$$\begin{aligned}
\frac{d^l}{dy_1^l}\left(\varepsilon^{2/3}\lambda(y_1)\right) &\underset{y_1\to-\infty}{\approx} \lambda_{-1}\mathbf{1}_{\{l=0\}} + \varepsilon^{2/3}\frac{d^l}{dy_1^l}\left(y_1/(2\alpha_2)\right) + \varepsilon^{2/3}\sum_{n=1}^{N}\varepsilon^{2n/3}\sum_{0\leqslant m\leqslant(n-2)/3}\widetilde{L}_{n,m}\frac{d^l}{dy_1^l}\left(y_1^{n-2-3m}\right)\\
&\underset{y_1\to-\infty}{\approx} \lambda_{-1}\mathbf{1}_{\{l=0\}} + \varepsilon^{2/3}\frac{d^l}{dy_1^l}\left(y_1/(2\alpha_2)\right) + \varepsilon^{2/3}\sum_{n=1}^{N}\varepsilon^{2n/3}\sum_{1\leqslant m\leqslant(n+1)/3}\widetilde{L}_{n,m-1}\frac{d^l}{dy_1^l}\left(y_1^{n+1-3m}\right)\\
&\underset{y_1\to-\infty}{\approx} \lambda_{-1}\mathbf{1}_{\{l=0\}} + \sum_{n=0}^{N}\varepsilon^{2(n+1)/3}\sum_{0\leqslant m\leqslant(n+1)/3}\tilde{\widetilde{L}}_{n,m}\frac{d^l}{dy_1^l}\left(y_1^{n+1-3m}\right)\\
&\underset{y_1\to-\infty}{\approx} \lambda_{-1}\mathbf{1}_{\{l=0\}} + \sum_{0\leqslant n\leqslant N,m\geqslant 0}\varepsilon^{2(n+1)/3}\tilde{\widetilde{L}}_{n,m}\frac{d^l}{dy_1^l}\left(y_1^{n+1-3m}\right), \qquad (3.59)
\end{aligned}$$

with

$$\tilde{\widetilde{L}}_{n,m} = \begin{cases} 1/(2\alpha_2) & \text{if} \quad n=m=0\\ 0 & \text{if} \quad n\geqslant 1 \text{ and } m=0 \text{ or } n=0 \text{ and } m\geqslant 1\\ \widetilde{L}_{n,m-1} & \text{if} \quad n\geqslant 1 \text{ and } 1\leqslant m\leqslant (n+1)/3\\ 0 & \text{if} \quad n\geqslant 1 \text{ and } m>(n+1)/3. \end{cases}$$

Thus, for $x\in D_1\cap D_2$, throwing away the smallest terms,

$$\frac{d^l}{dy_1^l}\left(\varepsilon^{2/3}\lambda(y_1)\right) \underset{\varepsilon\to 0}{=} \lambda_{-1}\mathbf{1}_{\{l=0\}} + \varepsilon^{2l/3}\sum_{\substack{(m,n)\in\mathbb{N}^2\\(2-3\beta)m+\beta n\leqslant\beta N}}\varepsilon^{2m}\tilde{\widetilde{L}}_{n,m}\frac{d^l}{dz^l}\left(z^{1+n-3m}\right) + o_{L^\infty(D_1\cap D_2)}(\varepsilon^{\beta(N+1)+(2/3-\beta)l}).$$

At this point, the calculation becomes similar to the one which was performed for $y_1\to+\infty$ in the proof of Lemma 3.8. Indeed, we can deduce like in (3.31) that for $l=0$,

$$\varepsilon^{1/3}\lambda(y_1)^{1/2} \underset{\varepsilon\to 0}{=} \lambda_{-1}^{1/2} + \sum_{\substack{(n,m)\in\mathbb{N}^2\\(2-3\beta)m+\beta n\leqslant\beta N}}\varepsilon^{2m}z^{n-3m+1}\widetilde{l}_{m,n} + o_{L^\infty(D_1\cap D_2)}(\varepsilon^{\beta(N+1)}), \qquad (3.60)$$

where

$$\widetilde{l}_{m,n} = \sum_{k=1}^{n+1}c_k\sum_{\substack{((n_1,m_1),\cdots,(n_k,m_k))\in(\mathbb{N}^2)^k\\n_1+\cdots+n_k=n-k+1\\m_1+\cdots+m_k=m}}\prod_{j=1}^{k}\tilde{\widetilde{L}}_{n_j,m_j} \qquad (3.61)$$

for the same coefficients c_k as in (3.30). Note in particular that $\widetilde{l}_{m,n}=0$ if $m>(n+1)/3$. Indeed, under this condition, for every $k\in\{1,\cdots,N+1\}$, if $n_1,\cdots,n_k,m_1,\cdots,m_k$ are indices like in the second sum in (3.61), we have

$$m_1+\cdots+m_k = m > \frac{n+1}{3} = \frac{(n_1+1)+\cdots+(n_k+1)}{3},$$

therefore at least for one of the indices $j\in\{1,\cdots,k\}$, we have $m_j>(n_j+1)/3$, which implies

$$\prod_{j=1}^{k}\tilde{\widetilde{L}}_{n_j,m_j}=0,$$

for every $k\in\{1,\cdots,N+1\}$, and therefore $\widetilde{l}_{m,n}=0$. This is the reason why we can add without changing the result the condition $1+n-3m\geqslant 0$ in the sum that appears in (3.58) for $l=0$. The proof of (3.58) for $l=1$ and $l=2$ is similar to the one which was done on $D_0\cap D_1$ in the proof of Lemma 3.8. ∎

The next lemma provides an asymptotic expansion of $(0,\varepsilon^{1/3}\mu)$ into powers of ε in $D_1\cap D_2$ as $\varepsilon\to 0$.

Lemma 3.12 *Let $L \geqslant 1$, and μ given by (3.1). Then there exists a family of numbers $(\alpha_{m,n})_{m,n\geqslant 0}$ such that for every $l \in \{0,1,2\}$,*

$$\left\| \frac{d^l}{dz^l}\left(\varepsilon^{1/3}\mu(y_2)\right) - \lambda_{-1}^{1/2}\mathbf{1}_{\{l=0\}} - \sum_{\substack{m,n\geqslant 0\\ \beta n+(2-3\beta)m\leqslant 2L-\beta\\ 1+n-3m\geqslant 0}} \alpha_{m,n}\varepsilon^{2m}\frac{d^l}{dz^l}\left(z^{1+n-3m}\right)\right\|_{L^\infty(D_1\cap D_2)} = o(\varepsilon^{2L-\beta l}) \quad (3.62)$$

Proof. For $x \in D_1\cap D_2$, $(R_2^2-R_1^2)\varepsilon^{-2/3}-\varepsilon^{\beta-2/3} \geqslant y_2 \geqslant (R_2^2-R_1^2)\varepsilon^{-2/3}-2\varepsilon^{\beta-2/3} \to +\infty$ as $\varepsilon\to 0$. Thus, for $l=0,1,2$, thanks to (2.19), (2.25) and Proposition 2.6, using for convenience the notations $g_{0,m}=a_m$, $g^{(0)}_{n,m}=g_{n,m}$, $g^{(1)}_{n,m}=(1/2-2n-3m)g_{n,m}$ and $g^{(2)}_{n,m}=(-1/2-2n-3m)g^{(1)}_{n,m}$, we infer

$$\begin{aligned}
&\frac{d^l}{dz^l}\left(\varepsilon^{1/3}\mu(y_2)\right)\\
&= \varepsilon^{1/3}\sum_{n=0}^{L}\varepsilon^{2n/3}\frac{d^l}{dz^l}\left(\mu_n(y_2)\right)\\
&= \frac{\varepsilon^{1/3}}{(2\alpha_2)^{1/2}}\sum_{n=0}^{L}\varepsilon^{2n/3}\sum_{m=0}^{\infty}g^{(l)}_{n,m}R_2^{2m}\varepsilon^{-2l/3}y_2^{1/2-2n-3m-l}\\
&= \frac{\varepsilon^{1/3}}{(2\alpha_2)^{1/2}}\sum_{n=0}^{L}\varepsilon^{2n/3}\sum_{m=0}^{L-n}g^{(l)}_{n,m}R_2^{2m}\varepsilon^{-2l/3}y_2^{1/2-2n-3m-l}+\varepsilon^{1/3}\sum_{n=0}^{L}\varepsilon^{2(n-l)/3}o_{L^\infty(D_1\cap D_2)}(y_2^{1/2+n-3L-l})\\
&= \frac{\varepsilon^{1/3}}{(2\alpha_2)^{1/2}}\sum_{n=0}^{L}\varepsilon^{2n/3}\sum_{m=0}^{L-n}g^{(l)}_{n,m}R_2^{2m}\varepsilon^{-2l/3}\frac{(R_2^2-R_1^2)^{1/2-2n-3m-l}}{\varepsilon^{1/3-4n/3-2m-2l/3}}\left(1+\frac{z}{R_2^2-R_1^2}\right)^{1/2-2n-3m-l}+o_{L^\infty(D_1\cap D_2)}(\varepsilon^{2L})\\
&= \lambda_{-1}^{1/2}\sum_{n=0}^{L}\sum_{m=0}^{L-n}\varepsilon^{2(n+m)}g^{(l)}_{n,m}R_2^{2m}(R_2^2-R_1^2)^{-2n-3m-l}\left(1+\frac{z}{R_2^2-R_1^2}\right)^{1/2-2n-3m-l}+o_{L^\infty(D_1\cap D_2)}(\varepsilon^{2L})\\
&= \lambda_{-1}^{1/2}\sum_{j=0}^{L}\varepsilon^{2j}\sum_{\substack{m,n\geqslant 0,\\ n+m=j}}g^{(l)}_{n,m}R_2^{2m}(R_2^2-R_1^2)^{-2n-3m-l}\sum_{\substack{k\geqslant 0,\\ \beta k+2j\leqslant 2L}}c_{k,l,m,n}z^k+o_{L^\infty(D_1\cap D_2)}(\varepsilon^{2L}) \quad (3.63)\\
&= \lambda_{-1}^{1/2}g^{(l)}_{0,0}(R_2^2-R_1^2)^{-l}+\lambda_{-1}^{1/2}\sum_{\substack{j,k\geqslant 0\\ \beta k+2j\leqslant 2L,\\ (j,k)\neq(0,0)}}\varepsilon^{2j}\sum_{\substack{m,n\geqslant 0,\\ n+m=j}}g^{(l)}_{n,m}R_2^{2m}(R_2^2-R_1^2)^{-2n-3m-l}c_{k,l,m,n}z^k+o_{L^\infty(D_1\cap D_2)}(\varepsilon^{2L}),
\end{aligned}$$

for some coefficients $(c_{k,l,m,n})_{k\geqslant 0}$ (with $c_{0,l,m,n}=1$, $\forall l,m,n$). Then, we change the variable k in the sum into $p=3j+k-1$. Note that $p\in\mathbb{N}$ since $(j,k)\in\mathbb{N}^2\backslash\{(0,0)\}$. Thus,

$$\begin{aligned}
\frac{d^l}{dz^l}\left(\varepsilon^{1/3}\mu(y_2)\right) &= \lambda_{-1}^{1/2}g^{(l)}_{0,0}(R_2^2-R_1^2)^{-l} \quad (3.64)\\
&+\lambda_{-1}^{1/2}\sum_{\substack{j,p\geqslant 0\\ \beta(p+1)+(2-3\beta)j\leqslant 2L\\ p\geqslant 3j-1}}c_{1+p-3j}\sum_{\substack{m,n\geqslant 0,\\ n+m=j}}g^{(l)}_{n,m}R_2^{2m}(R_2^2-R_1^2)^{-2n-3m-l}\varepsilon^{2j}z^{1+p-3j}+o_{L^\infty(D_1\cap D_2)}(\varepsilon^{2L}).
\end{aligned}$$

The result follows for $l=0$, since $g_{0,0}=1$, with

$$\alpha_{m,n}=\lambda_{-1}^{1/2}c_{1+n-3m}\sum_{\substack{k,i\geqslant 0,\\ k+i=m}}g^{(0)}_{i,k}R_2^{2k}(R_2^2-R_1^2)^{-2i-3k}.$$

For $l=1$, (3.64) gives the existence of some coefficients $(\alpha'_{m,n})_{m,n}$ such that

$$\frac{d}{dz}\left(\varepsilon^{1/3}\mu(y_2)\right) = \alpha'_{0,0}+\sum_{\substack{m,n\geqslant 0\\ \beta(n+1)+(2-3\beta)m\leqslant 2L\\ 1+n-3m\geqslant 0}}\alpha'_{m,n+1}\varepsilon^{2m}\frac{d}{dz}\left(z^{2+n-3m}\right)+o_{L^\infty(D_1\cap D_2)}(\varepsilon^{2L}).$$

Thus,

$$\begin{aligned}\frac{d}{dz}\left(\varepsilon^{1/3}\mu(y_2)\right) &= \sum_{\substack{m,n\geqslant 0\\ \beta n+(2-3\beta)m\leqslant 2L\\ n-3m\geqslant 0}} \alpha'_{m,n}\varepsilon^{2m}\frac{d}{dz}\left(z^{1+n-3m}\right)+o_{L^\infty(D_1\cap D_2)}(\varepsilon^{2L})\\ &= \sum_{\substack{m,n\geqslant 0\\ \beta(n+1)+(2-3\beta)m\leqslant 2L\\ n-3m\geqslant 0}} \alpha'_{m,n}\varepsilon^{2m}\frac{d}{dz}\left(z^{1+n-3m}\right)+o_{L^\infty(D_1\cap D_2)}(\varepsilon^{2L-\beta})\\ &= \sum_{\substack{m,n\geqslant 0\\ \beta(n+1)+(2-3\beta)m\leqslant 2L\\ 1+n-3m\geqslant 0}} \alpha'_{m,n}\varepsilon^{2m}\frac{d}{dz}\left(z^{1+n-3m}\right)+o_{L^\infty(D_1\cap D_2)}(\varepsilon^{2L-\beta}). \quad (3.65)\end{aligned}$$

where in the first equality, we have changed the index of summation n by $n+1$, in the second equality, we have neglected some terms in the sum, and in the last equality, the extra term we write in the sum is in fact equal to 0. In order to prove that (3.62) also holds for $l=1$, it remains to prove that for every pair of indices (m,n) appearing in the sum in (3.62) (except for $1+n-3m=0$, for which the corresponding term in (3.62) for $l=1$ is anyway equal to 0), we have $\alpha'_{m,n}=\alpha_{m,n}$. This can be done by using the same trick as in the proof of Lemma 3.8. Namely, we have on the one side thanks to (3.62)

$$\begin{aligned}&\varepsilon^{1/3}\mu\left(\frac{R_2^2-R_1^2}{\varepsilon^{2/3}}-\varepsilon^{\beta-2/3}\right)-\varepsilon^{1/3}\mu\left(\frac{R_2^2-R_1^2}{\varepsilon^{2/3}}-2\varepsilon^{\beta-2/3}\right)\\ &= \sum_{\substack{m,n\geqslant 0\\ \beta n+(2-3\beta)m\leqslant 2L-\beta\\ 1+n-3m\geqslant 0}} \alpha_{m,n}\varepsilon^{2m}(-1)^{1+n-3m}\varepsilon^{\beta(1+n-3m)}\left(1-2^{1+n-3m}\right)+o_{L^\infty(D_1\cap D_2)}(\varepsilon^{2L}) \quad (3.66)\end{aligned}$$

and on the other side, by integration of (3.65) between $z=-2\varepsilon^\beta$ and $z=-\varepsilon^\beta$, we have the same equality with $\alpha_{m,n}$ replaced by $\alpha'_{m,n}$. Since β has been chosen irrational, the linear independance of the functions $\varepsilon\mapsto\varepsilon^{(2-3\beta)m+\beta(n+1)}$ implies that for all the indices (m,n) appearing in the sum (except for $1+n-3m=0$), we have $\alpha_{m,n}=\alpha'_{m,n}$. The proof of (3.62) for $l=2$ is similar. ∎

The next lemma shows that the expansions of $\varepsilon^{1/3}(\nu,\lambda^{1/2})$ and $(0,\varepsilon^{1/3}\mu)$ calculated respectively in Lemmata 3.11 and 3.12 are in fact the same.

Lemma 3.13 *Let $N\geqslant 1$ be an integer, $\varepsilon_0>0$, and $\beta\in(0,2/3)\backslash\mathbb{Q}$. Let $(\theta)_{0<\varepsilon\leqslant\varepsilon_0}$ be a sequence of regular functions defined for $z\in[-2\varepsilon^\beta,-\varepsilon^\beta]$ such that*

$$\left\|\varepsilon^2\Delta\theta+(R_2^2-R_1^2)\theta+z\theta-2\alpha_2\theta^3\right\|_{L^\infty(D_1\cap D_2)}=o\left(\varepsilon^{\beta(N+1)}\right). \quad (3.67)$$

We assume that there exists a family of real numbers $q_{m,n}$, defined for every $(m,n)\in\mathbb{N}^2$ such that $(2-3\beta)m+\beta n\leqslant\beta N$, such that for $l\in\{0,1,2\}$, we have

$$\left\|\theta^{(l)}-\lambda_{-1}^{1/2}\mathbf{1}_{\{l=0\}}-\sum_{\substack{(m,n)\in\mathbb{N}^2\\ (2-3\beta)m+\beta n\leqslant\beta N}}\varepsilon^{2m}q_{m,n}\,\frac{d^l}{dz^l}\left(z^{1+n-3m}\right)\right\|_{L^\infty(D_1\cap D_2)}\underset{\varepsilon\to 0}{=}o(\varepsilon^{\beta(N+1-l)}). \quad (3.68)$$

Then, equations (3.68) and (3.67) entirely determine the values of the $q_{m,n}$'s for $(2-3\beta)m+\beta n\leqslant\beta(N-1)$. Moreover, these coefficients do not depend on N or β.

Proof. For convenience, for every $(m,n)\in\mathbb{N}^2$, we denote $q'_{m,n}=(1+n-3m)q_{m,n}$ and $q''_{m,n}=(n-3m)(1+n-3m)q_{m,n}$. For a function θ that satisfies (3.68), let us calculate the function that

appears in the left hand side of (3.67), evaluated at $z = -\varepsilon^\beta$. We have

$$\begin{aligned}
&\varepsilon^2\Delta\theta + (R_2^2 - R_1^2)\theta + z\theta - 2\alpha_2\theta^3 \\
&= -2 \sum_{\substack{m,n\geqslant 0\\ (2-3\beta)m+\beta n\leqslant \beta N}} (dq'_{m,n} + 2q''_{m,n})(-1)^{n-3m}\varepsilon^{(2-3\beta)(m+1)+\beta(n+2)+\beta} \\
&\quad +4R_1^2 \sum_{\substack{m,n\geqslant 0\\ (2-3\beta)m+\beta n\leqslant \beta N}} q''_{m,n}(-1)^{n-3m-1}\varepsilon^{(2-3\beta)(m+1)+\beta(n+1)+\beta} \\
&\quad +(R_2^2 - R_1^2)\lambda_{-1}^{1/2} + (R_2^2 - R_1^2) \sum_{\substack{m,n\geqslant 0\\ (2-3\beta)m+\beta n\leqslant \beta N}} q_{m,n}(-1)^{n-3m+1}\varepsilon^{(2-3\beta)m+\beta n+\beta} \\
&\quad -\varepsilon^\beta\lambda_{-1}^{1/2} + \sum_{\substack{m\geqslant 0,n\geqslant 0\\ (2-3\beta)m+\beta n\leqslant \beta N}} q_{m,n}(-1)^{n-3m}\varepsilon^{(2-3\beta)m+\beta(n+1)+\beta} \\
&\quad -2\alpha_2\lambda_{-1}^{3/2} - 6\alpha_2\lambda_{-1} \sum_{\substack{m\geqslant 0,n\geqslant 0\\ (2-3\beta)m+\beta n\leqslant \beta N}} q_{m,n}(-1)^{n-3m+1}\varepsilon^{(2-3\beta)m+\beta n+\beta} \\
&\quad -6\alpha_2\lambda_{-1}^{1/2} \sum_{\substack{m,n\geqslant 0\\ (2-3\beta)m+\beta n\leqslant \beta N}} \Big(\sum_{\substack{m_1,m_2,n_1,n_2\geqslant 0\\ m_1+m_2=m\\ n_1+n_2=n}} q_{m_1,n_1}q_{m_2,n_2}\Big)(-1)^{n-3m}\varepsilon^{(2-3\beta)m+\beta(n+1)+\beta} \\
&\quad -2\alpha_2 \sum_{\substack{m\geqslant 0,n\geqslant 0\\ (2-3\beta)m+\beta n\leqslant \beta N}} \Big(\sum_{\substack{m_1,m_2,m_3,n_1,n_2,n_3\geqslant 0,\\ n_1+n_2+n_3=n\\ m_1+m_2+m_3=m}} q_{m_1,n_1}q_{m_2,n_2}q_{m_3,n_3}\Big)(-1)^{n-3m+1}\varepsilon^{(2-3\beta)m+\beta(n+2)+\beta}
\end{aligned}$$

Thus, changing the indices and throwing away all the terms that can be incorporated in the rest,

$$\begin{aligned}
&\varepsilon^2\Delta\theta + (R_2^2 - R_1^2)\theta + z\theta - 2\alpha_2\theta^3 - 2\alpha_0\theta_1^2\theta \\
&= -2 \sum_{\substack{m\geqslant 1,n\geqslant 2\\ (2-3\beta)m+\beta n\leqslant \beta N}} (dq'_{m-1,n-2} + 2q''_{m-1,n-2})(-1)^{n-3m+1}\varepsilon^{(2-3\beta)m+\beta n+\beta} \\
&\quad +4R_1^2 \sum_{\substack{m,n\geqslant 1\\ (2-3\beta)m+\beta n\leqslant \beta N}} q''_{m-1,n-1}(-1)^{n-3m+1}\varepsilon^{(2-3\beta)m+\beta n+\beta} \\
&\quad +(R_2^2 - R_1^2) \sum_{\substack{m,n\geqslant 0\\ (2-3\beta)m+\beta n\leqslant \beta N}} q_{m,n}(-1)^{n-3m+1}\varepsilon^{(2-3\beta)m+\beta n+\beta} \\
&\quad -\varepsilon^\beta\lambda_{-1}^{1/2} + \sum_{\substack{m\geqslant 0,n\geqslant 1\\ (2-3\beta)m+\beta n\leqslant \beta N}} q_{m,n-1}(-1)^{n-3m+1}\varepsilon^{(2-3\beta)m+\beta n+\beta} \\
&\quad -6\alpha_2\lambda_{-1} \sum_{\substack{m\geqslant 0,n\geqslant 0\\ (2-3\beta)m+\beta n\leqslant \beta N}} q_{m,n}(-1)^{n-3m+1}\varepsilon^{(2-3\beta)m+\beta n+\beta} \\
&\quad -6\alpha_2\lambda_{-1}^{1/2} \sum_{\substack{m\geqslant 0,n\geqslant 1\\ (2-3\beta)m+\beta n\leqslant \beta N}} \Big(\sum_{\substack{m_1,m_2,n_1,n_2\geqslant 0\\ m_1+m_2=m\\ n_1+n_2=n-1}} q_{m_1,n_1}q_{m_2,n_2}\Big)(-1)^{n-3m+1}\varepsilon^{(2-3\beta)m+\beta n+\beta} \\
&\quad -2\alpha_2 \sum_{\substack{m\geqslant 0,n\geqslant 2\\ (2-3\beta)m+\beta n\leqslant \beta N}} \Big(\sum_{\substack{m_1,m_2,m_3,n_1,n_2,n_3\geqslant 0,\\ n_1+n_2+n_3=n-2\\ m_1+m_2+m_3=m}} q_{m_1,n_1}q_{m_2,n_2}q_{m_3,n_3}\Big)(-1)^{n-3m+1}\varepsilon^{(2-3\beta)m+\beta n+\beta}.
\end{aligned}$$

According to (3.67), the right hand side of (3.69) is equal to 0, up to the rest term $o(\varepsilon^{\beta(N+1)})$. Thus, the linear independance of the family of functions of ε, $\left(\varepsilon^{(2-3\beta)m+\beta n}\right)_{m,n\geqslant 0}$ yields:

- for $m = 0$, $n = 0$, thanks to (2.30), we get

$$2(R_2^2 - R_1^2)q_{0,0} = \lambda_{-1}^{1/2}, \tag{3.69}$$

- for $m = 0$ and $1 \leqslant n \leqslant N$,

$$(R_2^2 - R_1^2)q_{0,n} + q_{0,n-1} - 6\alpha_2\lambda_{-1}q_{0,n} - 6\alpha_2\lambda_{-1}^{1/2} \sum_{\substack{n_1,n_2\geqslant 0,\\ n_1+n_2=n-1}} q_{0,n_1}q_{0,n_2} - 2\alpha_2 \sum_{\substack{n_1,n_2,n_3\geqslant 0,\\ n_1+n_2+n_3=n-2}} q_{0,n_1}q_{0,n_2}q_{0,n_3} = 0,$$

which, using (2.30), can be rewritten as

$$-2(R_2^2 - R_1^2)q_{0,n} = -q_{0,n-1} + 6\alpha_2\lambda_{-1}^{1/2} \sum_{\substack{n_1,n_2\geqslant 0,\\ n_1+n_2=n-1}} q_{0,n_1}q_{0,n_2} + 2\alpha_2 \sum_{\substack{n_1,n_2,n_3\geqslant 0,\\ n_1+n_2+n_3=n-2}} q_{0,n_1}q_{0,n_2}q_{0,n_3}, \quad (3.70)$$

- for $1 \leqslant m \leqslant \beta N/(2-3\beta)$ and $n = 0$, we get

$$q_{m,0} = 0. \quad (3.71)$$

- for $m \geqslant 1$ and $n \geqslant 1$ such that $(2-3\beta)m + \beta n \leqslant \beta N$,

$$-2(dq'_{m-1,n-2} + 2q''_{m-1,n-2})\mathbf{1}_{\{n\geqslant 2\}} + 4R_1^2 q''_{m-1,n-1} + (R_2^2 - R_1^2)q_{m,n} + q_{m,n-1}$$
$$-6\alpha_2\lambda_{-1}q_{m,n} - 6\alpha_2\lambda_{-1}^{1/2} \sum_{\substack{m_1,m_2,n_1,n_2\geqslant 0\\ m_1+m_2=m,\\ n_1+n_2=n-1}} q_{m_1,n_1}q_{m_2,n_2} - 2\alpha_2 \sum_{\substack{m_1,m_2,m_3,n_1,n_2,n_3\geqslant 0,\\ n_1+n_2+n_3=n-2\\ m_1+m_2+m_3=m}} q_{m_1,n_1}q_{m_2,n_2}q_{m_3,n_3} = 0,$$

which can be rewritten as

$$-2(R_2^2 - R_1^2)q_{m,n} = 2(dq'_{m-1,n-2} + 2q''_{m-1,n-2})\mathbf{1}_{\{n\geqslant 2\}} - 4R_1^2 q''_{m-1,n-1} - q_{m,n-1}$$
$$+6\alpha_2\lambda_{-1}^{1/2} \sum_{\substack{0\leqslant m_1,m_2,n_1,n_2\\ n_1+n_2=n-1\\ m_1+m_2=m}} q_{m_1,n_1}q_{m_2,n_2} + 2\alpha_2 \sum_{\substack{m_1,m_2,m_3,n_1,n_2,n_3\geqslant 0\\ n_1+n_2+n_3=n-2\\ m_1+m_2+m_3=m}} q_{m_1,n_1}q_{m_2,n_2}q_{m_3,n_3}. \quad (3.72)$$

From (3.69), (3.70), (3.71) and (3.72), it clearly follows that all the $q_{m,n}$'s for indices (m,n) that satisfy $(2-3\beta)m + \beta n \leqslant \beta N$ are completely determined. ∎

Finally, we show that $(\varepsilon^{1/3}\nu(y_1), \varepsilon^{1/3}\lambda(y_1)^{1/2})$ and $(0, \varepsilon^{1/3}\mu(y_2))$ are close one from another on $D_1 \cap D_2$.

Lemma 3.14 *Let $N \geqslant 1$ be an integer, $L \geqslant \beta(N+1)/2$ and ν, λ, μ given by (3.1). Then for $l \in \{0,1,2\}$,*

$$\forall \alpha > 0, \quad \left\| \frac{d^l}{dz^l}\left(\varepsilon^{1/3}\nu(y_1)\right)\right\|_{L^\infty(D_1\cap D_2)} = o(\varepsilon^\alpha) \quad (3.73)$$

and

$$\left\| \frac{d^l}{dz^l}\left(\varepsilon^{1/3}\lambda(y_1)^{1/2} - \varepsilon^{1/3}\mu(y_2)\right)\right\|_{L^\infty(D_1\cap D_2)} = o(\varepsilon^{\beta(N+1-l)}). \quad (3.74)$$

Proof. (3.73) has already been proved in Lemma 3.11. $\theta = \varepsilon^{1/3}\lambda(y_1)^{1/2}$ satisfies assumption (3.68) in Lemma 3.13 thanks to Lemma (3.11) (with $q_{m,n} = 0$ if $1 + n - 3m < 0$). $\varepsilon^{1/3}\lambda^{1/2}$ also satisfies the assumption (3.67) thanks to Lemma 3.3 and (3.73). The two assumptions (3.68) and (3.67) of Lemma 3.13 are also satisfied by $\theta = \varepsilon^{1/3}\mu(y_2)$, thanks respectively to Lemma 3.12 and Corollary 3.6. Therefore, thanks to Lemma 3.13, (3.58) and (3.62), we deduce (3.74). ∎

4 Proof of Theorem 1.5

4.1 Derivation of the equations

We look for solutions of (1.13) under the form given by the ansatz (1.20), where $\beta \in (0, 2/3)\backslash\mathbb{Q}$, N is a large integer, $M \geqslant \max(1, \beta N/(2-3\beta))$ and $L \geqslant \max(1, \beta(N+1)/2$. For the sake of simplicity, we rewrite this ansatz as

$$\eta_1 = \varepsilon^{1/3}\left(\rho_1 + \varepsilon^{2(N+1)/3}P\right), \quad (4.1)$$

$$\eta_2 = \varepsilon^{1/3}\left(\rho_2 + \varepsilon^{2(N+1)/3}Q\right), \quad (4.2)$$

where

$$\rho_1 = \Phi_\varepsilon \varepsilon^{-1/3}\omega + \chi_\varepsilon \nu, \tag{4.3}$$
$$\rho_2 = \Phi_\varepsilon \varepsilon^{-1/3}\tau + \chi_\varepsilon \lambda^{1/2} + \Psi_\varepsilon \mu. \tag{4.4}$$

Implicitely, ρ_1, ρ_2, P and Q are functions of $x \in \mathbb{R}^d$, ω and τ are functions of $z = R_1^2 - |x|^2$, ν and λ are functions of $y_1 = z/\varepsilon^{2/3}$ and μ is a function of the variable $y_2 = (R_2^2 - |x|^2)/\varepsilon^{2/3}$. ∇ and Δ refer to derivatives with respect to $x \in \mathbb{R}^d$, whereas primes refer to derivatives with respect to variables z, y_1 or y_2, depending on the function which is concerned. For instance, we note $\nabla\omega$ for $\nabla\omega = -2x\omega'(R_1^2 - |x|^2) = -2x\omega'(z)$. Using this ansatz and these notations, the first equation in (1.13) becomes

$$\begin{aligned}\varepsilon^{4/3}\Delta\rho_1 + \varepsilon^{2N/3+2}\Delta P + \left(\frac{\alpha_0}{\alpha_2}\frac{R_2^2 - R_1^2}{\varepsilon^{2/3}} + y_1\right)(\rho_1 + \varepsilon^{2(N+1)/3}P) \\ -2\alpha_1(\rho_1 + \varepsilon^{2(N+1)/3}P)^3 - 2\alpha_0(\rho_2 + \varepsilon^{2(N+1)/3}Q)^2(\rho_1 + \varepsilon^{2(N+1)/3}P) = 0.\end{aligned}$$

Reorganizing the different terms, we get

$$\begin{aligned}\varepsilon^{4/3}\Delta\rho_1 + \frac{\alpha_0}{\alpha_2}\frac{R_2^2 - R_1^2}{\varepsilon^{2/3}}\rho_1 + y_1\rho_1 - 2\alpha_1\rho_1^3 - 2\alpha_0\rho_2^2\rho_1 \\ +\varepsilon^{2(N+1)/3}\left(\varepsilon^{4/3}\Delta P + \frac{\alpha_0}{\alpha_2}\frac{R_2^2 - R_1^2}{\varepsilon^{2/3}}P + y_1P - 6\alpha_1\rho_1^2P - 2\alpha_0\rho_2^2P - 4\alpha_0\rho_1\rho_2Q\right) \\ +\varepsilon^{4(N+1)/3}\left(-6\alpha_1\rho_1P^2 - 4\alpha_0\rho_2PQ - 2\alpha_0\rho_1Q^2\right) + \varepsilon^{2(N+1)}\left(-2\alpha_1P^3 - 2\alpha_0PQ^2\right) = 0.\end{aligned} \tag{4.5}$$

Similarly, the second equation in (1.13) writes

$$\begin{aligned}\varepsilon^{4/3}\Delta\rho_2 + y_2\rho_2 - 2\alpha_2\rho_2^3 - 2\alpha_0\rho_1^2\rho_2 \\ +\varepsilon^{2(N+1)/3}\left(\varepsilon^{4/3}\Delta Q + y_2Q - 6\alpha_2\rho_2^2Q - 2\alpha_0\rho_1^2Q - 4\alpha_0\rho_1\rho_2P\right) \\ +\varepsilon^{4(N+1)/3}\left(-6\alpha_2\rho_2Q^2 - 4\alpha_0\rho_1PQ - 2\alpha_0\rho_2P^2\right) + \varepsilon^{2(N+1)}\left(-2\alpha_2Q^3 - 2\alpha_0P^2Q\right) = 0.\end{aligned} \tag{4.6}$$

Equations (4.5) and (4.6) can be rewritten as the system

$$A_\varepsilon \begin{bmatrix} P \\ Q \end{bmatrix} = f_\varepsilon^0(x) + f_\varepsilon^2(x,P,Q) + f_\varepsilon^3(x,P,Q), \tag{4.7}$$

where

$$A_\varepsilon = \begin{bmatrix} -\varepsilon^{4/3}\Delta + p_\varepsilon(x) & r_\varepsilon(x) \\ r_\varepsilon(x) & -\varepsilon^{4/3}\Delta + q_\varepsilon(x) \end{bmatrix},$$
$$p_\varepsilon(x) = -\frac{\alpha_0}{\alpha_2}\frac{R_2^2 - R_1^2}{\varepsilon^{2/3}} - y_1 + 6\alpha_1\rho_1^2 + 2\alpha_0\rho_2^2,$$
$$q_\varepsilon(x) = -y_2 + 6\alpha_2\rho_2^2 + 2\alpha_0\rho_1^2,$$
$$r_\varepsilon(x) = 4\alpha_0\rho_1\rho_2,$$

$$f_\varepsilon^0(x) = \varepsilon^{-2(N+1)/3}\begin{bmatrix} \varepsilon^{4/3}\Delta\rho_1 + \frac{\alpha_0}{\alpha_2}\frac{R_2^2 - R_1^2}{\varepsilon^{2/3}}\rho_1 + y_1\rho_1 - 2\alpha_1\rho_1^3 - 2\alpha_0\rho_2^2\rho_1 \\ \varepsilon^{4/3}\Delta\rho_2 + y_2\rho_2 - 2\alpha_2\rho_2^3 - 2\alpha_0\rho_1^2\rho_2 \end{bmatrix},$$

$$f_\varepsilon^2(x,P,Q) = -2\varepsilon^{2(N+1)/3}\begin{bmatrix} 3\alpha_1\rho_1P^2 + 2\alpha_0\rho_2PQ + \alpha_0\rho_1Q^2 \\ 3\alpha_2\rho_2Q^2 + 2\alpha_0\rho_1PQ + \alpha_0\rho_2P^2 \end{bmatrix},$$

$$f_\varepsilon^3(x,P,Q) = -2\varepsilon^{4(N+1)/3}\begin{bmatrix} \alpha_1P^3 + \alpha_0PQ^2 \\ \alpha_2Q^3 + \alpha_0P^2Q \end{bmatrix}.$$

4.2 Estimate on the source term f_ε^0

Equation (4.7) will be solved thanks to a fixed point argument. For this purpose, we need to show that the source term f_ε^0 is small if functions ω, τ, ν, λ and μ are given by (3.1). The first component of f_ε^0 can be rewritten as

$$\begin{aligned}
\left[f_\varepsilon^0\right]_1 &= \varepsilon^{-2(N+1)/3}\left[\varepsilon^{4/3}\Delta\rho_1 + \frac{\alpha_0}{\alpha_2}\frac{R_2^2-R_1^2}{\varepsilon^{2/3}}\rho_1 + y_1\rho_1 - 2\alpha_1\rho_1^3 - 2\alpha_0\rho_2^2\rho_1\right] \\
&= \underbrace{\Phi_\varepsilon\,\varepsilon^{-2(N+1)/3}\varepsilon^{-1}\left[\varepsilon^2\Delta\omega + \frac{\alpha_0}{\alpha_2}(R_2^2-R_1^2)\omega + z\omega - 2\alpha_1\omega^3 - 2\alpha_0\tau^2\omega\right]}_{g_0} \\
&\quad \underbrace{+\chi_\varepsilon\,\varepsilon^{-2(N+1)/3}\left[\varepsilon^{4/3}\Delta\nu + \frac{\alpha_0}{\alpha_2}\frac{R_2^2-R_1^2}{\varepsilon^{2/3}}\nu + y_1\nu - 2\alpha_1\nu^3 - 2\alpha_0\lambda\nu\right]}_{g_1} \\
&\quad \underbrace{+2\varepsilon^{-2(N-1)/3}\left[\nabla\Phi_\varepsilon\nabla(\varepsilon^{-1/3}\omega) + \nabla\chi_\varepsilon\nabla\nu\right]}_{k_1} + \underbrace{\varepsilon^{-2(N-1)/3}\left[\Delta\Phi_\varepsilon(\varepsilon^{-1/3}\omega) + \Delta\chi_\varepsilon\nu\right]}_{k_2} \\
&\quad \underbrace{+2\alpha_1\varepsilon^{-2(N+1)/3}\left[\varepsilon^{-1}\Phi_\varepsilon\omega^3 + \chi_\varepsilon\nu^3 - \rho_1^3\right]}_{l_1} + \underbrace{2\alpha_0\varepsilon^{-2(N+1)/3}\left[\varepsilon^{-1}\Phi_\varepsilon\tau^2\omega + \chi_\varepsilon\lambda\nu - \rho_1\rho_2^2\right]}_{l_2} \qquad (4.8)
\end{aligned}$$

As for the second component of f_ε^0, we have

$$\begin{aligned}
\left[f_\varepsilon^0\right]_2 &= \varepsilon^{-2(N+1)/3}\left[\varepsilon^{4/3}\Delta\rho_2 + y_2\rho_2 - 2\alpha_2\rho_2^3 - 2\alpha_0\rho_1^2\rho_2\right] \\
&= \underbrace{\Phi_\varepsilon\,\varepsilon^{-2(N+1)/3}\varepsilon^{-1}\left[\varepsilon^2\Delta\tau + (R_2^2-R_1^2+z)\tau - 2\alpha_2\tau^3 - 2\alpha_0\omega^2\tau\right]}_{h_0} \qquad (4.9) \\
&\quad \underbrace{+\chi_\varepsilon\,\varepsilon^{-2(N+1)/3}\left[\varepsilon^{4/3}\Delta(\lambda^{1/2}) + y_2\lambda^{1/2} - 2\alpha_2\lambda^{3/2} - 2\alpha_0\nu^2\lambda^{1/2}\right]}_{h_1} + \Psi_\varepsilon\,\underbrace{\varepsilon^{-2(N+1)/3}\left[\varepsilon^{4/3}\Delta\mu + y_2\mu - 2\alpha_2\mu^3\right]}_{h_2} \\
&\quad \underbrace{+\varepsilon^{-2(N-1)/3}\left[\Delta\Phi_\varepsilon\varepsilon^{-1/3}\tau + \Delta\chi_\varepsilon\lambda^{1/2} + \Delta\Psi_\varepsilon\mu\right]}_{k_3} + \underbrace{2\varepsilon^{-2(N-1)/3}\left[\nabla\Phi_\varepsilon\varepsilon^{-1/3}\nabla\tau + \nabla\chi_\varepsilon\nabla\lambda^{1/2} + \nabla\Psi_\varepsilon\nabla\mu\right]}_{k_4} \\
&\quad \underbrace{+2\alpha_2\varepsilon^{-2(N+1)/3}\left[\varepsilon^{-1}\Phi_\varepsilon\tau^3 + \chi_\varepsilon\lambda^{3/2} + \Psi_\varepsilon\mu^3 - \rho_2^3\right]}_{l_3} + \underbrace{2\alpha_0\varepsilon^{-2(N+1)/3}\left[\varepsilon^{-1}\Phi_\varepsilon\omega^2\tau + \chi_\varepsilon\nu^2\lambda^{1/2} - \rho_1^2\rho_2\right]}_{l_4}.
\end{aligned}$$

Thanks to Lemma 3.2, for $x \in \mathrm{Supp}\Phi_\varepsilon \subset D_0$, we have

$$|g_0| \lesssim \varepsilon^{(2-3\beta)(M+1/2)-2N/3-2/3} \quad \text{and} \quad |h_0| \lesssim \varepsilon^{(2-3\beta)(M+1/2)-2N/3-2/3-\beta/2}. \qquad (4.10)$$

From Lemma 3.3, for $x \in \mathrm{Supp}\chi_\varepsilon \subset D_1$, we obtain

$$|g_1| \lesssim \varepsilon^{-(2-3\beta)(N/3-7/6)} \quad \text{and} \quad |h_1| \lesssim \varepsilon^{-(2-3\beta)N/3+1/3-\beta}. \qquad (4.11)$$

Lemma 3.4 yields, for $x \in \mathrm{Supp}\Psi_\varepsilon \subset D_2$,

$$|h_2| \lesssim \varepsilon^{2(L-N)/3}h(x). \qquad (4.12)$$

Next, let us estimate k_1. Note that $\nabla\Phi_\varepsilon$ is supported in $D_0 \cap D_1$, whereas $\nabla\chi_\varepsilon$ is supported in $(D_0 \cap D_1) \cup (D_1 \cap D_2)$. Moreover, for $x \in D_0 \cap D_1$, we have

$$\nabla\Phi_\varepsilon = -\nabla\chi_\varepsilon = -2x\varepsilon^{-\beta}\varphi'\left(\frac{z-\varepsilon^\beta}{2\varepsilon^\beta - \varepsilon^\beta}\right).$$

Thus,

$$|k_1| \lesssim \varepsilon^{-2N/3+1/3-\beta}\left\|\nabla(\omega - \varepsilon^{1/3}\nu)\right\|_{L^\infty(D_0\cap D_1)}\mathbf{1}_{D_0\cap D_1} + \varepsilon^{-2N/3+2/3-\beta}\left\|\nabla\nu\right\|_{L^\infty(D_1\cap D_2)}\mathbf{1}_{D_1\cap D_2}.$$

Then, thanks to Lemma 3.10 and Lemma 3.14,

$$k_1 = o_{L^\infty((D_0\cap D_1)\cup(D_1\cap D_2))}\left(\varepsilon^{-(2-3\beta)N/3-5\beta/2+1/3}\right). \tag{4.13}$$

Similarly,

$$\begin{aligned}
&\|k_4\|_{L^\infty((D_0\cap D_1)\cup(D_1\cap D_2))} &(4.14)\\
&\lesssim \varepsilon^{-2N/3+2/3}\left(\varepsilon^{-1/3-\beta}\left\|\frac{d}{dz}\left(\tau-\varepsilon^{1/3}\lambda^{1/2}\right)\right\|_{L^\infty(D_0\cap D_1)}+\varepsilon^{-1/3-\beta}\left\|\frac{d}{dz}\left(\varepsilon^{1/3}\lambda^{1/2}-\varepsilon^{1/3}\mu\right)\right\|_{L^\infty(D_1\cap D_2)}\right)\\
&= o\left(\varepsilon^{-(2-3\beta)N/3-2\beta+1/3}\right), &(4.15)
\end{aligned}$$

and we also get similar estimates for k_2 and k_3:

$$\begin{aligned}
k_2 &= o_{L^\infty((D_0\cap D_1)\cup(D_1\cap D_2))}(\varepsilon^{-(2-3\beta)N/3-5\beta/2+1/3}), &(4.16)\\
k_3 &= o_{L^\infty((D_0\cap D_1)\cup(D_1\cap D_2))}(\varepsilon^{-(2-3\beta)N/3-2\beta+1/3}). &(4.17)
\end{aligned}$$

Next, we estimate l_1. Clearly, l_1 is supported in $D_0\cap D_1$. Moreover, Lemma 3.10 implies

$$\varepsilon^{1/3}\nu = \omega + o_{L^\infty(D_0\cap D_1)}(\varepsilon^{\beta(N-1/2)}),$$

and since $\varepsilon^\beta \leqslant |z| \leqslant 2\varepsilon^\beta$ for $x\in D_0\cap D_1$, it follows from the definition of ω given by (3.1), (2.6) and (2.11) and from the asymptotics of the ω_m's as $z\to 0$ given in (2.6) and (2.11) that

$$\|\omega\|_{L^\infty(D_0\cap D_1)} = O(\varepsilon^{\beta/2}).$$

Thus, on $D_0\cap D_1$, we get

$$\begin{aligned}
l_1 &= 2\alpha_1\varepsilon^{-2N/3-5/3}\left[\Phi_\varepsilon\omega^3+(1-\Phi_\varepsilon)\left(\omega+o_{L^\infty(D_0\cap D_1)}(\varepsilon^{\beta(N-1/2)})\right)^3\right.\\
&\quad\left.-\left(\Phi_\varepsilon\omega+(1-\Phi_\varepsilon)(\omega+o_{L^\infty(D_0\cap D_1)}(\varepsilon^{\beta(N-1/2)})\right)^3\right]\\
&= 2\alpha_1\varepsilon^{-2N/3-5/3}o_{L^\infty(D_0\cap D_1)}(\varepsilon^{\beta(N+1/2)}) \quad = \quad o_{L^\infty(D_0\cap D_1)}\left(\varepsilon^{-(2-3\beta)N/3+\beta/2-5/3}\right). &(4.18)
\end{aligned}$$

As for l_2, it is supported in $(D_0\cap D_1)\cup(D_1\cap D_2)$. Taking into account Lemma 3.10 and Lemma 3.14, l_2 can be rewritten as

$$\begin{aligned}
l_2 &= \varepsilon^{-2(N+1)/3}\varepsilon^{-1}\left[\left\{\Phi_\varepsilon\tau^2\omega+(1-\Phi_\varepsilon)\left(\tau+o(\varepsilon^{\beta N})\right)^2\left(\omega+o(\varepsilon^{\beta(N-1/2)})\right)\right.\right.\\
&\quad\left.-\left(\Phi_\varepsilon\omega+(1-\Phi_\varepsilon)(\omega+o(\varepsilon^{\beta(N-1/2)}))\right)\left(\Phi_\varepsilon\tau+(1-\Phi_\varepsilon)(\tau+o(\varepsilon^{\beta N}))\right)^2\right\}\mathbf{1}_{D_0\cap D_1}\\
&\quad\left.+\varepsilon\left\{\chi_\varepsilon\lambda\nu-\chi_\varepsilon\nu\left(\chi_\varepsilon\lambda^{1/2}+(1-\chi_\varepsilon)\mu\right)^2\right\}\mathbf{1}_{D_1\cap D_2}\right]\\
&= o_{L^\infty(D_0\cap D_1)}(\varepsilon^{-(2-3\beta)N/3-5/3-\beta/2})\mathbf{1}_{D_0\cap D_1}+o_{L^\infty(D_1\cap D_2)}(\varepsilon^\alpha)\mathbf{1}_{D_1\cap D_2}, &(4.19)
\end{aligned}$$

where α is arbitrarily large. Similar calculations yield

$$l_3 = o_{L^\infty(D_0\cap D_1)}(\varepsilon^{-(2-3\beta)N/3-5/3})\mathbf{1}_{D_0\cap D_1}+o_{L^\infty(D_1\cap D_2)}(\varepsilon^{-(2-3\beta)N/3-5/3+\beta})\mathbf{1}_{D_1\cap D_2} \tag{4.20}$$

and

$$l_4 = o_{L^\infty(D_0\cap D_1)}(\varepsilon^{-(2-3\beta)N/3-5/3})\mathbf{1}_{D_0\cap D_1}+o_{L^\infty(D_1\cap D_2)}(\varepsilon^\alpha)\mathbf{1}_{D_1\cap D_2}. \tag{4.21}$$

Combining all these inequalities and noting that the measure of D_1 is of the order of ε^β, we deduce

$$\begin{aligned}
&\left\|\left[f_\varepsilon^0\right]_1\right\|_{L^2(\mathbb{R}^d)}\\
&\lesssim \varepsilon^{(2-3\beta)(M+1/2)-2N/3-2/3}+\varepsilon^{-(2-3\beta)N/3+7/3-3\beta}+\varepsilon^{-(2-3\beta)N/3+1/3-2\beta}+\varepsilon^{-(2-3\beta)N/3-5/3} &(4.22)
\end{aligned}$$

and

$$\begin{aligned}
\left\|\left[f_\varepsilon^0\right]_2\right\|_{L^2(\mathbb{R}^d)} &\lesssim \varepsilon^{(2-3\beta)(M+1/2)-2N/3-2/3-\beta/2}+\varepsilon^{-(2-3\beta)N/3+1/3-\beta/2}\\
&\quad+\varepsilon^{2(L-N)/3}+\varepsilon^{-(2-3\beta)N/3+1/3-3\beta/2}+\varepsilon^{-(2-3\beta)N/3-5/3+\beta/2}\\
&\lesssim \varepsilon^{(2-3\beta)(M+1/2)-2N/3-2/3-\beta/2}+\varepsilon^{2(L-N)/3}+\varepsilon^{-(2-3\beta)N/3-5/3+\beta/2}. &(4.23)
\end{aligned}$$

4.3 Estimate on the resolvent of A_ε

In order to solve equation (4.7) with the choice of ν, μ, λ given in (3.1), we have to invert A_ε. For this purpose, we prove that A_ε is a positive self-adjoint operator on $L^2(\mathbb{R}^d)$. It will be convenient to have an idea about the size of the functions p_ε, q_ε and r_ε appearing in the expression of A_ε, depending on x. As a preliminary, let us first simplify the expressions of ρ_1^2 and ρ_2^2, depending on whether $x \in D_0$, $x \in D_1 \backslash (D_0 \cup D_2)$ or $x \in D_2$. Thanks to Lemma 3.10, (2.6) and (2.11), we have, for $x \in D_0$,

$$\begin{aligned}
\rho_1^2 &= \frac{1}{\varepsilon^{2/3}} \left(\Phi_\varepsilon \omega + \chi_\varepsilon \varepsilon^{1/3} \nu \right)^2 \\
&= \frac{1}{\varepsilon^{2/3}} \left(\Phi_\varepsilon \omega + (1 - \Phi_\varepsilon)(\omega + o_{L^\infty(D_0 \cap D_1)}(\varepsilon^{\beta(N-1/2)})) \right)^2 \\
&= \frac{1}{\varepsilon^{2/3}} \left(\omega + o_{L^\infty(D_0 \cap D_1)}(\varepsilon^{\beta(N-1/2)}) \mathbf{1}_{D_0 \cap D_1} \right)^2 \\
&= \frac{\omega^2}{\varepsilon^{2/3}} + o_{L^\infty(D_0 \cap D_1)}(\varepsilon^{\beta N - 2/3}) \mathbf{1}_{D_0 \cap D_1} \\
&= \frac{\omega_0^2}{\varepsilon^{2/3}} + \frac{1}{\varepsilon^{2/3}} \sum_{m=1}^{2M} \varepsilon^{2m} \sum_{\substack{m_1+m_2=m \\ 0 \leqslant m_1, m_2 \leqslant M}} \omega_{m_1} \omega_{m_2} + o_{L^\infty(D_0 \cap D_1)}(\varepsilon^{\beta N - 2/3}) \mathbf{1}_{D_0 \cap D_1} \\
&= \frac{\Gamma_2 z}{2\alpha_1 \Gamma_{12} \varepsilon^{2/3}} + O_{L^\infty(D_0)}(\varepsilon^{4/3 - 2\beta}) + o_{L^\infty(D_0 \cap D_1)}(\varepsilon^{\beta N - 2/3}) \mathbf{1}_{D_0 \cap D_1}, \qquad (4.24)
\end{aligned}$$

where for the last equality, we have used (2.11) to infer that for $m \geqslant 1$, $m_1 + m_2 = m$ and $x \in D_0$, $\omega_{m_1} \omega_{m_2} \lesssim |z|^{1-3m} \leqslant \varepsilon^{\beta - 3\beta m}$, and that $2m + \beta - 3\beta m \geqslant 2 - 2\beta$. The same kind of calculation yields, still for $x \in D_0$,

$$\begin{aligned}
\rho_2^2 &= \frac{1}{\varepsilon^{2/3}} \left(\Phi_\varepsilon \tau + (1 - \Phi_\varepsilon)(\tau + o_{L^\infty(D_0 \cap D_1)}(\varepsilon^{\beta N})) \right)^2 \\
&= \frac{\tau_0^2}{\varepsilon^{2/3}} + \frac{1}{\varepsilon^{2/3}} \sum_{m=1}^{2M} \varepsilon^{2m} \sum_{\substack{m_1+m_2=m \\ 0 \leqslant m_1, m_2 \leqslant M}} \tau_{m_1} \tau_{m_2} + o_{L^\infty(D_0 \cap D_1)}(\varepsilon^{\beta N - 2/3}) \mathbf{1}_{D_0 \cap D_1} \\
&= \frac{R_2^2 - R_1^2}{2\alpha_2 \varepsilon^{2/3}} + \frac{\Gamma_1 z}{2\alpha_2 \Gamma_{12} \varepsilon^{2/3}} + O_{L^\infty(D_0)}(\varepsilon^{4/3 - 2\beta}) + o_{L^\infty(D_0 \cap D_1)}(\varepsilon^{\beta N - 2/3}) \mathbf{1}_{D_0 \cap D_1}. \qquad (4.25)
\end{aligned}$$

Next, we deduce from (2.39), (2.49), and (2.51) that for $x \in D_1 \backslash (D_0 \cup D_2)$,

$$\begin{aligned}
\rho_1^2 &= \nu^2 = \nu_0^2 + (\nu^2 - \nu_0^2) = \nu_0^2 + \sum_{n=1}^{2N} \varepsilon^{2n/3} \sum_{\substack{n_1+n_2=n \\ 0 \leqslant n_1, n_2 \leqslant N}} \nu_{n_1} \nu_{n_2} \\
&= \frac{R_1^{2/3} \Gamma_2^{2/3}}{2\alpha_1 \Gamma_{12}} \gamma_0 \left(\widetilde{y}_1 \right)^2 + O_{L^\infty(D_1 \backslash (D_0 \cup D_2))}(\varepsilon^{2/3}), \qquad (4.26)
\end{aligned}$$

and using (2.30), (2.34), (2.50) and (2.52), we get (again for $x \in D_1 \backslash (D_0 \cup D_2)$)

$$\rho_2^2 = \lambda = \frac{R_2^2 - R_1^2}{2\alpha_2 \varepsilon^{2/3}} + \frac{y_1}{2\alpha_2} - \frac{\alpha_0 R_1^{2/3} \Gamma_2^{2/3}}{2\alpha_1 \alpha_2 \Gamma_{12}} \gamma_0 \left(\widetilde{y}_1 \right)^2 + O_{L^\infty(D_1 \backslash (D_0 \cup D_2))}(\varepsilon^{2/3}), \qquad (4.27)$$

where

$$\widetilde{y}_1 = \frac{\Gamma_2^{1/3} y_1}{R_1^{2/3}}.$$

For $x \in D_2$, we use Lemma 3.14 to obtain, for $\alpha > 0$ arbitrarily large,

$$\rho_1^2 = \chi_\varepsilon^2 \nu^2 = o_{L^\infty(D_1 \cap D_2)}(\varepsilon^\alpha) \mathbf{1}_{L^\infty(D_1 \cap D_2)}, \qquad (4.28)$$

as well as, using also (2.19), (2.25) and Proposition 2.6,

$$\begin{aligned}
\rho_2^2 &= \frac{1}{\varepsilon^{2/3}}\left(\Psi_\varepsilon \varepsilon^{1/3}\mu + (1-\Psi_\varepsilon)(\varepsilon^{1/3}\mu + o_{L^\infty(D_1\cap D_2)}(\varepsilon^{\beta(N+1)}))\right)^2 \\
&= \frac{1}{\varepsilon^{2/3}}\left(\varepsilon^{1/3}\mu + o_{L^\infty(D_1\cap D_2)}(\varepsilon^{\beta(N+1)})\mathbf{1}_{L^\infty(D_1\cap D_2)}\right)^2 \\
&= \mu_0^2 + \sum_{n=1}^{2L}\varepsilon^{2n/3}\sum_{\substack{n_1+n_2=n\\ 0\leqslant n_1,n_2\leqslant L}}\mu_{n_1}\mu_{n_2} + o_{L^\infty(D_1\cap D_2)}(\varepsilon^{\beta(N+1)-2/3})\mathbf{1}_{L^\infty(D_1\cap D_2)} \\
&= \frac{R_2^{2/3}}{2\alpha_2}\gamma_0(\widetilde{y}_2)^2 + O_{L^\infty(D_2)}(\varepsilon^{2/3}) + o_{L^\infty(D_1\cap D_2)}(\varepsilon^{\beta(N+1)-2/3})\mathbf{1}_{L^\infty(D_1\cap D_2)}, \qquad (4.29)
\end{aligned}$$

where

$$\widetilde{y}_2 = \frac{y_2}{R_2^{2/3}}.$$

From (4.24), (4.25), (4.26), (4.27), (4.28), (4.29) and the definitions of p_ε and q_ε, we deduce the following expressions of p_ε and q_ε, depending on whether $x \in D_0$, $x \in D_1\backslash(D_0 \cup D_2)$ or $x \in D_2$. For each of this cases, we also calculate r_ε^2 and $-\Delta_\varepsilon = p_\varepsilon q_\varepsilon - r_\varepsilon^2$, a quantity which will play a key role in the sequel. A large integer N been fixed, We assume that $\beta \in (0, 2/3)$ satisfies $\beta N - 2/3 > 4/3 - 2\beta$ and $\beta(N+1) - 2/3 > 2/3$ (which are equivalent to $\beta > 2/(N+2)$ if N is large). For $x \in D_0$, we obtain

$$p_\varepsilon(x) = \frac{2\Gamma_2 y_1}{\Gamma_{12}} + O_{L^\infty(D_0)}(\varepsilon^{4/3-2\beta}), \qquad (4.30)$$

$$q_\varepsilon(x) = \tfrac{2(R_2^2-R_1^2)}{\varepsilon^{2/3}} + \tfrac{2\Gamma_1 y_1}{\Gamma_{12}} + O_{L^\infty(D_0)}(\varepsilon^{4/3-2\beta}), \qquad (4.31)$$

$$r_\varepsilon(x)^2 = \frac{4\alpha_0^2\Gamma_2 y_1}{\alpha_1\alpha_2\Gamma_{12}\varepsilon^{2/3}}\left(R_2^2 - R_1^2 + \frac{\Gamma_1}{\Gamma_{12}}z\right) + O_{L^\infty(D_0)}(\varepsilon^{2/3-2\beta}), \qquad (4.32)$$

$$-\Delta_\varepsilon(x) = \frac{4\Gamma_2 y_1}{\varepsilon^{2/3}}\left(R_2^2 - R_1^2 + \frac{\Gamma_1}{\Gamma_{12}}z\right) + O_{L^\infty(D_0)}(\varepsilon^{2/3-2\beta}). \qquad (4.33)$$

For $x \in D_1\backslash(D_0 \cup D_2)$,

$$p_\varepsilon(x) = R_1^{2/3}\Gamma_2^{2/3}\widetilde{W}_0(\widetilde{y}_1) + O_{L^\infty(D_1\backslash(D_0\cup D_2))}(\varepsilon^{2/3}), \qquad (4.34)$$

where

$$\widetilde{W}_0(y) = \left(1 + \frac{2}{\Gamma_{12}}\right)\gamma_0(y)^2 - y,$$

$$q_\varepsilon(x) = 2y_2 - \tfrac{2\alpha_0 R_1^{2/3}\Gamma_2^{2/3}}{\alpha_1\Gamma_{12}}\gamma_0(\widetilde{y}_1)^2 + O_{L^\infty(D_1\backslash(D_0\cup D_2))}(\varepsilon^{2/3}), \qquad (4.35)$$

$$r_\varepsilon(x)^2 = \frac{4\alpha_0^2 R_1^{2/3}\Gamma_2^{2/3}}{\alpha_1\alpha_2\Gamma_{12}}\gamma_0(\widetilde{y}_1)^2\left(y_2 - \frac{\alpha_0 R_1^{2/3}\Gamma_2^{2/3}}{\alpha_1\Gamma_{12}}\gamma_0(\widetilde{y}_1)^2\right) + O_{L^\infty(D_1\backslash(D_0\cup D_2))}(1), \qquad (4.36)$$

$$-\Delta_\varepsilon = 2R_1^{2/3}\Gamma_2^{2/3}\left(y_2 - \frac{\alpha_0 R_1^{2/3}\Gamma_2^{2/3}}{\alpha_1\Gamma_{12}}\gamma_0(\widetilde{y}_1)^2\right)W_0(\widetilde{y}_1) + O_{L^\infty(D_1\backslash(D_0\cup D_2))}(1). \qquad (4.37)$$

Finally, for $x \in D_2$,

$$p_\varepsilon(x) = -\Gamma_2 y_1 + \frac{\alpha_0}{\alpha_2}R_2^{2/3}V_0(\widetilde{y}_2) + O_{L^\infty(D_2)}(\varepsilon^{2/3}), \qquad (4.38)$$

where

$$V_0(y) = \gamma_0(y)^2 - y,$$

$$q_\varepsilon(x) \quad = R_2^{2/3} W_0(\widetilde{y}_2) + O_{L^\infty(D_2)}(\varepsilon^{2/3}), \tag{4.39}$$

$$r_\varepsilon(x)^2 = o_{L^\infty(D_2)}(\varepsilon^\alpha), \tag{4.40}$$

for α arbitrarily large, and

$$-\Delta_\varepsilon \quad = \quad \left(-\Gamma_2 y_1 + \frac{\alpha_0}{\alpha_2} R_2^{2/3} V_0(\widetilde{y}_2)\right) R_2^{2/3} W_0(\widetilde{y}_2) + O_{L^\infty(D_2)}(1). \tag{4.41}$$

Then, (4.30), (4.34) and (4.38) will provide us upper and lower bounds on p_ε. For this purpose, since the function $\widetilde{W}_0$ appears in (4.34), we first prove a lemma which gives informations about the size of this function.

Lemma 4.1 *For $y \in \mathbb{R}$,*

$$W_0(y) \leqslant \widetilde{W}_0(y) \lesssim W_0(y), \tag{4.42}$$

and

$$\max(1, |y|) \lesssim W_0(y) \lesssim \max(1, |y|). \tag{4.43}$$

Proof. We write

$$\widetilde{W}_0(y) = W_0(y) + 2\left(\frac{1}{\Gamma_{12}} - 1\right)\gamma(y)^2,$$

where $1/\Gamma_{12} - 1 > 0$, which directly provides the lower bound on $\widetilde{W}_0$. Moreover, the analysis of the continuous functions γ_0 and W_0 which was done in [GP] ensures that $W_0(y) > 0$ for every $y \in \mathbb{R}$, $W_0(y) \underset{y\to+\infty}{\sim} 2y$, $W_0(y) \underset{y\to-\infty}{\sim} -y$, $\gamma_0(y)^2 \underset{y\to+\infty}{\sim} y$ and $\gamma(y)^2 \underset{y\to-\infty}{\longrightarrow} 0$. We deduce (4.43) and the existence of $C_0 > 0$ such that $\gamma_0^2/W_0 \leqslant C_0$. Then, we obtain the upper bound

$$\widetilde{W}_0 \leqslant \left(1 + 2\left(\frac{1}{\Gamma_{12}} - 1\right) C_0\right) W_0.$$

∎

Then, we get lower and upper bounds on p_ε as stated in the lemma below.

Lemma 4.2 *For $x \in \mathbb{R}^d$ and $\varepsilon > 0$ sufficiently small,*

$$\max(1, |y_1|) \lesssim p_\varepsilon(x) \lesssim \max(1, |y_1|).$$

Proof. On D_0, the two estimates directly follow from (4.30), since for $x \in D_0$, $y_1 \geqslant \varepsilon^{\beta-2/3} \to +\infty$ as $\varepsilon \to 0$, whereas $\varepsilon^{4/3-2\beta} \to 0$. On $D_1 \backslash (D_0 \cup D_2)$, they are consequences from (4.34) and Lemma 4.1. On D_2, we deduce them from (4.38). Indeed, we know from the asymptotic expansion of γ_0 (2.26) that $V_0(y) \underset{y\to+\infty}{=} O(y^{-2})$, and $V_0(y) \underset{y\to-\infty}{\sim} -y \to +\infty$, therefore V_0 is bounded from below. Then, we have on the one side, for ε sufficiently small,

$$p_\varepsilon(x) \quad \geqslant \quad \frac{\alpha_0}{\alpha_2} R_2^{2/3} \inf_{y\in\mathbb{R}} V_0(y) + \Gamma_2 |y_1| - 1 \gtrsim \max(1, |y_1|) \gtrsim \varepsilon^{\beta-2/3} \underset{\varepsilon\to 0}{\longrightarrow} +\infty. \tag{4.44}$$

On the other side, the properties of γ_0 stated in Proposition 2.4 imply

$$\forall y \in \mathbb{R}, \quad V_0(y) \lesssim \max(1, -y).$$

Thus, using also Lemma 4.1 and the inequalities $y_1 < 0$ and $y_1 < y_2$, we get

$$p_\varepsilon(x) \lesssim \max(1, -y_2) + \max(1, -y_1) \lesssim \max(1, -y_1) = \max(1, |y_1|). \tag{4.45}$$

∎

As for q_ε, we infer similarly the next lemma from (4.31), (4.35) and (4.39).

Lemma 4.3 *For $x \in \mathbb{R}^d$ and $\varepsilon > 0$ sufficiently small,*

$$\max(1, |y_2|) \lesssim q_\varepsilon(x) \lesssim \max(1, |y_2|).$$

Proof. In order to prove the two inequalities for $x \in D_0$, we rewrite (4.31) as

$$q_\varepsilon(x) \quad = \tfrac{2}{\varepsilon^{2/3}} \left(R_2^2 - R_1^2 + \tfrac{\Gamma_1}{\Gamma_{12}} z \right) + O_{L^\infty(D_0)}(\varepsilon^{4/3-2\beta}). \tag{4.46}$$

As x describes D_0, z describes the interval $[\varepsilon^\beta, R_1^2] \subset [0, R_1^2]$. On this interval, $G(z) = R_2^2 - R_1^2 + \frac{\Gamma_1}{\Gamma_{12}} z$ reaches its extrema at $z = 0$ and at $z = R_1^2$. Since $G(0) = R_2^2 - R_1^2 > 0$ and $G(R_1^2) = R_2^2 - \frac{\alpha_0 \Gamma_2}{\alpha_1 \Gamma_{12}} R_1^2 > 0$ (thanks to (1.10)), and because $-2/3 < 0 < 4/3 - 2\beta$, there exists a constant $c > 1$ such that for every $x \in D_0$,

$$\frac{1}{c} \varepsilon^{-2/3} \leqslant q_\varepsilon(x) \leqslant c\varepsilon^{-2/3}.$$

The inequality follows for $x \in D_0$, since $(R_2^2 - R_1^2)\varepsilon^{-2/3} \leqslant y_2 \leqslant R_2^2 \varepsilon^{-2/3}$ on D_0. On $D_1 \backslash (D_1 \cup D_2)$, the inequalities clearly follow from (4.35), since on that set, $y_2 \gtrsim \varepsilon^{-2/3}$. Finally, on D_2, they are consequences of (4.39) and Lemma 4.1. ∎

We are now ready to prove positivity of the operator A_ε

Theorem 4.4 *A_ε defines a positive self-adjoint operator on $L^2(\mathbb{R}^d)^2$, with form domain $H^1_w(\mathbb{R}^d)^2$, where*

$$H^1_w(\mathbb{R}^d) = \left\{ P \in H^1(\mathbb{R}^d) \middle| \max(1, \min(|y_1|, |y_2|))^{1/2} P \in L^2(\mathbb{R}^d) \right\}.$$

Moreover, there exists $C > 0$ such that for every (P, Q) in the domain of A_ε,

$$\left\langle A_\varepsilon \begin{bmatrix} P \\ Q \end{bmatrix}, \begin{bmatrix} P \\ Q \end{bmatrix} \right\rangle \geqslant \varepsilon^{4/3} \int_{\mathbb{R}^d} \left(|\nabla P|^2 + |\nabla Q|^2 \right) dx + C \int_{\mathbb{R}^d} \max(1, \min(|y_1|, |y_2|))(|P|^2 + |Q|^2) dx.$$

Proof. For $P, Q \in \mathcal{C}_c^\infty(\mathbb{R}^d)$, we have

$$\left\langle A_\varepsilon \begin{bmatrix} P \\ Q \end{bmatrix}, \begin{bmatrix} P \\ Q \end{bmatrix} \right\rangle = \int_{\mathbb{R}^d} \left(\varepsilon^{4/3} |\nabla P|^2 + \varepsilon^{4/3} |\nabla Q|^2 + p_\varepsilon P^2 + q_\varepsilon Q^2 + 2 r_\varepsilon PQ \right) dx. \tag{4.47}$$

Taking into account the positivity of p_ε and q_ε shown in Lemmata 4.2 and 4.3,

$$p_\varepsilon P^2 + q_\varepsilon Q^2 + 2 r_\varepsilon PQ = \frac{1}{p_\varepsilon} (p_\varepsilon P + r_\varepsilon Q)^2 + \frac{p_\varepsilon q_\varepsilon - r_\varepsilon^2}{p_\varepsilon} Q^2 \geqslant \frac{-\Delta_\varepsilon}{p_\varepsilon} Q^2,$$

where

$$\Delta_\varepsilon = r_\varepsilon^2 - p_\varepsilon q_\varepsilon.$$

Symmetrically,

$$p_\varepsilon P^2 + q_\varepsilon Q^2 + 2 r_\varepsilon PQ \geqslant \frac{-\Delta_\varepsilon}{q_\varepsilon} P^2,$$

and thus

$$p_\varepsilon P^2 + q_\varepsilon Q^2 + 2 r_\varepsilon PQ \geqslant \frac{1}{2} \min \left(\frac{-\Delta_\varepsilon}{p_\varepsilon}, \frac{-\Delta_\varepsilon}{q_\varepsilon} \right) (P^2 + Q^2). \tag{4.48}$$

We shall see next that there exists $c > 0$ such that for every $x \in \mathbb{R}^d$,

$$-\Delta_\varepsilon(x) \geqslant c p_\varepsilon(x) q_\varepsilon(x), \tag{4.49}$$

which is equivalent to

$$-\Delta_\varepsilon(x) \geqslant c \max(1, |y_1|) \max(1, |y_2|) \tag{4.50}$$

thanks to Lemmata 4.2 and 4.3 (up to a change of the constant $c > 0$), and which implies

$$\min \left(\frac{-\Delta_\varepsilon}{p_\varepsilon}, \frac{-\Delta_\varepsilon}{q_\varepsilon} \right) \geqslant c \min \left(p_\varepsilon(x), q_\varepsilon(x) \right) \gtrsim \min \left(\max(1, |y_1|), \max(1, |y_2|) \right) = \max(1, \min(|y_1|, |y_2|)). \tag{4.51}$$

For $x \in D_0$, (4.49) comes from (4.33), because for such values of x, we have $y_1 \gtrsim \varepsilon^{\beta-2/3} \geqslant 1$ for ε sufficiently small (and therefore $y_1 = \max(1, |y_1|)$), we also have $R_2^2 \varepsilon^{-2/3} \geqslant y_2 \geqslant (R_2^2 - R_1^2)\varepsilon^{-2/3}$, and therefore $\varepsilon^{-2/3} \gtrsim \max(1, |y_2|)$, and finally, the remark we have done to bound q_ε from below in the proof of Lemma 4.3 implies that $R_2^2 - R_1^2 + \frac{\Gamma_1}{\Gamma_{12}} z$ is bounded from below by a positive constant as $x \in D_0$. For $x \in D_1 \backslash (D_0 \cup D_2)$ (4.49) follows from (4.37) and Lemma 4.1, since $y_2 = \max(1, |y_2|)$ on that domain, and $\gamma_0(\widetilde{y_1})^2) \lesssim \varepsilon^{\beta-2/3}$. For $x \in D_2$, note that $W_0(\widetilde{y_2}) \gtrsim \max(1, |y_2|)$ thanks to Lemma 4.1. Then, using (4.41) and the same arguments as to obtain (4.44), we complete the proof. ∎

We deduce classicaly from Theorem 4.4 the following corollary.

Corollary 4.5 *A_ε is invertible, and*

$$\|A_\varepsilon^{-1}\|_{\mathcal{L}(L^2(\mathbb{R}^d)^2, H_w^1(\mathbb{R}^d)^2)} \lesssim \varepsilon^{-4/3},$$

where $H_w^1(\mathbb{R}^d)^2$ is endowed by the norm

$$\|(P,Q)\|_{H_w^1(\mathbb{R}^d)^2} = \left(\int_{\mathbb{R}^d} \left(|\nabla P|^2 + |\nabla Q|^2 \right) dx + \int_{\mathbb{R}^d} \max(1, \min(|y_1|, |y_2|))(|P|^2 + |Q|^2) dx \right)^{1/2}.$$

Remark 4.6 *Note that the set $H_w^1(\mathbb{R}^d)^2$ does not depend on ε, even though it's norm does. However, our choice of the $H_w^1(\mathbb{R}^d)^2$-norm ensures that the norm of the embedding of $H_w^1(\mathbb{R}^d)^2$ into $H^1(\mathbb{R}^d)^2$ is uniformly bounded in ε.*

4.4 The fixed point argument

Let $1/3 > \delta > 0$, and N a large integer. We fix $\beta \in (0, 2/3)$ such that $(2-3\beta)N/3 < \delta$, and then L and M large enough such that $(2-3\beta)(M+1/2) - 2N/3 - 2/3 > -2$, $(2-3\beta)(M+1/2) - 2N/3 - 2/3 - \beta/2 > -2$ and $2(L-N)/3 > -2$, in such a way that (4.22) and (4.23) imply

$$\left\| f_\varepsilon^0 \right\|_{L^2(\mathbb{R}^d)^2} \quad \lesssim \quad \varepsilon^{-2}. \tag{4.52}$$

We are going to apply the Picard fix-point theorem to the map

$$\begin{array}{rccl} \Theta_\varepsilon & : & H_w^1(\mathbb{R}^d)^2 & \longrightarrow & H_w^1(\mathbb{R}^d)^2 \\ & & (P,Q) & \longrightarrow & A_\varepsilon^{-1} f_\varepsilon^0 + A_\varepsilon^{-1} f_\varepsilon^2(P,Q) + A_\varepsilon^{-1} f_\varepsilon^3(P,Q), \end{array}$$

in the ball $\mathcal{B}_R$ of $H_w^1(\mathbb{R}^d)^2$ centered at the origin, with radius $R = 2 \left\| A_\varepsilon^{-1} f_\varepsilon^0 \right\|_{H_w^1(\mathbb{R}^d)^2}$. Note that it follows from Corollary 4.5 and (4.52) that

$$R \lesssim \varepsilon^{-10/3}. \tag{4.53}$$

From (4.24), (4.25), (4.26), (4.27), (4.28), (4.29), it follows that for $x \in \mathbb{R}^d$,

$$|\rho_1| \lesssim \varepsilon^{-1/3} \quad \text{and} \quad |\rho_2| \lesssim \varepsilon^{-1/3}.$$

Thus, the Sobolev embedding $H_w^1(\mathbb{R}^d) \subset H^1(\mathbb{R}^d) \subset L^4(\mathbb{R}^d)$ ($d \leqslant 3$) implies that for every $(P,Q) \in H_w^1(\mathbb{R}^d)^2$, we have $f_\varepsilon^2 \in L^2(\mathbb{R}^d)^2$, and

$$\|f_\varepsilon^2(P,Q)\|_{L^2(\mathbb{R}^d)^2} \lesssim \varepsilon^{2N/3+1/3} \|(P,Q)\|_{H_w^1(\mathbb{R}^d)^2}^2. \tag{4.54}$$

Then, Corollary 4.5 yields

$$\left\| A_\varepsilon^{-1} f_\varepsilon^2(P,Q) \right\|_{H_w^1(\mathbb{R}^d)^2} \lesssim \varepsilon^{2N/3-1} \|(P,Q)\|_{H_w^1(\mathbb{R}^d)^2}^2. \tag{4.55}$$

Similarly, thanks to the Sobolev embedding $H_w^1(\mathbb{R}^d) \subset H^1(\mathbb{R}^d) \subset L^6(\mathbb{R}^d)$ ($d \leqslant 3$), we get, for $(P,Q) \in H_w^1(\mathbb{R}^d)^2$,

$$\left\| f_\varepsilon^3(P,Q) \right\|_{H_w^1(\mathbb{R}^d)^2} \lesssim \varepsilon^{4N/3+4/3} \|(P,Q)\|_{H_w^1(\mathbb{R}^d)^2}^3 \tag{4.56}$$

and

$$\left\| A_\varepsilon^{-1} f_\varepsilon^3(P,Q) \right\|_{H_w^1(\mathbb{R}^d)^2} \lesssim \varepsilon^{4N/3} \|(P,Q)\|_{H_w^1(\mathbb{R}^d)^2}^3. \tag{4.57}$$

From (4.55) and (4.57) we deduce that if $(P,Q) \in \mathcal{B}_R$, for some positive constants C_2 and C_3,

$$\|\Theta_\varepsilon(P,Q)\|_{H^1_w(\mathbb{R}^d)^2} \leqslant \frac{R}{2} + C_2\varepsilon^{2N/3-13/3}R + C_3\varepsilon^{4N/3-20/3}R.$$

Therefore, if $N \geqslant 7$ and ε is sufficiently small, $\mathcal{B}_R$ is stable by Θ_ε. Similar arguments prove that Θ_ε is a contraction on that ball. As a result, Θ_ε has a unique fixed point in $\mathcal{B}_R$.

4.5 Positivity of η_1 and η_2.

This section is devoted to the proof of the positivity of the solution (η_1,η_2) to the system (1.13) given by (4.1)-(4.2)-(4.3)-(4.4), which has just been constructed in the sections above. We proceed in three steps. First, we prove that for $j=1,2$, ρ_j (given by (4.3) or (4.4)) is bounded from below by a positive constant on the set $S_j = \left\{x \in \mathbb{R}^d, |x|^2 \leqslant R_j^2 + \varepsilon^{2/3}\right\}$, provided ε is sufficiently small. Second, we prove L^∞ estimates on P and Q, which ensure that η_1 and η_2 are positive on S_1 and S_2 respectively. Finally, we prove positivity of (η_1,η_2) on $\mathbb{R}^d$ thanks to the maximum principle.

1st step. For some integers $N, M, L \geqslant 1$, let ω, τ, ν, λ and μ be the functions given by (3.1). Then, we decompose the functions ρ_1 and ρ_2 given by (4.3)-(4.4) as

$$\rho_1 = \varepsilon^{-1/3}\omega \mathbf{1}_{D_0\setminus D_1} + \left(\Phi_\varepsilon\varepsilon^{-1/3}\omega + \chi_\varepsilon\nu\right)\mathbf{1}_{D_0\cap D_1} + \nu\mathbf{1}_{D_1\setminus D_0} \tag{4.58}$$

and

$$\begin{aligned}\rho_2 = {} & \varepsilon^{-1/3}\tau\mathbf{1}_{D_0\setminus D_1} + \left(\Phi_\varepsilon\varepsilon^{-1/3}\tau + \chi_\varepsilon\lambda^{1/2}\right)\mathbf{1}_{D_0\cap D_1} + \lambda^{1/2}\mathbf{1}_{D_1\setminus(D_0\cup D_2)} \\ & + \left(\chi_\varepsilon\lambda^{1/2} + \Psi_\varepsilon\mu\right)\mathbf{1}_{D_1\cap D_2} + \mu\mathbf{1}_{D_2\setminus D_1},\end{aligned} \tag{4.59}$$

and we are going to bound from below ω, τ, ν, λ and μ separately on the different sets appearing in the indicatrix functions above. According to Remark 2.3, ω and τ satisfy

$$\omega = \omega_0 + O_{L^\infty(D_0)}(\varepsilon^{2-5\beta/2}) \quad \text{and} \quad \tau = \tau_0 + O_{L^\infty(D_0)}(\varepsilon^{2-2\beta}). \tag{4.60}$$

Moreover, thanks to the explicit expressions of ω_0 and τ_0 (2.6) and (2.7), we deduce that for $x \in D_0$,

$$\omega_0 \geqslant \left(\frac{\Gamma_2}{2\alpha_1\Gamma_{12}}\right)^{1/2}\varepsilon^{\beta/2} \quad \text{and} \quad \tau_0 \geqslant \left(\frac{R_2^2-R_1^2}{2\alpha_2}\right)^{1/2} + O(\varepsilon^\beta). \tag{4.61}$$

Since $\beta < 2/3$, we have $2-5\beta/2 > \beta/2$ and $2-2\beta > \beta$, so we conclude that for $x \in D_0$,

$$\omega \geqslant \left(\frac{\Gamma_2}{2\alpha_1\Gamma_{12}}\right)^{1/2}\varepsilon^{\beta/2} + O(\varepsilon^{2-5\beta/2}) \quad \text{and} \quad \tau \geqslant \left(\frac{R_2^2-R_1^2}{2\alpha_2}\right)^{1/2} + O(\varepsilon^\beta). \tag{4.62}$$

We have already seen in the proof of Lemma 3.1 that for $x \in D_1$,

$$\lambda \geqslant \frac{R_2^2-R_1^2}{2\alpha_2\varepsilon^{2/3}} + O(\varepsilon^{\beta-2/3}). \tag{4.63}$$

Using similar arguments, we infer from Proposition 2.7 that

$$\nu = \nu_0 + O_{L^\infty(D_1)}(\varepsilon^{2/3}). \tag{4.64}$$

Then, (2.39), the fact that γ_0 is an increasing function and Proposition 2.4 imply

$$\nu \geqslant \frac{(R_1\Gamma_2)^{1/3}}{(2\alpha_1\Gamma_{12})^{1/2}}\gamma_0\left(\frac{\Gamma_2^{1/3}}{R_1^{2/3}}\varepsilon^{\beta-2/3}\right) + O(\varepsilon^{2/3}) = \frac{\Gamma_2^{1/2}}{(2\alpha_1\Gamma_{12})^{1/2}}\varepsilon^{\beta/2-1/3} + O(\varepsilon^{2/3}) \quad \text{for } x \in D_0\cap D_1, \tag{4.65}$$

whereas

$$\nu \geqslant \frac{(R_1\Gamma_2)^{1/3}}{(2\alpha_1\Gamma_{12})^{1/2}}\gamma_0\left(-\frac{\Gamma_2^{1/3}}{R_1^{2/3}}\right) + O(\varepsilon^{2/3}) \quad \text{for } x \in D_1\cap S_1. \tag{4.66}$$

From Proposition 2.6, we get in the same way

$$\mu = \mu_0 + O_{L^\infty(D_2)}(\varepsilon^{2/3}), \tag{4.67}$$

which implies thanks to (2.19) for $n = 0$

$$\mu \geqslant \frac{R_2^{1/3}}{(2\alpha_2)^{1/2}}\gamma_0\left(-\frac{1}{R_2^{2/3}}\right) + O(\varepsilon^{2/3}) \quad \text{for } x \in D_2 \cap S_2. \tag{4.68}$$

Combining (4.58), (4.62), (4.65) and (4.66), we deduce that

$$\begin{aligned}
\rho_1 &\geqslant \left(\varepsilon^{-1/3}\left(\frac{\Gamma_2}{2\alpha_1\Gamma_{12}}\right)^{1/2}\varepsilon^{\beta/2} + O(\varepsilon^{5/3-5\beta/2})\right)\mathbf{1}_{D_0\setminus D_1} \\
&\quad + \left(\Phi_\varepsilon\varepsilon^{-1/3}\left(\frac{\Gamma_2}{2\alpha_1\Gamma_{12}}\right)^{1/2}\varepsilon^{\beta/2} + \chi_\varepsilon\left(\frac{\Gamma_2}{2\alpha_1\Gamma_{12}}\right)^{1/2}\varepsilon^{\beta/2-1/3} + O(\varepsilon^{5/3-5\beta/2}) + O(\varepsilon^{2/3})\right)\mathbf{1}_{D_0\cap D_1} \\
&\quad + \left(\frac{(R_1\Gamma_2)^{1/3}}{(2\alpha_1\Gamma_{12})^{1/2}}\gamma_0\left(-\Gamma_2^{1/3}/R_1^{2/3}\right) + O(\varepsilon^{2/3})\right)\mathbf{1}_{S_1\setminus D_0}. \\
&\geqslant \left(\frac{\Gamma_2}{2\alpha_1\Gamma_{12}}\right)^{1/2}\varepsilon^{\beta/2-1/3}\mathbf{1}_{D_0} + \frac{(R_1\Gamma_2)^{1/3}}{(2\alpha_1\Gamma_{12})^{1/2}}\gamma_0\left(-\Gamma_2^{1/3}/R_1^{2/3}\right)\mathbf{1}_{S_1\setminus D_0} \\
&\quad + O(\varepsilon^{5/3-5\beta/2})\mathbf{1}_{D_0} + O(\varepsilon^{2/3})\mathbf{1}_{S_1\cap D_1}. \\
&\geqslant c_1\mathbf{1}_{S_1},
\end{aligned} \tag{4.69}$$

if $\varepsilon \leqslant 1$ is sufficiently small, where $c_1 = \frac{1}{2}\min\left[\left(\frac{\Gamma_2}{2\alpha_1\Gamma_{12}}\right)^{1/2}, \frac{(R_1\Gamma_2)^{1/3}}{(2\alpha_1\Gamma_{12})^{1/2}}\gamma_0\left(-\frac{\Gamma_2^{1/3}}{R_1^{2/3}}\right)\right]$. On the other side, using (4.59), (4.62), (4.63) and (4.68), we have

$$\begin{aligned}
\rho_2 &\geqslant \left(\left(\frac{R_2^2-R_1^2}{2\alpha_2}\right)^{1/2}\varepsilon^{-1/3} + O(\varepsilon^{\beta-1/3})\right)\mathbf{1}_{(D_0\cup D_1)\setminus D_2} \\
&\quad + \left(\min\left[\left(\frac{R_2^2-R_1^2}{2\alpha_2}\right)^{1/2}\varepsilon^{-1/3}, \frac{R_2^{1/3}}{(2\alpha_2)^{1/2}}\gamma_0\left(\frac{-1}{R_2^{2/3}}\right)\right] + O(\varepsilon^{\beta-1/3})\right)\mathbf{1}_{D_1\cap D_2} \\
&\quad + \left(\frac{R_2^{1/3}}{(2\alpha_2)^{1/2}}\gamma_0\left(\frac{-1}{R_2^{2/3}}\right) + O(\varepsilon^{2/3})\right)\mathbf{1}_{S_2\setminus D_1} \\
&\geqslant c_2\mathbf{1}_{S_2}
\end{aligned} \tag{4.70}$$

for $\varepsilon \leqslant 1$ sufficiently small, where $c_2 = \frac{1}{2}\min\left[\left(\frac{R_2^2-R_1^2}{2\alpha_2}\right)^{1/2}, \frac{R_2^{1/3}}{(2\alpha_2)^{1/2}}\gamma_0\left(\frac{-1}{R_2^{2/3}}\right)\right]$.

2^{nd} step. Let N be a large integer, $R > 0$ as in Section 4.4 and $(P,Q) \in (H^1_w)^2$ the unique fixed point of the map Θ_ε constructed in that section. In order to control the L^∞ norm of (P,Q), we will use the continuity of the embedding of $H^2(\mathbb{R}^d)$ into $L^\infty(\mathbb{R}^d)$ (remember that $d \leqslant 3$). Because of Remark 4.6 and (4.53), we know that the H^1 norms of P and Q are controled by

$$\|(P,Q)\|_{(H^1)^2} \lesssim \varepsilon^{-10/3},$$

such that in order to control the H^2 norms of P and Q, we only need to control the L^2 norms of ΔP and ΔQ. For this purpose, let us introduce a C^∞ function θ on $\mathbb{R}^d$, which is radial, positive, supported in $\{x \in \mathbb{R}^d, |x| \leqslant 2\}$ and such that $\theta(x) \equiv 1$ for $|x| \leqslant 1$. We also define, for integers $n \geqslant 1$,

$\theta_n(x) = \theta(x/n)$. After integrations by parts, the $(L^2)^2$ scalar product of (4.7) with $(\Delta P, \Delta Q)\theta_n$ yields

$$\begin{aligned}
&\varepsilon^{4/3}\int_{\mathbb{R}^d}(|\Delta P|^2+|\Delta Q|^2)\theta_n+\int_{\mathbb{R}^d}(p_\varepsilon|\nabla P|^2+q_\varepsilon|\nabla Q|^2+2r_\varepsilon\nabla P\cdot\nabla Q)\theta_n\\
&= -\int_{\mathbb{R}^d}(P\nabla p_\varepsilon\cdot\nabla P+Q\nabla q_\varepsilon\cdot\nabla Q)\theta_n-\int_{\mathbb{R}^d}(p_\varepsilon P\nabla P+q_\varepsilon Q\nabla Q)\cdot\nabla\theta_n\\
&\quad -\int_{\mathbb{R}^d}\nabla r_\varepsilon\cdot\nabla(PQ)\theta_n-\int_{\mathbb{R}^d}r_\varepsilon\nabla(PQ)\cdot\nabla\theta_n-\int_{\mathbb{R}^d}f_\varepsilon\cdot(\Delta P,\Delta Q)\theta_n\\
&= \frac12\int_{\mathbb{R}^d}(\Delta p_\varepsilon P^2+\Delta q_\varepsilon Q^2+2\Delta r_\varepsilon PQ)\theta_n+\frac12\int_{\mathbb{R}^d}(p_\varepsilon P^2+q_\varepsilon Q^2+2r_\varepsilon PQ)\Delta\theta_n\\
&\quad +\int_{\mathbb{R}^d}(\nabla p_\varepsilon P^2+\nabla q_\varepsilon Q^2+2\nabla r_\varepsilon PQ)\cdot\nabla\theta_n-\int_{\mathbb{R}^d}f_\varepsilon\cdot(\Delta P,\Delta Q)\theta_n. \qquad (4.71)
\end{aligned}$$

Thanks to Lemma 4.2, for $n\geqslant 1$ and $\varepsilon\leqslant 1$ sufficiently small,

$$\begin{aligned}
|p_\varepsilon\Delta\theta_n| &\lesssim \max(1,|y_1|)|\Delta\theta_n|\lesssim\frac{1}{\varepsilon^{2/3}}\max(1,|x|^2)\frac{1}{n^2}\left|\Delta\theta\left(\frac{x}{n}\right)\right|\\
&\lesssim \frac{1}{\varepsilon^{2/3}}\max\left(\frac{1}{n^2}\|\Delta\theta\|_{L^\infty},\||x|^2\Delta\theta\|_{L^\infty}\right)\lesssim\frac{1}{\varepsilon^{2/3}}. \qquad (4.72)
\end{aligned}$$

Similarly, Lemma 4.3 yields

$$|q_\varepsilon\Delta\theta_n| \lesssim \frac{1}{\varepsilon^{2/3}}, \qquad (4.73)$$

and since $\Delta_\varepsilon\leqslant 0$ thanks to (4.49), (4.72) and (4.73) also imply

$$|r_\varepsilon\Delta\theta_n| \lesssim \frac{1}{\varepsilon^{2/3}}. \qquad (4.74)$$

Next, we use the estimates

$$\max(|\nabla p_\varepsilon|,|\nabla q_\varepsilon|,|\nabla r_\varepsilon|)\lesssim\max(\varepsilon^{-4/3},|x|/\varepsilon^{2/3}),\quad \max(|\Delta p_\varepsilon|,|\Delta q_\varepsilon|,|\Delta r_\varepsilon|)\lesssim\varepsilon^{-2}, \qquad (4.75)$$

that will be proved later. Arguing like in (4.72), it follows from (4.75) that for $n\geqslant 1$,

$$|\nabla p_\varepsilon\cdot\nabla\theta_n|\lesssim\varepsilon^{-4/3},\quad |\nabla q_\varepsilon\cdot\nabla\theta_n|\lesssim\varepsilon^{-4/3},\quad |\nabla r_\varepsilon\cdot\nabla\theta_n|\lesssim\varepsilon^{-4/3}. \qquad (4.76)$$

Letting $n\to\infty$, and using the positivity of the quadratic form $a(P,Q)=p_\varepsilon P^2+q_\varepsilon Q^2+2r_\varepsilon PQ$, shown in (4.48) and (4.49), we deduce from (4.71), (4.72), (4.73), (4.74), (4.75), (4.76), the Young inequality and (4.53) that

$$\begin{aligned}
\varepsilon^{4/3}\int_{\mathbb{R}^d}(|\Delta P|^2+|\Delta Q|^2) &\lesssim \varepsilon^{-2}\left(\|P\|_{L^2}^2+\|Q\|_{L^2}^2\right)+\varepsilon^{-4/3}\|f_\varepsilon\|_{(L^2)^2}^2\\
&\lesssim \varepsilon^{-26/3}+\varepsilon^{-4/3}\|f_\varepsilon\|_{L^2(\mathbb{R}^d)^2}^2. \qquad (4.77)
\end{aligned}$$

Thanks to (4.52), (4.54), (4.56) and (4.53), the L^2 norm of f_ε can estimated as

$$\|f_\varepsilon\|_{L^2(\mathbb{R}^d)^2}\lesssim\varepsilon^{-2}+\varepsilon^{2N/3-19/3}+\varepsilon^{4N/3-26/3}\lesssim\varepsilon^{-2},$$

provided N is large enough. Thus, (4.77) yields

$$\|(\Delta P,\Delta Q)\|_{L^2(\mathbb{R}^d)^2}\lesssim\varepsilon^{-5},$$

which combined with (4.53), implies

$$\|(P,Q)\|_{L^\infty(\mathbb{R}^d)^2}\lesssim\|(P,Q)\|_{H^2(\mathbb{R}^d)^2}\lesssim\varepsilon^{-5}. \qquad (4.78)$$

In view of the ansatz (4.1)-(4.2) as well asthe estimates (4.69), (4.70) and (4.78), we conclude that if N is sufficiently large and if ε is small enough, η_1 and η_2 are strictly positive respectively on S_1 and

S_2. In order to complete the proof of this last statement, it remains to prove estimates (4.75). This is the issue we address now. First, we deduce from (2.6), (2.7) and Lemma 2.1 that

$$\|\omega\|_{L^\infty(D_0)} \lesssim 1, \quad \|\nabla\omega\|_{L^\infty(D_0)} \lesssim \varepsilon^{-\beta}, \quad \|\Delta\omega\|_{L^\infty(D_0)} \lesssim \varepsilon^{-2\beta}, \tag{4.79}$$

$$\|\tau\|_{L^\infty(D_0)} \lesssim 1, \quad \|\nabla\tau\|_{L^\infty(D_0)} \lesssim 1, \quad \|\Delta\tau\|_{L^\infty(D_0)} \lesssim 1, \tag{4.80}$$

where for the estimates on $\nabla\tau_0$ and $\Delta\tau_0$, we have used assumption (1.10). From (2.51), (2.49), (2.52) and (2.50), we infer

$$\|\nu\|_{L^\infty(D_1)} \lesssim \varepsilon^{\beta/2-1/3}, \quad \|\nabla\nu\|_{L^\infty(D_1)} \lesssim \varepsilon^{-2/3}, \quad \|\Delta\nu\|_{L^\infty(D_1)} \lesssim \varepsilon^{-4/3}, \tag{4.81}$$

$$\inf_{x\in D_1} \lambda^{1/2} \gtrsim \varepsilon^{-1/3}, \quad \|\lambda^{1/2}\|_{L^\infty(D_1)} \lesssim \varepsilon^{-1/3}, \quad \|\nabla(\lambda^{1/2})\|_{L^\infty(D_1)} \lesssim \varepsilon^{-1/3}, \quad \|\Delta(\lambda^{1/2})\|_{L^\infty(D_1)} \lesssim \varepsilon^{-1}, \tag{4.82}$$

Note that the first estimate in (4.82) has already been proved in Lemma 3.1. (2.19) and Propositions 2.4 and 2.6 imply

$$\|\mu\|_{L^\infty(D_2)} \lesssim \varepsilon^{-1/3}, \quad \|\nabla\mu\|_{L^\infty(D_2)} \lesssim \varepsilon^{-2/3}, \quad \|\Delta\mu\|_{L^\infty(D_2)} \lesssim \varepsilon^{-4/3}. \tag{4.83}$$

Moreover, it follows from their definitions that the truncation functions Φ_ε, χ_ε and Ψ_ε satisfy the estimates

$$\|\nabla\Phi_\varepsilon\|_{L^\infty},\ \|\nabla\chi_\varepsilon\|_{L^\infty}, \|\nabla\Psi_\varepsilon\|_{L^\infty} \lesssim \varepsilon^{-\beta}, \quad \|\Delta\Phi_\varepsilon\|_{L^\infty},\ \|\Delta\chi_\varepsilon\|_{L^\infty}, \|\Delta\Psi_\varepsilon\|_{L^\infty} \lesssim \varepsilon^{-2\beta} \tag{4.84}$$

Combining (4.79), (4.80), (4.81), (4.82), (4.83) and (4.84) and using Lemmata 3.10 and 3.14, we obtain

$$\|\rho_1\|_{L^\infty(\mathbb{R}^d)} \lesssim \varepsilon^{-1/3}, \quad \|\rho_2\|_{L^\infty(\mathbb{R}^d)} \lesssim \varepsilon^{-1/3}, \tag{4.85}$$

$$\|\nabla\rho_1\|_{L^\infty(\mathbb{R}^d)} = \|\Phi_\varepsilon\varepsilon^{-1/3}\nabla\omega + \chi_\varepsilon\nabla\nu + \nabla\Phi_\varepsilon(\varepsilon^{-1/3}\omega - \nu)\|_{L^\infty(\mathbb{R}^d)} \lesssim \varepsilon^{\min(-1/3-\beta,-2/3)}, \tag{4.86}$$

and

$$\begin{aligned}\|\nabla\rho_2\|_{L^\infty(\mathbb{R}^d)} &= \|\Phi_\varepsilon\varepsilon^{-1/3}\nabla\tau + \chi_\varepsilon\nabla(\lambda^{1/2}) + \Psi_\varepsilon\nabla\mu + \nabla\Phi_\varepsilon(\varepsilon^{-1/3}\tau - \lambda^{1/2}) + \nabla\Psi_\varepsilon(\mu - \lambda^{1/2})\|_{L^\infty(\mathbb{R}^d)} \\ &\lesssim \varepsilon^{-2/3}\end{aligned} \tag{4.87}$$

provided N is large enough, as well as

$$\begin{aligned}\|\Delta\rho_1\|_{L^\infty(\mathbb{R}^d)} &= \|\Phi_\varepsilon\varepsilon^{-1/3}\Delta\omega + \chi_\varepsilon\Delta\nu + 2\nabla\Phi_\varepsilon(\varepsilon^{-1/3}\nabla\omega - \nabla\nu) + \Delta\Phi_\varepsilon(\varepsilon^{-1/3}\omega - \nu)\|_{L^\infty(\mathbb{R}^d)} \\ &\lesssim \varepsilon^{\min(-1/3-2\beta,-4/3)},\end{aligned} \tag{4.88}$$

and

$$\begin{aligned}\|\Delta\rho_2\|_{L^\infty(\mathbb{R}^d)} &= \|\Phi_\varepsilon\varepsilon^{-1/3}\Delta\tau + \chi_\varepsilon\Delta(\lambda^{1/2}) + \Psi_\varepsilon\Delta\mu + 2\nabla\Phi_\varepsilon\nabla(\varepsilon^{-1/3}\tau - \lambda^{1/2}) + 2\nabla\Psi_\varepsilon\nabla(\mu - \lambda^{1/2}) \\ &\quad +\Delta\Phi_\varepsilon(\varepsilon^{-1/3}\tau - \lambda^{1/2}) + \Delta\Psi_\varepsilon(\mu - \lambda^{1/2})\|_{L^\infty(\mathbb{R}^d)} \lesssim \varepsilon^{-4/3},\end{aligned} \tag{4.89}$$

where we assume again that N is sufficiently large. (4.75) follows by differentiation of the definitions of p_ε, q_ε and r_ε given in Section 4.1.

3rd step. First, note that the functions η_1 and η_2 we have constructed are radial. Indeed, ρ_1 and ρ_2 are functions of the variables z, y_1 and y_2, which all depend on x only through $|x|$. On the other side, the equation (4.7) is radially symmetric, such that the uniqueness of its solution (P, Q) in the ball $\mathcal{B}_R$, which was proved in Section 4.4, ensures that both P and Q are radial. For convenience, we consider η_1 and η_2 as functions of $r = |x|$. At this point, according to the conclusion of the second step, we already know that for $j = 1, 2$, $\eta_j(r) > 0$ for $r \in [0, (R_j^2 + \varepsilon^{2/3})^{1/2}]$. So it remains to prove that $\eta_j(r) > 0$ for $r > (R_j^2 + \varepsilon^{2/3})^{1/2}$. We shall see that it is a consequence of Hopf's lemma (see for instance [E]). Indeed, the system of equations (1.13) satisfied by (η_1, η_2) can be rewritten as

$$\left(-\Delta + \frac{c_j}{\varepsilon^2}\right)(-\eta_j) = 0 \quad \text{for } j = 1, 2,$$

where

$$c_1 = 2\alpha_1\eta_1^2 + 2\alpha_0\eta_2^2 - \frac{\alpha_0}{\alpha_2}(R_2^2 - R_1^2) + (|x|^2 - R_1^2)$$

and

$$c_2 = 2\alpha_0\eta_1^2 + 2\alpha_2\eta_2^2 + |x|^2 - R_2^2.$$

Let us fix $j \in \{1,2\}$. We shall see in Lemmata 4.7 and 4.8 below that $c_j \geqslant 0$ for $|x| > (R_j^2 + \varepsilon^{2/3})^{1/2}$. Let us admit provisionnaly this fact. We know that $-\eta_j(r) < 0$ for $r < (R_j^2 + \varepsilon^{2/3})^{1/2}$. Assume by contradiction that there exists $r_0 \geqslant (R_j^2 + \varepsilon^{2/3})^{1/2}$ such that $-\eta_j(r_0) = 0$. Then, Hopf's Lemma applied on the ball of $\mathbb{R}^d$ centered at the origin and with radius r_0 ensures that $-\eta_j'(r_0) > 0$. In particular, $r \mapsto -\eta_j(r)$ is strictly increasing in a neighborhood of r_0, in such a way that we can define $r_1 \in (r_0, +\infty]$ by

$$r_1 = \sup\{r > r_0, -\eta_j \text{ is stricly increasing on } (r_0, r_1)\}.$$

If r_1 is finite, we can apply again Hopf's Lemma on the ball centered at 0, with radius r_1, and conclude that $-\eta_j$ is increasing on a neighborhood of r_1, which is a contradiction with the definition of r_1. Thus, $r_1 = +\infty$. Thus, $-\eta_j$ is strictly increasing on $[r_0, +\infty)$, with $-\eta_j(r_0) = 0$. This is a contradiction with the fact that $\eta_j(r) \to 0$ as $r \to +\infty$ (which is itself a consequence of the decay of the $\mu_n(y_2)$'s as $y_2 \to -\infty$ and of $(P,Q) \in H^2(\mathbb{R}^d)$). Therefore $-\eta_j(r) < 0$ for every $r > 0$.

Lemma 4.7 *For $\varepsilon > 0$ sufficiently small, $c_1(x) > 0$ for every $x \in \mathbb{R}^d \backslash S_1$.*

Proof. Note first that for $\varepsilon \leqslant 1$, since $\beta < 2/3$, $\mathbb{R}^d \backslash S_1$ is the disjoint union of the sets $D_1 \backslash S_1$ and $D_2 \backslash D_1$. We first consider the case where $x \in D_1 \backslash S_1$. Starting from (4.1) and (4.3), we have

$$\begin{aligned}
\eta_1 &= \varepsilon^{1/3}(\chi_\varepsilon \nu + \varepsilon^{2(N+1)/3} P) \\
&= \varepsilon^{1/3}(\nu + o_{L^\infty(D_1 \cap D_2)}(\varepsilon^\alpha) + O_{L^\infty(D_1 \backslash S_1)}(\varepsilon^{2N/3 - 13/3})) \\
&= \varepsilon^{1/3}(\nu + O_{L^\infty(D_1 \backslash S_1)}(\varepsilon^{2N/3 - 13/3})) \\
&= \varepsilon^{1/3}(\nu_0 + O_{L^\infty(D_1 \backslash S_1)}(\varepsilon^{2/3})),
\end{aligned} \tag{4.90}$$

where the first line holds because $x \notin \mathrm{Supp}(\Phi_\varepsilon)$, the second one because $\chi_\varepsilon \equiv 1$ on $D_1 \backslash D_2$ and thanks to Lemma 3.14 and (4.78), the third line holds provided α is chosen large enough, and the last line is true for N large enough, since $D_1 \backslash S_1 \subset \{x, y_1 \leqslant -1\}$ and thanks to the asymptotics of the ν_n's as $y_1 \to -\infty$ given in Proposition 2.7. The same kind of arguments yield, still for $x \in D_1 \backslash S_1$:

$$\begin{aligned}
\eta_2 &= \varepsilon^{1/3}(\chi_\varepsilon \lambda^{1/2} + \Psi_\varepsilon \mu + \varepsilon^{2(N+1)/3} Q) \\
&= \varepsilon^{1/3}(\lambda^{1/2} + o_{L^\infty(D_1 \cap D_2)}(\varepsilon^{\beta(N+1) - 1/3}) + O_{L^\infty(D_1 \backslash S_1)}(\varepsilon^{2N/3 - 13/3}) \\
&= \left(\lambda_{-1} + \varepsilon^{2/3}\lambda_0 + O_{L^\infty(D_1 \backslash S_1)}(\varepsilon^{4/3})\right)^{1/2} + O_{L^\infty(D_1 \backslash S_1)}(\varepsilon^{4/3}) = \left(\lambda_{-1} + \varepsilon^{2/3}\lambda_0 + O_{L^\infty(D_1 \backslash S_1)}(\varepsilon^{4/3})\right)^{1/2}.
\end{aligned} \tag{4.91}$$

As a result, thanks to (2.30), (2.34) and (2.39), for $x \in D_1 \backslash S_1$, we have

$$\begin{aligned}
c_1 &= 2\alpha_1 \varepsilon^{2/3}(\nu_0 + O_{L^\infty}(\varepsilon^{2/3}))^2 + 2\alpha_0 \left(\lambda_{-1} + \varepsilon^{2/3}\lambda_0 + O_{L^\infty}(\varepsilon^{4/3})\right) - \frac{\alpha_0}{\alpha_2}(R_2^2 - R_1^2) - \varepsilon^{2/3} y_1 \\
&= R_1^{2/3}\Gamma_2^{2/3}\varepsilon^{2/3}\gamma_0 \left(\frac{\Gamma_2^{1/3} y_1}{R_1^{2/3}}\right)^2 - \Gamma_2 \varepsilon^{2/3} y_1 + O_{L^\infty}(\varepsilon^{4/3}) \geqslant \Gamma_2 \varepsilon^{2/3} + O(\varepsilon^{4/3}) > 0,
\end{aligned} \tag{4.92}$$

for ε sufficiently small, where the inequality holds because $x \notin S_1$, which implies $y_1 \leqslant -1$. Let us now consider the case where $x \in D_2 \backslash D_1$. Then, using again (4.78),

$$\eta_1 = \varepsilon^{2N/3+1} P = O_{L^\infty(D_2)}(\varepsilon^{2N/3-4}) = O_{L^\infty(D_2)}(\varepsilon^{4/3})$$

and

$$\eta_2 = \varepsilon^{1/3}(\mu + \varepsilon^{2(N+1)/3} Q) = \varepsilon^{1/3}(\mu_0 + O_{L^\infty(D_2)}(\varepsilon^{2/3})). \tag{4.93}$$

We infer that

$$\begin{aligned} c_1 &= 2\alpha_0\varepsilon^{2/3}\mu_0^2 - \frac{\alpha_0}{\alpha_2}(R_2^2-R_1^2) + (|x|^2-R_1^2) + O_{L^\infty(D_2)}(\varepsilon^{4/3}) \\ &= \varepsilon^{2/3}\left(\frac{\alpha_0}{\alpha_2}R_2^{2/3}\gamma_0\left(\frac{y_2}{R_2^{2/3}}\right)^2 - \frac{\alpha_0}{\alpha_2}\frac{R_2^2-R_1^2}{\varepsilon^{2/3}} - y_2 + \frac{R_2^2-R_1^2}{\varepsilon^{2/3}}\right) + O_{L^\infty(D_2)}(\varepsilon^{4/3}) \\ &= \varepsilon^{2/3}\left(\frac{\alpha_0}{\alpha_2}R_2^{2/3}\underbrace{\left(\gamma_0\left(\frac{y_2}{R_2^{2/3}}\right)^2 - \frac{y_2}{R_2^{2/3}}\right)}_{\geqslant \inf_{y\in\mathbb{R}}[\gamma_0(y)^2-y]>-\infty} + \Gamma_2\underbrace{\left(\frac{R_2^2-R_1^2}{\varepsilon^{2/3}} - y_2\right)}_{\geqslant 2\varepsilon^{\beta-2/3}\underset{\varepsilon\to 0}{\longrightarrow}+\infty}\right) + O_{L^\infty(D_2)}(\varepsilon^{4/3}), \end{aligned} \tag{4.94}$$

and thus $c_1 > 0$ on $D_2\backslash D_1$ if ε is sufficiently small. ∎

Lemma 4.8 *$c_2(x) > 0$ for every $x \in \mathbb{R}^d\backslash S_2$.*

Proof. The lemma is a straightforward consequence of the definition of c_2, since the assumption $x \in \mathbb{R}^d\backslash S_2$ can be rewritten as $|x|^2 > R_2^2 + \varepsilon^{2/3}$. ∎

4.6 Uniqueness of the ground state

In this section, we prove that the solution of (1.13) constructed in the previous sections is the unique ground state of the system, that is the unique solution of (1.13) with two positive components. Uniqueness of the ground state of (1.13) was proved in [ANS]. We recall the arguments for the sake of completeness. First, the next lemma gives an a priori upper bound on positive solutions to (1.13).

Lemma 4.9 *Let $\varepsilon > 0$, and let (η_1, η_2) be a positive solution of (1.13). Then, for every $\theta \in (0,1)$ and $x \in \mathbb{R}^d$, for $j = 1, 2$,*

$$\eta_j(x) \leqslant M_j \min\left[1, \exp\left(-\frac{\theta}{2\varepsilon}\left(|x|^2 - r_j^2\right)\right)\right], \tag{4.95}$$

where $a_1 = \left(\frac{\alpha_0}{\alpha_2}(R_2^2-R_1^2)+R_1^2\right)^{1/2}$, $M_1 = \frac{a_1}{(2\alpha_1)^{1/2}}$, $r_1 = \frac{a_1}{(1-\theta^2)^{1/2}}$, $M_2 = \frac{R_2}{(2\alpha_2)^{1/2}}$ and $r_2 = \frac{R_2}{(1-\theta^2)^{1/2}}$.

Proof. We first prove that η_1 is uniformly bounded from above by the constant M_1 defined in the statement of the lemma. The proof follows an idea which is due to Farina [F], and which is also used in [IM] and [ANS]. Let us define

$$w_1 = \frac{1}{\varepsilon}\left((2\alpha_1)^{1/2}\eta_1 - a_1\right), \quad \text{and} \quad w_1^+ = \max(0, w_1).$$

Then, Kato's inequality yields

$$\begin{aligned} \Delta w_1^+ &\geqslant \mathbf{1}_{\{w_1\geqslant 0\}}\Delta w_1 \\ &= \frac{\mathbf{1}_{\{w_1\geqslant 0\}}}{\varepsilon^3}(2\alpha_1)^{1/2}\eta_1\left(2\alpha_1\eta_1^2 + 2\alpha_0\eta_2^2 - \frac{\alpha_0}{\alpha_2}(R_2^2-R_1^2) - (R_1^2-|x|^2)\right) \\ &\geqslant \frac{\mathbf{1}_{\{w_1\geqslant 0\}}}{\varepsilon^3}(\varepsilon w_1 + a_1)\left((\varepsilon w_1 + a_1)^2 - (a_1)^2\right) \\ &= \frac{\mathbf{1}_{\{w_1\geqslant 0\}}}{\varepsilon^3}(\varepsilon w_1 + a_1)\varepsilon w_1(\varepsilon w_1 + 2a_1) \\ &\geqslant (w_1^+)^3. \end{aligned} \tag{4.96}$$

From Lemma 2 in [B], it follows that $w_1^+ \leqslant 0$, which means $\eta_1 \leqslant M_1$.

Next, like it was done in [ANS], we prove that η_1 decays at least as fast as a gaussian as $|x|$ goes to infinity. Easy calculations show that

$$\left(-\Delta + \frac{\theta^2}{\varepsilon^2}|x|^2\right)\exp\left(-\frac{\theta|x|^2}{2\varepsilon}\right) = \frac{d\theta}{\varepsilon}\exp\left(-\frac{\theta|x|^2}{2\varepsilon}\right) > 0, \tag{4.97}$$

whereas

$$\left(-\Delta+\frac{\theta^2}{\varepsilon^2}|x|^2\right)\eta_1=\frac{1}{\varepsilon^2}\left(a_1^2-(1-\theta^2)|x|^2\right)\eta_1-\frac{2\alpha_1}{\varepsilon^2}\eta_1^3-\frac{2\alpha_0}{\varepsilon^2}\eta_2^2\eta_1<0 \tag{4.98}$$

for $x>r_1$. Then, we set $W_1=M_1\exp\left(-\frac{\theta(|x|^2-r_1^2)}{2\varepsilon}\right)-\eta_1$. We know from the first part of the proof that $W_1(x)\geqslant 0$ for $|x|=r_1$. Assume by contradiction that the inequality $W_1\geqslant 0$ does not hold for every $x\in\mathbb{R}^d$ such that $|x|\geqslant r_1$. Then, since $W_1(x)\to 0$ as $|x|\to\infty$, W_1 reaches a minimum at some $x_0\in\mathbb{R}^d$ such that $|x_0|>r_1$. In particular, $\Delta W_1(x_0)\geqslant 0$ and $W_1(x_0)<0$. This is in contradiction with the difference between (4.97) multiplied by M_1 and (4.98) evaluated at x_0. The proof of the estimate on η_2 is similar. ∎

The next lemma states the uniqueness of the ground state of (1.13) and is also proved in [ANS]. We give here a proof which is slightly simpler.

Lemma 4.10 *Let $\varepsilon>0$, and let (η_1,η_2), (ξ_1,ξ_2) be two positive solutions of (1.13). Then $\eta_1=\xi_1$ and $\eta_2=\xi_2$.*

Proof. Let $v_1=\xi_1/\eta_1$ and $v_2=\xi_2/\eta_2$. Since (η_1,η_2) and (ξ_1,ξ_2) solve (1.13), it follows that for $(i,j)=(1,2)$ or $(2,1)$, we have

$$\varepsilon^2\mathrm{div}\left(\eta_i^2\nabla v_i\right) \;=\; 2\alpha_i\eta_i^4v_i(v_i^2-1)+2\alpha_0\eta_i^2\eta_j^2v_i(v_j^2-1). \tag{4.99}$$

Let $\zeta\in\mathcal{C}^\infty(\mathbb{R}^d)$ be a non-negative function supported in $\{x\in\mathbb{R}^d,\ |x|\leqslant 2\}$ such that $\zeta(x)=1$ for $|x|\leqslant 1$. For $n\geqslant 1$, we also define $\zeta_n=\zeta(\cdot/n)$. Next, let us multiply (4.99) by $(v_i^2-1)\zeta_n^2/v_i$, sum over $\mathbb{R}^d$ and use integration by parts. We obtain

$$\begin{aligned}
&\int_{\mathbb{R}^d}\eta_i^2|\nabla v_i|^2\left(1+\frac{1}{v_i^2}\right)\zeta_n^2dx+\frac{2}{\varepsilon^2}\int_{\mathbb{R}^d}\left[\alpha_i\left(\eta_i^2(v_i^2-1)\right)^2+\alpha_0\eta_i^2(v_i^2-1)\eta_j^2(v_j^2-1)\right]\zeta_n^2dx\\
&\quad=\quad -\int_{\mathbb{R}^d}\eta_i^2\nabla v_i\left(v_i-\frac{1}{v_i}\right)\nabla\left(\zeta_n^2\right)dx\\
&\quad=\quad -2\int_{\mathbb{R}^d}\eta_i^2v_i\nabla v_i\zeta_n\nabla\zeta_ndx+2\int_{\mathbb{R}^d}\eta_i^2\frac{\nabla v_i}{v_i}\zeta_n\nabla\zeta_ndx.
\end{aligned} \tag{4.100}$$

Next, we estimate each integral in the right hand side of (4.100) thanks to the Cauchy-Schwarz inequality. For the first one, we have

$$\begin{aligned}
\left|\int_{\mathbb{R}^d}\eta_i^2v_i\nabla v_i\zeta_n\nabla\zeta_ndx\right| &= \left|\int_{\mathbb{R}^d}\eta_i\xi_i\nabla v_i\zeta_n\nabla\zeta_ndx\right|\\
&\leqslant \left(\int_{\mathbb{R}^d}\eta_i^2|\nabla v_i|^2\zeta_n^2dx\right)^{1/2}\left(\int_{\mathbb{R}^d}\xi_i^2|\nabla\zeta_n|^2dx\right)^{1/2}\\
&\leqslant \frac{1}{4}\int_{\mathbb{R}^d}\eta_i^2|\nabla v_i|^2\zeta_n^2dx+\int_{\mathbb{R}^d}\xi_i^2|\nabla\zeta_n|^2dx,
\end{aligned} \tag{4.101}$$

whereas for the second one, we get

$$\begin{aligned}
\left|\int_{\mathbb{R}^d}\eta_i^2\frac{\nabla v_i}{v_i}\zeta_n\nabla\zeta_ndx\right| &\leqslant \left(\int_{\mathbb{R}^d}\eta_i^2\frac{|\nabla v_i|^2}{v_i^2}\zeta_n^2dx\right)^{1/2}\left(\int_{\mathbb{R}^d}\eta_i^2|\nabla\zeta_n|^2dx\right)^{1/2}\\
&\leqslant \frac{1}{4}\int_{\mathbb{R}^d}\eta_i^2\frac{|\nabla v_i|^2}{v_i^2}\zeta_n^2dx+\int_{\mathbb{R}^d}\eta_i^2|\nabla\zeta_n|^2dx.
\end{aligned} \tag{4.102}$$

Combining (4.100), (4.101) and (4.102), we infer

$$\begin{aligned}
&\frac{1}{2}\int_{\mathbb{R}^d}\eta_i^2|\nabla v_i|^2\left(1+\frac{1}{v_i^2}\right)\zeta_n^2dx+\frac{2}{\varepsilon^2}\int_{\mathbb{R}^d}\left[\alpha_i\left(\eta_i^2(v_i^2-1)\right)^2+\alpha_0\eta_i^2(v_i^2-1)\eta_j^2(v_j^2-1)\right]\zeta_n^2dx\\
&\quad\leqslant\quad 2\int_{\mathbb{R}^d}(\xi_i^2+\eta_i^2)|\nabla\zeta_n|^2dx.
\end{aligned} \tag{4.103}$$

Finally, we sum the inequalities given by (4.103) for $(i,j)=(1,2)$ and for $(i,j)=(2,1)$. We deduce

$$\begin{aligned}\frac{1}{2}\int_{\mathbb{R}^d}\eta_1^2|\nabla v_1|^2\left(1+\frac{1}{v_1^2}\right)\zeta_n^2dx+\frac{1}{2}\int_{\mathbb{R}^d}\eta_2^2|\nabla v_2|^2\left(1+\frac{1}{v_2^2}\right)\zeta_n^2dx+\frac{2}{\varepsilon^2}\int_{\mathbb{R}^d}q\left[\eta_1^2(v_1^2-1),\eta_2^2(v_2^2-1)\right]\zeta_n^2dx\\ \leqslant\quad 2\int_{\mathbb{R}^d}(\xi_1^2+\eta_1^2+\xi_2^2+\eta_2^2)|\nabla\zeta_n|^2dx,\end{aligned}\tag{4.104}$$

where $q[u_1,u_2]=\alpha_1u_1^2+2\alpha_0u_1u_2+\alpha_2u_2^2$. Note that the assumption $\Gamma_{12}>0$ can be rewritten as $\alpha_0^2-\alpha_1\alpha_2<0$, which implies that there exists $c>0$ such that for every $u_1,u_2\in\mathbb{R}$, $q[u_1,u_2]\geqslant c(u_1^2+u_2^2)$. As a result, in order to conclude that $v_1\equiv v_2\equiv 1$, it is sufficient to prove that the right hand side of (4.104) converges to 0 as $n\to\infty$. It is the case thanks to Lemma 4.9. Indeed, for $n\geqslant\max(r_1,r_2)$, since $\nabla\zeta_n$ is supported in $\{x\in\mathbb{R}^d, n\leqslant|x|\leqslant 2n\}$, we have

$$\int_{\mathbb{R}^d}(\xi_1^2+\eta_1^2+\xi_2^2+\eta_2^2)|\nabla\zeta_n|^2dx\leqslant 2\left(M_1^2e^{\theta r_1^2/\varepsilon}+M_2^2e^{\theta r_2^2/\varepsilon}\right)\|\nabla\zeta\|_{L^\infty}^2\left|\{x\in\mathbb{R}^d,1\leqslant|x|\leqslant 2\}\right|n^{d-2}e^{-n^2\theta/\varepsilon},$$

where the right hand side goes to 0 as $n\to\infty$. ∎

4.7 End of the proof of Theorem 1.5

In section 4.4, we have constructed a solution (η_1,η_2) to (1.13) that converges to 0 at infinity. In section 4.5, we have checked that this solution is positive. In section 4.6, we have seen that (η_1,η_2) is in fact the unique such solution of (1.13). So the first part of the statement of Theorem 1.5 has already been proved. Let us now fix three integers M_0,N_0 and L_0, as well as $\beta\in(0,\varepsilon^{2/3})$. According to our construction of (η_1,η_2) explained in sections 4.1 and 4.4, provided $M>M_0$, $N>\max(N_0,2/\beta-2)$ and $L>L_0$ are large integers that satisfy the conditions listed at the beginning of sections 4.1 and 4.4, (η_1,η_2) can be written like in the ansatz (1.20)-(3.1). Thus, defining η_{1app} and η_{2app} as in the statement of Theorem 1.5, we have

$$\eta_1-\eta_{1app}=\Phi_\varepsilon\sum_{m=M_0+1}^{M}\varepsilon^{2m}\omega_m+\varepsilon^{1/3}\chi_\varepsilon\sum_{n=N_0+1}^{N}\varepsilon^{2n/3}\nu_n+\varepsilon^{2N/3+1}P\tag{4.105}$$

and

$$\begin{aligned}\eta_2-\eta_{2app}\quad=\quad&\Phi_\varepsilon\sum_{m=M_0+1}^{M}\varepsilon^{2m}\tau_m+\varepsilon^{1/3}\chi_\varepsilon\left(\left(\sum_{n=-1}^{N}\varepsilon^{2n/3}\lambda_n\right)^{1/2}-\left(\sum_{n=-1}^{N_0}\varepsilon^{2n/3}\lambda_n\right)^{1/2}\right)\\&+\varepsilon^{1/3}\Psi_\varepsilon\sum_{n=L_0+1}^{L}\varepsilon^{2n/3}\mu_n+\varepsilon^{2N/3+1}Q.\end{aligned}\tag{4.106}$$

The next step consists in evaluating the L^p and H^1 norms of each term in the right hand side of (4.105) and (4.106). Let us start with the L^p norm of $\Phi_\varepsilon\omega_m$, for $m\geqslant 1$ and $p\in[2,+\infty)$. Since $\mathrm{Supp}\Phi_\varepsilon\subset D_0$, we have

$$\begin{aligned}\|\Phi_\varepsilon\omega_m\|_{L^p(\mathbb{R}^d)}^p\quad&\leqslant\quad\int_{|x|^2\leqslant R_1^2-\varepsilon^\beta}|\omega_m(z)|^pdx=\int_{\mathbb{S}^{d-1}}\int_0^{(R_1^2-\varepsilon^\beta)^{1/2}}|\omega_m(R_1^2-r^2)|^pr^{d-1}drd\theta\\&=\quad|\mathbb{S}^{d-1}|\int_{\varepsilon^\beta}^{R_1^2}|\omega_m(z)|^p(R_1^2-z)^{d/2-1}\frac{dz}{2}.\end{aligned}\tag{4.107}$$

Since $d/2-1\geqslant -1/2$, the integral converges at $z=R_1^2$. Moreover, thanks to (2.11), we deduce

$$\int_{\varepsilon^\beta}^{R_1^2}|\omega_m(z)|^p(R_1^2-z)^{d/2-1}dz\underset{\varepsilon\to 0}{\sim}R_1^{d-2}|\omega_{m0}|^p\int_{\varepsilon^\beta}^{R_1^2}z^{p(1/2-3m)}dz\underset{\varepsilon\to 0}{\sim}\frac{R_1^{d-2}|\omega_{m0}|^p}{p(3m-1/2)-1}\varepsilon^{\beta(-p(3m-1/2)+1)}.$$

As a result,

$$\|\Phi_\varepsilon\omega_m\|_{L^p(\mathbb{R}^d)}=\mathcal{O}(\varepsilon^{-3\beta m+\beta/2+\beta/p}).\tag{4.108}$$

Similarly, (2.12) yields

$$\|\Phi_\varepsilon \tau_m\|_{L^p(\mathbb{R}^d)} = \mathcal{O}(\varepsilon^{-3\beta m+\beta+\beta/p}). \tag{4.109}$$

Note that (4.108) and (4.109) also hold for $p=+\infty$ thanks to (2.11) and (2.12). Note also that (4.108) and (4.109) are sharp. Indeed, since $\Phi_\varepsilon \equiv 1$ for $|x|^2 \leqslant R_1^2 - 2\varepsilon^\beta$, we deduce that $\|\Phi_\varepsilon \omega_m\|^p_{L^p(\mathbb{R}^d)}$ can be bounded from below by an integral similar to the one that appears in the right hand side of (4.107).

Next, let us estimate the L^p norm of $\chi_\varepsilon \nu_n$, for $n \geqslant 1$ and $p \in [2,+\infty)$. Since $\mathrm{Supp}\chi_\varepsilon \subset D_1$, we get

$$\begin{aligned}\|\chi_\varepsilon \nu_n\|^p_{L^p(\mathbb{R}^d)} &\leqslant \int_{R_1^2-2\varepsilon^\beta \leqslant |x|^2 \leqslant R_1^2+2\varepsilon^\beta} |\nu_n(y_1)|^p dx = \int_{\mathbb{S}^{d-1}} \int_{(R_1^2-2\varepsilon^\beta)^{1/2}}^{(R_1^2+2\varepsilon^\beta)^{1/2}} \left|\nu_n\left(\frac{R_1^2-r^2}{\varepsilon^{2/3}}\right)\right|^p r^{d-1} dr d\theta \\ &= |\mathbb{S}^{d-1}| \int_{-2\varepsilon^{\beta-2/3}}^{2\varepsilon^{\beta-2/3}} |\nu_n(y_1)|^p (R_1^2 - \varepsilon^{2/3} y_1)^{d/2-1} \frac{\varepsilon^{2/3} dy_1}{2}. \end{aligned} \tag{4.110}$$

For $y_1 \in [-2\varepsilon^{\beta-2/3}, 2\varepsilon^{\beta-2/3}]$, we have $1 \lesssim R_1^2 - \varepsilon^{2/3} y_1 \lesssim 1$, therefore according to the asymptotic behaviour of $\nu_n(y_1)$ as $y_1 \to \pm\infty$ given in Proposition 2.7, we obtain

$$\|\chi_\varepsilon \nu_n\|_{L^p(\mathbb{R}^d)} = \begin{cases} \mathcal{O}(\varepsilon^{\frac{2}{3p}}) & \text{if } n=1 \text{ or } (n=2 \text{ and } p>2) \\ \mathcal{O}(|\ln \varepsilon|^{\frac12} \varepsilon^{\frac13}) & \text{if } n=2 \text{ and } p=2 \\ \mathcal{O}(\varepsilon^{-\frac{2n}{3}+\beta(n-\frac52)+\frac{\beta}{p}+\frac53}) & \text{if } n \geqslant 3. \end{cases} \tag{4.111}$$

Similarly,

$$\|\chi_\varepsilon \lambda_n\|_{L^p(\mathbb{R}^d)} = \begin{cases} \mathcal{O}(\varepsilon^{\frac{2}{3p}}) & \text{if } n=1 \\ \mathcal{O}(\varepsilon^{-\frac{2n}{3}+\beta(n-2)+\frac{\beta}{p}+\frac43}) & \text{if } n \geqslant 2. \end{cases} \tag{4.112}$$

Again, it easily follows from Proposition 2.7 that (4.111) and (4.112) also hold for $p=+\infty$, and the two estimates are sharp. Next, since $\mathrm{Supp}\Psi_\varepsilon \subset D_2$, we infer

$$\begin{aligned}\|\Psi_\varepsilon \mu_n\|^p_{L^p(\mathbb{R}^d)} &\leqslant \int_{|x|^2 \geqslant R_1^2+\varepsilon^\beta} |\mu_n(y_2)|^p dx = \int_{\mathbb{S}^{d-1}} \int_{(R_1^2+\varepsilon^\beta)^{1/2}}^{+\infty} \left|\mu_n\left(\frac{R_2^2-r^2}{\varepsilon^{2/3}}\right)\right|^p r^{d-1} dr d\theta \\ &= |\mathbb{S}^{d-1}| \int_{-\infty}^{\frac{R_2^2-R_1^2}{\varepsilon^{2/3}}-\varepsilon^{\beta-2/3}} |\mu_n(y_2)|^p (R_2^2 - \varepsilon^{2/3} y_2)^{d/2-1} \frac{\varepsilon^{2/3} dy_2}{2}. \end{aligned} \tag{4.113}$$

In order to estimate the integral in the right hand side of (4.113), we split the integral into two pieces. First, for $y_2 \in (-R_2^2/\varepsilon^{2/3}, (R_2^2-R_1^2)/\varepsilon^{2/3} - \varepsilon^{\beta-2/3})$, we have $1 \lesssim R_1^2 + \varepsilon^\beta \leqslant R_2^2 - \varepsilon^{2/3} y_2 \leqslant 2R_2^2 \lesssim 1$. Therefore, according to (2.19) and Proposition 2.6,

$$\int_{-R_2^2/\varepsilon^{2/3}}^{\frac{R_2^2-R_1^2}{\varepsilon^{2/3}}-\varepsilon^{\beta-2/3}} |\mu_n(y_2)|^p (R_2^2 - \varepsilon^{2/3} y_2)^{d/2-1} dy_2 \underset{\varepsilon\to 0}{=} \mathcal{O}(1). \tag{4.114}$$

If $d=1,2$ and $y_2 \leqslant -R_2^2/\varepsilon^{2/3}$, we still have $(R_2^2 - \varepsilon^{2/3} y_2)^{d/2-1} \lesssim 1$, therefore

$$\int_{-\infty}^{-R_2^2/\varepsilon^{2/3}} |\mu_n(y_2)|^p (R_2^2 - \varepsilon^{2/3} y_2)^{d/2-1} dy_2 \underset{\varepsilon\to 0}{=} \mathcal{O}(1), \tag{4.115}$$

whereas if $d=3$ and $y_2 \leqslant -R_2^2/\varepsilon^{2/3}$, then $(R_2^2 - \varepsilon^{2/3} y_2)^{1/2} \leqslant \sqrt{2}\varepsilon^{1/3}|y_2|^{1/2}$ and since Proposition 2.6 implies $\mu_n(y_2) \underset{y_2\to-\infty}{=} \mathcal{O}(|y_2|^{-5/2})$, we deduce that (4.115) also holds. Combining (4.113), (4.114) and (4.115), we deduce

$$\|\Psi_\varepsilon \mu_n\|_{L^p(\mathbb{R}^d)} = \mathcal{O}(\varepsilon^{\frac{2}{3p}}). \tag{4.116}$$

Note again that (4.116) is sharp and that Proposition 2.6 implies that it is also true for $p=+\infty$.

We are now ready to estimate $\eta_1 - \eta_{1app}$ and $\eta_2 - \eta_{2app}$ in $L^p(\mathbb{R}^d)$. Remark first that since $\beta < 2/3$, (4.108) and (4.109) imply that the larger is $m \geqslant 1$, the smaller are $\varepsilon^{2m}\Phi_\varepsilon \omega_m$ and $\varepsilon^{2m}\Phi_\varepsilon \tau_m$ in $L^p(\mathbb{R}^d)$,

in the limit $\varepsilon \to 0$. Similarly, since $\beta > 0$, it follows from (4.111) and (4.112) that the larger is n, the smaller are $\varepsilon^{2n/3}\chi_\varepsilon\nu_n$ and $\varepsilon^{2n/3}\chi_\varepsilon\lambda_n$ in $L^p(\mathbb{R}^d)$. Thus,

$$\begin{aligned}\|\eta_1 - \eta_{1app}\|_{L^p(\mathbb{R}^d)} &= \mathcal{O}(\varepsilon^{(2-3\beta)(M_0+1)+\frac{\beta}{2}+\frac{\beta}{p}}) + \left\{\begin{array}{ll} \mathcal{O}(\varepsilon^{1+\frac{2}{3p}}) & \text{if } N_0 = 0 \\ \mathcal{O}(\varepsilon^{\frac{5}{3}+\frac{2}{3p}}) & \text{if } N_0 = 1 \text{ and } p > 2 \\ \mathcal{O}(|\ln\varepsilon|^{\frac{1}{2}}\varepsilon^2) & \text{if } N_0 = 1 \text{ and } p = 2 \\ \mathcal{O}(\varepsilon^{\beta(N_0-\frac{3}{2})+\frac{\beta}{p}+2}) & \text{if } N_0 \geqslant 2. \end{array}\right\} \\ &\quad + \|P\|_{L^p(\mathbb{R}^d)}\mathcal{O}(\varepsilon^{2N/3+1}). \end{aligned} \tag{4.117}$$

Now, remember that in (4.78), the $H^2(\mathbb{R}^d)$ norm of P is controled by some power of ε (namely, ε^{-5}) wich is independent of N. Thus, thanks to Sobolev embeddings, for fixed values of M_0, N_0 and L_0, if M, N and L are chosen sufficiently large (and such that they satisfy the conditions at the beginning of sections 4.1 and 4.4), for ε small, $\varepsilon^{2N/3+1}\|P\|_{L^p(\mathbb{R}^d)}$ becomes negligible in comparison with the other terms in the right hand side of (4.117). The estimate on $\eta_1 - \eta_{1app}$ in (1.22) follows in the case $E = L^p(\mathbb{R}^d)$.

As for the second component, using the same arguments, we infer from (4.109), and (4.116) that

$$\begin{aligned}\|\eta_2 - \eta_{2app}\|_{L^p(\mathbb{R}^d)} &= \mathcal{O}(\varepsilon^{(2-3\beta)(M_0+1)+\beta+\frac{\beta}{p}}) \\ &\quad + \chi_\varepsilon\left(\left(\lambda_{-1} + \sum_{n=1}^{N+1}\varepsilon^{2n/3}\lambda_{n-1}\right)^{1/2} - \left(\lambda_{-1} + \sum_{n=1}^{N_0+1}\varepsilon^{2n/3}\lambda_{n-1}\right)^{1/2}\right) \\ &\quad + \mathcal{O}(\varepsilon^{\frac{1}{3}+\frac{2(L_0+1)}{3}+\frac{2}{3p}}) + \|Q\|_{L^p(\mathbb{R}^d)}\mathcal{O}(\varepsilon^{2N/3+1}). \end{aligned} \tag{4.118}$$

In order to estimate the second term in the right hand side, note that thanks to the asymptotic behaviour of λ_0 given in Proposition 2.7 and (4.112) for $p = +\infty$, we have

$$\sum_{n=1}^{N+1}\varepsilon^{2n/3}\lambda_{n-1} = \mathcal{O}_{L^\infty(D_1)}(\varepsilon^\beta),$$

and the same property holds for N replaced by N_0. Thus, the mean value theorem applied to the function square root close to λ_{-1} and (4.112) imply

$$\begin{aligned}\|\eta_2 - \eta_{2app}\|_{L^p(\mathbb{R}^d)} &= \mathcal{O}(\varepsilon^{(2-3\beta)(M_0+1)+\beta+\frac{\beta}{p}}) + \left\{\begin{array}{ll} \mathcal{O}(\varepsilon^{\frac{4}{3}+\frac{2}{3p}}) & \text{if } N_0 = 0 \\ \mathcal{O}(\varepsilon^{2+\beta(N_0-1)+\frac{\beta}{p}}) & \text{if } N_0 \geqslant 1. \end{array}\right. \\ &\quad + \mathcal{O}(\varepsilon^{\frac{1}{3}+\frac{2(L_0+1)}{3}+\frac{2}{3p}}), \end{aligned} \tag{4.119}$$

under the same condition of largeness on M, N, L than for the estimate on $\eta_1 - \eta_{1app}$. We have proved the estimate on $\eta_2 - \eta_{2app}$ in (1.22) for $E = L^p(\mathbb{R}^d)$.

Next, let us prove (1.22) for $E = H^1(\mathbb{R}^d)$. For this purpose, we have to estimate the $L^2(\mathbb{R}^d)$ norms of $\nabla(\Phi_\varepsilon\omega_m)$, $\nabla(\Phi_\varepsilon\tau_m)$, $\nabla(\chi_\varepsilon\nu_n)$, $\nabla(\chi_\varepsilon\lambda_n)$ and $\nabla(\Psi_\varepsilon\mu_n)$ for $m, n \geqslant 1$. In view of the definitions of Φ_ε, χ_ε and Ψ_ε, it is clear that the $L^\infty(\mathbb{R}^d)^d$ norms of their gradients are all $\mathcal{O}(\varepsilon^{-\beta})$. Thus, performing calculations similar to the ones which were done to obtain (4.108), (4.109), (4.111),(4.112) and (4.116), we obtain

$$\|\nabla(\Phi_\varepsilon)\omega_m\|_{L^2(\mathbb{R}^d)} = \mathcal{O}(\varepsilon^{-3\beta m}), \tag{4.120}$$

$$\|\nabla(\Phi_\varepsilon)\tau_m\|_{L^2(\mathbb{R}^d)} = \mathcal{O}(\varepsilon^{-3\beta m+\beta/2}), \tag{4.121}$$

$$\|\nabla(\chi_\varepsilon)\nu_n\|_{L^2(\mathbb{R}^d)} = \left\{\begin{array}{ll} \mathcal{O}(\varepsilon^{\frac{1}{3}-\beta}) & \text{if } n = 1 \\ \mathcal{O}(|\ln\varepsilon|^{\frac{1}{2}}\varepsilon^{\frac{1}{3}-\beta}) & \text{if } n = 2 \\ \mathcal{O}(\varepsilon^{-\frac{2n}{3}+\beta(n-3)+\frac{5}{3}}) & \text{if } n \geqslant 3 \end{array}\right. \tag{4.122}$$

$$\|\nabla(\chi_\varepsilon)\lambda_n\|_{L^2(\mathbb{R}^d)} = \left\{\begin{array}{ll} \mathcal{O}(\varepsilon^{\frac{1}{3}-\beta}) & \text{if } n = 1 \\ \mathcal{O}(\varepsilon^{-\frac{2n}{3}+\beta(n-5/2)+\frac{4}{3}}) & \text{if } n \geqslant 2 \end{array}\right. \tag{4.123}$$

$$\|\nabla(\Psi_\varepsilon)\mu_n\|_{L^2(\mathbb{R}^d)} = \mathcal{O}(\varepsilon^{\frac{1}{3}-\beta}). \tag{4.124}$$

By differentiation of (2.11) and (2.12), since $\nabla = -2x\frac{d}{dz}$, similar calculations as the ones that gave (4.108) and (4.109) yield

$$\|\Phi_\varepsilon \nabla \omega_m\|_{L^2(\mathbb{R}^d)} = \mathcal{O}(\varepsilon^{-3\beta m}), \tag{4.125}$$

$$\|\Phi_\varepsilon \nabla \tau_m\|_{L^2(\mathbb{R}^d)} = \mathcal{O}(\varepsilon^{-3\beta m+\beta/2}). \tag{4.126}$$

Next, since $\nabla = \frac{-2x}{\varepsilon^{2/3}}\frac{d}{dy_1}$, a calculation similar to (4.110) yields

$$\|\chi_\varepsilon \nabla \nu_n\|^2_{L^2(\mathbb{R}^d)} \leqslant \frac{2|\mathbb{S}^{d-1}|}{\varepsilon^{2/3}} \int_{-2\varepsilon^{\beta-2/3}}^{2\varepsilon^{\beta-2/3}} |\nu_n'(y_1)|^2 (R_1^2 - \varepsilon^{2/3} y_1)^{d/2} dy_1. \tag{4.127}$$

Then, after differentiation of (2.49), we deduce that

$$\|\chi_\varepsilon \nabla \nu_n\|_{L^2(\mathbb{R}^d)} = \begin{cases} \mathcal{O}(\varepsilon^{-1/3}) & \text{if } n = 1 \text{ or } n = 2 \\ \mathcal{O}(|\ln \varepsilon|^{\frac{1}{2}} \varepsilon^{-\frac{1}{3}}) & \text{if } n = 3 \\ \mathcal{O}(\varepsilon^{-2n/3+\beta(n-3)+5/3}) & \text{if } n \geqslant 4. \end{cases} \tag{4.128}$$

Similarly, differentiation of (2.50) yields

$$\|\chi_\varepsilon \nabla \lambda_n\|_{L^2(\mathbb{R}^d)} = \begin{cases} \mathcal{O}(\varepsilon^{-1/3}) & \text{if } n = 1 \text{ or } n = 2 \\ \mathcal{O}(\varepsilon^{-2n/3+\beta(n-5/2)+4/3}) & \text{if } n \geqslant 3. \end{cases} \tag{4.129}$$

Using Proposition 2.19 like it was done to obtain (4.116), we infer

$$\|\Psi_\varepsilon \nabla \mu_n\|_{L^2(\mathbb{R}^d)} = \mathcal{O}(\varepsilon^{-1/3}). \tag{4.130}$$

Like in (4.117), taking M, N and L large enough, we deduce from (4.120), (4.122), (4.125) and (4.128) that

$$\|\nabla(\eta_1 - \eta_{1app})\|_{L^2(\mathbb{R}^d)} = \mathcal{O}(\varepsilon^{(2-3\beta)(M_0+1)}) + \begin{cases} \mathcal{O}(\varepsilon^{2N_0/3+2/3}) & \text{if } N_0 = 0 \text{ or } N_0 = 1 \\ \mathcal{O}(|\ln \varepsilon|^{\frac{1}{2}} \varepsilon^2) & \text{if } N_0 = 2 \\ \mathcal{O}(\varepsilon^{\beta(N_0-2)+2}) & \text{if } N_0 \geqslant 3. \end{cases} \tag{4.131}$$

Next, we write

$$\begin{aligned} \nabla(\eta_2 - \eta_{2app}) = {} & \underbrace{\nabla\left(\Phi_\varepsilon \sum_{m=M_0+1}^{M} \varepsilon^{2m}\tau_m\right)}_{=:T_1} + \underbrace{\varepsilon^{1/3}\nabla\chi_\varepsilon\left(\left(\sum_{n=-1}^{N}\varepsilon^{2n/3}\lambda_n\right)^{1/2} - \left(\sum_{n=-1}^{N_0}\varepsilon^{2n/3}\lambda_n\right)^{1/2}\right)}_{=:T_2} \\ & + \underbrace{\frac{1}{2}\varepsilon^{1/3}\chi_\varepsilon\left(\left(\sum_{n=-1}^{N}\varepsilon^{2n/3}\lambda_n\right)^{-1/2} - \left(\sum_{n=-1}^{N_0}\varepsilon^{2n/3}\lambda_n\right)^{-1/2}\right)\sum_{n=-1}^{N}\varepsilon^{2n/3}\nabla\lambda_n}_{=:T_3} \\ & + \underbrace{\frac{1}{2}\varepsilon^{1/3}\chi_\varepsilon\left(\sum_{n=-1}^{N_0}\varepsilon^{2n/3}\lambda_n\right)^{-1/2}\sum_{n=N_0+1}^{N}\varepsilon^{2n/3}\nabla\lambda_n}_{=:T_4} \\ & + \underbrace{\varepsilon^{1/3}\nabla\left(\Psi_\varepsilon \sum_{n=L_0+1}^{L}\varepsilon^{2n/3}\mu_n\right)}_{=:T_5} + \underbrace{\varepsilon^{2N/3+1}\nabla Q}_{=:T_6}. \end{aligned} \tag{4.132}$$

Thanks to (4.121) and (4.126), we have

$$\|T_1\|_{L^2(\mathbb{R}^d)} = \mathcal{O}(\varepsilon^{(2-3\beta)(M_0+1)+\beta/2}). \tag{4.133}$$

T_2 is estimated like the second term in the right hand side of (4.118) in (4.119), using (4.123) instead of (4.112). We obtain

$$\|T_2\|_{L^2(\mathbb{R}^d)} = \begin{cases} \mathcal{O}(\varepsilon^{5/3-\beta}) & \text{if } N_0 = 0 \\ \mathcal{O}(\varepsilon^{\beta(N_0-3/2)+2}) & \text{if } N_0 \geqslant 1. \end{cases} \tag{4.134}$$

In order to estimate T_3, note that thanks to (2.52) and (2.50), λ_n' is uniformly bounded on $\mathbb{R}$ for $n = 0, 1, 2, 3$, whereas for $n \geqslant 4$, $\lambda_n' = \mathcal{O}_{L^\infty(D_1)}(\varepsilon^{-(2/3-\beta)(n-3)})$. Therefore, since λ_{-1} is constant,

$$\sum_{n=-1}^{N} \varepsilon^{2n/3} \nabla\lambda_n \underset{\varepsilon\to 0}{=} -\frac{2x}{\varepsilon^{2/3}}\lambda_0' + \mathcal{O}_{L^\infty(D_1)}(1).$$

Applying the mean value theorem to the inverse of the square root close to λ_{-1}, we use the same arguments as to obtain (4.119) from (4.118) and we get thanks to (4.112)

$$\|T_3\|_{L^2(\mathbb{R}^d)} = \begin{cases} \mathcal{O}(\varepsilon^{5/3}) & \text{if } N_0 = 0 \\ \mathcal{O}(\varepsilon^{\beta(N_0-1/2)+2}) & \text{if } N_0 \geqslant 1. \end{cases} \tag{4.135}$$

Lemma 3.1 and (4.129) yield

$$\|T_4\|_{L^2(\mathbb{R}^d)} = \begin{cases} \mathcal{O}(\varepsilon^{2N_0/3+1}) & \text{if } N_0 = 0 \text{ or } 1 \\ \mathcal{O}(\varepsilon^{\beta(N_0-3/2)+2}) & \text{if } N_0 \geqslant 2. \end{cases} \tag{4.136}$$

It follows from (4.124) and (4.130) that

$$\|T_5\|_{L^2(\mathbb{R}^d)} = \mathcal{O}(\varepsilon^{2(L_0+1)/3}). \tag{4.137}$$

Finally, like in (4.117), we deduce from (4.53) that if M, N and L are chosen large enough, T_6 is neglectible in comparison with the sum of the five other terms. Therefore, combining (4.133), (4.134), (4.135), (4.136) and (4.137), we obtain

$$\begin{aligned} \|\nabla(\eta_2 - \eta_{2app})\|_{L^2(\mathbb{R}^d)} &= \mathcal{O}(\varepsilon^{(2-3\beta)(M_0+1)+\beta/2}) + \begin{cases} \mathcal{O}(\varepsilon^{5/3-\beta}) & \text{if } N_0 = 0 \\ \mathcal{O}(\varepsilon^{\beta(N_0-3/2)+2}) & \text{if } N_0 \geqslant 1. \end{cases} \\ &\quad + \begin{cases} \mathcal{O}(\varepsilon^{5/3}) & \text{if } N_0 = 0 \\ \mathcal{O}(\varepsilon^{\beta(N_0-1/2)+2}) & \text{if } N_0 \geqslant 1. \end{cases} + \begin{cases} \mathcal{O}(\varepsilon^{2N_0/3+1}) & \text{if } N_0 = 0 \text{ or } 1 \\ \mathcal{O}(\varepsilon^{\beta(N_0-3/2)+2}) & \text{if } N_0 \geqslant 2. \end{cases} \\ &\quad + \mathcal{O}(\varepsilon^{2(L_0+1)/3}). \\ &= \begin{cases} \mathcal{O}(\varepsilon^{(2-3\beta)(M_0+1)+\beta/2}) + \mathcal{O}(\varepsilon) + \mathcal{O}(\varepsilon^{2(L_0+1)/3}) & \text{if } N_0 = 0 \\ \mathcal{O}(\varepsilon^{(2-3\beta)(M_0+1)+\beta/2}) + \mathcal{O}(\varepsilon^{5/3}) + \mathcal{O}(\varepsilon^{2(L_0+1)/3}) & \text{if } N_0 = 1 \\ \mathcal{O}(\varepsilon^{(2-3\beta)(M_0+1)+\beta/2}) + \mathcal{O}(\varepsilon^{\beta(N_0-3/2)+2}) + \mathcal{O}(\varepsilon^{2(L_0+1)/3}) & \text{if } N_0 \geqslant 2. \end{cases} \end{aligned} \tag{4.138}$$

The estimate on $\eta_1 - \eta_{1app}$ in (1.22) for $E = H^1(\mathbb{R}^d)$ follows from (4.117) and (4.131), the estimate on $\eta_2 - \eta_{2app}$ comes from (4.119) and (4.138).

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