Nonlinear attitude estimation from biased vector and gyro measurements

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Abstract—This paper considers the problem of attitude estimation for rigid bodies using measurements from a triaxial rate gyro and two other vector sensors (e.g. accelerometers, magnetometers). The novelty is to take into account biases not only on the gyro, but also on one of the vector measurements. The attitude estimation is achieved by a nonlinear, "geometry-free", observer. We also study the observability of the system and obtain conditions under which the reconstruction of both the attitude and the biases is possible. Under these persistency-of-excitation conditions, and through an explicit Lyapunov analysis, we then establish the global asymptotic and local exponential convergence of the observer. The theoretical results are illustrated by a thorough numerical simulation.

I. INTRODUCTION

Through the last decades a plethora of works have been devoted to the problem of the reconstruction of the orientation of a rigid body. This line of research has been further accelerated by the increasing technological developments and applications in deploying rigid bodies, such as quadrotors, satellites and aircraft. Depending on the vehicle and the mission specifications, a number of sensors are used, for example triaxial rate gyros, accelerometers, magnetometers or barometers, to provide information that can be exploited to reconstruct the vehicle orientation, or attitude, in three-dimensional space.

Even though attitude can be modeled in different ways, the usual parameterizations adopted in the literature are in terms of rotation matrices, quaternions or Euler angles, each with its advantages and drawbacks. Using these parameterizations, and when at least two vector measurements are available, it has been well established that a variety of algorithms can provide an estimate of the attitude, see [1]–[9]. While a large number of the past contributions involves optimization-based methods or stochastic filtering, the most recent approaches that hinge upon the design of nonlinear observers have been particularly successful as they can provide a large domain of convergence along with a simple gain tuning. Additionally, when the inherent geometry of the problem (e.g. SO(3)) is not incorporated in the observer design, as in [6]–[9], it is possible to avoid the topological obstacles of the other approaches and achieve global stability results.

The aforementioned works propose observers based on different methodologies, but share the same hypotheses on the measurements, that is they consider biased angular velocity signals and unbiased vector measurements. However,

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and especially with low-cost sensors, the reality is that biases affect also the vector observations. Although there are various EKF-based implementations that consider such biases, with a noticeable reference being [10] which discusses informally that observability requires rotational motion, to our knowledge, this is the first analytical work on nonlinear observability and observer design that tackles explicitly biases in both angular velocity and vector measurements.

The objective of this work is to study the additional presence of biases in the vector observations in the attitude estimation problem. We focus here on the case of one unbiased and one biased measurement. More precisely, under the assumption of one persistent (unbiased) vector measurement, we design a nonlinear, "geometry-free" observer that ensures global asymptotic stability (GAS) and local exponential stability (LES) of the errors of all the estimated variables (the gyro and vector biases, as well as the two directions necessary attitude reconstruction). The stability analysis relies on an explicit Lyapunov function, with the advantage of not requiring an explicit knowledge of the bound on the angular velocity, as is often done in the literature. The proposed observer builds upon previous works of the authors presented in [9] that considered only bias in the angular velocity measurements.

The paper runs as follows: the model used in the observer design is described in section II; an obervability analysis then follows in III; the observer is presented in section IV, and its convergence is proved; finally, section V illustrates the excellent behavior of the observer on a detailed simulation.

II. THE DESIGN MODEL

We consider a moving rigid body subjected to the angular velocity ω (in body axes). The rotation matrix (from body to inertial axes) $R \in \mathrm{SO}(3)$ defining its orientation is related to ω by

$$\dot{R} = R\omega_\times, \tag{1}$$

where the skew-symmetric matrix $\omega_\times$ is defined by $\omega_\times x := \omega \times x$ whatever the vector x.

The rigid body is equipped with a triaxal rate gyro measuring the angular velocity ω, and two additional triaxial sensors (for example accelerometers or magnetometers) providing the measurements of two vectors α and β. These two vectors correspond to the expression in body axes of two known independent vectors α_i and β_i which are constant in inertial axes. In other words, $\alpha := R^T\alpha_i$, $\beta := R^T\beta_i$. We readily find, since α_i, β_i are constant,

$$\dot{\alpha} = \alpha \times \omega$$
$$\dot{\beta} = \beta \times \omega.$$

As usual, we consider that the rate gyro is biased, and rather provides the measurement $\omega_m := \omega + b$, where b is a slowly-varying (for instance with temperature) unknown bias. In addition, we consider that the measurement of α is also biased while β is not, that is

$$\alpha_m := \alpha + b_\alpha \qquad (2)$$
$$\beta_m := \beta. \qquad (3)$$

The effect of these biases on attitude estimation may be important when the observer gains are small, hence it is worth estimating their values. But being not exactly constant, these cannot be calibrated offline and must be estimated online together with the attitude.

Our objective consists is the design of an estimator that can reconstruct online the orientation matrix $R(t)$ and the biases b, b_α, using i) the measurements of the gyro and of the two vector sensors; ii) the knowledge of the constant vectors α_i and β_i.

The model on which the design will be based therefore consists of the dynamics

$$\dot{\alpha} = \alpha \times \omega \qquad (4)$$
$$\dot{\beta} = \beta \times \omega \qquad (5)$$
$$\dot{b} = 0 \qquad (6)$$
$$\dot{b}_\alpha = 0 \qquad (7)$$

together with the measurements

$$\omega_m := \omega + b \qquad (8)$$
$$\alpha_m := \alpha + b_\alpha \qquad (9)$$
$$\beta_m := \beta. \qquad (10)$$

Notice (4)-(5) can be written in terms of the known signals α_m, ω_m as

$$\dot{\alpha}_m = (\alpha_m - b_\alpha) \times (\omega_m - b) \qquad (11)$$
$$\dot{\beta}_m = \beta_m \times (\omega_m - b). \qquad (12)$$

III. Observability analysis

Contrary to the case where two vector measurements are available without biases as e.g. in [9], the design model (4)–(7) with the two biases is not necessarily observable. Indeed, assuming ω sufficiently differentiable, we can write for all $k \geq 1$

$$\dot{\alpha}_m = \alpha_m \times \omega_m - \alpha_m \times b + b_\alpha \times b - b_\alpha \times \omega_m$$
$$\alpha_m^{(k+1)} = b \times \alpha_m^{(k)} - b_\alpha \times \omega^{(k)}$$
$$\qquad + F_k(\omega_m, \cdots, \omega_m^{(k)}, \alpha_m, \cdots, \alpha_m^{(k)})$$
$$\dot{\beta}_m = \beta_m \times \omega_m - \beta_m \times b$$
$$\beta_m^{(k+1)} = b \times \beta_m^{(k)}$$
$$\qquad + G_k(\omega_m, \cdots, \omega_m^{(k)}, \beta_m, \cdots, \beta_m^{(k)})$$

where F_k, G_k are some (polynomial) maps. Stacking these relations, we find

$$b \times \begin{pmatrix} \alpha_m & \dot{\alpha}_m & \cdots \end{pmatrix} - b_\alpha \times \begin{pmatrix} \omega_m - b & \dot{\omega}_m & \cdots \end{pmatrix}$$
$$= \begin{pmatrix} \dot{\alpha}_m & \ddot{\alpha}_m & \cdots \end{pmatrix} + \mathcal{F}(\omega_m, \dot{\omega}_m, \cdots, \alpha_m, \dot{\alpha}_m, \cdots)$$
$$b \times \begin{pmatrix} \beta_m & \dot{\beta}_m & \ddot{\beta}_m & \cdots \end{pmatrix} = \begin{pmatrix} \dot{\beta}_m & \ddot{\beta}_m & \cdots \end{pmatrix}$$
$$\qquad + \mathcal{G}(\omega_m, \dot{\omega}_m, \cdots, \beta_m, \dot{\beta}_m, \cdots).$$

It is clear that the biases will be observable if they can be expressed in terms of α_m, β_m, ω_m and their derivatives. From the second relation we can observe for example that b is observable if and only if the matrix $\begin{pmatrix} \beta_m & \dot{\beta}_m & \ddot{\beta}_m & \cdots \end{pmatrix}$ is of rank 2. Then, from the first relation we have that b_α is observable if and only if the matrix $\begin{pmatrix} \omega & \dot{\omega} & \cdots \end{pmatrix}$ is also of rank 2.

To design an observer for this scenario, we will consider the following slightly stronger property than observability, namely persistence of excitation (PE) for $\beta(t)$ and $\omega(t)$.

Assumption 1: There exist some constants $T, \mu > 0$, such that for all $t \geq 0$,

$$-\frac{1}{T} \int_t^{t+T} \beta_\times^2(\sigma) d\sigma \geq \mu I. \qquad (13)$$

Similarly, there exist some constants $T_\omega, \mu_\omega > 0$, such that for all $t \geq 0$,

$$-\frac{1}{T_\omega} \int_t^{t+T_\omega} \omega_\times^2(\sigma) d\sigma \geq \mu_\omega I. \qquad (14)$$

Remark 1: Let us note that the two aforementioned PE conditions are intimately related. This can be seen by observing that in (5) ω acts as input and thus, affects the behavior of β and $\dot{\beta}$. Hence, it seems that in practical scenarios essentially only the PE condition of ω should be required.

IV. The observer

In this section, we show that the state of the design system (4)–(7) can be estimated by the observer

$$\dot{\hat{\alpha}}_m := (\hat{\alpha}_m - \hat{b}_\alpha) \times (\omega_m - \hat{b}) - k_\alpha(\hat{\alpha}_m - \alpha_m)$$
$$\dot{\hat{\beta}}_m := \hat{\beta}_m \times (\omega_m - \hat{b}_m) - k_\beta(\hat{\beta}_m - \beta_m)$$
$$\dot{\hat{b}} := l_\beta \hat{\beta}_m \times \beta_m$$
$$\dot{\hat{b}}_\alpha := m_\alpha(\omega_m - \hat{b}) \times (\hat{\alpha}_m - \alpha_m),$$

where $k_\alpha, k_\beta, l_\beta, m_\alpha$ are strictly positive constants.

Let us define the errors $e_\alpha := \hat{\alpha}_m - \alpha_m$, $e_\beta := \hat{\beta}_m - \beta_m$, $e_{b\alpha} := \hat{b}_\alpha - b_\alpha$, $e_b := \hat{b} - b$. Then, the error dynamics gives

$$\dot{e}_\alpha = (e_\alpha - e_{b\alpha}) \times \omega - k_\alpha e_\alpha - (e_\alpha - e_{b\alpha} + \alpha) \times e_b \quad (15)$$
$$\dot{e}_{b\alpha} = m_\alpha(\omega - e_b) \times e_\alpha \qquad (16)$$
$$\dot{e}_\beta = e_\beta \times \omega - k_\beta e_\beta - \hat{\beta}_m \times e_b \qquad (17)$$
$$\dot{e}_b = l_\beta e_\beta \times \hat{\beta}_m \qquad (18)$$

We can now state our main result.

Theorem 1: Assume $k_\alpha, l_\alpha, l_\beta, m_\alpha > 0$ and ω, $\dot{\omega}$ bounded. Then, provided the persistent excitation condition (13) holds, the origin of the error system (15)-(18) is globally (locally) asymptotically (exponentially) stable.

Proof: First, for the subsystem $(e_\alpha, e_{b\alpha})$ consider the candidate Lyapunov function $V_0(e_\alpha, e_{b\alpha}) = \frac{1}{2}(l_\alpha |e_\alpha|^2 + \frac{l_\alpha}{m_\alpha}|e_{b\alpha}|^2)$. After straightforward calculations, and using the property of the scalar triple product $\langle x, y \times z \rangle = \langle y, z \times x \rangle = \langle z, x \times y \rangle$ for any $x, y, z \in \mathbb{R}^3$, along with Young's inequality and the fact that $|\alpha| = 1$, its time-derivative along trajectories of the error dynamics gives

$$\dot{V}_0 = -l_\alpha \langle e_\alpha, k_\alpha e_\alpha \rangle - l_\alpha \langle e_\alpha, e_b \times e_{b\alpha} \rangle - l_\alpha \langle e_\alpha, \alpha \times e_b \rangle$$
$$- l_\alpha \langle e_\alpha, e_{b\alpha} \times \omega \rangle + l_\alpha \langle e_{b\alpha}, \omega \times e_\alpha \rangle - l_\alpha \langle e_{b\alpha}, e_b \times e_\alpha \rangle$$
$$\leq -l_\alpha (k_\alpha - \epsilon)|e_\alpha|^2 + \frac{l_\alpha}{4\epsilon}|e_b|^2.$$

For the second subsystem $(e_\beta, e_{b\beta})$ we can construct a strict Lyapunov function, under the assumption of a persistently exciting $\beta(t)$, as follows.

Consider the candidate Lyapunov function $V := \sigma_1 V_1 + \sigma_2 V_1^2 + V_3 + V_4$ where the coefficients $\sigma_1, \sigma_2 > 0$ are yet to be defined, and

$$V_1(e_\beta, e_b) := \frac{1}{2}|e_\beta|^2 + \frac{1}{2l_\beta}|e_b|^2$$

$$V_3(e_\beta, e_b) := \frac{1}{2}\left| \beta \times e_\beta - \frac{k_\beta}{l_\beta}e_b \right|^2$$

$$V_4(e_b, t) := \frac{k_\beta}{l_\beta}\langle e_b, \Psi(t)e_b \rangle,$$

$\Psi(t)$ is the 3×3 symmetric matrix defined by $\Psi(t) := (1 + c_\beta^2)TI + \frac{1}{T}\int_t^{t+T}\int_t^s \beta_\times^2(\tau)d\tau ds$, where $c_\beta \geq |\beta_\times| = |\beta_{i\times}|$ for the matrix norm induced by the 2-norm on $\mathbb{R}^3$. The form of V_4 is inspired by the construction of strict Lyapunov functions for persistently excited time-varying systems in [11], [12, p. 288]; notice $\Psi(t)$ satisfies

$$TI \preceq \Psi(t) \preceq (1 + c_\beta^2)TI. \qquad (19)$$

Clearly, V is positive definite and radially unbounded. We next compute the derivatives of its pieces along the trajectories of the error system (17)–(18). First, $\dot{V}_1 = -k_\beta |e_\beta|^2$, where we have used $\langle x, x \times y \rangle = 0$ and $\langle x, y \times z \rangle = \langle z, x \times y \rangle$.

$$\frac{d}{dt}V_1^2 = -k_\beta |e_\beta|^4 - \frac{k_\beta}{l_\beta}|e_\beta|^2 |e_b|^2$$

$$\dot{V}_3 = \langle \frac{k_\beta}{l_\beta}e_b - \beta \times e_\beta, \beta_\times^2 e_b + \beta \times (e_\beta \times e_b)$$
$$+ (e_\beta \times \beta) \times \omega \rangle$$
$$\leq \frac{k_\beta}{l_\beta}\langle e_b, \beta_\times^2 e_b \rangle + \frac{k_\beta}{l_\beta}|e_\beta||e_b|^2 + |e_\beta|^2|e_b|$$
$$+ \left(1 + \frac{k_\beta}{l_\beta}c_\omega\right)|e_\beta||e_b|$$
$$\leq \frac{k_\beta}{l_\beta}\langle e_b, \beta_\times^2 e_b \rangle + \frac{3\epsilon}{2}|e_b|^2 + \frac{1}{2\epsilon}\left(\frac{k_\beta}{l_\beta}|e_\beta||e_b|\right)^2$$
$$+ \frac{1}{2\epsilon}|e_\beta|^4 + \frac{1}{2\epsilon}\left(1 + \frac{k_\beta}{l_\beta}c_\omega\right)^2|e_\beta|^2;$$

for the first line of $\dot{V}_3$, we have used $(x \times y) \times z + (y \times z) \times x + (z \times x) \times y = 0$; for the second line, $\langle x, x \times y \rangle = 0$

and $|\beta| = |\beta_i| = 1$; for the third line, three times Young's inequality $xy \leq \frac{\epsilon x^2}{2} + \frac{y^2}{2\epsilon}$. Finally,

$$\dot{\Psi}(t) = \frac{1}{T}\int_t^{t+T}\beta_\times^2(\tau)d\tau - \beta_\times^2(t)$$

$$\dot{V}_4 = \frac{k_\beta}{l_\beta}\langle e_b, \frac{1}{T}\int_t^{t+T}\beta_\times^2(\tau)d\tau\, e_b \rangle$$
$$+ 2k_\beta \langle e_\beta \times \beta, \Psi(t)e_b \rangle - \frac{k_\beta}{l_\beta}\langle e_b, \beta_\times^2 e_b \rangle$$
$$\leq 2k_\beta(1 + c_\beta^2)T\,|e_\beta|\,|e_b| - \frac{k_\beta}{l_\beta}\mu\,|e_b|^2 - \frac{k_\beta}{l_\beta}\langle e_b, \beta_\times^2 e_b \rangle$$
$$\leq \frac{2}{\epsilon}k_\beta^2(1 + c_\beta^2)^2T^2\,|e_\beta|^2 - \left(\frac{k_\beta}{l_\beta}\mu - \frac{\epsilon}{2}\right)|e_b|^2$$
$$- \frac{k_\beta}{l_\beta}\langle e_b, \beta_\times^2 e_b \rangle;$$

for the second line of $\dot{V}_4$, we have used (19), $|\beta| = 1$, and assumption (13); for the third line, Young's inequality.

Collecting all the pieces, we eventually find

$$\dot{V} \leq -\mu'\,|e_b|^2 - \sigma_1'\,|e_\beta|^2 - \sigma_2'\,|e_\beta|^4 - \sigma_2''\,|e_\beta|^2\,|e_b|^2,$$

with $\sigma_1' := \sigma_1 k_\beta - \frac{1}{2\epsilon}\left(1 + \frac{k_\beta}{l_\beta}c_\omega\right)^2 - \frac{2}{\epsilon}k_\beta^2(1 + c_\beta^2)^2T^2\,|e_\beta|^2$, $\mu' := \frac{k_\beta}{l_\beta}\mu - \frac{\epsilon}{2}$, $\sigma_2' := \sigma_2 k_\beta - \frac{1}{2\epsilon}$, $\sigma_2'' := \sigma_2 - \frac{1}{2\epsilon}\frac{k_\beta}{l_\beta}$. All the above coefficients are strictly positive since ϵ and σ_1, σ_2 can be freely chosen, infinitely small and large respectively. As such V is a strict Lyapunov function, proving uniform global asymptotic stability of the equilibrium $(e_\beta, e_b) = (0, 0)$. Notice the bound c_ω need not be known, since σ_1 can always been chosen large enough to achieve $\sigma_1' > 0$.

Now, taking the time-derivative of the composite Lyapunov function $W := V_0(e_\alpha, e_{b\alpha}) + \sigma_3 V(e_\beta, e_b, t)$, with $\sigma_3 > 0$ to be defined, results in

$$\dot{W} \leq -\left(\sigma_3\mu' - \frac{l_\alpha}{4\epsilon}\right)|e_b|^2 - l_\alpha(k_\alpha - \epsilon)|e_\alpha|^2 - \sigma_1'\,|e_\beta|^2$$
$$= -\sigma_3'|e_b|^2 - \sigma_4|e_\alpha|^2 - \sigma_1'\,|e_\beta|^2$$

where $\sigma_3' := \sigma_3\mu' - \frac{l_\alpha}{4\epsilon}$, $\sigma_4 := l_\alpha(k_\alpha - \epsilon)$. The aforementioned coefficients are strictly positive as ϵ is chosen small enough and σ_3 is chosen large enough.

Since $W(t)$ is bounded from below and $\dot{W} \leq 0$, $W(t)$ reaches a finite limit as $t \to \infty$. On the other hand, $\ddot{W}$ is a polynomial in terms of $e_\alpha, e_\beta, e_b, e_{b\alpha}, \omega(t), \dot{\omega}(t)$; as W is bounded, since $\omega(t), \dot{\omega}(t)$ are also assumed bounded, so are $e_\alpha, e_\beta, e_b, e_{b\alpha}$, hence $\ddot{W}$. With W having a finite limit and $\dot{W}$ begin uniformly continuous, due to boundedness of $\ddot{W}$, we can conclude by Barbalat's lemma that $\dot{W}(t)$ tends to 0 as $t \to \infty$, and so do $e_\alpha(t), e_\beta(t), e_b(t)$.

The components of $\ddot{e}_\alpha, \ddot{e}_\beta, \ddot{e}_{b\alpha}$ are polynomials in terms of $e_\alpha, e_\beta, e_b, e_{b\alpha}$; as $e_\alpha, e_\beta, e_b, e_{b\alpha}, \omega$ are bounded, so are $\ddot{e}_\alpha, \ddot{e}_\beta, \ddot{e}_{b\alpha}$,. Now, $e_\alpha(t), e_\beta(t), e_b(t)$ have a finite limit, namely 0, and $\dot{e}_\alpha, \dot{e}_\beta, \dot{e}_b$ are uniformly continuous (since $\ddot{e}_\alpha, \ddot{e}_\beta, \ddot{e}_b$ are bounded): by Barbalat's lemma $\dot{e}_\alpha(t), \dot{e}_\beta(t), \dot{e}_b(t)$ tend to 0 as $t \to \infty$. Using the dynamics, this implies that $-e_{b\alpha}(t) \times \omega(t) \to 0$.

Proceeding similarly for higher derivatives of e_α, and putting everything together finally yields

$-e_{b\alpha}(t) \times \left(\omega(t)\dot{\omega}(t) \quad \ddot{\omega}(t) \cdots\right) \to 0$. Under persistence of ω the observability condition holds and hence, obtain $e_{b\alpha}(t) \times \omega(t) \to 0$. This concludes the GAS claim.

The LES claim can be established as follows. First, consider the first-order approximation of (17)-(18) around the equilibrium point $(\bar{e}_\beta, \bar{e}_b) := (0,0)$; it reads

$$\delta\dot{e}_\beta = \delta e_\beta \times \omega - \beta \times \delta e_b - k_\beta \delta e_\beta \qquad (20)$$

$$\delta\dot{e}_b = -l_\beta \beta \times \delta e_\beta. \qquad (21)$$

This is a linear time-varying system (LTV) in the skew-symmetric form of [13], that satisfies the conditions of Theorem 1 of [13] with $A(t) := -k_\beta I - \omega_\times(t), B(t) := \beta_\times(t), C(t) := l_\beta \beta_\times(t), P := l_\beta I$ for $\beta(t)$ persistently-exciting and $\beta(t), \dot{\beta}(t)$ bounded. Thus, we can conclude local exponential stability.

Now, let us consider the first-order approximation of (15)-(16) around the equilibrium point $(\bar{e}_\alpha, \bar{e}_{b\alpha}) := (0,0)$, that yields

$$\delta\dot{e}_\alpha = \delta e_\alpha \times \omega - k_\alpha \delta e_\alpha - \alpha \times \delta e_b \qquad (22)$$

$$\delta\dot{e}_{b\alpha} = -m_\alpha \omega \times \delta e_\alpha. \qquad (23)$$

This is again an LTV system in the skew-symmetric form but with an additional linear term, $\alpha \times \delta e_b$. Since the nominal system, i.e. for $\delta e_b = 0$, with $A(t) := -k_\alpha I - \omega_\times(t), B(t) := \omega_\times(t), C(t) := m_\alpha \omega_\times(t), P := m_\alpha I$, $\omega(t)$ persistently-exciting and $\omega(t), \dot{\omega}(t)$ bounded, is locally exponentially stable and the additional term satisfies a linear growth condition, from standard arguments on cascaded time-varying systems (e.g Proposition 2.3 of [14]) we can conclude LES of the origin of (20)-(23). ∎

Finally, we will be able to obtain the true orientation matrix R by using the above estimates and the knowledge of α_i and β_i.

Corollary 1: Under the assumptions of Theorem 1, the matrix $\tilde{R}$ defined by

$$\tilde{R}^T := \left(\frac{\hat{\alpha}}{|\alpha_i|} \quad \frac{\hat{\alpha} \times \hat{\beta}}{|\alpha_i \times \beta_i|} \quad \frac{\hat{\alpha} \times (\hat{\alpha} \times \hat{\beta})}{|\alpha_i \times (\alpha_i \times \beta_i)|} \right) \cdot R_i^T$$

$$R_i := \left(\frac{\alpha_i}{|\alpha_i|} \quad \frac{\alpha_i \times \beta_i}{|\alpha_i \times \beta_i|} \quad \frac{\alpha_i \times (\alpha_i \times \beta_i)}{|\alpha_i \times (\alpha_i \times \beta_i)|} \right)$$

globally converges to R.

Proof: By Theorem 1, $\hat{\alpha} \to \alpha$ and $\hat{\beta} \to \beta$. Hence,

$$\tilde{R}^T \to \left(\frac{\alpha}{|\alpha_i|} \quad \frac{\alpha \times \beta}{|\alpha_i \times \beta_i|} \quad \frac{\alpha \times (\alpha \times \beta)}{|\alpha_i \times (\alpha_i \times \beta_i)|} \right) \cdot R_i^T$$

$$= \left(\frac{R^T \alpha_i}{|\alpha_i|} \quad \frac{R^T \alpha_i \times R^T \beta_i}{|\alpha_i \times \beta_i|} \quad \frac{R^T \alpha_i \times (R^T \alpha_i \times R^T \beta_i)}{|\alpha_i \times (\alpha_i \times \beta_i)|} \right) \cdot R_i^T$$

$$= R^T R_i R_i^T$$

$$= R^T,$$

where we have used $R^T(u \times v) = R^T u \times R^T v$ since R is a rotation matrix. ∎

V. SIMULATIONS

The good behavior of the observer is now illustrated in simulation. All the units in this section are SI units (m,s,rad,...). The constant vectors α_i and β_i are respectively $(1,0,0)^T$ and $(0,0,1)^T$.
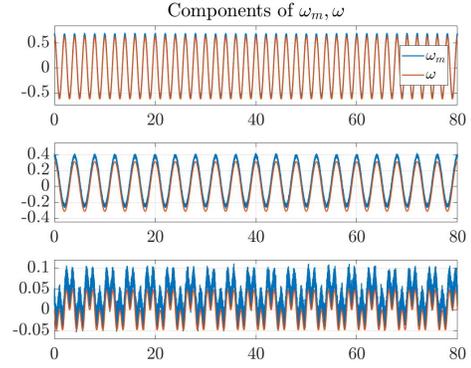

Fig. 1. Components of true ω (red) and measured ω_m (blue), in $\mathrm{rad\,s^{-1}}$.
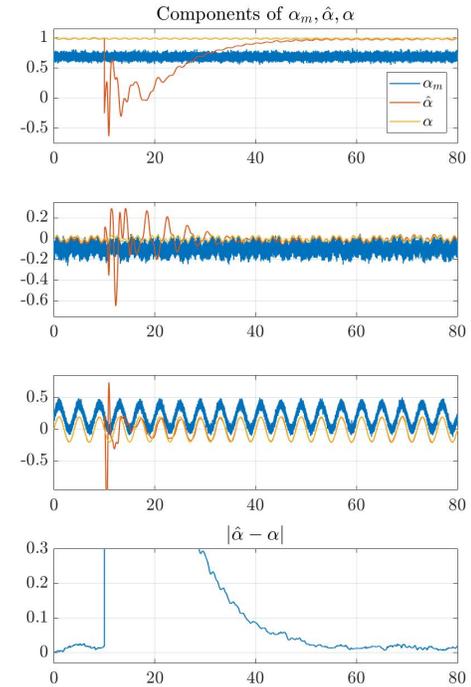

Fig. 2. Components of true α (red), measured α_m (blue) and estimated $\hat{\alpha}$ (orange).

The main practical problem is to enforce the persistent excitation condition (13), which may not hold for the desired trajectory to be followed. The idea is then to suitably modify this desired trajectory by oscillating around it. We illustrate the approach on a very simple situation, namely hovering, for which (13) is clearly not satisfied; indeed, $\beta(t) = \beta_i$ and $\omega(t) = 0$ for all t. Instead, we consider approximate hovering with $\beta(t) \approx \beta_i$ and $\omega(t) \approx 0$. In terms of the usual Euler angles ϕ, θ, ψ, this can be achieved by choosing: $\phi(t) \ll 1$ and T-periodic; $\theta(t) \ll 1$ and $\frac{T}{k}$-periodic for some positive integer k; ψ as prescribed by the desired trajectory (no approximation is needed), i.e., $\psi(t) = 0$ in our case.
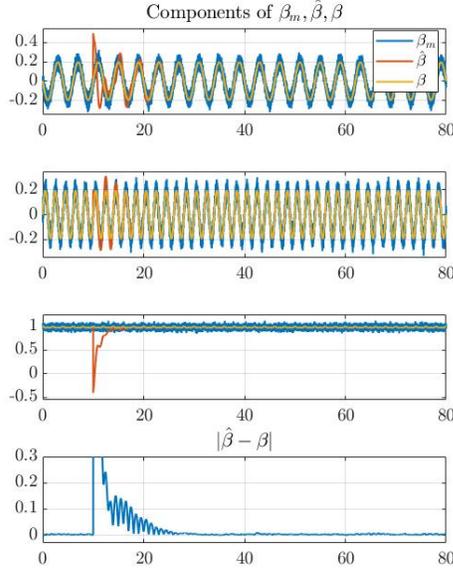
Fig. 3. Components of true β (red), measured β_m (blue) and estimated $\hat\beta$ (orange).
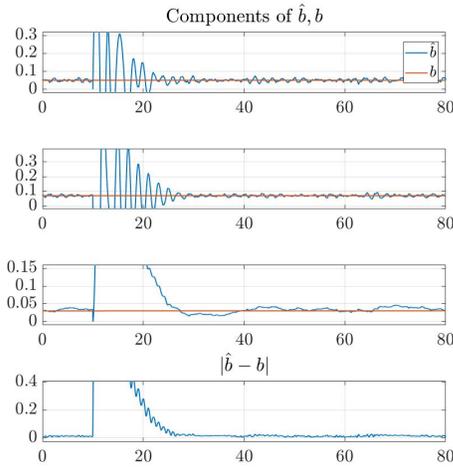

Fig. 4. Components of true b (red) and estimated $\hat b$ (blue).

Indeed, $\beta = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix} = \begin{pmatrix} -\sin\theta \\ \sin\phi\cos\theta \\ \cos\phi\cos\theta \end{pmatrix} \approx \begin{pmatrix} -\theta \\ \phi\theta \\ 1 \end{pmatrix} \approx \beta_i$. The angular velocity ω realizing the approximated hovering is then obtained by inverting the kinematic relation

$$\begin{pmatrix} \dot\phi\cos\theta \\ \dot\theta \\ \dot\psi\cos\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\phi\cos\theta & \cos\phi\cos\theta \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{pmatrix}\omega,$$

which is (1) expressed in terms of the Euler angles. Then,
$$-\beta_\times^2 \approx \begin{pmatrix} 1 & \phi\theta & \theta \\ \phi\theta & 1 & -\phi \\ \theta & -\phi & \theta^2+\phi^2 \end{pmatrix}.$$ Integrating over the period
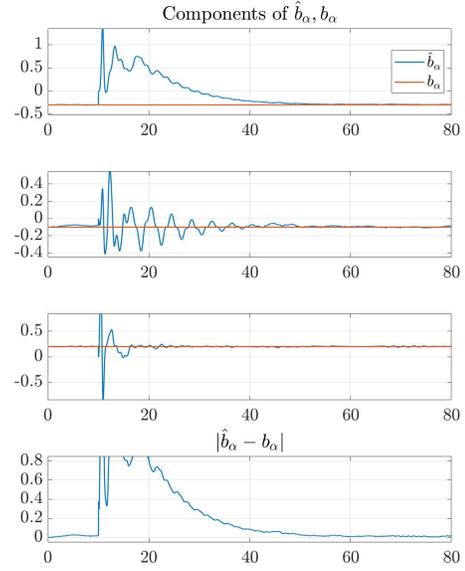

Fig. 5. Components of true b_α (red) and estimated $\hat b_\alpha$ (blue).

T yields

$$-\frac{1}{T}\int_t^{t+T}\beta_\times^2(\tau)d\tau \approx \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{T}\int(\theta^2+\phi^2) \end{pmatrix},\ (24)$$

the terms $\int\phi\theta \ll 1$ are not strictly zero, but can be neglected since they contribute little to the matrix positivity. In the simulation, we choose $\phi(t) := 0.2\sin(2\pi\frac{t}{2})$ and $\theta(t) := 0.2\sin(2\pi\frac{t}{4})$, which gives $\frac{1}{T}\int(\theta^2+\phi^2) = 0.08$; (13) is then satisfied with $T = 2$ and $\mu \approx 0.08$.

The system starts in the initial state $R(0) := $ I, i.e. $\big(\phi(0),\theta(0),\psi(0)\big) := (0,0,0)$, and then undergoes the angular velocity $\omega(t)$ displayed in Fig. 1. The observer is fed with the measured signals ω_m, α_m and β_m, see Fig. 1 to 3. The measured angular velocity ω_m is affected by the (unknown) slowly drifting bias b, see Fig. 1 and 4; though this is hardly visible on the figures, b does vary from $(5,7,3)^T \times 10^{-2}$ to $(5.15,7.15,3.15)^T \times 10^{-2}$. The measured vector α_m is affected by the (unknown) constant bias $b_\alpha = (-0.3,-0.1,0.2)^T$. All the measurement signals are corrupted by band-limited independent gaussian white noises (sample time 10^{-3}, noise powers 10^{-6} for the components of α_m, β_m and 10^{-7} for those of ω_m). The characteristics of the biases and noises are representative of MEMS sensors. Finally, the tuning gains are set to $(k_\alpha, m_\alpha, k_\beta, l_\beta) := (2,10,1,10)$. The observer is initialized with no error, but suddenly reinitialized to very different values at $t = 10$. As expected, the estimated quantities $\hat\alpha$, $\hat\beta$, $\hat b$ and $\hat b_\alpha$ converge to their true values after the reinitialization, see Fig. 2 to 5. Fig. 6-7-8 show the reconstruction of the Euler angles ϕ, θ, ψ from the estimated vectors $\hat\alpha, \hat\beta$. The trajectory is indeed an approximation of hovering.

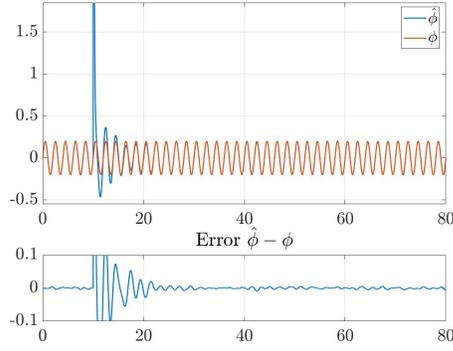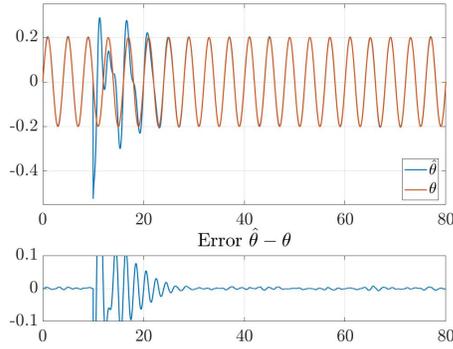Fig. 6. True ϕ (red) and estimated $\hat{\phi}$ (blue), in rad.

Fig. 8. True ψ (red) and estimated $\hat{\psi}$ (blue), in rad.

Fig. 7. True θ (red) and estimated $\hat{\theta}$ (blue), in rad.

VI. CONCLUSION

We have presented a simple nonlinear "geometry-free" observer for attitude, vector bias and gyro bias estimation. This seems to be the first work considering both a gyro bias and a vector bias in the attitude estimation. Through a Lyapunov analysis, the designed estimator is shown to guarantee global asymptotic convergence and local exponential convergence. In addition, simulations demonstrate that it performs very well, even in the presence of noise and slowly-varying biases.

The case of two vector biases b_α and b_β also seems tractable: though we have so far not been able to provide a convergence proof, we have checked in simulation the good behavior of the observer

$$\dot{\hat{\alpha}}_m = (\hat{\alpha}_m - \hat{b}_\alpha) \times (\omega_m - \hat{b}) - k_\alpha(\hat{\alpha}_m - \alpha_m)$$
$$\dot{\hat{\beta}}_m = (\hat{\beta}_m - \hat{b}_\beta) \times (\omega_m - \hat{b}) - k_\beta(\hat{\beta}_m - \beta_m)$$
$$\dot{\hat{b}} = l_\alpha \hat{\alpha}_m \times \alpha_m + l_\beta \hat{\beta}_m \times \beta_m$$
$$\dot{\hat{b}}_\alpha = m_\alpha(\omega_m - \hat{b}) \times (\hat{\alpha}_m - \alpha_m)$$
$$\dot{\hat{b}}_\beta = m_\beta(\omega_m - \hat{b}) \times (\hat{\beta}_m - \beta_m),$$

where the positive constants $k_\alpha, k_\beta, l_\alpha, l_\beta, m_\alpha, m_\beta$ are tuning gains. This will be the focus of future work.
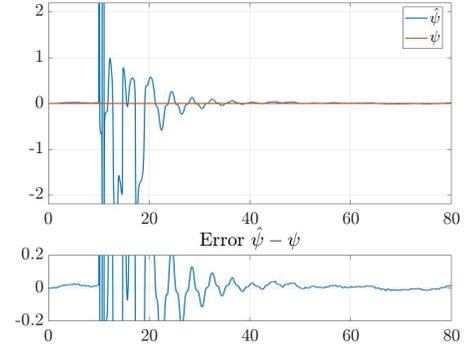
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