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► **To cite this version:**

Bénédicte Alziary, Jacqueline Fleckinger. Blow up of the solutions to a linear elliptic system involving Schrödinger operators. Publicaciones del Seminario Matematico Garcia Galdeano, 2018. hal-01960581

**HAL Id: hal-01960581**

**<https://hal.science/hal-01960581>**

Submitted on 10 Jan 2019

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# Blow up of the solutions to a linear elliptic system involving Schrödinger operators

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**Abstract** We show how the solutions to a  $2 \times 2$  linear system involving Schrödinger operators blow up as the parameter  $\mu$  tends to some critical value which is the principal eigenvalue of the system; here the potential is continuous positive with superquadratic growth and the square matrix of the system is with constant coefficients and may have a double eigenvalue.

## 1 Introduction

We study here the behavior of the solutions to a  $2 \times 2$  system (considered in its variational formulation):

$$(S) \quad LU := (-\Delta + q(x))U = AU + \mu U + F(x) \text{ in } \mathbb{R}^N,$$

$$U(x)|_{|x| \rightarrow \infty} \rightarrow 0$$

where  $q$  is a continuous positive potential tending to  $+\infty$  at infinity with superquadratic growth;  $U$  is a column vector with components  $u_1$  and  $u_2$  and  $A$  is a  $2 \times 2$  square matrix with constant coefficients.  $F$  is a column vector with components  $f_1$  and  $f_2$ .

Such systems have been intensively studied mainly for  $\mu = 0$  and for  $A$  with 2 distinct eigenvalues; here we consider also the case of a double eigenvalue. In both cases, we show the blow up of solutions as  $\mu$  tends to some critical value  $\nu$  which is the principal eigenvalue of System (S). This extends to systems involving Schrödinger operators defined on  $\mathbb{R}^N$  earlier results valid for systems involving the classical Laplacian defined on smooth bounded domains with Dirichlet boundary conditions.

This paper is organized as follows: In Section 2 we recall known results for one equation. In Section 3 we consider first the case where  $A$  has two different eigenvalues and then we study the case of a double eigenvalue.

## 2 The equation

We shortly recall the case of one equation

$$(E) \quad Lu := (-\Delta + q(x))u = \sigma u + f(x) \in \mathbb{R}^N,$$

$$\lim_{|x| \Rightarrow +\infty} u(x) = 0.$$

$\sigma$  is a real parameter.

### Hypotheses

$(H_q)$   $q$  is a positive continuous potential tending to  $+\infty$  at infinity.

$(H_f)$   $f \in L^2(\mathbb{R}^N)$ ,  $f \geq 0$  and  $f > 0$  on some subset with positive Lebesgue measure.

It is well known that if  $(H_q)$  is satisfied,  $L$  possesses an infinity of eigenvalues tending to  $+\infty$ :  $0 < \lambda_1 < \lambda_2 \leq \dots$ .

**Notation:**  $(\Lambda, \phi)$  Denote by  $\Lambda$  the smallest eigenvalue of  $L$ ; it is positive and simple and denote by  $\phi$  the associated eigenfunction, positive and with  $L^2$ -norm  $\|\phi\| = 1$ .

It is classical ([15], [19]) that if  $f > 0$  and  $\sigma < \Lambda$  the positivity is improved, or in other words, the maximum principle (**MP**) is satisfied:

$$(MP) \quad f \geq 0, \neq 0 \Rightarrow u > 0.$$

Lately, for potentials growing fast enough (faster than the harmonic oscillator), another notion has been introduced ([4], [5], [9], [10]) which improves the maximum (or antimaximum principle): the "groundstate positivity" (**GSP**) (resp. "negativity" (**GSN**)) which means that there exists  $k > 0$  such that

$$u > k\phi \text{ (GSP) (resp. } u < -k\phi \text{ (GSN))}.$$

We also say shortly "fundamental positivity" or "negativity", or also " $\phi$ -positivity" or " $\phi$ -negativity".

The first steps in this direction use a radial potential. Here we consider a small perturbation of a radial one as in [9].

**The potential  $q$**  We define first a class  $\mathcal{P}$  of radial potentials:

$$\mathcal{P} := \{Q \in \mathcal{C}(\mathbb{R}_+, (0, \infty)) / \exists R_0 > 0, Q' > 0 \text{ a.e. on } [R_0, \infty), \int_{R_0}^{\infty} Q(r)^{-1/2} < \infty\}.$$

(1)

The last inequality holds if  $Q$  is growing sufficiently fast ( $> r^2$ ). Now we give results of GSP or GSN for a potential  $q$  which is a small perturbation of  $Q$ ; we assume:

$(H'_q)$   $q$  satisfies  $(H_q)$  and there exists two functions  $Q_1$  and  $Q_2$  in  $\mathcal{P}$ , and two positive constants  $R_0$  and  $C_0$  such that

$$Q_1(|x|) \leq q(x) \leq Q_2(|x|) \leq C_0 Q_1(|x|), \quad \forall x \in \mathbb{R}^N, \quad (2)$$

$$\int_{R_0}^{\infty} (Q_2(s) - Q_1(s)) \int_{R_0}^s \exp\left(-\int_r^s [Q_1(t)^{1/2} + Q_2(t)^{1/2}] dt\right) dr ds < \infty. \quad (3)$$

Denoting by  $\Phi_1$  (resp.  $\Phi_2$ ) the groundstate of  $L_1 := -\Delta + Q_1$  (resp.  $L_2 = -\Delta + Q_2$ ), Corollary 3.3 in [9] says that all these groundstates are "comparable" that is there exists constants  $0 < k_1 \leq k_2 \leq \infty$  such that  $k_1 \phi \leq \Phi_1, \Phi_2 \leq k_2 \phi$ . Finally

**Theorem 1** (GSP) ([9]) *If  $(H'_q)$  and  $(H_f)$  are satisfied, then, for  $\sigma < \Lambda$ , there is a unique solution  $u$  to  $(E)$  which is positive, and there exists a constant  $c > 0$ , such that*

$$u > c\phi. \quad (4)$$

Moreover, if also  $f \leq C\phi$  with some constant  $C > 0$ , then

$$u \leq \frac{C}{\Lambda - \sigma} \phi. \quad (5)$$

**The space  $\mathcal{X}$**  It is convenient for several results to introduce the space of "groundstate bounded functions":

$$\mathcal{X} := \{h \in L^2(\mathbb{R}^N) : h/\phi \in L^\infty(\mathbb{R}^N)\}, \quad (6)$$

equipped with the norm  $\|h\|_{\mathcal{X}} = \text{ess sup}_{\mathbb{R}^N} (|h|/\phi)$ .

For a potential satisfying  $(H'_q)$  and a function  $f \in \mathcal{X}$ , there is also a result of "groundstate negativity" (**GSN**) for  $(E)$ ; it is an extension of the antimaximum principle, introduced by Clément and Peletier in 1978 ([13]) for the Laplacian when the parameter  $\sigma$  crosses  $\Lambda$ .

**Theorem 2** (GSN) ([9]) *Assume  $(H'_q)$  and  $(H_f)$  are satisfied and  $f \in \mathcal{X}$ ; then there exists  $\delta(f) > 0$  and a positive constant  $c' > 0$  such that for all  $\sigma \in (\Lambda, \Lambda + \delta)$ ,*

$$u \leq -c'\phi. \quad (7)$$

**Remark 1** *This holds also if we only assume  $f^1 := \int f\phi > 0$ .*

**Hypothesis** ( $H'_f$ ) We consider now functions  $f$  which are such that ( $H'_f$ ):  $f \in \mathcal{X}$  and  $f^1 := \int f\phi > 0$ .

**Theorem 3** Assume ( $H'_q$ ) and ( $H'_f$ ) are satisfied. Then there exists  $\delta > 0$  such that for  $\Lambda - \delta < \sigma < \Lambda$  there exists positive constants  $k'$  and  $K'$ , depending on  $f$  and  $\delta$  such that

$$0 < \frac{k'}{\Lambda - \sigma}\phi < u < \frac{K'}{\Lambda - \sigma}\phi. \quad (8)$$

If  $\Lambda < \sigma < \Lambda + \delta$ , there exists positive constants  $k''$  and  $K''$ , depending on  $f$  and  $\delta$  such that

$$-\frac{k''}{\Lambda - \sigma}\phi < -u < -\frac{K''}{\Lambda - \sigma}\phi < 0. \quad (9)$$

This result extends earlier one in [17] and a close result is Theorem 2.03 in [11]. It shows in particular that  $u \in \mathcal{X}$  and  $|u| \rightarrow \infty$  as  $|\nu - \mu| \rightarrow 0$ .

**Proof:** Decompose  $u$  and  $f$  on  $\phi$  and its orthogonal:

$$u = u^1\phi + u^\perp; \quad f = f^1\phi + f^\perp. \quad (10)$$

We derive from (E)  $Lu = \sigma u + f$  :

$$Lu^\perp = \sigma u^\perp + f^\perp \quad (11)$$

$$Lu^1\phi = \Lambda u^1\phi = \sigma u^1\phi + f^1\phi. \quad (12)$$

We notice that since  $q$  is smooth; so is  $u$ . Also, since  $f \in \mathcal{X}$ ,  $f^\perp$ ,  $u$  and  $u^\perp$  are also in  $\mathcal{X}$  and hence are bounded. Choose  $\sigma < \Lambda$  and assume ( $H'_f$ ). We derive from Equation (11) (by [6]Thm 3.2) that :  $\|u^\perp\|_{\mathcal{X}} < K_1$ . Therefore  $|u^\perp|$  is bounded by some  $cste.\phi > 0$ .

From Equation (12) we derive

$$u^1 = \frac{f^1}{(\Lambda - \sigma)} \rightarrow \pm\infty \text{ as } (\Lambda - \sigma) \rightarrow 0. \quad (13)$$

Choose  $\Lambda - \delta$  small enough and  $\sigma \in (\Lambda - \delta, \Lambda)$ . Hence

$$\frac{K'}{\Lambda - \sigma}\phi < u; \quad u < \frac{K''}{\Lambda - \sigma}\phi.$$

For  $\sigma > \Lambda$ . we do exactly the same, except that the signs are changed for  $u^1$  in (13).

### 3 A $2 \times 2$ Linear system

Consider now a linear system with constant coefficients.

$$(S) \quad LU = AU + \mu U + F(x) \text{ in } \mathbb{R}^N.$$

As above,  $L := -\Delta + q$  where the potential  $q$  satisfies  $(H'_q)$ , and where  $\mu$  is a real parameter.  $L$  can be detailed as 2 equations:

$$(S) \quad \begin{cases} Lu_1 &= au_1 + bu_2 + \mu u_1 + f_1(x) \\ Lu_2 &= cu_1 + du_2 + \mu u_2 + f_2(x) \end{cases} \text{ in } \mathbb{R}^N, .$$

$$u_1(x), u_2(x)|_{|x| \rightarrow \infty} \rightarrow 0.$$

Assume

$$(H_A) \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ with } b > 0 \text{ and } D := (a - d)^2 + 4bc \geq 0.$$

Note that  $b > 0$  does not play any role since we can always change the order of the equations.

The eigenvalues of  $A$  are

$$\xi_1 = \frac{a + d + \sqrt{D}}{2} \geq \xi_2 = \frac{a + d - \sqrt{D}}{2}.$$

As far as we know, all the previous studies suppose that the largest eigenvalue  $\xi_1$  is simple (i.e.  $D = (a - d)^2 + 4bc > 0$ ). Here we also study, in the second subsection, the case of a double eigenvalue  $\xi_1 = \xi_2$ , that is  $D = 0$ ; this implies necessarily  $bc < 0$  and necessarily the matrix is not cooperative.

#### 3.1 Case $\xi_1 > \xi_2$

This is the classical case where  $\xi_1$  is simple. Set  $\xi_1 > \xi_2$ . The eigenvectors are

$$X_k = \begin{pmatrix} b \\ \xi_k - a \end{pmatrix},$$

Set  $X := X_1$ .

As above, denote by  $(\Lambda, \phi)$ ,  $\phi > 0$ , the principal eigenpair of the operator  $L = (-\Delta + q(x))$ .

It is easy to see that

$$L(X\phi) - AX\phi = (\Lambda - \xi_1)X\phi.$$

Hence

$$\nu = \Lambda - \xi_1 \quad (14)$$

is the principal eigenvalue of  $(S)$  with associated eigenvector  $X\phi$ . Note that the components of  $X\phi$  do not change sign, but, in the case of a non cooperative matrix they are not necessarily both positive. We prove:

**Theorem 4** *Assume  $(H'_q)$  and  $(H_A)$ ;  $f_1$  and  $f_2$  satisfy  $(H'_f)$ ; assume also  $D > 0$  and  $d - a > 0$ . If*

$$(\xi_2 - a)f_1^1 < bf_2^1,$$

*there exists  $\delta > 0$ , independant of  $\mu$ , such that if  $\nu - \delta < \mu < \nu$ , there exists a positive constant  $\gamma$  depending only on  $F$  such that*

$$u_1, u_2 \geq \frac{\gamma}{\nu - \mu} \phi > 0. \quad (15)$$

*If  $\nu < \mu < \nu + \delta$ , the sign are reversed:*

$$u_1, u_2 \leq -\frac{\gamma}{\nu - \mu} \phi < 0. \quad (16)$$

**Remark 2** *If  $(H'_q)$  and  $(H_A)$  are satisfied; if  $f_1$  and  $f_2$  satisfy  $(H'_f)$  as in Theorem 4, but if  $d - a < 0$  we have, if  $(a - \xi_2)f_1^1 + bf_2^1 > 0$  and  $\nu - \delta < \mu < \nu$*

$$u_1 \geq \frac{\gamma}{\nu - \mu} \phi > 0, \quad u_2 \leq -\frac{\gamma}{\nu - \mu} \phi < 0$$

**Remark 3** *It is noticeable that for all these cases,  $|u_1|, |u_2| \rightarrow +\infty$  as  $|\nu - \mu| \rightarrow 0$ .*

These results extend Theorem 4.2 in [4].

**Proof:** As in [3], we use  $J$  the associated Jordan matrix (which in this case is diagonal) and  $P$  the change of basis matrix which are such that

$$A = PJP^{-1}.$$

Here

$$P = \begin{pmatrix} b & b \\ \xi_1 - a & \xi_2 - a \end{pmatrix}, \quad P^{-1} = \frac{1}{b(\xi_1 - \xi_2)} \begin{pmatrix} a - \xi_2 & b \\ \xi_1 - a & -b \end{pmatrix}. \quad (17)$$

$$J = \begin{pmatrix} \xi_1 & 0 \\ 0 & \xi_2 \end{pmatrix}.$$

Denoting  $\tilde{U} = P^{-1}U$  and  $\tilde{F} = P^{-1}F$ , we derive from System (S) (after multiplication by  $P^{-1}U$  to the left):

$$L\tilde{U} = J\tilde{U} + \mu\tilde{U} + \tilde{F}.$$

Since  $J$  is diagonal we have two independant equations:

$$L\tilde{u}_k = (\xi_k + \mu)\tilde{u}_k + \tilde{f}_k, \quad k = 1 \text{ or } 2. \quad (18)$$

The projection on  $\phi$  and on its orthogonal for  $k = 1$  and  $2$  gives

$$\tilde{u}_k = (\tilde{u}_k)^1 \phi + \tilde{u}_k^\perp, \quad \tilde{f}_k = (\tilde{f}_k)^1 \phi + \tilde{f}_k^\perp;$$

hence

$$L(\tilde{u}_k)^1 \phi = \Lambda(\tilde{u}_k)^1 \phi = \xi_k(\tilde{u}_k)^1 \phi + \mu(\tilde{u}_k)^1 \phi + (\tilde{f}_k)^1 \phi, \quad (19)$$

$$L\tilde{u}_k^\perp = \xi_k\tilde{u}_k^\perp + \mu\tilde{u}_k^\perp + \tilde{f}_k^\perp. \quad (20)$$

If both  $f_k$  verify  $(H'_f)$ , they are in  $\mathcal{X}$  and bounded and hence both  $\tilde{f}_k^\perp$  are bounded; therefore, by (20) both  $\tilde{u}_k^\perp$  are also bounded.

We derive from (19) that

$$(\tilde{u}_k)^1 = \frac{(\tilde{f}_k)^1}{\Lambda - \xi_k - \mu} = \frac{(\tilde{f}_k)^1}{\nu - \mu}.$$

Consider again Equation (19) for  $k = 2$ ; obviously,  $(\tilde{u}_2)^1$  stays bounded as  $\mu \rightarrow \nu = \Lambda - \xi_1$  and therefore  $\tilde{u}_2$  stays bounded. .

For  $k = 1$ ,  $(\tilde{u}_1)^1 = \frac{(\tilde{f}_1)^1}{\nu - \mu} \rightarrow \infty$  as  $\mu \rightarrow \nu = \Lambda - \xi_1$ , since  $(\tilde{f}_1)^1 = \frac{1}{\xi_1 - \xi_2}((a - \xi_2)f_1^1 + bf_2^1) > 0$ ; this is the condition which appears in Theorem 4. Then, we simply apply Theorem 3 to (18) for  $k = 1$  and deduce that there exists  $\delta > 0$ , such that, for  $|\Lambda - \xi_1 - \mu| = |\nu - \mu| < \delta$ , there exists a positive constant  $C > 0$  such that  $\tilde{u}_1\phi \geq \frac{C}{\nu - \mu}\phi$ . Now, it follows from  $U = P\tilde{U}$ , that

$$u_1 = b(\tilde{u}_1 + \tilde{u}_2), \quad u_2 = (\xi_1 - a)\tilde{u}_1 + (\xi_2 - a)\tilde{u}_2.$$

As  $\nu - \mu \rightarrow 0$ , since  $\tilde{u}_2$  stays bounded,  $u_1$  behaves as  $b(\tilde{u}_1)^1\phi$ ,  $u_2$  as  $(\xi_1 - a)(\tilde{u}_1)^1\phi$ . More precisely, if  $|\mu - \nu|$  small enough

$$(\tilde{u}_1)^1 \geq \frac{K}{\nu - \mu} \text{ if } \mu < \nu; \quad \tilde{u}_1^1 \leq -\frac{K}{\nu - \mu} \text{ if } \mu > \nu$$

where  $K$  is a positive constant depending only on  $F$ .

**Remark 4** *Indeed, we always assume that  $b > 0$ , hence  $u_1 > 0$  for  $\nu - \mu > 0$  small enough. For the sign of  $u_2$  we remark that  $(\xi_1 - a)$  and  $(d - a)$  have the same sign.*



### 3.2 Case $\xi_1 = \xi_2$

Consider now the case where the coefficients of the matrix  $A$  satisfy  $(H_A)$  and

$$D := (a - d)^2 + 4bc = 0.$$

Of course this implies  $bc < 0$  and since  $b > 0$ , then  $c < 0$ : we have a non cooperative system. Now  $\xi_1 = \xi_2 = \xi = \frac{a+d}{2}$ . We prove here

**Theorem 5** *Assume  $(H'_q)$  and  $(H_A)$  with  $(a - d)^2 + 4bc = 0$ ; assume also that  $f_1, f_2$  satisfy  $(H'_f)$  and :*

$$\frac{(a - d)}{2} f_1^1 + b f_2^1 > 0.$$

*If  $\mu < \nu = \Lambda - \xi$ ,  $\nu - \mu < \delta$ , small enough, there exists a positive constant  $\gamma$  such that*

$$u_1 \geq \frac{\gamma}{\nu - \mu} \phi, \quad u_2 \leq -\frac{\gamma}{\nu - \mu} \phi.$$

*If  $\nu = \Lambda - \xi < \mu < \nu + \delta$ , ( $\delta$  small enough), there exists a positive constant  $\gamma'$  such that*

$$u_1 \leq -\frac{\gamma'}{\nu - \mu} \phi, \quad u_2 \geq \frac{\gamma'}{\nu - \mu} \phi.$$

**Remark 5** *Note that the condition  $\frac{(a-d)}{2} f_1^1 + b f_2^1 > 0$  in the theorem above is the same than in theorem 4  $(\xi_2 - a) f_1^1 < b f_2^1$ , since in theorem 5  $\xi_2 = \xi = \frac{a+d}{2}$ .*

**Prrof** The eigenvector associated to eigenvalue  $\xi$  is

$$X = \begin{pmatrix} b \\ \frac{d-a}{2} \end{pmatrix}.$$

The vector  $X\phi$  is thus an eigenvector for  $L - A$ ,

$$L(X\phi) - AX\phi = (\Lambda - \xi)X\phi = \nu X\phi.$$

We use again  $J$  the associated Jordan matrix and  $P$  the change of basis matrix; we have

$$A = PJP^{-1}.$$

Here

$$P = \begin{pmatrix} b & \frac{2b}{a-d} \\ \frac{d-a}{2} & 0 \end{pmatrix}, \quad P^{-1} = \frac{1}{b} \begin{pmatrix} 0 & -\frac{2b}{a-d} \\ \frac{a-d}{2} & b \end{pmatrix}.$$

$$J = \begin{pmatrix} \xi & 1 \\ 0 & \xi \end{pmatrix}.$$

As above, setting  $\tilde{U} = P^{-1}U$  and  $\tilde{F} = P^{-1}F$ , we derive from System (S)

$$L\tilde{U} = J\tilde{U} + \mu\tilde{U} + \tilde{F}.$$

We do not have anymore a decoupled system but

$$\begin{cases} L\tilde{u}_1 &= (\xi + \mu)\tilde{u}_1 + \tilde{u}_2 + \tilde{f}_1 \\ L\tilde{u}_2 &= \phantom{(\xi + \mu)\tilde{u}_1} + (\xi + \mu)\tilde{u}_2 + \tilde{f}_2 \end{cases} \quad (21)$$

If  $\xi + \mu < \Lambda$  (that is  $\mu < \nu$ ) and if  $\tilde{f}_2 = \frac{(a-d)}{2b}f_1 + f_2$  and  $\tilde{f}_1 = \frac{-2}{a-d}f_2$  satisfies  $(H'_f)$ , hence are in  $\mathcal{X}$  and  $\frac{(a-d)}{2b}f_1^1 + f_2^1 > 0$ . By Theorem 3 applied to the second equation, there exists a constant  $K > 0$ , such that  $\tilde{u}_2 > \frac{K}{\nu-\mu}\phi$ . Hence, for  $\nu - \mu$  small enough fo any  $\tilde{f}_1 \in \mathcal{X}$ ,  $\tilde{u}_2 + \tilde{f}_1 > 0$  and is in  $X$ ; then again Theorem 3 for the first equation implies that there exists a constant  $K' > 0$ , such that  $\tilde{u}_1 > \frac{K'}{\nu-\mu}\phi$ .

Since here  $a > d$ ., there exists a constant  $\gamma > 0$ ,

$$U = P\tilde{U} = \begin{cases} u_1 = b\tilde{u}_1 + \frac{2b}{a-d}\tilde{u}_2 > \frac{\gamma}{\nu-\mu}\phi \\ u_2 = \frac{d-a}{2}\tilde{u}_1 < -\frac{\gamma}{\nu-\mu}\phi \end{cases}$$

Again  $u_1 \rightarrow +\infty$  as  $\nu - \mu \rightarrow 0$  and  $u_2 \rightarrow -\infty$  as  $\nu - \mu \rightarrow 0$ .

If  $\mu > \Lambda$  ( and  $\mu - \nu > 0$  small enough we have analogous calculation with signs reversed.

**Remark 6** *The results in theorem 5 coincide with those of theroem 4 in the case  $D = 0$ .*

## References

- [1] A.Abakhti-Machachti *Systèmes semilinéaires d'équations de Schrödinger* Université Paul Sabatier-Toulouse III, thèse N.1338, 1993.
- [2] B.Alziary, N.Besbas
- [3] Alziary, Fleckinger *Sign of the solution to a non-cooperative system* Ro-MaKo 2016

- [4] B.Alziary, J. Fleckinger, P. Takac, *Maximum and anti-maximum principles for some systems involving Schrödinger operator*, Operator Theory: Advances and applications, 110, 1999, p.13-21.
- [5] B.Alziary, J. Fleckinger, P. Takac, *An extension of maximum and anti-maximum principles to a Schrödinger equation in  $\mathbb{R}^N$*  **Positivity** , 5, (4), 2001, pp. 359-382
- [6] *Groundstate positivity, negativity, and compactness for Schrödinger operator in  $\mathbb{R}^N$* . Jal Funct. Anal. 245, 2007, p.213-248.
- [7] B.Alziary, L.Cardoulis, J. Fleckinger
- [8] B. Alziary, J. Fleckinger, M. H. Lecureux , N. Wei *Positivity and negativity of solutions to  $n \times n$  weighted systems involving the Laplace operator defined on  $\mathbb{R}^N$ ,  $N \geq 3$*  Electron. J. Diff. Eqns, 101, 2012, p.1-14.
- [9] B.Alziary, P.Takac *Compactness for a Schrödinger operator in the groundstate space over  $\mathbb{R}^N$*  Electr. J Diff. Eq., Conf. 16, (2007) p.35-58.
- [10] B.Alziary, P.Takac *Intrinsic ultracontractivity of a Schrödinger semigroup in  $\mathbb{R}^N$* . J. Funct. Anal. 256 (2009), no. 12, 40954127.
- [11] N.Besbas *Principe d'antimaximum pour des equations et des systemes de type Schrödinger dans  $\mathbb{R}^N$* . These de doctorat de l'Universit Toulouse I (2004).
- [12] L.Cardoulis *Problèmes elliptiques:applications de la théorie spectrale et étude des systèmes, existence des solutions* Thèse, Université Toulouse, 1997.
- [13] P. Clément, L. Peletier, *An anti-maximum principle for second order elliptic operators.*, **J. Diff. Equ.** **34** , 1979, p.218-229.
- [14] D.G.de Figueiredo, E.Mitidieri , *A Maximum Principle for an Elliptic System and Applications to semilinear Problems*, SIAM J. Math and Anal. N17 (1986), 836-849.
- [15] DE..Edmunds, W.D. Evans, Spectral Theory and Differential Operators Oxford Science Publ. Clarendon Press, 1987
- [16] M.H.Lécoreux, *Au-delà du principe du maximum pour des systèmes d'opérateurs elliptiques.* Thèse, Université de Toulouse, Toulouse 1, 13 juin 2008.

- [17] M.H.Lécureux, *Comparison with groundstate for solutions of non cooperative systems for Schrödinger operators in  $\mathbb{R}^N$* . Rostok Math. Kolloq. 65, 2010, p51-69.
- [18] M.H.Protter, H.Weinberger, Maximum Principles in Differential Equations, Springer-Verlag, 1984.
- [19] M.Reed, B.Simon *Methods of modern mathematical physics IV. Analysis of operators* Acad.Press, New York, 1978.