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KONSTANTIN PANKRASHKIN
Laboratoire de Mathématiques d'Orsay, Univ. Paris-Sud, CNRS,
Université Paris-Saclay, 91405 Orsay, France
E-mail: konstantin.pankrashkin@math.u-psud.fr
Webpage: http://www.math.u-psud.fr/~pankrashkin/

Abstract

Let $\Omega \subset \mathbb{R}^d$ be a domain and Σ a hypersurface cutting Ω into two parts $\Omega_\pm$. For $\mu > 0$, consider the function h whose value is $(-\mu)$ in Ω_- and 1 in Ω_+. In the present contribution we discuss the construction and some properties of the self-adjoint realizations of the operator $L = -\nabla \cdot (h\nabla)$ in $L^2(\Omega)$ with suitable (e.g. Dirichlet) on the exterior boundary. We give first a detailed study for the case when $\Omega_\pm$ are two rectangles touching along a side, which is based on operator-valued differential operators, in order to see in an elementary but an abstract level the principal effects such as a loss of regularity and unusual spectral properties. Then we give a review of available approaches and results for more general geometric configurations and formulate some open problems.

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1 Introduction

The present contribution discusses some approaches to the strict mathematical definition of non-elliptic self-adjoint differential operators of a special form. Namely, let $\Omega \subset \mathbb{R}^d$ be bounded open set and Σ be a hypersurface (called *interface*) splitting Ω into two parts Ω_- and Ω_+ (for the moment we do not discuss the precise regularity assumptions). Let $\mu > 0$ be a parameter and $h : \Omega \to \mathbb{R}$ takes the value $(-\mu)$ in Ω_- and is equal to 1 in Ω_+. We will be interested in operators L acting in $L^2(\Omega)$ and at least formally given by the differential expression $u \mapsto -\nabla \cdot (h\nabla)u$ in Ω and the Dirichlet boundary condition $u = 0$ on $\partial\Omega$. Such operators appear in the mathematical theory of negative-index metamaterials arised from the pioneering works by Veselago [43]. At the naive level, if one identifies $L^2(\Omega) \simeq L^2(\Omega_-) \oplus L^2(\Omega_+)$, $u \simeq (u_-, u_+)$ with $u_\pm$ being the restrictions of u to $\Omega_\pm$, then the above operator L is

expected to act as $(u_-, u_+) \mapsto (\mu\Delta u_-, -\Delta u_+)$ while the functions $u_\pm$ should satisfy the boundary conditions

$$u_\pm = 0 \text{ on } \partial\Omega, \quad u_- = u_+ \text{ on } \Sigma, \quad \mu\partial_- u_- = \partial_+ u_+ \text{ on } \Sigma,$$

with $\partial_\pm$ being the outward normal (with respect to $\Omega_\pm$) derivative on Σ. As will be explained below, understanding the precise regularity properties of the functions $u_\pm$ appears to be a non-trivial task depending on a combination of the value μ and the geometric properties of $\Omega_\pm$. In some cases, a very low regularity of $u_\pm$ is needed in order to have an operator with reasonable properties, and this is an essential feature of the problem which is of relevance for various effects such as the presence of an anomalous localized resonance and the cloaking, see [1, 8, 28, 35, 34, 33], and the critical value of the parameter $\mu = 1$ appears to be of a special importance.

Remark that if one has a self-adjoint operator A in a Hilbert space, then solving the equation $Au = f$ with a given f can be understood in terms of the domain of A and of its spectral properties. As the above differential expression for L is formally symmetric, it is natural to look for self-adjoint realizations of the differential expression, which then may provide a rigorous reformulation of the above equation. In the present text we collect some known results and approaches to the study of such self-adjoint differential operators.

It seems that the problem was first addressed in [6] for the case when $\overline{\Omega_-} \subset \Omega$, both $\Omega_\pm$ are smooth and $\mu \neq 1$, which was then extended to the case of domains with corners, see a detailed discussion in Section 5. The question of self-adjointness for the critical case $\mu = 1$ was first addressed in the very recent paper [2], in which the very particular case of $\Omega = (-1,1)\times(0,1)$, $\Omega_- = (-1,0)\times(0,1)$, $\Omega_+ = (0,1)\times(0,1)$, $h = \pm 1$ in $\Omega_\pm$. was considered. An interesting feature of the model is the fact that the resulting self-adjoint operator appears to have an infinitely degenerate zero eigenvalue, hence, it is not with compact resolvent, although the domain Ω is bounded. This comes from the fact that the functions in the operator domain can be very irregular near the interface $\{0\}\times(0,1)$, and a precise description of the resulting operator domain is given as well. On the other hand, the construction appears to be very sensible to the symmetries, in particular, the approach had no easy extension to configurations consisting of two non-congruent rectangles touching along a side.

In fact we use the paper [2] as our starting point and use the situation with two rectangles as a toy example illustrating the main features of the problem and the special role of the value $\mu = 1$, which allows for a rather complete spectral analysis and may provide us with some intuition for the general case. Therefore, we prefer to give first a detailed study of this situation, which is done in the following three sections. In Section 2 we recall the basic notions of the theory of self-adjoint extensions. In Section 3 we give some facts related to operator-valued differential equations, which represents an abstract version of the so-called modal analysis, see e.g. [44] and can also be obtained in a much more general setting, see e.g. [18, 19, 20, 32, 39], but we prefer to provide complete "manual" proofs in order to keep the presentation at an elementary level. In Section 4, the preceding constructions are used to study the self-adjointness of indefinite Laplacians on rectangles, which

gives the sought extension of the result of [2]. In the last section 5 we give a review of available approaches and results for more general geometries and state some open questions, in the hope that the present text will attract the attention of the spectral community to this relatively new class of problems.

2 Basic constructions for self-adjoint extensions

2.1 Notation

The scalar product in a Hilbert space $\mathscr{H}$ will be denoted as $\langle\cdot,\cdot\rangle_{\mathscr{H}}$ or simply as $\langle\cdot,\cdot\rangle$ if there is no ambiguity. The scalar produce will be assumed anti-linear with respect to the first argument. If B is a linear operator in a Hilbert space, then by $\operatorname{dom} B$, $\ker B$, $\operatorname{ran} B$, $\sigma(B)$ and $\rho(B)$ we denote its domain, kernel, range, spectrum and resolvent set, and $\overline{B}$ and B^* stand for the closure of B and for its adjoint, respectively. If B is self-adjoint, then $\sigma_{\mathrm{ess}}(B)$ and $\sigma_{\mathrm{p}}(B)$ stand respectively for its essential spectrum and point spectrum (defined as the set of eigenvalues). By $\mathscr{B}(\mathscr{G},\mathscr{H})$ we mean the Banach space of the bounded linear operators from a Hilbert space $\mathscr{G}$ to a Hilbert space $\mathscr{H}$, and we will denote $\mathscr{B}(\mathscr{H}) := \mathscr{B}(\mathscr{H},\mathscr{H})$.

2.2 Boundary triples

Let us briefly recall the key points of the boundary triple approach to self-adjoint extensions following the first sections of [9]. A detailed discussion can also be found in [17, 20].

Let S be a closed densely defined symmetric operator in a Hilbert space $\mathscr{H}$. A triple (Ξ,Γ,Γ'), where Ξ is a Hilbert space and $\Gamma,\Gamma' : \operatorname{dom} S^* \to \Xi$ are linear maps, is called a *boundary triple* for S if the following three conditions are satisfied:

$$\text{(a) } \langle f, S^*g\rangle_{\mathscr{H}} - \langle S^*f, g\rangle_{\mathscr{H}} = \langle \Gamma f, \Gamma' g\rangle_{\Xi} - \langle \Gamma' f, \Gamma g\rangle_{\Xi} \text{ for } f,g \in \operatorname{dom} S^*,$$
$$\text{(b) the map } \operatorname{dom} S^* \ni f \mapsto (\Gamma f, \Gamma' f) \in \Xi\times\Xi \text{ is surjective,}$$
$$\text{(c) } \ker\Gamma \cap \ker\Gamma' = \operatorname{dom} S.$$

A boundary triple for S exists if and only if S admits self-adjoint extensions, i.e. if its deficiency indices are equal, $\dim\ker(S^*-i) = \dim\ker(S^*+i) =: n(S)$. A boundary triple is not unique, but for any choice of a boundary triple (Ξ,Γ,Γ') for S one has $\dim\Xi = n(S)$, and the maps Γ,Γ' are bounded with respect to the graph norm of S^*.

Assume from now on that the deficiency indices of S are equal and pick a boundary triple (Ξ,Γ,Γ'). Let $\Pi : \Xi \to \Xi_\Pi := \operatorname{ran}\Pi \subseteq \Xi$ be an orthogonal projector in Ξ and Θ be a linear operator in the Hilbert space Ξ_Π carrying the induced scalar product. Denote by $A_{\Pi,\Theta}$ the restriction of S^* onto

$$\operatorname{dom} A_{\Pi,\Theta} = \big\{ f \in \operatorname{dom} S^* : \Gamma f \in \operatorname{dom}\Theta,\ \Pi\Gamma' f = \Theta\Gamma f \big\},$$

then the closedness, the symmetry and the self-adjointness of $A_{\Pi,\Theta}$ in $\mathscr{H}$ are equivalent to the closedness, the symmetry and the self-adjointness of the operator Θ in

Ξ_Π, and for the closures one has $\overline{A_{\Pi,\Theta}} = A_{\Pi,\overline{\Theta}}$. Finally, any self-adjoint extension of S writes as $A_{\Pi,\Theta}$.

The spectral analysis of the self-adjoint extensions can be carried out using the associated Weyl function. Namely, let A be the restriction of S^* to $\ker\Gamma$, i.e. corresponds to $(\Pi,\Theta) = (0,0)$ in the above notation, which is therefore a self-adjoint operator. For $z \in \rho(A)$ the restriction $\Gamma : \ker(S^* - z) \to \Xi$ is a bijection, and we denote its inverse by $G(z)$. In other word, for $\xi \in \Xi$, one has $G(z)\xi := f$ with $f \in \operatorname{dom} S^*$ uniquely determined by the conditions $(S^* - z)f = 0$ and $\Gamma f = \xi$. The map $z \mapsto G(z)$, called the associated γ-field, is then a holomorphic map from $\rho(A)$ to $\mathscr{B}(\Xi, \mathscr{H})$, and one has the identity

$$G(z) - G(w) = (z-w)(A-z)^{-1}G(w) \text{ for any } z,w \in \rho(A). \tag{1}$$

The associated *Weyl function* is then the holomorphic map $z \mapsto M(z) := \Gamma' G(z) \in \mathscr{B}(\Xi)$ defined for $z \in \rho(A)$. The above maps M and G satisfy a number of identities, in particular,

$$M'(z) = G(z)^* G(z) > 0, \quad 0 \in \sigma\big(M'(z)\big) \text{ for any } z \in \mathbb{R} \cap \rho(A). \tag{2}$$

To describe the spectral properties of the self-adjoint operators $A_{\Pi,\Theta}$ let us consider first the case $\Pi = 1$, then Θ is a self-adjoint operator in $\mathscr{H}$, and the following holds, see Theorems 1.29 and Theorem 3.3 in [9]:

Proposition 1. *For any $z \in \rho(A) \cap \rho(A_{1,\Theta})$ one has $0 \in \rho\big(\Theta - M(z)\big)$ and*

$$(A_{1,\Theta} - z)^{-1} = (A-z)^{-1} + G(z)\big(\Theta - M(z)\big)^{-1} G(\overline{z})^*. \tag{3}$$

For any $z \in \rho(A)$ one has

(1) $z \in \sigma(A_{1,\Theta})$ if and only if $0 \in \sigma\big(\Theta - M(z)\big)$,

(2) $z \in \sigma_{\mathrm{ess}}(A_{1,\Theta})$ if and only $0 \in \sigma_{\mathrm{ess}}\big(\Theta - M(z)\big)$,

(3) $G(z) : \ker\big(\Theta - M(z)\big) \to \ker(A_{1,\Theta} - z)$ is an isomorphism, hence,

$$z \in \sigma_{\mathrm{p}}(A_{1,\Theta}) \text{ if and only if } 0 \in \sigma_{\mathrm{p}}\big(\Theta - M(z)\big).$$

Now consider a self-adjoint extension $A_{\Pi,\Theta}$ for an arbitrary orthogonal projector Π. Denote by S_Π the restriction of S^* to $\operatorname{dom} S_\Pi = \{u \in \operatorname{dom} S^* : \Gamma u = \Pi\Gamma' u = 0\}$, which is a closed densely defined symmetric operator whose adjoint S_Π^* is the restriction of S^* to $\operatorname{dom} S_\Pi^* = \{u \in \operatorname{dom} S^* : \Gamma u \in \operatorname{ran}\Pi\}$, then $(\Xi_\Pi, \Gamma_\Pi, \Gamma_\Pi)$ with $\Xi_\Pi = \operatorname{ran}\Pi$, $\Gamma_\Pi := \Pi\Gamma$ and $\Gamma'_\Pi := \Pi\Gamma'$, is a boundary triple for S_Π, while the restriction of S_Π^* to $\ker\Gamma_\Pi$ is still the original operator $A = S^*|_{\ker\Gamma}$. The associated γ-field G_Π and Weyl function M_Π take the form $z \mapsto G_\Pi(z) := G(z)\Pi^*$ and $z \mapsto M_\Pi(z) := \Pi M(z)\Pi^*$, and $\operatorname{dom} A_{\Pi,\Theta} := \{u \in \operatorname{dom} S_\Pi^* : \Gamma'_\Pi u = \Theta\Gamma_\Pi u\}$, see [9, Remark 1.30]. A direct application of Proposition 1 gives the following assertions:

Corollary 2. *For any $z \in \rho(A) \cap \rho(A_{\Pi,\Theta})$ one has $0 \in \rho\big(\Theta - M_\Pi(z)\big)$ and*

$$(A_{\Pi,\Theta} - z)^{-1} = (A-z)^{-1} + G_\Pi(z)\big(\Theta - M_\Pi(z)\big)^{-1}\big(G_\Pi(\overline{z})\big)^*. \tag{4}$$

For any $z \in \rho(A)$ one has:

(1) $z \in \sigma(A_{\Pi,\Theta})$ if and only if $0 \in \sigma\big(\Theta - M_\Pi(z)\big)$,

(2) $z \in \sigma_{\mathrm{ess}}(A_{\Pi,\Theta})$ if and only if $0 \in \sigma_{\mathrm{ess}}\big(\Theta - M_\Pi(z)\big)$,

(3) $G_\Pi(z) : \ker\big(\Theta - M_\Pi(z)\big) \to \ker(A_{\Pi,\Theta} - z)$ is an isomorphism, hence, $z \in \sigma_{\mathrm{p}}(A_{\Pi,\Theta})$ if and only if $0 \in \sigma_{\mathrm{p}}\big(\Theta - M_\Pi(z)\big)$.

2.3 Construction of boundary triples using trace maps

In various situations one deals with a symmetric operator obtained by restricting a self-adjoint operator with well-known properties to the kernel of an explicitly given linear map. This observation may be used to construct a boundary triple and to study other self-adjoint extensions of the symmetric operator. We present here very briefly the respective construction, which is described in detail e.g. in [38] or in [9, Section 1.4.2].

Let A be a self-adjoint operator in a Hilbert space $\mathscr{H}$, then one denotes by $\mathscr{H}(A)$ the Hilbert space given by the linear space $\mathrm{dom}A$ equipped with the scalar product $\langle u,v\rangle_{\mathscr{H}(A)} = \langle u,v\rangle_{\mathscr{H}} + \langle Au,Av\rangle_{\mathscr{H}}$. Let a Hilbert space Ξ and a bounded *surjective* linear operator $\tau : \mathscr{H}(A) \to \Xi$ be such that the kernel $\ker\tau$ is a dense linear subspace of $\mathscr{H}$. Such a map τ will be referred to as a *trace map* for A. It follows that the restriction S of A to $\ker\tau$ (i.e. the restriction to the vectors with zero traces) is a closed densely defined symmetric operator in $\mathscr{H}$. To avoid some technicalities we additionally assume that $\rho(A) \cap \mathbb{R} \neq \emptyset$ and pick an arbitrary value $\lambda \in \rho(A) \cap \mathbb{R}$. For $z \in \rho(A)$ consider the maps

$$G(z) := \big(\tau(A - \overline{z})^{-1}\big)^* \in \mathscr{B}(\Xi, \mathscr{H}), \quad M(z) := \tau\big(G(z) - G(\lambda)\big) \in \mathscr{B}(\Xi),$$

then the adjoint S^* is given by

$$\mathrm{dom}S^* := \Big\{u = u_\lambda + G(\lambda)f_u : u_\lambda \in \mathrm{dom}A \text{ and } f_u \in \Xi\Big\},$$
$$(S^* - \lambda)u = (A - \lambda)u_\lambda.$$

Furthermore, the triple (Ξ, Γ, Γ') with $\Gamma u := f_u$ and $\Gamma' u := \tau u_\lambda$ is a boundary triple for S, and $z \mapsto G(z)$ and $z \mapsto M(z)$ are the associated γ-field and Weyl function, respectively.

2.4 Trace maps for direct sums

In what follows we will be interested in boundary triples for infinite direct sums of operators. In view of the preceding considerations it is useful to understand how to

construct a trace map for a direct sum of operators. The construction below follows [29, 37].

Let A_n be *non-negative* self-adjoint operators in Hilbert spaces $\mathscr{H}_n$, $n \in \mathbb{N}$. For each n we consider a trace map $\tau_n : \mathscr{H}(A_n) \to \Xi_n$ for A_n, where Ξ_n is a Hilbert space. Due to the constructions of the preceding subsection this gives rise to closed densely defined symmetric operators $S_n := A_n|_{\ker \tau_n}$ and the associated γ-fields and Weyl functions $z \mapsto G_n(z)$ and $z \mapsto M_n(z)$. To construct a trace map for the self-adjoint operator $A := \bigoplus_{n\in\mathbb{N}} A_n$ we use [37, Theorem 2.5], which gives the following result: For $n \in \mathbb{N}$ set $R_n := \sqrt{M_n'(-1)} \in \mathscr{B}(\Xi_n)$, then the linear operator

$$\tau : \mathscr{H}(A) \ni (u_n)_{n\in\mathbb{N}} \mapsto (R_n^{-1}\tau_n u_n)_{n\in\mathbb{N}} \in \bigoplus_{n\in\mathbb{N}} \Xi_n, \tag{5}$$

is bounded and surjective. This construction can be adjusted as follows:

Proposition 3. *Let $K_n \in \mathscr{B}(\Xi_n)$ be strictly positive and such that for some $a \geq 1$ one has*

$$a^{-1}K_n^2 \leq M_n'(-1) \leq aK_n^2 \text{ for any } n \in \mathbb{N}. \tag{6}$$

then the linear operator

$$\tau : \mathscr{H}(A) \ni (u_n) \mapsto (K_n^{-1}\tau_n u_n) \in \bigoplus_{n\in\mathbb{N}} \Xi_n$$

is bounded and surjective.

Proof. Representing $K_n^{-1} = (K_n^{-1}R_n)R_n^{-1}$ and comparing with (5) one sees that it is sufficient to show that for some $b > 0$ there holds

$$\|K_n^{-1}R_n\| \leq b \text{ and } \left\|(K_n^{-1}R_n)^{-1}\right\| \leq b \text{ for all } n \in \mathbb{N},$$

which is due to the self-adjointness of R_n and B_n equivalent to

$$\|R_nK_n^{-1}\| \leq b \text{ and } \left\|(R_nK_n^{-1})^{-1}\right\| \leq b \text{ for all } n \in \mathbb{N}, \tag{7}$$

Remark that the condition (6) takes the form $a^{-1}K_n^2 \leq R_n^2 \leq aK_n^2$. Therefore, for any $\xi \in \Xi_n$ one has

$$\begin{aligned}
\|R_nK_n^{-1}\xi\|_{\Xi_n}^2 &= \langle K_n^{-1}\xi, R_n^2K_n^{-1}\xi\rangle_{\Xi_n} \leq a\langle K_n^{-1}, K_n^2K_n^{-1}\xi\rangle_{\Xi_n} = a\|\xi\|_{\Xi_n}^2,\\
\|R_nK_n^{-1}\xi\|_{\Xi_n}^2 &= \langle K_n^{-1}\xi, R_n^2K_n^{-1}\xi\rangle_{\Xi_n} \geq \frac{1}{a}\langle K_n^{-1}, K_n^2K_n^{-1}\xi\rangle_{\Xi_n} = \frac{1}{a}\|\xi\|_{\Xi_n}^2,
\end{aligned}$$

which gives the estimates (7) with $b := \sqrt{a}$. □

3 Sign-changing operator-valued Sturm-Liouville operators

3.1 Sturm-Liouville operator on a bounded interval

In order to illustrate the constructions of the preceding section let us consider first a simple one-dimensional situation. Let $(a,b) \subset \mathbb{R}$ be a non-empty bounded interval.

In the Hilbert space $\mathscr{H} = L^2(a,b)$ consider the self-adjoint Dirichlet Laplacian

$$A : f \mapsto -f'', \quad \mathrm{dom}A = \big\{ f \in H^2(a,b) : f(a) = f(b) = 0 \big\},$$

whose spectrum consists of the simple eigenvalues $\pi^2 n^2/(b-a)^2$, $n \in \mathbb{N}$. For the subsequent computations we choose $\lambda = 0 \in \rho(A) \cap \mathbb{R}$. The resolvent $(A-z)^{-1}$ is an integral operator,

$$\big[(A-z)^{-1} f\big](t) = \int_a^b K_z(t,s) f(s) ds,$$

where the integral kernel K_z is given by

$$K_z(t,s) = \frac{1}{\sqrt{-z}\sinh\big(\sqrt{-z}(b-a)\big)} \times \begin{cases} \sinh\big(\sqrt{-z}(t-a)\big)\sinh\big(\sqrt{-z}(b-s)\big), & t < s, \\ \sinh\big(\sqrt{-z}(s-a)\big)\sinh\big(\sqrt{-z}(b-t)\big), & t > s. \end{cases}$$

The map

$$\tau : \mathscr{H}(A) \ni f \mapsto \begin{pmatrix} f'(a) \\ -f'(b) \end{pmatrix} \in \mathbb{C}^2$$

is surjective and bounded due to the Sobolev embedding theorem, and the restriction of A to $\ker\tau$ is the closed symmetric operator

$$S : f \mapsto -f'', \quad \mathrm{dom}S = H_0^2(a,b).$$

A direct computation shows that the operators $G(z) := \big(\tau(A-\bar{z})^{-1}\big)^* : \mathbb{C}^2 \to \mathscr{H}$ are given by

$$\begin{aligned} G(z)\begin{pmatrix} \xi_a \\ \xi_b \end{pmatrix}(s) &= \overline{\partial_t K_{\bar{z}}(a,s)}\xi_a - \overline{\partial_t K_{\bar{z}}(b,s)}\xi_b \\ &= \frac{\sinh\big(\sqrt{-z}(b-s)\big)}{\sinh\big(\sqrt{-z}(b-a)\big)}\xi_a + \frac{\sinh\big(\sqrt{-z}(s-a)\big)}{\sinh\big(\sqrt{-z}(b-a)\big)}\xi_b, \end{aligned}$$

and the associated map $M(z) = \tau\big(G(z) - G(0)\big) : \mathbb{C}^2 \to \mathbb{C}^2$ is

$$\begin{aligned} M(z) &= m(z,b-a) - m(0,b-a) \\ \text{with} \quad m(z,\ell) &:= \frac{\sqrt{-z}}{\sinh(\ell\sqrt{-z})}\begin{pmatrix} -\cosh(\ell\sqrt{-z}) & 1 \\ 1 & -\cosh(\ell\sqrt{-z}) \end{pmatrix}. \end{aligned} \tag{8}$$

In order to construct a boundary triple for S we remark first that the adjoint S^* acts as $f \mapsto -f''$ on the domain $\mathrm{dom}S^* = H^2(a,b)$. Each $f \in \mathrm{dom}S^*$ can be uniquely represented as $f = f_0 + G(0)(\xi_a, \xi_b)$ with $f_0 \in \mathrm{dom}A$ and $(\xi_a, \xi_b) \in \mathbb{C}^2$, i.e.

$$f(s) = f_0(s) + \frac{b-s}{b-a}\xi_a + \frac{s-a}{b-a}\xi_b, \quad s \in (a,b),$$

and as a boundary triple $(\mathbb{C}^2, \Gamma, \Gamma')$ for S one can take

$$\Gamma f = \begin{pmatrix} \xi_a \\ \xi_b \end{pmatrix} = \begin{pmatrix} f(a) \\ f(b) \end{pmatrix},$$
$$\Gamma' f = \tau f_0 = \begin{pmatrix} f_0'(a) \\ -f_0'(b) \end{pmatrix} = \begin{pmatrix} f'(a) \\ -f'(b) \end{pmatrix} - m(0, b-a) \begin{pmatrix} f(a) \\ f(b) \end{pmatrix}.$$

For a later use we remark that, by a direct computation,

$$M'(z) = \frac{1}{2\sqrt{-z}} \cdot \frac{1}{\sinh^2 \zeta} \begin{pmatrix} \sinh\zeta\cosh\zeta - \zeta & \zeta\cosh\zeta - \sinh\zeta \\ \zeta\cosh\zeta - \sinh\zeta & \sinh\zeta\cosh\zeta - \zeta \end{pmatrix}, \qquad (9)$$
$$\zeta := (b-a)\sqrt{-z}.$$

3.2 Direct sums of Sturm-Liouville operators on a bounded interval

Consider an infinite number sequence $(\lambda_n)_{n\in\mathbb{N}}$ such that $\lambda_n \geq 0$ for all $n \in \mathbb{N}$ and that

$$\lim_{n\to+\infty} \lambda_n = +\infty.$$

Let $(a,b) \subset \mathbb{R}$ be a non-empty bounded interval. For each $n \in \mathbb{N}$, in the Hilbert space $\mathscr{H}_n := L^2(a,b)$ consider the closed densely defined symmetric operators

$$S_n : f_n \mapsto -f_n'' + \lambda_n f_n, \quad \operatorname{dom} S_n = H_0^2(a,b).$$

For each S_n one can construct a boundary triple as in the preceding subsection, and the associated γ-fields G_n and the Weyl functions M_n are then given by $G_n(z) = G(z-\lambda_n)$ and $M_n(z) = m(z-\lambda_n, b-a) - m(-\lambda_n, b-a)$ with m as in (8). We would like to construct a boundary triple for the direct sum $S = \bigoplus_n S_n$. Remark that we clearly have $S^* = \bigoplus_n S_n^*$. By A_n we denote the self-adjoint extension of S_n defined by the Dirichlet boundary condition $f_n(a) = f_n(b) = 0$, and B_n will stand for the self-adjoint extension of S_n defined by the Neumann boundary condition $f_n'(a) = f_n'(b) = 0$. In addition, denote

$$A := \bigoplus_{n\in\mathbb{N}} A_n, \quad B := \bigoplus_{n\in\mathbb{N}} B_n,$$

then both A and B are self-adjoint with compact resolvents.

Lemma 4. *The linear map*

$$\tau : \mathscr{H}(A) \to \ell^2(\mathbb{N}) \otimes \mathbb{C}^2, \quad \tau(f_n) = \left(\left(1+\lambda_n\right)^{\frac{1}{4}} \begin{pmatrix} f_n'(a) \\ -f_n'(b) \end{pmatrix} \right)_{n\in\mathbb{N}},$$

is bounded and surjective.

Proof. An elementary computation with the help of (9) shows that

$$(\lambda_n+1)^{\frac{1}{2}}M_n'(-1)\equiv(\lambda_n+1)^{\frac{1}{2}}M'\big(-(\lambda_n+1)\big)\to\frac{1}{2}\,\mathrm{Id}\quad\text{for } n\to+\infty,$$

and it is sufficient to use Proposition 3 with $K_n=(\lambda_n+1)^{-\frac{1}{4}}$. □

Using the map τ from Lemma 4 and the constructions of Subsection 2.3 one then easily computes a boundary triple $(\mathscr{G},\Gamma,\Gamma')$ for the operator $S\equiv A|_{\ker\tau}$,

$$\mathscr{G}:=\ell^2(\mathbb{N})\otimes\mathbb{C}^2,\quad \Gamma(f_n)=\left((1+\lambda_n)^{-\frac{1}{4}}\begin{pmatrix}f_n(a)\\ f_n(b)\end{pmatrix}\right)_{n\in\mathbb{N}},$$
$$\Gamma'(f_n)=\left((1+\lambda_n)^{\frac{1}{4}}\left[\begin{pmatrix}f_n'(a)\\ -f_n'(b)\end{pmatrix}-m(-\lambda_n,b-a)\begin{pmatrix}f_n(a)\\ f_n(b)\end{pmatrix}\right]\right)_{n\in\mathbb{N}}.$$

In particular, in view of the asymptotic behavior $m(-\lambda)\simeq-\sqrt{\lambda}\,\mathrm{Id}$ for $\lambda\to+\infty$ it follows that the linear maps

$$f\mapsto\left((1+\lambda_n)^{-\frac{1}{4}}\begin{pmatrix}f_n(a)\\ f_n(b)\end{pmatrix}\right)_{n\in\mathbb{N}}\in\ell^2(\mathbb{N})\otimes\mathbb{C}^2,$$
$$f\mapsto\left((1+\lambda_n)^{-\frac{3}{4}}\begin{pmatrix}f_n'(a)\\ -f_n'(b)\end{pmatrix}\right)_{n\in\mathbb{N}}\in\ell^2(\mathbb{N})\otimes\mathbb{C}^2,$$

are bounded with respect to the graph norm of S^*.

It is instructive to look at the self-adjoint operator B from the point of view of the above boundary triple:

Lemma 5. *The linear map*

$$\tau:\mathscr{H}(B)\to\ell^2(\mathbb{N})\otimes\mathbb{C}^2,\quad \tau(f_n)=\left(\big(1+\lambda_n\big)^{\frac{3}{4}}\begin{pmatrix}f_n(a)\\ f_n(b)\end{pmatrix}\right)_{n\in\mathbb{N}},$$

is bounded and surjective.

Proof. In terms of the above boundary triple, the boundary condition for B take the form $\Gamma' f=L\Gamma f$ with $\Gamma f\in\mathrm{dom}\,L$, where

$$L(\xi_n)=-\big[(1+\lambda_n)^{\frac{1}{4}}m(-\lambda_n,b-a)\xi_n\big]_{n\in\mathbb{N}},$$

i.e. the map $\mathscr{H}(B)\ni f\mapsto\Gamma f\in\mathscr{H}(L)$ is bounded and surjective. Using the asymptotics $\lim_{\lambda\to+\infty}\lambda^{-\frac{1}{2}}m(-\lambda,b-a)=-\mathrm{Id}$ we obtain the result. □

For a later use we give two additional estimates:

Proposition 6. *On* $\mathrm{dom}A$ *and* $\mathrm{dom}B$*, the graph norm of* S^* *is equivalent to the norm*

$$\|f\|_{H^2}:=\sum_{n\in\mathbb{N}}\Big(\|f_n''\|^2_{L^2(a,b)}+(1+\lambda_n)^2\|f_n\|^2_{L^2(a,b)}\Big)<\infty.$$

Proof. The graph norm of S^* for $f=(f_n)\in\operatorname{dom}S^*$ is given by

$$\sum_{n\in\mathbb{N}}\Big(\|f_n''+\lambda_n f_n\|^2_{L^2(a,b)}+\|f_n\|^2_{L^2(a,b)}\Big),$$

and $\|f_n''+\lambda_n f_n\|^2_{L^2(a,b)}=\|f_n''\|^2_{L^2(a,b)}+\lambda_n^2\|f_n\|^2_{L^2(a,b)}+2\lambda_n\Re\langle f_n'',f_n\rangle_{L^2(a,b)}$. Using the integration by parts and the Cauchy-Schwarz inequality, for $f\in\operatorname{dom}A$ or $f\in\operatorname{dom}B$ we obtain

$$0\le 2\lambda_n\|f_n'\|^2_{L^2(a,b)}=2\lambda_n\Re\langle f_n'',f_n\rangle_{L^2(a,b)}\le\|f_n''\|^2_{L^2(a,b)}+\lambda_n^2\|f_n\|^2_{L^2(a,b)},$$

which gives the result. □

3.3 Sturm-Liouville operator with an operator-valued potential

Let $\mathscr{G}$ be a Hilbert space and T be a non-negative self-adjoint operator in $\mathscr{G}$ with a compact resolvent. Let us pick an orthonormal basis (e_n) of $\mathscr{G}$ consisting of eigenfunctions of T:

$$Te_n=\lambda_n e_n,\quad \lambda_n\in\mathbb{R},\quad n\in\mathbb{N},\quad \langle e_k,e_n\rangle=\delta_{k,n}.$$

For $s\ge 0$ we will consider the Hilbert spaces $\mathscr{G}_s\equiv\mathscr{G}_s(T)=\operatorname{dom}(T+1)^{\frac{s}{2}}$ equipped with the scalar product $\langle u,v\rangle_s=\big\langle(T+1)^{\frac{s}{2}}u,(T+1)^{\frac{s}{2}}v\big\rangle_{\mathscr{G}}$, then, in particular, $\mathscr{G}_0=\mathscr{G}$, and $\mathscr{G}_2$ coincides with $\mathscr{H}(T)$ algebraically and topologically. By $\mathscr{G}_{-s}\equiv\mathscr{G}_{-s}(T)$ we denote then the dual of $\mathscr{G}_s$ realized as the completion of $\mathscr{G}$ with respect to the scalar product $\langle u,v\rangle_{-s}=\big\langle(T+1)^{-\frac{s}{2}}u,(T+1)^{-\frac{s}{2}}\big\rangle_{\mathscr{G}}$ and introduce the subspace $\mathscr{G}_\infty:=\bigcap_{t>0}\mathscr{G}_t$ which is then dense in any $\mathscr{G}_s$, $s\in\mathbb{R}$. With these definitions, the operator $T:\mathscr{G}_\infty\to\mathscr{G}_\infty$ extends uniquely to a bounded linear map $T:\mathscr{G}_s\to\mathscr{G}_{s-2}$ for any $s\in\mathbb{R}$, while the operator $1+T:\mathscr{G}_s\to\mathscr{G}_{s-2}$ becomes unitary.

Furthermore, let $I=(a,b)\subset\mathbb{R}$ be a non-empty bounded interval and $\mathscr{H}=L^2(I,\mathscr{G})$. Each function $f\in\mathscr{H}$ can then be uniquely represented as

$$f(t)=\sum_{n\in\mathbb{N}}f_n(t)e_n,\quad f_n(\cdot):=\big\langle f(\cdot),e_n\big\rangle_{\mathscr{G}}\in L^2(a,b).$$

Denote by S_0 the operator $f\mapsto -f''+Tf$ defined on the domain

$$\operatorname{dom}S_0=\Big\{f:f(t)=\sum_{n=1}^N f_n(t)e_n:\quad N\in\mathbb{N},\quad f_n\in H^2_0(a,b)\Big\}$$

and let S be its closure, which will be called the *minimal operator* generated by the differential expression $-d^2/dt^2+T$ on (a,b), and its adjoint S^* will be called the associated *maximal operator*. Using the unitary transform $U:\mathscr{H}\to\oplus_n L^2(a,b)$, $f\mapsto(f_n)$, one easily sees that S is unitarily equivalent to $\oplus S_n$ with S_n defined as in the preceding section 3.2, which gives a choice of a boundary triple. For what follows is will be more instructive to not to use directly the eigenbasis (e_n) and the unitary transform U associated with T and to reformulate the preceding constructions using the above spaces $\mathscr{G}_s$ as follows:

Lemma 7. *The domain of the adjoint S^* consists of the functions $f \in L^2(I,\mathscr{G})$ such that $(-f''+Tf) \in L^2(I,\mathscr{G})$, where f'' is computed in $\mathscr{G}_{-2}$, and the operator S^* acts by $f \mapsto -f''+Tf$. For any $f \in \operatorname{dom} S^*$, there exist boundary values*

$$f(a), f(b) \in \mathscr{G}_{-\frac{1}{2}}, \quad f'(a), f'(b) \in \mathscr{G}_{-\frac{3}{2}},$$

which are bounded with respect to the graph norm of S^*, and $(\mathscr{G} \otimes \mathbb{C}^2, \Gamma, \Gamma')$ with

$$\Gamma f = D^{-1} \begin{pmatrix} f(a) \\ f(b) \end{pmatrix}, \quad \Gamma' f = D \left[\begin{pmatrix} f'(a) \\ -f'(b) \end{pmatrix} - m(-T, b-a) \begin{pmatrix} f(a) \\ f(b) \end{pmatrix} \right],$$

is a boundary triple for S, and the respective Weyl function is

$$M(z) = D\Big(m(z-T, b-a) - m(-T, b-a)\Big)D,$$
$$D := \begin{pmatrix} (T+1)^{\frac{1}{4}} & 0 \\ 0 & (T+1)^{\frac{1}{4}} \end{pmatrix}.$$

The minimal operator S is exactly the restriction of S^* to the functions f satisfying $f(a) = f(b) = f'(a) = f'(b) = 0$.

Let A be the extension of S corresponding to the boundary condition $f(a) = f(b) = 0$, which will be called the *Dirichlet realization* of $-d^2/dt^2 + T$ on (a,b), and B be the extension corresponding to the boundary condition $f'(a) = f'(b) = 0$, called then the *Neumann realization*. By construction of the previous subsection, the both operators are self-adjoint and with compact resolvents.

We introduce the operator Sobolev space $H^2_T(I) := L^2(I,\mathscr{G}_2) \cap H^2(I,\mathscr{G})$. In other words, $H^2_T(I)$ consists of the functions $f \in L^2(I,\mathscr{G})$ satisfying

$$\|f\|^2_{H^2_T(I)} := \sum_{n\in\mathbb{N}} \big(\|f_n''\|^2_{L^2(I)} + (1+\lambda_n)^2 \|f_n\|^2_{L^2(I)}\big) < +\infty,$$

and one has the obvious inclusion $H^2_T(I) \subset \operatorname{dom} S^*$.

Lemma 8. *The linear maps*

$$H^2_T(I) \ni f \mapsto \big(f(a), f(b)\big) \in \mathscr{G}_{\frac{3}{2}} \times \mathscr{G}_{\frac{3}{2}},$$
$$H^2_T(I) \ni f \mapsto \big(f'(a), -f'(b)\big) \in \mathscr{G}_{\frac{1}{2}} \times \mathscr{G}_{\frac{1}{2}}$$

are surjective. Furthermore, if $f \in \operatorname{dom} S^*$, then $f \in H^2_T(I)$ if and only if $f(a), f(b) \in \mathscr{G}_{\frac{3}{2}}$.

Proof. By Proposition 6, one has $\operatorname{dom} A \subset H^2_T(I)$ and $\operatorname{dom} B \subset H^2_T(I)$, and the surjectivity follows from Lemmas 4 and 5. Now let $f \in \operatorname{dom} S^*$ with $f(a), f(b) \in \mathscr{G}_{\frac{3}{2}}$, then due to the surjectivity there exists $g \in H^2_T(I)$ with $g(a) = f(a)$ and $g(b) = f(b)$. The function $f_0 := f - g$ satisfies $f_0(a) = f_0(b) = 0$, hence, $f_0 \in \operatorname{dom} A \subset H^2_T(I)$. □

As a direct consequence we obtain:

Corollary 9. $\operatorname{dom} S = H^2_{T,0}(I) := \big\{ f \in H^2_T(I) : f(a) = f(b) = 0, f'(a) = f'(b) = 0 \big\}$.

3.4 Sign-changing operator on an interval

Let T be as in the preceding subsection 3.3. Consider three parameters $a>0$, $b>0$, $\mu\neq 0$, and the intervals

$$I_- := (-a,0), \quad I_+ := (0,b), \quad I := (-a,b).$$

We would like to construct and study self-adjoint realizations of the operator L in $L^2(I,\mathscr{G})$ formally given by the differential expression

$$(Lf)(t) = \begin{cases} -\mu\big[& -f''(t)+Tf(t)\big], & t & \in I_-, \\ & -f''(t)+Tf(t)\ , & t & \in I_+, \end{cases}$$

the transmission conditions $f(0^-)=f(0^+)$ and $-\mu f'(0^-)=f'(0+)$ at zero and the Dirichlet boundary condition $f(-a)=f(b)=0$ at the endpoints. To be more precise, one uses the natural identification $L^2(I,\mathscr{G})\simeq L^2(I_-,\mathscr{G})\oplus L^2(I_+,\mathscr{G})$ and considers the following linear operator L in $L^2(I_-,\mathscr{G})\oplus L^2(I_+,\mathscr{G})$:

$$L\begin{pmatrix} f_- \\ f_+ \end{pmatrix} = \begin{pmatrix} -\mu(-f_-''+Tf_-) \\ -f_+''+Tf_+ \end{pmatrix},$$
$$\operatorname{dom} L = \big\{ f_\pm \in H^2_T(I_\pm) : f_-(-a)=f_-(0)-f_+(0)=f_+(b)=0, \\ -\mu f_-'(0)=f_+'(0)\big\}.$$

We will consider the operator L as an extension of another closed densely defined symmetric operator. Namely, let $A_\pm$ be the Dirichlet realizations of $-d^2/dt^2+T$ in $I_\pm$, which are both self-adjoint in $L^2(I_\pm,\mathscr{G})$ and with compact resolvents. Consider the operator $A:=(-\mu A_-)\oplus A_+$ and the linear map

$$\tau : \mathscr{H}(A)\to\mathscr{G}^4, \quad \tau\begin{pmatrix} f_- \\ f_+ \end{pmatrix} = \begin{pmatrix} D & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} -\mu & 0 & 0 & 0 \\ 0 & -\mu & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} f_-'(-a) \\ -f_-'(0) \\ f_+'(0) \\ -f_+'(b) \end{pmatrix},$$

where

$$D := \begin{pmatrix} (T+1)^{\frac{1}{4}} & 0 \\ 0 & (T+1)^{\frac{1}{4}} \end{pmatrix}$$

which is bounded and surjective due to the above constructions, and consider the restriction S of A to $\ker\tau$. Indeed, the operator S decomposes as $S=(-\mu S_-)\oplus S_+$, where $S_\pm$ are closed densely defined symmetric operators in $L^2(I_\pm,\mathscr{G})$ covered by the constructions of Lemma 7, and the adjoint is similarly decomposed as $S^*=$

$(-\mu S_-^*)\oplus S_+^*$. As a result, we obtain a boundary triple $(\mathscr{G}^4,\Gamma,\Gamma')$ for S with

$$\Gamma f=\begin{pmatrix}(T+1)^{-\frac14}f_-(-a)\\(T+1)^{-\frac14}f_-(0)\\(T+1)^{-\frac14}f_+(0)\\(T+1)^{-\frac14}f_+(b)\end{pmatrix},$$

$$\Gamma' f=\begin{pmatrix}-\mu D\left[\begin{pmatrix}f_-'(-a)\\-f_-'(0)\end{pmatrix}-m(-T,a)\begin{pmatrix}f_-(-a)\\f_-(0)\end{pmatrix}\right]\\ D\left[\begin{pmatrix}f_+'(0)\\-f_+'(b)\end{pmatrix}-m(-T,b)\begin{pmatrix}f_+(0)\\f_+(b)\end{pmatrix}\right]\end{pmatrix},$$

the associated Weyl function is

$$M(z)=\begin{pmatrix}D&0\\0&D\end{pmatrix}N(z)\begin{pmatrix}D&0\\0&D\end{pmatrix},$$

$$N(z):=\begin{pmatrix}-\mu\left[m(-\frac{z}{\mu}-T,a)-m(-T,a)\right]&0\\0&m(z-T,b)-m(-T,b)\end{pmatrix},$$

and the expression of the respective γ-fields $z\mapsto G(z)$ will have no importance, we just remark that due to (1) the only possible singularities of $z\mapsto G(z)$ at the points of $\sigma(A)$ are poles with finite-dimensional residues.

In view of Lemma 8, the domain of L can be rewritten as

$$\operatorname{dom}L=\Big\{f=(f_-,f_+)\in\operatorname{dom}S^*:\quad f_\pm(0)\in\mathscr{G}_{\frac32},\quad -\mu f_-'(0)=f_+'(0),\\ f_-(-a)=f_-(0)-f_+(0)=f_+(b)=0\Big\}.\quad(10)$$

The last boundary condition in (10) can be rewritten as $\Gamma f=\Pi\Gamma f$, or, equivalently, as $\Gamma f\in\operatorname{ran}\Pi$, where $\Pi:\mathscr{G}^4\to\mathscr{G}^4$ is the orthogonal projector given by

$$\Pi(f_1,f_2,f_3,f_4)=\frac12\big(0,f_2+f_3,f_2+f_3,0\big).$$

In the subsequent constructions it will be convenient to use the unitary map $U:\mathscr{G}\to\operatorname{ran}\Pi$ given by $Ug=\frac{1}{\sqrt2}(0,g,g,0)$, then $U^*(0,g,g,0)=\sqrt{2}g$, and (10) takes the form

$$\operatorname{dom}L=\big\{f=(f_-,f_+)\in\operatorname{dom}S^*:\Gamma f\in U(\operatorname{dom}\Theta),\\ U^*\Pi\Gamma' f=\Theta U^*\Gamma f\big\}.\quad(11)$$

where Θ is a linear operator in $\mathscr{G}$ given by

$$\Theta=\frac12(T+1)^{\frac14}\sqrt{T}\Big(\coth(b\sqrt{T})-\mu\coth(a\sqrt{T})\Big)(T+1)^{\frac14},\qquad(12)$$

$$\operatorname{dom}\Theta=\mathscr{G}_2.$$

Therefore, in terms of boundary triples one represents $L = A_{\Pi,U\Theta U^*}$ (see Subsection 2.2), and we remark that

$$\begin{aligned}\mathscr{M}(z) &:= U^*\Pi M(z)\Pi^* U \\ &= \frac{1}{2}(T+1)^{\frac{1}{4}}\sqrt{T}\Big(\coth(b\sqrt{T}) - \mu\coth(a\sqrt{T})\Big)(T+1)^{\frac{1}{4}} \\ &\qquad - \frac{1}{2}(T+1)^{\frac{1}{4}}\Bigg[\sqrt{T-z}\coth(b\sqrt{T-z}) \\ &\qquad\qquad - \mu\sqrt{T+\frac{z}{\mu}}\coth\Big(a\sqrt{T+\frac{z}{\mu}}\Big)\Bigg](T+1)^{\frac{1}{4}}.\end{aligned}$$

Proposition 10. *For $\mu \neq 1$, the operator L is self-adjoint and has a compact resolvent.*

Proof. According to the discussion of Subsection 2.2, the self-adjointness of L is equivalent to the self-adjointness of Θ in $\mathscr{G}$. One easily sees that on $\operatorname{dom}\Theta$ one has $\Theta = \frac{1-\mu}{2}T + C$ with a bounded self-adjoint operator C. As T defined on $\mathscr{G}_2 \equiv \operatorname{dom} T$ is self-adjoint, the operators Θ and then L are self-adjoint too. Furthermore, for non-real z the operator $\Theta - \mathscr{M}(z)$ has a compact inverse. As A is with compact resolvent as well, it follows from the resolvent formula (4) that the resolvent of L is a compact operator. □

Proposition 11. *For $\mu = 1$, the operator L is not closed, but it is essentially self-adjoint. Its closure $\mathscr{L}$ is the restriction of S^* to the domain*

$$\begin{aligned}\operatorname{dom}\mathscr{L} = \big\{f = (f_-, f_+) \in \operatorname{dom} S^* : \quad & f_-(-a) = f(b) = 0, \\ & f_-(0) = f_+(0),\ -f_-'(0) = f_+'(0)\big\}. \quad (13)\end{aligned}$$

and the essential spectrum of $\mathscr{L}$ is $\{0\}$. If $a = b$, the zero is an isolated infinitely degenerate eigenvalue of $\mathscr{L}$. If $a \neq b$, then 0 is not an eigenvalue of $\mathscr{L}$. In this case, there exist $\varepsilon > 0$ and $N > 0$ such that there exist a bijection E between the set $\{n : n \geq N\}$ and the set of the eigenvalues of $\mathscr{L}$ in $(-\varepsilon, \varepsilon)$ such that for $n \to +\infty$ there holds

$$E(n) \sim -2\lambda_n e^{-2a\sqrt{\lambda_n}} \text{ if } a < b, \quad E(n) \sim 2\lambda_n e^{-2b\sqrt{\lambda_n}} \text{ if } a > b. \qquad (14)$$

Proof. One easily sees that $\Theta = \Phi(T)|_{\mathscr{G}_2}$ with a bounded function $\Phi : \mathbb{R}_+ \to \mathbb{R}$ satisfying $\Phi(+\infty) = 0$. Therefore, Θ is a compact operator in $\mathscr{G}$. As its domain $\mathscr{G}_2$ is dense in $\mathscr{G}$, it has a unique self-adjoint extension, which is just the closure $\overline{\Theta}$ defined on the the whole space. According to the constructions of subsection 2.2 it implies that L is essentially self-adjoint, and the domain of the closure $\mathscr{L} = \overline{L} = A_{\Pi,U\overline{\Theta}U^*}$ is given as in (11) with Θ replaced by $\overline{\Theta}$, and by using the explicit expressions for Γ and Γ' one arrives at (13).

In order to study the essential spectrum of $\mathscr{L}$ let us remark first that one has $0 \in \sigma_{\mathrm{ess}}(\overline{\Theta})$ due to the compactness of $\overline{\Theta}$, hence, by Corollary 2 one has $0 \in \sigma_{\mathrm{ess}}(\mathscr{L})$. Therefore, it remains to show that $\mathscr{L}$ has no essential spectrum in $\mathbb{R} \setminus \{0\}$. To

see this, remark first that $\mathscr{M}(z)-\frac{z}{2}\mathrm{Id}$ is a compact operator for $z\notin\sigma(A)$. Denote $\Sigma:=\sigma(A)\cup\{0\}$, then for $z\in\mathbb{C}\setminus\Sigma$ one can represent $\mathscr{M}(z)-\overline{\Theta}=\frac{1}{2}z\big(\mathrm{Id}+\mathscr{K}(z)\big)$, where $\mathscr{K}(z)$ are compact operators meromorphically depending on $z\in\mathbb{C}\setminus\{0\}$ and having at most simple poles with finite-dimensional residues at the points of $\sigma(A)$. Due to the meromorphic Fredholm alternative, see e.g. [40, Theorem XIII.13], only two situations are possible:

(a) $0\in\sigma(\mathscr{M}(z)-\overline{\Theta})$ for all $z\in\mathbb{C}\setminus\Sigma$,

(b) there exists a subset $B\subset\mathbb{C}\setminus\{0\}$, without accumulation points in $\mathbb{C}\setminus\{0\}$, such that the inverse $(\mathscr{M}(z)-\overline{\Theta})^{-1}$ exists and is bounded for $z\in\mathbb{C}\setminus\big(\{0\}\cup B\cup\sigma(A)\big)$ and extends to a meromorphic function in $\mathbb{C}\setminus\big(\{0\}\cup B\big)$ such that the coefficients in the Laurent series of $(\mathscr{M}(z)-\overline{\Theta})^{-1}$ at the points of B are finite-dimensional operators.

The case (a) is impossible, in fact, by Corollary 2 this would imply the presence of a non-empty non-real spectrum for the self-adjoint operator $\mathscr{L}$. Therefore, the case (b) is realized. It follows then from the resolvent formula (4) that the only possible singularities of the resolvent of $\mathscr{L}\equiv A_{\Pi,U\overline{\Theta}U^*}$ in $\mathbb{C}\setminus\{0\}$ are poles with finite-dimensional residues, which shows that $\mathscr{L}$ has no essential spectrum in $\mathbb{C}\setminus\{0\}$.

By Corollary 2 one has $\dim\ker\mathscr{L}=\dim\ker\overline{\Theta}$. For $a=b$ one has simply $\overline{\Theta}=0$, which gives $\dim\ker\overline{\Theta}=\infty$. For z close to 0 one can represent $\mathscr{M}(z)-\overline{\Theta}\equiv\mathscr{M}(z)=z\mathscr{M}'(0)+z^2B(z)$ and the norms of $B(z)$ are uniformly bounded, and $\mathscr{M}'(0)$ has a bounded inverse by (2). It follows that there exists $\varepsilon>0$ such that $0\in\rho\big(\mathscr{M}(z)-\overline{\Theta}\big)$ for $0<|z|<\varepsilon$, and in view of Corollary 2 this shows that 0 is an isolated point in the spectrum of $\mathscr{L}$.

For $a\neq b$ we represent

$$\overline{\Theta}=\frac{\sqrt{T}\sinh\big((b-a)\sqrt{T}\big)}{\sinh(a\sqrt{T})\sinh(b\sqrt{T})},$$

which shows that $\dim\ker\overline{\Theta}=0$. Now it remains to show the asymptotics (14). Let us take a small $\rho>0$, then by Corollary 2 the eigenvalues of $\mathscr{L}$ are exactly the values $E\in(-\rho,\rho)$ for which 0 is an eigenvalue of $\mathscr{M}(E)-\overline{\Theta}$ (with same multiplicities), i.e. iff for some $n\in\mathbb{N}$ one has

$$\sqrt{\lambda_n-E}\coth(b\sqrt{\lambda_n-E})-\sqrt{\lambda_n+E}\coth\Big(a\sqrt{\lambda_n+E}\Big).$$

Remark for each fixed n this equation admits at most finitely many solutions in $(-\rho,\rho)$, hence, in order to study the accumulation it is sufficient to look at large values of n. Then one can assume without loss of generality that $\lambda_n+E\neq0$, and the equation rewrites as $F_{\lambda_n}(E)=0$, where

$$F_\lambda(z)=\sqrt{\frac{\lambda-z}{\lambda+z}}\coth(b\sqrt{\lambda-z})-\coth(a\sqrt{\lambda+z}).$$

An elementary analysis shows that there exists λ_0 such that for $\lambda > \lambda_0 > 0$ one has $(F_\lambda)' < 0$ in $(-\rho,\rho)$ with $F_\lambda(-\rho) > 0$ and $F_\lambda(-\rho) < 0$. Therefore, for $\lambda > \lambda_0$ there is a unique $z_\lambda \in (-\rho,\rho)$ with $F_\lambda(z_\lambda) = 0$. Then one has

$$\sqrt{\frac{\lambda - z_\lambda}{\lambda + z_\lambda}} = \frac{\coth(a\sqrt{\lambda + z_\lambda})}{\coth(b\sqrt{\lambda - z_\lambda})}$$

and taking Taylor expansions with respect to z_λ/λ for large λ one obtains

$$1 - \frac{z_\lambda}{\lambda} + O\Big(\Big(\frac{z_\lambda}{\lambda}\Big)^2\Big) = \frac{1 + 2e^{-2a\sqrt{\lambda}} + o(e^{-2a\sqrt{\lambda}})}{1 + 2e^{-2b\sqrt{\lambda}} + o(e^{-2b\sqrt{\lambda}})},$$

which shows that $z_\lambda = 2\lambda(e^{-2b\sqrt{\lambda}} - e^{-2a\sqrt{\lambda}}) + o(e^{-2a\sqrt{\lambda}} + e^{-2b\sqrt{\lambda}})$ as $\lambda \to +\infty$. Using $E(n) = z_{\lambda_n}$ we arrive at the result. □

4 Indefinite Laplacians with separated variables

4.1 Indefinite Laplacian on a cylinder

Let us see how the preceding constructions apply to a simple two-dimensional example. Let a, b, ℓ be strictly positive constants. Let $\mathscr{C}$ be a circle of length $2\ell > 0$, i.e. $\mathscr{C} = \mathbb{R}/(2\ell\mathbb{Z})$, and $\Omega := (-a,b) \times \mathscr{C}$. We define a function $h : \Omega \to \mathbb{R}$ by $h(t,s) = -\mu$ for $t < 0$ and $h(t,s) = 1$ for $t > 0$. Our objective is to find self-adjoint realizations of the operator $u \mapsto -\nabla \cdot (h\nabla)u$ in Ω with the Dirichlet boundary condition $u = 0$ on $\partial\Omega$.

We denote $I_- = (-a,0)$, $I_+ := (0,b)$, $I := (-a,b)$ and $\Omega_\pm := I_\pm \times \mathscr{C}$. By setting $\mathscr{G} := L^2(\mathscr{C})$, we obtain the identifications $L^2(\Omega_\pm) \simeq L^2(I_\pm, \mathscr{G})$. Furthermore, we will identify $L^2(\Omega) \simeq L^2(\Omega_-) \times L^2(\Omega_+)$, $u \simeq (u_-, u_+)$, where $u_\pm$ is the restriction of u to $\Omega_\pm$. With these conventions, let us consider the following operator L in $L^2(\Omega_\pm) \simeq L^2(I_\pm, \mathscr{G})$ acting as $L(u_-, u_+) = (\mu\Delta u_-, -\Delta u_+)$ on the domain

$$\begin{aligned}\operatorname{dom} L = \Big\{(u_-,u_+) :\quad & u_\pm \in H^2(\Omega_\pm),\\ & u_-(-a,\cdot) = u_+(b,\cdot) = 0, \quad u_-(0,\cdot) = u_+(0,\cdot),\\ & \qquad -\mu\frac{\partial u_-}{\partial t}(0,\cdot) = \frac{\partial u_+}{\partial t}(0,\cdot)\Big\}.\end{aligned}$$

In order to make a link with the constructions of the preceding section we denote by T the self-adjoint Laplacian in $L^2(\mathscr{C})$ acting as $f \mapsto -f''$ on the domain $\operatorname{dom} T = H^2(\mathscr{C})$. The associated spaces $\mathscr{G}_s(T)$ are then the usual Sobolev spaces $H^s(\mathscr{C})$. Furthermore, we introduce the minimal operators $S_\pm$ generated by $-d^2/dt^2 + T$ in $L^2(I_\pm, \mathscr{G})$, defined on $H^2_{T,0}(I_\pm, \mathscr{G})$, and their adjoints, i.e. the maximal operators, $S_\pm^*$.

Proposition 12. *There holds $H^2_{T,0}(I_\pm) = H^2_0(\Omega_\pm)$, and the operators $S_\pm$ act by $u_\pm \mapsto -\Delta u_\pm$.*

Proof. As $\Omega_\pm$ have smooth boundaries, the space $H^2_0(\Omega_\pm)$ is the closure of $C^\infty_0(\Omega_\pm)$ in the norm

$$\|u\|^2_{H^2(\Omega_\pm)} = \int_{\Omega_\pm} \big(|\Delta u|^2 + |u|^2\big) dx.$$

On the other hand, the space $H^2_{T,0}(I_\pm)$ is the closure, with respect to the graph norm $\|\cdot\|_\pm$ of $S_\pm$, of the set

$$\mathscr{D}_{\pm,0} := \Big\{ u(t,s) = \sum_{n=-N}^{N} u_n^\pm(t) e^{2\pi i n s/\ell} : \\ N \in \mathbb{N},\ u_n^\pm \in C^\infty_0(I_\pm) \Big\} \subset C^\infty_0(\Omega_\pm),$$

and for $u \in \mathscr{D}_{\pm,0}$ one has $\|u\|^2_\pm = \|u\|^2_{H^2(\Omega_\pm)}$. The classical theory of Fourier series shows that each function from $C^\infty_0(\Omega_\pm)$ can be approximated by functions from $\mathscr{D}_{\pm,0}$ in any C^k-norm, hence, also in the $\|\cdot\|_\pm$ norms, which shows the equality between the spaces. □

In view of Proposition 12, the maximal operators $S^*_\pm$ are defined as in the classical PDE theory, i.e. they act as

$$\operatorname{dom} S^*_\pm = \big\{ u_\pm \in L^2(\Omega_\pm) : \Delta u_\pm \in L^2(\Omega_\pm) \big\}, \quad S^*_\pm u_\pm = -\Delta u_\pm,$$

and the functions $u_\pm \in \operatorname{dom} S^*_\pm$ admit boundary values

$$\begin{aligned} u_\pm|_{\partial\Omega_\pm} &\in H^{-\frac{1}{2}}(\partial\Omega) \simeq H^{-\frac{1}{2}}(\mathscr{C}) \times H^{-\frac{1}{2}}(\mathscr{C}), \\ \partial_\pm u_\pm|_{\partial\Omega_\pm} &\in H^{-\frac{3}{2}}(\partial\Omega_\pm) \simeq H^{-\frac{3}{2}}(\mathscr{C}) \times H^{-\frac{3}{2}}(\mathscr{C}), \end{aligned} \tag{15}$$

where $\partial_\pm$ is the outward normal derivative on the boundary of $\Omega_\pm$, and these boundary values are bounded with respect to the graph norm of $S^*_\pm$, see [31, Chapter 2, Section 6.5].

Lemma 13. *For $u_\pm \in \operatorname{dom} S^*_\pm$, the values of $u_\pm$ and $\partial u_\pm/\partial t$ at the endpoints of $I_\pm$ defined as in Lemma 7 coincide with the Sobolev boundary values in* (15).

Proof. Recall that by construction the sets

$$\mathscr{D}_\pm := \Big\{ u_\pm(t,s) = \sum_{n=-N}^{N} u_n^\pm(t) e^{2\pi i n s/\ell} : \\ N \in \mathbb{N}, \quad u_n^\pm \in C^\infty(\overline{I_\pm}) \Big\} \subset C^\infty(\overline{\Omega_\pm}).$$

are dense in $\operatorname{dom} S^*_\pm$ in the respective graph norms. For $u_\pm \in \mathscr{D}_\pm$, the two trace versions coincide. As the traces are bounded with respect to the graph norm of $S^*_\pm$, the result follows. □

Finally we arrive at the following identification:

Lemma 14. $H^2_T(I_\pm) = H^2(\Omega_\pm)$.

Proof. As the boundary of $\partial\Omega_\pm$ is smooth, it is a standard elliptic regularity result that, if $u_\pm \in \operatorname{dom} S^*_\pm$, then $u \in H^2(\Omega_\pm)$ iff $u_\pm|_{\partial\Omega_\pm} \in H^{\frac{3}{2}}(\partial\Omega_\pm)$. Now it is sufficient to substitute the result of Lemma 13 into the second assertion of Lemma 8. □

With the above Lemmas at hand, the study of the operator L is reduced to the constructions of the subsection 3.4 by considering it as an extension of the operator $S := (-\mu S_-) \oplus S_+$, and by using Propositions 10 and 11 one arrives at the following results:

Proposition 15. *If $\mu \neq 1$, then the above operator L is self-adjoint and has a compact resolvent. For $\mu = 1$, the above operator L is essentially self-adjoint, and its closure $\mathscr{L}$ acts as $(u_-, u_+) \mapsto (\mu\Delta u_-, -\Delta u_+)$ on the domain*

$$\begin{aligned}\operatorname{dom}\mathscr{L} = \Big\{(u_-,u_+) : u_\pm \in L^2(\Omega_\pm),\ \Delta u_\pm \in L^2(\Omega_\pm),\\ u_-(-a,\cdot) = u_+(b,\cdot) = 0,\, u_-(0,\cdot) = u_+(0,\cdot),\\ -\frac{\partial u_-}{\partial t}(0,\cdot) = \frac{\partial u_+}{\partial t}(0,\cdot)\Big\},\end{aligned}$$

where the boundary values are understood as the Sobolev traces. One has $\sigma_{\mathrm{ess}}(\mathscr{L}) = \{0\}$. If $a = b$, then 0 is an isolated infinitely degenerate eigenvalue of $\mathscr{L}$. If $a \neq b$, then 0 is not an eigenvalue of $\mathscr{L}$, and the eigenvalues accumulate to the zero from below (respectively, from above) if $a < b$ (respectively, $a > b$).

4.2 Indefinite Laplacian on a rectangle

Let us modify the example of the preceding section. Let a, b, ℓ be strictly positive constants and $R := (-a, b) \times (0, \ell)$. We define a function $h : R \to \mathbb{R}$ by $h(t,s) = -\mu$ for $t < 0$ and $h(t,s) = 1$ for $t > 0$. Our objective is to find self-adjoint realizations of the operator $u \mapsto -\nabla \cdot (h\nabla)u$ in R with the Dirichlet boundary condition $u = 0$ on ∂R.

We denote $I_- = (-a, 0)$, $I_+ := (0, b)$, $I := (-a, b)$, $R_\pm := I_\pm \times (0, \ell)$ and use the natural identification $L^2(R) \simeq L^2(R_-) \times L^2(R_+)$, $u \simeq (u_-, u_+)$, where $u_\pm$ is the restriction of u to $R_\pm$. Let us consider the following operator L_0 in $L^2(R_-) \times L^2(R_+)$ acting as $L_0(u_-, u_+) = (\mu\Delta u_-, -\Delta u_+)$ on the domain

$$\begin{aligned}\operatorname{dom} L_0 = \Big\{(u_-,u_+) :\quad & u_\pm \in H^2(R_\pm),\\ & u_-(-a,\cdot) = u_+(b,\cdot) = 0, \quad u_\pm(\cdot,0) = u_\pm(\cdot,\ell) = 0,\\ & u_-(0,\cdot) = u_+(0,\cdot), \quad -\mu\frac{\partial u_-}{\partial t}(0,\cdot) = \frac{\partial u_+}{\partial t}(0,\cdot)\Big\}.\end{aligned}$$

In order to simplify the construction, we can reduce the study to the case of a cylinder. Namely, let $\mathscr{C} := \mathbb{R}/(2\ell\mathbb{Z})$ and $\Omega := I \times \mathscr{C}$. In $L^2(\Omega)$ consider the orthogonal

projector P onto the subspace $\Lambda := \{u : \ u(t,s) = -u(t,2\ell - s)\} \subset L^2(\Omega)$ and the unitary operator

$$U : L^2(R) \to \Lambda, \quad (Uu)(t,s) = \frac{1}{\sqrt{2}} \begin{cases} u(t,s), & s \in (0,\ell), \\ -u(t,2\ell - s), & s \in (\ell, 2\ell) \end{cases}.$$

One easily checks that Λ is an invariant subspace of the operator L from the preceding subsection and that $L_0 = U^* PLP^* U$. Furthermore, if one denotes by T_0 the Dirichlet Laplacian in $L^2(0,\ell)$ and by $S_{0,\pm}$ the minimal realizations of $-d^2/dt^2 + T_0$ on $I_\pm$ and $S_0 := (-\mu S_{0,-}) \oplus S_{0,+}$, then one also has $S_0 = U^* PSP^* U$ with the operator S from the preceding subsection, and the similar representation holds for the maximal operator as well. Therefore, a minor variation of the preceding constructions gives the following result:

Proposition 16. *If $\mu \neq 1$, then the above operator L_0 is self-adjoint and has a compact resolvent. For $\mu = 1$, the operator L_0 is not closed but is essentially self-adjoint, and its closure $\mathscr{L}_0$ acts as*

$$(u_-, u_+) \mapsto (\mu \Delta u_-, -\Delta u_+)$$

on the domain

$$\begin{aligned} \operatorname{dom} \mathscr{L}_0 = \Big\{ (u_-, u_+) : \quad & u_\pm \in L^2(R_\pm), \ \Delta u_\pm \in L^2(R_\pm), \\ & u_-(-a,\cdot) = u_+(b,\cdot) = 0, \ u(\cdot,0) = u(\cdot,\ell) = 0, \\ & u_-(0,\cdot) = u_+(0,\cdot), \quad -\frac{\partial u_-}{\partial t}(0,\cdot) = \frac{\partial u_+}{\partial t}(0,\cdot) \Big\}, \end{aligned}$$

where the boundary values are understood as the Sobolev traces. One has $\sigma_{\mathrm{ess}}(\mathscr{L}_0) = \{0\}$. If $a = b$, then 0 is an isolated infinitely degenerate eigenvalue of $\mathscr{L}_0$. If $a \neq b$, then 0 is not an eigenvalue of $\mathscr{L}_0$, and the eigenvalues accumulate to zero from below (respectively, from above) if $a < b$ (respectively, $a > b$).

Remark that for $a = b = \ell = 1$ and $\mu = 1$ one recovers exactly the result of [2].

5 Self-adjoint indefinite Laplacians in general domains: a review

5.1 Smooth domains

The construction of the preceding discussion looks heavily depending on the presence of the special geometry and of the separation of variables. Nevertheless, the general construction and the use of boundary triples appear to useful in a much more general context.

Let $d \geq 2$ and $\Omega \subset \mathbb{R}^d$ be a bounded open set. Furthermore, let Ω_- be a subset of Ω having a boundary Σ and such that $\overline{\Omega}_- \subset \Omega$, and we set $\Omega_+ := \Omega \setminus \overline{\Omega_-}$, whose

boundary is $\partial\Omega_+ = \Sigma \cup \partial\Omega$, and we denote by $\partial_\pm$ the outward normal derivatives on $\partial\Omega_\pm$. For $\mu > 0$, consider the function $h : \Omega \to \mathbb{R}$, $h = -\mu$ in Ω_- and $h = 1$ in Ω_+. It seems that the first result discussing the self-adjointness of $-\nabla \cdot (h\nabla)$ in Ω in such a setting was obtained in [6]:

Proposition 17. *Assume that $d = 2$, that the boundaries of $\partial\Omega_\pm$ are C^2, and that $\mu \neq 1$, then the operator $L = -\nabla \cdot (h\nabla)$ with*

$$\operatorname{dom} L = \left\{ u \in H_0^1(\Omega) : \nabla \cdot (h\nabla) u \in L^2(\Omega) \right\}$$

is self-adjoint with compact resolvent in $L^2(\Omega)$.

The study was based on a reformulation using a boundary integral equation and used a compactness result from [15]. The case $\mu = 1$ was not covered by the machinery, but it was clearly seen that the above operator L is not self-adjoint for this case. In the author's joint work [10] the study was extended to a more general combination of parameters using the machinery of boundary triples. Namely, for $s \geq 0$ introduce the sets $\mathscr{D}^s \subset L^2(\Omega_-) \oplus L^2(\Omega_+)$,

$$\mathscr{D}^s := \Big\{ u = (u_-, u_+) \in H^s(\Omega_-) \oplus H^s(\Omega_+) : \Delta u_\pm \in L^2(\Omega_\pm), \\ u_- = u_+ \text{ and } \mu \partial_- u_- = \partial_+ u_+ \text{ on } \Sigma, \quad u_+ = 0 \text{ on } \partial\Omega \Big\},$$

and the operator L in $L^2(\Omega_-) \oplus L^2(\Omega_+)$ acting as

$$\operatorname{dom} L = \mathscr{D}^2, \quad L(u_-, u_+) = (\mu \Delta u_-, -\Delta u_+),$$

then the following results were obtained:

Proposition 18. *Assume that the boundaries of $\Omega_\pm$ are C^∞. If $\mu \neq 1$, the operator L is self-adjoint with a compact resolvent. Assume now that $\mu = 1$, then L is not closed but is essentially self-adjoint, and the following assertions hold for its closure $\mathscr{L}$:*

(a) If $d = 2$, then $\operatorname{dom} \mathscr{L} = \mathscr{D}^0$ and $\sigma_{\mathrm{ess}}(\mathscr{L}) = \{0\}$.

(b) If $d \geq 3$, then $\mathscr{D}^1 \subset \operatorname{dom} \mathscr{L}$, and

- *If on each maximal connected component of Σ the principal curvatures are either all strictly positive or all strictly negative (in particular, if each maximal connected component of Σ is strictly convex), then $\operatorname{dom} \mathscr{L} = \mathscr{D}^1$, and $\mathscr{L}$ has compact resolvent.*
- *If a subset of the interface Σ is isometric to a non-empty open subset of $\mathbb{R}^{d-1}$, then $\operatorname{dom} \mathscr{L} \not\subset \mathscr{D}^s$ for any $s > 0$, and $\{0\} \subseteq \sigma_{\mathrm{ess}}(\mathscr{L})$.*

At first sight, the domain of L given in Proposition 17 is strictly larger than the set $\mathscr{D}^2$ appearing as the self-adjointness domain in the first part of Proposition 18, but in fact they coincide due to the maximality property of self-adjoint operators. A more direct proof of the equality can be found in [16].

Let us compare the scheme of proof of Proposition 18 given in [10] with the above proof for rectangles. The construction starts by considering the Dirichlet Laplacians $A_\pm$ in $L^2(\Omega_\pm)$ and the direct sum $A = (-\mu A_-) \oplus A_+$. Let $\Delta_\Sigma \leq 0$ be the Laplace-Beltrami operator on Σ and $\Delta_{\partial\Omega} \leq 0$ be the Laplace-Beltrami operator on $\partial\Omega$. We set $\Lambda := \sqrt{1-\Delta_\Sigma}$, $\Lambda_\partial := \sqrt{1-\Delta_{\partial\Omega}}$ considered first on smooth functions and then extended to unitary operators $H^s(\Sigma) \to H^{s-1}(\Sigma)$ and $H^s(\partial\Omega) \to H^{s-1}(\partial\Omega)$. Consider now the trace maps

$$\tau : \mathscr{H}(A) \to L^2(\Sigma) \oplus L^2(\Sigma) \oplus L^2(\partial\Omega), \quad \tau(u_-, u_+) = \begin{pmatrix} -\mu\Lambda^{\frac{1}{2}}\big(\partial_- u_-|_\Sigma\big) \\ \Lambda^{\frac{1}{2}}\big(\partial_+ u_+|_\Sigma\big) \\ \Lambda_\partial^{\frac{1}{2}}\big(\partial_+ u_+|_{\partial\Omega}\big) \end{pmatrix},$$

and the closed symmetric operator $S := A|_{\ker\tau}$. Proceeding as above one constructs a boundary triple for S and then represents the above operator L as an extension of S by representing in in the form $L = A_{\Pi,\Theta}$. We remark that the orthogonal projector Π in question is of the form $\Pi(\varphi_-, \varphi_+, \varphi_\partial) = \frac{1}{2}(\varphi_- + \varphi_+, \varphi_- + \varphi_+, 0)$, and its range will be identified with $L^2(\Sigma)$ with the help of the unitary operator $U : \operatorname{ran}\Pi \to L^2(\Sigma)$, $U(\varphi, \varphi, 0) = \sqrt{2}\,\varphi$. In order to introduce the respective operator parameter Θ one needs additional objects. Namely, for $z \in \rho(A)$ introduce the Dirichlet-to-Nuemann maps $D_z^\pm : H^s(\Sigma) \to H^{s-1}(\Sigma)$ by setting $D_z^\pm f = \partial_\pm u_\pm|_\Sigma$, where $u_\pm$ are the unique solutions of the following problems:

$$\begin{cases} (-\Delta - z)u_- = 0 \text{ in } \Omega_-, \\ \qquad\qquad u_- = f \text{ on } \Sigma, \end{cases} \qquad \begin{cases} (-\Delta - z)u_+ = 0 \text{ in } \Omega_+, \\ \qquad\qquad u_+ = f \text{ on } \Sigma, \\ \qquad\qquad u_+ = 0 \text{ on } \partial\Omega, \end{cases}$$

then $\Theta := U^*\Theta_0 U$, $\Theta_0 := \frac{1}{2}\Lambda^{\frac{1}{2}}(D_0^+ - \mu D_0^-)\Lambda^{\frac{1}{2}}$, $\operatorname{dom}\Theta_0 = H^2(\Sigma)$, and the (essential) self-adjointness of L is then equivalent to the (essential) self-adjointness of Θ_0. Up to this point one may observe some similarity with the case of rectangles, see e.g. (12). But, contrary to the construction for separated variables, the further analysis is less explicit and uses the pseudodifferential calculus: for $\mu \neq 1$ the operator Θ_0 is second order elliptic, hence is self-adjoint. For $\mu = 1$ it is at most first order and its closure is self-adjoint and it defined at least on $H^1(\Sigma)$. Furthermore, the principal symbol can be then computed explicitly, and it depends on the dimension and the principal curvatures of Σ, which allows for the study of the two particular cases mentioned (strictly convex or partially flat interface). One should mention that similar geometric conditions on the interface appeared previously in the context of the well-posedness of related transmission problems, see e.g. [36, 27].

To our knowledge, there are no more precise results on the spectrum of $\mathscr{L}$ for $\mu = 1$. The following aspects seem to be relevant for the original applications, and we formulate them as open questions:

Open question 1 Assume that $\mu = 1$ and $d = 2$. Describe the accumulation of the eigenvalues near 0. In particular, under which conditions do the eigenvalue accumulate to zero from above/from below only?

Open question 2 Assume that $\mu = 1$ and $d \geq 3$. Are there $\Omega_\pm$ such that the essential spectrum of the associated $\mathscr{L}$ is strictly larger than $\{0\}$? Can the essential spectrum contain an interval or cover the whole real axis?

It seems that so far there were no works dealing with self-adjoint realizations in the case of unbounded Ω, in particular, simply with $\Omega = \mathbb{R}^d$ and a bounded Ω_-. While the results concerning the Sobolev regularity of the functions in the domain seem to be easily transferable, it would be interesting to understand whether the loss of regularity for $\mu = 1$ has any consequence for the spectral properties. In fact, the point 0 would always be in the essential spectrum of the resulting operator $\mathscr{L}$ due to the presence of the non-compact part Ω_+, so the study of the density of states of $\mathscr{L}$ near 0 could be a more appropriate tool.

Indeed, one can study the dependence of the eigenvalues of L and $\mathscr{L}$ on parameters of a particular configuration. In this connection we mention the paper [13] studying the asymptotics of the eigenvalues of L when the domain Ω_- contracts to a point.

5.2 Non-smooth domains

In [6, 16] the case of planar domains with non-smooth interfaces was studied. Namely, let $d = 2$ and $\Omega_\pm$ and h be as in the preceding section, and assume that the interface Σ is C^2 smooth except a single point O (vertex). Denote by (r, θ) the polar coordinates centered at O, then we assume in addition that in a neighborhood of O the domain Ω_- coincides with the sector $0 < \theta < \omega$ with some $\omega \neq \pi$, and denote

$$\mu_\omega = \max\left\{\frac{\omega}{2\pi - \omega}, \frac{2\pi - \omega}{\omega}\right\}.$$

Consider the operator

$$L = -\nabla \cdot (h\nabla), \quad \operatorname{dom} L = \left\{u \in H^1_0(\Omega) : \nabla \cdot (h\nabla)u \in L^2(\Omega)\right\},$$

then the following results were obtained:

Proposition 19. *If $\mu \notin [\mu_\omega^{-1}, \mu_\omega]$, then the operator L is self-adjoint with compact resolvent. For $\mu \in [\mu_\omega^{-1}, \mu_\omega]$ the operator L is closed, has deficiency indices $(1,1)$, and any of its self-adjoint extensions has compact resolvent.*

In addition, it is shown in [16] that if $(u_-, u_+) \in \operatorname{dom} L$, then the functions $u_\pm$ are in fact H^2 near each regular point of Σ. As shown in [6, 16], if $\mu \in (\mu_\omega^{-1}, \mu_\omega)$, then the domain of each self-adjoint extension of L contains functions behaving near O as $r^{\pm i\eta}$ with some non-zero $\eta \in \mathbb{R}$, and we refer to [5] for an interpretation of such a behavior. Furthermore, this very special singularity is responsible for an unusual behavior of eigenvalues if one smoothens $\Omega_\pm$ near the vertex, see [12, 14].

A number of 3D situations with a non-smooth interface Σ were studied in [3]. While the self-adjointness in the L^2-setting was not addressed explicitly, the analysis suggests that for a large class of domains Ω_- with corners there is a set M_Σ such that the above operator L is self-adjoint iff $\mu \notin M_\Sigma$. It would be interesting to carry out

a precise analysis in such a setting and to charactzerize the critical set M_Σ in terms of geometric quantities. To our knowledge, the (essential) self-adjointness of the above operator L for $\mu = 1$ was never studied for the case of non-smooth interfaces.

5.3 Further results

The approaches presented were mostly based on the PDE machinery. In order to mention alternative ways of dealing with the problem, let us return back to the situation discussed in the introduction (Section 1) and consider the sesquilinear form

$$q(u,u) = \int_\Omega h|\nabla u|^2\,dx, \quad \operatorname{dom} q = H^1_0(\Omega).$$

Remark that by Lax-Milgram theorem for $h > c > 0$ there would exist a unique self-adjoint operator L in $L^2(\Omega)$ associated with the form q, i.e. such that $\operatorname{dom} L \subset \operatorname{dom} q$ and $\langle u, Lv\rangle_{L^2(\Omega)} = q(u,v)$ for all $u \in \operatorname{dom} q$ and $v \in \operatorname{dom} L$, and under suitable regularity assumption this operator would act as $-\nabla\cdot(h\nabla)$. The initial assumption about the positivity of h is not satisfied in our case, i.e. the form q is not semibounded from below, and the classical theory is not applicable, but a manual adaptation may work in some special situations, see e.g. [7]. On the other hand, there are a number of works dealing with systematic extensions of the classical theory to indefinite sesquilinear forms, see e.g. [22, 42] and references there-in. The theses [25] and [41] contain a number of results to be presented in the papers in preparation [23, 26]. At the moment we are now aware of the publication of the announced papers, but we mention a part of the results in order to have a more complete vision of the state of art in the domain.

Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$ be a bounded Lipschitz domain and $D : H^1_0(\Omega) \to L^2(\Omega)^d$ be defined by $Du = \nabla u$. We denote $\Lambda := \operatorname{ran} D \subset L^2(\Omega)^d$, which appears to be a closed subspace, and let $Q : L^2(\Omega)^d \to \Lambda$ be the orthogonal projector, then as a particular case of Theorem 8.2.2 in [41] one has the following result:

Proposition 20. *Assume that*

$$QhQ^* : \Lambda \to \Lambda \text{ has a bounded inverse}, \tag{16}$$

then there exists a unique self-adjoint operator L associated with the above form q. This operator L is boundedly invertible and has compact resolvent.

In fact, the results presented in [41] are more general and cover functions h taking more than two distinct values as well as some matrix operators, and it also gives Schatten-type estimates for the eigenvalues of L. On the other hand, they do not cover all possible cases: as already seen in some situations the resulting operator $\mathscr{L}$ does not have compact resolvent. Furthermore, the initial assumption on QhQ^* appears to be rather involved as it needs some information on the Dirichlet-to-Neumann maps, see [41, Theorem 8.2.8]. Nevertheless, we mention one of the situations which can be handled and which complements the examples considered in the preceding sections, see Corollary 8.4.9 in [41]:

Corollary 21. *Let $\Omega \subset \mathbb{R}^d$ be a Lipschitz domain symmetric with respect to the hyperplane $x_1 = 0$ and*

$$\Omega_\pm := \Omega \cap \big\{(x_1, x_2) : \pm x_1 > 0\big\},$$

then the condition (16) *is satsfied iff $\mu \neq 1$.*

We remark that counterparts of the two above results for the operators with *Neumann* boundary condition on the exterior boundary $\partial\Omega$ can be found in Chapter 9 of [41].

For the situations in which Proposition 20 is applicable, some results on the spectral properties of L were obtained in [25, Section 6.2]. Namely, for $\lambda > 0$ denote by $N^\pm(\lambda)$ the number of the eigenvalues of L in $(-\lambda, \lambda) \cap \mathbb{R}_\pm$, counting with multilplicities, then it is shown that $N^\pm(\lambda) \simeq c_\pm \lambda^{d/2}$ with some $c_\pm > 0$ as $\lambda \to +\infty$. Some estimates for $c_\pm$ were also obtained, and the main conjecture is that the main term of the asymptotics should be the same as in the Weyl asymptotics for the Dirichlet Laplacian in Ω_+ for N^+ and for μ times the Dirichlet Laplacian in Ω_+ for N^-. This conjecture holds true at least for the cases when Ω is convex or has a C^2 boundary, see [30] and [25, Proposition 6.13].

At last we mention that the above class of sign-changing operators can be extended and modified in various directions, which gives rise to new classes of non-classical spectral problems, see e.g. [4, 11, 21, 24].

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