Principal series representations of Iwahori-Hecke algebras for Kac-Moody groups over local fields

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Auguste HÉBERT
École normale supérieure de Lyon
UMR 5669 CNRS, auguste.hebert@ens-lyon.fr

Abstract

Recently, Iwahori-Hecke algebras were associated to Kac-Moody groups over non-Archimedean local fields. We introduce principal series representations for these algebras. We study these representations and partially generalize Kato and Matsumoto irreducibility criteria.

1 Introduction

1.1 The reductive case

Let G be a reductive group over a non-Archimedean local field $\mathcal{K}$. To each open compact subgroup K of G is associated a Hecke algebra $\mathcal{H}_K$. There exists a strong link between the smooth representations of G and the representations of the Hecke algebras of G. Let I be the Iwahori subgroup of G. Then the Hecke algebra $\mathcal{H}_{\mathbb{C}}$ associated with I is called the Iwahori-Hecke algebra of G and plays an important role in the representation theory of G. Its representations have been extensively studied. Let Y be the cocharacter lattice of G and W^v be the vectorial (i.e finite) Weyl group of G. Then by the Bernstein-Lusztig relations, $\mathcal{H}_{\mathbb{C}}$ admits a basis $(Z^\lambda H_w)_{\lambda\in Y, w\in W^v}$ such that $\bigoplus_{\lambda\in Y}\mathbb{C}Z^\lambda$ is a subalgebra of $\mathcal{H}_{\mathbb{C}}$ isomorphic to the group algebra $\mathbb{C}[Y]$ of Y. We identify $\bigoplus_{\lambda\in Y}\mathbb{C}Z^\lambda$ and $\mathbb{C}[Y]$. Let $\tau \in T_{\mathbb{C}} = \mathrm{Hom}(Y, \mathbb{C}^*)$. Then τ induces a representation $\tau : \mathbb{C}[Y] \to \mathbb{C}$. Inducing τ to $\mathcal{H}_{\mathbb{C}}$, one gets a representation I_τ of $\mathcal{H}_{\mathbb{C}}$. These representations were introduced by Matsumoto in [Mat77] and are called **principal series representations**. We refer to [Sol09, Section 3.2] for a survey on this subject.

Matsumoto and Kato gave criteria for the irreducibility of I_τ. The group W^v acts on Y and thus it acts on $T_{\mathbb{C}}$. If $\tau \in T_{\mathbb{C}}$, we denote by W_τ the fixer of τ in W^v. Let $\Phi^\vee$ be the coroot lattice of G. Let q be the residue cardinal of $\mathcal{K}$. Suppose that G is of adjoint type. Let $W_{(\tau)}$ be the subgroup of W_τ generated by the reflections that it contains. Then Kato proved the following theorem (see [Kat81, Theorem 2.4]):

Theorem 1. Let $\tau \in T_{\mathbb{C}}$. Then I_τ is irreducible if and only if it satisfies the following conditions:

1. $W_\tau = W_{(\tau)}$,
2. for all $\alpha^\vee \in \Phi^\vee$, $\tau(\alpha^\vee) \neq q$.

When τ is **regular**, that is when $W_\tau = \{1\}$, condition (1) is satisfied and this is a result by Matsumoto (see [Mat77, Théorème 4.3.5]).

1.2 The Kac-Moody case

Let G be a split Kac-Moody group over a non-Archimedean local field $\mathcal{K}$. There is up to now no definition of smoothness for the representations of G. However one can define certain Hecke algebras in this framework. In [BK11] and [BKP16], Braverman, Kazhdan and Patnaik defined the spherical Hecke algebra and the Iwahori-Hecke $\mathcal{H}_{\mathbb{C}}$ of G when G is affine. Bardy-Panse, Gaussent and Rousseau generalized these constructions to the case where G is a general Kac-Moody group. They achieved this construction by using masures (also known as hovels), which are an analogue of Bruhat-Tits buildings (see [GR08]). Together with Abdellatif, we attached Hecke algebras to subgroups slightly more general than the Iwahori subgroup (see [AH19]).

Let Y be the cocharacter lattice of G and W^v be the Weyl group of G. The Iwahori-Hecke algebra $\mathcal{H}_{\mathbb{C}}$ of G admits a Bernstein-Lusztig presentation but it is no more indexed by Y. Let $\mathcal{T} \subset \mathbb{A} = Y \otimes \mathbb{R}$ be the Tits cone of G. Then $\mathcal{T}$ is a convex cone and it satisfies $\mathcal{T} = \mathbb{A}$ if and only if G is reductive. Then $\mathcal{H}_{\mathbb{C}}$ can be embedded in an algebra ${}^{\mathrm{BL}}\mathcal{H}_{\mathbb{C}}$ called the **Bernstein-Lusztig-Hecke algebra of** G. The algebra ${}^{\mathrm{BL}}\mathcal{H}_{\mathbb{C}}$ admits a basis $(Z^\lambda H_w)_{\lambda \in Y, w \in W^v}$ such that $\bigoplus_{\lambda \in Y} \mathbb{C} Z^\lambda$ is isomorphic to the group algebra $\mathbb{C}[Y]$ of Y. We identify $\bigoplus_{\lambda \in Y} \mathbb{C} Z^\lambda$ and $\mathbb{C}[Y]$. Then $\mathcal{H}_{\mathbb{C}}$ is isomorphic to the subalgebra $\bigoplus_{w \in W^v, \lambda \in Y^+} \mathbb{C} Z^\lambda H_w$. Let $\tau \in T_{\mathbb{C}} = \mathrm{Hom}(Y, \mathbb{C}^*)$. Then τ induces a map $\tau : \mathbb{C}[Y] \to \mathbb{C}$ and we can define the representation I_τ of ${}^{\mathrm{BL}}\mathcal{H}_{\mathbb{C}}$ induced by τ. By restriction, this also defines a representation I_τ^+ of $\mathcal{H}_{\mathbb{C}}$. As I_τ admits a basis indexed by the Weyl group of G, I_τ is infinite dimensional unless G is reductive. The aim of this paper is to study these representations and in particular to study their irreducibility. As we shall see (Proposition 4.2), I_τ is irreducible if and only if I_τ^+ is irreducible and we will mainly study I_τ. We prove the following theorem, generalizing Matsumoto irreducibility criterion (see Corollary 5.10):

Theorem 2. Let τ be a regular character. Then I_τ is irreducible if and only if for all $\alpha \in \Phi^\vee$,

$$\tau(\alpha^\vee) \neq q.$$

We also generalize one implication of Kato's criterion (see Proposition 5.17). Suppose that G is of adjoint type and let $W_{(\tau)}$ be the subgroup of W_τ generated by the reflections that it contains. Then:

Theorem 3. Let $\tau \in T_{\mathbb{C}}$. Assume that I_τ is irreducible. Then:

1. $W_\tau = W_{(\tau)}$,
2. for all $\alpha^\vee \in \Phi^\vee$, $\tau(\alpha^\vee) \neq q$.

We then obtain Kato's criterion when the Kac-Moody group G is associated with a size 2 Kac-Moody matrix (see Theorem 6.40):

Theorem 4. Assume that G is associated with a size 2 Kac-Moody matrix. Let $\tau \in T_{\mathbb{C}}$. Then I_τ is irreducible if and only if it satisfies the following conditions:

1. $W_\tau = W_{(\tau)}$,
2. for all $\alpha^\vee \in \Phi^\vee$, $\tau(\alpha^\vee) \neq q$.

In order to prove these theorems, we first establish the following irreducibility criterion. For $\tau \in T_{\mathbb{C}}$ set $I_\tau(\tau) = \{x \in I_\tau | \theta.x = \tau(\theta).x \ \forall \theta \in \mathbb{C}[Y]\}$. Then:

Theorem 5. (see Theorem 5.8) I_τ is irreducible if and only if:

- $\tau(\alpha^\vee) \neq q$ for all $\alpha^\vee \in \Phi^\vee$
- $\dim I_\tau(\tau) = 1$.

Frameworks Actually, following [BPGR16] we study Iwahori-Hecke algebras associated to abstract masures. In particular our results also apply when G is an almost-split Kac-Moody group over a non-Archimedean local field.

Organization of the paper In Section 2, we recall the definition of the Iwahori-Hecke algebras and of the Bernstein-Lusztig-Hecke algebras, introduce principal series representations and define an algebra ${}^{\mathrm{BL}}\mathcal{H}(T_{\mathcal{F}})$ containing ${}^{\mathrm{BL}}\mathcal{H}_{\mathcal{F}}$, where $\mathcal{F}$ is the field of coefficients of ${}^{\mathrm{BL}}\mathcal{H}_{\mathcal{F}}$.

In Section 3, we study the $\mathcal{F}[Y]$-module I_τ and we study the intertwining operators from I_τ to $I_{\tau'}$, for $\tau, \tau' \in T_{\mathcal{F}}$.

In Section 4, we study principal series representations of $\mathcal{H}_{\mathcal{F}}$ and their links with principal series representations of ${}^{\mathrm{BL}}\mathcal{H}_{\mathcal{F}}$.

In Section 5, we establish Theorem 5. We then apply it to obtain Theorem 2 and Theorem 3.

In Section 6 we regard the weights vectors of I_τ and use it to prove Kato's irreductibility criterion for size 2 Kac-Moody matrices.

There is an index of notations at the end of the paper.

Acknowledgments I would like to thank Ramla Abdellatif and Stéphane Gaussent for the discussions we had on this topic. I also would like to thank Anne-Marie Aubert for her advice concerning references and Olivier Taïbi for correcting the statement of the main theorem and for discussing this subject with me.

Funding The author was supported by the ANR grant ANR-15-CE40-0012.

Contents

2 Iwahori-Hecke algebras

Let G be a Kac-Moody group over a non-archimedean local field. Then Gaussent and Rousseau constructed a space $\mathcal{I}$, called a masure on which G acts, generalizing the construction of the Bruhat-Tits buildings (see [GR08], [Rou16] and [Rou17]). Rousseau then defined in [Rou11] an axiomatic definition of masures inspired by the axiomatic definition of Bruhat-Tits buildings. We simplified it in [Héb19]. Masures satisfying these axiomatics are called abstract masures because they might not be associated with some Kac-Moody group.

In [BPGR16], Bardy-Panse, Gaussent and Rousseau attached an Iwahori-Hecke algebra $\mathcal{H}_{\mathcal{R}}$ to each abstract masure satisfying certain conditions and to each ring $\mathcal{R}$. The algebra $\mathcal{H}_{\mathcal{R}}$ is an algebra of functions defined on some pairs of chambers of the masure, equipped with a convolution product. Then they prove that under some additional hypothesis on the ring $\mathcal{R}$ (which are satisfied by $\mathbb{R}$ and $\mathbb{C}$), $\mathcal{H}_{\mathcal{R}}$ admits a Bernstein-Lusztig presentation. In this paper, we will only use the Bernstein-Lusztig presentation of $\mathcal{H}_{\mathcal{R}}$ and we do not introduce masures (see [Héb18, Appendix A] for a definition). We however introduce the standard apartment of a masure. We restrict our study to the case where $\mathcal{R} = \mathcal{F}$ is a field.

2.1 Standard apartment of a masure

A **Kac-Moody matrix** (or generalized Cartan matrix) is a square matrix $A = (a_{i,j})_{i,j\in I}$ indexed by a finite set I, with integral coefficients, and such that :

(i) $\forall\ i \in I,\ a_{i,i} = 2$;

(ii) $\forall\ (i,j) \in I^2, (i \neq j) \Rightarrow (a_{i,j} \leq 0)$;

(iii) $\forall\ (i,j) \in I^2,\ (a_{i,j} = 0) \Leftrightarrow (a_{j,i} = 0)$.

A **root generating system** is a 5-tuple $\mathcal{S} = (A, X, Y, (\alpha_i)_{i\in I}, (\alpha_i^\vee)_{i\in I})$ made of a Kac-Moody matrix A indexed by the finite set I, of two dual free $\mathbb{Z}$-modules X and Y of finite rank, and of a free family $(\alpha_i)_{i\in I}$ (respectively $(\alpha_i^\vee)_{i\in I}$) of elements in X (resp. Y) called **simple roots** (resp. **simple coroots**) that satisfy $a_{i,j} = \alpha_j(\alpha_i^\vee)$ for all i, j in I. Elements of X (respectively of Y) are called **characters** (resp. **cocharacters**).

Fix such a root generating system $\mathcal{S} = (A, X, Y, (\alpha_i)_{i\in I}, (\alpha_i^\vee)_{i\in I})$ and set $\mathbb{A} := Y \otimes \mathbb{R}$. Each element of X induces a linear form on $\mathbb{A}$, hence X can be seen as a subset of the dual $\mathbb{A}^*$. In particular, the α_i's (with $i \in I$) will be seen as linear forms on $\mathbb{A}$. This allows us to define, for any $i \in I$, an involution r_i of $\mathbb{A}$ by setting $r_i(v) := v - \alpha_i(v)\alpha_i^\vee$ for any $v \in \mathbb{A}$. Let $\mathscr{S} = \{r_i | i \in I\}$ be the (finite) set of **simple reflections**. One defines the **Weyl group of** $\mathcal{S}$ as the subgroup W^v of $\mathrm{GL}(\mathbb{A})$ generated by $\mathscr{S}$. The pair $(W^v, \mathscr{S})$ is a Coxeter system, hence we can consider the length $\ell(w)$ with respect to $\mathscr{S}$ of any element w of W^v. If $s \in \mathscr{S}$, $s = r_i$ for some unique $i \in I$. We set $\alpha_s = \alpha_i$ and $\alpha_s^\vee = \alpha_i^\vee$.

The following formula defines an action of the Weyl group W^v on $\mathbb{A}^*$:

$$\forall\, x \in \mathbb{A}, w \in W^v, \alpha \in \mathbb{A}^*,\ (w.\alpha)(x) := \alpha(w^{-1}.x).$$

Let $\Phi := \{w.\alpha_i | (w, i) \in W^v \times I\}$ (resp. $\Phi^\vee = \{w.\alpha_i^\vee | (w, i) \in W^v \times I\}$) be the set of **real roots** (resp. **real coroots**): then Φ (resp. $\Phi^\vee$) is a subset of the **root lattice** $Q := \bigoplus_{i\in I} \mathbb{Z}\alpha_i$ (resp. **coroot lattice** $Q^\vee = \bigoplus_{i\in I} \mathbb{Z}\alpha_i^\vee$). By [Kum02, 1.2.2 (2)], one has $\mathbb{R}\alpha^\vee \cap \Phi^\vee = \{\pm\alpha^\vee\}$ and $\mathbb{R}\alpha \cap \Phi = \{\pm\alpha\}$ for all $\alpha^\vee \in \Phi^\vee$ and $\alpha \in \Phi$.

Fundamental chamber, Tits cone and vectorial faces As in the reductive case, define the **fundamental chamber** as $C_f^v := \{v \in \mathbb{A} \mid \forall s \in \mathscr{S}, \alpha_s(v) > 0\}$.

Let $\mathcal{T} := \bigcup_{w\in W^v} w.\overline{C_f^v}$ be the **Tits cone**. This is a convex cone (see [Kum02, 1.4]).

For $J \subset \mathscr{S}$, set $F^v(J) = \{x \in \mathbb{A} | \alpha_j(x) = 0 \forall j \in J \text{ and } \alpha_j(x) > 0 \forall j \in \mathscr{S} \setminus J\}$. A **positive vectorial face** is a set of the form $w.F^v(J)$ for some $w \in W^v$ and $J \subset \mathscr{S}$. Then by [Rém02, 5.1 Théorème (ii)], the family of positive vectorial faces of $\mathbb{A}$ is a partition of $\mathcal{T}$ and the stabilizer of $F^v(J)$ is $W_J = \langle J \rangle$.

One sets $Y^+ = Y \cap \mathcal{T}$.

Remark 2.1. *By [Kac94, §4.9] and [Kac94, § 5.8] the following conditions are equivalent:*

1. *the Kac-Moody matrix A is of finite type (i.e. is a Cartan matrix),*
2. *$\mathbb{A} = \mathcal{T}$*
3. *W^v is finite.*

2.2 Recalls on Coxeter groups

2.2.1 Bruhat order

Let $(W_0, \mathscr{S}_0)$ be a Coxeter system. We equip it with the Bruhat order $\leq_{W_0}$ (see [BB05, Definition 2.1.1]). We have the following characterization (see [BB05, Corollary 2.2.3]): let $u, w \in W_0$. Then $u \leq_{W_0} w$ if and only if every reduced expression for w has a subword that is a reduced expression for u. By [BB05, Proposition 2.2.9], $(W_0, \leq_{W_0})$ is a **directed poset**, i.e for every finite set $E \subset W_0$, there exists $w \in W_0$ such that $v \leq_{W_0} w$ for all $v \in E$.

We write $\leq$ instead of $\leq_{W^v}$. For $u, v \in W^v$, we denote by $[u, v]$, $[u, v)$, ... the sets $\{w \in W^v | u \leq w \leq v\}$, $\{w \in W^v | u \leq w < v\}$, ...

2.2.2 Reflections and coroots

Let $\mathscr{R} = \{wsw^{-1} | w \in W^v, s \in \mathscr{S}\}$ be the set of **reflections** of W^v. Let $r \in \mathscr{R}$. Write $r = wsw^{-1}$, where $w \in W^v$, $s \in \mathscr{S}$ and $ws > w$ (which is possible because if $ws < w$, then $r = (ws)s(ws)^{-1}$). Then one sets $\alpha_r = w.\alpha_s \in \Phi_+$ (resp. $\alpha_r^\vee = w.\alpha_s^\vee \in \Phi_+^\vee$). This is well defined by the lemma below.

Lemma 2.2. *Let $w, w' \in W^v$ and $s, s' \in \mathscr{S}$ be such that $wsw^{-1} = w's'w'^{-1}$ and $ws > w$, $w's' > w'$. Then $w.\alpha_s = w'.\alpha_{s'} \in \Phi_+$ and $w.\alpha_s^\vee = w'.\alpha_{s'}^\vee \in \Phi_+^\vee$.*

Proof. One has $r(x) = x - w.\alpha_s(x)w.\alpha_s^\vee = x - w'.\alpha_{s'}(x)w'.\alpha_{s'}^\vee$ for all $x \in \mathbb{A}$ and thus $w.\alpha_s \in \mathbb{R}^* w'.\alpha_{s'}$ and $w.\alpha_s^\vee \in \mathbb{R}^* w'.\alpha_{s'}^\vee$. As Φ and $\Phi^\vee$ are reduced, $w.\alpha_s = \pm w'.\alpha_{s'}$ and $w.\alpha_s^\vee = \pm w'.\alpha_s^\vee$. By [Kum02, Lemma 1.3.13], $w.\alpha_s, w'.\alpha_{s'} \in \Phi_+$ and $w.\alpha_s^\vee, w'.\alpha_{s'}^\vee \in \Phi_+^\vee$, which proves the lemma. □

Lemma 2.3. *Let $r, r' \in \mathscr{R}$ and $w \in W^v$ be such that $w.\alpha_r = \alpha_{r'}$ or $w.\alpha_r^\vee = \alpha_{r'}^\vee$. Then $wrw^{-1} = r'$.*

Proof. Write $r = vsv^{-1}$ and $r = v's'v'^{-1}$ for $s, s' \in \mathscr{S}$ and $v, v' \in W^v$. Then $v'^{-1}wv.\alpha_s = \alpha_{s'}$. Thus by [Kum02, Theorem 1.3.11 (b5)], $v'^{-1}wvsv^{-1}w^{-1}v' = s'$ and hence $wrw^{-1} = r'$. □

Let $r \in \mathscr{R}$. Then for all $x \in \mathbb{A}$, one has:

$$r(x) = x - \alpha_r(x)\alpha_r^\vee.$$

Let $\alpha^\vee \in \Phi^\vee$. One sets $r_{\alpha^\vee} = wsw^{-1}$ where $(w, s) \in W^v \times \mathscr{S}$ is such that $\alpha^\vee = w.\alpha_s^\vee$. This is well defined, by Lemma 2.3. Thus $\alpha^\vee \mapsto r_{\alpha^\vee}$ and $r \mapsto \alpha_r^\vee$ induce bijections $\Phi_+^\vee \to \mathscr{R}$ and $\mathscr{R} \to \Phi_+^\vee$. If $r \in \mathscr{R}$, $r = wsw^{-1}$, one sets $\sigma_r = \sigma_s$, which is well defined by assumption on the σ_t, $t \in \mathscr{S}$ (see Subsection 2.3).

For $w \in W^v$, set $N_{\Phi^\vee}(w) = \{\alpha^\vee \in \Phi_+^\vee | w.\alpha^\vee \in \Phi_-^\vee\}$.

Lemma 2.4. *([Kum02, Lemma 1.3.14]) Let $w \in W^v$. Then $|N_{\Phi^\vee}(w)| = \ell(w)$ and if $w = s_1 \dots s_r$ is a reduced expression, then $N_{\Phi^\vee}(w) = \{\alpha_{s_r}^\vee, s_r.\alpha_{s_{r-1}}^\vee, \dots, s_r \dots s_2.\alpha_{s_1}^\vee\}$.*

2.2.3 Reflections subgroups of a Coxeter group

If W_0 is a Coxeter group, a **Coxeter generating set** is a set $\mathscr{S}_0$ such that $(W_0, \mathscr{S}_0)$ is a Coxeter system. Let $(W_0, \mathscr{S}_0)$ be a Coxeter system and $\mathscr{R}_0 = \{w.s.w^{-1} | w \in W_0, s \in \mathscr{S}_0\}$ be its set of reflections. A **reflection subgroup of** W_0 is a group of the form $W_1 = \langle \mathscr{R}_1 \rangle$ for some $\mathscr{R}_1 \subset \mathscr{R}_0$. For $w \in W_0$, set $N_{\mathscr{R}_0}(w) = \{r \in \mathscr{R}_0 | rw^{-1} < w^{-1}\}$. By [Dye90, 3.3] or [Dye91, 1], if $\mathscr{S}(W_1) = \{r \in \mathscr{R}_0 | N_{\mathscr{R}_0}(r) \cap W_1 = \{r\}\}$, then $(W_1, \mathscr{S}(W_1))$ is a Coxeter system.

Let $(W_0, \mathscr{S}_0)$ be a Coxeter system. The **rank of** $(W_0, \mathscr{S}_0)$ is $|\mathscr{S}_0|$.

Remark 2.5.

1. *The rank of a Coxeter group is not well defined. For example, by [Müh05, 3], if $k \in \mathbb{Z}_{\geq 1}$ and $n = 4(2k+1)$ then the dihedral group of order n admits Coxeter generating sets of order 2 and 3. However by [Rad99], all the Coxeter generating sets of the infinite dihedral group have cardinal 2.*

2. *Using [Bou81, IV 1.8 Proposition 7] we can prove that if $(W_0, \mathscr{S}_0)$ is a Coxeter system of infinite rank, then every Coxeter generating set of W_0 is infinite.*

3. *Reflection subgroups of finite rank Coxeter groups are not necessarily of finite rank. Indeed, let W_0 be the Coxeter group generated by the involutions s_1, s_2, s_3, with $s_i s_j$ of infinite order when $i \neq j \in [\![1,3]\!]$. Let $W_0' = \langle s_1, s_2\rangle \subset W_0$ and $\mathscr{R}_1 = \{ws_3w^{-1} | w \in W_0'\} \subset \mathscr{R}_0$. Then $W_1 = \langle \mathscr{R}_1\rangle$ has infinite rank. Indeed, let $\psi : W_0 \to W_0'$ be the group morphism defined by $\psi_{|W_0'} = \mathrm{Id}_{W_0'}$ and $\psi(s_3) = 1$. Then $\mathscr{R}_1 \subset \ker\psi$. Thus s_3 appears in the reduced writing of every nontrivial element of W_1. By [BB05, Corollary 1.4.4] if $r \in \mathscr{R}_1$, then the unique element of $N_{\mathscr{R}_0}(r)$ containing an s_3 in its reduced writing is r. Thus $\mathscr{S}(W_1) \supset \mathscr{R}_1$ is infinite.*

2.3 Iwahori-Hecke algebras

In this subsection, we give the definition of the Iwahori-Hecke algebra via its Bernstein-Lusztig presentation, as done in [BPGR16, Section 6.6].

Let $\mathcal{R}_1 = \mathbb{Z}[(\sigma_s)_{s\in\mathscr{S}}, (\sigma_s')_{s\in\mathscr{S}}]$, where $(\sigma_s)_{s\in\mathscr{S}}, (\sigma_s')_{s\in\mathscr{S}}$ are two families of indeterminates satisfying the following relations:

- if $\alpha_s(Y) = \mathbb{Z}$, then $\sigma_s = \sigma_s'$;
- if $s, t \in \mathscr{S}$ are conjugate (i.e. such that $\alpha_s(\alpha_t^\vee) = \alpha_t(\alpha_s^\vee) = -1$), then $\sigma_s = \sigma_t = \sigma_s' = \sigma_t'$.

To define the Iwahori-Hecke algebra $\mathcal{H}_{\mathcal{R}_1}$ associated with $\mathbb{A}$ and $(\sigma_s, \sigma_s')_{s\in\mathscr{S}}$, we first introduce the Bernstein-Lusztig-Hecke algebra. Let ${}^{BL}\mathcal{H}_{\mathcal{R}_1}$ be the free $\mathcal{R}_1$-vector space with basis $(Z^\lambda H_w)_{\lambda\in Y, w\in W^v}$. For short, one sets $H_w = Z^0 H_w$ for $w \in W^v$ and $Z^\lambda = Z^\lambda H_1$ for $\lambda \in Y$. The **Bernstein-Lusztig-Hecke algebra** ${}^{BL}\mathcal{H}_{\mathcal{R}_1}$ is the module ${}^{BL}\mathcal{H}_{\mathcal{R}_1}$ equipped with the unique product $*$ that turns it into an associative algebra and satisfies the following relations (known as the **Bernstein-Lusztig relations**):

- (BL1) $\forall\ (\lambda, w) \in Y \times W^v$, $Z^\lambda * H_w = Z^\lambda H_w$;
- (BL2) $\forall\ s \in \mathscr{S}, \forall\ w \in W^v$, $H_s * H_w = \begin{cases} H_{sw} & \text{if } \ell(sw) = \ell(w) + 1 \\ (\sigma_s - \sigma_s^{-1})H_w + H_{sw} & \text{if } \ell(sw) = \ell(w) - 1 \end{cases}$;
- (BL3) $\forall\ (\lambda, \mu) \in Y^2$, $Z^\lambda * Z^\mu = Z^{\lambda+\mu}$;
- (BL4) $\forall\ \lambda \in Y,\ \forall\ i \in I$, $H_s * Z^\lambda - Z^{s.\lambda} * H_s = Q_s(Z)(Z^\lambda - Z^{s.\lambda})$, where $Q_s(Z) = \frac{(\sigma_s - \sigma_s^{-1}) + (\sigma_s' - \sigma_s'^{-1})Z^{-\alpha_s^\vee}}{1 - Z^{-2\alpha_s^\vee}}$.

The existence and uniqueness of such a product $*$ comes from [BPGR16, Theorem 6.2].

Definition 2.6. *Let $\mathcal{F}$ be a field of characteristic 0 and $f : \mathcal{R}_1 \to \mathcal{F}$ be a ring morphism such that $f(\sigma_s)$ and $f(\sigma_s')$ are invertible in $\mathcal{F}$ for all $s \in \mathscr{S}$. Then the **Bernstein-Lusztig-Hecke algebra of** $(\mathbb{A}, (\sigma_s)_{s\in\mathscr{S}}, (\sigma_s')_{s\in\mathscr{S}})$ over $\mathcal{F}$ is the algebra ${}^{\mathrm{BL}}\mathcal{H}_{\mathcal{F}} = {}^{\mathrm{BL}}\mathcal{H}_{\mathcal{R}_1} \otimes_{\mathcal{R}_1} \mathcal{F}$. Following [BPGR16, Section 6.6], the **Iwahori-Hecke algebra** $\mathcal{H}_{\mathcal{F}}$ associated with $\mathcal{S}$ and $(\sigma_s, \sigma_s')_{s\in\mathscr{S}}$ is now defined as the $\mathcal{F}$-subalgebra of ${}^{BL}\mathcal{H}_{\mathcal{F}}$ spanned by $(Z^\lambda H_w)_{\lambda\in Y^+, w\in W^v}$ (recall that $Y^+ = Y \cap \mathcal{T}$ with $\mathcal{T}$ being the Tits cone). Note that for G reductive, we recover the usual Iwahori-Hecke algebra of G, since $Y \cap \mathcal{T} = Y$.*

In certain proofs, when $\mathcal{F} = \mathbb{C}$, we will make additional assumptions on the σ_s and σ_s', $s \in \mathscr{S}$. To avoid these assumptions, we can assume that $\sigma_s, \sigma_s' \in \mathbb{C}$ and $|\sigma_s| > 1, |\sigma_s'| > 1$ for all $s \in \mathscr{S}$.

Remark 2.7. *1. Let $s \in \mathscr{S}$. Then if $\sigma_s = \sigma'_s$, $Q_s(Z) = \frac{(\sigma_s - \sigma_s^{-1})}{1 - Z^{-\alpha_s^\vee}}$.*

2. Let $s \in \mathscr{S}$ and $\lambda \in Y$. Then $Q_s(Z)(Z^\lambda - Z^{s.\lambda}) \in \mathcal{F}[Y]$. Indeed, $Q_s(Z)(Z^\lambda - Z^{s.\lambda}) = Q_s(Z).Z^\lambda(1 - Z^{-\alpha_s(\lambda)\alpha_s^\vee})$. Assume that $\sigma_s = \sigma'_s$. Then

$$\frac{1 - Z^{-\alpha_s(\lambda)\alpha_s^\vee}}{1 - Z^{-\alpha_s^\vee}} = \begin{cases} \sum_{j=0}^{\alpha_s(\lambda)-1} Z^{-j\alpha_s^\vee} & \text{if } \alpha_s(\lambda) \geq 0 \\ -Z^{\alpha_s^\vee} \sum_{j=0}^{-\alpha_s(\lambda)-1} Z^{j\alpha_s^\vee} & \text{if } \alpha_s(\lambda) \leq 0, \end{cases}$$

and thus $Q_s(Z)(Z^\lambda - Z^{s.\lambda}) \in \mathcal{F}[Y]$. Assume $\sigma'_s \neq \sigma_s$. Then $\alpha_s(Y) = 2\mathbb{Z}$ and a similar computation enables to conclude.

3. From (BL4) we deduce that for all $s \in \mathscr{S}$, $\lambda \in Y$,

$$Z^\lambda * H_s - H_s * Z^{s.\lambda} = Q_s(Z)(Z^\lambda - Z^{s.\lambda}).$$

4. When G is a split Kac-Moody group over a non-Archimedean local field $\mathcal{K}$ with residue cardinal q, we can choose $\mathcal{F}$ to be a field containing $\mathbb{Z}[\sqrt{q}^{\pm 1}]$ and take $f(\sigma_s) = f(\sigma'_s) = \sqrt{q}$ for all $s \in \mathscr{S}$.

5. By (BL4), the family $(H_w * Z^\lambda)_{w \in W^v, \lambda \in Y}$ is also a basis of ${}^{\mathrm{BL}}\mathcal{H}_{\mathcal{F}}$.

We equip $\mathcal{F}[Y]$ with an action of W^v. For $\theta = \sum_{\lambda \in Y} a_\lambda Z^\lambda \in \mathcal{F}[Y]$ and $w \in W^v$, set $\theta^w := \sum_{\lambda \in Y} a_\lambda Z^{w.\lambda}$.

Lemma 2.8. *Let $\theta \in \mathcal{F}[Y]$ and $w \in W^v$. Then $\theta * H_w - H_w * \theta^{w^{-1}} \in {}^{\mathrm{BL}}\mathcal{H}_{\mathcal{F}}^{<w} := \bigoplus_{v<w} H_v \mathcal{F}[Y]$. In particular, ${}^{\mathrm{BL}}\mathcal{H}_{\mathcal{F}}^{\leq w} := \bigoplus_{v \leq w} H_v \mathbb{C}[Y]$ is a left finitely generated $\mathcal{F}[Y]$-submodule of ${}^{\mathrm{BL}}\mathcal{H}_{\mathcal{F}}$.*

Proof. We do it by induction on $\ell(w)$. Let $\theta \in \mathcal{F}[Y]$ and $w \in W^v$ be such that $u := \theta H_w - H_w \theta^{w^{-1}} \in {}^{\mathrm{BL}}\mathcal{H}(T_{\mathcal{F}})^{<w}$. Let $s \in \mathscr{S}$ and assume that $\ell(ws) = \ell(w) + 1$. Then by (BL4):

$$\theta * H_{ws} = (H_w \theta^{w^{-1}} + u) * H_s = H_{ws} \theta^{sw^{-1}} + a H_w + u H_s,$$

for some $a \in \mathcal{F}$. Moreover, by [Kum02, Corollary 1.3.19] and (BL2), $u * H_s \in {}^{\mathrm{BL}}\mathcal{H}(T_{\mathcal{F}})^{<ws}$ and the lemma follows. □

Definition 2.9. *Let $\mathcal{H}_{\mathcal{F},W^v} = \bigoplus_{w \in W^v} \mathcal{F} H_w \subset \mathcal{H}_{\mathcal{F}}$. Then $\mathcal{H}_{\mathcal{F},W^v}$ is a subalgebra of $\mathcal{H}_{\mathcal{F}}$. This is the Hecke algebra of the Coxeter group $(W^v, \mathscr{S})$.*

2.4 Principal series representations

In this subsection, we introduce the principal series representations of ${}^{\mathrm{BL}}\mathcal{H}_{\mathcal{F}}$.

We now fix $(\mathbb{A}, (\sigma_s)_{s \in \mathscr{S}}, (\sigma'_s)_{s \in \mathscr{S}})$ as in Subsection 2.3 and a field $\mathcal{F}$ as in Definition 2.6. Let $\mathcal{H}_{\mathcal{F}}$ and ${}^{\mathrm{BL}}\mathcal{H}_{\mathcal{F}}$ be the Iwahori-Hecke and the Bernstein-Lusztig Hecke algebras of $(\mathbb{A}, (\sigma_s)_{s \in \mathscr{S}}, (\sigma'_s)_{s \in \mathscr{S}})$ over $\mathcal{F}$.

Let $T_{\mathcal{F}} = \mathrm{Hom}_{\mathrm{Gr}}(Y, \mathcal{F}^\times)$ be the group of group morphism from Y to $\mathcal{F}^*$. Let $\tau \in T_{\mathcal{F}}$. Then τ induces an algebra morphism $\tau : \mathcal{F}[Y] \to \mathcal{F}$ by the formula $\tau(\sum_{y \in Y} a_y e^y) = \sum_{y \in Y} a_y \tau(y)$, for $\sum a_y e^y \in \mathcal{F}[Y]$. This equips $\mathcal{F}$ with the structure of a $\mathcal{F}[Y]$-module.

Let $I_\tau = \mathrm{Ind}^{{}^{\mathrm{BL}}\mathcal{H}_\mathcal{F}}(\tau) = {}^{\mathrm{BL}}\mathcal{H}_\mathcal{F} \otimes_{\mathcal{F}[Y]} \mathcal{F}$. For example if $\lambda \in Y$, $w \in W^v$ and $s \in \mathscr{S}$, one has:

$$Z^\lambda.1 \otimes_\tau 1 = \tau(\lambda)1 \otimes_\tau 1, H_w * Z^\lambda \otimes_\tau 1 = \tau(\lambda)H_w \otimes_\tau 1 \text{ and}$$

$$Z^\lambda.H_s \otimes_\tau 1 = H_s * Z^{s.\lambda} \otimes_\tau 1 + Q_s(Z)(Z^\lambda - Z^{s.\lambda}) \otimes_\tau 1 = \tau(s.\lambda)H_i \otimes_\tau 1 + \tau\big(Q_s(Z)(Z^\lambda - Z^{s.\lambda})\big) \otimes_\tau 1.$$

Let $h \in I_\tau$. Write $h = \sum_{\lambda \in Y, w \in W^v} h_{w,\lambda} H_w Z^\lambda \otimes_\tau c_{w,\lambda}$, where $(h_{w,\lambda}), (c_{w,\lambda}) \in \mathcal{F}^{(W^v \times Y)}$, which is possible by Remark 2.7. Thus

$$h = \sum_{\lambda \in Y, w \in W^v} h_{w,\lambda} c_{w,\lambda} \tau(\lambda) H_w \otimes_\tau 1 = \big(\sum_{\lambda \in Y, w \in W^v} h_{w,\lambda} c_{w,\lambda} \tau(\lambda) H_w \big) 1 \otimes_\tau 1.$$

Thus I_τ is a principal ${}^{\mathrm{BL}}\mathcal{H}_\mathcal{F}$-module and $(H_w \otimes_\tau 1)_{w \in W^v}$ is a basis of I_τ. Moreover $I_\tau = \mathcal{H}_{W^v,\mathcal{F}}.1 \otimes_\tau 1$ (see Definition 2.9 for the definition of $\mathcal{H}_{W^v,\mathcal{F}}$).

2.5 The algebra ${}^{\mathrm{BL}}\mathcal{H}_\mathcal{F}(T_\mathcal{F})$

In this subsection, we introduce an algebra ${}^{\mathrm{BL}}\mathcal{H}(T_\mathcal{F})$ containing ${}^{\mathrm{BL}}\mathcal{H}_\mathcal{F}$. This algebra will enable us to regard the elements of I_τ as specializations at τ of certain elements of ${}^{\mathrm{BL}}\mathcal{H}(T_\mathcal{F})$. When $\mathcal{F} = \mathbb{C}$, this will enable us to make $\tau \in T_\mathbb{C}$ vary and to use density arguments and basic algebraic geometry to study the I_τ.

2.5.1 Description of ${}^{\mathrm{BL}}\mathcal{H}(T_\mathcal{F})$

Let ${}^{\mathrm{BL}}\mathcal{H}(T_\mathcal{F})$ be the right $\mathcal{F}(Y)$ vector space $\bigoplus_{w \in W^v} H_w \mathbb{C}(Y)$. We equip $\mathcal{F}(Y)$ with an action of W^v. For $\theta = \frac{\sum_{\lambda \in Y} a_\lambda Z^\lambda}{\sum_{\lambda \in Y} b_\lambda Z^\lambda} \in \mathcal{F}(Y)$ and $w \in W^v$, set $\theta^w := \frac{\sum_{\lambda \in Y} a_\lambda Z^{w.\lambda}}{\sum_{\lambda \in Y} b_\lambda Z^{w.\lambda}}$.

Proposition 2.10. *There exists a unique multiplication $*$ on ${}^{\mathrm{BL}}\mathcal{H}(T_\mathcal{F})$ which equips ${}^{\mathrm{BL}}\mathcal{H}(T_\mathcal{F})$ with the structure of an associative algebra and such that:*

- *$\mathcal{F}(Y)$ embeds into ${}^{\mathrm{BL}}\mathcal{H}(T_\mathcal{F})$ as an algebra,*
- *(BL2) is satisfied,*
- *the following relation (BL4') is satisfied:*

$$\text{for all } \theta \in \mathcal{F}(Y) \text{ and } s \in \mathscr{S}, \theta * H_s - H_s * \theta^s = Q_s(Z)(\theta - \theta^s).$$

The proof of this proposition is postponed to 2.5.2.

We regard the elements of $\mathcal{F}[Y]$ as polynomial functions on $T_\mathcal{F}$ by setting:

$$\tau(\sum_{\lambda \in Y} a_\lambda Z^\lambda) = \sum_{\lambda \in Y} a_\lambda \tau(\lambda),$$

for all $(a_\lambda) \in \mathcal{F}^{(Y)}$. The ring $\mathcal{F}[Y]$ is a unique factorization domain. Let $\theta \in \mathcal{F}(Y)$ and $(f, g) \in \mathcal{F}[Y] \times \mathcal{F}[Y]^*$ be such that $\theta = \frac{f}{g}$ and f and g are coprime. Set $\mathcal{D}(\theta) = \{\tau \in T_\mathcal{F} | \theta(g) \neq 0\}$. Then we regard θ as a map from $\mathcal{D}(\theta)$ to $\mathcal{F}$ by setting $\theta(\tau) = \frac{f(\tau)}{g(\tau)}$ for all $\tau \in \mathcal{D}(\theta)$.

For $w \in W^v$, let $\pi_w^H : {}^{\mathrm{BL}}\mathcal{H}(T_\mathcal{F}) \to \mathcal{F}(Y)$ be defined by $\pi_w^H(\sum_{v \in W^v} H_w \theta_v) = \theta_w$. If $\tau \in T_\mathcal{F}$, let $\mathcal{F}(Y)_\tau = \{\frac{f}{g} | f, g \in \mathbb{C}[Y] \text{ and } g(\tau) \neq 0\} \subset \mathcal{F}(Y)$. Let ${}^{\mathrm{BL}}\mathcal{H}(T_\mathcal{F})_\tau = \bigoplus_{w \in W^v} H_w \mathcal{F}(Y)_\tau \subset {}^{\mathrm{BL}}\mathcal{H}(T_\mathcal{F})$. This is a not a subalgebra of ${}^{\mathrm{BL}}\mathcal{H}(T_\mathcal{F})$ (consider for example $\frac{1}{Z^\lambda - 1} * H_s = H_s * \frac{1}{Z^{s.\lambda} - 1} + \ldots$ for some well chosen $\lambda \in Y$, $s \in \mathscr{S}$ and $\tau \in T_\mathbb{C}$). It is however an $\mathcal{H}_{W^v,\mathcal{F}} - \mathcal{F}(Y)_\tau$ bimodule. For $\tau \in T_\mathcal{F}$, we define $\mathrm{ev}_\tau : {}^{\mathrm{BL}}\mathcal{H}(T_\mathcal{F})_\tau \to \mathcal{H}_{W^v,\mathcal{F}}$ by $\mathrm{ev}_\tau(h) = h(\tau) = \sum_{w \in W^v} H_w \theta_w(\tau)$ if $h = \sum_{w \in W^v} H_w \theta_w \in \mathcal{H}(Y)_\tau$. This is a morphism of $\mathcal{H}_{W^v,\mathcal{F}} - \mathcal{F}(Y)_\tau$-bimodule.

2.5.2 Construction of ${}^{\mathrm{BL}}\mathcal{H}(T_{\mathcal{F}})$

We now prove the existence of ${}^{\mathrm{BL}}\mathcal{H}(T_{\mathcal{F}})$. For this we use the theory of Asano and Ore of rings of fractions: ${}^{\mathrm{BL}}\mathcal{H}(T_{\mathcal{F}})$ will be the ring ${}^{\mathrm{BL}}\mathcal{H}_{\mathcal{F}} * (\mathcal{F}[Y] \setminus \{0\})^{-1}$.

Let $V = {}^{\mathrm{BL}}\mathcal{H}_{\mathcal{F}} \otimes_{\mathcal{F}[Y]} \mathcal{F}(Y) \supset {}^{\mathrm{BL}}\mathcal{H}_{\mathcal{F}}$, where ${}^{\mathrm{BL}}\mathcal{H}_{\mathcal{F}}$ is equipped with its structure of a right $\mathcal{F}[Y]$-module. As a right $\mathcal{F}(Y)$-vector space, $V = \bigoplus_{w\in W^v} H_w\mathbb{C}(Y)$. The left action of $\mathcal{F}[Y]$ on ${}^{\mathrm{BL}}\mathcal{H}_{\mathcal{F}}$ extends to an action of $\mathcal{F}[Y]$ on V by setting $\theta.\sum_{w\in W^v} H_w f_w = \sum_{w\in W^v}(\theta.H_w)f_w$, for all $(f_w) \in \mathcal{F}(Y)^{(W^v)}$ and $\theta \in \mathcal{F}[Y]$. This equips V with the structure of a $(\mathcal{F}[Y] - \mathcal{F}(Y))$-bimodule.

Lemma 2.11. *The left action of $\mathcal{F}[Y]$ on V extends uniquely to a left action of $\mathcal{F}(Y)$ on V. This equips V with the structure of a $(\mathcal{F}(Y)$-$\mathcal{F}(Y))$-bimodule.*

Proof. Let $w \in W^v$ and $P \in \mathcal{F}[Y] \setminus \{0\}$. Let $V^{\leq w} = \bigoplus_{v\in[1,w]} H_v\mathcal{F}(Y)$. By Lemma 2.8, the map $m_P : V^{\leq w} \to V^{\leq w}$ defined by $m_P(h) = P.h$ is well defined. Thus the left action of $\mathcal{F}[Y]$ on $V^{\leq w}$ induces a ring morphism $\phi_w : \mathcal{F}[Y] \to \mathrm{End}_{v.s}(V^{\leq w})$, where $\mathrm{End}_{v.s}(V^{\leq w})$ is the space of endormophism of the $\mathbb{C}(Y)$-vector space $V^{\leq w}$.

Let us prove that $\phi_w(P)$ is injective. Let $h \in V^{\leq w}$. Write $h = \sum_{v\in W^v} H_w\theta_w$, with $\theta_w \in \mathcal{F}(Y)$ for all $w \in W^v$. Suppose that $h \neq 0$. Let $v \in W^v$ be such that $\theta_v \neq 0$ and such that v is maximal for this property for the Bruhat order. By Lemma 2.8, $P * h \neq 0$ and thus $\phi_w(P)$ is injective. Therefore it is invertible for all $P \in \mathcal{F}[Y]$. Thus ϕ_w extends uniquely to a ring morphism $\widetilde{\phi_w} : \mathcal{F}(Y) \to V_w$. As $(W^v, \leq)$ is a directed poset, there exists an increasing sequence $(w_n)_{n\in\mathbb{Z}_{\geq 0}}$ (for the Bruhat order) such that $\bigcup_{n\in\mathbb{Z}_{\geq 0}}[1, w_n] = W^v$. Let $m, n \in \mathbb{Z}_{\geq 0}$ be such that $m \leq n$. Let $P \in \mathcal{F}[Y]$ and $f^{(m)} = \widetilde{\phi_{w_m}}(P)$ and $f^{(n)} = \widetilde{\phi_{w_n}}(P)$. Then $f^{(n)}_{|V^{\leq w_m}} = f^{(m)}$ and thus for all $\theta \in \mathcal{F}(Y)$ and $x \in {}^{\mathrm{BL}}\mathcal{H}(T_{\mathcal{F}})$, $\theta.x := \widetilde{\phi}_{w_k}(\theta)(x)$ is well defined, independently of $k \in \mathbb{Z}_{\geq 0}$ such that $x \in V^{\leq w_k}$. This defines an action of $\mathcal{F}(Y)$ on V.

Let $h \in V$, $\theta \in \mathcal{F}(Y)$ and $P \in \mathcal{F}[Y] \setminus \{0\}$. Let $x = \frac{1}{P}.h$. Then as V is a $(\mathcal{F}[Y]$-$\mathcal{F}(Y))$-bimodule, $(P * x) * \theta = h * \theta = P * (x * \theta)$ and thus $x * \theta = \frac{1}{P} * (h * \theta) = (\frac{1}{P} * h) * \theta$. Thus V is a $(\mathcal{F}(Y) - \mathcal{F}(Y))$-bimodule. □

Lemma 2.12. *The set $\mathcal{F}[Y] \subset {}^{\mathrm{BL}}\mathcal{H}_{\mathcal{F}}$ satisfies the right Ore condition: for all $P \in \mathcal{F}[Y]$ and $h \in {}^{\mathrm{BL}}\mathcal{H}_{\mathcal{F}}$, $P * {}^{\mathrm{BL}}\mathcal{H}_{\mathcal{F}} \cap h * \mathbb{C}[Y] \neq \emptyset$.*

Proof. Let $P \in \mathcal{F}[Y]$ and $h \in {}^{\mathrm{BL}}\mathcal{H}_{\mathcal{F}}$. Then by definition, $P * (\frac{1}{P} * h) = h \in V$. Moreover, $V = \bigoplus_{w\in W^v} H_w\mathbb{C}(Y)$ and thus there exists $\theta \in \mathcal{F}[Y]$ such that $\frac{1}{P} * h * \theta \in {}^{\mathrm{BL}}\mathcal{H}_{\mathcal{F}}$. Then $P * \frac{1}{P} * h * \theta = h * \theta \in P * {}^{\mathrm{BL}}\mathcal{H}_{\mathcal{F}} \cap h * \mathcal{F}[Y]$, which proves the lemma. □

Definition 2.13. *Let R be a ring and r in R. Then r is said to be **regular** if for all $r' \in R \setminus \{0\}$, $rr' \neq 0$ and $r'r \neq 0$.*

*Let R be a ring and $X \subset R$ a multiplicative set of regular elements. A **right ring of fractions for R with respect to X** is any overring $S \supset R$ such that:*

- *Every element of X is invertible in S.*
- *Every element of S can be expressed in the form ax^{-1} for some $a \in R$ and $x \in X$.*

We can now prove Proposition 2.10. The uniqueness of such a product follows from (BL4'). By Lemma 2.8, the elements of $\mathcal{F}[Y] \setminus \{0\}$ are regular. By Lemma 2.12 and [GW04, Theorem 6.2], there exists a right ring of fractions ${}^{\mathrm{BL}}\mathcal{H}(T_{\mathcal{F}})$ for ${}^{\mathrm{BL}}\mathcal{H}_{\mathcal{F}}$ with respect to $\mathcal{F}[Y] \setminus \{0\}$. Then ${}^{\mathrm{BL}}\mathcal{H}(T_{\mathcal{F}})$ is an algebra over $\mathcal{F}$ and as a vector space, ${}^{\mathrm{BL}}\mathcal{H}(T_{\mathcal{F}}) = \bigoplus_{w\in W^v}(H_w\mathcal{F}[Y])(\mathcal{F}[Y] \setminus \{0\})^{-1} = \bigoplus_{w\in W^v} H_w\mathbb{C}(Y)$.

Let $(f,g) \in \mathcal{F}[Y] \times (\mathcal{F}[Y] \setminus \{0\})$. Then it is easy to check that $g * \big(H_s * \frac{1}{g^s} + Q_s(Z)\big)(\frac{1}{g} - \frac{1}{g^s})\big) = H_s$ and thus $\frac{1}{g} * H_s = (H_s * \frac{1}{g^s} + Q_s(Z)(\frac{1}{g} - \frac{1}{g^s})$. Let $f \in \mathcal{F}[Y]$. A straightforward computation yields the formula $\frac{f}{g} * H_s = H_s * (\frac{f}{g})^s + Q_s(Z)(\frac{f}{g} - (\frac{f}{g})^s)$ which finishes the proof of Proposition 2.10.

Remark 2.14.

- *Inspired by the proof of [BPGR16, Theorem 6.2] we could try to define $*$ on V as follows. Let $\theta_1, \theta_2 \in \mathcal{F}[Y]$ and $w_1, w_2 \in W^v$. Write $\theta_1 * H_{w_2} = \sum_{w \in W^v} H_w \theta_w$, with $(\theta_w) \in \mathcal{F}(Y)^{(W^v)}$. Then $(H_{w_1} * \theta_1) * (H_{w_2} * \theta_2) = \sum_{w \in W} (H_{w_1} * H_w) * (\theta_2 \theta_w)$. However it is not clear a priori that the so defined law is associative.*

- *Suppose that $\mathcal{H}_\mathcal{F}$ is the Iwahori-Hecke algebra associated with some masure defined in [BPGR16, Definition 2.5]. Using the same procedure as above (by taking $S = \{Y^\lambda | \lambda \in Y^+\}$), we can construct the algebra ${}^{\mathrm{BL}}\mathcal{H}_\mathcal{F}$ from the algebra $\mathcal{H}_\mathcal{F}$. In this particular case, this gives an alternative proof of [BPGR16, Theorem 6.2].*

3 Weight decompositions and intertwining operators

Let $\tau \in T_\mathcal{F}$. In this section, we study the structure of I_τ as a $\mathcal{F}[Y]$-module and the set $\mathrm{Hom}_{\mathrm{BL}_{\mathcal{H}_\mathcal{F}-\mathrm{mod}}}(I_\tau, I_{\tau'})$ for $\tau' \in T_\mathcal{F}$.

In Subsection 3.1, we study the weights of I_τ and decompose every ${}^{\mathrm{BL}}\mathcal{H}_\mathcal{F}$-submodule of I_τ as a sum of generalized weight spaces (see Lemma 3.3).

In Subsection 3.2, we relate intertwining operators and weight spaces. We then prove the existence of nontrivial intertwining operators $I_\tau \to I_{w.\tau}$ for all $w \in W^v$.

In Subsection 3.3, we prove that when W^v is infinite, then every nontrivial submodule of I_τ is infinite dimensional. We deduce that contrary to the reductive case, there exist irreducible representations of ${}^{\mathrm{BL}}\mathcal{H}_\mathcal{F}$ which does not embed in any I_τ.

3.1 Generalized weight spaces of I_τ

Let $\tau \in T_\mathcal{F}$. Let $x \in I_\tau$. Write $x = \sum_{w \in W^v} x_w H_w \otimes_\tau 1$, with $(x_w) \in \mathcal{F}^{(W^v)}$. Set $\mathrm{supp}(x) = \{w \in W^v |\ x_w \neq 0\}$. Equip W^v with the Bruhat order. If E is a finite subset of W^v, $\max(E)$ is the set of elements of E that are maximal for the Bruhat order. Let R be a binary relation on W^v (for example $R =$"$\leq$", $R =$"$\not\geq$", ...) and $w \in W^v$. One sets

$$I_\tau^{Rw} = \bigoplus_{v \in W^v | vRw} \mathcal{F} H_v \otimes_\tau 1, \mathcal{H}_{W^v,\mathcal{F}}{}^{Rw} = \bigoplus_{vRw} \mathcal{F} H_v,\ {}^{\mathrm{BL}}\mathcal{H}(T_\mathcal{F})^{Rw} = \bigoplus_{vRw} H_v \mathcal{F}(Y)$$

and ${}^{\mathrm{BL}}\mathcal{H}_\mathcal{F}^{Rw} = {}^{\mathrm{BL}}\mathcal{H}(T_\mathcal{F})^{Rw} \cap {}^{\mathrm{BL}}\mathcal{H}_\mathcal{F} = \bigoplus_{vRw} H_v \mathcal{F}[Y]$.

Let V be a vector space over $\mathcal{F}$ and $E \subset \mathrm{End}(V)$. For $\tau \in \mathcal{F}^E$ set $V(\tau) = \{v \in V | e.v = \tau(e).v \forall e \in E\}$ and $V(\tau, \mathrm{gen}) = \{v \in V | \exists k \in \mathbb{Z}_{\geq 0} | (e - \tau(e)\mathrm{Id})^k.v = 0, \forall e \in E\}$. Let $\mathrm{Wt}(E) = \{\tau \in \mathcal{F}^E | V(\tau) \neq \{0\}\}$.

The following lemma is well known.

Lemma 3.1. *Let V be a finite dimensional vector space over $\mathcal{F}$. Let $E \subset \mathrm{End}(V)$ be a subset such that for all $e, e' \in E$,*

1. *e is triangularizable*
2. *$ee' = e'e$.*

Then $V = \bigoplus_{\tau \in \mathrm{Wt}(E)} V(\tau, \mathrm{gen})$ *and in particular* $\mathrm{Wt}(E) \neq \emptyset$.

For $\tau \in T_{\mathcal{F}}$, set $W_\tau = \{w \in W^v |\ w.\tau = \tau\}$.

Remark 3.2. *Let* $\tau \in T_{\mathcal{F}}$. *By Lemma 2.8,* $I_\tau^{\leq w}$ *and* $I_\tau^{\not\geq w}$ *are* $\mathcal{F}[Y]$*-submodules of* I_τ. *In particular* $\mathcal{F}[Y].x$ *is finite dimensional for all* $x \in I_\tau$.

Let M be a ${}^{\mathrm{BL}}\mathcal{H}_{\mathcal{F}}$-module. For $\tau \in T_{\mathcal{F}}$, set

$$M(\tau) = \{m \in M | P.m = \tau(P).m \ \forall P \in \mathcal{F}[Y]\}$$

and

$$M(\tau, \mathrm{gen}) = \{m \in M | \exists k \in \mathbb{Z}_{\geq 0} | \forall P \in \mathcal{F}[Y], (P - \tau(P))^k.m = 0\} \supset M(\tau).$$

Let $\mathrm{Wt}(M) = \{\tau \in T_{\mathcal{F}} | M(\tau) \neq \{0\}\}$ and $\mathrm{Wt}(M, \mathrm{gen}) = \{\tau \in T_{\mathcal{F}} | M(\tau, \mathrm{gen}) \neq \{0\}\}$.

Lemma 3.3. *1. Let* $\tau, \tau' \in T_{\mathcal{F}}$. *Let* $x \in I_\tau(\tau', gen)$. *Then if* $x \neq 0$,

$$\max \mathrm{supp}(x) \subset \{w \in W^v |\ w.\tau = \tau'\}.$$

In particular, if $I_\tau(\tau', gen) \neq \{0\}$, *then* $\tau' \in W^v.\tau$ *and thus*

$$\mathrm{Wt}(I_\tau) \subset W^v.\tau.$$

2. Let $\tau \in T_{\mathcal{F}}$. *Let* $M \subset I_\tau$ *be a* $\mathcal{F}[Y]$*-submodule of* I_τ. *Then* $\mathrm{Wt}(M) = \mathrm{Wt}(M, gen) \subset W^v.\tau$ *and* $M = \bigoplus_{\chi \in \mathrm{Wt}(M)} M(\chi, gen)$. *In particular,* $\mathrm{Wt}(M) \neq \emptyset$.

Proof. (1) Let $x \in I_\tau(\tau', \mathrm{gen}) \setminus \{0\}$. Let $w \in \max \mathrm{supp}(x)$. Write $x = a_w H_w \otimes_\tau 1 + y$, where $a_w \in \mathcal{F} \setminus \{0\}$ and $y \in I_\tau^{\not\geq w}$. Then by Lemma 2.8,

$$Z^\lambda.x = a_w H_w Z^{w^{-1}.\lambda} \otimes_\tau 1 + y' = \tau(w^{-1}.\lambda) a_w H_w \otimes_\tau 1 + y' = \tau'(\lambda) a_w H_w \otimes_\tau 1 + \tau'(\lambda) y,$$

where $y' \in I_\tau^{\not\geq w}$. Therefore $w.\tau = \tau'$.

(2) Let $w \in W^v$. Let $P \in \mathcal{F}[Y]$ and $m_P : I_\tau^{\leq w} \to I_\tau^{\leq w}$ be defined by $m_P(x) = P.x$ for all $x \in I_\tau^{\leq w}$. Then by Lemma 2.8, $(m_P - w.\tau(P)\mathrm{Id})(I_\tau^{\leq w}) \subset I_\tau^{<w}$. By induction on $\ell(w)$ we deduce that m_P is triangularizable on $I_\tau^{\leq w}$ and $\mathrm{Wt}(I_\tau^{\leq w}) \subset [1, w].\tau \subset W^v.\tau$.

Let $x \in M$ and $M_x = \mathcal{F}[Y].x$. By the fact that $(W^v, \leq)$ is a directed poset and by Lemma 2.8, there exists $w \in W^v$ such that $M_x \subset I_\tau^{\leq w}$. Therefore, for all $P \in \mathcal{F}[Y]$, $m_P : M_x \to M_x$ is triangularizable. Thus by Lemma 3.1, $\mathcal{F}[Y].x = \bigoplus_{\chi \in \mathrm{Wt}(M_x, \mathrm{gen})} M_x(\chi, \mathrm{gen}) = \bigoplus_{\chi \in W^v.\tau} M_x(\chi, \mathrm{gen})$. Consequently, $M = \sum_{x \in M} M_x = \bigoplus_{\chi \in \mathrm{Wt}(M, \mathrm{gen})} M(\chi, \mathrm{gen})$ and $\mathrm{Wt}(M) \subset \bigcup_{w \in W^v} \mathrm{Wt}(I_\tau^{\leq w}) \subset W^v.\tau$.

Let $\chi \in \mathrm{Wt}(M, \mathrm{gen})$. Let $x \in M(\chi, \mathrm{gen}) \setminus \{0\}$ and $N = \mathcal{F}[Y].x$. Then by Lemma 2.8, N is a finite dimensional submodule of I_τ. By Lemma 3.1, $\mathrm{Wt}(N) \neq \emptyset$. As $\mathrm{Wt}(N) \subset \{\chi\}$, $\chi \in \mathrm{Wt}(M)$. Thus $\mathrm{Wt}(M, \mathrm{gen}) \subset \mathrm{Wt}(M)$ and as the other inclusion is clear, we get the lemma.

□

Proposition 3.4. *(see [Mat77, 4.3.3 Théorème (iii)]) Let* $\tau, \tau' \in T_{\mathcal{F}}$ *and* M *(resp.* M'*) be a* ${}^{\mathrm{BL}}\mathcal{H}_{\mathcal{F}}$*-submodule of* I_τ *(resp.* $I_{\tau'}$*). Assume that* $\mathrm{Hom}_{\mathrm{BL}\mathcal{H}_{\mathcal{F}}-\mathrm{mod}}(M, M') \setminus \{0\}$. *Then* $\tau' \in W^v.\tau$.

Proof. Let $f \in \mathrm{Hom}_{\mathrm{BL}\mathcal{H}_{\mathcal{F}}}(M, M') \setminus \{0\}$. Then by Lemma 3.3 (2), there exists $w \in W^v / W_\tau$ such that $f\big(M(w.\tau, \mathrm{gen})\big) \neq \{0\}$. Then $w.\tau \in \mathrm{Wt}(I_{\tau'})$ and by Lemma 3.3 (1) the proposition follows. □

An element $\tau \in T_{\mathcal{F}}$ is said to be **regular** if $w.\tau \neq \tau$ for all $w \in W^v \setminus \{1\}$. We denote by $T_{\mathcal{F}}^{\mathrm{reg}}$ the set of regular elements of $T_{\mathcal{F}}$.

Proposition 3.5. *(see [Kat81, Proposition 1.17]) Let $\tau \in T_{\mathcal{F}}$.*

1. *There exists a basis $(\xi_w)_{w\in W^v}$ of I_τ such that for all $w \in W^v$:*
   - *$\xi_w \in I_\tau^{\leq w}$ and $\pi_w^H(\xi_w) = 1$*
   - *$\xi_w \in I_\tau(w.\tau, \mathrm{gen})$.*

   Moreover, if $w \in W^v$ is minimal for $\leq$ among $\{v \in W^v | v.\tau = w.\tau\}$, then $\xi_w \in I_\tau(w.\tau)$. In particular, $\mathrm{Wt}(I_\tau) = W^v.\tau$.
2. *If τ is regular, then $I_\tau(w.\tau, \mathrm{gen}) = I_\tau(w.\tau)$ is one dimensional for all $w \in W^v$ and $I_\tau = \bigoplus_{w\in W^v} I_\tau(w.\tau)$.*

Proof. (1) Let $w \in W^v$. Then by Lemma 2.8, Lemma 3.1 and Lemma 3.3, $I_\tau^{\leq w} = \bigoplus_{\overline{v}\in W^v/W_\tau} I_\tau^{\leq w}(\overline{v}.\tau, \mathrm{gen})$. Write $H_w \otimes_\tau 1 = \sum_{\overline{v}\in W^v/W_\tau} x_{\overline{v}}$, where $x_{\overline{v}} \in I_\tau^{\leq w}(v.\tau, \mathrm{gen})$ for all $\overline{v} \in W^v/W_\tau$. Let $\overline{v} \in W^v/W_\tau$ be such that $\pi_w^H(x_{\overline{v}}) \neq 0$. Then $\max \mathrm{supp}(x_{\overline{v}}) = \{w\}$ and by Lemma 3.3, $w.\tau = \overline{v}.\tau$. Set $\xi_w = \frac{1}{\pi_w^H(x_{\overline{v}})} x_{\overline{v}}$. Then $(\xi_u)_{u\in W^v}$ is a basis of I_τ and has the desired properties. Let $w \in W^v$ be minimal for $\leq$ among $\{v \in W^v | v.\tau = w.\tau\}$. Let $\lambda \in Y$. Then by Lemma 2.8, $(Z^\lambda - w.\tau(\lambda).\xi_w) \in I_\tau(w.\tau, \mathrm{gen}) \cap I_\tau^{<w}$. By Lemma 3.3, we deduce that $(Z^\lambda - w.\tau(\lambda)).\xi_w = 0$ and thus that $\xi_w \in I_\tau(w.\tau)$. Thus $w.\tau \in \mathrm{Wt}(I_\tau)$ and by Lemma 3.3, $\mathrm{Wt}(I_\tau) = I_\tau$.

(2) Suppose that τ is regular. Let $w \in W^v$, $\lambda \in Y$ and $x \in I_\tau(\tau, \mathrm{gen})$. Then by Lemma 3.3 (1), $x - \pi_w^H(x)\xi_w \in I_\tau(\tau, \mathrm{gen}) \cap I_\tau^{<w} = \{0\}$. By (1), $\xi_w \in I_\tau(w.\tau)$ and thus $I_\tau(\tau) = I_\tau(\tau, \mathrm{gen})$ is one dimensional. By Lemma 3.3, we deduce that $I_\tau = \bigoplus_{w\in W^v} I_\tau(w.\tau)$. □

3.2 Intertwining operators and weight spaces

In this subsection, we relate intertwining operators and weight spaces and study some consequences. Let $\tau \in T_{\mathcal{F}}$. Using Subsection 3.1, we prove the existence of nonzero morphisms $I_\tau \to I_{w.\tau}$ for all $w \in W^v$. We will give a more precise construction of such morphisms in Subsection 5.4.

Let M be a ${}^{\mathrm{BL}}\mathcal{H}_{\mathcal{F}}$-module and $\tau \in T_{\mathcal{F}}$. For $x \in M(\tau)$ define $\Upsilon_x : I_\tau \to M$ by $\Upsilon_x(u.1 \otimes_\tau 1) = u.x$, for all $u \in {}^{\mathrm{BL}}\mathcal{H}_{\mathcal{F}}$. Then Υ_x is well defined. Indeed, let $u \in {}^{\mathrm{BL}}\mathcal{H}_{\mathcal{F}}$ be such that $u.1 \otimes_\tau 1 = 0$. Then $u \in \mathcal{F}[Y]$ and $\tau(u) = 0$. Therefore $u.x = 0$ and hence Υ_x is well defined. The following lemma is then easy to prove.

Lemma 3.6. *(Frobenius reciprocity, see [Kat81, Proposition 1.10]) Let M be a ${}^{\mathrm{BL}}\mathcal{H}_{\mathcal{F}}$-module, $\tau \in T_{\mathcal{F}}$ and $x \in M(\tau)$. Then the map $\Upsilon : M(\tau) \to \mathrm{Hom}_{\mathrm{BL}\,\mathcal{H}_{\mathcal{F}}-mod}(I_\tau, M)$ mapping each $x \in M(\tau)$ to Υ_x is a vector space isomorphism and $\Upsilon^{-1}(f) = f(1 \otimes_\tau 1)$ for all $f \in \mathrm{Hom}_{\mathrm{BL}\,\mathcal{H}_{\mathcal{F}}-mod}(I_\tau, M)$.*

Proposition 3.7. *(see [Mat77, (4.1.10)]) Let M be a ${}^{\mathrm{BL}}\mathcal{H}_{\mathcal{F}}$-module such that there exists $\xi \in M$ satisfying:*

1. *there exists $\tau \in T_{\mathcal{F}}$ such that $\xi \in M(\tau)$,*
2. *$M = {}^{\mathrm{BL}}\mathcal{H}_{\mathcal{F}}.\xi$.*

Then there exists a surjective morphism $\phi : \mathcal{I}_\tau \twoheadrightarrow M$ of ${}^{\mathrm{BL}}\mathcal{H}_{\mathcal{F}}$-modules.

Proof. One can take $\phi = \Upsilon_\xi$, where Υ is as in Lemma 3.6. □

Proposition 3.8. *(see [Mat77, Théorème 4.2.4]) Let M be an irreducible representation of ${}^{\mathrm{BL}}\mathcal{H}_{\mathcal{F}}$ containing a finite dimensional $\mathcal{F}[Y]$-submodule $M' \neq \{0\}$. Then there exists $\tau \in T_{\mathcal{F}}$ such that there exists a surjective morphism of ${}^{\mathrm{BL}}\mathcal{H}_{\mathcal{F}}$-modules $\phi : I_\tau \twoheadrightarrow M$.*

Proof. By Lemma 3.1, there exists $\xi \in M' \setminus \{0\}$ such that $Z^\mu.\xi \in \mathcal{F}.\xi$ for all $\mu \in Y$. Let $\tau \in T_{\mathcal{F}}$ be such that $\xi \in M(\tau)$. Then we conclude with Proposition 3.7. □

Remark 3.9. *Let $\mathcal{Z}({}^{\mathrm{BL}}\mathcal{H}_{\mathcal{F}})$ be the center of ${}^{\mathrm{BL}}\mathcal{H}_{\mathcal{F}}$. When W^v is finite, it is well known that ${}^{\mathrm{BL}}\mathcal{H}_{\mathcal{F}}$ is a finitely generated $\mathcal{Z}({}^{\mathrm{BL}}\mathcal{H}_{\mathcal{F}})$ module and thus every irreducible representation of ${}^{\mathrm{BL}}\mathcal{H}_{\mathcal{F}}$ is finite dimensional. Assume that W^v is infinite. Using the same reasoning as in [AH19, Remark 4.32] we can prove that ${}^{\mathrm{BL}}\mathcal{H}_{\mathcal{F}}$ is not a finitely generated $\mathcal{Z}({}^{\mathrm{BL}}\mathcal{H}_{\mathcal{F}})$-module. As we shall see (see Remark 5.11), when $\mathcal{F} = \mathbb{C}$, there exist irreducible infinite dimensional representations of ${}^{\mathrm{BL}}\mathcal{H}_{\mathcal{F}}$. However we do not know if there exist an irreducible representation V of ${}^{\mathrm{BL}}\mathcal{H}_{\mathcal{F}}$ such that for all $x \in V \setminus \{0\}$, $\mathcal{F}[Y].x$ is infinite dimensional or equivalently, a representation which is not a quotient of a principal series representation.*

Proposition 3.10. *(see [Kat81, (1.21)]) Let $\tau \in T_{\mathcal{F}}$ and $w \in W^v$. Then* $\mathrm{Hom}_{\mathrm{BL}\mathcal{H}_{\mathcal{F}}-\mathrm{mod}}(I_\tau, I_{w.\tau}) \neq \{0\}$.

Proof. By Proposition 3.5 $w.\tau \in \mathrm{Wt}(I_\tau)$ and we conclude with Lemma 3.6. □

3.3 Nontrivial submodules of I_τ are infinite dimensional

In this subsection, we prove that when W^v is infinite, then every submodule of I_τ is infinite dimensional. We then deduce that there can exist an irreducible representation of ${}^{\mathrm{BL}}\mathcal{H}_{\mathbb{C}}$ such that V does not embed in any I_τ, for $\tau \in T_{\mathbb{C}}$.

Lemma 3.11. *Assume that W^v is infinite. Let $w \in W^v$. Then there exists $s \in \mathscr{S}$ such that $sw > w$.*

Proof. Let $D_L(w) = \{s \in \mathscr{S} | sw < w\}$. By the proof of [BB05, Lemma 3.2.3], $\mathscr{S} \nsubseteq D_L(w)$, which proves the lemma. □

Proposition 3.12. *(compare [Mat77, 4.2.4]) Let $\tau \in T_{\mathcal{F}}$. Let $M \subset I_\tau$ be a nonzero $\mathcal{H}_{W^v,\mathcal{F}}$-submodule. Then the dimension of M is infinite. In particular, if V is a finite dimensional irreducible representation of ${}^{\mathrm{BL}}\mathcal{H}_{\mathcal{F}}$, then* $\mathrm{Hom}_{\mathcal{F}}(V, I_\tau) = \{0\}$ *for all $\tau \in T_{\mathcal{F}}$.*

Proof. Let $m \in M \setminus \{0\}$. Let $\ell(m) = \max\{\ell(v) | v \in \mathrm{supp}(m)\}$. Let $w \in \mathrm{supp}(m)$ be such that $\ell(w) = \ell(m)$. By Lemma 3.11 there exists $(s_n) \in \mathscr{S}^{\mathbb{Z}_{\geq 1}}$ such that if $w_1 = w$ and $w_{n+1} = s_n w_n$ for all $n \in \mathbb{Z}_{\geq 1}$, one has $\ell(w_{n+1}) = \ell(w_n) + 1$ for all $n \in \mathbb{Z}_{\geq 1}$. Let $m_1 = m$ and $m_{n+1} = H_{s_n}.m_n$ for all $n \in \mathbb{Z}_{\geq 1}$. Then for all $n \in \mathbb{Z}_{\geq 1}$, $w_n \in \max\big(\mathrm{supp}(m_n)\big)$, which proves that M is infinite dimensional. □

As we shall see in Appendix A, there can exist finite dimensional representations of ${}^{\mathrm{BL}}\mathcal{H}_{\mathbb{C}}$.

4 Principal series representations of ${}^{\mathrm{BL}}\mathcal{H}_{\mathcal{F}}$ and $\mathcal{H}_{\mathcal{F}}$

In this section, we study principal series representations of $\mathcal{H}_{\mathcal{F}}$.

In Subsection 4.1, we prove that when $\tau \in T_{\mathcal{F}}$, the $\mathcal{H}_{\mathcal{F}}$-submodules of I_τ are exactly the ${}^{\mathrm{BL}}\mathcal{H}_{\mathcal{F}}$ submodules of I_τ (see Proposition 4.2).

In Subsection 4.2, we study the existence of principal series representations of $\mathcal{H}_{\mathcal{F}}$ that do not extend to representations of ${}^{\mathrm{BL}}\mathcal{H}_{\mathcal{F}}$.

4.1 Principal series representations regarded as modules over $\mathcal{H}_{\mathcal{F}}$

One sets $Y^{\emptyset} = Y$, $T_{\mathcal{F}}^{\emptyset} = T_{\mathcal{F}}$. Let $Y^+ = Y \cap \mathcal{T}$ (see Subsection 2.1 for the definition of $\mathcal{T}$). Let $\mathrm{Hom}_{\mathrm{Mon}}(Y^+, \mathcal{F})$ be the set of monoid morphisms from Y^+ to $\mathcal{F}$ and $T_{\mathcal{F}}^+ = \mathrm{Hom}_{\mathrm{Mon}}(Y^+, \mathcal{F}) \setminus \{0\}$. Let $\eta \in \{\emptyset, +\}$. If $\tau \in T_{\mathcal{F}}$, $\chi \in T_{\mathcal{F}}^{\eta}$ and M is a $\mathcal{F}[Y^{\eta}]$-submodule of I_{τ}, one sets $M(\chi, \mathcal{F}[Y^{\eta}]) = \{m \in M | P.m = \chi(P),\ \forall P \in \mathcal{F}[Y^{\eta}]\}$ and $M(\chi, \mathcal{F}[Y^{\eta}], \mathrm{gen}) = \{m \in M | \exists k \in \mathbb{Z}_{\geq 0} |\ \big(P - \chi(P)\mathrm{Id}\big)^k.m = 0,\ \forall P \in \mathcal{F}[Y^{\eta}]\}$. Let $\mathrm{Wt}(M, \mathcal{F}[Y^{\eta}]) = \{\tau \in T_{\mathcal{F}}^{\eta} | M(\tau, \mathcal{F}[Y^{\eta}]) \neq \{0\}\}$.

Lemma 4.1. *Let $\tau \in T_{\mathcal{F}}$. Let $M \subset I_{\tau}$ be a finite dimensional $\mathcal{F}[Y^+]$-submodule of I_{τ}. Then M is an $\mathcal{F}[Y]$-submodule of I_{τ}.*

Proof. For $n \in \mathbb{Z}_{\geq 0}$ set $\mathcal{P}(n)$: "every n-dimensional $\mathcal{F}[Y^+]$ submodule of I_{τ} is an $\mathcal{F}[Y]$-submodule". Let $n \in \mathbb{Z}_{\geq 0}$ be such that $\mathcal{P}(n)$ is true. Let M' be an $n+1$-dimensional $\mathcal{F}[Y^+]$-submodule of I_{τ}.

Assume $|\mathrm{Wt}(M, \mathcal{F}[Y^+])| \geq 2$. Let $\tau_1 \in \mathrm{Wt}(M)$. Then by a lemma similar to Lemma 3.3,

$$M = M(\tau_1, \mathcal{F}[Y^+], \mathrm{gen}) \oplus \bigoplus_{\tau' \in \mathrm{Wt}(M) \setminus \{\tau_1\}} M(\tau', \mathcal{F}[Y^+], \mathrm{gen}).$$

Then $M(\tau_1, \mathcal{F}[Y^+], \mathrm{gen})$ and $\bigoplus_{\tau' \in \mathrm{Wt}(M) \setminus \{\tau_1\}} M(\tau', \mathcal{F}[Y^+], \mathrm{gen})$ are $\mathcal{F}[Y^+]$-submodules of M and have dimension at most n. Thus M is a $\mathcal{F}[Y]$-submodule.

Assume that $\mathrm{Wt}(M) = \{\tau'\}$ for some $\tau' \in \mathrm{Hom}_{\mathrm{Gr}}(Y, \mathcal{F}^*)$. Let $x \in M \setminus \{0\}$ and $k = \min\{k' \in \mathbb{Z}_{\geq 1} | (Z^{\lambda} - \tau'(\lambda)\mathrm{Id})^{k'} = 0,\ \forall \lambda \in Y^+\}$. Let $M' = \{m \in M | (Z^{\lambda} - \tau'(\lambda)\mathrm{Id})^{k-1}.m = 0,\ \forall \lambda \in Y^+\}$. Then M' is a proper $\mathcal{F}[Y^+]$-submodule of M and thus it is an $\mathcal{F}[Y]$-submodule. Let $\lambda \in Y^+$. Then $(Z^{\lambda} - \tau'(\lambda)\mathrm{Id}).x \in M'$ and hence $Z^{\lambda}.x = \tau'(\lambda)x + m'_{\lambda}$, where $m'_{\lambda} \in M'$. Let $\mu \in Y$ and $\nu \in C_f^v \cap Y$ (see Subsection 2.1 for the definition of C_f^v) be such that $\mu + \nu \in Y^+$. Then $x = \frac{1}{\tau'(\nu)}(Z^{\nu}.x - m'_{\nu})$ and thus $Z^{\mu}.x = Z^{\mu}.\frac{1}{\tau'(\nu)}(Z^{\nu}.x - m'_{\nu}) = \frac{1}{\tau'(\nu)}(Z^{\mu+\nu}.x - Z^{\mu}.m'_{\nu}) \in M$. Therefore M is an $\mathcal{F}[Y]$-submodule of I_{τ}, which proves the lemma. □

Proposition 4.2. *Let $\tau \in T_{\mathcal{F}}$ and $M \subset I_{\tau}$. Then M is an $\mathcal{H}_{\mathcal{F}}$-submodule of I_{τ} if and only if M is a ${}^{BL}\mathcal{H}_{\mathcal{F}}$-submodule of I_{τ}. In particular, I_{τ} is irreducible as a ${}^{\mathrm{BL}}\mathcal{H}_{\mathcal{F}}$-module if and only if I_{τ} is irreducible as an $\mathcal{H}_{\mathcal{F}}$-module.*

Proof. Let $M \subset I_{\tau}$ be a $\mathcal{H}_{\mathcal{F}}$-submodule. Then M is an $\mathcal{F}[Y^+]$ submodule of I_{τ}. Thus $M = \sum_{x \in M} \mathcal{F}[Y^+].x$ and by Lemma 4.1, M is an $\mathcal{F}[Y]$-submodule of I_{τ}. As ${}^{\mathrm{BL}}\mathcal{H}_{\mathcal{F}}$ is generated as an algebra by $\mathcal{H}_{\mathcal{F}}$ and $\mathcal{F}[Y]$, we deduce the proposition. □

4.2 Degenerate principal series representations of $\mathcal{H}_{\mathcal{F}}$

Let $\tau : Y^+ \to \mathcal{F}$ be a monoid morphism. Then τ induces an algebra morphism $\tau : \mathcal{F}[Y^+] \to \mathcal{F}$ and thus this defines a representation $I_{\tau}^+ = \mathrm{Ind}^{\mathcal{H}_{\mathcal{F}}}(\tau) = \mathcal{H}_{\mathcal{F}} \otimes_{\mathcal{F}[Y^+]} F$. Then if I_{τ}^+ is not the restriction of a representations of ${}^{\mathrm{BL}}\mathcal{H}_{\mathcal{F}}$ we call I_{τ}^+ a **degenerate principal series representation of $\mathcal{H}_{\mathcal{F}}$**. In this section we study the existence of degenerate principal series representation of $\mathcal{H}_{\mathcal{F}}$. We prove that in some cases - for example when $\mathcal{H}_{\mathcal{F}}$ is associated with an affine root generating system or to a size 2 Kac-Moody matrix - there exists no degenerate principal series representations of $\mathcal{H}_{\mathcal{F}}$ (see Lemma 4.5). We prove that there exist Kac-Moody matrices such that there exist degenerate principal series representations of $\mathcal{H}_{\mathcal{F}}$ (see Lemma 4.9).

Let $\mathrm{res}_{Y^+} : \mathrm{Hom}_{\mathrm{Mon}}(Y, \mathcal{F}) \to \mathrm{Hom}_{\mathrm{Mon}}(Y^+, \mathcal{F})$ be defined by $\mathrm{res}_{Y^+}(\tau) = \tau_{|Y^+}$ for all $\tau \in \mathrm{Hom}_{\mathrm{Mon}}(Y, \mathcal{F})$.

Lemma 4.3. *The map* $\mathrm{res}_{Y^+} : \mathrm{Hom}_{\mathrm{Gr}}(Y, \mathcal{F}^*) = \mathrm{Hom}_{\mathrm{Mon}}(Y, \mathcal{F}^*) \to \mathrm{Hom}_{\mathrm{Mon}}(Y^+, \mathcal{F}^*)$ *is a bijection.*

Proof. Let $\tau \in \mathrm{Hom}_{\mathrm{Mon}}(Y, \mathcal{F}^*)$. Let $\nu \in C_f^v$. Let $\lambda \in Y$ and $n \in \mathbb{Z}_{\geq 0}$ be such that $\lambda + n\nu \in \mathcal{T}$. Then $\tau(\lambda) = \frac{\tau(\lambda+n\nu)}{\tau(n\nu)}$ and thus $\mathrm{res}_{|Y^+}$ is injective.

Let $\tau^+ \in \mathrm{Hom}_{\mathrm{Mon}}(Y^+, \mathcal{F}^*)$. Let $\lambda \in Y$. Write $\lambda = \lambda_+ - \lambda_-$, with $\lambda_+, \lambda_- \in Y^+$. Set $\tau(\lambda) = \frac{\tau^+(\lambda_+)}{\tau^+(\lambda_-)}$, which does not depend on the choices of λ_- and λ_+. Then $\tau \in \mathrm{Hom}_{\mathrm{Mon}}(Y, \mathcal{F}^*)$ is well defined and $\mathrm{res}_{|Y^+}(\tau) = \tau^+$, which finishes the proof. □

Lemma 4.4. *Let* $\tau \in \mathrm{Hom}_{\mathrm{Mon}}(Y^+, \mathcal{F})$ *and* $\chi \in T_{\mathcal{F}}$.

1. *Suppose* $\mathrm{Hom}_{\mathcal{H}_{\mathcal{F}}-\mathrm{mod}}(I_\tau^+, I_\chi) \neq \{0\}$. *Then there exists* $w \in W^v$ *such that* $\tau = w.\chi_{|Y^+}$.

2. *Suppose* $\mathrm{Hom}_{\mathcal{H}_{\mathcal{F}}-\mathrm{mod}}(I_\chi, I_\tau^+) \neq \{0\}$. *Then there exists* $w \in W^v$ *such that* $\tau = w.\chi_{|Y^+}$.

Proof. (1) Let $\phi \in \mathrm{Hom}_{\mathcal{H}_{\mathcal{F}}-\mathrm{mod}}(I_\tau^+, I_\chi) \setminus \{0\}$. Let $x = \phi(1 \otimes_{\tau^+} 1)$. Then $Z^\lambda.x = \tau(\lambda).x$ for all $\lambda \in Y^+$. By Lemma 2.8, $Z^\lambda.x \neq 0$ for all $\lambda \in Y^+$. Thus $\tau(\lambda) \neq 0$ for all $\lambda \in Y^+$.

Let $\mu \in Y$. Let $\nu \in C_f^v \cap Y$ be such that $\mu + \nu \in Y^+$. Then $Z^\mu.x = \frac{\tau(\mu+\nu)}{\tau(\nu)}.x$. Therefore there exists $\chi' \in T_{\mathcal{F}}$ such that $x \in I_\chi(\chi')$. By Lemma 3.3, $\chi' \in W^v.\chi$. Moreover, $\chi'_{|Y^+} = \tau$, which proves (1).

(2) Let $\phi \in \mathrm{Hom}_{\mathcal{H}_{\mathcal{F}}-\mathrm{mod}}(I_\chi, I_\tau^+) \setminus \{0\}$. Let $x = \phi(1 \otimes_\chi 1)$. Then $Z^\lambda.x = \chi(\lambda).x$ for all $\lambda \in Y^+$. By a lemma similar to Lemma 3.3 we deduce that $\chi_{|Y^+} \in W^v.\tau$, which proves the lemma. □

One has $\mathrm{Hom}_{\mathrm{Mon}}\big(Y, (\mathcal{F}, .)\big) = \mathrm{Hom}_{\mathrm{Gr}}(Y, \mathcal{F}^*) \cup \{0\}$. Set $\mathbb{A}_{in} = \bigcap_{s \in \mathscr{S}} \ker(\alpha_s)$. Let $\mathring{\mathcal{T}}$ be the interior of the Tits cone.

Lemma 4.5. *Let* $\tau^+ \in \mathrm{Hom}_{\mathrm{Mon}}\big(Y, (\mathcal{F}, .)\big)$. *Assume that there exists* $\lambda \in Y^+$ *such that* $\tau^+(\lambda) = 0$. *Then* $\tau^+(\mathring{\mathcal{T}} \cap Y) = \{0\}$. *In particular, if* $\mathcal{T} = \mathring{\mathcal{T}} \cup \mathbb{A}_{in}$, *then* $\mathrm{Hom}_{\mathrm{Mon}}\big(Y^+, (\mathcal{F}, .)\big) = \mathrm{Hom}_{\mathrm{Mon}}(Y, \mathcal{F}^*) \cup \{0\}$.

Proof. Let $\mu \in \mathring{\mathcal{T}} \cap Y$. Then for $n \gg 0$, $n\mu \in \lambda + \mathcal{T}$. Indeed, $n\mu - \lambda = n(\mu - \frac{\lambda}{n}) \in \mathcal{T}$ for $n \gg 0$. Hence $\tau^+(n\mu) = (\tau^+(\mu))^n = 0$. □

A face $F^v \subset \mathcal{T}$ is called **spherical** if its fixer in W^v is finite.

Remark 4.6.

1. *If* $\mathbb{A}$ *is associated to an affine Kac-Moody matrix, then* $\mathcal{T} = \mathring{\mathcal{T}} \cup \mathbb{A}_{in}$ *(see [Héb18, Corollary 2.3.8] for example).*

2. *If* $\mathbb{A}$ *is associated to a size* 2 *indefinite Kac-Moody matrix, then* $\mathcal{T} = \mathring{\mathcal{T}} \cup \mathbb{A}_{in}$. *Indeed, by [Rém02, Théorème 5.2.3],* $\mathring{\mathcal{T}}$ *is the union of the spherical vectorial faces. By [Rou11, 1.3], if* $J \subset \mathscr{S}$ *and* $w \in W^v$, *the fixer of* $w.F^v$ *is* $w.W^v(J).w^{-1}$. *Therefore the only non-spherical face of* $\mathcal{T}$ *is* $\mathbb{A}_{in}$ *and hence* $\mathcal{T} = \mathring{\mathcal{T}} \cup \mathbb{A}_{in}$.

3. *Let* $A = (a_{i,j})_{i,j \in [\![1,3]\!]}$ *be a Kac-Moody matrix such that for all* $i \neq j$, $a_{i,j}a_{j,i} \geq 4$. *Then by [Kum02, Proposition 1.3.21],* W^v *is the free group with 3 generators* s_1, s_2, s_3 *of order* 2. *Thus for all* $J \subset \mathscr{S}$ *such that* $|J| = 2$, $F^v(J)$ *is non-spherical. Hence* $\mathcal{T} \supsetneq \mathring{\mathcal{T}} \cup \mathbb{A}_{in}$.

4.2.1 Construction of an element of $\mathrm{Hom}_{\mathrm{Mon}}(Y^+,\mathcal{F})\setminus \mathrm{Hom}_{\mathrm{Mon}}(Y,\mathcal{F})$

We now prove that there exist Kac-Moody matrices for which $\mathrm{Hom}_{\mathrm{Mon}}(Y^+,\mathcal{F})\neq \mathrm{Hom}_{\mathrm{Mon}}(Y,\mathcal{F})$. Assume that $\mathbb{A}$ is associated to an invertible indefinite size 3 Kac-Moody matrix (see [Kac94, Theorem 4.3] for the definition of indefinite). Then one has $\mathbb{A}=\mathbb{A}'\oplus\mathbb{A}_{in}$, where $\mathbb{A}'=\bigoplus_{i\in I}\mathbb{R}\alpha_i^\vee$. Maybe considering $\mathbb{A}/\mathbb{A}_{in}$, we may assume that $\mathbb{A}_{in}=\{0\}$.

Recall that $\mathcal{T}$ is the disjoint union of the positive vectorial faces of $\mathbb{A}$.

Lemma 4.7. *Assume that there exists a non-spherical vectorial face $F^v\neq\{0\}$. Let $x\in\mathcal{T}$ and $y\in\mathcal{T}\setminus\overline{F^v}$. Then $[x,y]\cap F^v\subset\{x\}$.*

Proof. Assume that $y\in\mathring{\mathcal{T}}$. Then $(x,y]\subset\mathring{\mathcal{T}}$ and thus $[x,y]\cap F^v\subset\{x\}$.

Assume that $y\notin\mathring{\mathcal{T}}$. For $a\in\mathcal{T}$, we denote by F_a^v the vectorial face of $\mathcal{T}$ containing a. If $F_x^v=F_y^v$, then $[x,y]\subset F_x^v$. As $F_y^v\neq F^v$, we deduce that $[x,y]\cap F^v=\emptyset$. We now assume that $F_x^v\neq F_y^v$. As W^v is countable, the number of positive vectorial faces is countable and thus there exist $u\neq u'\in[x,y]$ such that $F_u^v=F_{u'}^v$. Then the dimension of the vector space spanned by F_u^v is at least 2. Thus there exists $w\in W^v$ such that $F_u^v=w.F^v(J)$, for some $J\subset\mathscr{S}$ such that $|J|\leq 1$. Then the fixer of F_u^v is $w.W_J.w^{-1}$, where $W_J=\langle J\rangle$. Then W_J is finite and thus F_u^v is spherical. Consequently, $(x,y)=(x,u]\cup[u,y)\subset\mathring{\mathcal{T}}$ and the lemma follows. □

Lemma 4.8. *Assume that there exists a non-spherical vectorial face $F^v\neq\{0\}$. Then $\mathcal{T}\setminus\overline{F^v}$ and $\mathcal{T}\setminus\{0\}$ are convex.*

Proof. Let $x,y\in\mathcal{T}\setminus F^v$. Suppose that $[x,y]\cap F^v\neq\emptyset$. By Lemma 4.7, $y\in\overline{F^v}=F^v\cup\{0\}$ and hence $y=0$. Let F_x^v be the vectorial face containing x. Then $[x,y)\subset F_x^v$ and hence $[x,y)\cap F^v=\emptyset$: a contradiction. Thus $\mathcal{T}\setminus F^v$ is convex.

By [GR14, 2.9 Lemma], there exists a basis $(\delta_s)_{s\in\mathscr{S}}$ of $\bigoplus_{s\in\mathscr{S}}\mathbb{R}\alpha_s^\vee$ such that $\delta_s(\mathcal{T})\geq 0$ for all $s\in\mathscr{S}$. Thus $\mathcal{T}\setminus\{0\}$ is convex and hence $\mathcal{T}\setminus\overline{F^v}=\mathcal{T}\setminus F^v\cap\mathcal{T}\setminus\{0\}$ is convex. □

Lemma 4.9. *Assume that $\mathbb{A}$ is associated to an indefinite Kac-Moody matrix of size 3 such that there exists a non-spherical face different from $\mathbb{A}_{in}$. Assume moreover that $(\alpha_s^\vee)_{s\in\mathscr{S}}$ is a basis of $\mathbb{A}$. Then $\mathrm{Hom}_{\mathrm{Mon}}\big(Y^+,(\mathcal{F},.)\big)\supsetneq\mathrm{Hom}_{\mathrm{Mon}}\big(Y^+,\mathcal{F}^*\big)\cup\{0\}$.*

Proof. Let $\tau^+=\mathbb{1}_{\overline{F^v}}:\mathcal{T}\to\mathcal{F}$. Let us prove that $\tau^+\in\mathrm{Hom}_{\mathrm{Mon}}\big(\mathcal{T},(\mathcal{F},.)\big)$.

Let $x,y\in\mathcal{T}$. If $x,y\in\mathcal{T}\setminus\overline{F^v}$, then $x+y=2.\frac{1}{2}(x+y)\in\mathcal{T}\setminus\overline{F^v}$ by Lemma 4.8 and thus $\tau^+(x+y)=0=\tau^+(x)\tau^+(y)$.

Suppose $x\in F^v$ and $y\in\mathcal{T}\setminus\overline{F^v}$, then $x+y=2.\frac{1}{2}(x+y)\in\mathcal{T}\setminus\overline{F^v}$ by Lemma 4.7. Thus $\tau^+(x+y)=0=\tau^+(x)\tau^+(y)$.

Suppose $x=\{0\}$ and $y\in\mathcal{T}\setminus\overline{F^v}$. Let F_y^v be the vectorial face containing y. Then $(x,y]\subset F_y^v$ and hence $x+y\in F_y^v$: $\tau^+(x+y)=0=\tau^+(x)\tau^+(y)$. Consequently, $\tau^+\in\mathrm{Hom}_{\mathrm{Mon}}\big(\mathcal{T},(\mathcal{F},.)\big)$.

Maybe considering $w.F^v$, for some $w\in W^v$, we can assume $F^v\subset\overline{C_f^v}$. Then there exist $s_1,s_2,s_3\in\mathscr{S}$ such that $\mathscr{S}=\{s_1,s_2,s_3\}$ and $F^v=\alpha_{s_1}^{-1}(\{0\})\cap\alpha_{s_2}^{-1}(\{0\})\cap\alpha_{s_3}^{-1}(\mathbb{R}_+^*)$. Let $\lambda\in\mathbb{A}$ be such that $\alpha_{s_1}(\lambda)=\alpha_{s_2}(\lambda)=0$ and $\alpha_{s_3}(\lambda)=1$. There exists $n\in\mathbb{Z}_{\geq 1}$ such that $\lambda\in\frac{1}{n}Y$. Thus $\tau^+_{|Y^+}\in\mathrm{Hom}_{\mathrm{Mon}}\big(Y^+,(\mathcal{F},.)\big)\setminus\big(\mathrm{Hom}_{\mathrm{Mon}}\big(Y^+,\mathcal{F}^*\big)\cup\{0\}\big)$.

□

5 Study of the irreducibility of I_τ

In this section, we study the irreducibility of I_τ.

In Subsection 5.1, we describe certain intertwining operators between I_τ and $I_{s.\tau}$, for $s \in \mathscr{S}$ and $\tau \in T_{\mathcal{F}}$. For this, we introduce elements $F_s \in {}^{\mathrm{BL}}\mathcal{H}(T_{\mathcal{F}})$ such that $F_s(\chi) \otimes_\chi 1 \in I_\chi(s.\chi)$ for all $\chi \in \mathcal{T}_{\mathcal{F}}$ for which this is well defined.

In Subsection 5.2, we establish that the condition (2) appearing in Theorems 1, 2 and 3 is a necessary condition for the irreducibility of I_τ. This conditions comes from the fact that when I_τ is irreducible, certain intertwinners have to be isomorphisms.

In Subsection 5.3, we prove an irreducibility criterion for I_τ involving the dimension of I_τ and the values of τ (see Theorem 5.8). We then deduce Matsumoto criterion.

In Subsection 5.4 we introduce and study, for every $w \in W^v$, an element $F_w \in {}^{\mathrm{BL}}\mathcal{H}(T_{\mathcal{F}})$ such that $F_w(\chi) \otimes_\chi 1 \in I_\chi(w.\chi)$ for every $\chi \in T_{\mathbb{C}}$ for which this is well defined.

In Subsection 5.5 we prove one implication of Kato's criterion (see Proposition 5.17).

The definition we gave for I_τ is different from the definition of Matsumoto (see [Mat77, (4.1.5)]). It seems to be well known that these definitions are equivalent. We justify this equivalence in Subsection 5.6. We also explain why it seems difficult to adapt Kato's proof in our framework.

5.1 Intertwining operators associated with simple reflections

Let $s \in \mathscr{S}$. In this subsection we define and study an element $F_s \in {}^{\mathrm{BL}}\mathcal{H}(T_{\mathcal{F}})$ such that $F_s(\chi) \otimes_\chi 1 \in I_\chi(s.\chi)$ for all χ such that $F_s(\chi)$ is well defined.

Let $s \in \mathscr{S}$ and $T_s = \sigma_s H_s$. Let $w \in W^v$ and $w = s_1 \dots s_k$ be a reduced writing. Set $T_w = T_{s_1} \dots T_{s_k}$. This is independent of the choice of the reduced writing by [BPGR16, 6.5.2].

Set $B_s = T_s - \sigma_s^2 \in \mathcal{H}_{W^v,\mathcal{F}}$. One has $B_s^2 = -(1+\sigma_s^2)B_s$. Let $\zeta_s = -\sigma_s Q_s(Z) + \sigma_s^2 \in \mathcal{F}(Y) \subset {}^{\mathrm{BL}}\mathcal{H}(T_{\mathcal{F}})$. When $\sigma_s = \sigma_s' = \sqrt{q}$ for all $s \in \mathscr{S}$, we have $\zeta_s = \frac{1-qZ^{-\alpha_s^\vee}}{1-Z^{-\alpha_s^\vee}} \in \mathcal{F}(Y)$. Let $F_s = B_s + \zeta_s \in {}^{\mathrm{BL}}\mathcal{H}(T_{\mathcal{F}})$.

Let $\alpha^\vee \in \Phi^\vee$. Write $\alpha^\vee = w.\alpha_s^\vee$ for $w \in W^v$ and $s \in \mathscr{S}$. We set $\zeta_{\alpha^\vee} = (\zeta_s)^w$.

Let $\alpha^\vee \in \Phi^\vee$. Write $\alpha = w.\alpha_s^\vee$, with $w \in W^v$ and $s \in \mathscr{S}$. We set $\sigma_{\alpha^\vee} = \sigma_s$ and $\sigma'_{\alpha^\vee} = w.\sigma'_s$. This is well defined by Lemma 2.4 and by the relations on the σ_t, $t \in \mathscr{S}$ (see Subsection 2.3).

The ring $\mathcal{F}[Y]$ is a unique factorization domain. For $\alpha^\vee$, write $\zeta_{\alpha^\vee} = \frac{\zeta_{\alpha^\vee}^{\mathrm{num}}}{\zeta_{\alpha^\vee}^{\mathrm{den}}}$ where $\zeta_{\alpha^\vee}^{\mathrm{num}}, \zeta_{\alpha^\vee}^{\mathrm{den}} \in \mathbb{C}[Y]$ are pairwise coprime. For example if $\alpha^\vee \in \Phi^\vee$ is such that $\sigma_{\alpha^\vee} = \sigma'_{\alpha^\vee}$ we can take $\zeta_{\alpha^\vee}^{\mathrm{den}} = 1 - Z^{-\alpha^\vee}$ and in any case we will choose $\zeta_{\alpha^\vee}^{\mathrm{den}}$ among $\{1-Z^{-\alpha^\vee}, 1+Z^{-\alpha^\vee}, 1-Z^{-2\alpha^\vee}\}$.

Remark 5.1. *Let $\tau \in T_{\mathcal{F}}$ and $r = r_{\alpha^\vee} \in \mathscr{R}$. Suppose that $r.\tau \neq \tau$. Then $\zeta_{\alpha^\vee}^{\mathrm{den}}(\tau) \neq 0$. Indeed, let $\lambda \in Y$ be such that $\tau(r.\lambda) \neq \tau(\lambda)$. Then $\tau(r.\lambda - \lambda) = \tau(\alpha_r^\vee)^{\alpha_r(\lambda)} \neq 1$. Suppose $\sigma_{\alpha^\vee} = \sigma'_{\alpha^\vee}$, then $\zeta_{\alpha^\vee}^{\mathrm{den}} = 1 - Z^{-\alpha_r^\vee}$ and thus $\tau(\zeta_{\alpha^\vee}^{\mathrm{den}}) \neq 0$. Suppose $\sigma_r = \sigma'_r$. Then $\alpha_r(\lambda) \in 2\mathbb{Z}$ thus $\tau(\alpha_r^\vee) \notin \{-1, 1\}$ and hence $\tau(\zeta_{\alpha^\vee}^{\mathrm{den}}) \neq 0$.*

Lemma 5.2. *Let $s \in \mathscr{S}$ and $\theta \in \mathbb{C}(Y)$. Then*

$$\theta * F_s = F_s * \theta^s.$$

In particular, for all $\tau \in T_{\mathcal{F}}$ such that $\tau(\zeta_s^{\mathrm{den}}) \neq 0$, $F_s(\tau) \otimes_\tau 1 \in I_\tau(s.\tau)$ and $F_s(\tau) \otimes_{s.\tau} 1 \in I_{s.\tau}(\tau)$.

Proof. Let $\lambda \in Y$. Then

$$\begin{aligned} Z^\lambda * B_s - B_s * Z^{s.\lambda} =& \sigma_s(Z^\lambda * H_s - H_s * Z^{s.\lambda}) + \sigma_s^2(Z^{s.\lambda} - Z^\lambda) \\ =& -\sigma_s Q_s(Z)(Z^{s\lambda} - Z^\lambda) + \sigma_s^2(Z^{s.\lambda} - Z^\lambda) \\ =& \zeta_s(Z^{s.\lambda} - Z^\lambda). \end{aligned}$$

Thus $Z^\lambda * F_s = Z^\lambda * (B_s + \zeta_s) = F_s * Z^{s.\lambda}$ and hence $\theta * F_s = F_s * \theta^s$ for all $\theta \in \mathbb{C}[Y]$.
Let $\theta \in \mathbb{C}[Y] \setminus \{0\}$. Then $\theta * (F_s * \frac{1}{\theta^s}) = F_s$ and thus $\frac{1}{\theta} * F_s = F_s * \frac{1}{\theta^s}$. Lemma follows. □

Lemma 5.3. *Let $s \in \mathscr{S}$. Then $F_s^2 = \zeta_s \zeta_s^s \in \mathcal{F}(Y) \subset {}^{\mathrm{BL}}\mathcal{H}(T_\mathcal{F})$.*

Proof. By Lemma 5.2, one has:

$$F_s^2 = (B_s+\zeta_s)*F_s = B_s*F_s+F_s*\zeta_s^s = B_s^2+B_s\zeta_s+B_s\zeta_s^s+\zeta_s\zeta_s^s = B_s(-1-\sigma_s^2+\zeta_s+\zeta_s^s)+\zeta_s\zeta_s^s = \zeta_s\zeta_s^s.$$

□

5.2 A necessary condition for irreducibility

In this subsection, we establish that the condition (2) appearing in Theorems 1, 2 and 3 is a necessary condition for the irreducibility of I_τ.

Recall the definition of Υ from Subsection 3.2.

Lemma 5.4. *Let $\tau \in T_\mathcal{F}$ and $s \in \mathscr{S}$ be such that $\tau(\zeta_s^{\mathrm{den}})\tau((\zeta_s^{\mathrm{den}})^s) \neq 0$. Let $\phi(\tau, s.\tau) = \Upsilon_{F_s(\tau)\otimes_{s.\tau}1} : I_\tau \to I_{s.\tau}$ and $\phi(s.\tau, \tau) = \Upsilon_{F_s(\tau)\otimes_\tau 1} : I_{s.\tau} \to I_\tau$. Then*

$$\phi(s.\tau, \tau) \circ \phi(\tau, s.\tau) = \tau(\zeta_s\zeta^s)\mathrm{Id}_{I_\tau} \text{ and } \phi(\tau, s.\tau) \circ \phi(s.\tau, \tau) = \tau(\zeta_s\zeta^s)\mathrm{Id}_{I_{s.\tau}}.$$

Proof. By Lemma 5.2 and Lemma 3.6, $\phi(s.\tau, \tau)$ and $\phi(\tau, s.\tau)$ are well defined. Let $f = \phi(s.\tau, \tau) \circ \phi(\tau, s.\tau) \in \mathrm{End}_{\mathrm{BL}\mathcal{H}_\mathcal{F}-\mathrm{mod}}(I_\tau)$. Then by Lemma 5.2 and Lemma 5.3:

$$f(1 \otimes_\tau 1) = \phi(s.\tau, \tau)\big(F_s(\tau) \otimes_{s.\tau} 1\big) = F_s(\tau).\phi(s.\tau, \tau)\big(1 \otimes_{s.\tau} 1\big) = F_s(\tau)^2 \otimes_\tau 1 = \tau(\zeta_s\zeta_s^s) \otimes_\tau 1.$$

By symmetry, we get the lemma.

□

Let $\mathcal{U}_\mathcal{F}$ be the set of $\tau \in T_\mathcal{F}$ such that for all $\alpha^\vee \in \Phi^\vee$, $\tau(\zeta_{\alpha^\vee}^{\mathrm{num}}) \neq 0$. When $\sigma_s = \sigma_s' = \sqrt{q}$ for all $s \in \mathscr{S}$, then $\mathcal{U}_\mathcal{F} = \{\tau \in T_\mathcal{F} | \tau(\alpha^\vee) \neq q,\ \forall \alpha^\vee \in \Phi^\vee\}$.

We assume that for all $s \in \mathscr{S}$, $\sigma_s' \notin \{\sigma_s^{-1}, -\sigma_s, -\sigma_s^{-1}\}$. Under this condition, if $\alpha^\vee \in \Phi^\vee$ and $\tau \in T_\mathcal{F}$ are such that $\tau(\zeta_{\alpha^\vee}^{\mathrm{den}}) = 0$, then $\tau(\zeta_{\alpha^\vee}^{\mathrm{num}}) \neq 0$.

Lemma 5.5. *1. Let $\tau \in \mathcal{U}_\mathcal{F}$. Then for all $w \in W^v$, I_τ and $I_{w.\tau}$ are isomorphic as ${}^{\mathrm{BL}}\mathcal{H}_\mathcal{F}$-modules.*

2. Let $\tau \in T_\mathcal{F}$ be such that I_τ is irreducible. Then $\tau \in \mathcal{U}_\mathcal{F}$.

Proof. Let $\tau \in \mathcal{U}_\mathcal{F}$. Let $w \in W^v$ and $\tilde{\tau} = w.\tau$. Let $s \in \mathscr{S}$. Assume that $s.\tilde{\tau} \neq \tilde{\tau}$. Then by Remark 5.1 and Lemma 5.4, $I_{\tilde{\tau}}$ is isomorphic to $I_{s.\tilde{\tau}}$ and (1) follows by induction.

Let $\tau \in T_\mathcal{F}$ be such that I_τ is irreducible. Let $s \in \mathscr{S}$.

Suppose $\tau(\zeta_s^{\mathrm{den}}) = 0$. Then by assumption, $\tau(\zeta_s^{\mathrm{num}}) \neq 0$. Moreover by Remark 5.1, $I_{s.\tau} = I_\tau$.

Suppose now $\tau(\zeta_s^{\mathrm{den}}) \neq 0$. Then (with the same notations as in Lemma 5.4), $\phi(s.\tau, \tau) \neq 0$ and $\mathrm{Im}\big(\phi(s.\tau, \tau)\big)$ is a ${}^{\mathrm{BL}}\mathcal{H}_\mathcal{F}$-submodule of I_τ: $\mathrm{Im}\big(\phi(s.\tau, \tau)\big) = I_\tau$. Therefore $\phi(\tau, s.\tau) \circ \phi(s.\tau, \tau) \neq 0$. Thus by Lemma 5.4, $\phi(\tau, s.\tau)$ is an isomorphism and $\tau(\zeta_s\zeta_s^s) \neq 0$. In particular, $\tau(\zeta_s^{\mathrm{num}}) \neq 0$.

Therefore in any cases, I_τ is isomorphic to $I_{s.\tau}$ and $\tau(\zeta_s^{\mathrm{num}}) \neq 0$. By induction we deduce that $I_{w.\tau}$ is isomorphic to I_τ. Thus $I_{w.\tau}$ is irreducible for all $w \in W^v$. Thus $w.\tau(\zeta_s^{\mathrm{num}}) \neq 0$ for all $w \in W^v$ and $s \in \mathscr{S}$, which proves that $\tau \in \mathcal{U}_\mathcal{F}$. □

Lemma 5.6. *Let $\tau \in T_{\mathcal{F}}$ be such that $I_{w.\tau} \simeq I_\tau$ (as a ${}^{\mathrm{BL}}\mathcal{H}_{\mathcal{F}}$-module) for all $w \in W^v$. Then for all $w \in W^v$, there exists a vector space isomorphism $I_\tau(\tau) \simeq I_\tau(w.\tau)$.*

Proof. Let $w \in W^v$. Then by hypothesis, $\mathrm{Hom}_{\mathrm{BL}_{\mathcal{H}_{\mathcal{F}}-\mathrm{mod}}}(I_\tau, I_\tau) \simeq \mathrm{Hom}_{\mathrm{BL}_{\mathcal{H}_{\mathcal{F}}-\mathrm{mod}}}(I_{w.\tau}, I_{w.\tau})$. Let $\phi : I_\tau \to I_{w.\tau}$ be a ${}^{\mathrm{BL}}\mathcal{H}_{\mathcal{F}}$-module isomorphism. Then ϕ induces an isomorphism of vector spaces $I_\tau(w.\tau) \simeq I_{w.\tau}(w.\tau)$. By Lemma 3.6,

$$I_\tau(\tau) \simeq \mathrm{Hom}_{\mathrm{BL}_{\mathcal{H}_{\mathcal{F}}-\mathrm{mod}}}(I_\tau, I_\tau) \simeq \mathrm{Hom}_{\mathrm{BL}_{\mathcal{H}_{\mathcal{F}}-\mathrm{mod}}}(I_{w.\tau}, I_{w.\tau}) \simeq I_{w.\tau}(w.\tau) \simeq I_\tau(w.\tau).$$

□

5.3 An irreducibility criterion for I_τ

In this subsection, we give a characterization of irreducibility for I_τ, for $\tau \in T_{\mathbb{C}}$.

If $\mathcal{B}$ is a $\mathbb{C}$-algebra with unity e and $a \in \mathcal{B}$, one sets $\mathrm{Spec}(a) = \{\lambda \in \mathbb{C} |\ a - \lambda e \text{ is not invertible}\}$. Recall the following theorem of Amitsur (see Théorème B.I of [Ren10]):

Theorem 5.7. *Let $\mathcal{B}$ be a $\mathbb{C}$-algebra with unity e. Assume that the dimension of $\mathcal{B}$ over $\mathbb{C}$ is countable. Then for all $a \in \mathcal{B}$, $\mathrm{Spec}(a) \neq \emptyset$.*

Recall that $\mathcal{U}_{\mathbb{C}}$ is the set of $\tau \in T_{\mathbb{C}}$ such that for all $\alpha^\vee \in \Phi^\vee$, $\tau(\zeta^{\mathrm{num}}_{\alpha^\vee}) \neq 0$.

Theorem 5.8. *Let $\tau \in T_{\mathbb{C}}$. Then the following are equivalent:*

1. *I_τ is irreducible,*
2. *$I_\tau(\tau) = \mathbb{C}.1 \otimes_\tau 1$ and $\tau \in \mathcal{U}_{\mathbb{C}}$,*
3. *$\mathrm{End}_{\mathrm{BL}_{\mathcal{H}_{\mathbb{C}}-\mathrm{mod}}}(I_\tau) = \mathbb{C}.\mathrm{Id}$ and $\tau \in \mathcal{U}_{\mathbb{C}}$.*

Proof. Assume that $\mathcal{B} = \mathrm{End}_{\mathrm{BL}_{\mathcal{H}_{\mathbb{C}}-\mathrm{mod}}}(I_\tau) \neq \mathbb{C}\mathrm{Id}$. By Lemma 3.6 and the fact that I_τ has countable dimension, $\mathcal{B}$ has countable dimension. Let $\phi \in \mathcal{B} \setminus \mathbb{C}\mathrm{Id}$. Then by Amitsur Theorem, there exists $\gamma \in \mathrm{Spec}(\phi)$. Then $\phi - \gamma\mathrm{Id}$ is non-injective or non-surjective and therefore $\mathrm{Ker}(\phi - \gamma\mathrm{Id})$ or $\mathrm{Im}(\phi - \gamma\mathrm{Id})$ is a non-trivial ${}^{\mathrm{BL}}\mathcal{H}_{\mathbb{C}}$-module, which proves that I_τ is reducible. Using Lemma 5.5 we deduce that (1) implies (3).

By Lemma 3.6, (2) is equivalent to (3).

Let $\tau \in T_{\mathbb{C}}$ satisfying (2). Then by Lemma 5.5 and Lemma 5.6, $\dim I_\tau(w.\tau) = 1$ for all $w \in W^v$. By Lemma 5.5, for all $w \in W^v$, there exists an isomorphism of ${}^{\mathrm{BL}}\mathcal{H}_{\mathbb{C}}$-modules $f_w : I_{w.\tau} \to I_\tau$. As $\mathbb{C}.f_w(1 \otimes_{w.\tau} 1) \subset I_\tau(w.\tau)$ we deduce that $I_\tau(w.\tau) = \mathbb{C}.f_w(1 \otimes_{w.\tau} 1)$ for all $w \in W^v$.

Let $M \neq \{0\}$ be a ${}^{\mathrm{BL}}\mathcal{H}_{\mathbb{C}}$-submodule of I_τ. Let $x \in M \setminus \{0\}$. Then $M' = \mathbb{C}[Y].x$ is a finite dimensional $\mathbb{C}[Y]$-module. Thus by Lemma 3.1), there exists $\xi \in M' \setminus \{0\}$ such that $Z^\lambda.\xi \in \mathbb{C}.\xi$ for all $\lambda \in Y$. Then $\xi \in I_\tau(\tau')$ for some $\tau' \in T_{\mathbb{C}}$. By Lemma 3.3, $\tau' = w.\tau$, for some $w \in W^v$. Thus $\xi \in \mathbb{C}^* f_w(1 \otimes_{w.\tau} 1)$. One has ${}^{\mathrm{BL}}\mathcal{H}_{\mathbb{C}}.\xi = f_w({}^{\mathrm{BL}}\mathcal{H}_{\mathbb{C}}.1 \otimes_{w.\tau} 1) = f_w(I_{w.\tau}) = I_\tau \subset M$. Hence I_τ is irreducible, which finishes the proof of the theorem. □

Remark 5.9. *Actually, our proof of the equivalence between (2) and (3), and of the fact that (2) implies (1) is valid when $\mathcal{F}$ is a field, without assuming $\mathcal{F} = \mathbb{C}$.*

Recall that an element $\tau \in T_{\mathcal{F}}$ is called regular if $w.\tau \neq \tau$ for all $w \in W^v$.

Corollary 5.10. *(see [Mat77, Théorème 4.3.5]) Let $\tau \in T_{\mathcal{F}}$ be regular. Then I_τ is irreducible if and only if $\tau \in \mathcal{U}_{\mathcal{F}}$.*

Proof. By Lemma 5.5, if I_τ is irreducible, then $\tau \in \mathcal{U}_\mathcal{F}$.

Assume that $\tau \in \mathcal{U}_\mathcal{F}$. Then by Proposition 3.5 (2), $\dim I_\tau(\tau) = 1$ and we conclude with Theorem 5.8 and Remark 5.9. □

Remark 5.11. *Assume that $\mathcal{F} = \mathbb{C}$ and that $\sigma_s = \sigma'_s = \sqrt{q}$ for all $s \in \mathscr{S}$, for some $q \in \mathbb{Z}_{\geq 2}$. Let $(y_j)_{j\in J}$ be a $\mathbb{Z}$-basis of Y. Then the map $T_\mathbb{C} \to (\mathbb{C}^*)^J$ defined by $\tau \in T_\mathbb{C} \mapsto (\tau(y_j))_{j\in J}$ is a group isomorphism. We equip $T_\mathbb{C}$ with a Lebesgue measure through this isomorphism. Then the set of measurable subsets of $T_\mathbb{C}$ having full measure does not depend on the choice of the $\mathbb{Z}$-basis of Y. Then $\mathcal{U}_\mathbb{C} = \bigcap_{\alpha^\vee\in\Phi^\vee}\{\tau \in T_\mathbb{C} | \tau(\alpha^\vee) \neq q\}$ has full measure in $T_\mathbb{C}$. Moreover $T_\mathbb{C}^{\mathrm{reg}} \supset \bigcap_{\lambda\in Y\setminus\{0\}}\{\tau \in T_\mathbb{C} | \tau(\lambda) \neq 1\}$ has full measure in $T_\mathbb{C}$ and thus $\{\tau \in T_\mathbb{C} | I_\tau$ is irreducible$\}$ has full measure in $T_\mathbb{C}$.*

Recall that $\mathscr{R} = \{wsw^{-1} | w \in W^v, s \in \mathscr{S}\}$ is the set of reflections of W^v. For $\tau \in T_\mathbb{C}$, set $W_\tau = \{w \in W^v |\ w.\tau = \tau\}$, $\Phi^\vee_{(\tau)} = \{\alpha^\vee \in \Phi^\vee_+ | \zeta^{\mathrm{den}}_{\alpha^\vee}(\tau) = 0\}$, $\mathscr{R}_{(\tau)} = \{r = r_{\alpha^\vee} \in \mathscr{R} | \alpha^\vee \in \Phi^\vee_{(\tau)}\}$ and

$$W_{(\tau)} = \langle \mathscr{R}_{(\tau)} \rangle = \langle \{r = r_{\alpha^\vee} \in \mathscr{R} | \zeta^{\mathrm{den}}_{\alpha^\vee}(\tau) = 0\} \rangle \subset W^v.$$

By Remark 5.1, $W_{(\tau)} \subset W_\tau$. It is moreover normal in W_τ. When $\alpha_s(Y) = \mathbb{Z}$ for all $s \in \mathscr{S}$, then $W_{(\tau)} = \langle W_\tau \cap \mathscr{R} \rangle$.

Corollary 5.12. *Let $\tau \in T_\mathcal{F}$ be such that $W_\tau = W_{(\tau)} = \{1, t\}$ for some reflection t. Then I_τ is irreducible if and only if $\tau \in \mathcal{U}_\mathcal{F}$.*

Proof. By Lemma 5.5, if I_τ is irreducible, then $\tau \in \mathcal{U}_\mathcal{F}$. Reciprocally, let $\tau \in \mathcal{U}_\mathcal{F}$ be such that $W_\tau = W_{(\tau)} = \{1, t\}$, for some $t \in \mathscr{R}$. Write $t = v^{-1}sv$ for $s \in \mathscr{S}$ and $v \in W^v$. Let $\tilde{\tau} = v.\tau$. One has $s.\tilde{\tau} = \tilde{\tau}$ and $W_{\tilde{\tau}} = \{1, s\}$. By Lemma 3.3, $I_{\tilde{\tau}}(\tilde{\tau}) \subset I_{\tilde{\tau}}^{\leq s}$.

Let $\lambda \in Y$. Then $Z^\lambda.H_s \otimes_{\tilde{\tau}} 1 = \tilde{\tau}(\lambda)H_s \otimes_{\tilde{\tau}} 1 + \tilde{\tau}(Q_s(Z)(Z^\lambda - Z^{s.\lambda}))1 \otimes_{\tilde{\tau}} 1$.

Suppose $\sigma_s = \sigma'_s$. Then as $W_{(\tilde{\tau})} = v.W_{(\tau)}.v^{-1} = \{1, s\}$, one has $\tilde{\tau}(\alpha_s^\vee) = 1$. By Remark 2.7, $\tilde{\tau}((Q_s(Z)(Z^\lambda - Z^{s.\lambda})) = (\sigma_s - \sigma_s^{-1})\alpha_s(\lambda)$. As there exists $\lambda \in Y$ such that $\alpha_s(\lambda) \neq 0$, we deduce that $H_s \otimes_{\tilde{\tau}} 1 \notin I_{\tilde{\tau}}(\tilde{\tau})$ and thus $I_{\tilde{\tau}}(\tilde{\tau}) = \mathcal{F}.1 \otimes_{\tilde{\tau}} 1$. Similarly, if $\sigma_s \neq \sigma'_s$ then $I_{\tilde{\tau}}(\tilde{\tau}) = \mathcal{F}.1 \otimes_{\tilde{\tau}} 1$. By Theorem 5.8 and Remark 5.9, we deduce that $I_{\tilde{\tau}}$ is irreducible. By Lemma 5.5 we deduce that I_τ is isomorphic to $I_{\tilde{\tau}}$ and thus I_τ is irreducible.

□

5.4 Weight vectors regarded as rational functions

In this subsection, we introduce and study elements $F_w \in {}^{\mathrm{BL}}\mathcal{H}(T_\mathcal{F})$, $w \in W^v$, such that for all $\chi \in T_\mathcal{F}$ such that $F_w(\chi)$ is well defined, $F_w(\chi) \otimes_\chi 1 \in I_\chi(w.\chi)$.

For $w \in W^v$, let $\pi_w^T : {}^{\mathrm{BL}}\mathcal{H}(T_\mathcal{F}) \to \mathcal{F}(Y)$ be the right $\mathcal{F}(Y)$-module morphism defined by $\pi_w^T(T_v) = \delta_{v,w}$ for all $v \in W^v$.

Lemma 5.13. *Let $\mathcal{F}'$ be a uncountable field containing $\mathcal{F}$. Let $P \in \mathcal{F}[Y]$ be such that $P(\tau) = 0$ for all $\tau \in T_{\mathcal{F}'}^{\mathrm{reg}}$. Then $P = 0$.*

Proof. Let $\mathcal{F}_0 \subset \mathcal{F}$ be a countable field (one can take $\mathcal{F}_0 = \mathbb{Q}$ or $\mathcal{F}_0 = \mathbb{F}_\ell$ for some prime power ℓ). Write $P = \sum_{\lambda\in Y} a_\lambda Z^\lambda$, with $a_\lambda \in \mathcal{F}$ for all $\lambda \in Y$. Let $(y_j)_{j\in J}$ be a $\mathbb{Z}$-basis of Y and $X_j = Z^{y_j}$ for all $j \in J$. Let $\mathcal{F}_1 = \mathcal{F}(a_\lambda | \lambda \in Y)$. Let $(x_j)_{j\in J} \in (\mathcal{F}')^J$ be algebraically independent over $\mathcal{F}_1$. Let $\tau \in T_{\mathcal{F}'}$ be defined by $\tau(y_j) = x_j$ for all $j \in J$.

Let us prove that $\tau \in T_\mathcal{F}^{\mathrm{reg}}$. Let $w \in W^v \setminus \{1\}$. Let $\lambda \in Y$ be such that $w^{-1}.\lambda - \lambda \neq 0$. Write $w^{-1}.\lambda - \lambda = \sum_{j\in J} n_j y_j$ with $n_j \in \mathbb{Z}$ for all $j \in J$. Let $Q = \prod_{j\in J} Z_j^{n_j} \in \mathcal{F}_1[Z_j, j \in J]$. Then $Q \neq 1$ and thus $\tau(w^{-1}.\lambda - \lambda) = Q((x_j)_{j\in J}) \neq 1$. Thus $w.\tau \neq \tau$ and $\tau \in T_{\mathcal{F}'}^{\mathrm{reg}}$. Thus $P(\tau) = 0$ and by choice of $(x_j)_{j\in J}$ this implies $P = 0$. □

Let $w \in W^v$. Let $w = s_1 \dots s_r$ be a reduced expression of w. Set $F_w = F_{s_r} \dots F_{s_1} = (B_{s_r} + \zeta_{s_r}) \dots (B_{s_1} + \zeta_{s_1}) \in {}^{\mathrm{BL}}\mathcal{H}(T_{\mathcal{F}})$. By the lemma below, this does not depend on the choice of the reduced expression of w.

Lemma 5.14. *(see [Ree97, Lemma 4.3]) Let $w \in W^v$.*

1. *The element $F_w \in {}^{\mathrm{BL}}\mathcal{H}(T_{\mathcal{F}})$ is well defined, i.e it does not depend on the choice of a reduced expression for w.*
2. *One has $F_w - T_w \in {}^{\mathrm{BL}}\mathcal{H}(T_{\mathcal{F}})^{<w}$.*
3. *If $\theta \in \mathcal{F}(Y)$, then $\theta * F_w = F_w * \theta^{w^{-1}}$.*
4. *If $\tau \in T_{\mathcal{F}}$ is such that $\zeta_{\beta^\vee} \in \mathcal{F}(Y)_\tau$ for all $\beta^\vee \in N_{\Phi^\vee}(w)$, then $F_w \in {}^{\mathrm{BL}}\mathcal{H}(T_{\mathcal{F}})_\tau$ and $F_w(\tau).1 \otimes_\tau 1 \in I_\tau(w.\tau)$.*
5. *Let $\tau \in T_{\mathcal{F}}^{\mathrm{reg}}$. Then $F_w \in {}^{\mathrm{BL}}\mathcal{H}(T_{\mathcal{F}})_\tau$.*

Proof. Let us prove (4) by induction on $\ell(w)$. By Lemma 5.2, $\theta * F_w = F_w * \theta^{w^{-1}}$ for all $\theta \in \mathcal{F}(Y)$. Let $n \in \mathbb{Z}_{\geq 0}$ and assume that (4) is true for all $w \in W^v$ such that $\ell(w) \leq n$. Let $w \in W^v$ be such that $\ell(w) \leq n+1$. Write $w = sv$, with $s \in \mathscr{S}$ and $\ell(v) \leq n$. By Lemma 2.4, $N_{\Phi^\vee}(sv) = N_{\Phi^\vee}(v) \cup \{v^{-1}.\alpha_s^\vee\}$. Let $\tau \in T_{\mathcal{F}}$ be such that be such that $\zeta_{\alpha^\vee} \in \mathcal{F}(Y)_\tau$ for all $\alpha^\vee \in N_{\Phi^\vee}(w)$. One has $F_w = (B_s + \zeta_s) * F_v$. As $F_v \in {}^{\mathrm{BL}}\mathcal{H}(T_{\mathcal{F}})_\tau$ and ${}^{\mathrm{BL}}\mathcal{H}(T_{\mathcal{F}})_\tau$ is a left $\mathcal{H}_{W^v,\mathcal{F}}$-submodule of ${}^{\mathrm{BL}}\mathcal{H}(T_{\mathcal{F}})$, $B_s * F_v \in {}^{\mathrm{BL}}\mathcal{H}(T_{\mathcal{F}})_\tau$. One has $\zeta_s * F_v = F_v * \zeta_s^{v^{-1}} \in {}^{\mathrm{BL}}\mathcal{H}(T_{\mathcal{F}})_\tau$ and hence $F_w \in {}^{\mathrm{BL}}\mathcal{H}(T_{\mathcal{F}})_\tau$.

Let $\tau \in T_{\mathcal{F}}$ be such that $\zeta_{\alpha^\vee} \in \mathcal{F}(Y)_\tau$ for all $\alpha^\vee \in N_{\Phi^\vee}(w)$. Let $\theta \in \mathcal{F}[Y]$. Then $(\theta * F_w)(\tau) = (F_w * \theta^{w^{-1}})(\tau) = \tau(\theta^{w^{-1}})\tau(F_w(\tau))$, which finishes the proof of (4).

Let $\tau \in T_{\mathcal{F}}^{\mathrm{reg}}$ and $\alpha^\vee \in \Phi^\vee$. Write $\alpha^\vee = w.\alpha_s^\vee$ for $w \in W^v$ and $s \in \mathscr{S}$. Then $s.w^{-1}.\tau \neq w^{-1}.\tau$ and by Remark 5.1, $w^{-1}.\tau(\zeta_s^{\mathrm{den}}) \neq 0$ or equivalently $\tau(\zeta_{\alpha^\vee}^{\mathrm{den}}) \neq 0$. By (4) we deduce that $F_w \in {}^{\mathrm{BL}}\mathcal{H}(T_{\mathcal{F}})_\tau$ for all $\tau \in T_{\mathcal{F}}^{\mathrm{reg}}$, which proves (5).

Let us prove (2). Let $v \in W^v$ be such that $h := F_v - T_v \in {}^{\mathrm{BL}}\mathcal{H}(T_{\mathcal{F}})^{<v}$ and $s \in \mathscr{S}$ be such that $sv > v$. Then

$$F_{sv} = (T_s - \sigma_s^2 + \zeta_s) * (T_v + h) = T_{sv} + (-\sigma_s^2 + \zeta_s) * T_v + (-\sigma_s^2 + \zeta_s) * h + T_s * h.$$

By Lemma 2.8, $(-\sigma_s^2 + \zeta_s) * T_v, (-\sigma_s^2 + \zeta_s) * h \in {}^{\mathrm{BL}}\mathcal{H}(T_{\mathcal{F}})^{\leq v}$. By [Kum02, Corollary 1.3.19], $s.[1, v) \subset [1, sv)$ and thus $T_s * h \in {}^{\mathrm{BL}}\mathcal{H}(T_{\mathcal{F}})^{<sw}$ thus $F_{sv} - T_{sv} \in {}^{\mathrm{BL}}\mathcal{H}(T_{\mathcal{F}})^{<sv}$. By induction we deduce (2).

Let $w = s_1 \dots s_r = s_1' \dots s_r'$ be reduced expressions of w. Let F_w be associated to $s_1 \dots s_r$ and F_w' be associated to $s_1' \dots s_r'$. Let $\mathcal{F}'$ be a uncountable field containing $\mathcal{F}$. Then by Proposition 3.5 (2), for all $\tau \in T_{\mathcal{F}'}^{\mathrm{reg}}$ there exists $\theta(\tau) \in \mathcal{F}'^*$ such that $F_w(\tau) = \theta(\tau)F_w'(\tau)$. Let $v \in W^v$ be such that $\pi^v(F_w') \neq 0$ and $\theta_v = \frac{\pi_v^H(F_w)}{\pi_v^H(F_w')} \in \mathcal{F}(Y)$. Then $\theta_v(\tau) = \theta(\tau)$ for all $\tau \in T_{\mathcal{F}'}^{\mathrm{reg}}$. But by (2), $\theta(\tau) = 1$ for all $\tau \in T_{\mathcal{F}'}^{\mathrm{reg}}$. Thus by Lemma 5.13, $\theta = 1 = \theta_v$ and $F_w' = F_w$. □

Remark 5.15. 1. *When $\sigma_s = \sigma_s'$ for all $s \in \mathscr{S}$, the condition (4) is equivalent to $\tau(\beta^\vee) \neq 1$ for all $\beta^\vee \in N_{\Phi^\vee}(w)$.*

5.5 One implication of Kato's criterion

Recall the definition of $W_{(\tau)}$ from Subsection 5.3.

In this subsection, we prove that if I_τ is irreducible, then $W_\tau = W_{(\tau)}$.

Lemma 5.16. *Let $\tau \in T_{\mathbb{C}}$ be such that $W_\tau \neq W_{(\tau)}$. Let $w \in W_\tau \setminus W_{(\tau)}$ be of minimal length. Then $F_w \in {}^{\mathrm{BL}}\mathcal{H}(T_{\mathcal{F}})_\tau$.*

Proof. Write $w = s_k \dots s_1$, where $k = \ell(w)$ and $s_1, \dots, s_k \in \mathscr{S}$. Let $j \in [\![0, k-1]\!]$. Set $w_j = s_j \dots s_1$. Suppose that $w_j.\zeta^{\mathrm{den}}_{s_{j+1}}(\tau) = 0$. Then $r_{w_j.\alpha^\vee_{s_{j+1}}} = s_1 \dots s_j s_{j+1} s_j \dots s_1 \in W_{(\tau)}$. Moreover as $W_{(\tau)} \subset W_\tau$, we have $s_{j+1} \dots s_1.\tau = s_j \dots s_1.\tau$. Therefore

$$\tau = w.\tau = s_k \dots s_j \dots s_1.\tau = s_k \dots \hat{s}_{j+1} \dots s_1.\tau,$$

and $w' = s_k \dots \hat{s}_{j+1} \dots s_1 \in W_\tau$. By definition of w, $w' \in W_{(\tau)}$. Consequently

$$w = s_k \dots \hat{s}_{j+1} \dots s_1.s_1 \dots s_j.s_{j+1}.s_j \dots s_1 = w' r_{w_j.\alpha^\vee_{s_{j+1}}} \in W_{(\tau)} :$$

a contradiction. Therefore $w_j.\zeta^{\mathrm{den}}_{s_{j+1}}(\tau) \neq 0$ and by Lemma 2.4 and Lemma 5.14, $F_w \in {}^{\mathrm{BL}}\mathcal{H}(T_{\mathcal{F}})_\tau$. □

Proposition 5.17. *Let $\tau \in T_{\mathbb{C}}$ be such that $W_\tau \neq W_{(\tau)}$. Then I_τ is reducible.*

Proof. Let $w \in W_\tau \setminus W_{(\tau)}$ be of minimal length. Then by Lemma 5.16 and Lemma 5.14, $F_w(\tau) \otimes_\tau 1 \in I_\tau(\tau)$. Moreover, $\pi^T_w\big(F_w(\tau) \otimes_\tau 1\big) = 1$ and thus $F_w(\tau) \otimes_\tau 1 \notin \mathbb{C}1 \otimes_\tau 1$. We conclude with Theorem 5.8.

□

5.6 Link with the works of Matsumoto and Kato

Assume that W^v is finite. Then $\mathcal{H}_{\mathbb{C}} = {}^{\mathrm{BL}}\mathcal{H}_{\mathbb{C}}$. Let $\tau \in T_{\mathbb{C}}$. Then by Subsection 2.4, $\dim_{\mathbb{C}} I_\tau = |W^v|$. One has $Z^\lambda.1 \otimes_\tau 1 = \tau(\lambda)1 \otimes_\tau 1$ for all $\lambda \in Y$ and $\mathcal{H}_{\mathbb{C}}.1 \otimes_\tau 1 = I_\tau$. Thus by [Mat77, Théorème 4.1.10] the definition we used is equivalent to Matsumoto's one.

Assume that $\mathcal{H}_{\mathbb{C}}$ is associated with a split reductive group over a field with residue cardinal q. Then by (BL2), one has:

$$\forall\, s \in \mathscr{S}, \forall\, w \in W^v,\ T_s * T_w = \begin{cases} T_{sw} & \text{if } \ell(sw) = \ell(w) + 1 \\ (q-1)T_w + qT_{sw} & \text{if } \ell(sw) = \ell(w) - 1. \end{cases}$$

Set $1'_\tau = \sum_{w \in W^v} T_w \otimes_\tau 1$. Then if $s \in \mathscr{S}$, $T_s.1'_\tau = q1'_\tau$. Then by [Kat81, (1.19)], $1'_\tau$ is proportional to the vector 1_τ defined in [Kat81]. Kato proves Theorem 1 by studying whether the following property is satisfied: "for all $w \in W^v$, $\mathcal{H}_{\mathbb{C}}.1'_{w.\tau} = I_{w.\tau}$" (see [Kat81, Lemma 2.3]). When W^v is infinite, we do not know how to define an analogue of $1'_\tau$ and thus we do not know how to adapt Kato's proof.

6 Description of generalized weight spaces

In this section, we describe $I_\tau(\tau, \mathrm{gen})$, when $\tau \in T_{\mathbb{C}}$ is such that $W_{(\tau)} = W_\tau$. We then deduce Kato's criterion for size 2 matrices.

Let us sketch our proof of this criterion. By Theorem 5.8 and Proposition 5.17, it suffices to study $I_\tau(\tau)$ when $\tau \in \mathcal{U}_{\mathbb{C}}$ is such that $W_\tau = W_{(\tau)}$. For this, we begin by describing

$I_\tau(\tau, \mathrm{gen})$. Let $\tau \in T_{\mathbb{C}}$ satisfying the above condition. By Dyer's theorem, $(W_{(\tau)}, \mathscr{S}_\tau)$ is a Coxeter system, for some $\mathscr{S}_\tau \subset W_{(\tau)}$. Let $r \in \mathscr{S}_\tau$. We study the singularity of F_r at τ, that is, we determine an (explicit) element $\theta \in \mathbb{C}(Y)$ such that $F_r - \theta$ is defined at τ (see Lemma 6.20). Using this, we then describe $I_\tau(\tau, \mathrm{gen})$. We then deduce that when $W_\tau = W_{(\tau)}$ is the infinite dihedral group then $I_\tau(\tau)$ is irreducible. After classifying the subgroups of the infinite dihedral group (see Lemma 6.39), we deduce Kato's criterion for size 2 matrices.

In Subsection 6.1, we study the torus $T_{\mathbb{C}}$.

In Subsection 6.2, we introduce a new basis of $\mathcal{H}_{W^v,\mathbb{C}}$ which enables us to have information on the poles of the coefficients of the F_w.

In Subsection 6.3, we give a recursive formula which enables us to have information on the poles of the coefficients of the F_w.

In Subsection 6.4, we study the singularity of F_r at τ, for $r \in \mathscr{S}_\tau$.

In Subsection 6.5, we give a description of $I_\tau(\tau, \mathrm{gen})$, when $W_\tau = W_{(\tau)}$.

In Subsection 6.6, we prove that when $W_\tau = W_{(\tau)}$ is the infinite dihedral group and $\tau \in \mathcal{U}_{\mathbb{C}}$, then I_τ is irreducible.

In Subsection 6.7, we prove Kato's criterion for size 2 Kac-Moody matrices.

This section is strongly inspired by [Ree97].

In certain proofs, when $\mathcal{F} = \mathbb{C}$, we will make additional assumptions on the σ_s and σ'_s, $s \in \mathscr{S}$. To avoid these assumptions, we can assume that $\sigma_s, \sigma'_s \in \mathbb{C}$ and $|\sigma_s| > 1, |\sigma'_s| > 1$ for all $s \in \mathscr{S}$.

6.1 The complex torus $T_{\mathbb{C}}$

We assume that $|\sigma_s| \in \mathbb{R}_{>1}$ for all $s \in \mathscr{S}$. Let $(y_j)_{j\in J}$ be a $\mathbb{Z}$-basis of Y. The map $T_{\mathbb{C}} \to (\mathbb{C}^*)^J$ mapping each $\tau \in T_{\mathbb{C}}$ on $(\tau(y_j))_{j\in J}$ is a bijection. We identify $T_{\mathbb{C}}$ and $(\mathbb{C}^*)^J$. We equip $T_{\mathbb{C}}$ with the usual topology on $(\mathbb{C}^*)^J$. This does not depend on the choice of a basis $(y_j)_{j\in J}$.

Lemma 6.1. *The set $\{\tau \in T_{\mathbb{C}} | \forall (w, \lambda) \in W^v \setminus \{1\} \times (C_f^v \cap Y),\ w.\tau(\lambda) \neq \tau(\lambda)\}$ is dense in $T_{\mathbb{C}}$. In particular, $T_{\mathbb{C}}^{\mathrm{reg}}$ is dense in $T_{\mathbb{C}}$.*

Proof. Let $\lambda \in C_f^v \cap Y$. By [Bou81, V.Chap 4 §6 Proposition 5], for all $w \in W^v \setminus \{1\}$, $w.\lambda \neq \lambda$. Let $(\gamma_j)_{j\in J} \in (\mathbb{C}^*)^J$ be algebraically independent over $\mathbb{Q}$ and $\tau_\gamma \in T_{\mathbb{C}}$ be defined by $\tau_\gamma(y_j) = \gamma_j$ for all $j \in J$. Then $w.\tau_\gamma(\lambda) \neq \tau_\gamma(\lambda)$ for all $w \in W^v \setminus \{1\}$. Let $\tau \in T_{\mathbb{C}}$. Let $(\gamma^{(n)}) \in \left((\mathbb{C}^*)^J\right)^{\mathbb{Z}_{\geq 0}}$ be such that $\gamma^{(n)}$ is algebraically independent over $\mathbb{Q}$ for all $n \in \mathbb{Z}_{\geq 0}$ and such that $\gamma^{(n)} \to (\tau(y_j))_{j\in J}$. Then $\tau_{\gamma^{(n)}} \to \tau$ and we get the lemma. □

Let $A \subset \mathbb{R}$ be a ring. We set $Q_A^\vee = \bigoplus_{s\in\mathscr{S}} A\alpha_s^\vee \subset \mathbb{A}$.

Lemma 6.2. *Let $(\gamma_s) \in (\mathbb{C}^*)^{\mathscr{S}}$. Then there exists $\tau \in T_{\mathbb{C}}$ such that $\tau(\alpha_s^\vee) = \gamma_s$ for all $s \in \mathscr{S}$.*

Proof. Let us prove that there exists $n \in \mathbb{Z}_{\geq 1}$ such that $\frac{1}{n}Q_{\mathbb{Z}}^\vee \supset Y \cap Q_{\mathbb{Q}}^\vee$. The module $Y \cap Q_{\mathbb{Q}}^\vee$ is a $\mathbb{Z}$-submodule of the free module Y. Thus it is a free module and its rank is lower or equal to the rank of Y. Let $(y_j)_{j\in J}$ be a $\mathbb{Z}$-basis of $Y \cap Q_{\mathbb{Q}}^\vee$. As $\alpha_s^\vee \in Y \cap Q_{\mathbb{Q}}^\vee$ for all $s \in \mathscr{S}$, we have we have $\mathrm{vect}_{\mathbb{Q}}(Y \cap Q_{\mathbb{Q}}^\vee) = Q_{\mathbb{Q}}^\vee$. Therefore for all $j \in J$, there exists $(m_{j,s}) \in \mathbb{Q}^{\mathscr{S}}$ such that $y_j = \sum_{j\in J} m_{j,s}\alpha_s^\vee$ and thus there exists $n \in \mathbb{Z}_{\geq 1}$ such that $\frac{1}{n}Q_{\mathbb{Z}}^\vee \supset Y \cap Q_{\mathbb{Q}}^\vee$.

Let S be a complement of $Y \cap Q_{\mathbb{Q}}^\vee$ in $Y \otimes \mathbb{Q}$. For $s \in \mathscr{S}$, choose $\gamma_s^{\frac{1}{n}} \in \mathbb{C}^*$ such that $(\gamma_s^{\frac{1}{n}})^n = \gamma_s$. Let $\tilde{\tau} : \frac{1}{n}Q_{\mathbb{Z}}^\vee \oplus S \to \mathbb{C}^*$ be defined by $\tilde{\tau}(\sum_{s\in\mathscr{S}} \frac{a_s}{n}\alpha_s^\vee + x) = \prod_{s\in\mathscr{S}} (\gamma_s^{\frac{1}{n}})^{a_s}$ for all $(a_s) \in \mathbb{Z}^{\mathscr{S}}$ and $x \in S$. Let $\tau = \tilde{\tau}_{|Y}$. Then $\tau \in T_{\mathbb{C}}$ and $\tau(\alpha_s^\vee) = \gamma_s$ for all $s \in \mathscr{S}$. □

6.2 A new basis of $\mathcal{H}_{W^v,\mathbb{C}}$

Following [Ree97, 5], we now define a new basis $(B_w)_{w\in W^v}$ of $\mathcal{H}_{W^v,\mathbb{C}}$ which has "good properties" with respect to multiplication (see Lemma 6.6). This will enable us to have information on the coefficient $\pi_1^H(F_w) \in \mathbb{C}(Y)$, for $w \in W^v$ (see Lemma 6.7 and Lemma 6.20). In the split reductive case, we could use the Kazhdan-Lusztig basis of $\mathcal{H}_{W^v,\mathbb{C}}$. In the general Kac-Moody case however, there is up to now no Kazhdan-Lusztig basis of $\mathcal{H}_{W^v,\mathbb{C}}$.

Let $\tau, \tau' \in T_{\mathbb{C}}$ and $x \in I_{\tau'}(\tau)$. Let $\Upsilon_x^{\tau,\tau'}$ be the ${}^{\mathrm{BL}}\mathcal{H}_{\mathbb{C}}$-modules morphism from I_τ to $I_{\tau'}$ sending $h.1 \otimes_\tau 1$ to $h.x$ for all $h \in {}^{\mathrm{BL}}\mathcal{H}_{\mathbb{C}}$. This is well defined by Lemma 3.6.

Lemma 6.3. *Let $v \in W^v$, $s \in \mathscr{S}$ be such that $vs > v$ and $\tau \in T_{\mathbb{C}}^{\mathrm{reg}}$. Then:*

$$F_{vs}(s.\tau) = F_v(s.\tau) * F_s(\tau) \in \mathcal{H}_{W^v,\mathbb{C}}.$$

Proof. By Lemma 5.14 (5), $F_{vs}(s.\tau)$, $F_v(s.\tau)$ and $F_s(\tau)$ are well defined.

Let $\tilde{\tau} = s.\tau$. By Lemma 5.14 $x := F_v(\tilde{\tau}) \in I_{\tilde{\tau}}(v.\tilde{\tau})$, $y := F_s(\tilde{\tau}) \in I_\tau(\tilde{\tau})$ and $z := F_{vs}(\tau) \otimes_\tau 1 \in I_\tau(vs.\tau)$. We have the following diagram:

$$I_{vs.\tau} = I_{v.\tilde{\tau}} \xrightarrow{\Upsilon_x^{v.\tilde{\tau},\tilde{\tau}}} I_{\tilde{\tau}} \xrightarrow{\Upsilon_y^{\tilde{\tau},\tau}} I_\tau.$$
$$\Upsilon_z^{v.\tilde{\tau},\tau}$$

By Proposition 3.5 (2), $\Upsilon_y^{\tilde{\tau},\tau} \circ \Upsilon_x^{v.\tilde{\tau},\tilde{\tau}} \in \mathbb{C}\Upsilon_z^{vs.\tau,\tau}$. By evaluating at $1 \otimes_{vs.\tau} 1$ we deduce that $F_v(s.\tau) * F_s(\tau) \in \mathbb{C}F_{vs}(s.\tau)$. Moreover $\pi_{vs}^T(F_v(s.\tau) * F_s(\tau)) = 1 = \pi_{vs}^T(F_{vs}(\tau) \otimes_\tau 1)$, which proves the lemma. □

Lemma 6.4. *Let $w \in W^v$ and $s \in \mathscr{S}$ be such that $ws > w$. Let $m_{B_s} : \mathcal{H}_{W^v,\mathbb{C}}^{\leq ws} \to \mathcal{H}_{W^v,\mathbb{C}}^{\leq ws}$ be the right multiplication by B_s. Then m_{B_s} is well defined, diagonalizable and its eigenvalues are $-(1+\sigma_s^2)$ and 0, both with multiplicity $\frac{|[1,ws]|}{2}$.*

Proof. Let $v \in [1, ws]$. Then by [Kum02, Corollary 1.3.19], $vs \leq \max(w, ws) = ws$. Thus $\mathcal{H}_{W^v,\mathbb{C}}^{\leq ws} * H_s \subset \mathcal{H}_{W^v,\mathbb{C}}^{\leq ws}$ and hence m_{B_s} is well defined.

We have $\mathcal{H}_{W^v,\mathbb{C}}^{\leq ws} = \mathcal{H}_{W^v,\mathbb{C}}^{\leq w} * \mathcal{H}_{W^v,\mathbb{C}}^{\leq s}$ and $\mathcal{H}_{W^v,\mathbb{C}}^{\leq s} = \mathbb{C}B_s \oplus \mathbb{C}(T_s + 1)$ and thus $\mathcal{H}_{W^v,\mathbb{C}}^{\leq ws} = \mathcal{H}_{W^v,\mathbb{C}}^{\leq w} * B_s + \mathcal{H}_{W^v,\mathbb{C}}^{\leq w} * (T_s + 1)$. The families $(T_v * B_s)_{v<vs\leq ws}$ and $(T_v * (T_s + 1))_{v<vs\leq ws}$ are bases of $\mathcal{H}_{W^v,\mathbb{C}}^{\leq w} * B_s$ and $\mathcal{H}_{W^v,\mathbb{C}}^{\leq w} * (T_s + 1)$. Moreover, $B_s^2 = -(1+\sigma_s^2)B_s$ and $(T_s+1) * B_s = 0$, which proves the lemma. □

Lemma 6.5. *Let $\tau \in T_{\mathbb{C}}$ be such that $|\tau(\alpha_s^\vee)| > 1$ for all $s \in \mathscr{S}$. Then $\tau \in T_{\mathbb{C}}^{\mathrm{reg}}$.*

Proof. Let $w \in W^v \setminus \{1\}$. Let $\lambda \in C_f^v \cap Y$. By [Bou81, V.Chap 4 §6 Proposition 5] $w^{-1}.\lambda \neq \lambda$ and by [GR14, Lemma 2.4 a)], $w^{-1}.\lambda - \lambda \in (\bigoplus_{s\in\mathscr{S}} \mathbb{Z}_{\leq 0}\alpha_s^\vee) \setminus \{0\}$. Thus $|\frac{w.\tau(\lambda)}{\tau(\lambda)}| < 1$ and hence $w.\tau \neq \tau$. □

Lemma 6.6. *There exists a basis $(B_w)_{w\in W^v}$ of $\mathcal{H}_{W^v,\mathbb{C}}$ such that :*

1. *$B_s = T_s - \sigma_s^2$ for all $s \in \mathscr{S}$,*
2. *$B_w - T_w \in \mathcal{H}_{W^v,\mathbb{C}}^{<w}$ for all $w \in W^v$,*
3. *For all $w \in W^v$ and $s \in \mathscr{S}$ we have:*

$$B_w B_s = \begin{cases} -(1+\sigma_s^2)B_w & \text{if } ws < w \\ B_{ws} + \sum_{vs<v<w} b(v,w)B_v & \text{if } ws > w, \end{cases}$$

for some $b(v, w) \in \mathbb{C}$.

Proof. Let $\tau_0 \in T_{\mathbb{C}}$ be such that $\tau_0(\alpha_s^\vee) = \sigma_s$ for all $s \in \mathscr{S}$, whose existence is provided by Lemma 6.2. For all $s \in \mathscr{S}$, $|\tau_0(\alpha_s^\vee)| > 1$ and thus by Lemma 6.5, τ_0 is regular. By Lemma 5.14, $B_w := F_w(\tau_0)$ is well defined for all $w \in W^v$. Then $(B_w)_{w\in W^v}$ satisfies (1) and (2).

Let $w \in W^v$ and $s \in \mathscr{S}$ be such that $ws < w$. By Lemma 6.3, one has

$$B_w = F_w(\tau_0) = F_{ws}(s.\tau_0) * F_s(\tau_0) = F_{ws}(s.\tau_0) * B_s.$$

Moreover, $B_s^2 = -(1+\sigma_s^2)B_s$ and thus $B_w * B_s = -(1+\sigma_s^2)B_w$, hence (3) holds in this case.

Let $w \in W^v$ and $s \in \mathscr{S}$ be such that $ws > w$. By (2), $\mathcal{H}_{W^v,\mathbb{C}}^{\leq ws} = \bigoplus_{v\in[1,ws]} \mathbb{C}B_s$. By Lemma 6.4 and the first part of (3), if $h \in \mathcal{H}_{W^v,\mathbb{C}}^{\leq ws}$ is such that $h * B_s = -(1+\sigma_s^2)h$, then $h \in \bigoplus_{vs<v<ws} B_v$. Thus $B_w * B_s - B_{ws} \in \bigoplus_{vs<v\leq w} B_v$. Moreover, $\pi_{ws}^T(B_w * B_s - B_{ws}) = 0$, and therefore $B_w * B_s - B_{ws} \in \bigoplus_{vs<v<w} B_v$, which concludes the proof of the lemma. □

As $(B_w)_{w\in W^v}$ is a $\mathbb{C}$-basis of $\mathcal{H}_{W^v,\mathbb{C}}$, $(B_w)_{w\in W^v}$ is a $\mathbb{C}(Y)$-basis of the right module ${}^{\mathrm{BL}}\mathcal{H}(T_{\mathbb{C}})$.

Let $w \in W^v$. Write $F_w = \sum_{v\in W^v} B_v p_{v,w}$, where $(p_{v,w}) \in \mathbb{C}(Y)^{(W^v)}$. By an induction on $\ell(w)$ using Lemma 6.6 (2) we have $\bigoplus_{v\leq w} H_v\mathbb{C}(Y) = \bigoplus_{v\leq w} B_v\mathbb{C}(Y)$ for all $w \in W^v$. Thus for all $v \in W^v$ such that $v \not\leq w$, one has $p_{v,w} = 0$. In [Ree97, 5.3], Reeder gives recursive formulae for the $p_{v,w}$. The following lemma is a particular case of them.

For $v \in W^v$, define $\pi_v^B : \mathcal{H}_{W^v,\mathbb{C}} \to \mathbb{C}(Y)$ by $\pi_v^B(\sum_{u\in W^v} B_u f_u) = f_v$ for all $(f_u) \in \mathbb{C}(Y)^{(W^v)}$.

Lemma 6.7. *Let $w \in W^v$. Then $p_{1,w} = \zeta_w := \prod_{\beta^\vee\in N_{\Phi^\vee}(w)} \zeta_{\beta^\vee}$.*

Proof. We prove it by induction on $\ell(w)$.

Let $v \in W^v$ and assume that $p_{1,v} = \zeta_v$. Let $s \in \mathscr{S}$ be such that $vs > v$. By Lemma 5.2 one has

$$\begin{aligned} F_{vs} =& F_v * F_s \\ =& (\sum_{u\in W^v} B_u p_{u,v}) * F_s \\ =& \sum_{u\in W^v} B_u * F_s p_{u,v}^s = \sum_{u\in W^v} B_u * B_s p_{u,v}^s + \sum_{u\in W^v} B_u p_{u,v}^s \zeta_s. \end{aligned}$$

By Lemma 6.6, we have $\pi_1^B(\sum_{u\in W^v} B_u * B_s p_{u,v}^s) = 0$ and $\pi_1^B(\sum_{u\in W^v} B_u p_{u,v}^s \zeta_s) = p_{1,v}^s \zeta_s$. By Lemma 2.4, $N_{\Phi^\vee}(vs) = s.N_{\Phi^\vee}(v) \sqcup \{\alpha_s^\vee\}$ and thus $\pi_1^B(F_{vs}) = p_{1,vs} = p_{1,v}^s \zeta_s = \zeta_{vs}$ which proves the lemma. □

6.3 An expression for the coefficients of the F_w in the basis (T_v)

In this subsection, we give a recursive formula for the coefficients of the F_w in the basis $(T_v)_{v\in W^v}$ (see formula (1) below and Lemma 6.9). We will deduce information concerning the elements $v \in W^v$ such that $\pi_v^T(F_w)$ is well defined at τ, for a given $\tau \in T_{\mathbb{C}}$ (see Lemma 6.10).

Let $\lambda \in Y$ and $w \in W^v$. By (BL4), Remark 2.7 (2) and an induction on $\ell(w)$, there exists $(P_{v,w,\lambda}(Z))_{v\in W^v} \in \mathbb{C}[Y]^{(W^v)}$ such that $Z^\lambda * T_w = \sum_{v\in W^v} T_v * P_{v,w,\lambda}(Z)$. Moreover $P_{w,w,\lambda} = Z^{w^{-1}.\lambda}$ and for all $v \in W^v \setminus [1,w]$, $P_{v,w,\lambda} = 0$.

Let $\lambda \in C_f^v \cap Y$. Then by [Bou81, V.Chap 4 §6 Proposition 5], for all $v, w \in W^v$ such that $v \neq w$, one has $v.\lambda \neq w.\lambda$. Let $w \in W^v$. Let $w = s_1 \dots s_k$ be a reduced expression. Set $Q_{w,w,\lambda}(Z) = 1 \in \mathbb{C}(Y)$. For $v \in W^v \setminus [1,w]$, set $Q_{v,w,\lambda}(Z) = 0$. Define $(Q_{v,w,\lambda}(Z))_{v\in[1,w]}$ by decreasing induction by setting:

$$Q_{v,w,\lambda}(Z) = \frac{1}{Z^{w^{-1}.\lambda} - Z^{v^{-1}.\lambda}} \sum_{w\geq u>v} Q_{u,w,\lambda} P_{v,u,\lambda} \in \mathbb{C}(Y). \tag{1}$$

Lemma 6.8. *Let $\lambda \in C_f^v \cap Y$, $w \in W^v$ and $\tau \in T_{\mathbb{C}}^{\mathrm{reg}}$ be such that $v.\tau(\lambda) \neq \tau(\lambda)$ for all $v \in W^v \setminus \{1\}$. Let $x \in I_\tau$ be such that $Z^\lambda.x = w.\tau(\lambda).x$. Then $x \in I_\tau(w.\tau)$.*

Proof. By Proposition 3.5 (2), we can write $x = \sum_{v \in W^v} x_v$ where $x_v \in I_\tau(v.\tau)$ for all $v \in W^v$. One has $Z^\lambda.x - w.\tau(\lambda).x = 0 = \sum_{v \in W^v} (v.\tau(\lambda) - w.\tau(\lambda))x_v$. As $v.\tau(\lambda) \neq w.\tau(\lambda)$ for all $v \neq w$, we deduce that $x = x_w$. □

Lemma 6.9. *Let $v, w \in W^v$. Then $\pi_v^T(F_w) = Q_{v,w,\lambda}$, for any $\lambda \in C_f^v \cap Y$. In particular, $Q_{v,w,\lambda}$ does not depend on the choice of λ.*

Proof. Let $\lambda \in C_f^v$ and $h = \sum_{v \in W^v} T_v Q_{v,w,\lambda} \in {}^{\mathrm{BL}}\mathcal{H}(T_{\mathbb{C}})$. One has:

$$\begin{aligned} Z^\lambda * h &= Z^\lambda * \sum_{v \in W^v} T_v Q_{v,w,\lambda} \\ &= \sum_{u,v \in W^v} T_u P_{u,v,\lambda} Q_{v,w,\lambda} \\ &= \sum_{u \in W^v} T_u \sum_{v \in W^v} P_{u,v,\lambda} Q_{v,w,\lambda}. \end{aligned}$$

Let $u \in W^v$. Then:

$$\sum_{v \in W^v} P_{u,v,\lambda} Q_{v,w,\lambda} = P_{u,u,\lambda} Q_{u,w,\lambda} + \sum_{v > u} P_{u,v,\lambda} Q_{v,w,\lambda} = Z^{u^{-1}.\lambda} + (Z^{w^{-1}.\lambda} - Z^{u^{-1}.\lambda}) Q_{u,w,\lambda} = Z^{w^{-1}.\lambda} Q_{u,w,\lambda},$$

and therefore $Z^\lambda.h = h.Z^{w^{-1}.\lambda}$.

Let $\lambda \in C_f^v \cap Y$ and $\tau \in T_{\mathbb{C}}^{\mathrm{reg}}$ be such that $u.\tau(\lambda) \neq \tau(\lambda)$ for all $u \in W^v \setminus \{1\}$. Then $\mathrm{ev}_\tau(Z^\lambda * h) = \mathrm{ev}_\tau(h * Z^{v^{-1}.\lambda}) = w.\tau(\lambda).h(\tau)$. By Lemma 6.8 we deduce that $h(\tau) \in I_\tau(w.\tau)$. By Proposition 3.5 (2) and Lemma 5.14 we deduce that $h(\tau) = F_w(\tau)$. By Lemma 6.1, we deduce that $h = F_w$, which proves the lemma. □

Lemma 6.10. *Let $w \in W^v$, $\tau \in T_{\mathbb{C}}$ and $v \in [1, w]$. Assume that for all $u \in [v, w)$, $u.\tau \neq w.\tau$. Then for all $u \in [v, w]$, $\pi_u^T(F_w) \in \mathbb{C}(Y)_\tau$.*

Proof. We do it by decreasing induction on v. Suppose that for all $u \in (v, w)$, $\pi_u^T(F_w) \in \mathbb{C}(Y)_\tau$. Let $\lambda \in C_f^v \cap Y$ be such that $v.\tau(\lambda) \neq w.\tau(\lambda)$, which exists because $C_f^v \cap Y$ generates Y. By Lemma 6.9 we have

$$\pi_v^T(F_w) = Q_{v,w,\lambda} = \frac{1}{Z^{w^{-1}.\lambda} - Z^{v^{-1}.\lambda}} \sum_{w \geq u > v} Q_{u,w,\lambda} P_{v,u,\lambda}.$$

We deduce that $\pi_v^T(F_w) \in \mathbb{C}(Y)_\tau$ because by assumption $Q_{u,w,\lambda} \in \mathbb{C}(Y)_\tau$ for all $u \in (v, w]$. Lemma follows. □

6.4 τ-simple reflections and intertwining operators

Let $\tau \in T_{\mathbb{C}}$. Following [Ree97, 14], we introduce τ-simple reflections (see Definition 6.11). If $\mathscr{S}_\tau$ is the set of τ-simple reflections, then $(W_{(\tau)}, \mathscr{S}_\tau)$ is a Coxeter system. We study, for such a reflection r, the singularity of F_r at τ: we prove that $F_r - \zeta_r$ is in ${}^{\mathrm{BL}}\mathcal{H}(T_{\mathbb{C}})_\tau$ (see Lemma 6.20). This enables us to define $K_r(\tau) = (F_r - \zeta_r)(\tau) \in \mathcal{H}_{W^v,\mathbb{C}}$. This will be useful to describe $I_\tau(\tau, \mathrm{gen})$.

We now define τ-simple reflections. Our definition slightly differs from [Ree97, Definition 14.2]. In many cases, these definitions are equivalent (see 6.4.3) but we do not know if they are always equivalent.

Definition 6.11. *Let $\tau \in T_{\mathbb{C}}$. A coroot $\beta^\vee \in \Phi_\tau^\vee$ and its corresponding reflection $r_{\beta^\vee}$ are said to be τ-**simple** if $N_{\mathscr{R}}(r_{\beta^\vee}) \cap W_{(\tau)} = \{\beta^\vee\}$. We denote by $\mathscr{S}_\tau$ the set of τ-simple reflections.*

Recall that $\Phi_{(\tau)}^\vee = \{\alpha^\vee \in \Phi_+^\vee | \zeta_{\alpha^\vee}^{\mathrm{den}}(\tau) = 0\}$ and $\mathscr{R}_{(\tau)} = \{r_{\alpha^\vee} | \alpha^\vee \in \Phi_{(\tau)}^\vee\}$.

6.4.1 Coxeter structure of $W_{(\tau)}$

We use the same notation as in 2.2.3. Then $\mathscr{S}_\tau = \mathscr{S}(W_{(\tau)})$ and thus $(W_{(\tau)}, \mathscr{S}_\tau)$ is a Coxeter system.

Let $\leq_\tau$ and ℓ_τ be the Bruhat order and the length on $(W_{(\tau)}, \mathscr{S}_\tau)$.

Lemma 6.12. *Let $x, y \in W_{(\tau)}$ be such that $x \leq_\tau y$. Then $x \leq y$.*

Proof. By definition, if $x, y \in W_{(\tau)}$, then $x \leq_\tau y$ (resp. $x \leq y$) if there exist $n \in \mathbb{Z}_{\geq 0}$ and $x_0 = x, x_1, \ldots, x_n = y \in W_{(\tau)}$ (resp . W^v) such that (x_i, x_{i+1}) is an arrow of the graph of [Dye91, Definition 1.1] for all $i \in [\![0, n-1]\!]$. We conclude with [Dye91, Theorem 1.4] □

Remark 6.13. *The orders $\leq$ and $\leq_\tau$ can be different on $W_{(\tau)}$: there can exist $v, w \in W_{(\tau)}$ such that v and w are not comparable for $\leq_\tau$ and $v < w$. For example if $W^v = \{s_1, s_2\}$ is the infinite dihedral group, $r_1 = s_1$ and $r_2 = s_2 s_1 s_2$ (see Lemma B.2), then $r_1 < r_2$ but r_1 and r_2 are not comparable for $<_\tau$.*

6.4.2 Singularity of F_r at τ for a τ-simple reflection

Lemma 6.14. *Let $r = r_{\beta^\vee} \in \mathscr{S}_\tau$. Then $N_{\Phi^\vee}(r_{\beta^\vee}) \cap \Phi_{(\tau)}^\vee \subset \{\beta^\vee\}$.*

Proof. Let $f : \Phi_+^\vee \to \mathscr{R}$ be the map defined by $f(\alpha^\vee) = r_{\alpha^\vee}$ for all $\alpha^\vee \in \Phi_+^\vee$. Then by Subsection 2.2, f is a bijection. One has $f\big(N_{\Phi^\vee}(r) \cap \Phi_{(\tau)}^\vee\big) = N_{\mathscr{R}}(r) \cap \mathscr{R}_{(\tau)}$. Moreover, $\mathscr{R}_{(\tau)} \subset W_{(\tau)} \cap \mathscr{R}$. Thus

$$f^{-1}\big(N_{\mathscr{R}}(r) \cap W_{(\tau)}\big) = \{\beta^\vee\} \supset f^{-1}\big(N_{\mathscr{R}}(r) \cap \mathscr{R}_{(\tau)}\big) = N_{\Phi^\vee}(r) \cap \Phi_{(\tau)}^\vee.$$

□

Lemma 6.15. *Let $\tau \in T_{\mathbb{C}}$ and $r_{\beta^\vee} \in \mathscr{S}_\tau$. Then there exists $h' \in {}^{\mathrm{BL}}\mathcal{H}(T_{\mathbb{C}})_\tau$ such that $F_{r_{\beta^\vee}} = h'.\zeta_{\beta^\vee}^{\mathrm{den}}$.*

Proof. Using [BB05, 1. Exercise 10], we write $r_{\beta^\vee} = wsw^{-1}$ with $w \in W^v$, $s \in \mathscr{S}$ and $\ell(wsw^{-1}) = 2\ell(w) + 1$. One has $\beta^\vee = w.\alpha_s^\vee$. Let $r_{\beta^\vee} = s_m \ldots s_1$ be a reduced expression of $r_{\beta^\vee}$, with $m \in \mathbb{Z}_{\geq 0}$ and $s_1, \ldots, s_m \in \mathscr{S}$. Let $k \in [\![0, m-1]\!]$ and $v = s_k \ldots s_1$. Suppose that $F_v = h'_k.(\zeta_{\beta^\vee}^{\mathrm{den}})^{\eta(k)}$ where $h'_k \in {}^{\mathrm{BL}}\mathcal{H}(T_{\mathbb{C}})_\tau$ and $\eta(k) \in \mathbb{Z}_{\geq 0}$. Then $F_{s_{k+1}v} = F_{s_{k+1}} * F_v = (B_{s_{k+1}} + \zeta_{s_{k+1}}) * F_v$. One has $\zeta_{s_{k+1}} * F_v = F_v.\zeta_{s_{k+1}}^{v^{-1}}$ by Lemma 5.14.

By Lemma 6.14 if $\zeta_{s_{k+1}}^{v^{-1}}$ is not defined in τ then $k = \ell(w)$. As $B_{s_{k+1}} \in \mathcal{H}_{W^v,\mathbb{C}}$ and ${}^{\mathrm{BL}}\mathcal{H}(T_{\mathbb{C}})_\tau$ is a left $\mathcal{H}_{W^v,\mathbb{C}}$-module, we can write $F_{s_{k+1}v} = h'_{k+1}.(\zeta_{\beta^\vee}^{\mathrm{den}})^{\eta(k+1)}$ where $h'_{k+1} \in {}^{\mathrm{BL}}\mathcal{H}(T_{\mathbb{C}})_\tau$ and $\eta(k+1) \leq \eta(k)$ if $k \neq \ell(w)$ and $\eta(k+1) \leq \eta(k) + 1$ if $k = \ell(w)$, which proves the lemma. □

Lemma 6.16. *Let $h \in {}^{\mathrm{BL}}\mathcal{H}(T_{\mathbb{C}})$ and $\tau \in T_{\mathbb{C}}$. Then*

$$\max\{u \in W^v | \pi_u^H(h) \notin \mathbb{C}(Y)_\tau\} = \max\{u \in W^v | \pi_u^B(h) \notin \mathbb{C}(Y)_\tau\}.$$

Proof. Let $v \in \max\{u \in W^v | \pi_u^H(h) \notin \mathbb{C}(Y)_\tau\}$. By 6.6 (2),

$$\pi_v^B(h) = \sum_{u \geq v} \pi_v^B(H_u)\pi_u^H(h) = \pi_v^B(H_v)\pi_v^H(h) + \sum_{u > v} \pi_v^B(H_u)\pi_u^H(h).$$

Moreover, by Lemma 6.6 (1) $\pi_v^B(H_v) \in \mathbb{C}^*$. Thus $\pi_v^B(h) \notin \mathbb{C}(Y)_\tau$. Similarly if $v' \in \max\{u \in W^v, u \geq v | \pi_u^B(h) \notin \mathbb{C}(Y)_\tau\}$, then $\pi_{v'}^H(h) \notin \mathbb{C}(Y)_\tau$. Hence $v \in \max\{u \in W^v | \pi_u^B(h) \notin \mathbb{C}(Y)_\tau\}$ and consequently $\max\{u \in W^v | \pi_u^H(h) \notin \mathbb{C}(Y)_\tau\} \subset \max\{u \in W^v | \pi_u^B(h) \notin \mathbb{C}(Y)_\tau\}$. By a similar reasoning we get the other inclusion. □

Lemma 6.17. *Let $w \in W^v$. Suppose that for some $s \in \mathscr{S}$, we have $w.\lambda - \lambda \in \mathbb{R}\alpha_s^\vee$ for all $\lambda \in Y$. Then $w \in \{\mathrm{Id}, s\}$.*

Proof. Let $\beta^\vee \in N_{\Phi^\vee}(w)$. Write $\beta^\vee = \sum_{t \in \mathscr{S}} n_t \alpha_t^\vee$, with $n_t \in \mathbb{Z}_{\geq 0}$ for all $t \in \mathscr{S}$. Then $w.\beta^\vee \in \Phi_-^\vee$ and by assumption, $n_t = 0$ for all $t \in \mathscr{S} \setminus \{s\}$. Therefore $\beta^\vee \in \mathbb{Z}_{\geq 0}\alpha_s^\vee \cap \Phi^\vee = \{\alpha_s^\vee\}$. We conclude with Lemma 2.4. □

Lemma 6.18. *Let $\chi \in T_\mathbb{C}$. Assume that there exists $\beta^\vee \in \Phi_+^\vee$ such that $r_{\beta^\vee} \in W_\tau$. Then there exists $(\chi_n) \in (T_\mathbb{C})^{\mathbb{Z}_{\geq 0}}$ such that:*

- *$\chi_n \to \chi$,*
- *$W_{\chi_n} = \langle r_{\beta^\vee} \rangle$ for all $n \in \mathbb{Z}_{\geq 0}$,*
- *$\chi_n(\beta^\vee) = \chi(\beta^\vee)$ for all $\in \mathbb{Z}_{\geq 0}$.*

Proof. We first assume that $\beta^\vee = \alpha_s^\vee$, for some $s \in \mathscr{S}$. Let $(y_j)_{j \in J}$ be a $\mathbb{Z}$-basis of Y. For all $j \in J$, choose $z_j \in \mathbb{C}$ such that $\chi(y_j) = \exp(z_j)$. Let $g : \mathbb{A} \to \mathbb{C}$ be the linear map such that $g(y_j) = z_j$ for all $j \in J$. Let V be a complement of $Q_\mathbb{R}^\vee$ in $\mathbb{A}$. Let $n \in \mathbb{Z}_{\geq 1}$. Let $b_s^{(n)} = g(\alpha_s^\vee)$ and $(b_t^{(n)}) \in \mathbb{C}^{\mathscr{S} \setminus \{s\}}$ be such that $|b_t^{(n)} - g(\alpha_t^\vee)| < \frac{1}{n}$ and such that the $\exp(b_t^{(n)})$, $t \in \mathscr{S} \setminus \{s\}$ are algebraically independent over $\mathbb{Q}$. Let $g_n : \mathbb{A} \to \mathbb{C}$ be the linear map such that $g_n(\alpha_t^\vee) = b_t^{(n)}$ for all $t \in \mathscr{S}$ and $g_n(v) = g(v)$ for all $v \in V$. For $n \in \mathbb{Z}_{\geq 0}$ set $\chi_n = (\exp \circ g_n)_{|Y} \in T_\mathbb{C}$. For all $x \in \mathbb{A}$, $g_n(x) \to g(x)$ and thus $\chi_n \to \chi$.

Let $n \in \mathbb{Z}_{\geq 1}$. Then $\chi(\alpha_s^\vee) = \chi_n(\alpha_s^\vee)$ and thus $s \in W_{\chi_n}$. Let $w \in W_{\chi_n}$. Then $w^{-1}.\lambda - \lambda \in \mathbb{Z}\alpha_s^\vee$ for all $\lambda \in Y$. By Lemma 6.17 we deduce that $w \in \{\mathrm{Id}, s\}$. Therefore $W_{\chi_n} = \{\mathrm{Id}, s\}$.

We no more assume that $\beta^\vee = \alpha_s^\vee$ for some $s \in \mathscr{S}$. Write $\beta^\vee = w.\alpha_s^\vee$ for some $w \in W^v$ and $s \in \mathscr{S}$. Let $\tilde{\chi} = w^{-1}.\chi$. Then $s \in W_{\tilde{\chi}}$. Thus there exists $(\tilde{\chi}_n) \in (T_\mathbb{C})^{\mathbb{Z}_{\geq 0}}$ such that $\tilde{\chi}_n \to \tilde{\chi}$ and $W_{\tilde{\chi}_n} = \{\mathrm{Id}, s\}$ for all $n \in \mathbb{Z}_{\geq 0}$. Let $(\chi_n) = (w.\tilde{\chi}_n)$. Then $\chi_n \to \chi$ and $W_{\chi_n} = \{1, r_{\beta^\vee}\}$ for all $n \in \mathbb{Z}_{\geq 0}$.

Moreover, $\chi(\beta^\vee) \in \{-1, 1\}$ and $\chi_n(\beta^\vee) \in \{-1, 1\}$ for all $n \in \mathbb{Z}_{\geq 0}$. Maybe considering a subsequence of (χ_n), we may assume that there exists $\epsilon \in \{-1, 1\}$ such that $\chi_n(\beta^\vee) = \epsilon$ for all $n \in \mathbb{Z}_{\geq 0}$. As $\chi_n \to \chi$, $\chi_n(\beta^\vee) = \epsilon \to \chi(\beta^\vee)$, which proves the lemma. □

Let $\mathbb{C}[Q_\mathbb{Z}^\vee] = \bigoplus_{\lambda \in Q_\mathbb{Z}^\vee} \mathbb{C}Z^\lambda \subset \mathbb{C}[Y]$. This is the group algebra of $Q_\mathbb{Z}^\vee$. Let $\mathbb{C}(Q_\mathbb{Z}^\vee) \subset \mathbb{C}(Y)$ be the field of fractions of $\mathbb{C}[Q_\mathbb{Z}^\vee]$ and $\mathcal{H}(Q_\mathbb{Z}^\vee) = \bigoplus_{w \in W^v} H_w \mathbb{C}(Q_\mathbb{Z}^\vee) \subset {}^{\mathrm{BL}}\mathcal{H}(T_\mathbb{C})$. This is a $(\mathcal{H}_{W^v,\mathbb{C}} - \mathbb{C}(Q_\mathbb{Z}^\vee))$-bimodule of ${}^{\mathrm{BL}}\mathcal{H}(T_\mathbb{C})$ and a left $\mathbb{C}(Q_\mathbb{Z}^\vee)$-submodule of ${}^{\mathrm{BL}}\mathcal{H}(T_\mathbb{C})$. Consequently $F_w \in \mathcal{H}(Q_\mathbb{Z}^\vee)$ for all $w \in W^v$.

Let $A = \mathbb{C}[Z^{\alpha_s^\vee} | s \in \mathscr{S}] \subset \mathbb{C}[Q_\mathbb{Z}^\vee]$. This is a unique factorization domain and $\mathbb{C}(Q_\mathbb{Z}^\vee)$ is the field of fractions of A.

Lemma 6.19. *Let $\beta^\vee \in \Phi^\vee$. Then $Z^{\beta^\vee} - 1$ and $Z^{\beta^\vee} + 1$ are irreducible in A.*

Proof. Write $\beta^\vee = w.\alpha_s^\vee$, where $w \in W^v$ and $s \in \mathscr{S}$. Then $Z^{\beta^\vee} = (Z^{\alpha_s^\vee})^w$. □

Lemma 6.20. *(see [Ree97, Proposition 14.3]) Let $\tau \in T_{\mathbb{C}}$ and $r = r_{\beta^\vee} \in \mathscr{S}_\tau$. Then $F_{r_\beta^\vee} - \zeta_{\beta^\vee} \in {}^{\mathrm{BL}}\mathcal{H}(T_{\mathbb{C}})_\tau$.*

Proof. One has $F_{r_\beta^\vee} - \zeta_{\beta^\vee} \in \mathcal{H}(Q_{\mathbb{Z}}^\vee)$. Write $F_{r_\beta^\vee} - \zeta_{\beta^\vee} = \sum_{u \in W^v} H_u \frac{f_u}{g_u}$, with $f_u, g_u \in A$ and $f_u \wedge g_u = 1$ for all $u \in W^v$. Let $u \in (1, r_{\beta^\vee})$. Let us prove that $\zeta_{\beta^\vee}^{\mathrm{den}} \wedge g_u = 1$. Suppose that $\zeta_{\beta^\vee}^{\mathrm{den}} \wedge g_u \neq 1$. Then there exists $\eta \in \{-1, 1\}$ such that $Z^{\beta^\vee} + \eta$ divides g_u.

Let $\chi \in T_{\mathbb{C}}$ be such that $\chi(\beta^\vee) = -\eta$. By Remark 5.1, $r_{\beta^\vee} \in W_\chi$. Let $(\chi_n) \in (T_{\mathbb{C}})^{\mathbb{Z}_{\geq 0}}$ be such that $\chi_n \to \chi$ and $W_{\chi_n} = \{1, r_{\beta^\vee}\}$ for all $n \in \mathbb{Z}_{\geq 0}$, and $\chi_n(\beta^\vee) = -\eta$ for all $n \in \mathbb{Z}_{\geq 0}$. whose existence is provided by Lemma 6.18. One has $g_u(\chi_n) = 0$ for all $n \in \mathbb{Z}_{\geq 0}$. Moreover by Lemma 6.10, $\pi_u^H(F_{r_\beta^\vee}) = \frac{f_u}{g_u} \in \mathbb{C}(Y)_{\chi_n}$ for all $n \in \mathbb{Z}_{\geq 0}$. Therefore, $f_u(\chi_n) = 0$ for all $n \in \mathbb{Z}_{\geq 0}$ and thus $f_u(\chi) = 0$.

By the Nullstellensatz (see [Lan02, IX, Theorem 1.5] for example), there exists $n \in \mathbb{Z}_{\geq 0}$ such that $Z^{\beta^\vee} + \eta$ divides f_u^n in A. By Lemma 6.19, $Z^{\beta^\vee} + \eta$ is irreducible in A and thus $Z^{\beta^\vee} + \eta$ divides f_u: a contradiction. Therefore $\zeta_{\beta^\vee}^{\mathrm{den}} \wedge g_u = 1$. By Lemma 6.15, $g_u(\tau) \neq 0$.

Therefore $\{u \in W^v | \pi_u^H(F_{r_{\beta^\vee}} - \zeta_{r_{\beta^\vee}}) \notin \mathbb{C}(Y)_\tau\} \subset \{1\}$. By Lemma 6.16 we deduce that $\{u \in W^v | \pi_u^B(F_{r_{\beta^\vee}} - \zeta_{r_{\beta^\vee}}) \notin \mathbb{C}(Y)_\tau\} \subset \{1\}$. Using Lemma 6.7 we deduce that $\{u \in W^v | \pi_u^B(F_{r_{\beta^\vee}} - \zeta_{r_{\beta^\vee}}) \notin \mathbb{C}(Y)_\tau\} = \emptyset$. By Lemma 6.16, $\{u \in W^v | \pi_u^H(F_{r_{\beta^\vee}} - \zeta_{r_{\beta^\vee}}) \notin \mathbb{C}(Y)_\tau\} = \emptyset$, which proves the lemma. □

6.4.3 Comparison of definitions of τ-simplicity

We now compare our definition of τ-simplicity and the one of Reeder (see [Ree97, Definition 14.2]). Let $\mathscr{S}_\tau^{\mathrm{Ree}}$ be the set of τ-simple reflections $r = r_{\beta^\vee} \in W_{(\tau)} \cap \mathscr{R}$ such that $N_{\Phi^\vee}(r_{\beta^\vee}) \cap \Phi_{(\tau)}^\vee = \{\beta^\vee\}$.

The advantage of our definition is that it is well adapted to the Coxeter structure of $W_{(\tau)}$ and the advantage of Reeder's one is that it is well adapted to the study of the singularity F_r at τ. Indeed, suppose that there exists $r \in \mathscr{S}_\tau \setminus \mathscr{S}_\tau^{\mathrm{Ree}}$. Then $\zeta_r \in {}^{\mathrm{BL}}\mathcal{H}(T_{\mathbb{C}})_\tau$, thus by Lemma 6.20, $F_r \in {}^{\mathrm{BL}}\mathcal{H}(T_{\mathbb{C}})_\tau$ and hence $F_r(\tau) \otimes_\tau 1 \in I_\tau(\tau) \setminus \mathbb{C}1 \otimes_\tau 1$. However, when $\tau \in \mathcal{U}_{\mathbb{C}}$, and $r \in \mathscr{S}_\tau^{\mathrm{Ree}}$, then one can prove that $F_r \notin {}^{\mathrm{BL}}\mathcal{H}(T_{\mathbb{C}})_\tau$. Thus in order to understand $I_\tau(\tau)$, we need to compare the two definitions, see Lemma 6.22.

Lemma 6.21. *1. One has $\mathscr{R}_{(\tau)} = \bigcup_{w \in W_{(\tau)}} w.(\mathscr{S}_\tau \cap \mathscr{R}_{(\tau)}).w^{-1}$. In particular, $\mathscr{R}_{(\tau)} = \mathscr{R} \cap W_{(\tau)}$ if and only if $\mathscr{S}_\tau \subset \mathscr{R}_{(\tau)}$.*

2. One has $\mathscr{S}_\tau^{\mathrm{Ree}} \subset \mathscr{R}_{(\tau)}$. Moreover, $\mathscr{S}_\tau = \mathscr{S}_\tau^{\mathrm{Ree}}$ if and only if $\mathscr{S}_\tau \subset \mathscr{R}_{(\tau)}$.

Proof. Let $w \in W_{(\tau)}$ and $r \in \mathscr{R}_{(\tau)}$. Let us prove that $r' := wrw^{-1} \in \mathscr{R}_{(\tau)}$. One has $\alpha_{r'}^\vee \in \{w.\alpha_r^\vee, wr.\alpha_r^\vee\}$. Let $v \in \{w, wr\}$ be such that $\alpha_{r'}^\vee = v.\alpha_r^\vee$. One has $\zeta_{\alpha_{r'}^\vee}^{\mathrm{den}} = (\zeta_{\alpha_r^\vee}^{\mathrm{den}})^v$ and hence $\zeta_{\alpha_{r'}^\vee}^{\mathrm{den}}(\tau) = (\zeta_{\alpha_r^\vee}^{\mathrm{den}})^v(\tau) = (\zeta_{\alpha_r^\vee}^{\mathrm{den}})(v^{-1}.\tau) = 0$ because $v \in W_{(\tau)} \subset W_\tau$. Thus $r' = wrw^{-1} \in \mathscr{R}_{(\tau)}$. Therefore $\bigcup_{w \in W_{(\tau)}} w.(\mathscr{S}_\tau \cap \mathscr{R}_{(\tau)}).w^{-1} \subset \mathscr{R}_{(\tau)}$. By [Dye90, Theorem 3.3 (i)], $\mathscr{R} \cap W_{(\tau)} = \bigcup_{w \in W_{(\tau)}} w\mathscr{S}_\tau w^{-1}$ and thus $\mathscr{R}_{(\tau)} \subset \bigcup_{w \in W_{(\tau)}} w.\mathscr{S}_\tau.w^{-1}$. Let $r \in \mathscr{R}_{(\tau)}$. Write $r = wr'w^{-1}$, with $w \in W_{(\tau)}$ and $r' \in \mathscr{S}_\tau$. Then $r' = w^{-1}rw \in \mathscr{R}_{(\tau)}$, which proves that $\mathscr{R}_{(\tau)} = \bigcup_{w \in W_{(\tau)}} w.(\mathscr{S}_\tau \cap \mathscr{R}_{(\tau)}).w^{-1}$.

One has $\mathscr{S}_\tau \subset \mathscr{R} \cap W_{(\tau)}$ and thus if $\mathscr{R}_{(\tau)} = \mathscr{R} \cap W_{(\tau)}$, then $\mathscr{S}_\tau \subset \mathscr{R}_{(\tau)}$. Suppose that $\mathscr{S}_\tau \subset \mathscr{R} \cap W_{(\tau)}$. Then by [Dye90, Theorem 3.3 (i)],

$$\mathscr{R} \cap W_{(\tau)} = \bigcup_{w \in W_{(\tau)}} w\mathscr{S}_\tau w^{-1} \subset \mathscr{R}_{(\tau)} \subset \mathscr{R} \cap W_{(\tau)},$$

which proves (1).

Let $r = r_{\beta^\vee} \in \mathscr{S}_\tau^{\mathrm{Ree}}$. Then $\beta^\vee \in \Phi^\vee_{(\tau)}$ and thus $r \in \mathscr{R}_{(\tau)}$. Consequently $\mathscr{S}_\tau^{\mathrm{Ree}} \subset \mathscr{R}_{(\tau)}$. Thus if $\mathscr{S}_\tau = \mathscr{S}_\tau^{\mathrm{Ree}}$ one has $\mathscr{R}_{(\tau)} = \mathscr{R} \cap W_{(\tau)}$ by (1).

Let $f : \Phi^\vee_+ \to \mathscr{R}$ be the bijection defined by $f(\alpha^\vee) = r_{\alpha^\vee}$ for all $\alpha^\vee \in \Phi^\vee$. Let $r \in \mathscr{R}$. Then $f\big(N_{\Phi^\vee}(r) \cap \Phi^\vee_{(\tau)}\big) = N_{\mathscr{R}}(r) \cap \mathscr{R}_{(\tau)}$. Thus if $\mathscr{R}_{(\tau)} = \mathscr{R} \cap W_{(\tau)}$, one has $f\big(N_{\Phi^\vee}(r) \cap \Phi^\vee_{(\tau)}\big) = N_{\mathscr{R}}(r) \cap W_{(\tau)}$. Thus if $r \in \mathscr{R}$, $r \in \mathscr{S}_\tau^{\mathrm{Ree}}$ if and only if $r \in \mathscr{S}_\tau$, which concludes the proof of the lemma. □

Lemma 6.22. *Let $\tau \in T_{\mathbb{C}}$ be such that I_τ is irreducible. Then $\mathscr{S}_\tau = \mathscr{S}_\tau^{\mathrm{Ree}}$.*

Proof. Let $\tau \in T_{\mathbb{C}}$ be such that $\mathscr{S}_\tau \neq \mathscr{S}_\tau^{\mathrm{Ree}}$. Then if $\mathscr{S}_\tau \subset \mathscr{S}_\tau^{\mathrm{Ree}}$, one has $\mathscr{S}_\tau \subset \mathscr{R}_{(\tau)}$ by Lemma 6.21 and thus $\mathscr{S}_\tau = \mathscr{S}_\tau^{\mathrm{Ree}}$ by Lemma 6.21: a contradiction. Thus there exists $r = r_{\beta^\vee} \in \mathscr{S}_\tau \setminus \mathscr{S}_\tau^{\mathrm{Ree}}$. Then by the proof of Lemma 6.14, $N_{\mathscr{R}}(r) \cap \Phi^\vee_{(\tau)} = \emptyset$. Therefore $\zeta_r \in {}^{\mathrm{BL}}\mathcal{H}(T_{\mathbb{C}})_\tau$. By Lemma 6.20 we deduce that $F_r \in {}^{\mathrm{BL}}\mathcal{H}(T_{\mathbb{C}})_\tau$. By Lemma 5.14, $F_r(\tau) \otimes_\tau 1 \in I_\tau(\tau) \setminus \mathbb{C}1 \otimes_\tau 1$ and by Theorem 5.8, I_τ is reducible. □

By the Lemma above and Kato's criterion ([Kat81, Theorem 2.2]), if $\tau \in \mathcal{U}_{\mathbb{C}}$ and $W_{(\tau)} = W_\tau$, then $\mathscr{S}_\tau = \mathscr{S}_\tau^{\mathrm{red}}$.

Lemma 6.23. *Assume that $\alpha_s(Y) = \mathbb{Z}$ for all $s \in \mathscr{S}$. Then $\mathscr{S}_\tau = \mathscr{S}_\tau^{\mathrm{Ree}}$.*

Proof. Let $r = r_{\beta^\vee} \in W_\tau \cap \mathscr{R}$. Then $r.\tau(\lambda) = \tau(\lambda - \beta(\lambda)\beta^\vee) = \tau(\lambda) = \tau(\lambda)\tau(\beta^\vee)^{\beta(\lambda)}$ for all $\lambda \in Y$. By assumption, there exists $\lambda \in Y$ such that $\beta(\lambda) = 1$ and thus $\tau(\beta^\vee) = 1$. Moreover, $\sigma_{\beta^\vee} = \sigma'_{\beta^\vee}$ for all $s \in \mathscr{S}$ and thus $\zeta^{\mathrm{den}}_{\beta^\vee} = 1 - Z^{\beta^\vee}$. Thus $r \in \mathscr{R}_{(\tau)}$. Consequently $\mathscr{R}_{(\tau)} = \mathscr{R} \cap W_{(\tau)}$ and thus by Lemma 6.21, $\mathscr{S}_\tau = \mathscr{S}_\tau^{\mathrm{Ree}}$. □

Lemma 6.24. *Let $\tau \in T_{\mathbb{C}}$ be such that $(W_{(\tau)}, \mathscr{S}_\tau)$ is the infinite dihedral group. Then $\mathscr{S}_\tau = \mathscr{S}_\tau^{\mathrm{Ree}}$.*

Proof. Write $\mathscr{S}_\tau = \{r_1, r_2\}$. Every element of $W_{(\tau)}$ has a unique reduced writing involving r_1 and r_2. By Lemma 6.21, $W_{(\tau)} = \langle \bigcup_{w \in W_{(\tau)}} w.(\mathscr{S}_\tau \cap \mathscr{R}_{(\tau)}).w^{-1} \rangle$ and thus $\mathscr{S}_\tau \cap \mathscr{R}_{(\tau)}$ is nonempty. Thus maybe exchanging the roles of r_1 and r_2, we may assume that $r_1 \in \mathscr{R}_{(\tau)}$. By definition, $r_2 \in W_{(\tau)} = \langle \mathscr{R}_{(\tau)} \rangle$. Write $r_2 = t_1 \dots t_k$, with $k \in \mathbb{Z}_{\geq 0}$ and $t_1, \dots, t_k \in \mathscr{R}_{(\tau)}$. Suppose $r_2 \notin \mathscr{R}_{(\tau)}$. Let $i \in [\![1, k]\!]$. Then by Lemma 6.21, one can write $t_i = w_i r_1 w_i^{-1}$ where $w_i \in W_{(\tau)}$. Thus the number of r_1 appearing in the reduced decomposition of t_i is odd. Therefore k is even. As $t_i \in \mathscr{R}$ for all $i \in [\![1, k]\!]$ we deduce that $\ell(t_i)$ is odd for all $i \in [\![1, k]\!]$ and $\ell(r_2)$ is even. We reach a contradiction with the fact that $r_2 \in \mathscr{R}$. Thus $r_2 \in \mathscr{R}_{(\tau)}$ and by Lemma 6.21, $\mathscr{S}_\tau = \mathscr{S}_\tau^{\mathrm{Ree}}$. □

6.5 Description of generalized weight spaces

In this subsection, we describe $I_\tau(\tau, \mathrm{gen})$ when $W_{(\tau)} = W_\tau$, using the $K_{r_1} \dots K_{r_k}(\tau)$, for $r_1, \dots, r_k \in \mathscr{S}_\tau$ (see Theorem 6.32).

For $r \in \mathscr{R}$, one sets $K_r = F_r - \zeta_{\alpha_r^\vee} \in {}^{\mathrm{BL}}\mathcal{H}(T_{\mathbb{C}})$. By Lemma 5.14 we have:

$$\theta * K_r = K_r * \theta^r + (\theta^r - \theta)\zeta_r \text{ for all } \theta \in \mathbb{C}(Y). \tag{2}$$

Lemma 6.25. *Let $w_1, w_2 \in W^v$. Then there exists $P \in \mathbb{C}(Y)^\times$ such that $F_{w_1} * F_{w_2} = F_{w_1 w_2} * P$. If moreover $\tau \in \mathcal{U}_{\mathbb{C}}$, then one can write $P = \frac{f}{g}$ with $f, g \in \mathbb{C}[Y]^\times$ and $f(w.\tau) \neq 0$ for all $w \in W^v$.*

Proof. Let $u, v \in W^v$. Let us prove that if $\chi \in T^{\mathrm{reg}}_{\mathbb{C}}$, then $F_u * F_v \in {}^{\mathrm{BL}}\mathcal{H}(T_{\mathbb{C}})_\chi$. Write $F_u = \sum_{u' \leq u} H_{u'}\theta_{u'}$, where $\theta_{u'} \in \mathbb{C}(Y)$ for all $u' \leq u$. Then by Lemma 5.14,

$$F_u * F_v = \sum_{u' \leq u} H_{u'}\theta_{u'} * F_v = \sum_{u' \leq u} H_{u'} * F_v * (\theta_{u'})^{v^{-1}}.$$

By Lemma 5.14, $\theta_{u'} \in {}^{\mathrm{BL}}\mathcal{H}(T_{\mathbb{C}})_\chi$ for all $\chi \in T^{\mathrm{reg}}_{\mathbb{C}}$ and thus $(\theta_{u'})^{v^{-1}} \in {}^{\mathrm{BL}}\mathcal{H}(T_{\mathbb{C}})_\chi$ for all $\chi \in T^{\mathrm{reg}}_{\mathbb{C}}$. Let $\chi \in T^{\mathrm{reg}}_{\mathbb{C}}$. As ${}^{\mathrm{BL}}\mathcal{H}(T_{\mathbb{C}})_\chi$ is an $\mathcal{H}_{W^v,\mathbb{C}} - \mathbb{C}(Y)_\chi$ bimodule, we deduce that $F_u * F_v \in {}^{\mathrm{BL}}\mathcal{H}(T_{\mathbb{C}})_\chi$.

Let $u, v \in W^v$. Let us prove that there exists $Q \in \mathbb{C}(Y)$ such that $F_u * F_v = F_{uv} * Q$. Let $\lambda \in Y$. Then by Lemma 5.14, one has $Z^\lambda F_u * F_v = F_u * F_v * Z^{(uv)^{-1}.\lambda}$. Therefore for all $\chi \in T^{\mathrm{reg}}_{\mathbb{C}}$, there exists $a(\chi) \in \mathbb{C}$ such that $F_u * F_v(\chi) = a(\chi)F_{uv}(\chi)$. Write $F_u * F_v = \sum_{w \in W^v} H_w * \theta_w$ and $F_{uv} = \sum_{w \in W^v} H_w * \tilde{\theta}_w$, where $(\theta_w), (\tilde{\theta}_w) \in \mathbb{C}(Y)^{(W^v)}$. Let $Q = \frac{\theta_{uv}}{\tilde{\theta}_{uv}} = \theta_{uv}$. Let $w \in W^v$ be such that $\tilde{\theta}_w = 0$. Then for all $\chi \in T^{\mathrm{reg}}_{\mathbb{C}}$, $\theta_w(\chi) = 0$ and by Lemma 6.1, $\theta_w = 0 = Q\tilde{\theta}_w$. Let $w \in W^v$ be such that $\theta_w \neq 0$. Then $U := \{\chi \in T_{\mathbb{C}} | \theta_w \in {}^{\mathrm{BL}}\mathcal{H}(T_{\mathbb{C}})_\chi \text{ and } \theta_w(\chi) \neq 0\}$ is open and dense in $T_{\mathbb{C}}$. By Remark 5.11, $T^{\mathrm{reg}}_{\mathbb{C}}$ has full measure in $T_{\mathbb{C}}$ and thus $U \cap T^{\mathrm{reg}}_{\mathbb{C}}$ is dense in $T_{\mathbb{C}}$. Moreover $\theta_w(\chi) = Q(\chi)\tilde{\theta}(\chi)$ for all $\chi \in U \cap T^{\mathrm{reg}}_{\mathbb{C}}$ and thus $\tilde{\theta}_w = Q\theta_w$. Consequently, there exists $Q \in \mathbb{C}(Y)$ such that $F_u * F_v = F_{uv} * Q$.

Let $\tau \in \mathcal{U}_{\mathbb{C}}$. Let $w_1 \in W^v$. Let $u \in W^v$ be such that there exists $\theta = \frac{f}{g} \in \mathbb{C}(Y)^\times$ such that $F_{w_1} * F_u = F_{w_1 u} * \theta$, with $f(w.\tau) \neq 0$ for all $w \in W^v$. Let $s \in \mathscr{S}$ be such that $us > u$. Then by Lemma 5.3,

$$F_{w_1} * F_{us} = F_{w_1 u} * \theta * F_s = F_{w_1 u} * F_s * \theta^s.$$

Suppose $w_1 us > w_1 u$. Then $F_{w_1 u} * F_s = F_{w_1 us}$ and thus $F_{w_1} * F_{us} = F_{w_1 us} * \theta^s$ and $f^s(w.\tau) \neq 0$ for all $w \in W^v$. Suppose $w_1 us < w_1 u$. Then $F_{w_1 u} * F_s = F_{w_1 us} * (F_s)^2$ and thus by Lemma 5.3, $F_{w_1} * F_{us} = F_{w_1 us} * (\theta^s \zeta_s \zeta_s^s)$. By definition of $\mathcal{U}_{\mathbb{C}}$, one can write $F_{w_1} * F_{us} = F_{w_1 us} * \frac{\tilde{f}}{\tilde{g}}$ with $\tilde{f}, \tilde{g} \in \mathbb{C}[Y]^\times$ such that $\tilde{f}(w.\tau) \neq 0$ for all $w \in W^v$ and the lemma follows. □

Remark 6.26. *In [Ree97, Lemma 4.3 (2)], Reeder gives an explicit expression of $F_u * F_v$, for $u, v \in W^v$.*

Let $r \in \mathscr{R}$. Let $\Omega_r : \mathbb{C}(Y) \to \mathbb{C}(Y)$ be defined by $\Omega_r(\theta) = \zeta_r(\theta^r - \theta)$ for all $\theta \in \mathbb{C}(Y)$.

Lemma 6.27. *Let $r \in \mathscr{S}_\tau$. Then $\Omega_r(\mathbb{C}(Y)_\tau) \subset \mathbb{C}(Y)_\tau$.*

Proof. Write $r = r_{\beta^\vee}$, where $\beta^\vee \in \Phi^\vee$. Then one has $r(\lambda) = \lambda - \beta(\lambda)\beta^\vee$ for all $\lambda \in Y$. Let $\lambda \in Y$. Then with the same computation as in Remark 2.7 (2), we have that $\Omega_r(Z^\lambda) \in \mathbb{C}(Y)_\tau$. Thus $\Omega_r(\theta) \in \mathbb{C}(Y)_\tau$ for all $\theta \in \mathbb{C}[Y]$.

Let $\theta \in \mathbb{C}(Y)_\tau$. Write $\theta = \frac{f}{g}$, where $f, g \in \mathbb{C}[Y]$ and $g(\tau) \neq 0$. Then $\zeta_r(\theta^r - \theta) = \zeta_r(\frac{f^r g - (f^r g)^r}{g g^r})$. Moreover, $g^r(\tau) = g(r.\tau) = g(\tau) \neq 0$ and as $f^r g \in \mathbb{C}[Y]$, we have that $\zeta_r(\theta^r - \theta) \in \mathbb{C}(Y)_\tau$. □

We now assume that $\tau \in \mathcal{U}_{\mathbb{C}}$.

For each $w \in W_{(\tau)}$ we fix a reduced writing $w = r_1 \dots r_k$, with $k = \ell(w)$ and $r_1, \dots, r_k \in \mathscr{S}_\tau$ and we set $\underline{w} = (r_1, \dots, r_k)$. Let $K_{\underline{w}} = K_{r_1} \dots K_{r_k} \in {}^{\mathrm{BL}}\mathcal{H}(T_{\mathbb{C}})$.

Lemma 6.28. *Let $r \in \mathscr{S}_\tau$. Then ${}^{\mathrm{BL}}\mathcal{H}(T_{\mathbb{C}})_\tau * K_r \subset {}^{\mathrm{BL}}\mathcal{H}(T_{\mathbb{C}})_\tau$. In particular, $K_{\underline{w}} \in {}^{\mathrm{BL}}\mathcal{H}(T_{\mathbb{C}})_\tau$ for all $w \in W_{(\tau)}$.*

Proof. Let $w \in W^v$ and $\theta \in \mathbb{C}(Y)_\tau$. Then $H_w\theta * K_r = H_w K_r \theta^r + H_w * \Omega_r(\theta)$. Using Lemma 6.20, Lemma 6.27 and the fact that ${}^{\mathrm{BL}}\mathcal{H}(T_\mathbb{C})_\tau$ is a $\mathcal{H}_{W^v,\mathbb{C}} - \mathbb{C}(Y)_\tau$-bimodule, we deduce that $H_w\theta * K_r \in {}^{\mathrm{BL}}\mathcal{H}(T_\mathbb{C})_\tau$. Hence ${}^{\mathrm{BL}}\mathcal{H}(T_\mathbb{C})_\tau * K_r \subset {}^{\mathrm{BL}}\mathcal{H}(T_\mathbb{C})_\tau$. □

Lemma 6.29. *Let $w \in W_{(\tau)}$. Then* $\max \mathrm{supp}\big(K_{\underline{w}}(\tau)\big) = \{w\}$.

Proof. Write $\underline{w} = (r_1, \dots, r_k)$ with $r_1, \dots, r_k \in \mathscr{S}_\tau$. Then

$$K_{\underline{w}} = (F_{r_{i_1}} - \zeta_{r_{i_1}}) \dots (F_{r_{i_k}} - \zeta_{r_{i_k}}) = F_{r_{i_1}} * F_{r_{i_2}} * \dots * F_{r_{i_k}} + \sum_{v <_\tau w} F_v P_v,$$

for some $P_v \in \mathbb{C}(Y)$. By Lemma 6.25, there exist $f, g \in \mathbb{C}[Y]^\times$ such that $F_{r_{i_1}} * F_{r_{i_2}} * \dots * F_{r_{i_k}} = F_w * \frac{f}{g}$ and $f(\tau) \neq 0$. One has $\pi_w^T(F_w) = 1$ and by Lemma 6.12, $\pi_v^T(F_v) = 0$ for all $v \in [1, w)_{\leq_\tau}$. Thus using Lemma 6.28, one can moreover assume $g(\tau) \neq 0$. Therefore $\pi_w^T(K_w) = \frac{f}{g} \in \mathbb{C}(Y)_\tau$ and $f(\tau) \neq 0$, which proves the lemma. □

Let $\mathcal{K}(W_{(\tau)}) = \bigoplus_{w \in W_{(\tau)}} F_w \mathbb{C}(Y)$. By Lemma 6.25 and Lemma 5.14, $\mathcal{K}(W_{(\tau)})$ is a subalgebra of ${}^{\mathrm{BL}}\mathcal{H}(T_\mathbb{C})$. Let $\mathcal{K}_\tau = \mathcal{K}(W_{(\tau)}) \cap {}^{\mathrm{BL}}\mathcal{H}(T_\mathbb{C})_\tau$. For $w \in W_{(\tau)}$, set $\mathcal{K}(W_{(\tau)})^{<_\tau w} = \bigoplus_{v \in W_{(\tau)}, v <_\tau w} F_w \mathbb{C}(Y)$ and $\mathcal{K}(W_{(\tau)})^{<_\tau w} = \bigoplus_{v <_\tau w} K_{\underline{v}} \mathbb{C}(Y)_\tau$.

Lemma 6.30. *Let $\theta \in \mathbb{C}(Y)_\tau$ and $w \in W_{(\tau)}$. Then there exists $k_{\underline{w}}(\theta) \in \mathcal{K}_\tau^{<_\tau w}$ such that $\theta * K_{\underline{w}} = K_{\underline{w}} * \theta^{w^{-1}} + k_{\underline{w}}(\theta)$.*

Proof. If $w = 1$, this is clear. Suppose $w > 1$. Write $w = vr$ with $v \in W_{(\tau)}$ and $r \in \mathscr{S}_\tau$ such that $v <_\tau w$. Suppose that $\theta * K_{\underline{v}} = K_{\underline{v}} * \theta^{v^{-1}} + k_{\underline{v}}(\theta)$ with $k_{\underline{v}}(\theta) \in \mathcal{K}_\tau^{<_\tau v}$. One has

$$\theta * K_{\underline{w}} = \theta * K_{\underline{v}} * K_r = \big(K_{\underline{v}} \theta^{v^{-1}} + k_{\underline{v}}(\theta)\big) * K_r = K_{\underline{w}} * \theta^{w^{-1}} + K_{\underline{v}} * \Omega_r(\theta^{v^{-1}}) + k_{\underline{v}}(\theta) * K_r.$$

The sets $\mathcal{K}(W_{(\tau)})^{\leq_\tau v} = \bigoplus_{v' \leq_\tau v} F_{v'} \mathbb{C}(Y)$ and ${}^{\mathrm{BL}}\mathcal{H}(T_\mathbb{C})_\tau$ are right $\mathbb{C}(Y)_\tau$-submodules of ${}^{\mathrm{BL}}\mathcal{H}(T_\mathbb{C})$ and thus by Lemma 6.28 and Lemma 6.27, $K_{\underline{v}} * \Omega_r(\theta^{v^{-1}}) \in \mathcal{K}_\tau^{\leq_\tau v} \subset \mathcal{K}_\tau^{<_\tau w}$.

By Lemma 6.28, $k_{\underline{v}}(\theta) * K_r \in {}^{\mathrm{BL}}\mathcal{H}(T_\mathbb{C})_\tau$. By Lemma 5.14 and [Kum02, Corollary 1.3.19], $k_{\underline{v}} F_r \in \mathcal{K}(W_{(\tau)})^{<_\tau \max(vr, v)} = \mathcal{K}(W_{(\tau)})^{<_\tau w}$. Consequently $k_{\underline{v}} * K_r \in \mathcal{K}_\tau^{<_\tau w}$ and $K_{\underline{v}} \Omega_r(\theta^{v^{-1}}) + k_{\underline{v}}(\theta) K_r \in \mathcal{K}_\tau^{<_\tau w}$, which proves the lemma. □

Lemma 6.31. *One has $\mathcal{K}_\tau = \bigoplus_{w \in W_{(\tau)}} K_{\underline{w}} \mathbb{C}(Y)_\tau$.*

Proof. By Lemma 6.28, $\mathcal{K}_\tau \supset \bigoplus_{w \in W_{(\tau)}} K_{\underline{w}} \mathbb{C}(Y)_\tau$.

For $w \in W_{(\tau)}$, set $\mathcal{K}(W_{(\tau)})^{\leq_\tau w} = \bigoplus_{v \leq_\tau w} F_v \mathbb{C}(Y) \subset \mathcal{K}(W_{(\tau)})$. Let $w \in W_{(\tau)}$. Suppose that for all $v \in [1, w)_{\leq_\tau}$, one has $\mathcal{K}_\tau^{\leq_\tau v} = \bigoplus_{v' \in [1, v]_{\leq_\tau}} K_{\underline{v'}} \mathbb{C}(Y)_\tau$. By Lemma 6.29, one can write $\pi_w^T(K_w) = \frac{f}{g}$, with $f, g \in \mathbb{C}[Y]$ such that $f(\tau) g(\tau) \neq 0$. Let $x \in \mathcal{K}_\tau^{\leq_\tau w}$ and $\theta = \pi_w^T(x) \in \mathbb{C}(Y)_\tau$. By Lemma 6.28, $\theta \frac{g}{f} K_w \in {}^{\mathrm{BL}}\mathcal{H}(T_\mathbb{C})_\tau$. Moreover, $x - \theta \frac{g}{f} K_w \in \sum_{v \in [1, w)_{\leq_\tau}} \mathcal{K}_\tau^{\leq_\tau v}$. Therefore, $x \in \bigoplus_{v \in [1, w]_{\leq_\tau}} K_{\underline{v}} \mathbb{C}(Y)_\tau$ and the lemma follows.

□

Theorem 6.32. *Let $\tau \in T_\mathbb{C}$ be such that $W_{(\tau)} = W_\tau$. Then $I_\tau(\tau, \mathrm{gen}) = \mathrm{ev}_\tau(\mathcal{K}_\tau) \otimes_\tau 1$.*

Proof. Let $w \in W_{(\tau)}$ and $\theta \in \mathbb{C}(Y)_\tau$. Then by Lemma 6.30, $(\theta - \tau(\theta)) K_w(\tau) \otimes_\tau 1 \in \mathcal{K}^{<_\tau w}(\tau) \otimes_\tau 1$. By an induction using Lemma 6.31 we deduce that $\mathcal{K}_w(\tau) \otimes_\tau 1 \subset I_\tau(\tau, \mathrm{gen})$.

Let $w \in W^v$ and $E_w = \big(\mathrm{ev}_\tau(\mathcal{K}_\tau) \otimes_\tau 1\big) \cap I_\tau^{\leq w}$. By Lemma 6.29, $\dim E_w = |W_{(\tau)} \cap \{v \in W^v | v \leq w\}|$. By Proposition 3.5, $\dim I_\tau(\tau, \mathrm{gen})^{\leq w} = |\{v \in W_\tau | v \leq w\}| = \dim E_w$. As $(W^v, \leq)$ is a directed poset, $I_\tau = \bigcup_{w \in W^v} I_\tau^{\leq w}$, which proves the theorem. □

6.6 Irreducibility of I_τ when $W_\tau = W_{(\tau)}$ is the infinite dihedral group

In this subsection, we prove that if $\tau \in \mathcal{U}_{\mathbb{C}}$ is such that $W_\tau = W_{(\tau)}$ and $W_{(\tau)}$ is isomorphic to the infinite dihedral group, then I_τ is irreducible (see Lemma 6.38). Let us sketch the proof of this lemma. We prove that $I_\tau(\tau) = \mathbb{C}1 \otimes_\tau 1$. For $w \in W_{(\tau)}$, let $\pi_w^K : \mathcal{I}_\tau(\tau, \mathrm{gen}) \to \mathbb{C}$ be defined $\pi_w^K\big(\sum_{v\in W^v} K_{\underline{v}}(\tau)x_v\big) = x_w$, for all $(x_v) \in \mathbb{C}^{(W_{(\tau)})}$, which is well defined by Lemma 6.29 and Theorem 6.32. We suppose that $I_\tau(\tau) \setminus \mathbb{C} \otimes_\tau 1$ is nonempty and we consider one of its elements x. We reach a contradiction by computing $\pi_w^K(x)$, where $w \in W_{(\tau)}$ is such that $\ell_\tau(w) = \max\{\ell_\tau(v) | v \in \mathrm{supp}(x) \cap W_{(\tau)}\} - 1$.

Let $\tau \in \mathcal{U}_{\mathbb{C}}$. Assume that $(W_{(\tau)}, \mathscr{S}_\tau)$ is isomorphic to the infinite dihedral group (in particular, $|\mathscr{S}_\tau| = 2$ and every element of $W_{(\tau)}$ admits a unique reduced writing).

The following lemma is easy to prove.

Lemma 6.33. *Let $w \in W_{(\tau)}$ and $r \in \mathscr{S}_\tau$ be such that $\ell_\tau(wr) = \ell_\tau(w) + 1$. Let $u \in [1, w)_{\leq_\tau}$. Then $ur \neq w$.*

Lemma 6.34. *Let $\tau \in \mathcal{U}_{\mathbb{C}}$. Let $r = r_{\beta^\vee} \in \mathscr{S}_\tau$, where $\beta^\vee \in \Phi^\vee$. Then there exists $a \in \mathbb{C}^*$ such that for all $\lambda \in Y$,*

$$\tau\big((Z^{r.\lambda} - Z^\lambda)\zeta_r\big) = a\tau(\lambda)\beta(\lambda).$$

Proof. One has

$$\zeta_r = \frac{1}{\zeta_{\beta^\vee}^{\mathrm{den}}} \cdot \prod_{\alpha^\vee \in N_{\Phi^\vee}(r)} \zeta_{\alpha^\vee}^{\mathrm{num}} \cdot \prod_{\alpha^\vee \in N_{\Phi^\vee}(r)\setminus\{\beta^\vee\}} \frac{1}{\zeta_{\alpha^\vee}^{\mathrm{den}}}.$$

By Lemma 6.14 and by definition of $\mathcal{U}_{\mathbb{C}}$, $\tau(\prod_{\alpha^\vee \in N_{\Phi^\vee}(r)\setminus\{\beta^\vee\}} \zeta_{\alpha^\vee}^{\mathrm{den}}) \neq 0$ and $\tau(\prod_{\alpha^\vee \in N_{\Phi^\vee}(r)} \zeta_{\alpha^\vee}^{\mathrm{num}}) \neq 0$.

If $\sigma_{\beta^\vee} = \sigma'_{\beta^\vee}$, one has $\frac{Z^{r.\lambda} - Z^\lambda}{\zeta_{\beta^\vee}^{\mathrm{den}}} = \frac{Z^{r.\lambda} - Z^\lambda}{1 - Z^{\beta^\vee}} = Z^\lambda \frac{Z^{-\beta(\lambda)\beta^\vee} - 1}{1 - Z^{\beta^\vee}}$. By Lemma 6.24, $r \in \mathscr{S}_\tau^{\mathrm{Ree}}$ and thus $\tau(\beta^\vee) = 1$. Thus by the same computation as in Remark 2.7, $\tau(\frac{Z^{r.\lambda} - Z^\lambda}{1 - Z^{\beta^\vee}}) = \beta(\lambda)\tau(\lambda)$. Using a similar computation when $\sigma_{\beta^\vee} \neq \sigma'_{\beta^\vee}$, we deduce the lemma. □

Lemma 6.35. *Let $w \in W_{(\tau)}$ and $r \in \mathscr{S}_\tau$ be such that $\ell_\tau(wr) = \ell_\tau(w) + 1$. Then there exists $a \in \mathbb{C}^*$ such that for all $\lambda \in Y$, one has:*

$$\pi_w^K\big(Z^\lambda * K_{\underline{wr}}(\tau) \otimes_\tau 1\big) = a\tau(\lambda)\alpha_r(w^{-1}.\lambda).$$

Proof. Let $\lambda \in Y$. Write $Z^\lambda * K_{\underline{w}} = K_{\underline{w}} * Z^{w^{-1}.\lambda} + k$, where $k \in \mathcal{K}_\tau^{<_\tau w}$, which is possible by Lemma 6.30. One has

$$Z^\lambda * K_{\underline{wr}} = (K_{\underline{w}} * Z^{w^{-1}.\lambda} + k) * K_r = K_{\underline{wr}} * Z^{rw^{-1}.\lambda} + K_{\underline{w}}\big((Z^{rw^{-1}.\lambda} - Z^{w^{-1}.\lambda})\zeta_r\big) + k * K_r.$$

Therefore, using Lemma 6.33 and Lemma 6.34 we deduce

$$\pi_w^K\big(Z^\lambda K_{\underline{wr}}(\tau) \otimes_\tau 1\big) = \tau\big((Z^{rw^{-1}.\lambda} - Z^{w^{-1}.\lambda})\zeta_r\big) = a\tau(\lambda)\beta(w^{-1}.\lambda),$$

for some $a \in \mathbb{C}^*$. □

Lemma 6.36. *Let $w \in W_{(\tau)}$ and $r \in \mathscr{S}_\tau$ be such that $\ell_\tau(rw) = \ell_\tau(w) + 1$.*

One has $\pi_w^K(K_r * \mathcal{K}(W_{(\tau)})^{\leq_\tau w}) = \{0\}$.

Proof. Let $u \in W_{(\tau)}$ and $r \in \mathscr{S}_\tau$ be such that $ru >_\tau u$. Then by Lemma 6.25 and [Kum02, Corollary 1.3.19], $F_r * \mathcal{K}(W_{(\tau)})^{\leq_\tau u} \subset \mathcal{K}(W_{(\tau)})^{\leq_\tau \max(u, ru)}$ and thus $K_r * \mathcal{K}(W_{(\tau)})^{\leq_\tau u} \subset \mathcal{K}(W_{(\tau)})^{\leq_\tau \max(u,ru)}$.

Let $v \in [1,w)_{\leq_\tau}$. If $rv >_\tau v$, then by Lemma 6.25, there exists $Q \in \mathbb{C}(Y)$ such that $F_r * F_v = F_{rv} * Q$ and thus $K_r * F_v \in F_{rv} * Q + F_v\mathbb{C}(Y)$. By Lemma 6.33, $rv \neq w$. Using Lemma 6.29 and the fact w and rv have the same length, we deduce that $\pi_w^K(K_r * F_v) = 0$.

If $rv <_\tau v$, then $K_r * F_v \in \mathcal{K}(W_{(\tau)})^{\leq_\tau v}$ and thus $\pi_w^K(K_r * F_v) = 0$ which finishes the proof of the lemma. □

Lemma 6.37. *Let $w \in W_\tau$, $r \in \{r_1, r_2\}$ be such that $\ell_\tau(rw) = \ell_\tau(w) + 1$. Then there exists $b \in \mathbb{C}^*$ such that for all $\lambda \in Y$:*

$$\pi_w^K(Z^\lambda.K_{rw}(\tau) \otimes_\tau 1) = b\tau(\lambda)\alpha_r(\lambda).$$

Proof. One has

$$Z^\lambda K_{rw} = (Z^\lambda * K_r) * K_w = \big(K_r.Z^{r.\lambda} + (Z^{r.\lambda} - Z^\lambda)\zeta_r\big) * K_w(\tau).$$

One has $Z^{r.\lambda} * K_w \in \mathcal{K}(W_{(\tau)})^{\leq_\tau w}$. Thus by Lemma 6.36, $\pi_w^K(K_r.Z^{r.\lambda} * K_w) = 0$. Moreover, by Lemma 6.34, there exists $b \in \mathbb{C}^*$ such that

$$\pi_w^K\big((Z^{r.\lambda} - Z^\lambda)\zeta_r K_w(\tau) \otimes_\tau 1\big) = w.\tau\big((Z^{r.\lambda} - Z^\lambda)\zeta_r\big) = b\tau(\lambda)\alpha_r(\lambda),$$

which proves the lemma. □

Lemma 6.38. *Let $\tau \in \mathcal{U}_\mathbb{C}$ be such that $W_\tau = W_{(\tau)}$ and that there exists $r_1, r_2 \in \mathscr{R}$ such that $(W_{(\tau)}, \{r_1, r_2\})$ is isomorphic to the infinite dihedral group and that r_1, r_2 are τ-simple. Then I_τ is irreducible.*

Proof. Let us prove that $I_\tau(\tau) = \mathbb{C}.1 \otimes_\tau 1$. Let $x \in I_\tau \setminus \mathbb{C}.1 \otimes_\tau 1$ and assume that $x \in I_\tau(\tau)$. Let $n = \max\{\ell_\tau(w) | w \in \mathrm{supp}(x)\}$. Let $w \in W_{(\tau)}$ be such that $\ell_\tau(w) = n - 1$. Then there exist $r, r' \in \mathscr{S}_\tau$ such that $\{v \in W_{(\tau)} | \ell_\tau(v) = n\} = \{rw, wr'\}$. By Theorem 6.32, $x \in \sum_{v \in W_{(\tau)}} \mathbb{C}K_{\underline{v}}(\tau) \otimes_\tau 1$. Let $\gamma = \pi_{rw}^K(x)$ and $\gamma' = \pi_{wr'}^K(x)$.

Set $\gamma_w = \pi_w^K(x)$. Then by Lemma 6.35 and Lemma 6.37, there exist $a, a' \in \mathbb{C}^*$ such that for all $\lambda \in Y$,

$$\pi_v^K(Z^\lambda.x) = \tau(\lambda)\big(a\gamma\alpha_r(\lambda) + a'\gamma' w.\alpha_{r'}(\lambda) + \gamma_w\big) = \tau(\lambda)\gamma_w.$$

Therefore $\{\alpha_r, w.\alpha_{r'}\}$ is lineraly dependent and hence $w.\alpha_{r'} \in \{\pm\alpha_r\} = \{\alpha_r, r.\alpha_r\}$. By Lemma 2.3 we deduce $rw = wr'$: a contradiction because $|\{rw, wr'\}| = |\{v \in W_{(\tau)} | \ell_\tau(v) = n\}| = 2$.

Therefore $I_\tau = \mathbb{C}1 \otimes_\tau 1$ and by Theorem 5.8, I_τ is irreducible. □

6.7 Kato's criterion when the Kac-Moody matrix has size 2

In this subsection, we prove Kato's irreducibility criterion when $|\mathscr{S}| = 2$ (see Theorem 6.40). As the case where W^v is finite is a particular case of Kato's theorem [Kat81, Theorem 2.2] we assume that W^v is infinite.

This is equivalent to assuming that the Kac-Moody matrix of the root generating system $\mathcal{S}$ is of the form $\begin{pmatrix} 2 & a \\ b & 2 \end{pmatrix}$, with $a, b \in \mathbb{Z}_{<0}$ and $ab \geq 4$ ([Kum02, Proposition 1.3.21]). The system $(W^v, \mathscr{S})$ is then the infinite dihedral group. Write $\mathscr{S} = \{s_1, s_2\}$. Then every element of W^v admits a unique reduced writing involving s_1 and s_2.

Let G be a group and $a, b \in G$. For $k \in \mathbb{Z}_{\geq 0}$, we define $P_k(a,b) = aba\ldots$ where the products has k terms.

Lemma 6.39. *The subgroups of W^v are exactly the ones of the following list:*

1. $\{1\}$
2. $\langle r\rangle = \{1, r\}$, *for some* $r \in \mathscr{R}$
3. $Z_k = \langle P_{2k}(s_1, s_2)\rangle = \langle P_{2k}(s_2, s_1)\rangle \simeq \mathbb{Z}$ *for* $k \in \mathbb{Z}_{\geq 1}$
4. $R_{k,m} = \langle P_{2k+1}(s_1, s_2), P_{2m+1}(s_2, s_1)\rangle \simeq W^v$ *for* $k, m \in \mathbb{Z}_{\geq 0}$.

Proof. Let $\{1\} \neq H \subset W^v$ be a subgroup. Let $n = \min\{\ell(w) | w \in H \setminus \{1\}\}$.

First assume that n is even and set $k = \frac{n}{2}$. Then $P(s_1, s_2, n) = P(s_2, s_1, n)^{-1}$ and as these are the only elements having length n in W^v, $H \supset Z_k$. Let $w = P_n(s_1, s_2)$. Let $h \in H \setminus \{1\}$. Write $\ell(h) = an + r$ with $a \in \mathbb{Z}_{\geq 1}$ and $r \in [\![0, r-1]\!]$. Then there exists $\epsilon \in \{-1, 1\}$ such that $h = w^{\epsilon a}.h'$, with $\ell(h') = r$. Moreover, $h' \in H$ and thus $h' = 1$. Therefore $H = Z_k$.

We now assume that n is odd. Maybe considering vHv^{-1} for some $v \in W^v$ and exchanging the roles of s_1 and s_2, we may assume that $s_1 \in H$. Assume $H \neq \langle s_1\rangle$. Let $n' = \min\{\ell(w) | w \in H \setminus \langle s_1\rangle\}$. Let $w \in H \setminus \langle s_1\rangle$ be such that $\ell(w) = n'$. Then the reduced writing of w begins and ends with s_2. Thus $n' = 2n'' + 1$ for some $n'' \in \mathbb{Z}_{\geq 0}$. Then it is easy to see that $H = R_{1,n''}$, which finishes the proof. □

We prove in Appendix B that there exists size 2 Kac-Moody matrices such that for each subgroup of W^v, there exists $\tau \in T_{\mathbb{C}}$ such that $W_{(\tau)}$ is isomorphic to this subgroup.

Theorem 6.40. *Assume that the matrix of the root generating system $\mathcal{S}$ is of size 2. Let $\tau \in T_{\mathbb{C}}$. Then I_τ is irreducible if and only if $\tau \in \mathcal{U}_{\mathbb{C}}$ and $W_\tau = W_{(\tau)}$.*

Proof. If W^v is finite, this is a particular case of Kato's theorem ([Kat81, Theorem 2.2]). Suppose that W^v is infinite. By Lemma 5.5 and Proposition 5.17, if I_τ is irreducible, then $\tau \in \mathcal{U}_{\mathbb{C}}$ and $W_\tau = W_{(\tau)}$. Reciprocally, suppose $\tau \in \mathcal{U}_{\mathbb{C}}$ and $W_\tau = W_{(\tau)}$. Then by Lemma 6.39, either $W_{(\tau)} = \{1\}$, or $W_{(\tau)} = \langle r\rangle$ for some $r \in \mathscr{R}$ or $W_{(\tau)} = \langle r_1, r_2\rangle$ for some $r_1, r_2 \in \mathscr{R}$ and $(W_{(\tau)}, \{r_1, r_2\})$ is isomorphic to the infinite dihedral group. In the first two cases, I_τ is irreducible by Corollary 5.10 or Corollary 5.12. Suppose $W_{(\tau)} = \langle r_1, r_2\rangle$. Then by Remark 2.5 (1), $(W_{(\tau)}, \mathscr{S}_\tau)$ is isomorphic to the infinite dihedral group and I_τ is irreducible by Lemma 6.38. □

Comments on the proofs of Kato's criterion There are several proofs of Kato's criterion in the litterature. In [Ree92], Reeder proves this criterion (see Corollary 8.7). In his proof, he uses the R-group $R_\tau = \{w \in W_\tau | w(\Phi^\vee_{(\tau)} \cap \Phi^\vee_+) = \Phi^\vee_{(\tau)} \cap \Phi^\vee_+\}$. This group is reduced to $\{1\}$ when $W_\tau = W_{(\tau)}$. His proof uses Harich-Chandra completeness theorem, which - under certain hypothesis on τ - majorizes the dimension of the space of intertwining operators of I_τ. Unfortunately, it seems that there exists up to now no equivalent of Harich-Chandra completeness theorem available in the Kac-Moody framework.

In [Rog85], Rogawski gives a proof of a particular case of Kato's criterion (see Corollary 3.2). However, it seems that its proof uses the fact that every element x of $I_\tau(\tau)$ can be written as a sum $x = \sum_{j\in J} x_j$ where J is a finite set and for all $j \in J$, $|\max \mathrm{supp}(x_j)| = 1$ and $x_j \in I_\tau(\tau)$. I do not know how to prove such a property.

In [Ree97], Reeder gives two proofs of Kato's criterion or of weak versions of it (see Corollary 4.6 and Theorem 14.7). Our proof of Theorem 6.40 is strongly inspired by the proof of [Ree97, Theorem 14.7]. However, there are some points that I do not understand in its proof. For example (with the same notation as in [Ree97]) I think that it is not

clear why the only terms contributing to the coefficient of $B_{rz}v_\tau$ are of the form r_1rz. For example, suppose that τ is the map $\mathbb{1} : Y \to \mathbb{C}$ defined by $\mathbb{1}(\lambda) = 1$ for all $\lambda \in Y$ and that $\sigma_s = \sigma'_s = q \in \mathbb{Z}_{\geq 2}$ for all $s \in \mathscr{S}$. Then $W_{\mathbb{1}} = W^v$ and the set of $\mathbb{1}$-simple reflections is $\mathscr{S}$. By definition, $K_s = B_s$ for all $s \in \mathscr{S}$. Let $s, t \in \mathscr{S}$ be such that $s \neq t$. By Lemma 6.6, one has $B_sB_t = B_{st}$ and $B_tB_s = B_{ts}$. Let $\lambda \in Y$. Then:

$$Z^\lambda * K_{st} = Z^\lambda B_sB_t = (B_sZ^{s.\lambda} + \Omega_s(Z^\lambda))B_t = B_{st}Z^{ts\lambda} + B_s\Omega_t(Z^{s.\lambda}) + B_t\Omega_s(Z^\lambda)^t + \Omega_t(\Omega_s(Z^\lambda)).$$

Therefore:

$$Z^\lambda * K_{st}v_{\mathbb{1}} = B_{st} + \alpha_t(s.\lambda)B_s + \alpha_s(\lambda)B_t + \dots \text{ and } Z^\lambda * K_{ts}v_{\mathbb{1}} = B_{ts} + \alpha_t(\lambda)B_s + \alpha_s(t.\lambda)B_t + \dots$$

Suppose that $z = st$ and $r = s$. Let $\lambda \in Y$ such that $\alpha_s(\lambda) \neq 0$ and $\alpha_t(\lambda) = 0$. Then if $a_{st}, a_{ts} \in \mathbb{C}$, the coefficient in front of $B_{rz} = B_t$ of $(Z^\lambda - 1)(a_{st}K_{st} + a_{ts}K_{ts})v_{\mathbb{1}}$ is $a_{st}\alpha_s(\lambda) + a_{ts}\alpha_s(t.\lambda) = (a_{st} + a_{ts})\alpha_s(\lambda)$. If the order of st is greater or equal to three, then $st \neq ts$ and $(a_{st} + a_{ts})\alpha_s(\lambda)$ is not a multiple of $\sum_{r_1rz>rz} a_{r_1rz}g_{r_1}(Z^\lambda - 1)$. The same kind of argument is used in the proof of Lemma 4.5, which implies Corollary 4.6.

A Existence of one dimensional representations of ${}^{\mathrm{BL}}\mathcal{H}_{\mathbb{C}}$

In this section, we prove the existence of one dimensional representations of ${}^{\mathrm{BL}}\mathcal{H}_{\mathbb{C}}$, when $\sigma_s = \sigma'_s = \sigma$, for all $s \in \mathscr{S}$.

Lemma A.1. *Assume that $\mathcal{F} = \mathbb{C}$ and that there exists $\sigma \in \mathbb{C}$ such that $\sigma_s = \sigma'_s = \sigma$ for all $s \in \mathscr{S}$ and such that $|\sigma| \neq 1$. Let $\epsilon \in \{-1, 1\}$ and $\tau \in T_{\mathbb{C}}$ be such that $\tau(\alpha_s^\vee) = \sigma^{2\epsilon}$ for all $s \in \mathscr{S}$. Then I_τ admits a unique maximal proper submodule M. Moreover, $I_\tau = M \oplus \mathbb{C}1 \otimes_\tau 1$ and if $x \in I_\tau/M$, then $Z^\lambda.x = \tau(\lambda).x$ and $H_w.x = (\epsilon\sigma^\epsilon)^{\ell(w)}.x$ for all $(w, \lambda) \in W^v \times Y$.*

Proof. By Lemma 6.2, such a τ exists. Let $q = \sigma^2$. Let $\mathrm{ht} : Y \to \mathbb{Q}$ be a $\mathbb{Z}$-linear map such that $\mathrm{ht}(\alpha_s^\vee) = 1$ for all $s \in \mathscr{S}$. Then one has $\tau(\alpha^\vee) = q^{\epsilon \mathrm{ht}(\alpha^\vee)}$ for all $\alpha^\vee \in \Phi^\vee$.

Let $s \in \mathscr{S}$. With the same notation as in Lemma 5.4, let $\phi_s = \phi(s.\tau, \tau) : I_{s.\tau} \to I_\tau$. Then by Lemma 5.4 $M_s := \mathrm{Im}(\phi_s)$ is a proper submodule of I_τ. Moreover, $H_s - \epsilon\sigma^\epsilon \otimes_\tau 1 \in M_s$. Let $M = \sum_{s\in\mathscr{S}} M_s$. Let $w \in W^v \setminus \{1\}$ and $w = s_1 \dots s_k$ be a reduced expression. Let $v = ws_k$. Then $H_v.(H_{s_k} - \epsilon\sigma^\epsilon) = H_w - \epsilon\sigma^\epsilon H_v \in M_{s_k}$. Therefore, for all $w \in W^v \setminus \{1\}$, there exists $x_w \in M$ such that $\pi_w^H(x_w) = 1$ and $x_w \in M \cap I_\tau^{\leq w}$. By induction on $\ell(w)$ we deduce that $M + \mathbb{C}1 \otimes_\tau 1 = I_\tau$.

By [GR14, Lemma 2.4 a)], $\tau \in T_{\mathbb{C}}^{\mathrm{reg}}$. Moreover, by Proposition 3.5 (2), $I_\tau = \bigoplus_{w\in W^v} I_\tau(w.\tau)$ and if we choose $\xi_v \in I_\tau(v.\tau) \setminus \{0\}$ for all $v \in W^v$, then $(\xi_v)_{v\in W^v}$ is a basis of I_τ. For $w \in W^v$, let $\pi_w^\xi : I_\tau \to \mathbb{C}$ be the linear map defined by $\pi_w^\xi(\xi_v) = \delta_{v,w}$ for all $v \in W^v$. As $\xi_1 \in \mathbb{C}1 \otimes_\tau 1$, one has $\pi_1^\xi(M_s) = \{0\}$ for all $s \in \mathscr{S}$. Thus $I_\tau = M \oplus \mathbb{C}1 \otimes_\tau 1$. Moreover, $M \subset (\pi_1^\xi)^{-1}(\{0\})$ and by dimension $M = \pi_1^\xi(\{0\})$. We deduce that M is the unique maximal proper submodule of I_τ and the lemma follows. □

B Examples of possibilities for W_τ for size 2 Kac-Moody matrices

In this section, we prove that there exist size 2 Kac-Moody matrices such that for each subgroup H of W^v, there exist $\tau \in T_{\mathbb{C}}$ such that W_τ is isomorphic to H. We assume that $\alpha_s(Y) = \mathbb{Z}$ for all $s \in \mathscr{S}$ and thus $W_{(\tau)} = W_\tau$. We already proved the existence of regular

elements in Lemma 6.1. If $\tau \in T_{\mathbb{C}}$ is such that $\tau(\alpha_{s_1}^\vee) = 1$ and $\tau(\alpha_{s_2}^\vee)$ is not a root of 1, then $W_\tau = \{1, s_1\}$.

Lemma B.1. *Let $A = (a_{i,j})_{(i,j)\in[\![1,2]\!]^2}$ be a Kac-Moody matrix. Assume that $a_{1,2}$ and $a_{2,1}$ are even and such that $a_{1,2}a_{2,1}$ is greater than 6. Let γ_2 be a primitive $\frac{1}{2}(a_{1,2}a_{2,1}-4)$-th root of 1. Let $\gamma_1 = \gamma_2^{\frac{1}{2}a_{1,2}}$. Let $\tau : Y = \mathbb{Z}\alpha_1^\vee \oplus \mathbb{Z}\alpha_2^\vee \to \mathbb{C}^*$ be the group morphism defined by $\tau(\alpha_i^\vee) = \gamma_i$ for both $i \in \{1,2\}$. Then $W_\tau = \langle s_1 s_2 \rangle \simeq \mathbb{Z}$.*

Proof. Let $\tau' \in T_{\mathbb{C}}$ and $\gamma_i' = \tau'(\alpha_i^\vee)$ for both $i \in \{1,2\}$. For $\lambda \in Y$, one has $(s_2 - s_1).\lambda = \alpha_1(\lambda)\alpha_1^\vee - \alpha_2(\lambda)\alpha_2^\vee$. Thus

$$\begin{aligned} s_1.\tau' = s_2.\tau' &\iff \forall \lambda \in Y, \tau'(\alpha_1(\lambda)\alpha_1^\vee - \alpha_2(\lambda)\alpha_2^\vee) = 1 \\ &\iff \forall \lambda \in Y, \gamma_1'^{\alpha_1(\lambda)} = \gamma_2'^{\alpha_2(\lambda)} \\ &\iff (\gamma_1')^2 = (\gamma_2')^{a_{1,2}} \text{ and } (\gamma_2')^2 = (\gamma_1')^{a_{2,1}}. \end{aligned}$$

Thus $s_1.s_2.\tau = \tau$. Moreover $s_2.\tau \neq \tau$ and hence $W_\tau = \langle s_1 s_2 \rangle$. □

If $\tau = \mathbb{1} : Y \to \{1\}$, then $W_\tau = 1$. The following lemma proves that W_τ can be a proper subgroup of W^v isomorphic to the infinite dihedral group.

Lemma B.2. *Let $A = (a_{i,j})_{(i,j)\in[\![1,2]\!]^2}$ be an irreducible Kac-Moody matrix which is not a Cartan matrix. One has $a_{1,2}a_{2,1} \geq 4$ and maybe considering ${}^tA$, one may assume $a_{1,2} \leq -2$. Write $W^v = \langle s_1, s_2 \rangle$. Let γ_2 be an $a_{1,2}$-th primitive root of 1 and $\tau \in T_{\mathbb{C}}$ be defined by $\tau(\alpha_{s_1}^\vee) = 1$ and $\tau(\alpha_{s_2}^\vee) = \gamma_2$. Then $W_\tau = \langle s_1, s_2 s_1 s_2 \rangle$.*

Proof. Let $\widetilde{\tau} = s_2.\tau$. Let us prove that $s_1.\widetilde{\tau} = \widetilde{\tau}$, i.e that $\widetilde{\tau}(\alpha_{s_1}^\vee) = 1$. One has $\widetilde{\tau}(\alpha_{s_1}^\vee) = \tau(s_2.\alpha_{s_1}^\vee) = \tau(\alpha_{s_1}^\vee - \alpha_{s_2}(\alpha_{s_1}^\vee)\alpha_{s_2}^\vee) = \tau(\alpha_{s_2}^\vee)^{-a_{1,2}} = 1$. Thus $W_\tau \ni \{s_1, s_2 s_1 s_2\}$. Therefore $W^v/W_\tau = \{W_\tau, t.W_\tau\}$. Moreover $t \notin W_\tau$, thus $[W^v : W_\tau] = 2$ and hence $W_\tau = \langle s_1, s_2 s_1 s_2 \rangle$. □

Index

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