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# Principal series representations of Iwahori-Hecke algebras for Kac-Moody groups over local fields

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## Abstract

Recently, Iwahori-Hecke algebras were associated to Kac-Moody groups over non-Archimedean local fields. We introduce principal series representations for these algebras. We study these representations and partially generalize Kato and Matsumoto irreducibility criteria.

## 1 Introduction

### 1.1 The reductive case

Let  $G$  be a reductive group over a non-Archimedean local field  $\mathcal{K}$ . To each open compact subgroup  $K$  of  $G$  is associated a Hecke algebra  $\mathcal{H}_K$ . There exists a strong link between the smooth representations of  $G$  and the representations of the Hecke algebras of  $G$ . Let  $I$  be the Iwahori subgroup of  $G$ . Then the Hecke algebra  $\mathcal{H}_I$  is called the Iwahori-Hecke algebra of  $G$  and plays an important role in the representation theory of  $G$ . Its representations have been extensively studied. Let  $Y$  be the cocharacter lattice of  $G$  and  $W^v$  be the Weyl group of  $G$ . Then by the Bernstein-Lusztig relations,  $\mathcal{H}_I$  admits a basis  $(Z^\lambda H_w)_{\lambda \in Y, w \in W^v}$  such that  $\bigoplus_{\lambda \in Y} \mathbb{C}Z^\lambda$  is a subalgebra of  $\mathcal{H}_I$  isomorphic to the group algebra  $\mathbb{C}[Y]$  of  $Y$ . We identify  $\bigoplus_{\lambda \in Y} \mathbb{C}Z^\lambda$  and  $\mathbb{C}[Y]$ . Let  $\chi \in T_{\mathbb{C}} = \text{Hom}(Y, \mathbb{C}^*)$ . Then  $\chi$  induces a representation  $\chi : \mathbb{C}[Y] \rightarrow \mathbb{C}$ . Inducing  $\chi$  to  $\mathcal{H}_I$ , one gets a representation  $I_\chi$  of  $\mathcal{H}_I$ . These representations were introduced by Matsumoto in [Mat77] and are called **principal series representations**. We refer to [Sol09, Section 3.2] for a survey on this subject.

Matsumoto and Kato gave criterion for the irreducibility of  $I_\chi$ . Let  $W^v$  be the vectorial (i.e finite) Weyl group of  $G$ . Then  $W^v$  acts on  $Y$  and thus it acts on  $T_{\mathbb{C}}$ . If  $\chi \in T_{\mathbb{C}}$ , we denote by  $W_\chi$  the fixer of  $\chi$  in  $W^v$ . Let  $\Phi^\vee$  be the coroot lattice of  $G$ . Let  $q$  be the residue cardinal of  $\mathcal{K}$ . Then Kato proved the following theorem (see [Kat81, Theorem 2.4]):

**Theorem 1.** Let  $\chi \in T_{\mathbb{C}}$ . Then  $I_\chi$  is irreducible if and only if it satisfies the following conditions:

1.  $W_\chi$  is generated by the reflections that it contains,
2. for all  $\alpha^\vee \in \Phi^\vee$ ,  $\chi(\alpha^\vee) + \chi^{-1}(\alpha^\vee) \neq q^{\frac{1}{2}} + q^{-\frac{1}{2}}$ .

When  $\chi$  is **regular**, that is when  $W_\chi = \{1\}$ , condition 1 is satisfied and this is a result by Matsumoto (see [Mat77, Théorème 4.3.5]).

## 1.2 The Kac-Moody case

Let  $G$  be a split Kac-Moody group over a non-Archimedean local field  $\mathcal{K}$ . There is up to now no definition of smoothness for the representations of  $G$ . However one can define certain Hecke algebras in this framework. In [BK11] and [BKP16], Braverman and Kazhdan and Patnaik defined the spherical Hecke algebra and the Iwahori-Hecke  $\mathcal{H}$  of  $G$  when  $G$  is affine. Bardy-Panse, Gaussen and Rousseau generalized these constructions to the case where  $G$  is a general Kac-Moody group. They achieved this construction by using masures (also known as hovels), which are an analogue of Bruhat-Tits buildings (see [GR08]). Together with Abdellatif, we attached in [AH17] Hecke algebras to subgroups slightly more general than the Iwahori subgroup.

Let  $Y$  be the cocharacter lattice of  $G$  and  $W^v$  be the Weyl group of  $G$ . The Iwahori-Hecke algebra  $\mathcal{H}_{\mathbb{C}}$  of  $G$  admits a Bernstein-Lusztig presentation but it is no more indexed by  $Y$ . Let  $\mathcal{T} \subset \mathbb{A} = Y \otimes \mathbb{R}$  be the Tits cone of  $G$ . Then  $\mathcal{T}$  is a convex cone and it satisfies  $\mathcal{T} = \mathbb{A}$  if and only if  $G$  is reductive. Then  $\mathcal{H}_{\mathbb{C}}$  can be embedded in an algebra  ${}^{\text{BL}}\mathcal{H}_{\mathbb{C}}$  called the **Bernstein-Lusztig-Hecke algebra of  $G$** . The algebra  ${}^{\text{BL}}\mathcal{H}_{\mathbb{C}}$  admits a basis  $(Z^\lambda H_w)_{\lambda \in Y, w \in W^v}$  such that  $\bigoplus_{\lambda \in Y} \mathbb{C}Z^\lambda$  is isomorphic to the group algebra  $\mathbb{C}[Y]$  of  $Y$ . We identify  $\bigoplus_{\lambda \in Y} \mathbb{C}Z^\lambda$  and  $\mathbb{C}[Y]$ . Then  $\mathcal{H}_{\mathbb{C}}$  is isomorphic to the subalgebra  $\bigoplus_{w \in W^v, \lambda \in Y^+} \mathbb{C}Z^\lambda H_w$ . Let  $\chi \in T_{\mathbb{C}} = \text{Hom}(Y, \mathbb{C}^*)$ . Then  $\chi$  induces a map  $\chi : \mathbb{C}[Y] \rightarrow \mathbb{C}$  and we can define the representation  $I_\chi$  of  ${}^{\text{BL}}\mathcal{H}_{\mathbb{C}}$  induced by  $\chi$ . By restriction, this also defines a representation  $I_\chi^+$  of  $\mathcal{H}_{\mathbb{C}}$ . As  $I_\chi$  admits a basis indexed by the Weyl group of  $G$ ,  $I_\chi$  is infinite dimensional unless  $G$  is reductive. The aim of this paper is to study these representations and in particular to study their irreducibility. As we shall see (Lemma 2.5),  $I_\chi$  is irreducible if and only if  $I_\chi^+$  is irreducible and we will mainly study  $I_\chi$ . We prove the following theorem, generalizing Matsumoto irreducibility criterion (see Corollary 4.5):

**Theorem 2.** Let  $\chi$  be a regular character. Then  $I_\chi$  is irreducible if and only if for all  $\alpha \in \Phi^\vee$ ,

$$\chi(\alpha^\vee) + \chi^{-1}(\alpha^\vee) \neq q^{\frac{1}{2}} + q^{-\frac{1}{2}}.$$

We also generalize one implication of Kato's criterion (see Theorem 4.7):

**Theorem 3.** Let  $\chi \in T_{\mathbb{C}}$ . Assume that  $I_\chi$  is irreducible. Then:

1.  $W_\chi$  is generated by the reflections that it contains,
2. for all  $\alpha^\vee \in \Phi^\vee$ ,  $\chi(\alpha^\vee) + \chi^{-1}(\alpha^\vee) \neq q^{\frac{1}{2}} + q^{-\frac{1}{2}}$ .

We then check the irreducibility of  $I_\chi$  for some particular  $\chi \in T_{\mathbb{C}}$  satisfying 1 and 2: when  $W_\chi$  is generated by one reflection, see Proposition 4.8 and when  $\chi(\lambda) = 1$  for all  $\lambda \in Y$ , when the Kac-Moody matrix defining  $G$  is of size 2, see Theorem 4.13).

**Frameworks** Actually, following [BPGR16] we study Iwahori-Hecke algebras associated to abstract masures. In particular our results also apply when  $G$  is an almost-split Kac-Moody group over a non-Archimedean local field.

**Organization of the paper** In Section 2, we recall the definition of the Iwahori-Hecke algebras and of the Bernstein-Lusztig-Hecke algebras and introduce principal series representation.

In Section 3, we study the  $\mathbb{C}[Y]$ -module induced by  $I_\chi$  by restriction and we study the intertwining operators from  $I_\chi$  to  $I_{\chi'}$ , for  $\chi, \chi' \in T_{\mathbb{C}}$ .

In Section 4, we establish an irreducibility criterion for  $I_\chi$  (see Theorem 4.2). We then apply it to obtain Theorem 2 and Theorem 3.

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## 2 Iwahori-Hecke algebras

Let  $G$  be a Kac-Moody group over a non-archimedean local field. Then Gaussent and Rousseau constructed a space  $\mathcal{I}$ , called a mesure on which  $G$  acts, generalizing the construction of the Bruhat-Tits buildings (see [GR08], [Rou16] and [Rou17]). Rousseau then defined in [Rou11] an axiomatic definition of mesures inspired by the axiomatic definition of Bruhat-Tits buildings. We simplified it in [Héb17]. Mesures satisfying these axiomatics are called abstract mesures because they might not be associated with some Kac-Moody group.

In [BPGR16], Bardy-Panse, Gaussent and Rousseau attached an Iwahori-Hecke algebra  $\mathcal{H}_{\mathcal{R}}$  to each abstract mesure satisfying certain conditions and to each ring  $\mathcal{R}$ . The algebra  $\mathcal{H}_{\mathcal{R}}$  is an algebra of functions defined on some pairs of chambers of the mesure, equipped with a convolution product. Then they prove that under some additional hypothesis on the ring  $\mathcal{R}$  (which are satisfied by  $\mathbb{R}$  and  $\mathbb{C}$ ),  $\mathcal{H}_{\mathcal{R}}$  admits a Bernstein-Lusztig presentation. In this paper, we will only use the Bernstein-Lusztig presentation of  $\mathcal{H}_{\mathcal{R}}$  and we do not introduce mesures (see [Héb18, Appendix A] for a definition). We however introduce the standard apartment of a mesure.

## 2.1 Standard apartment of a masure

A **Kac-Moody matrix** (or generalized Cartan matrix) is a square matrix  $A = (a_{i,j})_{i,j \in I}$  indexed by a finite set  $I$ , with integral coefficients, and such that :

$$(i) \quad \forall i \in I, a_{i,i} = 2;$$

$$(ii) \quad \forall (i, j) \in I^2, (i \neq j) \Rightarrow (a_{i,j} \leq 0);$$

$$(iii) \quad \forall (i, j) \in I^2, (a_{i,j} = 0) \Leftrightarrow (a_{j,i} = 0).$$

A **root generating system** is a 5-tuple  $\mathcal{S} = (A, X, Y, (\alpha_i)_{i \in I}, (\alpha_i^\vee)_{i \in I})$  made of a Kac-Moody matrix  $A$  indexed by the finite set  $I$ , of two dual free  $\mathbb{Z}$ -modules  $X$  and  $Y$  of finite rank, and of a free family  $(\alpha_i)_{i \in I}$  (respectively  $(\alpha_i^\vee)_{i \in I}$ ) of elements in  $X$  (resp.  $Y$ ) called **simple roots** (resp. **simple coroots**) that satisfy  $a_{i,j} = \alpha_j(\alpha_i^\vee)$  for all  $i, j$  in  $I$ . Elements of  $X$  (respectively of  $Y$ ) are called **characters** (resp. **cocharacters**).

Fix such a root generating system  $\mathcal{S} = (A, X, Y, (\alpha_i)_{i \in I}, (\alpha_i^\vee)_{i \in I})$  and set  $\mathbb{A} := Y \otimes \mathbb{R}$ . Each element of  $X$  induces a linear form on  $\mathbb{A}$ , hence  $X$  can be seen as a subset of the dual  $\mathbb{A}^*$ . In particular, the  $\alpha_i$ 's (with  $i \in I$ ) will be seen as linear forms on  $\mathbb{A}$ . This allows us to define, for any  $i \in I$ , an involution  $r_i$  of  $\mathbb{A}$  by setting  $r_i(v) := v - \alpha_i(v)\alpha_i^\vee$  for any  $v \in \mathbb{A}$ . Let  $\mathcal{S} = \{r_i | i \in I\}$  be the (finite) set of **simple reflections**. One defines the **Weyl group of  $\mathcal{S}$**  as the subgroup  $W^v$  of  $\text{GL}(\mathbb{A})$  generated by  $\mathcal{S}$ . The pair  $(W^v, \mathcal{S})$  is a Coxeter system, hence we can consider the length  $\ell(w)$  with respect to  $\mathcal{S}$  of any element  $w$  of  $W^v$ . If  $s \in \mathcal{S}$ ,  $s = r_i$  for some unique  $i \in I$ . We set  $\alpha_s = \alpha_i$  and  $\alpha_s^\vee = \alpha_i^\vee$ .

There is an action of the Weyl group  $W^v$  on  $\mathbb{A}^*$  given by the following formula:

$$\forall x \in \mathbb{A}, w \in W^v, \alpha \in \mathbb{A}^*, (w.\alpha)(x) := \alpha(w^{-1}.x).$$

Let  $\Phi := \{w.\alpha_i | (w, i) \in W^v \times I\}$  be the set of **real roots**: then  $\Phi$  is a subset of the **root lattice**  $Q := \bigoplus_{i \in I} \mathbb{Z}\alpha_i$ .

As in the reductive case, define the **fundamental chamber** as  $C_f^v := \{v \in \mathbb{A} \mid \forall s \in \mathcal{S}, \alpha_s(v) > 0\}$ .

Let  $\mathcal{T} := \bigcup_{w \in W^v} w.\overline{C_f^v}$  be the **Tits cone**. This is a convex cone.

One sets  $Y^+ = Y \cap \mathcal{T}$ .

**Remark 2.1.** By [Kac94, §4.9] and [Kac94, § 5.8] the following conditions are equivalent:

1. the Kac-Moody matrix  $A$  is of finite type (i.e. is a Cartan matrix),
2.  $\mathbb{A} = \mathcal{T}$
3.  $W^v$  is finite.

## 2.2 Iwahori-Hecke algebras

Let us first recall briefly the construction of the Iwahori-Hecke algebra via its Bernstein-Lusztig presentation, as done in [BPGR16, Section 6.6].

Let  $\mathcal{R}_1 = \mathbb{Z}[(\sigma_s)_{s \in \mathcal{S}}, (\sigma'_s)_{s \in \mathcal{S}}]$ , where  $(\sigma_s)_{s \in \mathcal{S}}, (\sigma'_s)_{s \in \mathcal{S}}$  are two families of indeterminates satisfying the following relations:

- if  $\alpha_s(Y) = \mathbb{Z}$ , then  $\sigma_s = \sigma'_s$ ;

- if  $s, t \in \mathcal{S}$  are conjugate (i.e. such that  $\alpha_s(\alpha_t^\vee) = \alpha_t(\alpha_s^\vee) = -1$ ), then  $\sigma_s = \sigma_t = \sigma'_s = \sigma'_t$ .

To define the Iwahori-Hecke algebra  $\mathcal{H}_{\mathcal{R}_1}$  associated with  $\mathbb{A}$  and  $(\sigma_s, \sigma'_s)_{s \in \mathcal{S}}$ , we first introduce the Bernstein-Lusztig-Hecke algebra. Let  ${}^{BL}\mathcal{H}_{\mathcal{R}_1}$  be the free  $\mathcal{R}_1$ -vector-space with basis  $(Z^\lambda H_w)_{\lambda \in Y, w \in W^v}$ . For short, one sets  $H_w = Z^0 H_w$  for  $w \in W^v$  and  $Z^\lambda = Z^\lambda H_1$  for  $\lambda \in Y$ . The **Bernstein-Lusztig-Hecke algebra**  ${}^{BL}\mathcal{H}_{\mathcal{R}_1}$  is the module  ${}^{BL}\mathcal{H}_{\mathcal{R}_1}$  equipped with the unique product  $*$  that turns it into an associative algebra and satisfies the following relations (known as the **Bernstein-Lusztig relations**):

- (BL1)  $\forall (\lambda, w) \in Y \times W^v, Z^\lambda * H_w = Z^\lambda H_w$ ;
- (BL2)  $\forall s \in \mathcal{S}, \forall w \in W^v, H_s * H_w = \begin{cases} H_{sw} & \text{if } \ell(sw) = \ell(w) + 1 \\ (\sigma_s - \sigma_s^{-1})H_w + H_{sw} & \text{if } \ell(sw) = \ell(w) - 1 \end{cases}$ ;
- (BL3)  $\forall (\lambda, \mu) \in Y^2, Z^\lambda * Z^\mu = Z^{\lambda+\mu}$ ;
- (BL4)  $\forall \lambda \in Y, \forall i \in I, H_s * Z^\lambda - Z^{s \cdot \lambda} * H_s = Q_s(Z)(Z^\lambda - Z^{s \cdot \lambda})$ , where  $Q_s(Z) = \frac{(\sigma_s - \sigma_s^{-1}) + (\sigma'_s - \sigma_s'^{-1})Z^{-\alpha_s^\vee}}{1 - Z^{-2\alpha_s^\vee}}$

The existence and unicity of such a product  $*$  comes from [BPGR16, Theorem 6.2].

**Definition 2.2.** Let  $\mathcal{R}$  be an integral domain containing  $\mathbb{Z}$  and  $f : \mathcal{R}_1 \rightarrow \mathcal{R}$  be a ring morphism such that  $f(\sigma_s)$  and  $f(\sigma'_s)$  are invertible in  $\mathcal{R}$  for all  $s \in \mathcal{S}$ . Then the **Bernstein-Lusztig-Hecke algebra of**  $(\mathbb{A}, (\sigma_s)_{s \in \mathcal{S}}, (\sigma'_s)_{s \in \mathcal{S}})$  over  $\mathcal{R}$  is the algebra  ${}^{BL}\mathcal{H}_{\mathcal{R}} = {}^{BL}\mathcal{H}_{\mathcal{R}_1} \otimes_{\mathcal{R}_1} \mathcal{R}$ . Following [BPGR16, Section 6.6], the **Iwahori-Hecke algebra**  $\mathcal{H}_{\mathcal{R}}$  associated with  $\mathcal{S}$  and  $(\sigma_s, \sigma'_s)_{s \in \mathcal{S}}$  is now defined as the  $\mathcal{R}$ -subalgebra of  ${}^{BL}\mathcal{H}_{\mathcal{R}}$  spanned by  $(Z^\lambda H_w)_{\lambda \in Y^+, w \in W^v}$  (recall that  $Y^+ = Y \cap \mathcal{T}$  with  $\mathcal{T}$  being the Tits cone). Note that for  $G$  reductive, we recover the usual Iwahori-Hecke algebra of  $G$ , since  $Y \cap \mathcal{T} = Y$ .

**Remark 2.3.** 1. Let  $s \in \mathcal{S}$ . Then if  $\sigma_s = \sigma'_s$ ,  $Q_s(Z) = \frac{(\sigma_s - \sigma_s^{-1})}{1 - Z^{-\alpha_s^\vee}}$ .

2. Let  $s \in \mathcal{S}$  and  $\lambda \in Y$ . Then  $Q_s(Z)(Z^\lambda - Z^{s \cdot \lambda}) \in \mathcal{R}[Y]$ . Indeed,  $Q_s(Z)(Z^\lambda - Z^{s \cdot \lambda}) = Q_s(Z) \cdot Z^\lambda (1 - Z^{-\alpha_s(\lambda)\alpha_s^\vee})$ . Assume that  $\sigma_s = \sigma'_s$ . Then

$$\frac{1 - Z^{-\alpha_s(\lambda)\alpha_s^\vee}}{1 - Z^{-\alpha_s^\vee}} = \begin{cases} \sum_{j=0}^{\alpha_s(\lambda)-1} Z^{-j\alpha_s^\vee} & \text{if } \alpha_s(\lambda) \geq 0 \\ -Z^{\alpha_s^\vee} \sum_{j=0}^{-\alpha_s(\lambda)-1} Z^{j\alpha_s^\vee} & \text{if } \alpha_s(\lambda) \leq 0, \end{cases}$$

and thus  $Q_s(Z)(Z^\lambda - Z^{s \cdot \lambda}) \in \mathcal{R}[Y]$ . Assume  $\sigma'_s \neq \sigma_s$ . Then  $\alpha_s(\mathbb{Z}) = 2\mathbb{Z}$  and a similar computation enables to conclude.

3. From (BL4) we deduce that for all  $s \in \mathcal{S}, \lambda \in Y$ ,

$$Z^\lambda * H_s - H_s * Z^{s \cdot \lambda} = Q_s(Z)(Z^\lambda - Z^{s \cdot \lambda}).$$

4. When  $G$  is a split Kac-Moody group over a non-Archimedean local  $\mathcal{K}$  with residue cardinal  $q$ , we can choose  $\mathcal{R}$  to be a ring containing  $\mathbb{Z}[\sqrt{q}^{\pm 1}]$  and take  $f(\sigma_s) = f(\sigma'_s) = \sqrt{q}$  for all  $s \in \mathcal{S}$ .

5. By (BL2), the family  $(H_w * Z^\lambda)_{w \in W^v, \lambda \in Y}$  is also a basis of  ${}^{BL}\mathcal{H}_{\mathcal{R}}$ .

**Definition 2.4.** Let  $\mathcal{H}_{\mathcal{R}, W^v} = \bigoplus_{w \in W^v} \mathcal{R}H_w \subset \mathcal{H}_{\mathcal{R}}$ . Then  $\mathcal{H}_{\mathcal{R}, W^v}$  is a subalgebra of  $\mathcal{H}_{\mathcal{R}}$ . This is the Hecke algebra of the Coxeter group  $(W^v, \mathcal{S})$ .

## 2.3 Principal series representations

We now fix  $(\mathbb{A}, (\sigma_s)_{s \in \mathcal{S}}, (\sigma'_s)_{s \in \mathcal{S}})$  as in Subsection 2.2 and a ring  $\mathcal{R}$  as in Definition 2.2. Let  $\mathcal{H}_{\mathcal{R}}$  and  ${}^{\text{BL}}\mathcal{H}_{\mathcal{R}}$  be the Iwahori-Hecke and the Bernstein-Lusztig Hecke algebras of  $(\mathbb{A}, (\sigma_s)_{s \in \mathcal{S}}, (\sigma'_s)_{s \in \mathcal{S}})$  over  $\mathcal{R}$ .

Set  $T_{\mathcal{R}} = \text{Hom}(Y, \mathcal{R}^\times)$ . Let  $\chi \in T_{\mathcal{R}}$ . Then  $\chi$  induces a morphism of algebra  $\chi : \mathcal{R}[Y] \rightarrow \mathcal{R}$  by the formula  $\chi(\sum_{y \in Y} a_y e^y) = \sum_{y \in Y} a_y \chi(y)$ , for  $\sum a_y e^y \in \mathcal{R}[Y]$ . This equips  $\mathcal{R}$  with a structure of an  $\mathcal{R}[Y]$ -module.

Let  $I_\chi = \text{Ind}^{{}^{\text{BL}}\mathcal{H}_{\mathcal{R}}}(\chi) = {}^{\text{BL}}\mathcal{H}_{\mathcal{R}} \otimes_{\mathcal{R}[Y]} \mathcal{R}$ . For example if  $\lambda \in Y$  and  $s \in \mathcal{S}$ , one has  $Z^\lambda \cdot 1 \otimes_\chi 1 = \chi(\lambda) 1 \otimes_\chi 1$  and

$$Z^\lambda \cdot H_s \otimes_\chi 1 = H_s * Z^{s, \lambda} \otimes_\chi 1 + Q_s(Z)(Z^\lambda - Z^{s, \lambda}) \otimes_\chi 1 = \chi(s, \lambda) H_s \otimes_\chi 1 + \chi(Q_s(Z)(Z^\lambda - Z^{s, \lambda})) \otimes_\chi 1.$$

Let  $h \in I_\chi$ . Write  $h = \sum_{\lambda \in Y, w \in W^v} h_{w, \lambda} H_w Z^\lambda \otimes_\chi c_{w, \lambda}$ , where  $(h_{w, \lambda}), (c_{w, \lambda}) \in \mathcal{R}^{(W^v \times Y)}$ , which is possible by Remark 2.3. Thus

$$h = \sum_{\lambda \in Y, w \in W^v} h_{w, \lambda} c_{w, \lambda} \chi(\lambda) H_w \otimes_\chi 1 = \left( \sum_{\lambda \in Y, w \in W^v} h_{w, \lambda} c_{w, \lambda} \chi(\lambda) H_w \right) 1 \otimes_\chi 1.$$

Thus  $I_\chi$  is a principal  ${}^{\text{BL}}\mathcal{H}_{\mathcal{R}}$ -module and  $I_\chi$  is a free  $\mathcal{R}$ -module with basis  $(H_w \otimes_\chi 1)_{w \in W^v}$ . If moreover  $\mathcal{R} = \mathcal{F}$  is a field  $I_\chi = \mathcal{H}_{W^v, \mathcal{F}} \cdot 1 \otimes_\chi 1$  (see Definition 2.4 for the definition of  $\mathcal{H}_{W^v, \mathcal{F}}$ ).

Assume that  $\mathcal{R} = \mathcal{F}$  is a field. Let  $\chi \in T_{\mathcal{F}}$ . Then  $I_\chi$  induces a representation  $I_\chi^+$  of  $\mathcal{H}_{\mathcal{F}}$  by restriction. As a vector space, one has  $I_\chi = I_\chi^+$ . By the following lemma, it suffices to study the irreducibility of  $I_\chi$

**Lemma 2.5.** *Let  $\chi \in T_{\mathcal{F}}$ . Then  $I_\chi$  is irreducible if and only if  $I_\chi^+$  is irreducible.*

*Proof.* Assume that  $I_\chi^+$  is irreducible. Let  $V \subset I_\chi$  be a  ${}^{\text{BL}}\mathcal{H}_{\mathcal{F}}$ -module. Then  $V$  is also an  $\mathcal{H}_{\mathcal{F}}$ -module and thus  $V = \{0\}$  or  $V = I_\chi = I_\chi^+$ . Hence  $I_\chi$  is irreducible.

Assume that  $I_\chi$  is irreducible. Let  $V^+ \subset I_\chi^+$ ,  $V^+ \neq \{0\}$  be an  $\mathcal{H}_{\mathcal{F}}$ -submodule. Let  $x \in V^+ \setminus \{0\}$ . Then  ${}^{\text{BL}}\mathcal{H}_{\mathcal{F}} \cdot x = I_\chi$  and thus there exists  $a \in {}^{\text{BL}}\mathcal{H}_{\mathcal{F}}$  such that  $a \cdot x = 1 \otimes_\chi 1$ . For  $\lambda \in C_{\mathcal{F}}^v$  sufficiently dominant,  $Z^\lambda \cdot a \in \mathcal{H}_{\mathcal{F}}$  and thus  $\frac{1}{\chi(\lambda)} Z^\lambda \cdot a \in \mathcal{H}_{\mathcal{F}}$ . Consequently  $1 \otimes_\chi 1 = \frac{1}{\chi(\lambda)} Z^\lambda \cdot a \cdot x \in V^+$  and hence  $V^+ = I_\chi^+$ :  $I_\chi^+$  is irreducible.  $\square$

## 2.4 Sketch of the proof of irreducibility criteria

Our proof of irreducibility criteria is based the fact that  $\mathbb{C}[Y] \cdot x$  is finite dimensional if  $x \in I_\chi$  (see Lemma 3.1) and on the following well known result.

**Theorem 2.6.** *(Frobenius) Let  $\mathcal{F}$  be a field,  $V$  be a finite dimensional vector space over  $\mathcal{F}$  and  $G \subset \text{GL}(V)$  be a commutative subgroup. Assume that for all  $g \in G$ ,  $g$  is triangularizable. Then there exists a basis  $B$  of  $V$  for which for every  $g \in G$ , the matrix of  $g$  in the basis  $B$  is triangular.*

Let us sketch our strategy to obtain irreducibility criteria for the  $I_\chi$ ,  $\chi \in T_{\mathbb{C}}$ .

Let  $M$  be a  ${}^{\text{BL}}\mathcal{H}_{\mathbb{C}}$ -module. For  $\chi \in T_{\mathbb{C}}$ , set  $M(\chi) = \{x \in M \mid Z^\lambda \cdot x = \chi(\lambda)x, \forall \lambda \in Y\}$ . One has  $\mathbb{C} \cdot 1 \otimes_\chi 1 \subset I_\chi(\chi)$  for all  $\chi \in T_{\mathbb{C}}$ .

Let  $\chi \in T_{\mathbb{C}}$ . Following Matsumoto, we define nontrivial intertwining operators  $\phi_w : I_\chi \rightarrow I_{w \cdot \chi}$  for all  $w \in W^v$  (see Proposition 3.12). If  $I_\chi$  is irreducible, these operators have to be isomorphisms. We prove that these operators are isomorphisms if and only if  $\chi$  is in some subset  $\mathcal{U}_{\mathbb{C}}$  of  $T_{\mathbb{C}}$  (see after Lemma 3.14 for the definition of  $\mathcal{U}_{\mathbb{C}}$  and see Lemma 3.16) and we deduce the condition (2) appearing in Theorem 1, 2 and 3.

We also prove that  $\{\chi' \in T_{\mathbb{C}} \mid I_{\chi}(\chi') \neq \{0\}\} = W^v \cdot \chi$  (see Proposition 3.13). If  $w \in W^v$   ${}^{\text{BL}}\mathcal{H}_{\mathbb{C}.1} \otimes_{w,\chi} 1 = I_{w,\chi}$ , and thus  $I_{w,\chi}$  is irreducible if and only if every  ${}^{\text{BL}}\mathcal{H}_{\mathbb{C}}$ -submodule of  $I_{w,\chi}$  contains  $1 \otimes_{w,\chi} 1$ . Using Frobenius Theorem and Schur Lemma, we deduce that  $I_{\chi}$  is irreducible if and only if  $\chi \in \mathcal{U}_{\mathbb{C}}$  and  $\dim I_{\chi}(\chi) = 1$  (see Theorem 4.2). We then apply this criterion to obtain Theorem 2 and Theorem 3.

### 3 Study of the $\mathcal{R}[Y]$ -module structure and of intertwining operators

Let  $\chi, \chi' \in T_{\mathcal{R}} = \text{Hom}(Y, \mathcal{R}^{\times})$ . Let  $M$  be a  ${}^{\text{BL}}\mathcal{H}_{\mathcal{R}}$ -module. For  $\chi \in T_{\mathcal{R}}$ , set  $M(\chi) = \{x \in M \mid Z^{\lambda} \cdot x = \chi(\lambda)x, \forall \lambda \in Y\}$ .

In this section, we study the morphisms of  ${}^{\text{BL}}\mathcal{H}_{\mathcal{R}}$ -modules from  $I_{\chi}$  to  $I_{\chi'}$ . We prove that when  $\mathcal{R} = \mathcal{F}$  is a field,  $\text{Hom}_{{}^{\text{BL}}\mathcal{H}_{\mathcal{F}}}(I_{\chi}, I_{\chi'}) \neq \{0\}$  implies  $\chi' \in W^v \cdot \chi$  (see Proposition 3.4). Reciprocally, we prove that  $\text{Hom}_{{}^{\text{BL}}\mathcal{H}_{\mathcal{F}}}(I_{\chi}, I_{w,\chi}) \neq \{0\}$  for all  $w \in W^v$  (see Proposition 3.12).

As we shall see,  $\text{Hom}_{{}^{\text{BL}}\mathcal{H}_{\mathcal{F}}}(I_{\chi}, I_{\chi'}) \simeq I_{\chi'}(\chi)$  (see Lemma 3.7). We thus study simultaneously the weight spaces  $I_{\chi}(w \cdot \chi)$  for  $\chi \in T_{\mathcal{F}}$ ,  $w \in W^v$  and the spaces  $\text{Hom}_{{}^{\text{BL}}\mathcal{H}_{\mathcal{F}}\text{-mod}}(I_{\chi}, I_{w,\chi})$ .

In Subsection 3.3 we prove that if  $I_{\chi}$  is irreducible, then  $I_{\chi}$  is isomorphic to  $I_{w,\chi}$  for all  $w \in W^v$ . We deduce that if  $I_{\chi}$  is irreducible, the values of  $\chi$  satisfy some conditions, see Lemma 3.16. This explains the condition (2) appearing in Theorems 1, 2 and 3 (see Remark 3.15).

The definition we gave for  $I_{\chi}$  is different from the definition of Matsumoto (see [Mat77, (4.1.5)]). It seems to be well known that these definitions are equivalent. We justify this equivalence in Subsection 3.4. We also explain why it seems difficult to adapt Kato's proof in our framework.

#### 3.1 Weights for the $\mathcal{R}[Y]$ -module structure

Let  $\chi \in T_{\mathcal{R}}$ . Let  $x \in I_{\chi}$ . Write  $x = \sum_{w \in W^v} x_w H_w \otimes_{\chi} 1$ , with  $(x_w) \in \mathcal{R}^{(W^v)}$ . Set  $\text{supp}(x) = \{w \in W^v \mid x_w \neq 0\}$ . Equip  $W^v$  with the Bruhat order. If  $w \in W^v$ , set  $[1, w] = \{v \in W^v \mid v \leq w\}$  and  $[1, w) = \{v \in W^v \mid v < w\}$ . If a finite set  $E$  is contained in  $W^v$ ,  $\max(E)$  is the set of elements of  $E$  that are maximal for the Bruhat order. Let  $R$  be a binary relation on  $W^v$  (for example  $R = \text{"}\leq\text{"}$ ,  $R = \text{"}\not\leq\text{"}$ , ...) and  $w \in W^v$ . One sets

$$I_{\chi}^{Rw} = \bigoplus_{v \in W^v \mid vRw} \mathcal{R}H_v \otimes_{\chi} 1 = \{x \in I_{\chi} \mid \text{supp}(x) \subset \{v \in W^v \mid vRw\}\} \subset I_{\chi},$$

and

$${}^{\text{BL}}\mathcal{H}_{\mathcal{R}}^{Rw} = \bigoplus_{v \in W^v \mid vRw, \lambda \in Y} \mathcal{R}H_v * Z^{\lambda} = \left\{ \sum_{v \in W^v, \lambda \in Y} a_{v,\lambda} H_v * Z^{\lambda} \mid a_{v,\lambda} \neq 0 \implies vRw \right\} \subset {}^{\text{BL}}\mathcal{H}_{\mathcal{R}}.$$

**Lemma 3.1.** *Let  $\lambda \in Y$  and  $w \in W^v$ . Then  $Z^{\lambda}H_w \in {}^{\text{BL}}\mathcal{H}_{\mathcal{R}}^{\leq w}$ . In particular, if  $\chi \in T_{\mathcal{R}}$ ,  $I_{\chi}^{\leq w}$  and  $I_{\chi}^{\not\leq w}$  are  $\mathcal{R}[Y]$ -submodules of  $I_{\chi}$ . In particular, if  $\mathcal{R} = \mathcal{F}$  is a field,  $\mathcal{F}[Y] \cdot x$  is finite dimensional for all  $x \in I_{\chi}$ .*

*Proof.* We prove it by induction on  $\ell(w)$ . If  $\ell(w) = 0$ , this is clear. Assume that  $\ell(w) > 0$  and that for all  $w' \in W^v$  such that  $\ell(w') < \ell(w)$ ,  $Z^{\lambda}H_{w'} \in {}^{\text{BL}}\mathcal{H}_{\mathcal{R}}^{\leq w'}$ . Write  $w = sw'$ , with  $s \in \mathcal{S}$  and  $w' \in W^v$  such that  $\ell(w') = \ell(w) - 1$ . Let  $\lambda \in Y$ . By (BL4), one has  $Z^{\lambda} * H_s = aH_s * Z^{s \cdot \lambda + b}$

for some  $a, b \in \mathcal{R}$ . Then  $Z^\lambda * H_s = (aH_s Z^{s.\lambda} + b) * H_{w'} \in H_s * {}^{\text{BL}}\mathcal{H}_{\mathcal{R}}^{\leq w'} + {}^{\text{BL}}\mathcal{H}_{\mathcal{F}}^{\leq w'}$  by the induction assumption. As  $s.[1, w'] \subset [1, w]$ , we deduce that  $H_s * {}^{\text{BL}}\mathcal{H}_{\mathcal{R}}^{\leq w'} + {}^{\text{BL}}\mathcal{H}_{\mathcal{R}}^{\leq w} \subset {}^{\text{BL}}\mathcal{H}_{\mathcal{R}}^{\leq w}$ , and the lemma follows.  $\square$

**Lemma 3.2.** *Let  $\lambda \in Y$  and  $w \in W^v$ . Then there exists  $u \in {}^{\text{BL}}\mathcal{H}_{\mathcal{R}}^{\leq w}$  such that  $Z^\lambda H_w = H_w Z^{w^{-1}.\lambda} + u$ . In particular if  $\chi \in T_{\mathcal{R}}$ ,*

$$(Z^\lambda . H_w - H_w . Z^{w^{-1}.\lambda}) \otimes_{\chi} 1 \in I_{\chi}^{\leq w} \subset I_{\chi}^{\neq w}.$$

*Proof.* We do it by induction on  $\ell(w)$ . Let  $w \in W^v$  be such that  $Z^\lambda H_w = H_w Z^{w^{-1}.\lambda} + u$ , with  $u \in {}^{\text{BL}}\mathcal{H}_{\mathcal{R}}^{\leq w}$ . Let  $s \in \mathcal{S}$  and assume that  $\ell(ws) = \ell(w) + 1$ . Then by (BL4):

$$Z^\lambda H_{ws} = (H_w Z^{w^{-1}.\lambda} + u) * H_s = H_{ws} Z^{sw^{-1}.\lambda} + aH_w + uH_s,$$

for some  $a \in \mathcal{R}$ . Moreover,  $u * H_s \in {}^{\text{BL}}\mathcal{H}_{\mathcal{R}}^{\leq ws}$  and the lemma follows.  $\square$

For  $\chi \in T_{\mathcal{R}}$ , set  $W_{\chi} = \{w \in W^v \mid w.\chi = \chi\}$ .

**Lemma 3.3.** *Let  $\chi, \chi' \in T_{\mathcal{R}}$ . Let  $x \in I_{\chi}(\chi')$ . Then if  $x \neq 0$ ,*

$$\max(\text{supp}(x)) \subset \{w \in W^v \mid w.\chi = \chi'\}.$$

*In particular, if  $I_{\chi}(\chi') \neq \{0\}$ , then  $\chi' \in W^v.\chi$  and thus*

$$\{\chi' \in T_{\mathcal{R}} \mid I_{\chi}(\chi') \neq \{0\}\} \subset W^v.\chi.$$

*Proof.* Assume  $x \neq 0$ . Let  $w \in \max(\text{supp}(x))$ . Write  $x = a_w H_w \otimes_{\chi} 1 + y$ , where  $a_w \in \mathcal{R} \setminus \{0\}$  and  $y \in I_{\chi}^{\neq w}$ . Then by Lemma 3.2,

$$Z^\lambda . x = a_w H_w Z^{w^{-1}.\lambda} \otimes_{\chi} 1 + y' = \chi(w^{-1}.\lambda) a_w H_w \otimes_{\chi} 1 + y' = \chi'(\lambda) a_w H_w \otimes_{\chi} 1 + \chi'(\lambda) y,$$

where  $y' \in I_{\chi}^{\neq w}$ . Therefore  $w.\chi = \chi'$ .  $\square$

**Proposition 3.4.** *(see 4.3.3 Théorème (iii) of [Mat77]) Let  $\chi, \chi' \in T_{\mathcal{R}}$  and  $M'$  be a  ${}^{\text{BL}}\mathcal{H}_{\mathcal{R}}$ -sub-module of  $I_{\chi}$ . Assume that there exists  $f \in \text{Hom}_{\text{BL}\mathcal{H}_{\mathcal{R}}\text{-mod}}(I_{\chi}, M') \setminus \{0\}$ . Then  $\chi' \in W^v.\chi$ .*

*Proof.* Let  $f \in \text{Hom}_{\text{BL}\mathcal{H}_{\mathcal{R}}}(I_{\chi}, M') \setminus \{0\}$ . As  $I_{\chi} = {}^{\text{BL}}\mathcal{H}_{\mathcal{R}}.1 \otimes_{\chi} 1$ ,  $f(1 \otimes_{\chi} 1) \neq 0$ . Therefore  $f(1 \otimes_{\chi} 1) \in M'(\chi) \setminus \{0\}$  and lemma 3.3 completes the proof.  $\square$

**Remark 3.5.** *Let  $\chi \in T_{\mathcal{R}}$ . Let  $\mathcal{F}$  be the field of fractions of  $\mathcal{R}$ . Then one can regard  $\chi$  as an element  $\chi_{\mathcal{F}}$  of  $T_{\mathcal{F}} = \text{Hom}(Y, \mathcal{F}^*)$ . There is a natural inclusion  $I_{\chi} \hookrightarrow I_{\chi_{\mathcal{F}}}$  of  ${}^{\text{BL}}\mathcal{H}_{\mathcal{R}}$ -modules and one has  $I_{\chi}(\chi') = I_{\chi_{\mathcal{F}}}(\chi'_{\mathcal{F}}) \cap I_{\chi}$  for all  $\chi' \in T_{\mathcal{R}}$ .*

The following lemma will be crucial to define an intertwining operators  $I_{\chi} \rightarrow I_{s.\chi}$ , for  $s \in \mathcal{S}$  and thus to define intertwining operators  $I_{\chi} \rightarrow I_{w.\chi}$  for all  $w \in W^v$ .

**Lemma 3.6.** *Let  $s \in \mathcal{S}$  and  $\chi \in T_{\mathcal{R}}$  be such that  $s.\chi \neq \chi$ . Then  $\chi(Q_s(Z)) \in \text{Frac}(\mathcal{R})$  is well defined and  $I_{\chi}^{\leq s} \cap I_{\chi}(s.\chi) = \mathcal{R} \cdot (H_s \otimes_{\chi} 1 - \chi(Q_s(Z)) \otimes_{\chi} 1) \cap I_{\chi}$ . In particular,  $I_{\chi}^{\leq s} \cap I_{\chi}(s.\chi) \neq \{0\}$ .*

*Proof.* By Remark 3.5, we may assume that  $\mathcal{R} = \mathcal{F}$  is a field. If  $\sigma_s = \sigma'_s$ , then  $Q_s(Z) = \frac{\sigma_s - \sigma_s^{-1}}{1 - Z^{-\alpha_s^\vee}}$ . Let  $\lambda \in Y$  be such that  $\chi(s.\lambda) \neq \chi(\lambda)$ . Then  $\chi(-\alpha_s(\lambda)\alpha_s^\vee) \neq 1$ , hence  $\chi(-\alpha_s^\vee) \neq 1$  and thus  $\chi(Q_s(Z))$  is well defined. If  $\sigma_s \neq \sigma'_s$ , then  $\alpha_s(Y) = 2\mathbb{Z}$  and thus  $\chi^2(\alpha_s^\vee) \neq 1$  by the same reasoning. In both cases,  $\chi(Q_s(Z))$  is well defined.

Let  $\lambda \in Y$ . Let  $a, b \in \mathcal{F}$ . Then

$$\begin{aligned} & Z^\lambda.(aH_s \otimes_\chi 1 + b \otimes_\chi 1) - \chi(s.\lambda)(aH_s \otimes_\chi + b \otimes_\chi 1) \\ &= (\chi(Q_s(Z))\chi(Z^\lambda - Z^{s.\lambda})a + (\chi(\lambda) - \chi(s.\lambda))b) \otimes_\chi 1 \\ &= (\chi(\lambda) - \chi(s.\lambda))(a\chi(Q_s(Z)) + b) \otimes_\chi 1. \end{aligned}$$

As  $\chi \neq s.\chi$ , the lemma follows.  $\square$

### 3.2 Intertwining operators

Let  $M$  be a  ${}^{\text{BL}}\mathcal{H}_{\mathcal{R}}$ -module and  $\chi \in T_{\mathcal{R}}$ . For  $x \in M(\chi)$  define  $\Upsilon_x : I_\chi \rightarrow M$  by  $\Upsilon_x(u.1 \otimes_\chi 1) = u.x$ , for all  $u \in {}^{\text{BL}}\mathcal{H}_{\mathcal{R}}$ . Then  $\Upsilon_x$  is well defined. Indeed, let  $u \in {}^{\text{BL}}\mathcal{H}_{\mathcal{R}}$  be such that  $u.1 \otimes_\chi 1 = 0$ . Then  $u \in \mathcal{R}[Y]$  and  $\chi(u) = 0$ . Therefore  $u.x = 0$  and hence  $\Upsilon_x$  is well defined. The following lemma, which is similar to the first form of ‘‘Frobenius reciprocity’’ (see [Kat81, Proposition 1.10]) is then easy to prove.

**Lemma 3.7.** *Let  $M$  be a  ${}^{\text{BL}}\mathcal{H}_{\mathcal{R}}$ -module,  $\chi \in T_{\mathcal{R}}$   $x \in M(\chi)$ . Then  $\Upsilon_x$  is well defined. Moreover the map  $\Upsilon : M(\chi) \rightarrow \text{Hom}_{{}^{\text{BL}}\mathcal{H}_{\mathcal{R}}\text{-mod}}(I_\chi, M)$  defined mapping each  $x \in M(\chi)$  to  $\Upsilon_x$  is an  $\mathcal{R}$ -module isomorphism and  $\Upsilon^{-1}(f) = f(1 \otimes_\chi 1)$  for all  $f \in \text{Hom}_{{}^{\text{BL}}\mathcal{H}_{\mathcal{R}}\text{-mod}}(I_\chi, M)$ .*

**Corollary 3.8.** (see [Mat77, (4.1.10)]) *Let  $M$  be a  ${}^{\text{BL}}\mathcal{H}_{\mathcal{R}}$ -module such that there exists  $\xi \in M$  satisfying:*

1. *there exists  $\chi \in T_{\mathcal{R}}$  such that  $\xi \in M(\chi)$ ,*
2.  *$M = {}^{\text{BL}}\mathcal{H}_{\mathcal{R}}.\xi$ .*

*Then there exists a surjective morphism  $\phi : I_\chi \rightarrow M$  of  ${}^{\text{BL}}\mathcal{H}_{\mathcal{R}}$ -modules.*

*Proof.* One can take  $\phi = \Upsilon_\xi$ , where  $\Upsilon$  is as in Lemma 3.7.  $\square$

**Proposition 3.9.** (see [Mat77, Théorème 4.2.4]) *Assume that  $\mathcal{R} = \mathcal{F}$  is a field. Let  $M$  be an irreducible representation of  ${}^{\text{BL}}\mathcal{H}_{\mathcal{F}}$  containing a finite dimensional  $\mathcal{F}[Y]$ -submodule  $M' \neq \{0\}$ . Then there exists  $\chi \in T_{\mathcal{F}}$  such that there exists a surjective morphism of  ${}^{\text{BL}}\mathcal{H}_{\mathcal{F}}$ -modules  $\phi : I_\chi \rightarrow M$ .*

*Proof.* By Frobenius Theorem (Theorem 2.6), there exists  $\xi \in M' \setminus \{0\}$  such that  $Z^\mu.\xi \in \mathcal{F}.\xi$  for all  $\mu \in Y$ . Let  $\chi \in T_{\mathcal{F}}$  be such that  $\xi \in M(\chi)$ . Then we conclude with Corollary 3.8.  $\square$

Recall that  $\mathcal{H}_{\mathcal{R}, W^v} = \bigoplus_{w \in W^v} \mathcal{R}H_w \subset {}^{\text{BL}}\mathcal{H}_{\mathcal{R}}$  is a subalgebra of  ${}^{\text{BL}}\mathcal{H}_{\mathcal{R}}$  (see Definition 2.4). If  $R$  is a binary relation on  $W^v$  and  $w \in W^v$ , set

$$\mathcal{H}_{\mathcal{R}, W^v}^{Rw} = \bigoplus_{v \in W^v | vRw} \mathcal{R}H_v = \left\{ \sum_{v \in W^v} a_v H_v \mid a_v \neq 0 \implies vRw \right\}.$$

If  $\chi \in T_{\mathcal{R}}$ , one has  $\mathcal{H}_{\mathcal{R}, W^v}.1 \otimes_\chi 1 = I_\chi$  and if  $w \in W^v$ ,  $(\mathcal{H}_{\mathcal{R}, W^v}^{\leq w} \setminus \mathcal{H}_{\mathcal{R}, W^v}^{< w}).1 \otimes_\chi 1 \subset I_\chi^{\leq w} \setminus I_\chi^{< w}$

**Lemma 3.10.** *Let  $w \in W^v$  and  $s \in \mathcal{S}$  be such that  $ws > w$ . Then*

$$(\mathcal{H}_{\mathcal{R}, W^v}^{\leq w} \setminus \mathcal{H}_{\mathcal{R}, W^v}^{\leq w}) * (\mathcal{H}_{\mathcal{R}, W^v}^{\leq s} \setminus \mathcal{H}_{\mathcal{R}, W^v}^{\leq s}) \subset \mathcal{H}_{\mathcal{R}, W^v}^{\leq ws} \setminus \mathcal{H}_{\mathcal{R}, W^v}^{\leq ws}.$$

*Proof.* This follows from the fact that  $[1, w].[1, s] \subset [1, ws]$  and that  $[1, w).s \cup [1, w] \subset [1, ws]$ .  $\square$

**Lemma 3.11.** *Let  $s_1, \dots, s_k \in \mathcal{S}$ . For  $j \in \llbracket 1, k \rrbracket$ , set  $w_j = s_{j-1} \dots s_1$  (where we set  $s_0 \dots s_1 = 1$ ) and  $\chi_j = w_j \cdot \chi$ . Set  $w = w_k$ . Assume that  $\chi_j \neq \chi_{j+1}$  for all  $j \in \llbracket 1, k-1 \rrbracket$ . Then there exists  $f \in \text{Hom}_{\text{BL}\mathcal{H}_{\mathcal{R}}\text{-mod}}(I_\chi, I_{w \cdot \chi})$  such that  $f(1 \otimes_\chi 1) \in I_{w \cdot \chi}^{\leq w^{-1}} \setminus I_{w \cdot \chi}^{\leq w^{-1}}$ .*

*Proof.* Let  $j \in \llbracket 1, k-1 \rrbracket$ . By Lemma 3.6, one can choose  $x_j \in I_{\chi_{j+1}}(\chi_j) \cap I_{\chi_{j+1}}^{\leq s_j} \setminus \{0\}$ . Set  $f_j = \Upsilon_{x_j} \circ \dots \circ \Upsilon_{x_1} \in \text{Hom}_{\text{BL}\mathcal{H}_{\mathcal{R}}\text{-mod}}(I_\chi, I_{\chi_{j+1}})$  (where the  $\Upsilon_{x_j} : I_{\chi_j} \rightarrow I_{\chi_{j+1}}$  are defined in Lemma 3.7). Let  $\mathcal{P}_j : "f_j(1 \otimes_\chi 1) \in I_{\chi_{j+1}}^{\leq w_j^{-1}} \setminus I_{\chi_{j+1}}^{\leq w_j^{-1}}"$ . Then  $\mathcal{P}_1$  is true by Lemma 3.6. Let  $j \in \llbracket 1, k-2 \rrbracket$  and assume that  $\mathcal{P}_j$  is true. Write  $f_j(1 \otimes_\chi 1) = h \cdot 1 \otimes_{\chi_{j+1}} 1$ , where  $h \in \mathcal{H}_{\mathcal{R}, W^v}^{\leq w_j^{-1}} \setminus \mathcal{H}_{\mathcal{R}, W^v}^{\leq w_j^{-1}}$ . Then one has  $\Upsilon_{x_{j+1}}(f_j(1 \otimes_\chi 1)) = h \cdot \Upsilon_{x_{j+1}}(1 \otimes_{\chi_{j+1}} 1)$ . Write  $x_{j+1} = h' \cdot 1 \otimes_{\chi_{j+2}} 1$ , where  $h' \in \mathcal{H}_{\mathcal{R}, W^v}^{\leq s_{j+1}} \setminus \mathcal{R} \cdot 1 \otimes_{\chi_{j+1}} 1$ . Then  $f_{j+1}(1 \otimes_\chi 1) = h \cdot h' \cdot 1 \otimes_{\chi_{j+1}} 1$ . By Lemma 3.10, we deduce that  $\mathcal{P}_{j+1}$  is true. Thus  $\mathcal{P}_{k-1}$  is true, which proves the proposition.  $\square$

**Proposition 3.12.** (see [Kat81, (1.21)]). *Let  $\chi \in T_{\mathcal{R}}$  and  $w \in W^v$ . Then one has  $\text{Hom}_{\text{BL}\mathcal{H}_{\mathcal{R}}\text{-mod}}(I_\chi, I_{w \cdot \chi}) \neq \{0\}$ . More precisely, let  $w_\chi \in W^v$  be such that  $w_\chi \cdot \chi = w \cdot \chi$  and having minimal length for this property. Then there exists  $f \in \text{Hom}_{\text{BL}\mathcal{H}_{\mathcal{R}}\text{-mod}}(I_\chi, I_{w \cdot \chi})$  such that  $f(1 \otimes_\chi 1) \in I_{w \cdot \chi}^{\leq (w_\chi)^{-1}} \setminus I_{w \cdot \chi}^{\leq (w_\chi)^{-1}}$ .*

*Proof.* Write  $w_\chi = s_k \dots s_1$ , where  $k = \ell(w_\chi)$  and  $s_1, \dots, s_k \in \mathcal{S}$ . For  $j \in \llbracket 1, k \rrbracket$  set  $w_j = s_{j-1} \dots s_1$  and  $\chi_j = w_j \cdot \chi$ . Let  $j \in \llbracket 1, k-1 \rrbracket$ . Then  $\chi_{j+1} \neq \chi_j$ . Indeed, suppose that  $\chi_{j+1} = \chi_j$ . Then  $w_\chi \cdot \chi = s_k \dots s_j \dots s_1 \cdot \chi = s_k \dots \hat{s}_j \dots s_1 \cdot \chi$ , which is absurd by choice of  $w_\chi$ . This is thus a consequence of Lemma 3.11.  $\square$

**Proposition 3.13.** *Let  $\chi \in T_{\mathcal{R}}$ . Then  $\{\chi' \in T_{\mathcal{R}} \mid I_\chi(\chi') \neq \{0\}\} = W^v \cdot \chi$ .*

*Proof.* By Lemma 3.3, we already know that  $\{\chi' \in T_{\mathcal{R}} \mid I_\chi(\chi') \neq \{0\}\} \subset W^v \cdot \chi$ . Let  $w \in W^v$  and  $f \in \text{Hom}_{\text{BL}\mathcal{H}_{\mathcal{R}}\text{-mod}}(I_{w \cdot \chi}, I_\chi) \setminus \{0\}$ . Then as  ${}^{\text{BL}}\mathcal{H}_{\mathcal{R}} \cdot 1 \otimes_{w \cdot \chi} 1 = I_{w \cdot \chi}$ ,  $f(1 \otimes_{w \cdot \chi} 1) \neq 0$ . As  $f(1 \otimes_{w \cdot \chi} 1) \in I_\chi(w \cdot \chi)$ , the lemma follows.  $\square$

### 3.3 A necessary condition for irreducibility

We now assume that  $\mathcal{R} = \mathcal{F}$  is a field. Let  $\chi \in \mathcal{T}_{\mathcal{F}}$  and  $s \in \mathcal{S}$  be such that  $s \cdot \chi \neq \chi$ . Recall that  $Q_s(Z) = \frac{(\sigma_s - \sigma_s^{-1}) + (\sigma'_s - \sigma_s'^{-1})Z^{-\alpha_s^\vee}}{1 - Z^{-2\alpha_s^\vee}}$ . Set  $x_{\chi, s} = H_s \otimes_{s \cdot \chi} 1 - \chi(Q_s(Z))1 \otimes_{s \cdot \chi} 1$ . By Lemma 3.6,  $x_{\chi, s} \in I_{s \cdot \chi}(\chi)$ . Set  $f_{\chi, s} = \Upsilon_{x_{\chi, s}} \in \text{Hom}_{\text{BL}\mathcal{H}_{\mathcal{F}}\text{-mod}}(I_\chi, I_{s \cdot \chi})$ .

**Lemma 3.14.** *Let  $\chi \in \mathcal{T}_{\mathcal{F}}$  and  $s \in \mathcal{S}$  be such that  $\chi' = s \cdot \chi \neq \chi$ . Then*

$$f_{\chi', s} \circ f_{\chi, s} = (1 + \chi(Q_s(Z))\chi'(Q_s(Z)))\text{Id}_{I_\chi}$$

and

$$f_{\chi, s} \circ f_{\chi', s} = (1 + \chi(Q_s(Z))\chi'(Q_s(Z)))\text{Id}_{I_\chi'}$$

*Proof.* Let  $g = f_{\chi',s} \circ f_{\chi,s}$ . Then  $g \in \text{Hom}_{\text{BL}\mathcal{H}_{\mathcal{F}}\text{-mod}}(I_{\chi}, I_{\chi})$ . Therefore it suffices to compute  $g(1 \otimes_{\chi} 1)$ . One has

$$\begin{aligned} g(1 \otimes_{\chi} 1) &= (H_s - \chi(Q_s(Z))) \cdot f_{\chi',s}(1 \otimes_{\chi'} 1) \\ &= (H_s - \chi(Q_s(Z))) * (H_s - \chi'(Q_s(Z))) \cdot 1 \otimes_{\chi} 1 \\ &= \left( (\sigma_i - \sigma_i^{-1})H_s - \chi(Q_s(Z)) - \chi'(Q_s(Z)) + 1 + \chi(Q_s(Z))\chi'(Q_s(Z)) \right) \cdot 1 \otimes_{\chi} 1 \\ &= (1 + \chi(Q_s(Z))\chi'(Q_s(Z))) \otimes_{\chi} 1, \end{aligned}$$

and the lemma follows by symmetry.  $\square$

Let  $\mathcal{U}_{\mathcal{F}}$  be the set of  $\chi \in T_{\mathcal{F}}$  such that for all  $w \in W^v$  and  $s \in \mathcal{S}$  such that  $sw.\chi \neq w.\chi$ ,  $1 + \chi(Q_s(w.Z))\chi(Q_s(sw.Z)) \neq 0$ .

**Remark 3.15.** Assume that  ${}^{\text{BL}}\mathcal{H}_{\mathcal{F}}$  is associated with a split Kac-Moody group over a local field of residue cardinal  $q$ . Let  $\Phi^{\vee} = W^v \cdot \{\alpha_s^{\vee} | s \in \mathcal{S}\}$  be the coroot system. Then

$$\mathcal{U}_{\mathcal{F}} = \{\chi \in T_{\mathcal{F}} | \forall w \in W^v, \forall \alpha^{\vee} \in \Phi^{\vee}, \chi(\alpha^{\vee}) + \chi^{-1}(\alpha^{\vee}) \neq q^{\frac{1}{2}} + q^{-\frac{1}{2}}\}.$$

Indeed, let  $\chi \in \mathcal{U}_{\mathcal{F}}$ . Let  $w \in W^v$  and  $s \in \mathcal{S}$ . Suppose that  $sw.\chi = w.\chi$ . Then  $w.\chi(\alpha_s^{\vee}) = 1$ . Hence  $w.\chi(\alpha_s^{\vee}) + (w.\chi(\alpha_s^{\vee}))^{-1} = 2 \neq q^{\frac{1}{2}} + q^{-\frac{1}{2}}$ . Suppose that  $sw.\chi \neq w.\chi$ . Then

$$\frac{(q^{\frac{1}{2}} - q^{-\frac{1}{2}})^2}{(1 - \chi(w^{-1}.\alpha_s^{\vee}))(1 - \chi(w^{-1}.\alpha_s^{\vee})^{-1})} \neq -1,$$

or equivalently,

$$\chi(w^{-1}.\alpha_s^{\vee}) + \chi^{-1}(w^{-1}.\alpha_s^{\vee}) \neq q^{\frac{1}{2}} + q^{-\frac{1}{2}}.$$

This proves one inclusion and similar computations yield the other inclusion.

**Lemma 3.16.** 1. Let  $\chi \in \mathcal{U}_{\mathcal{F}}$ . Then for all  $w \in W^v$ ,  $I_{\chi}$  and  $I_{w.\chi}$  are isomorphic as  ${}^{\text{BL}}\mathcal{H}_{\mathcal{F}}$ -modules.

2. Let  $\chi \in T_{\mathcal{F}}$  be such that  $I_{\chi}$  is irreducible. Then  $\chi \in \mathcal{U}_{\mathcal{F}}$ .

*Proof.* Let  $\chi \in \mathcal{U}_{\mathcal{F}}$ . Let  $w \in W^v$  and  $\tilde{\chi} = w.\chi$ . Let  $s \in \mathcal{S}$ . Assume that  $s.\tilde{\chi} \neq \tilde{\chi}$ . Then by Lemma 3.14,  $I_{\tilde{\chi}}$  is isomorphic to  $I_{s.\tilde{\chi}}$  and 1 follows by induction.

Let  $\chi \in T_{\mathcal{F}}$  be such that  $I_{\chi}$  is irreducible. Let  $s \in \mathcal{S}$  be such that  $s.\chi \neq \chi$ . Then  $f_{s,\chi,s} \neq 0$  and  $\text{Im}(f_{s,\chi,s})$  is an  ${}^{\text{BL}}\mathcal{H}_{\mathcal{F}}$ -submodule of  $I_{\chi}$ :  $\text{Im}(f_{s,\chi,s}) = I_{\chi}$ . Therefore  $f_{\chi,s} \circ f_{s,\chi,s} \neq 0$ . By Lemma 3.14,  $1 + \chi(Q_s(Z))\chi(Q_s(s.Z)) \neq 0$  and  $f_{\chi,s} : I_{\chi} \rightarrow I_{s.\chi}$  is an isomorphism. By induction we deduce that  $I_{w.\chi}$  is isomorphic to  $I_{\chi}$  and thus irreducible for all  $w \in W^v$  and that  $\chi \in \mathcal{U}_{\mathcal{F}}$ .  $\square$

**Lemma 3.17.** Let  $\chi \in T_{\mathcal{F}}$  be such that  $I_{w.\chi} \simeq I_{\chi}$  (as a  ${}^{\text{BL}}\mathcal{H}_{\mathcal{F}}$ -module) for all  $w \in W^v$ . Then for all  $w \in W^v$ , there exists a vector space isomorphism  $I_{\chi}(\chi) \simeq I_{\chi}(w.\chi)$ .

*Proof.* Let  $w \in W^v$ . Then by hypothesis,  $\text{Hom}_{\text{BL}\mathcal{H}_{\mathcal{F}}\text{-mod}}(I_{\chi}, I_{\chi}) \simeq \text{Hom}_{\text{BL}\mathcal{H}_{\mathcal{F}}\text{-mod}}(I_{w.\chi}, I_{w.\chi})$ . Let  $\phi : I_{\chi} \rightarrow I_{w.\chi}$  be a  ${}^{\text{BL}}\mathcal{H}_{\mathcal{F}}$ -module isomorphism. Then  $\phi$  induces an isomorphism of vector spaces  $I_{\chi}(w.\chi) \simeq I_{w.\chi}(w.\chi)$ . By Lemma 3.7,

$$I_{\chi}(\chi) \simeq \text{Hom}_{\text{BL}\mathcal{H}_{\mathcal{F}}\text{-mod}}(I_{\chi}, I_{\chi}) \simeq \text{Hom}_{\text{BL}\mathcal{H}_{\mathcal{F}}\text{-mod}}(I_{w.\chi}, I_{w.\chi}) \simeq I_{w.\chi}(w.\chi) \simeq I_{\chi}(w.\chi).$$

$\square$

### 3.4 Link with the works of Matsumoto and Kato

Assume that  $W^v$  is finite. Then  $\mathcal{H}_{\mathbb{C}} = {}^{\text{BL}}\mathcal{H}_{\mathbb{C}}$ . Let  $\chi \in T_{\mathbb{C}}$ . Then by Subsection 2.3,  $\dim_{\mathbb{C}} I_{\chi} = |W^v|$ . One has  $Z^{\lambda} \cdot 1 \otimes_{\chi} 1 = \chi(\lambda) 1 \otimes_{\chi} 1$  for all  $\lambda \in Y$  and  $\mathcal{H}_{\mathbb{C}} \cdot 1 \otimes_{\chi} 1 = I_{\chi}$ . Thus by [Mat77, Théorème 4.1.10] the definition we used is equivalent to Matsumoto's one.

Assume that  $\mathcal{H}_{\mathbb{C}}$  is associated with a split reductive group over a field with residue cardinal  $q$ . For  $w \in W^v$ , set  $T_w = q^{\frac{1}{2}\ell(w)} \cdot H_w$ . Then by (BL2), one has :  $\forall s \in \mathcal{S}, \forall w \in W^v$ ,

$$T_s * T_w = \begin{cases} T_{sw} & \text{if } \ell(sw) = \ell(w) + 1 \\ (q-1)T_w + qT_{sw} & \text{if } \ell(sw) = \ell(w) - 1. \end{cases}$$

Set  $1'_{\chi} = \sum_{w \in W^v} T_w \otimes_{\chi} 1$ . Then if  $s \in \mathcal{S}$ ,  $T_s \cdot 1_{\chi} = q 1_{\chi}$ . Then by [Kat81, (1.19)],  $1'_{\chi}$  is proportional to the vector  $1_{\chi}$  defined in [Kat81]. Kato proves Theorem 1 by studying whether the following property is satisfied: “for all  $w \in W^v$ ,  $\mathcal{H}_{\mathbb{C}} \cdot 1_{w,\chi} = I_{w,\chi}$ ” (see [Kat81, Lemma 2.3]). When  $W^v$  is infinite, we do not know how to define an analogue of  $1'_{\chi}$  and thus we do not know how to adapt Kato's proof.

## 4 Study of the reducibility of $I_{\chi}$

In this Section, we study the reducibility of  $I_{\chi}$ .

In Subsection 4.1, we prove that if  $\chi \in \mathcal{U}_{\mathcal{F}}$ ,  $I_{\chi}$  is irreducible if and only if  $\dim I_{\chi}(\chi) = 1$  (see Theorem 4.2).

In Subsection 4.2, we study the case where  $\chi$  is regular and prove Matsumoto's criterion (see Corollary 4.5).

In Subsection 4.3 we prove one implication of Kato's criterion (see Theorem 4.7).

In Subsection 4.4 and 4.5, we prove the irreducibility of  $I_{\chi}$  in some particular cases where  $\chi$  is non regular.

### 4.1 An irreducibility criterion for $I_{\chi}$

If  $\mathcal{B}$  is a  $\mathbb{C}$ -algebra with unity  $e$  and  $a \in \mathcal{B}$ , set  $\text{Spec}(a) = \{\lambda \in \mathbb{C} \mid a - \lambda e \text{ is not invertible}\}$ . Recall the following theorem of Amitsur (see Théorème B.I of [Ren10]):

**Theorem 4.1.** *Let  $\mathcal{B}$  be a  $\mathbb{C}$ -algebra with a unity  $e$ . Assume that the dimension of  $\mathcal{B}$  over  $\mathbb{C}$  is countable. Then for all  $a \in \mathcal{B}$ ,  $\text{Spec}(a) \neq \emptyset$ .*

Recall that  $\mathcal{U}_{\mathbb{C}}$  is the set of  $\chi \in T_{\mathcal{F}}$  such that for all  $w \in W^v$  and  $s \in \mathcal{S}$  with  $sw \cdot \chi \neq w \cdot \chi$ ,  $1 + w \cdot \chi(Q_s(Z)) w s \cdot \chi(Q_s(Z)) \neq 0$ .

**Theorem 4.2.** *Let  $\chi \in T_{\mathbb{C}}$ . Then the following are equivalent:*

1.  $I_{\chi}$  is irreducible,
2.  $I_{\chi}(\chi) = \mathbb{C} \cdot 1 \otimes_{\chi} 1$  and  $\chi \in \mathcal{U}_{\mathbb{C}}$ ,
3.  $\text{End}_{\text{BL}\mathcal{H}_{\mathbb{C}}\text{-mod}}(I_{\chi}) = \mathbb{C} \cdot \text{Id}$  and  $\chi \in \mathcal{U}_{\mathbb{C}}$ .

*Proof.* Assume that  $\mathcal{B} = \text{End}_{\text{BL}\mathcal{H}_{\mathbb{C}}\text{-mod}}(I_{\chi}) \neq \mathbb{C} \cdot \text{Id}$ . By Lemma 3.7 and the fact that  $I_{\chi}$  has countable dimension,  $\mathcal{B}$  has countable dimension. Let  $\phi \in \mathcal{B} \setminus \mathbb{C} \cdot \text{Id}$ . Then by Amitsur Theorem, there exists  $\gamma \in \text{Spec}(\mathcal{B})$ . Then  $\phi - \gamma \text{Id}$  is non-injective or non-surjective and therefore  $\text{Ker}(\phi - \gamma \text{Id})$  or  $\text{Im}(\phi - \gamma \text{Id})$  is a non-trivial  ${}^{\text{BL}}\mathcal{H}_{\mathbb{C}}$ -module, which proves that  $I_{\chi}$  is reducible. Using Lemma 3.16 we deduce that (1) implies (3).

By Lemma 3.7, (2) is equivalent to (3).

Let  $\chi \in T_{\mathbb{C}}$  satisfying (2). Then by Lemma 3.16 and Lemma 3.17,  $\dim I_{\chi}(w.\chi) = 1$  for all  $w \in W^v$ . By Lemma 3.16, for all  $w \in W^v$ , there exists an isomorphism of  ${}^{\text{BL}}\mathcal{H}_{\mathbb{C}}$ -modules  $f_w : I_{w.\chi} \rightarrow I_{\chi}$ . As  $\mathbb{C}.f_w(1 \otimes_{w.\chi} 1) \subset I_{\chi}(w.\chi)$  we deduce that  $I_{\chi}(w.\chi) = \mathbb{C}.f_w(1 \otimes_{w.\chi} 1)$  for all  $w \in W^v$ .

Let  $M \neq \{0\}$  be a  ${}^{\text{BL}}\mathcal{H}_{\mathbb{C}}$ -submodule of  $I_{\chi}$ . Let  $x \in M \setminus \{0\}$ . Then  $M' = \mathbb{C}[Y].x$  is a finite dimensional  $\mathbb{C}[Y]$ -module. Thus by Frobenius Theorem (Theorem 2.6), there exists  $\xi \in M' \setminus \{0\}$  such that  $Z^{\lambda}.\xi \in \mathbb{C}.\xi$  for all  $\lambda \in Y$ . Then  $\xi \in I(\chi')$  for some  $\chi' \in T_{\mathbb{C}}$ . By Proposition 3.13,  $\chi' = w.\chi$ , for some  $w \in W^v$ . Thus  $\xi \in \mathbb{C}^*.f_w(1 \otimes_{w.\chi} 1)$ . One has  ${}^{\text{BL}}\mathcal{H}_{\mathbb{C}}.\xi = f_w({}^{\text{BL}}\mathcal{H}_{\mathbb{C}}.1 \otimes_{w.\chi} 1) = f_w(I_{w.\chi}) = I_{\chi} \subset M$ . Hence  $I_{\chi}$  is irreducible, which finishes the proof of the theorem.  $\square$

**Remark 4.3.** *Actually, our proof of the equivalence between (2) and (3), and of the fact that (2) implies (1) is valid when  $\mathcal{R} = \mathcal{F}$  is a field.*

## 4.2 The regular case

Assume that  $\mathcal{R} = \mathcal{F}$  is a field. An element  $\chi \in T_{\mathcal{F}}$  is said to be **regular** if  $w.\chi \neq \chi$  for all  $w \in W^v$ .

**Proposition 4.4.** *(see [Kat81, Proposition 1.17]) Let  $\chi \in T_{\mathcal{F}}$  be regular. Then for all  $w \in W^v$ ,  $\dim I_{\chi}(w.\chi) = 1$  and  $I_{\chi} = \bigoplus_{w \in W^v} I_{\chi}(w.\chi)$ .*

*Proof.* By Lemma 3.3  $\dim I_{\chi}(w.\chi) \leq 1$  for all  $w \in W^v$  and by Proposition 3.13,  $\dim I_{\chi}(w.\chi) \geq 1$  for all  $w \in W^v$  and thus  $\dim I_{\chi}(w.\chi) = 1$  for all  $w \in W^v$ .

Let  $v \in W^v$ . By Lemma 3.3,  $I_{\chi}(v.\chi) \subset I_{\chi}^{\leq v}$ . Thus if  $w \in W^v$ , one has  $\bigoplus_{v \leq w} I_{\chi}(v.\chi) \subset I_{\chi}^{\leq w}$ . As these two vector-spaces have the same dimension,  $\bigoplus_{v \leq w} I_{\chi}(v.\chi) = I_{\chi}^{\leq w}$ . Let  $x \in I_{\chi} \setminus \{0\}$ . Then  $x \in \sum_{v \in \text{supp}(x)} I_{\chi}^{\leq v} \subset \bigoplus_{w \in W^v} I_{\chi}(w.\chi)$ , which concludes the proof of the proposition.  $\square$

**Corollary 4.5.** *(see [Mat77, Théorème 4.3.5]) Let  $\chi \in T_{\mathcal{F}}$  be regular. Then  $I_{\chi}$  is irreducible if and only if  $\chi \in \mathcal{U}_{\mathcal{F}}$ .*

*Proof.* By Lemma 3.16, if  $I_{\chi}$  is irreducible, then  $\chi \in \mathcal{U}_{\mathcal{F}}$ .

Assume that  $\chi \in \mathcal{U}_{\mathcal{F}}$ . Then by Proposition 4.4,  $\dim I_{\chi}(\chi) = 1$  and we conclude with Theorem 4.2 and Remark 4.3.  $\square$

## 4.3 One implication of Kato's criterion

Let  $\mathcal{R} = \{wsw^{-1} | w \in W^v, s \in \mathcal{S}\}$  be the set of reflections of  $W^v$ . For  $\chi \in T_{\mathbb{C}}$ , set  $W_{\chi} = \{w \in W^v | w.\chi = \chi\}$ .

**Lemma 4.6.** *Let  $\chi \in T_{\mathbb{C}}$  be such that  $W_{\chi}$  is not generated by  $\mathcal{R} \cap W_{\chi}$ . Let  $E = W_{\chi} \setminus \langle \mathcal{R} \cap W_{\chi} \rangle$ . Let  $w \in E$  be such that  $\ell(w) = \min\{\ell(v) | v \in E\}$ . Write  $w = s_k \dots s_1$ , where  $k = \ell(w)$  and  $s_1, \dots, s_k \in \mathcal{S}$ . Then for all  $j \in \llbracket 0, k-1 \rrbracket$ ,  $s_j \dots s_1.\chi \neq s_{j+1} \dots s_1.\chi$ , where we set  $s_0 \dots s_1 = 1$ .*

*Proof.* Suppose that for some  $j \in \llbracket 0, k-1 \rrbracket$ ,  $s_j \dots s_1.\chi = s_{j+1} \dots s_1.\chi$ . Then

$$w.\chi = s_k \dots s_j \dots s_1.\chi = s_k \dots \hat{s}_{j+1} \dots s_1.\chi.$$

By choice of  $w$ ,  $s_k \dots \hat{s}_{j+1} \dots s_1 \in \langle \mathcal{R} \cap W_{\chi} \rangle$ . Moreover,  $s_1 \dots s_j.s_{j+1}.s_j \dots s_1 \in \langle \mathcal{R} \cap W_{\chi} \rangle$ . Therefore  $w = s_k \dots \hat{s}_{j+1} \dots s_1.s_1 \dots s_j.s_{j+1}.s_j \dots s_1 \in \langle \mathcal{R} \cap W_{\chi} \rangle$ : a contradiction.  $\square$

**Theorem 4.7.** *Let  $\chi \in T_{\mathbb{C}}$  be such that  $W_{\chi}$  is not generated by  $\mathcal{R} \cap W_{\chi}$ . Then  $I_{\chi}$  is reducible.*

*Proof.* We take the same notations as in Lemma 4.6. For  $j \in \llbracket 1, k \rrbracket$ , set  $w_j = s_{j-1} \dots s_1$  and  $\chi_j = w_j \cdot \chi$ . Then by Lemma 4.6,  $\chi_j \neq \chi_{j+1}$  for all  $j \in \llbracket 1, k-1 \rrbracket$ . By Lemma 3.11, there exists  $f \in \text{Hom}_{\text{BL}\mathcal{H}_{\mathbb{C}}\text{-mod}}(I_{\chi}, I_{w \cdot \chi}) = \text{Hom}_{\text{BL}\mathcal{H}_{\mathbb{C}}\text{-mod}}(I_{\chi}, I_{\chi})$  such that  $f(1 \otimes_{\chi} 1) \in I_{\chi}^{\leq w^{-1}} \setminus I_{\chi}^{< w^{-1}}$ . Therefore  $f(1 \otimes_{\chi} 1) \in I_{\chi}(\chi) \setminus \mathbb{C}1 \otimes_{\chi} 1$ . By Theorem 4.2,  $I_{\chi}$  is reducible.  $\square$

#### 4.4 Case where the fixer of $\chi$ is generated by one reflection

Assume that  $\mathcal{R} = \mathcal{F}$  is a field.

**Proposition 4.8.** *Let  $\chi \in \mathcal{U}_{\mathcal{F}}$  be such that  $W_{\chi} = \{1, t\}$  for some reflection  $t$ . Then  $I_{\chi}$  is irreducible.*

*Proof.* Write  $t = s_1 \dots s_{j-1} \cdot s_j \cdot s_{j-1} \dots s_1$  for some  $s_1, \dots, s_j \in \mathcal{S}$ . Let  $v = s_{j-1} \dots s_1$ ,  $s = s_j$  and  $\tilde{\chi} = v \cdot \chi$ . One has  $s \cdot \tilde{\chi} = \tilde{\chi}$  and  $W_{\tilde{\chi}} = \{1, s\}$ . By Lemma 3.3,  $I_{\tilde{\chi}}(\tilde{\chi}) \subset I_{\tilde{\chi}}^{\leq s}$ .

Let  $\lambda \in Y$ . Then  $Z^{\lambda} \cdot H_s \otimes_{\tilde{\chi}} 1 = \tilde{\chi}(\lambda) H_s \otimes_{\tilde{\chi}} 1 + \tilde{\chi}(Q_s(Z)(Z^{\lambda} - Z^{s \cdot \lambda})) 1 \otimes_{\tilde{\chi}} 1$ . Suppose  $\sigma_s = \sigma'_s$ . By Remark 2.3,  $\tilde{\chi}((Q_s(Z)(Z^{\lambda} - Z^{s \cdot \lambda})) = \alpha_s(\lambda)$ . As there exists  $\lambda \in Y$  such that  $\alpha_s(\lambda) \neq 0$ , we deduce that  $H_s \otimes_{\tilde{\chi}} 1 \notin I_{\tilde{\chi}}(\tilde{\chi})$  and thus  $I_{\tilde{\chi}}(\tilde{\chi}) = \mathcal{F} \cdot 1 \otimes_{\tilde{\chi}} 1$ . Similarly, if  $\sigma_s \neq \sigma'_s$ ,  $I_{\tilde{\chi}}(\tilde{\chi}) = \mathcal{F} \cdot 1 \otimes_{\tilde{\chi}} 1$ . We conclude with Theorem 4.2 and Remark 4.3.  $\square$

#### 4.5 Irreducibility of $I_{\mathbb{1}}$ for size 2 Kac-Moody matrices

Assume that  $\mathcal{R} = \mathcal{F}$  is a field. Let  $\mathbb{1} \in T_{\mathcal{F}}$  be defined by  $\mathbb{1}(\lambda) = 1$  for all  $\lambda \in Y$ . One has  $W_{\mathbb{1}} = W^v$ , which is generated by  $\mathcal{S}$ . Thus when  $W^v$  is finite (i.e. in the reductive case),  $I_{\mathbb{1}}$  is irreducible, by Kato's Theorem. The aim of this subsection is to prove the irreducibility of  $I_{\mathbb{1}}$  in the case where the Kac-Moody matrix defining  $\mathbb{A}$  (see Subsection 2.1) is irreducible (see [Kac94, §1.1]) and of size 2 (see Theorem 4.13).

Assume that  $|\mathcal{S}| = 2$  and that the Kac-Moody matrix of the root generating system  $\mathcal{S}$  is irreducible. This is equivalent to assuming that the Kac-Moody matrix of  $\mathcal{S}$  is of the form  $\begin{pmatrix} 2 & a \\ b & 2 \end{pmatrix}$ , with  $a, b \in \mathbb{Z}_{<0}$ . As the case where  $W^v$  is finite is a particular case of Kato's Theorem we assume that  $W^v$  is infinite, which is equivalent to the assumption that  $a, b \leq -4$  by [Kum02, Proposition 1.3.21]. The group  $W^v$  is then the infinite dihedral group. Write  $\mathcal{S} = \{s_1, s_2\}$ . Then every element of  $W^v$  admits a unique reduced writing involving  $s_1$  and  $s_2$ .

For  $w \in W^v$  and  $x = \sum_{v \in W^v} a_v H_v \otimes_{\mathbb{1}} 1 \in I_{\mathbb{1}}$ , set  $\pi^w(x) = a_w$ .

The following lemma is easy to prove.

**Lemma 4.9.** *Let  $w \in W^v$  and  $s \in \mathcal{S}$  be such that  $\ell(ws) = \ell(w) + 1$ . Let  $v \in W^v$  be such that  $v < w$ , then  $vs \neq w$ .*

**Lemma 4.10.** *Let  $w \in W^v$ ,  $s \in \mathcal{S}$  be such that  $\ell(ws) = \ell(w) + 1$  and  $\lambda \in Y$ . Then*

$$\pi^w(Z^{\lambda} \cdot H_{ws} \otimes_{\mathbb{1}} 1) = \alpha_s(w^{-1} \cdot \lambda).$$

*Proof.* Write  $Z^{\lambda} H_w = H_w Z^{w^{-1} \cdot \lambda} + u$ , where  $u \in \text{BL}\mathcal{H}_{\mathbb{C}}^{< w}$ , which is possible by Lemma 3.2. One has

$$Z^{\lambda} H_{ws} = (H_w Z^{w^{-1} \cdot \lambda} + u) * H_s = H_{ws} Z^{sw^{-1} \cdot \lambda} + H_w Q_s(Z^{w^{-1} \cdot \lambda}) + u * H_s.$$

Therefore

$$Z^\lambda H_{ws} \otimes_{\mathbb{1}} 1 = H_{ws} \otimes_{\mathbb{1}} 1 + \mathbf{1}(Q_s(Z^{w^{-1}.\lambda}))H_w \otimes_{\mathbb{1}} 1 + u * H_s \otimes_{\mathbb{1}} 1.$$

By Lemma 4.9,  $\pi^w(Z^\lambda H_{ws} \otimes_{\mathbb{1}} 1) = \mathbf{1}(Q_s(Z^{w^{-1}.\lambda}))$ .

Assume for example that  $\alpha_s(w^{-1}.\lambda) \geq 0$ . Then  $Q_s(Z^{w^{-1}.\lambda}) = \sum_{j=0}^{\alpha_s(w^{-1}.\lambda)} Z^{w^{-1}.\lambda - j\alpha_s^\vee}$  and thus  $\mathbf{1}(Q_s(Z^{w^{-1}.\lambda})) = \alpha_s(w^{-1}.\lambda)$ . Similarly if  $\alpha_s(w^{-1}.\lambda) < 0$ ,  $\mathbf{1}(Q_s(Z^{w^{-1}.\lambda})) = \alpha_s(w^{-1}.\lambda)$ , and the lemma follows.  $\square$

**Lemma 4.11.** *Let  $w \in W^v$ ,  $s \in \mathcal{S}$  be such that  $\ell(sw) = \ell(w) + 1$  and  $\lambda \in Y$ . Then*

$$\pi^w(Z^\lambda.H_{sw} \otimes_{\mathbb{1}} 1) = \alpha_s(\lambda).$$

*Proof.* One has  $Z^\lambda H_{sw} = (H_s Z^{s\lambda} + Q_s(Z^\lambda)) * H_w$ . Moreover  $Z^{s.\lambda} * H_w \in {}^{\text{BL}}\mathcal{H}_{\mathbb{C}}^{\leq w}$ . By a reasoning similar to the one of the proof of Lemma 4.9,  $\pi^w(H_s Z^{s.\lambda} H_w) = 0$ . Moreover,  $\pi^w(Q_s(Z^\lambda)H_w) = \mathbf{1}(w.Q_s(Z^\lambda)) = \alpha(\lambda)$  (where  $w.Z^\mu = Z^{w^{-1}.\mu}$  for all  $\mu \in Y$ ).  $\square$

Recall that  $\mathcal{S} = \{s_1, s_2\}$ . Set  $\mathbb{A}_{in} = \alpha_{s_1}^{-1}(\{0\}) \cap \alpha_{s_2}^{-1}(\{0\}) = \bigcap_{\alpha \in \Phi} \alpha^{-1}(\{0\})$  (where  $\Phi = \{w.\alpha_i | w \in W^v, i \in I\}$ ).

**Lemma 4.12.** *Let  $w_1, w_2 \in W^v$ . Then  $w_1.\alpha_{s_1}^{-1}(\{0\}) \cap w_2.\alpha_{s_2}^{-1}(\{0\}) = \mathbb{A}_{in}$ .*

*Proof.* Recall that  $C_f^v = \{x \in \mathbb{A} | \alpha_{s_1}(x) > 0 \text{ and } \alpha_{s_2}(x) > 0\}$ . For  $J \subset \{1, 2\}$ , set

$$F^v(J) = \{x \in \mathbb{A} | \alpha_{s_i}(x) = 0 \forall i \in J \text{ and } \alpha_{s_j}(x) > 0 \forall i \in \{1, 2\} \setminus J\}.$$

By [Rou11, 1.3],  $w_1.F^v(\{1\}) \cap w_2.F^v(\{2\}) = \emptyset$  and the fixer  $W_i^v$  of  $F^v(\{i\})$  satisfies  $W_i^v = \langle s_i \rangle$ . Thus  $F^v(\{i\})$  is spherical (which means that the fixer of  $F^v(\{i\})$  in  $W^v$  is finite). Recall that  $\mathcal{T} = \bigcup_{w \in W^v} w.\overset{\circ}{C}_f^v$ . By [Rou11, 1.3],  $F^v(\{i\}) \subset \overset{\circ}{\mathcal{T}}$  for both  $i \in \{1, 2\}$  and  $\overset{\circ}{\mathcal{T}} \cap -\overset{\circ}{\mathcal{T}} = \emptyset$ . Therefore  $w_1.F^v(\{1\}) \cap -w_2.F^v(\{2\}) = \emptyset$ . Moreover if  $i \in \{1, 2\}$  and  $j = 2 - i$ ,  $\alpha_{s_i}^{-1}(\{0\}) = F^v(\{j\}) \sqcup \mathbb{A}_{in} \sqcup -F^v(\{j\})$  and the lemma follows.  $\square$

**Theorem 4.13.** *Assume that the matrix of the root generating system  $\mathcal{S}$  is irreducible of size 2 and that  $W^v$  is infinite. Then  $I_{\mathbb{1}}$  is irreducible.*

*Proof.* Let us prove that  $I_{\mathbb{1}}(\mathbb{1}) = \mathbb{C}.1 \otimes_{\mathbb{1}} 1$ . Let  $x \in I_{\mathbb{1}} \setminus \mathbb{C}.1 \otimes_{\mathbb{1}} 1$  and assume that  $x \in I_{\mathbb{1}}(\mathbb{1})$ . Let  $n = \max\{\ell(w) | w \in \text{supp}(x)\}$ . Let  $w_1, w_2 \in W^v$  be such that  $\{w \in W^v | \ell(w) = n\} = \{w_1, w_2\}$ . For  $i \in \{1, 2\}$ , set  $a_i = \pi^{w_i}(x)$ . Write  $\mathcal{S} = \{s_1, s_2\}$ .

First assume that  $n$  is odd. Maybe exchanging  $s_1$  and  $s_2$ , we may assume that the reduced writing of  $w_1$  begins (and ends) with  $s_1$ . Let  $v = s_1 w_1$ . Then  $\ell(v) = n - 1$ ,  $w_1 = s_1 v$  and  $w_2 = v s_2$ . Set  $a_v = \pi^v(x)$ . Then by Lemma 4.10 and Lemma 4.11,

$$\pi^v(Z^\lambda.x) = \pi^v(Z^\lambda.(a_1 H_{w_1} \otimes_{\mathbb{1}} 1 + a_2 H_{w_2} \otimes_{\mathbb{1}} 1 + a_v H_v \otimes_{\mathbb{1}} 1)) = a_1 \alpha_{s_1}(\lambda) + a_2 \alpha_{s_2}(v^{-1}.\lambda) + a_v.$$

As  $x \in I_{\mathbb{1}}(\mathbb{1})$ , we deduce that for all  $\lambda \in Y$ ,  $a_1 \alpha_{s_1}(\lambda) + a_2 \alpha_{s_2}(v^{-1}.\lambda) + a_v = a_v$  and thus

$$a_1 \alpha_{s_1}(\lambda) + a_2 \alpha_{s_2}(v^{-1}.\lambda) = 0.$$

As  $Y$  spans  $\mathbb{A}$ , we deduce that for all  $u \in \mathbb{A}$ ,  $a_1 \alpha_{s_1}(u) + a_2 \alpha_{s_2}(v^{-1}.u) = 0$ . By Lemma 4.12, this is absurd. Therefore  $n$  is even.

Let  $v = s_1.w_1$ . Maybe exchanging  $s_1$  and  $s_2$ , we may assume that  $\ell(v) = \ell(w_1) - 1$ . Then  $w_2 = v.s_1$ . Set  $a_v = \pi^v(x)$ . Then by Lemma 4.10 and Lemma 4.11, for all  $\lambda \in Y$ ,

$$\pi^v(Z^\lambda.x) = a_1\alpha_{s_1}(\lambda) + a_2\alpha_{s_1}(v^{-1}.\lambda) + a_v = a_v$$

and hence

$$a_1\alpha_{s_1}(\lambda) + a_2\alpha_{s_1}(v^{-1}.\lambda) = 0.$$

As  $Y$  spans  $\mathbb{A}$ , we deduce that for all  $u \in \mathbb{A}$ ,  $a_1\alpha_{s_1}(u) + a_2\alpha_{s_2}(v^{-1}.u) = 0$ . By hypothesis,  $a_1 \neq 0$  or  $a_2 \neq 0$  and thus  $a_1 \neq 0$  and  $a_2 \neq 0$

Let  $u \in F^v(\{1\})$ . Then  $\alpha_{s_2}(v^{-1}.u) = 0$ . Consequently,  $v^{-1}.F^v(\{1\}) = F^v(\{1\})$  and hence  $v = s_1$ . But then  $w_1 = 1$ : a contradiction. Therefore  $I_{\mathbb{1}}(1) = \mathbb{C}.1 \otimes_{\mathbb{1}} 1$  and by Theorem 4.2,  $I_{\mathbb{1}}$  is irreducible.  $\square$

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