HAL
open science

A spectral approach for quenched limit theorems for random hyperbolic dynamical systems

Davor Dragičević, Gary Froyland, Cecilia González-Tokman, Sandro Vaienti

To cite this version:

Davor Dragičević, Gary Froyland, Cecilia González-Tokman, Sandro Vaienti. A spectral approach for quenched limit theorems for random hyperbolic dynamical systems. Transactions of the American Mathematical Society, 2020, 373 (1), pp.629-664. 10.1090/tran/7943 . hal-01959792

HAL Id: hal-01959792

https://hal.science/hal-01959792v1

Submitted on 23 Jan 2020

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

arXiv:1812.07340v1 [math.DS] 18 Dec 2018

A SPECTRAL APPROACH FOR QUENCHED LIMIT THEOREMS FOR RANDOM HYPERBOLIC DYNAMICAL SYSTEMS

Abstract. We extend the recent spectral approach for quenched limit theorems developed for piecewise expanding dynamics under general random driving [9] to quenched random piecewise hyperbolic dynamics including some classes of billiards. For general ergodic sequences of maps in a neighbourhood of a hyperbolic map we prove a quenched large deviations principle (LDP), central limit theorem (CLT), and local central limit theorem (LCLT).

D. Dragičević [1], G. Froyland[2], C. González-Tokman[3], S. Vaienti[4]

Contents

1. Introduction 2
2. Preliminaries 4
3. Building the cocycle $\mathcal{L}$ 6
3.1. Strong measurability of $\omega \mapsto \mathcal{L}_\omega$ 6
3.2. Quasi-compactness of the cocycle $\mathcal{L}$ and existence of Oseledets splitting 7
3.3. One-dimensionality of the top Oseledets space 8
4. Quasi-compactness of the twisted cocycle $\mathcal{L}^\theta$ 9
5. Regularity of the top Oseledets space of the twisted cocycle 11
5.1. Regularity of the cocycles 12
5.2. An auxiliary function F and its regularity 12
6. Properties of $\Lambda(\theta)$ 14
6.1. A differentiable lower bound for $\Lambda(\theta)$ 14
6.2. One-dimensionality of $Y_1^\theta(\omega)$ and differentiability of Λ 15
6.3. Convexity of $\Lambda(\theta)$ 15
7. Large deviation principle and central limit theorem 15
7.1. Large deviation principle 16
7.2. Central limit theorem 16
8. Local central limit theorem 17
9. Piecewise hyperbolic dynamics 23
9.1. Preliminaries 23
9.2. Building the cocycle 25
9.3. The twisted cocycle 26
9.4. Regularity of the top Oseledets space, convexity of Λ 29
9.5. Large deviation principle and central limit theorem 30
9.6. Local central limit theorem 30

Date: December 19, 2018.

[1]Department of Mathematics, University of Rijeka, Rijeka Croatia. E-mail: ddragicevic@math.uniri.hr.

[2]School of Mathematics and Statistics, University of New South Wales, Sydney NSW 2052, Australia. Email: G.Froyland@unsw.edu.au .

[3]School of Mathematics and Physics, The University of Queensland, St Lucia QLD 4072, Australia. E-mail: cecilia.gt@uq.edu.au.

[4]Aix Marseille Université, Université de Toulon, CNRS, CPT, 13009 Marseille, France. E-mail: vaienti@cpt.univ-mrs.fr.

9.7. Billiards 30
10. Acknowledgements 31
References 31

1. Introduction

In our previous paper [9] we extended the Nagaev-Guivarc'h spectral method to obtain limit theorems, such as the Central Limit Theorem (CLT), the Large Deviation Principle (LDP) and the Local Central Limit Theorem (LCLT), for random dynamical systems governed by a *cocycle* of maps $T_\omega^{(n)} := T_{\sigma^{n-1}\omega} \circ \cdots \circ T_{\sigma\omega} \circ T_\omega$, assuming uniform-in-$\omega$ eventual expansivity conditions on the maps T_ω. The random driving was a general ergodic, invertible transformation $\sigma : \Omega \circlearrowleft$ on a probability space $(\Omega, \mathbb{P})$, and the real observable g was defined on the product space $\Omega \times X \to \mathbb{R}$.

Before introducing our new results, we briefly recap the essence of the Nagaev-Guivarc'h spectral method in the deterministic setting, where one deals with a single map T, deferring to the original articles by Nagaev [25, 26] and Guivarc'h [28, 16] and to the excellent survey [14] for more details. The spectral method uses the transfer operator $\mathcal{L} : \mathcal{B} \circlearrowleft$ acting on a Banach space $\mathcal{B}$, and in particular, the *twisted* transfer operator $\mathcal{L}^\theta f := \mathcal{L}(e^{\theta g} f)$, for f and $g \in \mathcal{B}$. In the situation where $\mathcal{L}^\theta$ is quasi-compact for θ near zero, regularity of the leading eigenvalues and eigenprojectors have been used to prove limit theorems [19, 18, 27, 28, 19, 3, 28, 24, 18, 29, 12] and more, namely Berry-Esseen theorems [16, 12] and almost-sure invariance principles [13]. The key equality was $\mathbb{E}(e^{\theta S_n g} f) = \mathbb{E}((\mathcal{L}^\theta)^n f)$, where $S_n g$ denotes the Birhkoff sum of the observable g and the expectation is taken with respect to the unique eigenmeasure m of the adjoint of $\mathcal{L}$. Since the map $\theta \mapsto \mathcal{L}^\theta$ is holomorphic, classical perturbation theory allows one to obtain $\mathbb{E}(e^{\theta S_n g}) = c(\theta)\lambda(\theta)^n + d_n(\theta)$, where $\lambda(\theta)$ is the leading eigenvalue of $\mathcal{L}^\theta$, with c and λ analytic in θ, and $\sup_\theta |d_n(\theta)| \to 0$. We can therefore easily compute the characteristic function and the log generating function of the process $g \circ T^n$ with respect to the invariant probability measure of T, which has density dh/dm where h is the eigenfunction of $\mathcal{L}$ corresponding to the simple eigenvalue 1.

In the quenched random setting we must replace the n-th power of the twisted operator with the twisted transfer operator cocycle $\mathcal{L}_\omega^{\theta,(n)} := \mathcal{L}^\theta_{\sigma^{n-1}\omega} \circ \cdots \circ \mathcal{L}^\theta_{\sigma\omega} \circ \mathcal{L}^\theta_\omega$. By using the multiplicative ergodic theorem adapted to the study of such cocycles and generalizing a theorem of Hennion and Hérve [18] to the random setting, we were able in our previous paper [9] to show that the cocycle $\mathcal{L}_\omega^{\theta,(n)}$ is quasi-compact for θ near to 0. We therefore thus obtained that for such values of θ and for $\mathbb{P}$-a.e. $\omega \in \Omega$, the top Lyapunov exponent $\Lambda(\theta)$ (analogous to the logarithm of $\lambda(\theta)$ in the deterministic setting) of the cocycle is analytic and given by

$$\lim_{n\to\infty} \frac{1}{n} \log |\mathbb{E}_{\mu_\omega}(e^{\theta S_n g(\omega,\cdot)})| = \Lambda(\theta),$$

where μ_ω is the equivariant probability measure on the ω-fiber (see below). This result together with the exponential decay of the norm of the elements in the complement of the top Oseledets space, which handled the error corresponding to quantity d_n above, allowed us to achieve the desired limit theorems.

In the present paper we move from cocycles of piecewise expanding maps to cocycles of hyperbolic maps (both smooth and piecewise smooth), including some classes of billiards. To our knowledge, this is the first time that this setting has been investigated with multiplicative ergodic theory tools. One of the primary differences with [9] is the use of

anisotropic Banach spaces here in place of the space of functions of bounded variation in [9]. Specifically, in the smooth hyperbolic setting and in any dimension, we use the functional analytic setup of Gouëzel and Liverani [15], and in the piecewise hyperbolic case in dimension two we use the spaces from Demers and Liverani [5] (as well as Demers and Zhang [6, 7]). This increased technicality in the underlying spaces necessitates a certain amount of checking of relevant conditions, however, we wish to highlight the fact that a wholesale change of the theory of [9] is not required, which demonstrates the power and flexibility of our approach. The use of transfer operators in the study of statistical properties and limit theorems for hyperbolic dynamical systems has flourished in the last years, and [2] presents a thorough discussion of the various spaces that have been used in the literature. Our intention in this work has not been to find the most general version of the results, but rather to illustrate the applicability of the methods. In fact, we expect the methods presented here to remain applicable in some (or all) of these functional analytic scenarios.

We first consider cocycles $T_\omega^{(n)}$ where the family of maps $\{T_\omega\}_{\omega\in\Omega}$ are selected from a C^{r+1}-neighbourhood of a topologically transitive Anosov map T of class C^{r+1} (in Section 9 we consider piecewise hyperbolic maps also describing periodic Lorentz gas). The random driving $\sigma : \Omega \circlearrowleft$ is a general (ergodic, invertible) automorphism preserving a probability measure $\mathbb{P}$. If $d_{C^{r+1}}(T_\omega, T) < \Delta$ for $\mathbb{P}$-a.e. $\omega \in \Omega$ and Δ is sufficiently small, the random dynamical system generated by the cocycle $T_\omega^{(n)}$ supports a measure μ, invariant under the skew product $\tau(\omega, x) = (\sigma\omega, T_\omega x)$. We obtain this measure by explicitly constructing the family μ_ω along the marginal $\mathbb{P}$, namely $\mu = \int_\Omega \mu_\omega \, d\mathbb{P}(\omega)$, and satisfying the usual equivariance condition $\mu_\omega \circ T_\omega^{-1} = \mu_{\sigma\omega}$. Our observable g satisfies $g(\omega, \cdot) \in C^r$ for $\mathbb{P}$-a.e. ω, $\operatorname{ess\,sup}_{\omega\in\Omega} \|g(\omega, \cdot)\|_{C^r} < \infty$, and is fiberwise centred: $\int_X g(\omega, x) \, d\mu_\omega(x) = 0$ for $\mathbb{P}$-a.e. ω. Our limit theorems concern random Birkhoff sums

$$S_n g(\omega, x) := \sum_{i=0}^{n-1} g(\tau^i(\omega, x)) = \sum_{i=0}^{n-1} g(\sigma^i, T_\omega^{(i)} x), \quad (\omega, x) \in \Omega \times X, n \in \mathbb{N}. \tag{1}$$

Our main theorems are:

Theorem A (Quenched large deviations theorem)**.** *In the above setting, there exists $\epsilon_0 > 0$ and a non-random function $c\colon (-\epsilon_0, \epsilon_0) \to \mathbb{R}$ which is nonnegative, continuous, strictly convex, vanishing only at 0 and such that*

$$\lim_{n\to\infty} \frac{1}{n} \log \mu_\omega(S_n g(\omega, \cdot) > n\epsilon) = -c(\epsilon), \quad \textit{for } 0 < \epsilon < \epsilon_0 \textit{ and } \mathbb{P}\textit{-a.e. } \omega \in \Omega.$$

We now define the non-random quantity

$$\Sigma^2 := \int_{\Omega\times X} g(\omega, x)^2 \, d\mu(\omega, x) + 2 \sum_{n=1}^{\infty} \int_{\Omega\times X} g(\omega, x) g(\tau^n(\omega, x)) \, d\mu(\omega, x). \tag{2}$$

It is clear that $\Sigma^2 \geq 0$.

Theorem B (Quenched central limit theorem)**.** *In the above setting, assume that the non-random variance Σ^2, defined in (2) satisfies $\Sigma^2 > 0$. Then, for every bounded and continuous function $\phi\colon \mathbb{R} \to \mathbb{R}$ and $\mathbb{P}$-a.e. $\omega \in \Omega$, we have*

$$\lim_{n\to\infty} \int \phi\left(\frac{S_n g(\omega, x)}{\sqrt{n}}\right) d\mu_\omega(x) = \int \phi \, d\mathcal{N}(0, \Sigma^2).$$

(The discussion in §6.3 deals with the degenerate case $\Sigma^2 = 0$).

One of the main achievements of our previous paper was the proof of the local central limit theorem (LCLT) in the non-arithmetic and arithmetic cases. Our basic assumption, which for convenience we simply call (L), expresses the exponential decay of the strong norm of the twisted operator when the parameter $\theta = it$ has $t \neq 0$. Moreover we showed under additional assumptions that we will recall in section 8, that hypothesis (L) was equivalent to a co-boundary condition which is better known as the *aperiodicity condition*. In the present paper we prove the LCLT in the non-arithmetic case by assuming (L). Recently Hafouta and Kifer [17] proposed a new set of assumptions which allow us to check condition (L). We will see that some of these assumptions can be verified easily for our systems, provided we restrict the class of the driving maps.

Theorem C (Quenched local central limit theorem)**.** *In the above setting suppose condition (L) holds. Then, for $\mathbb{P}$-a.e. $\omega \in \Omega$ and every bounded interval $J \subset \mathbb{R}$, we have*

$$\lim_{n\to\infty} \sup_{s\in\mathbb{R}} \left| \Sigma\sqrt{n}\mu_\omega(s + S_n g(\omega, \cdot) \in J) - \frac{1}{\sqrt{2\pi}} e^{-\frac{s^2}{2n\Sigma^2}} |J| \right| = 0.$$

In Section 9 we consider random cocycles of *piecewise* hyperbolic maps of the type considered in [5] on two-dimensional compact Riemannian manifolds, and billiard maps associated to periodic Lorentz gas [6, 7]. As we will explain later on and in order to apply the multiplicative ergodic theorem, we have now less choice for the random distribution of the maps, but for instance we can deal with countably many maps. All the preceding theorems A, B and C still hold.

Apart from [9] there are some quenched limit theorems (LDP and CLT) that have been obtained using different methods. Kifer derives a large deviation principle [20, 21, 22] for occupational measures using theory of equilibrium states, and a central limit theorem via martingale methods; in both cases, he treats random subshifts of finite type and random smooth expanding maps. Recently, Hafouta and Kifer [17] proved limit theorems for these systems in the more general "nonconventional setting". They used (complex) cone techniques, where the cones were defined in the functional space upon which the transfer operator acts. We emphasize that they don't consider the case of hyperbolic dynamics studied in the present paper. In fact, is not clear if their cone techniques can be adapted to the present setting. Bakhtin [1] is probably the closest to our work; he proves a central limit theorem and large deviation estimates for mixing sequences of smooth hyperbolic maps with common expanding and contracting directions, under a variance growth condition on the Birkhoff sums. He also used cones, but living on the tangent space of the manifold. In comparison to Bakhtin, we can additionally treat the case of random piecewise hyperbolic maps (with singularities and including billiards), and moreover we exhibit explicitly the rate function which produces asymptotic large deviation bounds; the local CLT is also new in this setting.

2. Preliminaries

Let X be a d-dimensional C^∞ compact connected Riemannian manifold and let T be a topologically transitive Anosov map of class C^{r+1}, where $r > 2$. We follow the setup of [15]. Replacing the Riemannian metric by an adapted metric [23], we use hyperbolicity constants $0 < \nu < 1 < \lambda$, where λ is less than the minimal expansion along the unstable directions, ν is greater than the minimal contraction along the stable directions, and the angles between the stable and unstable spaces (of dimensions d_s, d_u, respectively) are close to $\pi/2$. A collection of C^∞ coordinate charts $\psi_i : (-r_i, r_i)^d \to X$, $i = 1, \ldots, N$ are defined so that $\bigcup_{i=1}^N \psi_i((-r_i/2, r_i/2)^d)$ cover X, with the r_i small enough that $D\psi_i(0) \cdot (\mathbb{R}^{d_s} \times$

$\{0\}) = E^s(\psi_i(0))$, $|\psi_i|_{C^{r+1}}, |\psi_i^{-1}|_{C^{r+1}} \leq 1+\kappa$, and κ small enough in such a way that the stable cone at x in $\mathbb{R}^d$ is compatibly mapped to the stable cone at $\psi_i(x)$ in X (see [15] for details). Let $G_i(K)$ denote the set of graphs of C^{r+1} functions $\chi : (-r_i, r_i)^{d_s} \to (-r_i, r_i)^{d_u}$ with $|\chi|_{C^{r+1}} \leq K$ (and with $|D\chi| \leq c_i$ so that the tangent space of the graph belongs to the stable cone in $\mathbb{R}^d$ mentioned above). For large enough K, the coordinate map $\psi_j^{-1} \circ T^{-1} \circ \psi_i$ maps $G_i(K)$ into $G_j(K')$ for some $K' < K$. For A sufficiently large, (depending on κ and ν) and δ small enough that $A\delta < \min_i r_i/6$, an admissible graph is a map $\chi : \bar{B}(x, A\delta) \to (-2r_i/3, 2r_i/3)^{d_u}$, $\bar{B}(x, A\delta) \subset (-2r_i/3, 2r_i/3)^{d_s}$; the collection of admissible graphs is denoted Ξ_i.

For $p \in \mathbb{N}$, $p \leq r$, $q \geq 0$ and $h \in C^r(X, \mathbb{C})$, $\varphi \in C^q(X, \mathbb{C})$ we define (using the notation in [15])

$$\|h\|_{p,q}^{\sim} := \sup_{\substack{|\alpha|=p \\ 1\leq i\leq N}} \sup_{\substack{\chi\colon \overline{B}(x,A\delta)\to\mathbb{R}^{d_u} \\ \chi\in\Xi_i}} \sup_{\substack{\varphi\in C_0^q(\overline{B}(x,\delta),\mathbb{C}) \\ |\varphi|_{C^q}\leq 1}} \left| \int_{B(x,\delta)} \left[\partial^\alpha(h\circ\psi_i)\right] \circ (\mathrm{Id}, \chi) \cdot \varphi \right|. \tag{3}$$

Finally, for p and q as above satisfying $p + q < r$, we set

$$\|h\|_{p,q} := \sup_{0\leq k\leq p} \|h\|_{k,q+k}^{\sim} = \sup_{p'\leq p, q'\geq q+p'} \|h\|_{p',q'}^{\sim}. \tag{4}$$

The space $\mathcal{B}^{p,q}$ is defined to be the completion of $C^r(X, \mathbb{C})$ with respect to the norm $\|\cdot\|_{p,q}$.

The following proposition will be useful when applying the multiplicative ergodic theorem.

Proposition 2.1. *The space $\mathcal{B}^{p,q}$ is separable.*

Proof. The desired conclusion follows directly from [15, Remark 4.3] after we note that $C^\infty(X, \mathbb{C})$ has a countable subset which is dense with respect to the C^r norm. □

We recall from [15, Section 4] that the elements of $\mathcal{B}^{p,q}$ are distributions of order at most q. More precisely, there exists $C > 0$ such that any $h \in \mathcal{B}^{p,q}$ induces a linear functional $\varphi \to h(\varphi)$ with the property that

$$|h(\varphi)| \leq C\|h\|_{p,q}|\varphi|_{C^q}, \quad \text{for } \varphi \in C^q(X, \mathbb{C}). \tag{5}$$

In particular, for $h \in C^r$ we have that

$$h(\varphi) = \int_X h\varphi, \quad \text{for } \varphi \in C^q(X, \mathbb{C}). \tag{6}$$

We say that $h \in \mathcal{B}^{p,q}$ is nonnegative and write $h \geq 0$ if $h(\varphi) \geq 0$ for any $\varphi \in C^q(X, \mathbb{R})$ such that $\varphi \geq 0$.

Let $\mathcal{L}_T \colon \mathcal{B}^{p,q} \to \mathcal{B}^{p,q}$ be the transfer operator associated to T defined by

$$(\mathcal{L}_T h)(\varphi) = h(\varphi \circ T), \quad \text{for } h \in \mathcal{B}^{p,q} \text{ and } \varphi \in C^q(X, \mathbb{C}). \tag{7}$$

We recall that for $h \in C^r(X, \mathbb{C})$, $\mathcal{L}_T$ is the function given by

$$\mathcal{L}_T h = \left(\frac{h}{|\det T|}\right) \circ T^{-1}. \tag{8}$$

Take $g \in C^r(X, \mathbb{C})$ and $h \in \mathcal{B}^{p,q}$. Then, there exists a sequence $(h_n)_n \subset C^r(X, \mathbb{C})$ that converges to h in $\mathcal{B}^{p,q}$. It follows that $(gh_n)_n \subset C^r(X, \mathbb{C})$ is a Cauchy sequence in $\mathcal{B}^{p,q}$ and therefore it converges to some element of $\mathcal{B}^{p,q}$ which we denote by $g \cdot h$. It is straightfoward to verify that the above construction does not depend on the particular choice of the sequence $(h_n)_n$. Moreover, the action of $g \cdot h$ as a distribution is given by

$$(g \cdot h)(\varphi) = h(g\varphi), \quad \varphi \in C^q(X, \mathbb{C}). \tag{9}$$

We will need the following result.

Lemma 2.2. *For $h \in \mathcal{B}^{p,q}, g \in C^r(X, \mathbb{C})$ one has $\mathcal{L}_T(g \circ T \cdot h) = g \cdot \mathcal{L}_T h$.*

Proof. Let $\varphi \in C^q(X, \mathbb{C})$. It follows from (7) and (9) that $[\mathcal{L}_T(g \circ T \cdot h)](\varphi) = (g \circ T \cdot h)(\varphi \circ T) = h(g \circ T \cdot \varphi \circ T) = \mathcal{L}_T h(g \cdot \varphi) = (g \cdot \mathcal{L}_T h)(\varphi)$, which yields the desired result. □

3. Building the cocycle $\mathcal{L}$

In the sequel we will consider the case $p = q = 1$ and $r > 2$, but we will also require T to be C^{r+1}, to be in a suitable framework for perturbations. Using the fact that the unit ball in $\mathcal{B}^{1,1}$ is relatively compact in $\mathcal{B}^{0,2}$ [15, Lemma 2.1], it follows from [15, Theorem 2.3] that the associated transfer operator $\mathcal{L}_T$ is quasicompact on $\mathcal{B}^{1,1}$, 1 is a simple eigenvalue and there are no other eigenvalues of modulus 1. This in particular implies (using the terminology as in [4, Definition 2.6]) that $\mathcal{L}_T$ is exact in $\{h \in \mathcal{B}^{1,1} : h(1) = 0\}$. Let

$$\mathcal{M}_\epsilon(T) = \{S : \ S \text{ is an Anosov map of class } C^{r+1} \text{ satisfying } d_{C^{r+1}}(S, T) < \epsilon\}.$$

We also recall (see [15, Lemmas 2.1. and 2.2] and the discussion at the beginning of §7 [15]) that there exist $\epsilon, A > 0$ and $c \in (0, 1)$ such that for any $T' \in \mathcal{M}_\epsilon(T)$, one has

- $\|\mathcal{L}_{T'}^n h\|_{0,2} \leq A\|h\|_{0,2}$ for each $n \in \mathbb{N}$ and $h \in \mathcal{B}^{1,1}$;
- $\|\mathcal{L}_{T'}^n h\|_{1,1} \leq Ac^n\|h\|_{1,1} + A\|h\|_{0,2}$ for each $n \in \mathbb{N}$ and $h \in \mathcal{B}^{1,1}$.

Consider the family of (bounded, linear) transfer operators, acting on the Banach space $(\mathcal{B}^{1,1}, \|\cdot\|)$,

$$\mathcal{O}_\epsilon(T, \mathcal{B}^{1,1}) = \{\mathcal{L}_S : \mathcal{B}^{1,1} \to \mathcal{B}^{1,1} \text{ such that } S \in \mathcal{M}_\epsilon(T)\}.$$

It follows from [4, Proposition 2.10] (applied to the case where $\|\cdot\| = \|\cdot\|_{0,2}$ and $|\cdot|_v = \|\cdot\|_{1,1}$) that there exists $0 < \epsilon_0 \leq \epsilon$, $D, \lambda > 0$ such that for any $\mathcal{L}_{T_1}, \ldots, \mathcal{L}_{T_n} \in \mathcal{O}_{\epsilon_0}(T, \mathcal{B}^{1,1})$, one has

$$\|\mathcal{L}_{T_n} \circ \cdots \circ \mathcal{L}_{T_2} \circ \mathcal{L}_{T_1} h\|_{1,1} \leq De^{-\lambda n}\|h\|_{1,1} \quad \text{for } h \in \mathcal{B}^{1,1} \text{ satisfying } h(1) = 0, \tag{10}$$

where $\mathcal{L}_{T_i}$ denotes the transfer operator associated with T_i. From now on, we replace ϵ_0 by ϵ so that (10) holds on $\mathcal{O}_\epsilon(T, \mathcal{B}^{1,1})$.

We now build the cocycle $\mathcal{R} := (\Omega, \mathcal{F}, \mathbb{P}, \sigma, \mathcal{B}^{1,1}, \mathcal{L})$, simply referred to as $\mathcal{L}$, as follows:

(1) Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, where Ω is a Borel subset of a separable, complete metric space and $\sigma : \Omega \to \Omega$ an ergodic, invertible $\mathbb{P}$-preserving transformation.

(2) Let $\mathcal{T} : \Omega \to \mathcal{M}_\epsilon(T)$ be a measurable map given by $\omega \mapsto T_\omega$. By applying [15, Lemma 7.1] we find that there exists $C > 0$ such that for any $S \in \mathcal{M}_\epsilon(T)$,

$$\sup_{\|h\|_{1,1} \leq 1} \|(\mathcal{L}_S - \mathcal{L}_T)h\|_{0,2} \leq C\epsilon.$$

3.1. **Strong measurability of $\omega \mapsto \mathcal{L}_\omega$.** In this section we demonstrate strong measurability of the map $\mathcal{L} : \Omega \to \mathcal{O}_\epsilon(T, \mathcal{B}^{1,1})$ given by $\omega \mapsto \mathcal{L}_\omega := \mathcal{L}_{T_\omega}$; this is required to establish the existence of measurable Oseledets spaces for the cocycle. To prove strong measurability of $\omega \mapsto \mathcal{L}_\omega := \mathcal{L}_{T_\omega}$, we will show that the map from $\mathcal{M}_\epsilon(T)$ to $\mathcal{B}^{1,1}$ defined by $S \mapsto \mathcal{L}_S$ is strongly continuous. For this, let $S \in \mathcal{M}_\epsilon(T)$ and $h \in \mathcal{B}^{1,1}$. We must show that $\|\mathcal{L}_{\tilde{S}} h - \mathcal{L}_S h\|_{1,1} \to 0$ as $d_{C^{r+1}}(\tilde{S}, S) \to 0$. First, assume $h \in C^r$. Then, we need to estimate differences of the form

$$\left| \int_{B(x,\delta)} \left[\partial^\alpha(\mathcal{L}_S h \circ \psi_i)\right] \circ (\mathrm{Id}, \chi) \cdot \varphi - \int_{B(x,\delta)} \left[\partial^\alpha(\mathcal{L}_{\tilde{S}} h \circ \psi_i)\right] \circ (\mathrm{Id}, \chi) \cdot \varphi \right|,$$

where α, χ and φ vary as in the definition in (3), with $p = q = 1$. Arguing as in [15, Lemma 7.1], and employing the corresponding notation, we write

$$\int_{B(x,\delta)} \left[\partial^\alpha(\mathcal{L}_S h \circ \psi_i)\right] \circ (\mathrm{Id}, \chi) \cdot \varphi = \sum_{|\beta| \le |\alpha|} \sum_{j=1}^{l} \int_{B(x_j,\delta)} \partial^\beta \tilde{h}_j \circ (\mathrm{Id}, \chi_j) \cdot F_{\alpha,\beta,S,j} \cdot \rho_j, \quad (11)$$

where $\chi_1, \dots, \chi_l$ are γ-admissible graphs whose corresponding γ-admissible leaves cover $S^{-1}(W)$, with W an admissible leaf corresponding to the graph of χ; $\tilde{h}_j = h \circ \psi_{i(j)}$; $\{\rho_j\}_{j=1,\dots,l}$ is a partition of unity subordinated to the γ-admissible leaves of χ_j; and $F_{\alpha,\beta,S,j}$ are functions bounded in $C^{q+|\beta|}$. A similar expression holds for $\int_{B(x,\delta)} \left[\partial^\alpha(\mathcal{L}_{\tilde{S}} h \circ \psi_i)\right] \circ (\mathrm{Id}, \chi) \cdot \varphi$, with $F_{\alpha,\beta,S,j}$ replaced by $F_{\alpha,\beta,\tilde{S},j}$ and χ_j replaced by $\tilde{\chi}_j$, the graph corresponding to $\psi_{i(j)}^{-1} \circ \tilde{S}^{-1} \circ S \circ \psi_{i(j)} \circ (Id, \chi_j)(B(x_j, \gamma A\delta))$. Furthermore, if $d_{C^{r+1}}(S, \tilde{S})$ is small enough, each $\tilde{\chi}_j$ is a graph in $\Xi_{i(j)}$, and $|\chi_j - \tilde{\chi}_j|_{C^2(\bar{B}(x_j, A\delta))} < C d_{C^{r+1}}(S, \tilde{S})$. Also, $\|F_{\alpha,\beta,S,j}\|_{C^{q+|\beta|}}, \|F_{\alpha,\beta,\tilde{S},j}\|_{C^{q+|\beta|}}$ are uniformly bounded for $S, \tilde{S} \in \mathcal{M}_\epsilon(T)$ and $\|F_{\alpha,\beta,S,j} - F_{\alpha,\beta,\tilde{S},j}\|_{C^{q+|\beta|}} \to 0$ as $d_{C^{r+1}}(\tilde{S}, S) \to 0$, uniformly over φ as in (3). Hence, as $d_{C^{r+1}}(\tilde{S}, S) \to 0$, we get

$$\Big| \int_{B(x_j,\delta)} \partial^\beta(\tilde{h}_j) \circ (\mathrm{Id}, \tilde{\chi}_j) \cdot F_{\alpha,\beta,\tilde{S},j} \cdot \rho_j - \partial^\beta(\tilde{h}_j) \circ (\mathrm{Id}, \chi_j) \cdot F_{\alpha,\beta,S,j} \cdot \rho_j \Big| \to 0,$$

uniformly over χ (and so χ_j) and φ as in (3). It then follows from (11) that $\|\mathcal{L}_{\tilde{S}} h - \mathcal{L}_S h\|_{1,1} \to 0$ as $d_{C^{r+1}}(\tilde{S}, S) \to 0$, as claimed.

The result for general $h \in \mathcal{B}^{1,1}$ follows from an approximation argument by C^r functions, because if $d_{C^{r+1}}(S, \tilde{S})$ is sufficiently small, then $\|\mathcal{L}_{\tilde{S}}\|_{1,1} \le 1 + \|\mathcal{L}_S\|_{1,1} =: M$. Indeed, let $\{h_j\}_{j\in\mathbb{N}}$ be a sequence of C^r functions such that $\lim_{j\to\infty} h_j = h$ in $B^{1,1}$, and let $\epsilon > 0$. Then, there exists $n \in \mathbb{N}$ such that $\|h - h_n\|_{1,1} < \frac{\epsilon}{3M}$. Hence, $\|\mathcal{L}_{\tilde{S}} h - \mathcal{L}_S h\|_{1,1} \le \|\mathcal{L}_{\tilde{S}} h - \mathcal{L}_{\tilde{S}} h_n\|_{1,1} + \|\mathcal{L}_{\tilde{S}} h_n - \mathcal{L}_S h_n\|_{1,1} + \|\mathcal{L}_S h_n - \mathcal{L}_S h\|_{1,1} \le \frac{2\epsilon}{3} + \|\mathcal{L}_{\tilde{S}} h_n - \mathcal{L}_S h_n\|_{1,1}$. Since $h_n \in C^r$, we have that $\limsup_{d_{C^{r+1}}(\tilde{S},S)\to 0} \|\mathcal{L}_{\tilde{S}} h - \mathcal{L}_S h\|_{1,1} \le \frac{2\epsilon}{3}$. Since the choice of $\epsilon > 0$ is arbitrary, the result follows.

3.2. **Quasi-compactness of the cocycle $\mathcal{L}$ and existence of Oseledets splitting.** We may apply Kingman's subadditive ergodic theorem to form the following limits, which are constant for $\mathbb{P}$-a.e. $\omega \in \Omega$:

$$\Lambda(\mathcal{R}) := \lim_{n\to\infty} \frac{1}{n} \log \|\mathcal{L}_\omega^{(n)}\|_{1,1}, \text{ and}$$

$$\kappa(\mathcal{R}) := \lim_{n\to\infty} \frac{1}{n} \log ic(\mathcal{L}_\omega^{(n)}), \quad \text{where}$$

$$ic(A) := \inf\{r > 0 : A(B_{\mathcal{B}_{1,1}}) \text{ can be covered with finitely many balls of radius } r\},$$

and $B_{\mathcal{B}_{1,1}}$ is the unit ball in $\mathcal{B}_{1,1}$. The cocycle $\mathcal{R}$ is called *quasi-compact* if $\Lambda(\mathcal{R}) > \kappa(\mathcal{R})$.

For each $\omega \in \Omega, n \in \mathbb{N}$, let $\mathcal{L}_\omega^{(n)} := \mathcal{L}_{\sigma^{n-1}\omega} \circ \cdots \circ \mathcal{L}_{\sigma\omega} \circ \mathcal{L}_\omega$. It follows readily from (10) that

$$\|\mathcal{L}_\omega^{(n)} h\|_{1,1} \le D e^{-\lambda n} \|h\|_{1,1} \quad \text{for any } \omega \in \Omega,\ n \in \mathbb{N} \text{ and } h \in \mathcal{B}^{1,1},\ h(1) = 0. \quad (12)$$

After possibly decreasing ϵ we can also assume[5] that there exist $a \in (0,1)$ and $B, K > 0$ such that for every $\omega \in \Omega$, $n \in \mathbb{N}$ and $h \in \mathcal{B}^{1,1}$,

$$\|\mathcal{L}_\omega^{(n)} h\|_{0,2} \le B\|h\|_{0,2}, \quad \|\mathcal{L}_\omega^{(n)} h\|_{1,1} \le B a^n \|h\|_{1,1} + B\|h\|_{0,2}, \quad (13)$$

[5]See the discussion at the beginning of §7 in [15].

which in particular implies that

$$\|\mathcal{L}_\omega h\|_{1,1} \leq K\|h\|_{1,1}, \tag{14}$$

where $K := Ba + B > 0$. By [9, Lemma 2.1], the inequalities (13) and (14) imply that the cocycle $\mathcal{R}$ is quasi-compact. By separability of $\mathcal{B}^{1,1}$, and quasi-compactness and strong measurability of $\mathcal{L}$, the multiplicative ergodic theorem (Theorem A, [11]) yields the existence of a measurable Oseledets splitting

$$\mathcal{B}^{1,1} = \left(\bigoplus_{j=1}^{l} Y_j(\omega)\right) \oplus V(\omega),$$

where each component of the splitting is equivariant under $\mathcal{L}_\omega$. The $Y_j(\omega)$ are finite-dimensional and by $0 = \lambda_1 > \lambda_2 > \dots$ we denote the corresponding (finite or infinite) sequence of Lyapunov exponents.

3.3. **One-dimensionality of the top Oseledets space.**

Proposition 3.1. *There exists a unique family $(h^0_\omega)_{\omega\in\Omega} \subset \mathcal{B}^{1,1}$ such that:*

(1) $\mathcal{L}_\omega h^0_\omega = h^0_{\sigma\omega}$ for $\mathbb{P}$-a.e. $\omega \in \Omega$;
(2) h^0_ω is nonnegative and $h^0_\omega(1) = 1$ for $\mathbb{P}$-a.e. $\omega \in \Omega$;
(3) $\omega \to h^0_\omega$ is a measurable map from Ω to $\mathcal{B}^{1,1}$;
(4)

$$\text{ess sup}_{\omega\in\Omega}\|h^0_\omega\|_{1,1} < \infty. \tag{15}$$

Proof. Let

$$Y = \{v\colon \Omega \to \mathcal{B}^{1,1} :\ v \text{ is measurable and } \|v\|_\infty := \text{ess sup}_{\omega\in\Omega}\|v(\omega)\|_{1,1} < \infty\}.$$

Then, $Y = (Y, \|\cdot\|_\infty)$ is a Banach space. Furthermore, let Z be the subset of Y that consists of $v \in Y$ with the property that $v(\omega)$ is nonnegative and $v(\omega)(1) = 1$ for $\mathbb{P}$-a.e. $\omega \in \Omega$. It is easy to verify that Z is a closed subset of Y. Indeed, assume that $(v_n)_{n\in\mathbb{Z}}$ is a sequence in Z converging to $v \in Y$. It follows from (5) that

$$|v(\omega)(\varphi) - v_n(\omega)(\varphi)| \leq C\|v_n(\omega) - v(\omega)\|_{1,1}|\varphi|_{C^1} \leq C\|v_n - v\|_\infty|\varphi|_{C^1},$$

and thus $v_n(\omega)(\varphi) \to v(\omega)(\varphi)$ for $\varphi \in C^1$ and $\mathbb{P}$-a.e. $\omega \in \Omega$. Thus, $v(\omega)(\varphi) \geq 0$ for $\varphi \geq 0$ and $v(\omega)(1) = 1$ for $\mathbb{P}$-a.e. $\omega \in \Omega$ and we conclude that $v \in Y$.

We define $\mathbb{L}\colon Z \to Z$ by

$$(\mathbb{L}v)(\omega) = \mathcal{L}_{\sigma^{-1}\omega}v(\sigma^{-1}\omega) \quad \omega \in \Omega,\ v \in Z.$$

It follows from (7) and (14) that $\mathbb{L}$ is a well-defined and continuous map on Z. Using (12), one can easily verify (see [8, Proposition 1]) that there exists $n_0 \in \mathbb{N}$ such that $\mathbb{L}^{n_0}$ is a contraction on Z. Thus, $\mathbb{L}$ has a unique fixed point $\bar{v} \in Z$. It is easy to verify that the family h^0_ω, $\omega \in \Omega$ defined $h^0_\omega = \bar{v}(\omega)$, $\omega \in \Omega$ satisfies the desired properties. Conversely, each family satisfying properties (1)-(4) induces a fixed point of $\mathbb{L}$ which then must coincide with $\bar{v}$. □

Proposition 3.2. *Let $(h^0_\omega)_{\omega\in\Omega}$ be as in Proposition 3.1. Then h^0_ω is a probability measure on $\mathcal{B}^{1,1}$ for $\mathbb{P}$-a.e. $\omega \in \Omega$.*

Proof. Using Lemma 2.2, we have that

$$h^0_\omega(\varphi) = \mathcal{L}^{(n)}_{\sigma^{-n}\omega}h^0_{\sigma^{-n}\omega}(\varphi) = h^0_{\sigma^{-n}\omega}(\varphi \circ T^{(n)}_{\sigma^{-n}\omega}), \quad \text{for } \omega \in \Omega \text{ and } \varphi \in C^1.$$

Hence, using the arguments as in [5, Lemma 5.3], and equations (5) and (15), we find that there exists a constant $D > 0$ such that

$$|h^0_\omega(\varphi)| \leq D|\varphi|_\infty \quad \text{for } \mathbb{P}\text{-a.e. } \omega \in \Omega \text{ and } \varphi \in C^1.$$

Since C^1 is dense in C^0, we conclude that for $\mathbb{P}$-a.e. $\omega \in \Omega$, h^0_ω can be extended to a bounded linear functional on C^0. By the Riesz representation theorem, h^0_ω is a signed measure. By invoking the nonnegativity of h^0_ω together with $h^0_\omega(1) = 1$, we conclude that h^0_ω is a probability measure for $\mathbb{P}$-a.e. $\omega \in \Omega$. □

Proposition 3.3. *The top Oseledets space $Y_1(\omega)$ of the cocycle $\mathcal{L}$ is one-dimensional, and spanned by h^0_ω.*

Proof. Proposition 3.1 and (13) imply that the top Lyapunov exponent of the cocycle $\mathcal{L}$ is equal to 0. Furthermore, it follows from (15) that

$$\lim_{n\to\infty} \frac{1}{n} \log\|\mathcal{L}^{(n)}_\omega h^0_\omega\|_{1,1} = \lim_{n\to\infty} \frac{1}{n} \log\|h^0_{\sigma^n\omega}\|_{1,1} \leq 0 \quad \text{for } \mathbb{P}\text{-a.e. } \omega \in \Omega.$$

On the other hand, it follows from (5) that $1 = h^0_\omega(1) \leq C\|h^0_\omega\|_{1,1}$ for $\mathbb{P}$-a.e. $\omega \in \Omega$. Thus,

$$\lim_{n\to\infty} \frac{1}{n} \log\|\mathcal{L}^{(n)}_\omega h^0_\omega\|_{1,1} = \lim_{n\to\infty} \frac{1}{n} \log\|h^0_{\sigma^n\omega}\|_{1,1} \geq \lim_{n\to\infty} \frac{1}{n} \log C^{-1} = 0 \quad \text{for } \mathbb{P}\text{-a.e. } \omega \in \Omega.$$

We conclude that $h^0_\omega \in Y_1(\omega)$ for $\mathbb{P}$-a.e. $\omega \in \Omega$. We now claim that h^0_ω spans $Y_1(\omega)$ for $\mathbb{P}$-a.e. $\omega \in \Omega$. Indeed, assume that there exists $g_\omega \notin span\{h^0_\omega\}$, $g_\omega \in Y_1(\omega)$ and choose α, β scalars (that depend on ω) such that $|\alpha| + |\beta| > 0$ and $(\alpha h^0_\omega + \beta g_\omega)(1) = 0$. Then, it follows from (12) that

$$\lim_{n\to\infty} \frac{1}{n} \log\|\mathcal{L}^{(n)}_\omega(\alpha h^0_\omega + \beta g_\omega)\|_{1,1} \leq -\lambda < 0.$$

On the other hand, since $\alpha h^0_\omega + \beta g_\omega \in Y_1(\omega) \setminus \{0\}$ we have

$$\lim_{n\to\infty} \frac{1}{n} \log\|\mathcal{L}^{(n)}_\omega(\alpha h^0_\omega + \beta g_\omega)\|_{1,1} = 0,$$

which yields a contradiction. We conclude that $Y_1(\omega) = span\{h^0_\omega\}$ and thus $Y_1(\omega)$ is one-dimensional for $\mathbb{P}$-a.e. $\omega \in \Omega$.

□

4. Quasi-compactness of the twisted cocycle $\mathcal{L}^\theta$

We build a twisted cocycle $\mathcal{L}^\theta$, by setting

$$\mathcal{L}^\theta_\omega(h) = \mathcal{L}_\omega(e^{\theta g(\omega,\cdot)} \cdot h), \quad \text{for } \omega \in \Omega,\ \theta \in \mathbb{C}, \text{ and } h \in \mathcal{B}^{1,1}.$$

We will from now write $e^{\theta g(\omega,\cdot)}h$ instead of $e^{\theta g(\omega,\cdot)} \cdot h$. Our (centered) observable g will be a map $g\colon \Omega \times X \to \mathbb{R}$ such that $g(\omega, \cdot) \in C^r$ for $\omega \in \Omega$,

$$\text{ess sup}_{\omega\in\Omega}\|g(\omega,\cdot)\|_{C^r} < \infty, \tag{16}$$

and for $\mathbb{P}$-a.e. $\omega \in \Omega$,

$$h^0_\omega(g(\omega,\cdot)) = 0. \tag{17}$$

This twisted cocycle gives us access to an ω-wise moment-generating function for Birkhoff sums of g.

Lemma 4.1. *For $\mathbb{P}$-a.e. $\omega \in \Omega$, $h \in \mathcal{B}^{1,1}$ and $\varphi \in C^1(X, \mathbb{C})$ one has*

$$(\mathcal{L}^{\theta,(n)}_\omega h)(\varphi) = h(e^{\theta S_n g(\omega,\cdot)}(\varphi \circ T^n_\omega)). \tag{18}$$

Proof. One can follow the proof of Lemma 3.3 (part 2) [9], using the definition of the untwisted transfer operator (7) and Lemma 2.2. □

The following lemma is required as an auxiliary result in the proof of quasi-compactness of the twisted cocycle (Proposition 4.4).

Lemma 4.2. *For $\theta_1, \theta_2 \in B_{\mathbb{C}}(0,1) := \{\theta \in \mathbb{C} : |\theta| < 1\}$, we have that*

$$\text{ess sup}_{\omega\in\Omega}\|e^{\theta_1 g(\sigma^{-1}\omega,\cdot)} - e^{\theta_2 g(\sigma^{-1}\omega,\cdot)}\|_{C^2} \le C|\theta_1 - \theta_2|.$$

Proof. By applying the mean value theorem for the map $g(z) = e^{zg(\sigma^{-1}\omega,x)}$, where $x \in X$ is fixed and using (16), we find that

$$\text{ess sup}_{\omega\in\Omega}\|e^{\theta_1 g(\sigma^{-1}\omega,\cdot)} - e^{\theta_2 g(\sigma^{-1}\omega,\cdot)}\|_{C^0} \le C|\theta_1 - \theta_2|. \tag{19}$$

Furthermore, for $j = 1, \dots, d$

$$\begin{aligned}\|\partial^j(e^{\theta_1 g(\sigma^{-1}\omega,\cdot)} - e^{\theta_2 g(\sigma^{-1}\omega,\cdot)})\|_{C^0} &= \|e^{\theta_1 g(\sigma^{-1}\omega,\cdot)}\theta_1\partial^j g(\sigma^{-1}\omega,\cdot) - e^{\theta_2 g(\sigma^{-1}\omega,\cdot)}\theta_2\partial^j g(\sigma^{-1}\omega,\cdot)\|_{C^0}\\ &\le |\theta_1 - \theta_2| \cdot \|e^{\theta_1 g(\sigma^{-1}\omega,\cdot)}\partial^j g(\sigma^{-1}\omega,\cdot)\|_{C^0}\\ &\quad + |\theta_2| \cdot \|e^{\theta_1 g(\sigma^{-1}\omega,\cdot)} - e^{\theta_2 g(\sigma^{-1}\omega,\cdot)}\|_{C^0} \cdot \|\partial^j g(\sigma^{-1}\omega,\cdot)\|_{C^0}.\end{aligned}$$

It now follows from (16) and (19) that

$$\text{ess sup}_{\omega\in\Omega}\|\partial^j(e^{\theta_1 g(\sigma^{-1}\omega,\cdot)} - e^{\theta_2 g(\sigma^{-1}\omega,\cdot)})\|_{C^0} \le C|\theta_1 - \theta_2|.$$

One can now proceed and obtain the same estimates for the second derivatives of the map $e^{\theta_1 g(\sigma^{-1}\omega,\cdot)} - e^{\theta_2 g(\sigma^{-1}\omega,\cdot)}$ which implies the desired conclusion. □

We need the following basic regularity result for the operators $\mathcal{L}_\omega^\theta$.

Proposition 4.3. *There exists a continuous function $K\colon \mathbb{C} \to (0,\infty)$ such that*

$$\|\mathcal{L}_\omega^\theta h\|_{1,1} \le K(\theta)\|h\|_{1,1}, \quad \textit{for } h \in \mathcal{B}^{1,1},\ \theta \in \mathbb{C} \textit{ and } \mathbb{P}\textit{-a.e. } \omega \in \Omega. \tag{20}$$

Proof. We first note that it follows from (14) that

$$\|\mathcal{L}_\omega^\theta h\|_{1,1} = \|\mathcal{L}_\omega(e^{\theta g(\omega,\cdot)}h)\|_{1,1} \le K\|e^{\theta g(\omega,\cdot)}h\|_{1,1}, \quad \text{for } h \in \mathcal{B}^{1,1},\ \theta \in \mathbb{C} \text{ and } \mathbb{P}\text{-a.e. } \omega \in \Omega.$$

Hence, we need to estimate $\|e^{\theta g(\omega,\cdot)}h\|_{1,1}$. Note that by (4),

$$\|e^{\theta g(\omega,\cdot)}h\|_{1,1} = \max\{\|e^{\theta g(\omega,\cdot)}h\|_{0,1}^{\sim}, \|e^{\theta g(\omega,\cdot)}h\|_{1,2}^{\sim}\}.$$

It follows easily from (3) that

$$\|e^{\theta g(\omega,\cdot)}h\|_{0,1}^{\sim} \le \Big(\max_{1\le i\le N} \sup_{\substack{\chi\colon \overline{B}(x,A\delta)\to\mathbb{R}^{d_u}\\ \chi\in\Xi_i}} \|(e^{\theta g(\omega,\cdot)} \circ \psi_i) \circ (\mathrm{Id},\chi)\|_{C^1}\Big) \cdot \|h\|_{0,1}^{\sim}$$

and

$$\begin{aligned}\|e^{\theta g(\omega,\cdot)}h\|_{1,2}^{\sim} \le{}& \Big(\max_{1\le i\le N} \sup_{\substack{\chi\colon \overline{B}(x,A\delta)\to\mathbb{R}^{d_u}\\ \chi\in\Xi_i}} \|(e^{\theta g(\omega,\cdot)} \circ \psi_i) \circ (\mathrm{Id},\chi)\|_{C^2}\Big) \cdot \|h\|_{1,2}^{\sim}\\ &+ \left(\max_{\substack{1\le j\le d\\ 1\le i\le N}} \sup_{\substack{\chi\colon \overline{B}(x,A\delta)\to\mathbb{R}^{d_u}\\ \chi\in\Xi_i}} \|\big[\partial^j(e^{\theta g(\omega,\cdot)} \circ \psi_i)\big] \circ (\mathrm{Id},\chi)\|_{C^1}\right) \cdot \|h\|_{0,1}^{\sim},\end{aligned}$$

which together with (16) implies the desired conclusion. □

We can now state the main result of this section on quasi-compactness.

Proposition 4.4. *For θ close to 0, the cocycle $(\mathcal{L}_\omega^\theta)_{\omega\in\Omega}$ is quasi-compact.*

Proof. We follow closely [9, Lemma 3.13]. Observe (13) and choose $N \in \mathbb{N}$ such that $\gamma := Ba^N < 1$. Hence,

$$\begin{aligned}\|\mathcal{L}_\omega^{\theta,(N)} h\|_{1,1} &\leq \|\mathcal{L}_\omega^{(N)} h\|_{1,1} + \|\mathcal{L}_\omega^{\theta,(N)} - \mathcal{L}_\omega^{(N)}\|_{1,1} \cdot \|h\|_{1,1} \\ &\leq \gamma \|h\|_{1,1} + B\|h\|_{0,2} + \|\mathcal{L}_\omega^{\theta,(N)} - \mathcal{L}_\omega^{(N)}\|_{1,1} \cdot \|h\|_{1,1}.\end{aligned}$$

On the other hand, we have that

$$\mathcal{L}_\omega^{\theta,(N)} - \mathcal{L}_\omega^{(N)} = \sum_{j=0}^{N-1} \mathcal{L}_{\sigma^{N-j}\omega}^{\theta,(j)} (\mathcal{L}_{\sigma^{N-1-j}\omega}^{\theta} - \mathcal{L}_{\sigma^{N-1-j}\omega}) \mathcal{L}_\omega^{(N-1-j)}.$$

It follows from (14) and (20) that

$$\|\mathcal{L}_\omega^{(N-1-j)}\|_{1,1} \leq K^{N-1-j} \quad \text{and} \quad \|\mathcal{L}_{\sigma^{N-j}\omega}^{\theta,(j)}\|_{1,1} \leq K(\theta)^j.$$

Furthermore, using (14), we have that for any $h \in \mathcal{B}^{1,1}$ and $\mathbb{P}$-a.e. $\omega \in \Omega$,

$$\|(\mathcal{L}_\omega^\theta - \mathcal{L}_\omega)(h)\|_{1,1} = \|\mathcal{L}_\omega(e^{\theta g(\omega,\cdot)} h - h)\|_{1,1} \leq K\|(e^{\theta g(\omega,\cdot)} - 1)h\|_{1,1}.$$

Moreover,

$$\|(e^{\theta g(\omega,\cdot)} - 1)h\|_{1,1} = \max\{\|(e^{\theta g(\omega,\cdot)} - 1)h\|_{0,1}^{\sim}, \|(e^{\theta g(\omega,\cdot)} - 1)h\|_{1,2}^{\sim}\}.$$

Now Lemma 4.2 (applied for $\theta_1 = \theta$ and $\theta_2 = 0$) implies that there exists $C > 0$ such for $\theta \in B_{\mathbb{C}}(0,1)$,

$$\|(e^{\theta g(\omega,\cdot)} - 1)h\|_{1,1} \leq C|\theta|\|h\|_{1,1} \quad \text{for } h \in \mathcal{B}^{1,1}.$$

We conclude that

$$\|\mathcal{L}_\omega^{\theta,(N)} - \mathcal{L}_\omega^{(N)}\|_{1,1} \leq C|\theta| \sum_{j=0}^{N-1} K^{N-1-j} K(\theta)^j,$$

and therefore there exists $\tilde{\gamma} \in (0,1)$ such that for any θ sufficiently close to 0 and $h \in \mathcal{B}^{1,1}$,

$$\|\mathcal{L}_\omega^{\theta,(N)} h\|_{1,1} \leq \tilde{\gamma}\|h\|_{1,1} + B\|h\|_{0,2}. \tag{21}$$

Similarly, one can show that there exists $\tilde{B} > 0$ such that for any θ sufficiently close to 0 and $h \in \mathcal{B}^{1,1}$,

$$\|\mathcal{L}_\omega^\theta h\|_{0,2} \leq \tilde{B}\|h\|_{0,2}. \tag{22}$$

The conclusion of the proposition follows from (21) and (22) by arguing as in the quasi-compactness part of the proof of [9, Theorem 3.12]. □

5. Regularity of the top Oseledets space of the twisted cocycle

Let $\mathcal{S}'$ be the space of measurable maps $\mathcal{V}\colon \Omega \to \mathcal{B}^{1,1}$ with the property that

$$\|\mathcal{V}\|_\infty := \text{ess sup}_{\omega\in\Omega} \|\mathcal{V}(\omega)\|_{1,1} < \infty.$$

Then, $(\mathcal{S}', \|\cdot\|_\infty)$ is a Banach space. Furthermore, let $\mathcal{S}$ be the set of all $\mathcal{V} \in \mathcal{S}'$ such that $\mathcal{V}(\omega)(1) = 0$ for $\mathbb{P}$-a.e. $\omega \in \Omega$. Arguing as in the proof of Proposition 3.1, it is easy to verify that $\mathcal{S}$ is a closed subspace of $\mathcal{S}'$. For $\mathcal{V} \in \mathcal{S}'$ and $\omega \in \Omega$ we will often write $\mathcal{V}_\omega$ instead of $\mathcal{V}(\omega)$.

5.1. **Regularity of the cocycles.**

Lemma 5.1.

(1) For $\mathbb{P}$-a.e. $\omega \in \Omega$, the map $\theta \mapsto \mathcal{L}_\omega^\theta$ is analytic in the norm topology of $\mathcal{B}^{1,1}$.
(2) The map $\mathcal{P} : B_{\mathbb{C}}(0,1) \times \mathcal{S} \to \mathcal{S}$, given by $\mathcal{P}(\theta, \mathcal{V})_\omega = \mathcal{L}^\theta_{\sigma^{-1}\omega}(\mathcal{V}_{\sigma^{-1}\omega})$ is analytic in θ and bounded, linear in $\mathcal{V}$. In particular, $\mathcal{P}$ is C^∞.
(3) The map $\mathcal{P}_1 : B_{\mathbb{C}}(0,1) \times \mathcal{S} \to \mathcal{L}^\infty(\Omega)$, given by $\mathcal{P}_1(\theta, \mathcal{V})_\omega = (\mathcal{L}^\theta_{\sigma^{-1}\omega}(\mathcal{V}_{\sigma^{-1}\omega}))(1)$ is analytic in θ and bounded, linear in $\mathcal{V}$. In particular, $\mathcal{P}_1$ is C^∞.

Proof. We claim that for every $h \in \mathcal{B}^{1,1}$, the following holds:

$$\mathcal{L}_\omega^\theta(h) = \sum_{k=0}^{\infty} \frac{\theta^k}{k!} \mathcal{L}_\omega(g(\omega,\cdot)^k h), \quad \text{in } \mathcal{B}^{1,1}. \tag{23}$$

To verify this, note that [15, Lemma 3.2] implies that

$$\|\mathcal{L}_\omega(g(\omega,\cdot)^k h)\|_{1,1} \le C\|g(\omega,\cdot)^k\|_{C^2}\|h\|_{1,1} \le C\|g(\omega,\cdot)\|_{C^2}^k\|h\|_{1,1},$$

so by (16), the RHS of (23) is a well defined element of $\mathcal{B}^{1,1}$. The fact that it coincides with $\mathcal{L}_\omega^\theta(h)$ is straightforward to check, using linearity of $\mathcal{L}_\omega$, the power series expansion of $e^{\theta g(\omega,\cdot)}$, and testing against functions $\varphi \in \mathcal{C}^1$. This concludes the proof of Lemma 5.1(1).

Let us prove Lemma 5.1(2). For each $k \ge 0$ and $\mathcal{V} \in \mathcal{S}$, let $(g^k \cdot \mathcal{V})(\omega,\cdot) := g(\omega,\cdot)^k \mathcal{V}(\omega,\cdot)$. Then, $g^k \cdot \mathcal{V} \in \mathcal{S}$, because of (16) and [15, Lemma 3.2]. We claim that

$$\mathcal{P}(\theta, \mathcal{V}) = \sum_{k=0}^{\infty} \frac{\theta^k}{k!} \mathcal{P}(0, g^k \cdot \mathcal{V}) \quad \text{in } \mathcal{S}. \tag{24}$$

Indeed, (23) implies that

$$\mathcal{P}(\theta, \mathcal{V})_\omega = \sum_{k=0}^{\infty} \frac{\theta^k}{k!} \mathcal{P}(0, g^k \cdot \mathcal{V})_\omega \quad \text{in } \mathcal{B}^{1,1}. \tag{25}$$

Furthermore, using once again [15, Lemma 3.2], in combination with the uniform over ω bounds (14) and (16), we have that there exists $C > 0$ such that for $\mathbb{P}$-a.e. $\omega \in \Omega$,

$$\|\mathcal{P}(\theta, \mathcal{V})_\omega\|_{1,1} \le \sum_{k=0}^{\infty} \frac{\theta^k}{k!} \|\mathcal{P}(0, g^k \cdot \mathcal{V})_\omega\|_{1,1} \le C \sum_{k=0}^{\infty} \frac{\theta^k}{k!} \operatorname{ess\,sup}_{\omega\in\Omega} \|g(\omega,\cdot)\|_{C^2}^k \|\mathcal{V}\|_\infty. \tag{26}$$

Hence, the series in (24) indeed converges in $\mathcal{S}$ and yields analyticity as required. The fact that $\mathcal{V} \mapsto \mathcal{P}(\theta, \mathcal{V})$, and also $\mathcal{V} \mapsto \mathcal{P}(0, g^k \cdot \mathcal{V})$ is linear and bounded is straightforward to check. Hence, the C^∞ claim follows immediately.

The proof of Lemma 5.1(3) is similar to that of Lemma 5.1(2). Indeed,

$$\mathcal{P}_1(\theta, \mathcal{V})_\omega = \sum_{k=0}^{\infty} \frac{\theta^k}{k!} \langle \mathcal{P}(0, g^k \cdot \mathcal{V})_\omega, 1\rangle, \tag{27}$$

and (5) implies that $|\langle \mathcal{P}(0, g^k \cdot \mathcal{V})_\omega, 1\rangle| \le C\|\mathcal{P}(0, g^k \cdot \mathcal{V})_\omega\|_{1,1}$, which was bounded uniformly over ω in (26). Hence, the series (27) converges to $\mathcal{P}_1(\theta, \mathcal{V})$ in $L^\infty(\Omega)$. □

5.2. **An auxiliary function F and its regularity.** For $\theta \in \mathbb{C}$ and $\mathcal{W} \in \mathcal{S}$, set

$$F(\theta, \mathcal{W})(\omega) = \frac{\mathcal{L}^\theta_{\sigma^{-1}\omega}(\mathcal{W}(\sigma^{-1}\omega) + h^0_{\sigma^{-1}\omega})}{\mathcal{L}^\theta_{\sigma^{-1}\omega}(\mathcal{W}(\sigma^{-1}\omega) + h^0_{\sigma^{-1}\omega})(1)} - \mathcal{W}(\omega) - h^0_\omega, \quad \omega \in \Omega. \tag{28}$$

We define two further auxiliary functions, which will be used in the sequel. Let $G\colon \mathbb{C}\times\mathcal{S}\to\mathcal{S}'$ and $H\colon \mathbb{C}\times\mathcal{S}\to L^\infty(\Omega)$ be given by

$$G(\theta,\mathcal{W})(\omega) := \mathcal{P}(\theta,\mathcal{W}+h^0)(\omega) = \mathcal{L}^\theta_{\sigma^{-1}\omega}(\mathcal{W}_{\sigma^{-1}\omega}+h^0_{\sigma^{-1}\omega}), \quad \omega\in\Omega, \tag{29}$$

$$H(\theta,\mathcal{W})(\omega) := \mathcal{P}_1(\theta,\mathcal{W}+h^0)(\omega) = \mathcal{L}^\theta_{\sigma^{-1}\omega}(\mathcal{W}_{\sigma^{-1}\omega}+h^0_{\sigma^{-1}\omega})(1), \quad \omega\in\Omega. \tag{30}$$

It follows readily from (15) and Lemma 5.1 that G and H are well defined, and in fact C^∞ functions. Direct calculations, analogous to those of [9, Appendix B], yield the following:

Lemma 5.2. *For $\omega\in\Omega;\theta,z\in\mathbb{C}$; $\mathcal{W},\mathcal{H}\in\mathcal{S}$, the following identities hold:*

$$D_1G(\theta,\mathcal{W})(z)_\omega = z\mathcal{L}_{\sigma^{-1}\omega}(g(\sigma^{-1}\omega,\cdot)e^{\theta g(\sigma^{-1}\omega,\cdot)}(\mathcal{W}_{\sigma^{-1}\omega}+h^0_{\sigma^{-1}\omega})), \tag{31}$$

$$D_2G(\theta,\mathcal{W})(\mathcal{H})_\omega = \mathcal{L}^\theta_{\sigma^{-1}\omega}(\mathcal{H}_{\sigma^{-1}\omega}), \tag{32}$$

$$D_{11}G(\theta,\mathcal{W})(z_1,z_2)_\omega = z_1z_2\mathcal{L}_{\sigma^{-1}\omega}(g(\sigma^{-1}\omega,\cdot)^2e^{\theta g(\sigma^{-1}\omega,\cdot)}(\mathcal{W}_{\sigma^{-1}\omega}+h^0_{\sigma^{-1}\omega})), \tag{33}$$

$$D_{12}G(\theta,\mathcal{W})(z,\mathcal{H})_\omega = D_{21}G(\theta,\mathcal{W})(\mathcal{H},z)_\omega = z\mathcal{L}_{\sigma^{-1}\omega}(g(\sigma^{-1}\omega,\cdot)e^{\theta g(\sigma^{-1}\omega,\cdot)}\mathcal{H}_{\sigma^{-1}\omega}), \tag{34}$$

$$D_{22}G = 0. \tag{35}$$

Moreover, the expressions for the derivatives of H are equal to the corresponding expression for G applied to the constant function 1.

Lemma 5.3. *There exist $\epsilon,R>0$ such that $F\colon\mathcal{D}\to\mathcal{S}$ is a well-defined map on*

$$\mathcal{D} := \{\theta\in\mathbb{C} : |\theta|<\epsilon\}\times B_{\mathcal{S}}(0,R),$$

where $B_{\mathcal{S}}(0,R)$ denotes the ball of radius R in $\mathcal{S}$ centered at 0.

Proof. Let G and H be defined as in (29) and (30). The function H is continuous on a neighborhood of $(0,0)$ in $\mathbb{C}\times\mathcal{S}$ and obviously $H(0,0)(\omega)=h^0_\omega(1)=1$ for $\mathbb{P}$-a.e. $\omega\in\Omega$. Hence,

$$|H(\theta,\mathcal{W})(\omega)| \geq 1-|H(0,0)(\omega)-H(\theta,\mathcal{W})(\omega)| \geq 1-\|H(0,0)-H(\theta,\mathcal{W})\|_{L^\infty},$$

for $\mathbb{P}$-a.e. $\omega\in\Omega$. Continuity of H implies that $\|H(0,0)-H(\theta,\mathcal{W})\|_{L^\infty}\leq\frac{1}{2}$ for all $(\theta,\mathcal{W})$ in a neighborhood of $(0,0)$ in $\mathbb{C}\times\mathcal{S}$ and hence, in such a neighborhood,

$$\text{ess inf}_{\omega\in\Omega}|H(\theta,\mathcal{W})(\omega)| \geq \frac{1}{2}.$$

This together with (15) and a simple observation that $F(\theta,\mathcal{W})(1)=0$ immediately yields the desired conclusion.

□

Notice that map F defined by (28) satisfies $F(\theta,\mathcal{W})(\omega)=G(\theta,\mathcal{W})(\omega)/H(\theta,\mathcal{W})(\omega)-\mathcal{W}(\omega)-h^0_\omega$. The proof of Lemma 5.3 ensures that for $(\theta,\mathcal{W})$ in a neighbourhood $\mathcal{D}$ of $(0,0)\in\mathbb{C}\times\mathcal{S}$, $\text{ess inf}_{\omega\in\Omega}|H(\theta,\mathcal{W})(\omega)|\geq\frac{1}{2}$. Thus, the following result is a direct consequence of Lemma 5.2.

Proposition 5.4. *The map F defined by (28) is of class C^∞ on the neighborhood $\mathcal{D}$ of $(0,0)\in\mathbb{C}\times\mathcal{S}$ from Lemma 5.3. Moreover, for $\omega\in\Omega,(\theta,\mathcal{W})\in\mathcal{D}$ and $\mathcal{H}\in\mathcal{S}$,*

$$D_2F(\theta,\mathcal{W})(\mathcal{H})_\omega = \frac{1}{H(\theta,\mathcal{W})(\omega)}\mathcal{L}^\theta_{\sigma^{-1}\omega}\mathcal{H}_{\sigma^{-1}\omega} - \frac{\mathcal{L}^\theta_{\sigma^{-1}\omega}\mathcal{H}_{\sigma^{-1}\omega}(1)}{[H(\theta,\mathcal{W})(\omega)]^2}G(\theta,\mathcal{W})_\omega - \mathcal{H}_\omega,$$

$$\begin{aligned} D_1F(\theta,\mathcal{W})_\omega = {} & \frac{1}{H(\theta,\mathcal{W})(\omega)}\mathcal{L}_{\sigma^{-1}\omega}(g(\sigma^{-1}\omega,\cdot)e^{\theta g(\sigma^{-1}\omega,\cdot)}(\mathcal{W}_{\sigma^{-1}\omega}+h^0_{\sigma^{-1}\omega})) \\ & - \frac{\mathcal{L}_{\sigma^{-1}\omega}(g(\sigma^{-1}\omega,\cdot)e^{\theta g(\sigma^{-1}\omega,\cdot)}(\mathcal{W}_{\sigma^{-1}\omega}+h^0_{\sigma^{-1}\omega}))(1)}{[H(\theta,\mathcal{W})(\omega)]^2}\mathcal{L}^\theta_{\sigma^{-1}\omega}(\mathcal{W}_{\sigma^{-1}\omega}+h^0_{\sigma^{-1}\omega}), \end{aligned}$$

where we have identified $D_1F(\theta, \mathcal{W})$ with its value at 1.

Lemma 5.5. *Let $\mathcal{D} = \{\theta \in \mathbb{C} : |\theta| < \epsilon\} \times B_{\mathcal{S}}(0, R)$ be as in Lemma 5.3. Then, $F : \mathcal{D} \to \mathcal{S}$ is C^∞ and the equation*

$$F(\theta, \mathcal{W}) = 0 \tag{36}$$

has a unique solution $O(\theta) \in \mathcal{S}$, for every θ in a neighborhood of 0. Furthermore, $O(\theta)$ is a C^∞ function of θ.

Proof. Note that $F(0, 0) = 0$. Furthermore, Proposition 5.4 implies that F is of class C^∞ on a neighborhood of $(0, 0)$. In addition, Lemma 5.2 implies that

$$(D_2F(0, 0)\mathcal{X})(\omega) = \mathcal{L}_{\sigma^{-1}\omega}\mathcal{X}(\sigma^{-1}\omega) - \mathcal{X}(\omega), \quad \omega \in \Omega,\ \mathcal{X} \in \mathcal{S}.$$

Using (12) and proceeding as in [9, Lemma 3.5], one can show that $D_2F(0, 0)$ is invertible and that

$$(D_2F(0, 0)^{-1}\mathcal{X})(\omega) = -\sum_{j=0}^{\infty} \mathcal{L}^{(j)}_{\sigma^{-j}\omega}\mathcal{X}(\sigma^{-j}\omega) \quad \omega \in \Omega,\ \mathcal{X} \in S. \tag{37}$$

The conclusion of the lemma now follows directly from the implicit function theorem. □

6. Properties of $\Lambda(\theta)$

Let $0 < \epsilon < 1$ be as in Lemma 5.3 and $O(\theta)$ be as in Lemma 5.5. Let

$$h^\theta_\omega := h^0_\omega + O(\theta)(\omega) \in \mathcal{B}^{1,1}, \quad \omega \in \Omega. \tag{38}$$

We notice that $h^\theta_\omega(1) = 1$ and by Lemma 5.5, $\theta \mapsto h^\theta$ is continuously differentiable.

Let us define

$$\hat{\Lambda}(\theta) := \int \log \left| h^\theta_\omega(e^{\theta g(\omega,\cdot)}) \right| d\mathbb{P}(\omega), \tag{39}$$

and

$$\lambda^\theta_\omega := h^\theta_\omega(e^{\theta g(\omega,\cdot)}) = \mathcal{L}^\theta_\omega h^\theta_\omega(1). \tag{40}$$

6.1. A differentiable lower bound for $\Lambda(\theta)$.

Lemma 6.1 deals with differentiability properties of $\hat{\Lambda}(\theta)$.

Lemma 6.1.

(1) For every $\theta \in B_{\mathbb{C}}(0, \epsilon)$, $\hat{\Lambda}(\theta) \le \Lambda(\theta)$.

(2) $\hat{\Lambda}$ is differentiable on a neighborhood of 0, and

$$\hat{\Lambda}'(\theta) = \Re\left(\int \frac{\overline{\lambda^\theta_\omega}((O(\theta)(\omega) + h^0_\omega)(g(\omega, \cdot)e^{\theta g(\omega,\cdot)}) + O'(\theta)(\omega)(e^{\theta g(\omega,\cdot)}))}{|\lambda^\theta_\omega|^2}\, d\mathbb{P}(\omega) \right),$$

where $\Re(z)$ denotes the real part of z and $\overline{z}$ the complex conjugate of z.

(3) For $\mathbb{P}$-a.e. $\omega \in \Omega$, and θ in a neighborhood of 0, the map $\theta \mapsto Z_\omega(\theta) := Z(\theta, \omega)$ is differentiable. Moreover,

$$Z'_\omega(\theta) = \frac{\Re\left(\overline{\lambda^\theta_\omega}((O(\theta)(\omega) + h^0_\omega)(g(\omega, \cdot)e^{\theta g(\omega,\cdot)}) + O'(\theta)(\omega)(e^{\theta g(\omega,\cdot)}))\right)}{|\lambda^\theta_\omega|^2}.$$

(4) $\hat{\Lambda}'(0) = 0$.

Proof. The proof of part 1 is identical to the proof of Lemma 3.8 [9] replacing $\|\cdot\|_{\mathcal{B}}$ with $\|\cdot\|_{1,1}$ and $\|\mathcal{L}_\omega^{\theta,(n)} v_\omega^\theta\|_1$ with $|\mathcal{L}_\omega^{\theta,(n)} h_\omega^\theta(1)|$.

The proof of part 2 is identical to the proof of Lemma 3.9 [9], using Lemma 5.5 in place of Lemma 3.5 [9] and replacing the final two equation blocks with:

$$\begin{aligned}|(O(\theta)(\omega)+h_\omega^0)(g(\omega,\cdot)e^{\theta g(\omega,\cdot)})| &\le C\|O(\theta)(\omega)+h_\omega^0\|_{1,1}\cdot\|g(\omega,\cdot)e^{\theta g(\omega,\cdot)}\|_{C^1}\\ &\le C\|O(\theta)\|_\infty + C,\end{aligned}$$

and

$$|O'(\theta)(\omega)(e^{\theta g(\omega,\cdot)})| \le C\|O'(\theta)(\omega)\|_{1,1}\cdot\|e^{\theta g(\omega,\cdot)}\|_{C^1} \le C\|O'(\theta)\|_\infty.$$

The proof of part 3 is identical to the proof of Lemma 3.10 [9], using differentiability of H and O in Lemmas 5.2 and 5.5.

The proof of part 4 is identical to proof of Lemma 3.11 [9]. □

6.2. **One-dimensionality of $Y_1^\theta(\omega)$ and differentiability of Λ.** Let $Y_1^\theta(\omega)$ denote the top Oseledets subspace of the cocycle $(\mathcal{L}_\omega^\theta)_{\omega\in\Omega}$. The proof of part 1 of the following result can be obtained by repeating the argument as in [9, Theorem 3.12], using Proposition 4.4. Part 2 follows by arguing as in [9, Corollary 3.14].

Proposition 6.2. *For $\theta\in\mathbb{C}$ near 0*

(1) $\dim Y_1^\theta(\omega)=1$.

(2) $\Lambda(\theta)=\hat{\Lambda}(\theta)$. *In particular,* $\Lambda(\theta)$ *is differentiable near* 0 *and* $\Lambda'(0)=0$.

6.3. **Convexity of $\Lambda(\theta)$.** By Proposition 3.2, we can regard h_ω^0 as Borel probability measure on X which we will denote by μ_ω. The family $(\mu_\omega)_{\omega\in\Omega}$ induces a probability measure on $\Omega\times X$ given by

$$\mu(A\times B)=\int_A \mu_\omega(B)\,d\mathbb{P}(\omega),\quad \text{for measurable sets } A\subset\Omega \text{ and } B\subset X.$$

Then, μ is invariant for the skew-product transformation $\tau\colon\Omega\times X\to\Omega\times X$ defined by

$$\tau(\omega,x)=(\sigma\omega,T_\omega(x)),\quad \omega\in\Omega,\ x\in X.$$

Obviously, the variance Σ^2 defined in (2) is nonnegative. From now on we shall assume that $\Sigma^2>0$. Otherwise we can invoke and adapt the coboundary case proved in Proposition 3 in our paper [8], which says that $\Sigma^2=0$ if and only if there exists $r\in L^2_\mu(\Omega\times X)$ such that $g=r-r\circ\tau$.

Proposition 6.3. *On a neighbourhood of 0,*

(1) Λ *is of class* C^2 *and* $\Lambda''(0)=\Sigma^2$.

(2) Λ *is strictly convex.*

Proof. The proof of part 1 is identical to the proof of Lemma 3.15 [9] and part (ii) is a direct consequence of part 2. □

7. Large deviation principle and central limit theorem

For $\theta\in\mathbb{C}$ sufficiently close to 0, we have that $\dim Y_1^\theta(\omega)=1$. Choose $h_\omega^\theta\in Y_1^\theta(\omega)$ such that $h_\omega^\theta(1)=1$. We note that h_ω^θ is actually given by (38). Furthermore, let $\lambda_\omega^\theta\in\mathbb{C}$ be such that

$$\mathcal{L}_\omega^\theta h_\omega^\theta=\lambda_\omega^\theta h_{\sigma\omega}^\theta. \tag{41}$$

Note that

$$\lambda_\omega^\theta=h_\omega^\theta(e^{\theta g(\omega,\cdot)}), \tag{42}$$

which coincides with (40). Next, let us fix $\phi^\theta_\omega \in Y^{*\theta}_\omega$ so that $\phi^\theta_\omega(h^\theta_\omega) = 1$. Furthermore, one can show (see [9, p. 30]) that

$$(\mathcal{L}^\theta_\omega)^* \phi^\theta_{\sigma\omega} = \lambda^\theta_\omega \phi^\theta_\omega. \tag{43}$$

Remark 7.1. *The differentiability of $\theta \mapsto \phi^\theta$ follows similarly to the presentation in* [9, Appendix C]. *The proofs of Lemmas C.4 and C.6 make use of regularity estimates (90) and (99) in terms of variation; in the present work, these estimates may be replaced with C^1 estimates. In the proof of Lemma C.2, the expression $\|v^0_\omega\|_1$ may be replaced with $|h^0_\omega(1)|$ and bounded by (5) in the present work.*

In addition, let

$$\mathcal{B}^{1,1} = Y^\theta_\omega \oplus H^\theta_\omega \quad \text{and} \quad (\mathcal{B}^{1,1})^* = Y^{*\theta}_\omega \oplus H^{*\theta}_\omega$$

be the Oseledets splitting of cocycles $(\mathcal{L}^\theta_\omega)_{\omega\in\Omega}$ and $((\mathcal{L}^\theta_\omega)^*)_{\omega\in\Omega}$ respectively into a direct sum of the top space and the sum of all other Oseledets subspaces.

7.1. **Large deviation principle.** The following lemmas link the limits of characteristic functions of Birkhoff sums to the function Λ.

Lemma 7.2. *Let $\theta \in \mathbb{C}$ be sufficiently close to 0 and $h \in \mathcal{B}^{1,1}$ be such that $h \notin H^\theta_\omega$, i.e. $\phi^\theta_\omega(h) \neq 0$. Then,*

$$\lim_{n\to\infty} \frac{1}{n} \log \left| h(e^{\theta S_n g(\omega,\cdot)}) \right| = \Lambda(\theta).$$

Proof. Identical to the proof of Lemma 4.2 [9] with $h \in \mathcal{B}^{1,1}$ in the present paper playing the role of $\int f \cdot \, dm$ in the proof of Lemma 4.2 [9], and Lemma 18 replacing (43) [9]. □

Lemma 7.3. *For all complex θ in a neighborhood of 0, and $\mathbb{P}$-a.e. $\omega \in \Omega$, we have that*

$$\lim_{n\to\infty} \frac{1}{n} \log \left| \int e^{\theta S_n g(\omega,x)} \, d\mu_\omega(x) \right| = \Lambda(\theta).$$

Proof. We follow the proof of Lemma 4.3 [9], observing that

$$\int e^{\theta S_n g(\omega,x)} \, d\mu_\omega(x) = h^0_\omega(e^{\theta S_n g(\omega,\cdot)}),$$

and recalling the differentiability of the map $\theta \mapsto \phi^\theta$ in Remark 7.1. □

Proof of Theorem A. Following the proof of Theorem A [9], by applying Proposition 6.3 and Lemma 7.3, together with the Gärtner-Ellis theorem ([18] and Theorem 4.1 [9]), we obtain the large deviation principle. □

7.2. **Central limit theorem.** The proof of the following result is completely analogous to the proof of [9, Lemma 4.4].

Lemma 7.4. *There exist $C > 0, 0 < r < 1$ such that for every $\theta \in \mathbb{C}$ sufficiently close to 0, every $n \in \mathbb{N}$ and $\mathbb{P}$-a.e. $\omega \in \Omega$, we have*

$$\left| \mathcal{L}^{\theta,(n)}_\omega (h^0_\omega - \phi^\theta_\omega(h^0_\omega) h^\theta_\omega)(1) \right| \leq C r^n. \tag{44}$$

Proof of Theorem B. The proof is identical to the proof of Theorem B [9], with the same modifications as those listed in the proof of Lemma 7.2. Differentiability of $\theta \mapsto \phi^\theta$ is used (see Remark 7.1) as well as Lemma 4.1 to obtain the coding of the Birkhoff sums via the twisted transfer operator. Lemma 4.5 [9] is proved is proved in an identical way. □

8. Local central limit theorem

We begin by recalling the concept of $\mathbb{P}$-continuity which we will also use in section 9.2.1. We say that our cocycle is $\mathbb{P}$-continuous (a concept introduced in [30]) if the map $\omega \mapsto T_\omega$ has $\mathbb{P}$-a.e. a countable range (besides being measurable). This implies that $\omega \mapsto \mathcal{L}_\omega$ is continuous on each of countably many Borel subsets of Ω, whose union has full $\mathbb{P}$ measure. We refer to [10] for details.

In our earlier paper [9] we proved the local central limit theorem in the non-arithmetic case (we also separately treated the arithmetic case) under the condition that we called (L) in the introduction, namely

- (L) For $\mathbb{P}$-a.e. $\omega \in \Omega$ and for every compact interval $J \subset \mathbb{R}\backslash\{0\}$ there exists $C = C(\omega) > 0$ and $\rho \in (0,1)$ such that

$$||\mathcal{L}^{it,(n)}||_{\mathcal{B}} \leq C\rho^n, \text{ for } t \in J \text{ and } n \geq 0. \tag{45}$$

Moreover under the assumption that the cocycle is $\mathbb{P}$-continuous, we proved [9, Lemma 4.7.] that (L) is equivalent to the following *aperiodicity* condition

- For every $t \in \mathbb{R}$, either $\Lambda(it) < 0$ or the cocycle $\mathcal{L}_\omega^{it}$ is quasicompact and the equation

$$e^{itg(\omega,x)}\mathcal{L}_\omega^* \psi_{\sigma\omega} = \gamma_\omega^{it}\psi_\omega,$$

  where $\gamma_\omega^{it} \in S^1$, $\mathcal{L}_\omega^*$ denotes the adjoint of $\mathcal{L}_\omega$ and $\psi_\omega \in \mathcal{B}^*$, only has a measurable non-zero solution $\psi := \{\psi_\omega\}_{\omega\in\Omega}$ when $t = 0$. In this case $\gamma_\omega^0 = 1$ and $\psi_\omega(f) = \int f dm$ (up to a scalar multiplicative factor) for $\mathbb{P}$-a.e. $\omega \in \Omega$.

In our present Anosov setting the bound (45) will be replaced with the following:

$$\|\mathcal{L}_\omega^{it,(n)}\|_{1,1} \leq C\rho^n, \quad \text{for } t \in J \text{ and } n \geq 0. \tag{46}$$

Still in the present setting, we can not prove at the moment the equivalence between (L) and the aperiodicity condition although several of the technical steps which formed the skeleton of our proof of [9, Lemma 4.7.] for expanding maps and functions of bounded variation can be transferred to Anosov maps and the anisotropic Banach spaces used in this work.

Proof of Theorem C. The proof assuming (L) follows now exactly as in the proof of Theorem C [9], with the following minor modifications. We use Lemma 4.1 to obtain the coding of the Birkhoff sums through powers of the twisted transfer operator. The control of term (III) in the proof of Theorem C [9] uses Lemma 7.4 in place of Lemma 4.4 [9]. The control of term (IV) in the proof of Theorem C [9] uses (L1) in place of the analogous condition (C5) in [9]. □

As mentioned in the introduction, Hafouta and Kifer [17], Section 2.10, formulated a sort of classical aperiodicity condition in the random setting which allowed them to recover condition (L). We now state their assumptions and verify that some of them hold in our setting. This and additional hypothesis on the driving map σ will give us a new proof of the local central limit theorem.

We first recall the Hafouta and Kifer assumptions adapted to our situations:

- HK A1: The probability measure $\mathbb{P}$ assigns positive measure to open sets, σ is a homeomorphism and there exist $\omega_0 \in \Omega$ and $m_0 \in \mathbb{N}$ so that $\sigma^{m_0}\omega_0 = \omega_0$. Moreover for each $i \in \{0, 1, \ldots, m_0 - 1\}$, there exists a neighborhood of $\sigma^i\omega_0$ on which the map $\omega \mapsto T_\omega$ is constant (note that on this neighborhood we also have that the map $\omega \mapsto \mathcal{L}_\omega$ is constant).

- HK A2: For each compact interval $J \subset \mathbb{R}$, the family of maps $\omega \to \mathcal{L}_\omega^{it}$, where $t \in J$, is equicontinuous at the points $\omega = \sigma^i \omega_0, 0 \le i \le m_0$ with respect to the operator norm, and there exists a constant $B = B(J) \ge 1$, such that $\mathbb{P}$-a.e.,
$$\|\mathcal{L}_\omega^{it,(n)}\|_{1,1} \le B, \tag{47}$$
for any $n \in \mathbb{N}$ and $t \in J$.
- HK A3: For any compact interval $J \subset \mathbb{R}$ that does not contain the origin, there exists constants $c = c(J) > 0$ and $b = b(J) \in (0,1)$ such that
$$\|\mathbf{L}_{it}^s\|_{1,1} \le cb^s, \tag{48}$$
for any $s \in \mathbb{N}$ and $t \in J$, where the (deterministic) operator $\mathbf{L}_{it}$ is defined as $\mathbf{L}_{it} := \mathcal{L}_{\omega_0}^{it,(m_0)}$.

Under these three assumptions, it was proved in Lemma 2.10.4 [17] that condition (L) holds.

We now adapt the previous requirements to our setting. Assumption HK A1 is easily satisfied by requiring that σ is a homeomorphism that has at least one periodic point ω_0 and by building the cocycle in a way that $\omega \mapsto T_\omega$ is locally constant at all points that belong to the orbit of ω_0. Of course, we also need to work with $\mathbb{P}$ that assigns positive measure to all open nonempty subsets of Ω.

Assumption HK A3 is equivalent to requiring the classical aperiodicity condition for deterministic systems (see [18]). Namely, it is sufficient to require that the spectral radius of $\mathcal{L}_{\omega_0}^{it,(m_0)}$ is strictly less than 1 for all $t \neq 0$. Then, taking $J \subset \mathbb{R}$ a compact interval, we can find $c = c(J) > 0$ and $b = b(J) \in (0,1)$ such that (48) holds by arguing as in [9, Proof of Lemma 4.7] (although the argument is simpler in our present setting since we deal with a deterministic situation).

Assumption HK A2 should instead be checked under suitable conditions since it relies on the properties of the maps, of the observable and of the functional spaces: we now show that it holds for our Anosov maps. This plus the necessary conditions in HK A1 and HK A3, will give us a proof of the LCLT in the non-arithmetic case. To verify HK A2 we first prove the equicontinuity property (under suitable conditions) in Lemma 8.1 and then in Lemma 8.3 we will develop a Lasota-Yorke inequality from which we can obtain the bound (47). The latter will be obtained from a Lasota–Yorke inequality for the twisted operator $\mathcal{L}_\omega^{it,(n)}$ and with respect to the norm $\|\cdot\|_{1,1}$ (strong) and $\|\cdot\|_{0,2}$ (weak), and the fact that $\|h\|_{0,2} \le \|h\|_{1,1}$ for any $h \in \mathcal{B}^{1,1}$, as we did in (13) for the non-twisted operator.

Lemma 8.1. *Let us suppose $(\Omega, \mathcal{F}, \mathbb{P}, \sigma)$ is an invertible and ergodic measure-preserving dynamical system verifying HK A1 and moreover for each $i \in \{0, 1, \dots, m_0 - 1\}$, the observable g satisfies*
$$\lim_{\omega \to \sigma^i \omega_0} \|g(\omega, \cdot) - g(\sigma^i \omega_0, \cdot)\|_{C^2} = 0. \tag{49}$$
Furthermore, let $J \subset \mathbb{R}$ be a compact interval. Then, the family of maps $\{\omega \mapsto \mathcal{L}_\omega^{it} : t \in J\}$ is equicontinuous in all points ω that belong to the orbit of ω_0.

Remark 8.2. *Observe that it is not enough to simply prescribe that* (49) *holds since we also need to make sure that this requirement is not spoiled when we center our observable (see* (17)*). It turns out that under suitable conditions we can make sure that* (49) *is preserved under centering. Let us first recall that* (12) *and* (13) *hold for every $\omega \in \Omega$. We assume that Ω is a metric space, $\sigma \colon \Omega \to \Omega$ is a homeomorphism and that*

$\omega \mapsto T_\omega$ is locally constant at points that belong to the orbit of ω_0.

Let us now modify slightly Proposition 3.1 to ensure that under these additional assumptions we can say more about the top Oseledets space of our cocycle.

Set

$$\mathcal{Y} = \{v\colon \Omega \to \mathcal{B}^{1,1} : v \text{ measurable and } \|v\|_\infty := \sup_{\omega\in\Omega}\|v(\omega)\|_{1,1} < \infty\}.$$

Then, $(\mathcal{Y}, \|\cdot\|_\infty)$ is a Banach space. Let Y be a set of all $v \in \mathcal{Y}$ that are continuous at points $\sigma^i\omega_0$, $i = 0, 1, \dots, m_0 - 1$. We claim that Y is a closed subset of $\mathcal{Y}$. Indeed, take a sequence $(v_n)_n \subset Y$ such that $v_n \to v$ in $\mathcal{Y}$ and fix $i \in \{0, 1, \dots, m_0 - 1\}$. Then, we have that

$$\begin{aligned}\|v(\omega) - v(\sigma^i\omega_0)\|_{1,1} &\le \|v(\omega) - v_n(\omega)\|_{1,1} + \|v_n(\omega) - v_n(\sigma^i\omega_0)\|_{1,1}\\ &\quad + \|v_n(\sigma^i\omega_0) - v(\sigma^i\omega_0)\|_{1,1}\\ &\le 2\|v - v_n\|_\infty + \|v_n(\omega) - v_n(\sigma^i\omega_0)\|_{1,1}.\end{aligned}$$

Take $\varepsilon > 0$ and choose n such that

$$\|v - v_n\|_\infty < \frac{\varepsilon}{3}.$$

Since $v_n \in Y$, we have that

$$\|v_n(\omega) - v_n(\sigma^i\omega_0)\|_{1,1} < \frac{\varepsilon}{3},$$

whenever ω is sufficiently close to $\sigma^i\omega_0$. Hence,

$$\|v(\omega) - v(\sigma^i\omega_0)\|_{1,1} < \varepsilon,$$

whenever ω is sufficiently close to $\sigma^i\omega_0$. Therefore, $v \in Y$ and Y is closed.

Set

$$Z := \{v \in Y; v(\omega) \ge 0 \text{ and } v(\omega)(1) = 1 \text{ for } \omega \in \Omega\}.$$

Then, Z is a closed subset of Y (see the argument in the proof of Proposition 3.1) and hence it is a Banach space. We consider $\mathbb{L}\colon Z \to Z$ defined by

$$(\mathbb{L}v)(\omega) = \mathcal{L}_{\sigma^{-1}\omega} v(\sigma^{-1}\omega), \quad \omega \in \Omega,\ v \in Z.$$

In order to show that $\mathbb{L}$ is well-defined, we only need to note that

$$\omega \mapsto \mathcal{L}_{\sigma^{-1}\omega} v(\sigma^{-1}\omega)$$

is continuous at $\sigma^i\omega_0$, $i \in \{0, \dots, m_0 - 1\}$. However, this follows from the fact that $v \in Z$ (and thus $v \in Y$) and our assumption that $\omega \mapsto T_\omega$ (and thus also $\omega \mapsto \mathcal{L}_\omega$) is locally constant along the orbit of ω_0. It follows from (12) that $\mathbb{L}$ has the unique fixed point $h^0 \in Z$. This easily implies that

$$\omega \mapsto h^0_\omega(g(\omega, \cdot)), \quad h^0_\omega := h^0(\omega)$$

is continuous at $\sigma^i\omega_0$, $i = 0, \dots, m_0 - 1$ and therefore (49) will remain valid even after centering.

Proof of Lemma 8.1. We will prove the desired equicontinuity property in ω_0. The argument for all other points in the orbit of ω_0 is completely analogous. Observe that for all $\omega \in \Omega$ sufficiently close to ω_0, we have that $\mathcal{L}_\omega = \mathcal{L}_{\omega_0}$. Therefore, for all ω close to ω_0, we have that

$$(\mathcal{L}^{it}_\omega - \mathcal{L}^{it}_{\omega_0})(h) = \mathcal{L}_{\omega_0}((e^{itg(\omega,\cdot)} - e^{itg(\omega_0,\cdot)})h),$$

and thus

$$\|(\mathcal{L}^{it}_\omega - \mathcal{L}^{it}_{\omega_0})(h)\|_{1,1} \le \|\mathcal{L}_{\omega_0}\|_{1,1} \cdot \|(e^{itg(\omega,\cdot)} - e^{itg(\omega_0,\cdot)})h\|_{1,1},$$

for each $h \in \mathcal{B}^{1,1}$. Observe that

$$\|(e^{itg(\omega,\cdot)} - e^{itg(\omega_0,\cdot)})h\|_{1,1} =$$
$$= \max\{\|(e^{itg(\omega,\cdot)} - e^{itg(\omega_0,\cdot)})h\|_{0,1}^{\sim}, \|(e^{itg(\omega,\cdot)} - e^{itg(\omega_0,\cdot)})h\|_{1,2}^{\sim}\}.$$

As in the proofs of Propositions 4.3 and 4.4, we need to estimate

$$\|e^{itg(\omega,\cdot)} - e^{itg(\omega_0,\cdot)}\|_{C^2}.$$

Take $x \in X$. By applying the mean-value theorem for the map $z \mapsto e^{itz}$, we see that

$$|e^{itg(\omega,x)} - e^{itg(\omega_0,x)}| \leq |t| \cdot |g(\omega, x) - g(\omega_0, x)|.$$

Thus,

$$\|e^{itg(\omega,\cdot)} - e^{itg(\omega_0,\cdot)}\|_{C^0} \leq |t| \cdot \|g(\omega, \cdot) - g(\omega_0, \cdot)\|_{C^0},$$

which implies that

$$\|e^{itg(\omega,\cdot)} - e^{itg(\omega_0,\cdot)}\|_{C^0} \leq \max\{|t| : t \in J\}\|g(\omega, \cdot) - g(\omega_0, \cdot)\|_{C^2}. \tag{50}$$

Moreover, we have that

$$\begin{aligned}\partial^j(e^{itg(\omega,\cdot)} - e^{itg(\omega_0,\cdot)}) &= ite^{itg(\omega,\cdot)}\partial^j(g(\omega, \cdot)) - ite^{itg(\omega_0,\cdot)}\partial^j(g(\omega_0, \cdot)) \\ &= ite^{itg(\omega,\cdot)}\partial^j(g(\omega, \cdot)) - ite^{itg(\omega_0,\cdot)}\partial^j(g(\omega, \cdot)) \\ &\quad + ite^{itg(\omega_0,\cdot)}\partial^j(g(\omega, \cdot)) - ite^{itg(\omega_0,\cdot)}\partial^j(g(\omega_0, \cdot)),\end{aligned}$$

for each $j \in \{1, \ldots, d\}$. By (50), we have that

$$\begin{aligned}\|\partial^j(e^{itg(\omega,\cdot)} - e^{itg(\omega_0,\cdot)})\|_{C^0} &\leq \max\{|t|^2 : t \in J\}\|g(\omega, \cdot) - g(\omega_0, \cdot)\|_{C^2} \cdot \|g(\omega, \cdot)\|_{C^2} \\ &\quad + \max\{|t| : t \in J\}\|g(\omega, \cdot) - g(\omega_0, \cdot)\|_{C^2},\end{aligned}$$

for every $j \in \{1, \ldots, d\}$. Thus,

$$\max_{1\leq j\leq d}\|\partial^j(e^{itg(\omega,\cdot)} - e^{itg(\omega_0,\cdot)})\|_{C^0} \leq C\|g(\omega, \cdot) - g(\omega_0, \cdot)\|_{C^2}, \tag{51}$$

for some $C > 0$ which is independent on t and ω. Finally, for each $k, j \in \{1, \ldots, d\}$, we have that

$$\begin{aligned}\partial^k\partial^j(e^{itg(\omega,\cdot)} - e^{itg(\omega_0,\cdot)}) &= -t^2e^{itg(\omega,\cdot)}\partial^k(g(\omega, \cdot))\partial^j(g(\omega, \cdot)) \\ &\quad + ite^{itg(\omega,\cdot)}\partial^k\partial^j(g(\omega, \cdot)) \\ &\quad + t^2e^{itg(\omega_0,\cdot)}\partial^k(g(\omega_0, \cdot))\partial^j(g(\omega_0, \cdot)) \\ &\quad - ite^{itg(\omega_0,\cdot)}\partial^k\partial^j(g(\omega_0, \cdot)).\end{aligned}$$

Observe that

$$\begin{aligned}ite^{itg(\omega,\cdot)}\partial^k\partial^j(g(\omega, \cdot)) - ite^{itg(\omega_0,\cdot)}\partial^k\partial^j(g(\omega_0, \cdot)) &= ite^{itg(\omega,\cdot)}\partial^k\partial^j(g(\omega, \cdot)) \\ &\quad - ite^{itg(\omega,\cdot)}\partial^k\partial^j(g(\omega_0, \cdot)) \\ &\quad + ite^{itg(\omega,\cdot)}\partial^k\partial^j(g(\omega_0, \cdot)) \\ &\quad - ite^{itg(\omega_0,\cdot)}\partial^k\partial^j(g(\omega_0, \cdot)).\end{aligned}$$

Thus (using (50)),

$$\begin{aligned}&\|ite^{itg(\omega,\cdot)}\partial^k\partial^j(g(\omega, \cdot)) - ite^{itg(\omega_0,\cdot)}\partial^k\partial^j(g(\omega_0, \cdot))\|_{C^0} \\ &\leq \max\{|t| : t \in J\}\|g(\omega, \cdot) - g(\omega_0, \cdot)\|_{C^2} \\ &\quad + \max\{|t|^2 : t \in J\}\|g(\omega, \cdot) - g(\omega_0, \cdot)\|_{C^2}\|g(\omega_0, \cdot)\|_{C^2}.\end{aligned}$$

On the other hand,

$$
\begin{aligned}
&- t^2 e^{itg(\omega,\cdot)}\partial^k(g(\omega,\cdot))\partial^j(g(\omega,\cdot)) + t^2 e^{itg(\omega_0,\cdot)}\partial^k(g(\omega_0,\cdot))\partial^j(g(\omega_0,\cdot)) \\
&= -t^2 e^{itg(\omega,\cdot)}\partial^k(g(\omega,\cdot))\partial^j(g(\omega,\cdot)) + t^2 e^{itg(\omega_0,\cdot)}\partial^k(g(\omega,\cdot))\partial^j(g(\omega,\cdot)) \\
&\quad - t^2 e^{itg(\omega_0,\cdot)}\partial^k(g(\omega,\cdot))\partial^j(g(\omega,\cdot)) + t^2 e^{itg(\omega_0,\cdot)}\partial^k(g(\omega_0,\cdot))\partial^j(g(\omega,\cdot)) \\
&\quad - t^2 e^{itg(\omega_0,\cdot)}\partial^k(g(\omega_0,\cdot))\partial^j(g(\omega,\cdot)) + t^2 e^{itg(\omega_0,\cdot)}\partial^k(g(\omega_0,\cdot))\partial^j(g(\omega_0,\cdot))
\end{aligned}
$$

Hence, (50) implies that

$$
\begin{aligned}
&\|-t^2 e^{itg(\omega,\cdot)}\partial^k(g(\omega,\cdot))\partial^j(g(\omega,\cdot)) + t^2 e^{itg(\omega_0,\cdot)}\partial^k(g(\omega_0,\cdot))\partial^j(g(\omega_0,\cdot))\|_{C^0} \\
&\le \max\{|t|^3 : t \in J\}\|g(\omega,\cdot) - g(\omega_0,\cdot)\|_{C^2} \cdot \|g(\omega,\cdot)\|_{C^2}^2 \\
&\quad + t^2\|g(\omega,\cdot) - g(\omega_0,\cdot)\|_{C^2} \cdot \|g(\omega,\cdot)\|_{C^2} \\
&\quad + t^2\|g(\omega,\cdot) - g(\omega_0,\cdot)\|_{C^2} \cdot \|g(\omega_0,\cdot)\|_{C^2}.
\end{aligned}
$$

We conclude that (by increasing C) we have that

$$
\begin{aligned}
&\sup_{1\le k,j\le d}\|-t^2 e^{itg(\omega,\cdot)}\partial^k(g(\omega,\cdot))\partial^j(g(\omega,\cdot)) + t^2 e^{itg(\omega_0,\cdot)}\partial^k(g(\omega_0,\cdot))\partial^j(g(\omega_0,\cdot))\|_{C^0} \\
&\le C\|g(\omega,\cdot) - g(\omega_0,\cdot)\|_{C^2}.
\end{aligned}
$$

Thus,

$$
\|e^{itg(\omega,\cdot)} - e^{itg(\omega_0,\cdot)}\|_{C^2} \le C\|g(\omega,\cdot) - g(\omega_0,\cdot)\|_{C^2},
$$

and

$$
\|\mathcal{L}_\omega^{it} - \mathcal{L}_{\omega_0}^{it}\| \le C\|g(\omega,\cdot) - g(\omega_0,\cdot)\|_{C^2},
$$

for $t \in J$ and ω in a neighborhood of ω_0. The conclusion of the lemma follows directly from (49). □

As we already announced, we now prove a Lasota–Yorke inequality for the twisted operator.

Lemma 8.3. *For each $t \in \mathbb{R}$, there exist $A_t, B_t > 0$, $0 < \gamma_t < 1$ such that for every $n \ge 0, h \in \mathcal{B}^{1,1}$ and $\mathbb{P}$-a.e. $\omega \in \Omega$,*

$$
\|\mathcal{L}_\omega^{it,(n)} h\|_{1,1} \le A_t \gamma_t^n \|h\|_{1,1} + B_t \|h\|_{0,2}.
$$

Moreover, for each $J \subset \mathbb{R}$ compact interval, we have that

$$
\sup_{t\in J} \max\{A_t, B_t\} < \infty.
$$

Proof. (Sketch). The proof follows verbatim the proof of Lemma 6.3 in [15], with two differences. First, we work with composition of maps, but if they are close enough we can easily adapt the *deterministic* arguments (we recall that this was explicitly emphasized in [15, Section 7], in particular allowing for a *random version* of [15, Lemma 3.3] to be applied). Second, since we use the twisted operator instead of the usual one, in the various estimates in the proof of Lemma 6.3 [15] we find the extra multiplicative factor $e^{itS_n g(\omega,\cdot)}$. The proof of Lemma 6.3 [15] is done by induction on the index p and the first step is to get a weak version of the Lasota–Yorke, namely for each $t \in \mathbb{R}$, there exists $C_t \ge 0$ such that for every $n \ge 0$ and $\mathbb{P}$-a.e. $\omega \in \Omega$,

$$
\|\mathcal{L}_\omega^{it,(n)}\|_{0,q} \le C_t. \tag{52}
$$

We now prove (52) (with $q = 1$) to show how to handle the additional multiplicative factor. We use the notation as in [15, p.202]. Recall that

$$\mathcal{L}_\omega^{(n)}(h)(x) = \frac{h((T_\omega^{(n)})^{-1}(x))}{|\det DT_\omega^{(n)}((T_\omega^{(n)})^{-1}(x))|},$$

and therefore

$$\mathcal{L}_\omega^{it,(n)}(h)(x) = \frac{e^{itS_n g(\omega,(T_\omega^{(n)})^{-1}(x))} h((T_\omega^{(n)})^{-1}(x))}{|\det DT_\omega^{(n)}((T_\omega^{(n)})^{-1}(x))|},$$

for each $h \in C^r$ and $x \in X$. Thus,

$$\int_W \mathcal{L}_\omega^{it,(n)} h \cdot \varphi = \int_{(T_\omega^{(n)})^{-1}(W)} \bar{h}_n \cdot \varphi \circ T_\omega^{(n)} \cdot J_W T_\omega^{(n)},$$

where $J_W T_\omega^{(n)}$ is the Jacobian of $T_\omega^{(n)} \colon (T_\omega^{(n)})^{-1}(W) \to W$ and

$$\bar{h}_n := \frac{h e^{itS_n g(\omega,\cdot)}}{|\det DT_\omega^{(n)}|}.$$

Let $\varphi_j = \varphi \circ T_\omega^{(n)} \cdot \rho_j$, where $\rho_1, \ldots, \rho_\ell$ is a partition of unity on $(T_\omega^{(n)})^{-1}W$, as provided by (the random analogue of) Lemma 3.3 [15] (using $\gamma = 1$), and $W_1, \ldots, W_\ell$ the corresponding admissible leaves such that $(T_\omega^{(n)})^{-1}(W) \subset \cup_{j=1}^\ell W_j$. Hence, [15, (6.2)] becomes

$$\left| \int_{W_j} \bar{h}_n \cdot \varphi_j \cdot J_W T_\omega^{(n)} \right| \le C\|h\|_{0,1} \left| |\det DT_\omega^{(n)}|^{-1} \cdot e^{itS_n g(\omega,\cdot)} \cdot \varphi_j \cdot J_W T_\omega^{(n)} \right|_{C^1(W_j)}.$$

Note that

$$\left| |\det DT_\omega^{(n)}|^{-1} \cdot e^{itS_n g(\omega,\cdot)} \cdot \varphi_j \cdot J_W T_\omega^{(n)} \right|_{C^1(W_j)} \le \left| |\det DT_\omega^{(n)}|^{-1} \cdot J_W T_\omega^{(n)} \right|_{C^1(W_j)} \cdot |\varphi_j|_{C^1(W_j)} \cdot |e^{itS_n g(\omega,\cdot)}|_{C^1(W_j)}.$$

It follows from [15, Lemma 6.2] that

$$\sum_{j \le \ell} ||\det DT_\omega^{(n)}|^{-1} \cdot J_W T_\omega^{(n)}|_{C^1(W_j)} \le C.$$

In addition, from the argument at the bottom of [15, p. 203], it follows that

$$|\varphi_j|_{C^1(W_j)} \le |\varphi \circ T_\omega^{(n)}|_{C^1(W_j)} \cdot |\rho_j|_{C^1(W_j)} \le C.$$

Hence, in order to complete the proof of the weak Lasota–Yorke inequality, it is sufficient to show that

$$|e^{itS_n g(\omega,\cdot)}|_{C^1(W_j)} \le C. \tag{53}$$

Note that

$$|e^{itS_n g(\omega,\cdot)}|_{C^0(W_j)} = 1 \quad \text{and}$$

$$|\partial^\alpha(e^{itS_n g(\omega,\cdot)})|_{C^0(W_j)} = |t| \cdot |\partial^\alpha(S_n g(\omega,\cdot))|_{C^0(W_j)} \le |t| \sum_{i=0}^{n-1} |\partial^\alpha(g(\sigma^i\omega, T_\omega^{(i)}(\cdot)))|_{C^0(W_j)}.$$

In order to bound $|\partial^\alpha(g(\sigma^i\omega, T_\omega^{(i)}(\cdot)))|_{C^0(W_j)}$, we proceed as in [5, (4.3)]. For each i and $x, y \in W_j$, we have

$$\begin{aligned} \frac{|g(\sigma^i\omega, T_\omega^{(i)}x) - g(\sigma^i\omega, T_\omega^{(i)}y)|}{d(x,y)} &= \frac{|g(\sigma^i\omega, T_\omega^{(i)}x) - g(\sigma^i\omega, T_\omega^{(i)}y)|}{d(T_\omega^{(i)}x, T_\omega^{(i)}y)} \cdot \frac{d(T_\omega^{(i)}x, T_\omega^{(i)}y)}{d(x,y)} \\ &\le C\nu^i \operatorname{ess\,sup}_{\omega\in\Omega} \|g(\omega,\cdot)\|_{C^1}, \end{aligned}$$

and thus

$$|\partial^\alpha(g(\sigma^i\omega, T_\omega^{(i)}(\cdot)))|_{C^0(W_j)} \le C\nu^i \operatorname{ess\,sup}_{\omega\in\Omega}\|g(\omega,\cdot)\|_{C^1}.$$

In view of (16), (53) holds. Now one can repeat arguments in [15] to obtain the weak Lasota–Yorke inequality for the twisted cocycle, (52). The proof of the strong Lasota–Yorke inequality can be obtained in a similar manner. □

Using Lemmas 8.1 and 8.3 we get:

Theorem 8.4. *Suppose that conditions HK A1 and HK A3 hold and the observable g satisfies* (49)*. Then the same conclusions as those of Theorem C hold.*

9. Piecewise hyperbolic dynamics

In this section, we apply the previous theory to obtain statistical laws for the random compositions $T_\omega^{(n)} = T_{\sigma^{n-1}\omega} \circ \cdots \circ T_{\sigma\omega} \circ T_\omega$ of piecewise uniformly hyperbolic maps T_ω of the type studied in [5]. The class of maps T_ω considered contains piecewise toral automorphisms and piecewise hyperbolic maps with bounded derivatives; see Remark 2.2 [5].

9.1. **Preliminaries.** We follow the construction of [5]. Let X be a two-dimensional compact Riemannian manifold, possibly with boundary and not necessarily connected and let $T\colon X \to X$ be a piecewise hyperbolic map in the sense of [5]. That is, the domain X is broken into a finite number of pairwise disjoint open regions $\{X_i^+\}$ with piecewise C^1 boundary curves of finite length, such that $\bigcup_i \overline{X_i^+} = X$. The image of each X_i^+ under T is denoted $X_i^- = T(X_i^+)$; we assume that $\bigcup \overline{X_i^-} = X$. The sets $\mathcal{S}^\pm := X \setminus \bigcup_i X_i^\pm$ are the "singularity sets" for T and T^{-1}, respectively. Assume that T is a C^2 diffeomorphism from the complement of $\mathcal{S}^+$ to the complement of $\mathcal{S}^-$, and that for each i, there is a C^2 extension of T to $\overline{X_i^+}$. On each X_i, the map T is uniformly hyperbolic: there are two continuous, strictly DT-invariant families of cones C^s and C^u defined on $X \setminus (\mathcal{S}^+ \cup \partial X)$ satisfying

$$\begin{aligned}
\lambda &:= \inf_{x\in X\setminus\mathcal{S}^+} \inf_{v\in C^u} \frac{\|DTv\|}{\|v\|} > 1,\\
\mu &:= \inf_{x\in X\setminus\mathcal{S}^+} \inf_{v\in C^s} \frac{\|DTv\|}{\|v\|} < 1,\\
\mu_+^{-1} &:= \inf_{x\in X\setminus\mathcal{S}^-} \inf_{v\in C^s} \frac{\|DT^{-1}v\|}{\|v\|} > 1.
\end{aligned}$$

Assume that vectors tangent to the singularity curves in $\mathcal{S}^-$ are bounded away from C^s. The singularity curves and their images and preimages should not intersect at too many points. Denote by $\mathcal{S}_n^-$ (resp. $\mathcal{S}_n^+$) the set of singularity curves for T^{-n} (resp. T^n), and let $M(n)$ denote the maximum number of singularity curves that meet at a single point. Assume that there is an α_0 and an integer $n_0 > 0$ such that $\lambda\mu^{\alpha_0} > 1$ and $(\lambda\mu^{\alpha_0})^{n_0} > M(n_0)$; this condition is satisfied if $M(n)$ has polynomial growth, for example.

For each $n \in \mathbb{N}$, let $\mathcal{K}_n$ be the set of connected components of $X \setminus \mathcal{S}_n^+$, and let $C^1(\overline{K}, \mathbb{R})$ be the set of functions $\varphi \in C^1(\mathring{K}, \mathbb{R})$ with C^1 extension in a neighbourhood of $\overline{K}$. Let $(C^1_{\mathcal{S}_n^+})' := \{\varphi \in L^\infty(X) : \varphi \in C^1(\overline{K}, \mathbb{R})\ \forall K \in \mathcal{K}_n\}$. If $h \in (C^1_{\mathcal{S}_n^+})'$ is an element of the dual of $C^1_{\mathcal{S}_n^+}$, then $\mathcal{L} : (C^1_{\mathcal{S}_n^+})' \to (C^1_{\mathcal{S}_{n-1}^+})'$ acts on h by

$$\mathcal{L}h(\varphi) = h(\varphi \circ T) \quad \forall \varphi \in C^1_{\mathcal{S}_{n-1}^+}.$$

In order to obtain useful spectral information from $\mathcal{L}$, its action is restricted to a Banach space $\mathcal{B}$, analogous to the space $\mathcal{B}^{p,q} = \mathcal{B}^{1,1}$ in Section 2. We now briefly outline the construction of the norms on $\mathcal{B}$ and an associated "weak" space $\mathcal{B}_w$; see [5] for details. The norms are defined using "admissible leaves" W in a set of admissible leaves Σ. These leaves are smooth curves in approximately the stable direction, and are analogues of the $\psi_i \circ (\mathrm{Id}, \chi)$ defined in Section 2. Since we are going to recall several times estimates in [5], we intend to comply with the notation there. In particular the functions χ defined on the charts will now become F and the image of the graph of F, namely the *admissible leaves*, will be denoted with G_F. For $\alpha, \beta, q < 1$ such that $0 < \beta \leq \alpha \leq 1 - q \leq \alpha_0$ let $C^\alpha(W, \mathbb{C})$ denote the set of continuous complex-valued functions on W with Hölder exponent α and define the norm

$$|\varphi|_{W,\alpha,q} := |W|^\alpha \cdot |\varphi|_{C^q(W,\mathbb{C})}, \tag{54}$$

where $|W|$ denotes unnormalised induced Riemannian volume of W. For $h \in C^1(X, \mathbb{C})$ we define the weak norm of h by

$$|h|_w = \sup_{W \in \Sigma} \sup_{\substack{\varphi \in C^1(W,\mathbb{C}) \\ |\varphi|_{C^1(W,\mathbb{C})} \leq 1}} \left| \int_W h\varphi \, dm \right|$$

and the strong norm by

$$\|h\| = \|h\|_s + b\|h\|_u,$$

where the strong stable norm is

$$\|h\|_s = \sup_{W \in \Sigma} \sup_{\substack{\varphi \in C^1(W,\mathbb{C}) \\ |\varphi|_{W,\alpha,q} \leq 1}} \left| \int_W h\varphi \, dm \right| \tag{55}$$

and the strong unstable norm is

$$\|h\|_u = \sup_{\epsilon \leq \epsilon_0} \sup_{\substack{W_1, W_2 \in \Sigma \\ d_\Sigma(W_1, W_2) \leq \epsilon}} \sup_{\substack{|\varphi_i|_{C^1(W_i,\mathbb{C})} \leq 1 \\ d_q(\varphi_1, \varphi_2) \leq \epsilon}} \frac{1}{\epsilon^\beta} \left| \int_{W_1} h\varphi_1 \, dm - \int_{W_2} h\varphi_2 \, dm \right|, \tag{56}$$

where d_Σ and d_q are defined precisely in §3.1 [5]. In comparison to the setting in Section 2, the norm $|\cdot|_w$ plays the role of $\|\cdot\|_{p-1,q+1} = \|\cdot\|_{0,2}$, and the norm $\|\cdot\|$ plays the role of $\|\cdot\|_{p,q} = \|\cdot\|_{1,1}$.

Let $\mathcal{B}$ be the completion of $C^1(X, \mathbb{C})$ with respect to the norm $\|\cdot\|$. Similarly, we define $\mathcal{B}_w$ to the completion of $C^1(X, \mathbb{C})$ with respect to the norm $|\cdot|_w$.

We recall that the elements of $\mathcal{B}$ are distributions. More precisely, there exists $C > 0$ such that any $h \in \mathcal{B}$ induces a linear functional $\varphi \to h(\varphi)$ with the property that

$$|h(\varphi)| \leq C|h|_w |\varphi|_{C^1}, \quad \text{for } \varphi \in C^1(X, \mathbb{C}), \tag{57}$$

see [5, Remark 3.4] for details. In particular, for $h \in C^1(X, \mathbb{C})$ we have that (see [5, Remark 2.5])

$$h(\varphi) = \int_X h\varphi, \quad \text{for } \varphi \in C^1(X, \mathbb{C}). \tag{58}$$

We say that $h \in \mathcal{B}$ is nonnegative and write $h \geq 0$ if $h(\varphi) \geq 0$ for any $\varphi \in C^1(X, \mathbb{R})$ such that $\varphi \geq 0$. Finally, we recall (see [5, Section 2.1]) that for $h \in L^1(X, \mathbb{C})$,

$$\mathcal{L}h = \left(\frac{h}{|\det DT|} \right) \circ T^{-1}. \tag{59}$$

Proposition 9.1. *We have that*

$$(\mathcal{L}h)(\varphi) = h(\varphi \circ T), \quad \text{for } h \in \mathcal{B} \text{ and } \varphi \in C^1(X, \mathbb{C}).$$

Proof. For $h \in C^1(X, \mathbb{C})$ the desired conclusion can be easily obtained from (58) and (59) by using a change of variables. This immediately implies that the conclusion holds for any $h \in \mathcal{B}$. □

9.2. **Building the cocycle.** This section follows the material in Section 3, replacing $(\mathcal{B}^{1,1}, \|\cdot\|_{1,1})$ with $(\mathcal{B}, \|\cdot\|)$ and $(\mathcal{B}^{0,2}, \|\cdot\|_{0,2})$ with $(\mathcal{B}_w, \|\cdot\|_w)$. We have included this material to make the relevant references to [5] transparent.

[5, Theorem 2.8] implies that the associated transfer operator $\mathcal{L}_T$ is quasicompact on $\mathcal{B}$, 1 is a simple eigenvalue and there are no other eigenvalues of modulus 1. This in particular implies (using the terminology as in [4, Definition 2.6]) that $\mathcal{L}_T$ is exact in $\{h \in \mathcal{B} : h(1) = 0\}$.

Let Γ_{B_*} and X_ϵ be the sets of maps as defined in [5, Section 2.4]. By applying [5, Lemma 6.1], we find that there exists $C > 0$ such that

$$\sup_{\|h\| \le 1} |(\mathcal{L}_{T'} - \mathcal{L}_T)h|_w \le C\epsilon^\beta \quad \text{for } T' \in X_\epsilon.$$

It then follows from [5, Lemma 3.5] and the discussion on [5, Section 2.4] that there exist $\epsilon, A > 0$ and $c \in (0, 1)$ such that for any $T' \in \Gamma_\epsilon$, we have that

- the unit ball in $\mathcal{B}$ is relatively compact in $\mathcal{B}_w$;
- $|\mathcal{L}_{T'}^n h|_w \le A|h|_w$ for each $n \in \mathbb{N}$ and $h \in \mathcal{B}$;
- $\|\mathcal{L}_{T'}^n h\| \le Ac^n\|h\| + A|h|_w$ for each $n \in \mathbb{N}$ and $h \in \mathcal{B}$.

Consider now a family of operators

$$\mathcal{P} = \{\mathcal{L}_{T'} : \ T' \in X_\epsilon\}.$$

It then follows from [4, Proposition 2.10] (applied to the case where $\|\cdot\| = |\cdot|_w$ and $|\cdot|_v = \|\cdot\|$) that there exists $0 < \epsilon' \le \epsilon$, $D, \lambda > 0$ such that for any $T_1, \dots, T_n \in X_{\epsilon'}$, we have that

$$\|\mathcal{L}_{T_n} \cdots \mathcal{L}_{T_2}\mathcal{L}_{T_1} h\| \le De^{-\lambda n}\|h\| \quad \text{for } h \in \mathcal{B} \text{ satisfying } h(1) = 0, \tag{60}$$

where $\mathcal{L}_{T_i}$ denotes the transfer operator associated with T_i. From now on, we replace ϵ' with ϵ so that (60) holds for $T_1, \dots, T_n \in X_\epsilon$.

We now build our cocycle by prescribing that for each $\omega \in \Omega$, $T_\omega \in X_{\epsilon'}$ and we consider $\mathcal{L}_\omega$ which is the transfer operator associated to T_ω. Then, it follows readily from (60) that

$$\|\mathcal{L}_\omega^{(n)} h\| \le De^{-\lambda n}\|h\| \quad \text{for any } \omega \in \Omega,\ n \in \mathbb{N} \text{ and } h \in \mathcal{B},\ h(1) = 0. \tag{61}$$

In addition, by decreasing ϵ if necessary, we have (see the proof of [5, Lemma 6.3]) that there exist $a \in (0, 1)$ and $B > 0$ such that

$$|\mathcal{L}_\omega^{(n)} h|_w \le B|h|_w \quad \text{and} \quad \|\mathcal{L}_\omega^{(n)} h\| \le Ba^n\|h\| + B|h|_w, \tag{62}$$

for every $\omega \in \Omega$, $n \in \mathbb{N}$ and $h \in \mathcal{B}$. In particular, there exists $K > 0$ such that

$$\|\mathcal{L}_\omega h\| \le K\|h\| \quad \text{for } \omega \in \Omega \text{ and } h \in \mathcal{B}. \tag{63}$$

9.2.1. $\mathbb{P}$-*continuity of* $\omega \mapsto \mathcal{L}_\omega$. We assume Ω is a Borel subset of a complete separable metric space, $\mathcal{F}$ is the Borel sigma-algebra and σ is a homeomorphism. Unfortunately, in this (piecewise-hyperbolic) setting we are unable to establish strong measurability of the map $\omega \mapsto \mathcal{L}_\omega$ under the assumption that $\omega \mapsto T_\omega$ is measurable. In order to be able to apply the weaker version of MET from [10], we ask instead that $\omega \mapsto T_\omega$ is measurable and that it has a countable range.

9.2.2. *Quasi-compactness of the cocycle $\mathcal{L}$ and existence of Oseledets splitting.* Similarly to the description at the end of Section 3.2, by Lemma 2.1 [9], the inequalities (62) and (63) imply that the cocycle $\mathcal{R}$ is quasi-compact. By quasi-compactness and $\mathbb{P}$-continuity of $\mathcal{L}$, the multiplicative ergodic theorem (Theorem 17, [10]) yields the existence of a unique $\mathbb{P}$-continuous Oseledets splitting

$$\mathcal{B}_{1,1} = \left(\bigoplus_{j=1}^{l} Y_j(\omega) \right) \oplus V(\omega),$$

where each component of the splitting is equivariant under $\mathcal{L}_\omega$. The $Y_j(\omega)$ are finite-dimensional and have corresponding (finite or infinite) sequence of Lyapunov exponents $0 = \lambda_1 > \lambda_2 > \dots$

9.2.3. *One-dimensionality of the top Oseledets space.* The material in section 3.3 of the present work can be reused verbatim in the piecewise hyperbolic setting, replacing $(\mathcal{B}^{1,1}, \|\cdot\|_{1,1})$ with $(\mathcal{B}, \|\cdot\|)$ and $(\mathcal{B}^{0,2}, \|\cdot\|_{0,2})$ with $(\mathcal{B}_w, \|\cdot\|_w)$. In particular we can construct a unique family of probability measures $(h^0_\omega)_{\omega\in\Omega} \subset \mathcal{B}$ such that for $\mathbb{P}$-a.e. $\omega \in \Omega$, $\mathcal{L}_\omega h^0_\omega = h^0_{\sigma\omega}$.

9.3. **The twisted cocycle.** Our observable will be a map $g\colon \Omega \times X \to \mathbb{R}$ such that $g(\omega, \cdot) \in C^1$ for $\omega \in \Omega$ and

$$M := \text{ess sup}_{\omega\in\Omega} \|g(\omega, \cdot)\|_{C^1} < \infty. \tag{64}$$

We assume that g is ω-fibrewise centred: for $\mathbb{P}$-a.e. $\omega \in \Omega$, $h^0_\omega(g(\omega, \cdot)) = 0$.

For $g \in C^1(X, \mathbb{C})$ and $h \in \mathcal{B}$, we can introduce $g \cdot h \in \mathcal{B}$ as in Section 2. Furthermore, for $\omega \in \Omega$, $\theta \in \mathbb{C}$, and $h \in \mathcal{B}$ set $\mathcal{L}^\theta_\omega(h) = \mathcal{L}_\omega(e^{\theta g(\omega,\cdot)} h)$. We will need the following lemma, which is analogous to Lemma 3.2 [15].

Lemma 9.2. *For $h \in \mathcal{B}$ and $g \in C^1(X, \mathbb{C})$, we have that*

$$\|gh\| \le C|g|_{C^1}\|h\|,$$

for some $C > 0$, independent of g and h.

Proof. It is sufficient to establish the desired conclusion for $h \in C^1(X, \mathbb{C})$. Note that

$$\|gh\| = \|gh\|_s + b\|gh\|_u. \tag{65}$$

We have

$$\|gh\|_s = \sup_{W\in\Sigma} \sup_{\substack{\varphi\in C^1(W,\mathbb{C}) \\ |\varphi|_{W,\alpha,q}\le 1}} \left| \int_W h\varphi g \, dm \right| \le |g|_{C^1} \cdot \|h\|_s, \tag{66}$$

since

$$|\varphi g|_{W,\alpha,q} = |W|^\alpha |\varphi g|_{C^q(W,\mathbb{C})} \le |W|^\alpha |\varphi|_{C^q(W,\mathbb{C})} |g|_{C^1} = |\varphi|_{W,\alpha,q} |g|_{C^1} \le |g|_{C^1}.$$

Furthermore,

$$\|gh\|_u = \sup_{\epsilon\le\epsilon_0} \sup_{\substack{W_1,W_2\in\Sigma \\ d_\Sigma(W_1,W_2)\le\epsilon}} \sup_{\substack{|\varphi_i|_{C^1(W_i,\mathbb{C})}\le 1 \\ d_q(\varphi_1,\varphi_2)\le\epsilon}} \frac{1}{\epsilon^\beta} \left| \int_{W_1} h\varphi_1 g \, dm - \int_{W_2} h\varphi_2 g \, dm \right|.$$

Using the notation as in [5, p.12] we have that

$$\begin{aligned}\frac{1}{\epsilon^\beta}\left|\int_{W_1} h\varphi_1 g\,dm - \int_{W_2} h\varphi_2 g\,dm\right| &\le \frac{1}{\epsilon^\beta}\left|\int_{W_1} h\varphi_1 g\,dm - \int_{W_2} h((\varphi_1 g)\circ\Phi)\,dm\right| \\ &\quad + \frac{1}{\epsilon^\beta}\left|\int_{W_2} h((\varphi_1 g)\circ\Phi)\,dm - \int_{W_2} h\varphi_2 g\,dm\right| \\ &=: (I) + (II),\end{aligned}$$

where $\Phi := G_{F_1}\circ G_{F_2}^{-1}$, and F_1 and F_2 are respectively the parametrization of W_1 and W_2 in the local charts.

Let us first estimate term (I). Note that

$$|\varphi_1 g|_{C^1(W_1,\mathbb{C})} \le |\varphi_1|_{C^1(W_1,\mathbb{C})}\cdot|g|_{C^1(W_1,\mathbb{C})} \le |g|_{C^1}. \tag{67}$$

We now bound the term $|(\varphi_1 g)\circ\Phi|_{C^1(W_2,\mathbb{C})}$. In the estimates that follow $C>0$ will denote an arbitrary positive number independent of g and h. Observe that

$$|(\varphi_1 g)\circ\Phi|_{C^0(W_2,\mathbb{C})} \le |\varphi_1 g|_{C^0(W_1,\mathbb{C})} \le |\varphi_1|_{C^0(W_1,\mathbb{C})}\cdot|g|_{C^0} \le |\varphi_1|_{C^1(W_1,\mathbb{C})}\cdot|g|_{C^1} \le |g|_{C^1}.$$

Furthermore, [6]

$$Lip_{1,W_2}((\varphi_1 g)\circ\Phi) \le Lip_{1,W_1}(\varphi_1 g)\cdot Lip_{1,W_2}(\Phi) \le C Lip_{1,W_1}(\varphi_1 g),$$

since $\sup_F \max\{|G_F|_{C^1}, |G_F^{-1}|_{C^1}\} < \infty$. Moreover, since

$$Lip_{1,W_1}(\varphi_1 g) \le |\varphi_1|_{C^0(W_1,\mathbb{C})} Lip_{1,W_1}(g) + |g|_{C^0(W_1,\mathbb{C})} Lip_{1,W_1}(\varphi_1),$$

it follows that

$$Lip_{1,W_1}(\varphi_1 g) \le C|g|_{C^1}.$$

Consequently,

$$|(\varphi_1 g)\circ\Phi|_{C^1(W_2,\mathbb{C})} \le C|g|_{C^1}. \tag{68}$$

Finally, we observe that

$$d_q(\varphi_1 g, (\varphi_1 g)\circ\Phi) = |(\varphi_1 g)\circ G_{F_1} - (\varphi_1 g)\circ\Phi\circ G_{F_2}|_{C^q(I_{r_1},\mathbb{C})} = 0, \tag{69}$$

since $\Phi\circ G_{F_2} = G_{F_1}$. Hence, (67), (68) and (69) imply that

$$(I) \le C|g|_{C^1}\|h\|_u. \tag{70}$$

In order to estimate (II), we need to bound

$$|(\varphi_1 g)\circ\Phi - \varphi_2 g|_{W_2,\alpha,q} = |W_2|^\alpha\cdot|(\varphi_1 g)\circ\Phi - \varphi_2 g|_{C^q(W_2,\mathbb{C})} \le C|(\varphi_1 g)\circ\Phi - \varphi_2 g|_{C^q(W_2,\mathbb{C})}.$$

Note that

$$\begin{aligned}|(\varphi_1 g)\circ\Phi - \varphi_2 g|_{C^q(W_2,\mathbb{C})} &= |(\varphi_1 g)\circ\Phi\circ G_{F_2}\circ G_{F_2}^{-1} - (\varphi_2 g)\circ G_{F_2}\circ G_{F_2}^{-1}|_{C^q(W_2,\mathbb{C})} \\ &\le C|(\varphi_1 g)\circ\Phi\circ G_{F_2} - (\varphi_2 g)\circ G_{F_2}|_{C^q(I_{r_1},\mathbb{C})} \\ &= |(g\varphi_1)\circ G_{F_1} - (g\varphi_2)\circ G_{F_2}|_{C^q(I_{r_1},\mathbb{C})} \\ &\le |(g\circ G_{F_1})(\varphi_1\circ G_{F_1}) - (g\circ G_{F_1})(\varphi_2\circ G_{F_2})|_{C^q(I_{r_1},\mathbb{C})} \\ &\quad + |(g\circ G_{F_1})(\varphi_2\circ G_{F_2}) - (g\circ G_{F_2})(\varphi_2\circ G_{F_2})|_{C^q(I_{r_1},\mathbb{C})} \\ &\le |g\circ G_{F_1}|_{C^q(I_{r_1},\mathbb{C})}\cdot d_q(\varphi_1,\varphi_2) \\ &\quad + |g\circ G_{F_1} - g\circ G_{F_2}|_{C^q(I_{r_1},\mathbb{C})}\cdot|\varphi_2\circ G_{F_2}|_{C^q(I_{r_1},\mathbb{C})} \\ &\le \epsilon|g\circ G_{F_1}|_{C^1(I_{r_1},\mathbb{C})} \\ &\quad + |g\circ G_{F_1} - g\circ G_{F_2}|_{C^q(I_{r_1},\mathbb{C})}\cdot|\varphi_2\circ G_{F_2}|_{C^1(I_{r_1},\mathbb{C})}.\end{aligned}$$

[6] $Lip_{q,W}(f) := \sup_{\substack{x\neq y\\ x,y\in W}} \frac{|f(x)-f(y)|}{|x-y|^q}$.

Since $\sup_F |G_F|_{C^1(I_r,\mathbb{C})} < \infty$, we have that

$$|g \circ G_{F_1}|_{C^1(I_{r_1},\mathbb{C})} \leq C|g|_{C^1} \quad \text{and} \quad |\varphi_2 \circ G_{F_2}|_{C^1(I_{r_1},\mathbb{C})} \leq C.$$

Finally, it remains to estimate

$$|g \circ G_{F_1} - g \circ G_{F_2}|_{C^q(I_{r_1},\mathbb{C})}.$$

Since $d_\Sigma(W_1, W_2) \leq \epsilon$, it follows from the mean-value theorem that and setup from [5, p.12] that

$$|g \circ G_{F_1}(t) - g \circ G_{F_2}(t)| \leq C\epsilon|g|_{C^1} \quad \text{for } t \in I_{r_1}. \tag{71}$$

Indeed, we have that

$$\begin{aligned} |g \circ G_{F_1}(t) - g \circ G_{F_2}(t)| &\leq |g|_{C^1}|G_{F_1}(t) - G_{F_2}(t)| \\ &= |g|_{C^1}|\chi_j(x_1 + (t, F_1(t))) - \chi_j(x_2 + (t, F_2(t)))| \\ &\leq C|g|_{C^1}(|x_1 - x_2| + |F_1(t) - F_2(t)|) \\ &\leq C|g|_{C^1} d_\Sigma(W_1, W_2), \end{aligned}$$

for $t \in I_{r_1}$ which implies (71). Take now $t, s \in I_{r_1}$. Then, (71) implies that

$$\frac{|g \circ G_{F_1}(t) - g \circ G_{F_2}(t) - g \circ G_{F_1}(s) + g \circ G_{F_2}(s)|}{|t-s|^q} \leq \frac{2C\epsilon|g|_{C^1}}{|t-s|^q}.$$

On the other hand, by applying the mean-value theorem we have

$$\frac{|g \circ G_{F_1}(t) - g \circ G_{F_2}(t) - g \circ G_{F_1}(s) + g \circ G_{F_2}(s)|}{|t-s|^q} \leq \frac{2C|g|_{C^1}|t-s|}{|t-s|^q}.$$

One can now proceed as in [5, p.20] to show that

$$\sup_{t \neq s} \frac{|g \circ G_{F_1}(t) - g \circ G_{F_2}(t) - g \circ G_{F_1}(s) + g \circ G_{F_2}(s)|}{|t-s|^q} \leq C|g|_{C^1}\epsilon^{1-q}.$$

This together with (71) implies that

$$|g \circ G_{F_1} - g \circ G_{F_2}|_{C^q(I_{r_1},\mathbb{C})} \leq C|g|_{C^1}\epsilon^{1-q}.$$

We conclude that

$$|(\varphi_1 g) \circ \Phi - \varphi_2 g|_{C^q(W_2,\mathbb{C})} \leq C\epsilon^{1-q}|g|_{C^1} \leq C\epsilon^\beta|g|_{C^1}.$$

Thus,

$$(II) \leq C|g|_{C^1}\|h\|_s. \tag{72}$$

The conclusion of the lemma follows directly from (65), (66), (70) and (72). □

The following proposition is analogous to Proposition 4.3.

Proposition 9.3. *There exists a continuous function $K\colon \mathbb{C} \to (0, \infty)$ such that*

$$\|\mathcal{L}_\omega^\theta h\| \leq K(\theta)\|h\|, \quad \textit{for } h \in \mathcal{B},\ \theta \in \mathbb{C} \textit{ and } \mathbb{P}\textit{-a.e. } \omega \in \Omega. \tag{73}$$

Proof. Note that it follows from (63) and Lemma 9.2 that

$$\|\mathcal{L}_\omega^\theta h\| = \|\mathcal{L}_\omega(e^{\theta g(\omega,\cdot)}h)\| \leq K\|e^{\theta g(\omega,\cdot)}h\| \leq CK|e^{\theta g(\omega,\cdot)}|_{C^1}\|h\|,$$

for $h \in \mathcal{B}$, $\theta \in \mathbb{C}$ and $\mathbb{P}$-a.e. $\omega \in \Omega$. Furthermore, observe that (64) implies that

$$|e^{\theta g(\omega,\cdot)}|_{C^0} \leq e^{M|\theta|} \quad \text{for } \mathbb{P}\text{-a.e. } \omega \in \Omega.$$

Similarly, it follows from the mean-value theorem (applied for a map $z \mapsto e^{\theta z}$) and (64) that

$$\sup_{x \neq y} \frac{|e^{\theta g(\omega,x)} - e^{\theta g(\omega,y)}|}{|x-y|} \leq |\theta|e^{2M|\theta|} \sup_{x \neq y} \frac{|g(\omega,x) - g(\omega,y)|}{|x-y|} \leq M|\theta|e^{2M|\theta|}.$$

The desired conclusion follows directly from the above estimates.

□

Analogously to Proposition 4.4 we have:

Proposition 9.4. *For θ close to 0, the cocycle $(\mathcal{L}_\omega^\theta)_{\omega\in\Omega}$ is quasicompact.*

Proof. We follow closely [9, Lemma 3.13]. Observe (62) and choose $N \in \mathbb{N}$ such that $\gamma := Ba^N < 1$. Hence,

$$\begin{aligned}\|\mathcal{L}_\omega^{\theta,(N)}h\| &\le \|\mathcal{L}_\omega^{(N)}h\| + \|\mathcal{L}_\omega^{\theta,(N)} - \mathcal{L}_\omega^{(N)}\| \cdot \|h\| \\ &\le \gamma\|h\| + B|h|_w + \|\mathcal{L}_\omega^{\theta,(N)} - \mathcal{L}_\omega^{(N)}\| \cdot \|h\|.\end{aligned}$$

On the other hand, we have that

$$\mathcal{L}_\omega^{\theta,(N)} - \mathcal{L}_\omega^{(N)} = \sum_{j=0}^{N-1} \mathcal{L}_{\sigma^{N-j}\omega}^{\theta,(j)}(\mathcal{L}_{\sigma^{N-1-j}\omega}^{\theta} - \mathcal{L}_{\sigma^{N-1-j}\omega})\mathcal{L}_\omega^{(N-1-j)}.$$

It follows from (63) and (73) that

$$\|\mathcal{L}_\omega^{(N-1-j)}\| \le K^{N-1-j} \quad \text{and} \quad \|\mathcal{L}_{\sigma^{N-j}\omega}^{\theta,(j)}\| \le K(\theta)^j.$$

Furthermore, using (63) and Lemma 9.2, we have that for any $h \in \mathcal{B}$ and $\mathbb{P}$-a.e. $\omega \in \Omega$,

$$\|(\mathcal{L}_\omega^\theta - \mathcal{L}_\omega)(h)\| = \|\mathcal{L}_\omega(e^{\theta g(\omega,\cdot)}h - h)\| \le K\|(e^{\theta g(\omega,\cdot)} - 1)h\| \le CK|e^{\theta g(\omega,\cdot)} - 1|_{C^1}\|h\|.$$

On the other hand, using (64) and applying the mean value theorem for the map $z \mapsto e^{\theta z}$, it is easy to verify that there exists $C' > 0$ such that for $\theta \in B_{\mathbb{C}}(0,1)$,

$$|e^{\theta g(\omega,\cdot)} - 1|_{C^1} \le C'|\theta| \quad \text{for } \mathbb{P}\text{-a.e. } \omega \in \Omega. \tag{74}$$

Hence, there exists $\tilde{C} > 0$ such that

$$\|\mathcal{L}_\omega^\theta - \mathcal{L}_\omega\| \le \tilde{C}|\theta|, \quad \text{for } \mathbb{P}\text{-a.e. } \omega \in \Omega.$$

We conclude that

$$\|\mathcal{L}_\omega^{\theta,(N)} - \mathcal{L}_\omega^{(N)}\| \le \tilde{C}|\theta| \sum_{j=0}^{N-1} K^{N-1-j}K(\theta)^j,$$

and therefore there exists $\tilde{\gamma} \in (0,1)$ such that for any θ sufficiently close to 0 and $h \in \mathcal{B}$,

$$\|\mathcal{L}_\omega^{\theta,(N)}h\| \le \tilde{\gamma}\|h\| + B|h|_w. \tag{75}$$

Similarly, one can show that there exists $\tilde{B} > 0$ such that for any θ sufficiently close to 0 and $h \in \mathcal{B}$,

$$|\mathcal{L}_\omega^\theta h|_w \le \tilde{B}|h|_w. \tag{76}$$

The conclusion of the proposition follows from (75) and (76) by arguing as in [9, Theorem 3.12]. □

9.4. **Regularity of the top Oseledets space, convexity of Λ.** The regularity of the top Oseledets space of the twisted cocycles follows identically as in Section 5, with Lemma 9.2 used in place of Lemma 3.2 [15] in the proof of Lemma 5.1. Moreover the family of probability measures h_ω^0 will allow us to define the fibred measure μ_ω as we did in Section 6.3.

9.5. **Large deviation principle and central limit theorem.** The results of Sections 6 and 7 follow verbatim with the obvious modifications. We thus obtain our main results for piecewise hyperbolic dynamics.

Theorem D (Quenched large deviations theorem)**.** *In the setting of Section 9, there exists $\epsilon_0 > 0$ and a non-random function $c\colon (-\epsilon_0, \epsilon_0) \to \mathbb{R}$ which is nonnegative, continuous, strictly convex, vanishing only at 0 and such that*

$$\lim_{n\to\infty} \frac{1}{n} \log \mu_\omega(S_n g(\omega, \cdot) > n\epsilon) = -c(\epsilon), \quad \textit{for } 0 < \epsilon < \epsilon_0 \textit{ and } \mathbb{P}\textit{-a.e. } \omega \in \Omega.$$

Theorem E (Quenched central limit theorem)**.** *In the setting of Section 9, assume that the non-random variance Σ^2, defined in (2) satisfies $\Sigma^2 > 0$. Then, for every bounded and continuous function $\phi\colon \mathbb{R} \to \mathbb{R}$ and $\mathbb{P}$-a.e. $\omega \in \Omega$, we have*

$$\lim_{n\to\infty} \int \phi\left(\frac{S_n g(\omega, x)}{\sqrt{n}}\right) d\mu_\omega(x) = \int \phi \, d\mathcal{N}(0, \Sigma^2).$$

(The discussion in §6.3 deals with the degenerate case $\Sigma^2 = 0$).

9.6. **Local central limit theorem.**

Theorem F (Quenched Local central limit theorem)**.** *In the setting of Section 9, suppose that condition (L) holds, where the functional norm in (L) is now $\mathcal{B}$.*

Then, for $\mathbb{P}$-a.e. $\omega \in \Omega$ and every bounded interval $J \subset \mathbb{R}$, we have

$$\lim_{n\to\infty} \sup_{s\in\mathbb{R}} \left| \Sigma\sqrt{n}\mu_\omega(s + S_n g(\omega, \cdot) \in J) - \frac{1}{\sqrt{2\pi}} e^{-\frac{s^2}{2n\Sigma^2}} |J| \right| = 0.$$

The LCLT can also be obtained under the assumptions HK A1, A2, A3 and the hypothesis of Lemma 8.1 with the obvious change of the functional space which is now $\mathcal{B}$. The Lasota–Yorke inequality for the twisted operator follows now by adapting the analogous proof in [5] for the usual operator. So we have the analogous statement as in Theorem 8.4.

9.7. **Billiards.** The results of this section also apply to the billiard map associated with both a finite and infinite horizon Lorentz gas having smooth scatterers with strictly positive curvature: we refer in the following to the papers by Demers and Zhang [6, 7].

We first recall the setting. Let us consider on the bidimensional torus $\mathbb{T}^2$ a finite number of pairwise disjoint and simply connected convex regions $\{\Gamma\}_{i=1}^d$, which moreover have C^3 boundary curves $\partial\Gamma_i$ with strictly positive curvature. We denote by int$A$ the interior of the set A; then the billiard table Q is defined as $Q = \mathbb{T}^2 \backslash \cup_i \text{int}\Gamma_i$. On the phase space $\mathcal{M} = Q \times \mathbb{S}^1/\sim$, with the conventional identifications at the boundaries, we define the billiard flow, which is induced by a particle traveling at unit speed and undergoing elastic collisions at the boundaries. We will be concerned instead with the billiard map $T : M \to M$ as the Poincaré map corresponding to collisions with scatterers and defined on $M = \cup_i \partial\Gamma_i \times [-\pi/2, \pi/2]$. We put on M the coordinates (r, θ), where $r \in \cup_i \partial\Gamma_i$ is parametrized by arc length and θ is the angle formed by the unit tangent vector at r with the normal pointing into the domain Q. The map T preserves a probability measure μ defined by $d\mu = c \cos\theta dr d\theta$, where c is the normalizing constant. If we denote by $\tau(x)$ the first non-tangential collision time of the orbit flow starting at x, then T is defined when $\tau(x) < \infty$ and in this case it is uniformly hyperbolic. In particular T has a finite horizon if τ is bounded from above, otherwise T has infinite horizon. The map T shares the same properties of the piecewise hyperbolic maps studied in this section, with a relevant

difference: its derivative DT becomes infinite near singularities. This fact will induce a slight change in the definition of the norms. The latter are defined exactly as in section 9 but the norm on the test function φ (54) is now modified as

$$|\varphi|_{W,\alpha,q} = |W|^{\alpha} \cdot \cos W \cdot |\phi|_{C^q(W,\mathbb{C})},$$

where $\cos W = \frac{1}{m_W(W)} \int_W \cos\theta dm_W$, being m_W the unnormalized Lebesgue measure on the stable curve W. With this precaution, the Banach space $\mathcal{B}$ is defined as the completion of $C^1(M,\mathbb{C})$ with respect to the norm given by the sum of the strong norm defined in (55) and the strong unstable norm given by (56). In [6], Demers and Zhang established the Lasota-Yorke inequality and the associated spectral picture in the deterministic setting of a single billiard map T. The main technical difference with [5] was the control of distortion which required additional cuts at the boundaries of homogeneity strips with the consequence of generating a countably infinite number of curves in both the finite and infinite horizon cases. In [7] the same authors introduced a distance between maps, see section 3.4 in [7]. Then they consider a family $\mathcal{F}$ of billiard maps such that by taking composition of maps close in $\mathcal{F}$, that composition has the same hyperbolic and distortion properties of the iterates of a single map, see the discussion in section 5.3 in [7]. Moreover they defined a random walk on M by choosing the sequence of maps in $\mathcal{F}$ in an i.i.d. way with a prescribed density (see their $g(\omega,\cdot)$ in section 2.3), where ω belongs to the probability space (Ω,ν). By averaging the Perron-Fröbenius operators associated to the maps in the sequence over ν they finally defined an averaged transfer operator and applied to it standard perturbation theory, thus getting the *annealed* limit theorems stated in their Theorem 2.6. Our approach is devoted to quenched results; for that and by eventually reducing the family $\mathcal{F}$ by the Conze-Raugi criterion which we already employed in the previous sections, we get the quenched Lasota-Yorke inequality and the exponential decay expressed by the bound (61), which are the bases of our theory. Theorems A,B and C then follows for the billiards maps associated with perturbations of the periodic Lorentz gas described above.

10. Acknowledgements

DD was supported by Croatian Science Foundation under the project IP-2014-09-2285 and by the University of Rijeka under the project number 17.15.2.2.01. GF and CGT thank AMU, CPT and CIRM (Marseille) for hospitality during this research. GF is partially supported by an ARC Discovery project. CGT is supported by an ARC DECRA. SV was supported by the MATH AM-Sud Project Physeco, and by the project APEX Systèmes dynamiques: Probabilitès et Approximation Diophantienne PAD funded by the Région PACA (France). SV warmly thanks the LabEx Archiméde (AMU University, Marseille), and INdAM (Italy).

References

[1] V. I. Bakhtin. Random processes generated by a hyperbolic sequence of mappings. i. *Izvestiya Rossiiskoi Akademii Nauk. Seriya Matematicheskaya*, 58(2):40–72, 1994.

[2] V. Baladi. The quest for the ultimate anisotropic Banach space. *J. Stat. Phys.*, 166(3-4):525–557, 2017.

[3] A. Broise. Transformations dilatantes de l'intervalle et théorèmes limites. *Astérisque*, 238:1–109, 1996. Études spectrales d'opérateurs de transfert et applications.

[4] J.-P. Conze and A. Raugi. Limit theorems for sequential expanding dynamical systems on $[0,1]$. In *Ergodic theory and related fields*, volume 430 of *Contemp. Math.*, pages 89–121. Amer. Math. Soc., Providence, RI, 2007.

[5] M. F. Demers and C. Liverani. Stability of statistical properties in two-dimensional piecewise hyperbolic maps. *Trans. Amer. Math. Soc.*, 360(9):4777–4814, 2008.
[6] M. F. Demers and H. Zhang. Spectral analysis of the transfer operator for the lorentz gas. *Journal of Modern Dynamics*, 5(4):665–709, 2011.
[7] M. F. Demers and H. Zhang. A functional analytic approach to perturbations of the lorentz gas. *CMP*, 324(3):767–830, 2013.
[8] D. Dragičević, G. Froyland, C. González-Tokman, and S. Vaienti. Almost sure invariance principle for random piecewise expanding maps. *Nonlinearity*, 31(5):2252–2280, 2018.
[9] D. Dragičević, G. Froyland, C. González-Tokman, and S. Vaienti. A spectral approach for quenched limit theorems for random expanding dynamical systems. *Comm. Math. Phys.*, 360(3):1121–1187, 2018.
[10] G. Froyland, S. Lloyd, and A. Quas. A semi-invertible Oseledets theorem with applications to transfer operator cocycles. *Discrete and Continuous Dynamical Systems, Series A*, 33(9):3835–3860, 2013.
[11] C. González-Tokman and A. Quas. A semi-invertible operator Oseledets theorem. *Ergodic Theory and Dynamical Systems*, 34:1230–1272, 8 2014.
[12] S. Gouëzel. Berry–esseen theorem and local limit theorem for non uniformly expanding maps. In *Annales de l'Institut Henri Poincare (B) Probability and Statistics*, volume 41, pages 997–1024, 2005.
[13] S. Gouëzel. Almost sure invariance principle for dynamical systems by spectral methods. *The Annals of Probability*, 38(4):1639–1671, 2010.
[14] S. Gouëzel. Limit theorems in dynamical systems using the spectral method. *Hyperbolic dynamics, fluctuations and large deviations*, 89:161–193, 2015.
[15] S. Gouëzel and C. Liverani. Banach spaces adapted to Anosov systems. *Ergodic Theory Dynam. Systems*, 26(1):189–217, 2006.
[16] Y. Guivarc'h and J. Hardy. Théorèmes limites pour une classe de chaînes de markov et applications aux difféomorphismes d'anosov. *Annales de l'IHP Probabilités et statistiques*, 24(1):73–98, 1988.
[17] Y. Hafouta and Y. Kifer. *Nonconventional limit theorems and random dynamics*. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2018.
[18] H. Hennion and L. Hervé. *Limit theorems for Markov chains and stochastic properties of dynamical systems by quasi-compactness*, volume 1766 of *Lecture Notes in Mathematics*. Springer, 2001.
[19] G. Keller and T. Nowicki. Spectral theory, zeta functions and the distribution of periodic points for collet-eckmann maps. *Communications in mathematical physics*, 149(1):31–69, 1992.
[20] Y. Kifer. Large deviations in dynamical systems and stochastic processes. *Transactions of the American Mathematical Society*, 321(2):505–524, 1990.
[21] Y. Kifer. Equilibrium states for random expanding transformations. *Random Comput. Dynam*, 1(1):1–31, 1992.
[22] Y. Kifer. Limit theorems for random transformations and processes in random environments. *Transactions of the American Mathematical Society*, 350(4):1481–1518, 1998.
[23] J. Mather. Characterization of Anosov diffeomorphisms. *Indag. Math*, 30(5):479–483, 1968.
[24] T. Morita. Local limit theorem and distribution of periodic orbits of lasota-yorke transformations with infinite markov partition. *Journal of the Mathematical Society of Japan*, 46(2):309–343, 1994.
[25] S. V. Nagaev. Some limit theorems for stationary markov chains. *Theory of Probability & Its Applications*, 2(4):378–406, 1957.
[26] S. V. Nagaev. More exact statement of limit theorems for homogeneous markov chains. *Theory of Probability & Its Applications*, 6(1):62–81, 1961.
[27] L. Rey-Bellet and L.-S. Young. Large deviations in non-uniformly hyperbolic dynamical systems. *Ergodic Theory and Dynamical Systems*, 28(2):587–612, 2008.
[28] J. Rousseau-Egele. Un théoreme de la limite locale pour une classe de transformations dilatantes et monotones par morceaux. *The Annals of Probability*, 11:772–788, 1983.
[29] D. Szász and T. Varjú. Local limit theorem for the lorentz process and its recurrence in the plane. *Ergodic Theory and Dynamical Systems*, 24(1):257–278, 2004.
[30] P. Thieullen. Fibrés dynamiques asymptotiquement compacts exposants de lyapounov. entropie. dimension. *Annales de l'Institut Henri Poincare (C) Non Linear Analysis*, 4(1):49–97, 1987.