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# Radial solutions of scaling invariant nonlinear elliptic equations with mixed reaction terms

Marie-Françoise Bidaut-Véron\*  
Marta Garcia-Huidobro †  
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**Abstract** We study global properties of positive radial solutions of  $-\Delta u = u^p + M |\nabla u|^{\frac{2p}{p+1}}$  in  $\mathbb{R}^N$  where  $p > 1$  and  $M$  is a real number. We prove the existence or the non-existence of ground states and of solutions with singularity at 0 according to the values of  $M$  and  $p$ .

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*Key words.* elliptic equations; ground states; linearization; homoclinic and heteroclinic orbits; Floquet integral.

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# 1 Introduction

The aim of this article is to study local and global properties of positive radial solutions of the equation

$$-\Delta u = |u|^{p-1}u + M |\nabla u|^{\frac{2p}{p+1}}, \tag{1.1}$$

in  $\mathbb{R}^N$  or  $\mathbb{R}^N \setminus \{0\}$  where  $p > 1$  and  $M$  is a real parameter. This is a particular case of the following class of equations

$$-\Delta u = |u|^{p-1}u + M |\nabla u|^q, \tag{1.2}$$

where  $q > 1$  which has been the subject of many works in the radial case when  $M < 0$ , where a basic observation is that the two terms  $|u|^{p-1}u$  and  $M |\nabla u|^q$  are in competition. The first work in that case is due to Chipot and Weissler [14] who, in particular, solved completely the case  $N = 1$ , then Serrin and Zou [19] performed a very detailed analysis. Much less is known in the case  $M > 0$ . Under the scaling transformation  $T_k$  defined for  $k > 0$  by

$$u_k := T_k[u](x) = k^{\frac{2}{p-1}}u(kx), \tag{1.3}$$

(1.2) becomes

$$-\Delta u_k = |u_k|^{p-1}u_k + k^{\frac{2p-q(p+1)}{p-1}}M |\nabla u_k|^q, \tag{1.4}$$

Therefore, if  $q \neq \frac{2p}{p+1}$ , (1.2) can be reduced to

$$-\Delta u = |u|^{p-1}u \pm |\nabla u|^{\frac{2p}{p+1}}. \tag{1.5}$$

Moreover, when  $q < \frac{2p}{p+1}$ , the limit equation of (1.4) when  $k \rightarrow 0$  is the Lane-Emden equation

$$-\Delta u = |u|^{p-1}u, \tag{1.6}$$

and thus the exponent  $p$  is dominant. The other scaling transformation

$$v_k := S_k[u](x) = k^{\frac{2-q}{q-1}}u(kx), \tag{1.7}$$

transforms (1.2) into

$$-\Delta v_k = k^{\frac{q-p(2-q)}{q-1}} |v_k|^{p-1} v_k + M |\nabla v_k|^q, \quad (1.8)$$

and if  $q > \frac{2p}{p+1}$ , the limit equation of (1.8) when  $k \rightarrow 0$  is the Riccati equation

$$-\Delta v = M |\nabla v|^q, \quad (1.9)$$

therefore the exponent  $q$  is dominant. In [14] and [19] most of the study deals with the case  $q \neq \frac{2p}{p+1}$ . In the *critical case* i.e. when

$$q = \frac{2p}{p+1}, \quad (1.10)$$

then not only the sign of  $M$  but also its value plays a fundamental role, with a delicate interaction with the exponent  $p$ . Notice that an equivalent form of (1.1) is

$$-\Delta v = \lambda |v|^{p-1} v \pm |\nabla v|^q \quad (1.11)$$

with  $\lambda > 0$ . In the critical case first studies in the case  $M < 0$  are due to Chipot and Weissler [14] for  $N = 1$ . The case  $N \geq 2$  was left open by Serrin and Zou [19] and the first partial results are due to Fila and Quittner [16] and Voinov [22, 23]. The case  $M > 0$  was not considered.

The equation (1.1) is the stationary part of the associated parabolic equation

$$\partial_t u - \Delta u - |u|^{p-1} u - M |\nabla u|^q = 0. \quad (1.12)$$

which is studied in [14] and [15], where one of the aims was to find conditions for the blow-up of positive solutions. A general survey with several open problems can be found in [20].

In the non radial case an important contribution dealing with a priori estimates of local positive solutions of (1.2) and existence or non-existence of entire positive solution in  $\mathbb{R}^N$  is due to the authors [7]. In this paper we complete the results of [7] in presenting a quite exhaustive study of the radial solutions of (1.1) for any real number  $M$ .

The radial solutions of (1.1) are functions  $r \mapsto u(r)$  defined in  $(0, \infty)$  where they satisfy

$$-u_{rr} - \frac{N-1}{r} u_r = |u|^{p-1} u + M |u_r|^{\frac{2p}{p+1}}. \quad (1.13)$$

Because of the invariance of (1.13) under the transformation  $T_k$  there exists an autonomous variant of (1.1) obtained by setting

$$u(r) = r^{-\frac{2}{p-1}} x(t) \quad \text{with } t = \ln r. \quad (1.14)$$

Then (1.13) becomes

$$x_{tt} + Lx_t - \frac{2K}{p-1} x + |x|^{p-1} x + M \left| \frac{2K}{p-1} x - x_t \right|^{\frac{2p}{p+1}} = 0 \quad (1.15)$$

with

$$K = \frac{(N-2)p - N}{p-1} \quad \text{and} \quad L = \frac{(N-2)p - (N+2)}{p-1} = K - \frac{2}{p-1}. \quad (1.16)$$

Putting  $y(t) = -r^{\frac{p+1}{p-1}}u_r(r)$ , then  $(x(t), y(t))$  satisfies the system

$$\begin{aligned} x_t &= \frac{2}{p-1}x - y \\ y_t &= -Ky + |x|^{p-1}x + M|y|^{\frac{2p}{p+1}}. \end{aligned} \quad (1.17)$$

We are mainly interested in the trajectories of the system which remain in the first quarter  $\mathbf{Q} = \{(x, y) \in \mathbb{R}^2 : x > 0, y > 0\}$ . Indeed, among these trajectories, we find the ones corresponding to *ground states*, i.e. positive  $C^2$  solutions  $u$  of (1.13) which are defined on  $[0, \infty)$ . They verify  $u_r(0) = 0$  and actually they are  $C^\infty$  on  $(0, \infty)$ . Using the invariance of the equation under  $T_k$  all the ground states can be derived by scaling from a unique one which satisfies  $u(0) = 1$ . Since it is easy to prove that such a solution  $u$  is decreasing, in the variables  $(x, y)$ , a ground state is a trajectory of (1.17) in  $\mathbf{Q}$ , defined on  $\mathbb{R}$  and satisfying  $\lim_{t \rightarrow -\infty} \frac{y(t)}{x(t)} = 0$ . The corresponding trajectory is denoted by  $\mathbf{T}_{reg}$ .

Contrarily to the case of the Lane-Emden equation (1.6), there exists no natural Lyapunov function when  $M \neq 0$ . This makes the study much more delicate and it is based upon a phase plane analysis. The solutions of (1.13) invariant under  $T_k$  for any  $k > 0$  correspond to constant solutions of (1.15) and have the form

$$U(r) = Xr^{-\frac{2}{p-1}} \quad \text{for all } r > 0, \quad (1.18)$$

where  $X$  is a positive root of

$$X^{p-1} + M \left( \frac{2}{p-1} \right)^{\frac{2p}{p+1}} X^{\frac{p-1}{p+1}} - \frac{2K}{p-1} = 0. \quad (1.19)$$

This equation plays a fundamental role in the description of the set of solutions of (1.13). The following constant, defined for  $N = 1, 2$  and  $p > 1$  or  $N \geq 3$  and  $1 < p \leq \frac{N}{N-2}$  has an important role in the description of the set roots of (1.19),

$$\mu^*(N) = (p+1) \left( \frac{N - (N-2)p}{2p} \right)^{\frac{p}{p+1}}. \quad (1.20)$$

When there is no ambiguity we write  $\mu^* := \mu^*(N)$ . This set is described in the following proposition.

**Proposition 1** *1- If  $M \geq 0$ , equation (1.19) admits a positive root, necessarily unique, if and only if  $N \geq 3$  and  $p > \frac{N}{N-2}$ .*

*2- If  $M < 0$  and  $p \geq \frac{N}{N-2}$ , equation (1.19) admits a unique positive root  $X_M$ .*

*3- If  $M < 0$  and either  $N = 1, 2$  and  $p > 1$  or  $N \geq 3$  and  $1 < p \leq \frac{N}{N-2}$ , there exists no positive root of (1.19) if  $-\mu^* < M < 0$ , a unique positive root if  $M = -\mu^* < 0$  and two positive roots  $X_{1,M} < X_{2,M}$  if  $M < -\mu^*$ .*

We also set  $Y_M = \frac{2}{p-1}X_M$  and  $P_M = (X_M, Y_M)$  (resp.  $Y_{j,M} = \frac{2}{p-1}X_{j,M}$  and  $P_{j,M} = (X_{j,M}, Y_{j,M})$ , for  $j=1,2$ ) and define the corresponding singular solutions  $U_M(r) = X_M r^{-\frac{2}{p-1}}$  (resp.  $U_{j,M}(r) = X_{j,M} r^{-\frac{2}{p-1}}$ ).

Recall briefly the description of the positive solutions of the Lane-Emden equation (1.6), i.e.  $M = 0$ : there exists radial ground states if and only if  $N \geq 3$  and  $p \geq \frac{N+2}{N-2}$ . If  $p = \frac{N+2}{N-2}$  these ground states are explicit and they satisfy  $\lim_{r \rightarrow \infty} r^{N-2}u(r) = c > 0$ . There exist infinitely many singular solutions  $u$  undulating around  $U_M$ . Note that a ground state corresponds to a homoclinic orbit at 0 for system (1.17) and these singular solutions are cycles surrounding  $P_M$ . We recall that an orbit of (1.17) which connects two different equilibria (resp. the same equilibrium) when  $t \in \mathbb{R}$  is called *heteroclinic* (resp. *homoclinic*).

Without the radially assumptions, and using a delicate combination of refined Bernstein techniques and Keller-Osserman estimate we have obtained in [7, Theorems C, D] a series of general a priori estimates for any positive solution of (1.1), in an arbitrary domain of  $\mathbb{R}^N$  in the case  $N \geq 1$ ,  $p > 1$  and  $q = \frac{2p}{p+1}$  and  $M > 0$ . In particular we proved there that if  $p > 1$  and  $M > M_{\dagger} := \left(\frac{p-1}{p+1}\right)^{\frac{p-1}{p+1}} \left(\frac{N(p+1)^2}{4p}\right)^{\frac{p}{p+1}}$ , or if  $N \geq 2$ ,  $p < \frac{N+3}{N-1}$  and  $M > 0$  equation (1.1) admits no ground state.

In the sequel we describe the ground states and the singular global solutions of (1.13) in  $\mathbb{R}^N \setminus \{0\}$ . Concerning the ground states, we discuss according to the sign of  $M$  and the value of  $p$ . The next value of  $M$  appears when we linearize the system (1.17) at the equilibrium  $P_M$ ,

$$\bar{M} = \bar{M}(N, p) = \frac{(p+1)((N-2)p - N - 2)}{(4p)^{\frac{p}{p+1}}((N-2)(p-1)^2 + 4)^{\frac{1}{p+1}}}. \quad (1.21)$$

Then  $\bar{M}$  is positive (resp. negative) if  $p > \frac{N+2}{N-2}$  (resp.  $p < \frac{N+2}{N-2}$ ) and we set  $\bar{\mu} = |\bar{M}|$ . It is easy to see that if  $M = \bar{M}$  then the characteristic values of the linearized operator at  $P_M$  are purely imaginary. Notice that  $\bar{M}$  is positive (resp. negative) according  $p > \frac{N+2}{N-2}$  (resp.  $p < \frac{N+2}{N-2}$ ).

**Theorem A** *Let  $N \geq 1$ ,  $p > 1$  and  $M > 0$ .*

1- *For any  $1 < p \leq \frac{N+2}{N-2}$  if  $N \geq 3$ , and any  $p > 1$  if  $N = 1, 2$ , then equation (1.13) admits no ground state.*

2- *If  $N \geq 3$  and  $p > \frac{N+2}{N-2}$ , there exist constants  $\tilde{M}_{min}, \tilde{M}_{max}$  verifying*

$$0 < \bar{M} < \tilde{M}_{min} \leq \tilde{M}_{max},$$

*such that*

- *if  $0 < M < \tilde{M}_{min}$  there exist ground states  $u$  satisfying  $u(r) \sim U_M(r)$  when  $r \rightarrow \infty$ .*
- *if  $M = \tilde{M}_{min}$  or  $M = \tilde{M}_{max}$  there exists a ground state  $u$  minimal at infinity, that is satisfying  $\lim_{r \rightarrow \infty} r^{N-2}u(r) = c > 0$ .*
- *for  $M > \tilde{M}_{max}$  there exists no radial ground state.*

The values of  $\tilde{M}_{min}$  and  $\tilde{M}_{max}$  appear as transition values for which the ground state still exists but it is smaller than the others at infinity; it is of order  $r^{2-N}$  instead of  $r^{-\frac{2}{p-1}}$ . They are not explicit but they can be estimated in function of  $N$  and  $p$ . It is a numerical evidence that  $\tilde{M}_{min} = \tilde{M}_{max}$  in the phase plane analysis of system (1.17) and we conjecture that this is true. When  $M = \tilde{M}_{min}$  or  $\tilde{M}_{max}$ , the system (1.17) admits homoclinic trajectories. We prove that the system (1.17) admits a Hopf bifurcation when  $M = \bar{M}$ . When  $p > \frac{N+2}{N-2}$  we also prove the existence of different types of positive singular solutions

**Theorem A'** *Let  $N \geq 3$ .*

1- *If  $\frac{N}{N-2} < p \leq \frac{N+2}{N-2}$  for any  $M > 0$  there exist a unique (always up to a scaling transformation) positive singular solutions  $u$  of (1.13) satisfying  $\lim_{r \rightarrow 0} r^{\frac{2}{p-1}} u(r) = X_M$  and  $\lim_{r \rightarrow \infty} r^{N-2} u(r) = c > 0$*

2- *If  $p > \frac{N+2}{N-2}$ , then*

(i) *If  $M > \tilde{M}_{max}$ , there exists a unique singular solution  $u$  of (1.13) with the same behaviour as in 1.*

(ii) *If  $\bar{M} < M < \tilde{M}_{min}$  there exist positive singular solutions  $u$  undulating around  $U_M$  on  $\mathbb{R}$ .*

In terms of the system (1.17) the 1) and 2-(i) correspond to the existence of a heteroclinic orbit in  $\mathbf{Q}$  connecting  $P_M$  to  $(0, 0)$  and (ii) to the existence of a cycle in  $\mathbf{Q}$  surrounding  $P_M$ .

When  $M$  is negative, the precise description of the trajectories of (1.17) depends also on the value of  $p$  with respect to  $\frac{N}{N-2}$ . It is proved in [7, Th. B, E] that for  $N \geq 3$  and  $1 < p < \frac{N+2}{N-2}$  there exists  $\epsilon_0 > 0$  such that if  $|M| \leq \epsilon_0$  equation (1.1) admits no positive solution in  $\mathbb{R}^N$ . The same conclusion holds if  $N \geq 3$ ,  $1 < p \leq \frac{N}{N-2}$  (or  $N = 2$  and  $p > 1$ ) and  $M > -\mu^*$ . We first consider the case  $p \geq \frac{N}{N-2}$  for which there exists a unique explicit singular solution  $U_M$ , and the results present some similarity with the ones of Theorem A.

**Theorem B** *Let  $N \geq 3$ ,  $p \geq \frac{N}{N-2}$  and  $M < 0$ . Then*

1- *If  $p \geq \frac{N+2}{N-2}$ , then equation (1.13) admits ground states  $u$ . Moreover they satisfy  $u(r) \sim U_M(r)$  as  $r \rightarrow \infty$ .*

2- *If  $\frac{N}{N-2} \leq p < \frac{N+2}{N-2}$ , there exist numbers  $\tilde{\mu}_{min}$  and  $\tilde{\mu}_{max}$  verifying*

$$0 < \bar{\mu} < \tilde{\mu}_{min} \leq \tilde{\mu}_{max} < \mu^*(1),$$

*such that*

(i) *for  $M < -\tilde{\mu}_{max}$  there exist ground states  $u$  such that  $u(r) \sim U_M(r)$  when  $r \rightarrow \infty$ .*

(ii) *for  $M = -\tilde{\mu}_{min}$  or for  $M = -\tilde{\mu}_{max}$  there exist ground states minimal at infinity in the sense that  $u(r) \sim cr^{2-N}$  when  $r \rightarrow \infty$ ,  $c > 0$ .*

(iii) *for  $-\tilde{\mu}_{min} < M < 0$  there exists no radial ground state.*

Here also the value of  $\tilde{\mu}_{min}$ ,  $\tilde{\mu}_{max}$  are not explicit and we conjecture that they coincide. The next result presents some similarity with Theorem A'.

**Theorem B'** *Let  $N \geq 3$  and  $\frac{N}{N-2} < p < \frac{N+2}{N-2}$ .*

(i) *If  $\bar{M} < M < 0$  there exists a unique (up to scaling) positive singular solution  $u$  of (1.13), such that  $u(r) \sim U_M(r)$  when  $r \rightarrow 0$  and  $u(r) \sim cr^{2-N}$  when  $r \rightarrow \infty$  for some  $c > 0$ .*

(ii) *If  $-\tilde{\mu}_{min} < M < 0$  there exist positive singular solutions  $u$  undulating around  $U_M$  on  $[0, \infty)$  and singular solution undulating around  $U_M$  in a neighbourhood of 0 and satisfying  $u(r) \sim cr^{2-N}$  for some  $c > 0$  when  $r \rightarrow \infty$ .*

In terms of the system (1.17), (i) corresponds to a heteroclinic orbit connecting  $P_M$  and  $(0, 0)$ , while (ii) to the existence of a periodic solution in  $\mathbf{Q}$  around  $P_M$ , and the existence of a solution in  $\mathbf{Q}$  converging to  $(0, 0)$  at  $\infty$  and having a limit cycle at  $t = -\infty$  which is a periodic orbit around  $P_M$ .

The situation is more complicated when  $1 < p < \frac{N}{N-2}$  and  $M < -\mu^*$  because there exist two explicit singular solutions  $U_{1,M}$  and  $U_{2,M}$  which coincide when  $M = -\mu^*$ .

**Theorem C** *Let  $M < 0$ ,  $N \geq 3$  and  $1 < p < \frac{N}{N-2}$ , or  $N = 2$  and  $p > 1$ . Then there exist two constants  $\tilde{\mu}_{min}$  and  $\tilde{\mu}_{max}$  verifying*

$$\mu^* \leq \bar{\mu} < \tilde{\mu}_{min} \leq \tilde{\mu}_{max} < \mu^*(1),$$

such that

- 1- If  $M < -\tilde{\mu}_{max}$  then equation (1.13) admits ground states  $u$  either undulating around  $U_{2,M}$  or such that  $u(r) \sim U_{2,M}(r)$  as  $r \rightarrow \infty$ .
- 2- If  $M = -\tilde{\mu}_{min}$  or  $M = -\tilde{\mu}_{max}$  there exists a ground state  $u$  such that  $u(r) \sim U_{1,M}(r)$  as  $r \rightarrow \infty$ .
- 3- If  $-\tilde{\mu}_{min} < M < 0$  there exists no radial ground state.

Here again  $\tilde{\mu}_{min}$  and  $\tilde{\mu}_{max}$  appear as transition values for which the ground state still exists but it is smaller than the others at infinity: it behaves like  $U_1$  instead of  $U_2$ . The proof of this theorem is very elaborate in particular in the case  $N = 2$ . In the case  $N = 1$  the result is already proved in [14]. The nonexistence of a ground state, not necessarily radial for  $M > -\mu^*$  is proved in [1] and independently in [7] with a different method. In the radial case it was obtained much before in the case  $N = 1$  in [14] and then by Fila and Quittner [16] who raised the question whether the condition  $-\tilde{\mu}_{min} < M < 0$  is optimal for the non-existence of radial ground state. This question received a negative answer in the work of Voiron [22] who extended the domain of non-existence to  $-\mu^* - \epsilon < M \leq -\mu^*$ . The next result is the counterpart of Theorem C when dealing with singular solutions.

**Theorem C'** *Let  $M < 0$ ,  $N \geq 3$  and  $1 < p < \frac{N}{N-2}$ . (i) If  $M < -\mu^*$  there exist positive singular solutions  $u$  such that  $u(r) \sim U_{1,M}(r)$  as  $r \rightarrow \infty$  and  $u(r) \sim cr^{2-N}$  with  $c > 0$  when  $r \rightarrow 0$ .*

*(ii) If  $\bar{M} \leq M < -\mu^*$  there exists a unique up to scaling positive singular solution  $u$ , such that  $u(r) = U_{2,M}(r)$  as  $r \rightarrow 0$  and  $u(r) = U_{1,M}(r)$  as  $r \rightarrow \infty$ . Furthermore  $u(r) > U_{1,M}(r)$  for all  $r > 0$ .*

*(iii) If  $-\hat{\mu}_{min} < M < -\bar{\mu}$  there exist positive singular solutions  $u$  undulating around  $U_{2,M}$  at 0 and such that  $u(r) \sim U_{1,M}(r)$  as  $r \rightarrow \infty$ , and positive singular solutions  $u$  undulating around  $U_{2,M}$  on  $\mathbb{R}$ .*

*(iv) If  $M = -\tilde{\mu}_{min}$  or  $M = -\hat{\mu}_{max}$  there exists a positive singular solutions  $u$  different from  $U_{1,M}$  such that  $u(r) \sim U_{1,M}(r)$  when  $r \rightarrow 0$  and  $r \rightarrow \infty$ .*

*(v) If  $-\tilde{\mu}_{min} < M < -\hat{\mu}_{max}$  there exists a positive singular solution  $u$  such that  $\lim_{r \rightarrow 0} r^{N-2}u(r) = c > 0$  and either undulating around  $U_{2,M}$  or such that  $u(r) \sim U_{2,M}(r)$  when  $r \rightarrow \infty$ .*

*(vi) If  $N \geq 3$  and  $M = -\mu^*$ , there exist positive singular solutions  $u$  satisfying  $\lim_{r \rightarrow 0} r^{N-2}u(r) = c > 0$  and  $u(r) \sim U_{-\mu^*}(r)$  as  $r \rightarrow \infty$ .*

In terms of the system (1.17) (i) corresponds to a heteroclinic orbit connecting  $P_{1,M}$  to  $(0, 0)$ ; (ii) to a heteroclinic orbit connecting  $P_{2,M}$  to  $P_{1,M}$ ; (iii) to a trajectory having a periodic orbit around  $P_{2,M}$  for limit cycle at  $-\infty$  and converging to  $P_{2,M}$  at  $\infty$  and to a periodic orbit around  $P_{2,M}$ ; (iv) corresponds to homoclinic orbit at  $P_{1,M}$ ; (v) corresponds to a trajectory connecting



$(0, 0)$  at  $-\infty$  and either converging to  $P_{2,M}$  at  $\infty$  or having a periodic orbit around  $P_{2,M}$  for limit set at  $\infty$ ; (vi) corresponds to a heteroclinic orbit connecting from  $(0, 0)$  to  $P_{-\mu^*}$ .

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## 2 General properties of the system

### 2.1 Reduction to autonomous equation and system

#### 2.1.1 The standard reduction

We recall that if  $u$  is a  $C^3$  function defined on some interval  $I \subset [0, \infty)$  verifying (1.13) and if

$$u(r) = r^{-\frac{2}{p-1}} x(t) \quad \text{with } t = \ln r,$$

then  $x$  satisfies the autonomous equation

$$x_{tt} + Lx_t - \frac{2K}{p-1}x + |x|^{p-1}x + M \left| \frac{2x}{p-1} - x_t \right|^{\frac{2p}{p+1}} = 0, \quad (2.1)$$

on  $\ln(I)$  where  $K$  and  $L$  are defined in (1.16). Setting  $u_r = -r^{-\frac{p+1}{p-1}}y(t)$ , then  $(x(t), y(t))$  satisfies

$$\begin{aligned} x_t &= H_1(x, y) \\ y_t &= H_2(x, y), \end{aligned} \quad (2.2)$$

where

$$\begin{aligned} H_1(x, y) &= \frac{2x}{p-1} - y \\ H_2(x, y) &= -Ky + |x|^{p-1}x + M|y|^{\frac{2p}{p+1}}. \end{aligned} \quad (2.3)$$

and we denote by  $\mathbf{H}$  the vector field of  $\mathbb{R}^2$  with components  $H_1$  and  $H_2$ .

#### 2.1.2 The geometry of the vector field $\mathbf{H}$

Let us denote by  $\mathbf{Q} := \{(x, y) : x > 0, y > 0\}$  the first quadrant. The vector field is inward in (resp. outward of)  $\mathbf{Q}$  on the axis  $\{(x, y) : x > 0, y = 0\}$  (resp.  $\{(x, y) : x = 0, y > 0\}$ ). We set

$$\mathcal{L} := \left\{ (x, y) \in \mathbf{Q} : y = \frac{2x}{p-1} \right\} \quad \text{and} \quad \mathcal{C} := \left\{ (x, y) \in \mathbf{Q} : x = \left( Ky - My^{\frac{2p}{p+1}} \right)^{\frac{1}{p}} \right\} \quad (2.4)$$

and  $\psi(y) = \left( Ky - My^{\frac{2p}{p+1}} \right)^{\frac{1}{p}}$ . Then  $x_t = 0$  on  $\mathcal{L}$  and  $y_t = 0$  on  $\mathcal{C}$ . The curves  $\mathcal{L}$  and  $\mathcal{C}$  have zero, one or two intersections in  $\mathbf{Q}$  according the value of  $K$  and  $M$ . If  $M, K > 0$ , then  $\mathcal{C} \subset \left[ 0, \left( \frac{p-1}{2p} \right)^{\frac{1}{p}} K^{\frac{2}{p+1}} \left( \frac{p+1}{2pM} \right)^{\frac{p+1}{p(p-1)}} \right] \times \left[ 0, \left( \frac{K}{M} \right)^{\frac{p+1}{p-1}} \right]$ . The points  $(0, 0)$ ,  $P_M$  and  $\left( 0, \left( \frac{K}{M} \right)^{\frac{p+1}{p-1}} \right)$  belong

to  $\mathcal{C}$ . The function  $\psi$  is increasing on  $\left(0, \left(\frac{K}{qM}\right)^{\frac{p+1}{p-1}}\right)$  and decreasing on  $\left(\left(\frac{K}{qM}\right)^{\frac{p+1}{p-1}}, \left(\frac{K}{M}\right)^{\frac{p+1}{p-1}}\right)$ . If  $M < 0$  and  $K > 0$ ,  $\psi$  is concave and increasing on  $(0, \infty)$  with  $\psi(y) = (-M)^{\frac{1}{p}} y^{\frac{2}{p+1}} (1 + o(1))$  as  $y \rightarrow \infty$ . If  $M < 0$  and  $K < 0$ ,  $\psi$  is still concave and increasing on  $\left(\left(\frac{K}{M}\right)^{\frac{p+1}{p-1}}, \infty\right)$  with the same asymptotic as above. We quote below the possible connected components of  $\mathbf{Q} \setminus (\mathcal{L} \cup \mathcal{C})$ .

$$\mathbf{A} = \left\{ (x, y) : \frac{2x}{p-1} - y < 0 \right\} \cap \left\{ (x, y) : -Ky + x^p + My^{\frac{2p}{p+1}} < 0 \right\} = \{(x, y) : x_t < 0, y_t < 0\}.$$

$$\mathbf{B} = \left\{ (x, y) : \frac{2x}{p-1} - y > 0 \right\} \cap \left\{ (x, y) : -Ky + x^p + My^{\frac{2p}{p+1}} < 0 \right\} = \{(x, y) : x_t > 0, y_t < 0\}.$$

$$\mathbf{C} = \left\{ (x, y) : \frac{2x}{p-1} - y > 0 \right\} \cap \left\{ (x, y) : -Ky + x^p + My^{\frac{2p}{p+1}} > 0 \right\} = \{(x, y) : x_t > 0, y_t > 0\}.$$

$$\begin{aligned} \mathbf{D} &= \left\{ (x, y) : \frac{2x}{p-1} - y < 0 \right\} \cap \left\{ (x, y) : -Ky + x^p + My^{\frac{2p}{p+1}} > 0 \right\} \cap \left\{ (x, y) : x > X_{2,M} \right\} \\ &= \{(x, y) : x_t < 0, y_t > 0\} \cap \{(x, y) : x > X_{2,M}\}. \end{aligned}$$

$$\begin{aligned} \mathbf{E} &= \left\{ (x, y) : \frac{2x}{p-1} - y < 0 \right\} \cap \left\{ (x, y) : -Ky + x^p + My^{\frac{2p}{p+1}} > 0 \right\} \cap \left\{ (x, y) : x < X_{1,M} \right\} \\ &= \{(x, y) : x_t < 0, y_t > 0\} \cap \{(x, y) : x < X_{1,M}\}. \end{aligned}$$

These connected components are

**A, B, C, D** if  $K \geq 0, M < 0$  or  $K, M > 0$ .

**A, C, D** if  $K < 0$  and  $-\mu^* < M < 0$ .

**A, C, D, E** if  $K < 0$  and  $M = -\mu^*$ .

**A, B, C, D, E** if  $K < 0$  and  $M < -\mu^*$ .

### 2.1.3 Graphic representation of the vector field $\mathbf{H}$

We present below some graphics of the vector field  $H$  associated to system (1.17).

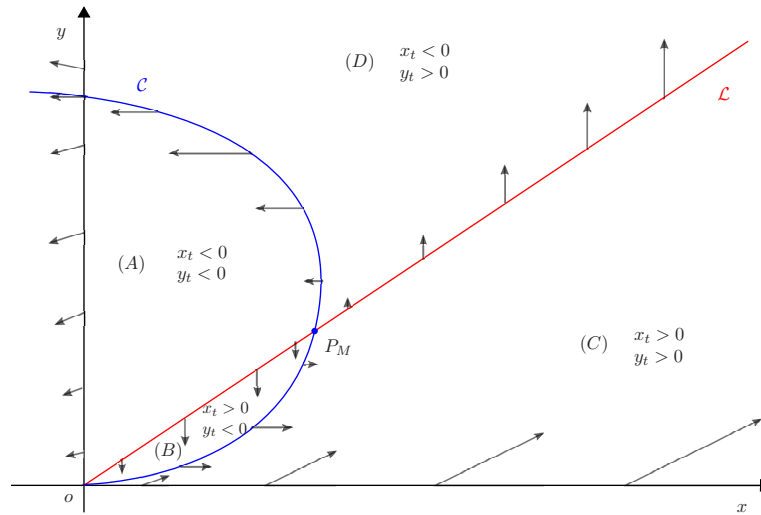


Figure 1:  $M > 0, K > 0$ .

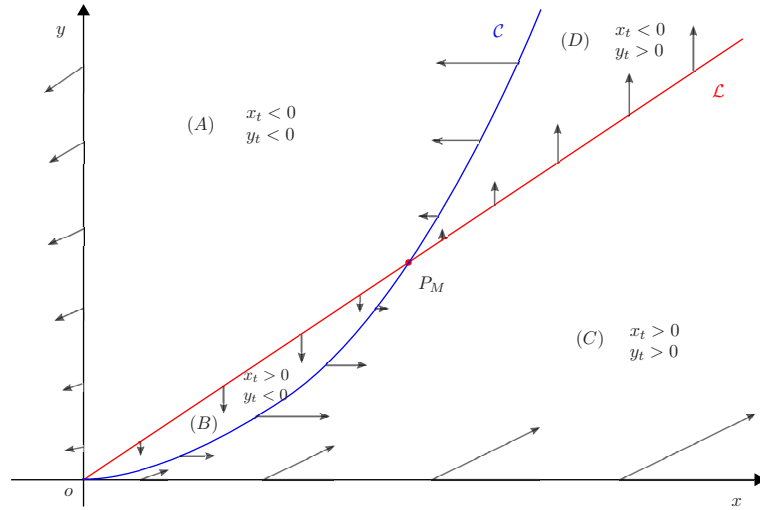


Figure 2:  $M < 0, K \geq 0$ .

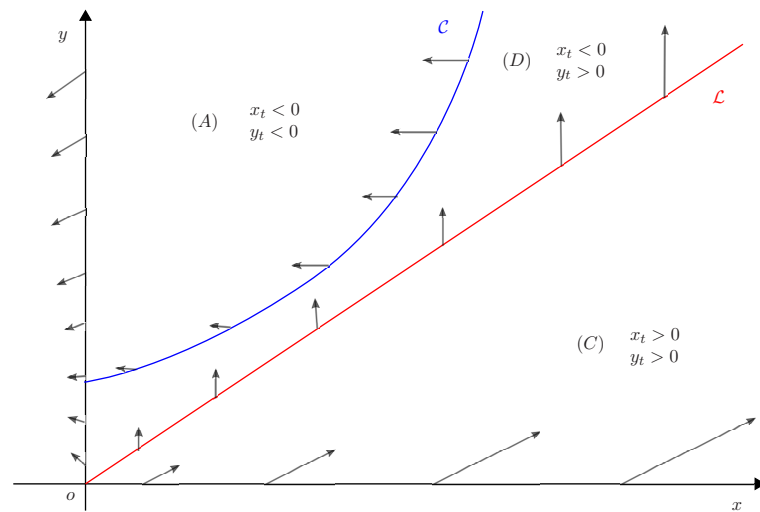


Figure 3:  $-\mu^* < M < 0, K < 0$ .

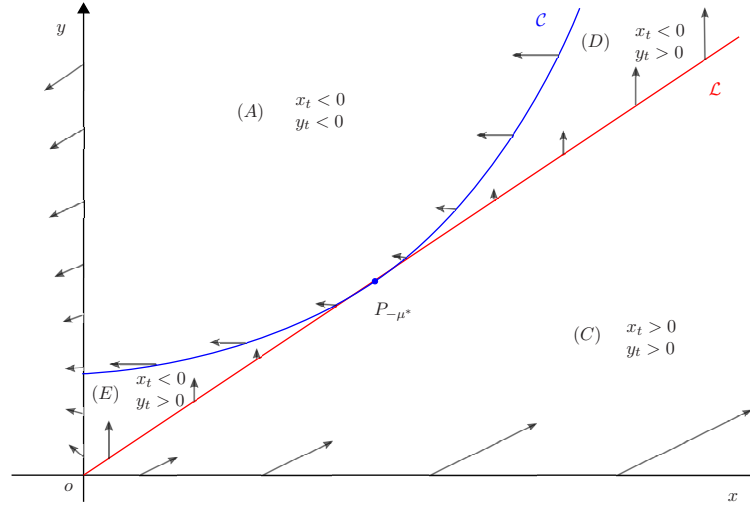


Figure 4:  $M = -\mu^*$ ,  $K < 0$ .

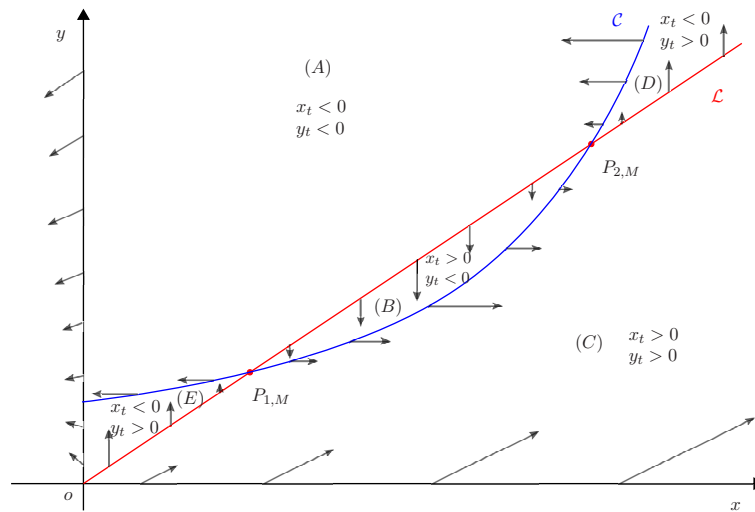


Figure 5:  $M < -\mu^*$ ,  $K < 0$ .

### 2.1.4 Other reduction

The following change of unknowns, already used in [9] when  $M = 0$ ,

$$\sigma(t) = \frac{y(t)}{x(t)} = -r \frac{u_r(r)}{u(r)} \quad \text{and} \quad z(t) = \frac{|x|^{p-1} x(t)}{y(t)} = -r \frac{|u|^{p-1} u(r)}{u_r(r)}, \quad (2.5)$$

and valid if  $u_r \neq 0$ , transforms (1.17) into a *Kolmogorov system* with vector field  $\mathbf{V} = (V_1, V_2)$

$$\begin{aligned} \sigma_t &= \sigma \left( \sigma + 2 - N + z + M \left| |\sigma|^{p-1} \sigma z \right|^{\frac{1}{p+1}} \right) = V_1(\sigma; z) \\ z_t &= z \left( N - p\sigma - z - M \left| |\sigma|^{p-1} \sigma z \right|^{\frac{1}{p+1}} \right) = V_2(\sigma; z). \end{aligned} \quad (2.6)$$

Since  $\sigma$  and  $z$  are in factor the two axis  $\{\sigma = 0\}$  and  $\{z = 0\}$  are trajectories, actually not admissible for (2.2) in view of (2.5). The system is singular on these two axis however it can be desingularized by setting  $\sigma = \tilde{\sigma}^{2k+1}$  and  $z = \tilde{z}^{2k+1}$  for some integer  $k > p + 1$ , which transforms (2.6) into a new nonsingular Kolmogorov system,

$$\begin{aligned} \tilde{\sigma}_t &= \frac{1}{2k+1} \tilde{\sigma} \left( \tilde{\sigma}^{2k+1} + 2 - N + \tilde{z}^{2k+1} + M \left| (|\tilde{\sigma}|^{p-1} \tilde{\sigma})^{2k+1} \tilde{z}^{2k+1} \right|^{\frac{1}{p+1}} \right) \\ \tilde{z}_t &= \frac{1}{2k+1} \tilde{z} \left( N - p\tilde{\sigma}^{2k+1} - \tilde{z}^{2k+1} - M \left| (|\tilde{\sigma}|^{p-1} \tilde{\sigma})^{2k+1} \tilde{z}^{2k+1} \right|^{\frac{1}{p+1}} \right). \end{aligned} \quad (2.7)$$

Therefore no other trajectory can intersect them in finite time and the quadrant  $\mathcal{Q} := \{(\sigma, z) : \sigma > 0, z > 0\}$  is invariant. Furthermore  $\sigma z = r^2 |u|^{p-1}$ . It is noticeable that if  $M = 0$  the initial system is quadratic and regular.

The system (2.6) will be used in the most delicate cases. It corresponds to the differentiation of the initial equation (2.1).

## 2.2 Regular solutions and ground states

**Definition 2.1** *A regular solution of (1.13) is a  $C^2$  solution defined on some maximal interval  $[0, r_0)$  satisfying  $u(0) = u_0 > 0$  and  $u_r(0) = 0$ . A ground state is a nonnegative  $C^2$  solution defined on  $[0, \infty)$ .*

The existence and uniqueness of a regular solution is standard by the Cauchy-Lipschitz integral method. If  $u$  is a  $C^2$  solution it satisfies  $u_r < 0$  on  $(0, r_0)$ . Indeed  $r^{N-1}u_r(r)$  is decreasing near 0, hence  $u_r < 0$  on some maximal interval  $(0, r_1) \subset (0, r_0)$  and  $u_r(r_1) = 0$  if  $r_1 < r_0$ . If  $u(r_1) = 0$  then  $u \equiv 0$  by uniqueness. If  $u(r_1) > 0$  then  $u_{rr} < 0$  near  $r_1$  which would imply that  $u(r) < u(r_1)$  for  $r_1 - \epsilon \leq r < r_1$  which contradict the negativity of  $u_r$  on  $(0, r_1)$ . Hence  $u(r_1) < 0$  which implies that  $u_r(r) < 0$  on the maximal interval  $(0, r_2)$  where  $u > 0$ . Thus, if  $u$  is a ground state  $u_r < 0$  on  $(0, \infty)$ . Hence the trajectory of a ground state expressed in the system (1.17) lies in  $\mathbf{Q}$  and expressed in the system (2.6) it lies in the quadrant  $\mathcal{Q}$ . It is easy to check that the regular solution  $u := u_{u_0}$ , such that  $u(0) = u_0$  satisfies

$$u(r) = u_0 \left( 1 - \frac{u_0^{p-1} r^2}{2N} - \frac{M(p+1)^2 (u_0^{p-1} r^2)^{\frac{2p+1}{p+1}}}{(4p+2)((N+2)p+N)N^q} (1 + o(1)) \right) \quad \text{as } r \rightarrow 0. \quad (2.8)$$

Under the scaling transformation  $T_k$ ,  $u$  can be transformed into the regular solution (1.13)  $u := u_1$  satisfying  $u(0) = 1$ . If one considers the system (1.17) the transformation  $T_k$  becomes the time shift which transforms  $t \mapsto (x(t), y(t))$  into  $t \mapsto (x(t + \ln k), y(t + \ln k))$ , and the trajectory  $(x(t), y(t))$  of (1.17) corresponding to a ground state is therefore uniquely determined and denoted by  $\mathbf{T}_{reg}$  and satisfies

$$\lim_{t \rightarrow \infty} (x(t), y(t)) = (0, 0), \quad \lim_{t \rightarrow \infty} \frac{y(t)}{x(t)} = 0 \quad \text{and} \quad \lim_{t \rightarrow \infty} \frac{y(t)}{x^p(t)} = \frac{1}{N}. \quad (2.9)$$

Hence in the system (2.5) there holds on the corresponding trajectory

$$\lim_{t \rightarrow \infty} \sigma(t) = 0, \quad \lim_{t \rightarrow \infty} z(t) = N. \quad (2.10)$$

### 2.3 Explicit singular solutions

Explicit self-similar solutions of (1.13), necessarily under the form  $u = Ar^{-\frac{2}{p-1}}$ , play a fundamental role in the study, whenever they exist. The following result covers Proposition 1.

**Proposition 2.2** *1- Let  $M \geq 0$ . Then there exists a unique self-similar solution of (1.13) if and only if  $N \geq 3$  and  $p > \frac{N}{N-2}$ . We denote it by  $U_M(r) = X_M r^{-\frac{2}{p-1}}$ , where  $X_M > 0$  depends also on  $M$ ,  $N$  and  $p$ . To this solution is associated the equilibrium  $(X_M, \frac{2}{p-1} X_M)$  of the system (1.17).*

*Furthermore the mappings  $M \mapsto X_M$  is continuous and decreasing on  $[0, \infty)$ ,  $M \mapsto M X_M^{\frac{p-1}{p+1}}$  is increasing and there holds*

$$(i) \quad X_0 = \left( \frac{2K}{p-1} \right)^{\frac{1}{p-1}}, \quad (2.11)$$

$$(ii) \quad \frac{p-1}{2} \left( \frac{K}{M} \right)^{\frac{p+1}{p-1}} \left( 1 - \frac{1}{M} \left( \frac{p-1}{2} \right)^p \left( \frac{K}{M} \right)^p \right)^{\frac{p+1}{p-1}} \leq X_M \leq \frac{p-1}{2} \left( \frac{K}{M} \right)^{\frac{p+1}{p-1}}.$$

*2- Let  $M < 0$ . If  $N \geq 3$  and  $p \geq \frac{N}{N-2}$  there exists a unique self-similar solution of (1.13)  $U_M(r)$ .*

*The mapping  $M \mapsto X_M$  is continuous and decreasing,  $M \mapsto M X_M^{\frac{p-1}{p+1}}$  is decreasing and there holds*

$$\max \left\{ \left( \frac{2K}{p-1} \right)^{\frac{1}{p-1}}, \left( \frac{2}{p-1} \right)^{\frac{2}{p-1}} |M|^{\frac{p+1}{p(p-1)}} \right\} \leq X_M$$

$$\leq 2^{\frac{2}{p-1}} \left( \left( \frac{2K}{p-1} \right)^{\frac{1}{p-1}} + \left( \frac{2}{p-1} \right)^{\frac{2}{p-1}} |M|^{\frac{p+1}{p(p-1)}} \right). \quad (2.12)$$

*3- Let  $M < 0$ . If  $N = 1, 2$  and  $p > 1$ , or  $N \geq 3$  and  $1 < p < \frac{N}{N-2}$ , there exists no self-similar solution of (1.13) if  $-\mu^* < M < 0$  where  $\mu^* = \mu^*(N) > 0$  is defined in (1.20). If  $M = -\mu^*$  there exists a unique self-similar solution  $U_{-\mu^*}(r)$ . If  $M < -\mu^* < 0$  there exist two-self-similar*

solutions  $U_{1,M}(r) = X_{1,M} r^{-\frac{2}{p-1}}$  and  $U_{2,M}(r) = X_{2,M} r^{-\frac{2}{p-1}}$  with  $X_{1,M} < X_{2,M}$ . Furthermore the mappings  $M \mapsto X_{1,M}$  and  $M \mapsto X_{2,M}$  are continuous, respectively increasing and decreasing on  $(-\infty, -\mu^*)$ , while  $M \mapsto MX_{1,M}^{\frac{p-1}{p+1}}$  and  $M \mapsto MX_{2,M}^{\frac{p-1}{p+1}}$  are respectively decreasing and increasing. Furthermore there holds for  $|M|$  large enough;

$$\begin{aligned}
 (i) \quad & X_{j,-\mu^*} = \left( \frac{2}{p-1} \right)^{\frac{p}{p-1}} \left( \frac{-K}{p} \right)^{\frac{1}{p-1}}, \\
 (ii) \quad & \frac{p-1}{2} \left( \frac{K}{M} \right)^{\frac{p+1}{p-1}} < X_{1,M} < \frac{p-1}{2} \left( \frac{K}{M} \right)^{\frac{p+1}{p-1}} \left( 1 - \frac{2}{K} \left( \frac{(p-1)K}{2M} \right)^{p+1} \right)^{\frac{p+1}{p-1}}, \\
 (iii) \quad & \left( \frac{2}{p-1} \right)^{\frac{2}{p-1}} (-M)^{\frac{p+1}{p(p-1)}} \left( 1 - \frac{K}{M|M|^{\frac{1}{p}}} \right)^{\frac{p+1}{p-1}} < X_{2,M} < \left( \frac{2}{p-1} \right)^{\frac{2}{p-1}} (-M)^{\frac{p+1}{p(p-1)}}.
 \end{aligned} \tag{2.13}$$

*Proof.* The function  $U_M = X_M r^{-\frac{2}{p-1}}$  is a self-similar solution of (1.13) if and only if  $X_M$  is a positive root of

$$\tilde{f}_M(x) := x^{p-1} + \left( \frac{2}{p-1} \right)^{\frac{2p}{p+1}} M x^{\frac{p-1}{p+1}} - \frac{2K}{p-1} = 0. \tag{2.14}$$

Equivalently  $P_M = (X_M, Y_M) = \left( \frac{p-1}{2} Y_M, Y_M \right)$  is a fixed point of system (1.17), where  $Y_M$  is the positive root of

$$f_M(y) = \left( \frac{p-1}{2} \right)^p y^{p-1} + M y^{\frac{p-1}{p+1}} - K = 0. \tag{2.15}$$

The use of the variable  $y$  is a little easier than  $x$ . Since  $X_0$  is explicit if  $M = 0$ , we shall study the cases  $M \neq 0$ .

1- *Case  $M > 0$ .* If  $K > 0$ , equivalently  $p > \frac{N}{N-2}$ , and  $M \geq 0$ ,  $f_M$  is an increasing function tending to  $\infty$  at  $\infty$  and negative at  $y = 0$ . Hence  $Y_M$  is the unique positive root of (2.15). If  $K < 0$ ,  $f_M$  is positive on  $[0, \infty)$ , hence no such solution exists. Since

$$\left( \frac{p-1}{2} \right)^p Y_M^{p-1} + M Y_M^{\frac{p-1}{p+1}} - K = 0,$$

by the implicit function theorem,  $M \mapsto Y_M$  is  $C^1$ . For  $M > M' > 0$ ,  $f_M(y) > f_{M'}(y)$  for all  $y > 0$ . Hence  $M \mapsto Y_M$  is decreasing on  $[0, \infty)$ . Actually the expression of  $f_M$  shows more, namely that  $M \mapsto M Y_M^{\frac{p-1}{p+1}}$  is increasing on  $[0, \infty)$ . Furthermore

$$M Y_M^{\frac{p-1}{p+1}} < K \implies Y_M < \left( \frac{K}{M} \right)^{\frac{p+1}{p-1}},$$

and

$$\begin{aligned}
 & \left( \frac{p-1}{2} \right)^p \left( \frac{K}{M} \right)^{p+1} + M Y_M^{\frac{p-1}{p+1}} > K \\
 & \implies Y_M > \left( \frac{K}{M} \right)^{\frac{p+1}{p-1}} \left( 1 - \left( \frac{p-1}{2} \right)^p \frac{1}{M} \left( \frac{K}{M} \right)^p \right)_+^{\frac{p+1}{p-1}},
 \end{aligned}$$

and we get (2.11).

2- *Case*  $M < 0$ . Clearly  $f_M$  has a minimum at  $y = y_{0,M}$  where

$$y_{0,M} = \left( \frac{2}{p-1} \left( \frac{-M}{p+1} \right)^{\frac{1}{p}} \right)^{\frac{p+1}{p-1}} \quad \text{and} \quad f_M(y_{0,M}) = -\frac{2p}{p-1} \left( \frac{-M}{p+1} \right)^{\frac{p+1}{p}} - K. \quad (2.16)$$

We encounter two possibilities:

2-a. If  $N \geq 3$  and  $p \geq \frac{N}{N-2}$ , then  $f_M(0) < 0$ ,  $f_M$  is decreasing on  $(0, y_{0,M})$  and increasing on  $(y_{0,M}, \infty)$ , hence  $f_M(y) = 0$  has a unique positive root  $Y_M > y_{0,M}$ . Since for  $M < M' < 0$  and  $y > 0$ ,  $f_{M'}(y) > f_M(y)$ , the mapping  $M \mapsto Y_M$  is continuous and decreasing and  $M \mapsto MY_M^{\frac{p-1}{p+1}}$  is increasing.

Then the left-hand side of (2.12) is clear. Next we put

$$A^p = \left( \frac{2}{p-1} \right)^{\frac{2p}{p+1}} |M|, \quad B^{p+1} = \frac{2K}{p-1}, \quad \xi = X^{\frac{p-1}{p+1}}, \quad a = \frac{B}{A} \quad \text{and} \quad \eta = \frac{\xi}{A}.$$

Then  $\phi(\eta) = \eta^{p+1} - \eta - a^{p+1} = 0$ . Since

$$\begin{aligned} \phi(1+a) &= (1+a)^{p+1} - 1 - a - a^{p+1} \geq (1+a)^{p+1} - 1 - a - a^p - a^{p+1} \\ &\geq (1+a) \left( (1+a)^p - (1+a^p) \right) \geq 0 \end{aligned}$$

we derive  $\eta \leq 1+a$ , which implies the right-hand side of (2.12).

2-b. If  $N = 1, 2$  or  $N \geq 3$  and  $1 < p < \frac{N}{N-2}$ , then  $K < 0$ . Hence, if  $f_M(y_{0,M}) > 0$ , or equivalently  $-\mu^* < M \leq 0$ , the equation  $f_M(y) = 0$  has no positive root, if  $M = -\mu^*$ , it has a double root  $Y_{-\mu^*}$  where

$$Y_{-\mu^*} = \left( \frac{2}{p-1} \right)^{\frac{p}{p-1}} \left( \frac{-K}{p} \right)^{\frac{1}{p-1}} \implies \mu^* Y_{-\mu^*}^{\frac{p-1}{p+1}} = -\frac{p+1}{p} K, \quad (2.17)$$

and if  $M < -\mu^*$  the equation (2.15) has two positive roots  $0 < Y_{1,M} < Y_{2,M}$  and since  $f'_M$  does not vanish at  $Y_{j,M}$ , they are  $C^1$  functions of  $M \in (-\infty, M^*)$ , respectively increasing and decreasing. Since  $M, K < 0$ , we obtain from  $f_M(Y_M) = 0$ ,

$$Y_{1,M} > \left( \frac{K}{M} \right)^{\frac{p+1}{p-1}},$$

and

$$\left( \frac{p-1}{2} \right)^p Y_{2,M}^{p-1} < -MY_{2,M}^{\frac{p-1}{p+1}} \implies Y_{2,M} < \left( \frac{2}{p-1} \right)^{\frac{p+1}{p-1}} (-M)^{\frac{p+1}{p(p-1)}}$$

For a sharper estimate, we have for  $M$  large enough,

$$\tilde{Y} = \left( \frac{K}{M} \right)^{\frac{p+1}{p-1}} \left( 1 - \frac{2}{K} \left( \frac{(p-1)K}{2M} \right)^{p+1} \right)^{\frac{p+1}{p-1}} \implies f_M(\tilde{Y}) < 0.$$



Hence

$$\left(\frac{K}{M}\right)^{\frac{p+1}{p-1}} < Y_{1,M} < \left(\frac{K}{M}\right)^{\frac{p+1}{p-1}} \left(1 - \frac{2}{K} \left(\frac{(p-1)K}{2M}\right)^{p+1}\right)^{\frac{p+1}{p-1}}.$$

Similarly

$$\left(\frac{2}{p-1}\right)^{\frac{p+1}{p-1}} (-M)^{\frac{p+1}{p(p-1)}} \left(1 - \frac{K}{M|M|^{\frac{1}{p}}}\right)^{\frac{p+1}{p-1}} < Y_{2,M} < \left(\frac{2}{p-1}\right)^{\frac{p+1}{p-1}} (-M)^{\frac{p+1}{p(p-1)}}.$$

The estimates (2.13) follow.  $\square$

### 2.3.1 Upper estimate of the regular solutions

We first recall the following estimate in the case  $M \geq 0$ , consequence of the fact that the positive solutions of (1.1) are superharmonic and proved in a more general setting in [7, Prop. 2.1].

**Proposition 2.3** 1- *There exists no positive solution of (1.13) in  $(R, \infty)$ ,  $R \geq 0$  if  $M \geq 0$  and either  $N = 1, 2$  and  $p > 1$  or  $N \geq 3$  and  $1 < p \leq \frac{N}{N-2}$ . In particular there exists no ground state.*

2- *If  $N \geq 3$ ,  $p > \frac{N}{N-2}$ ,  $M \geq 0$  and  $u$  is a positive solution of (1.13) in  $(R, \infty)$ ,  $R \geq 0$ . Then there exists  $\rho \geq R$  such that*

$$u(r) \leq \min \left\{ \left(\frac{2N}{p-1}\right)^{\frac{1}{p-1}}, \frac{p-1}{2} \left(\frac{p(N-2)-N}{(p-1)M}\right)^{\frac{p+1}{p-1}} \right\} r^{-\frac{2}{p-1}}, \quad \text{for all } r > \rho, \quad (2.18)$$

and

$$|u_r(r)| \leq \min \left\{ (N-2) \left(\frac{2N}{p-1}\right)^{\frac{1}{p-1}}, \left(\frac{p(N-2)-N}{(p-1)M}\right)^{\frac{p+1}{p-1}} \right\} r^{-\frac{p+1}{p-1}} \quad \text{for all } r > \rho. \quad (2.19)$$

Furthermore, if  $R = 0$ , inequalities (2.18), and (2.19) hold with  $\rho = 0$ .

The next estimate is verified by any ground state, independently of the sign of  $M$ .

**Proposition 2.4** *Let  $p > 1$  and  $N \geq 1$ . Then the ground state  $u$  of (1.13) with  $u(0) = 1$  satisfies*

$$u(r) \leq \min \left\{ 1, c_{N,p,M} r^{-\frac{2}{p-1}} \right\} \quad \text{and} \quad |u_r(r)| \leq c_{N,p,M} r^{-\frac{p+1}{p-1}} \quad \forall r > 0. \quad (2.20)$$

*Proof.* The trajectory  $\mathbf{T}_{reg}$  starts from  $(0, 0)$  and enters the region  $\mathbf{C}$ . If it stays in  $\mathbf{C}$ , then  $x$  is increasing on  $\mathbb{R}$ . Since  $y < \frac{2x}{p-1}$ , if  $x(t)$  tends to some finite limit as  $t \rightarrow \infty$ , it implies that the limit set of the trajectory exists. It cannot be a cycle since  $x$  is monotone, thus it is one of the equilibrium of the system. Hence  $x(t) \leq X_{j,M}$  for  $j = 1$  or  $2$  if  $K < 0$  and  $M < -\mu^*$  or  $x(t) \leq X_M$  if  $K \geq 0$  and either  $M > 0$  or  $-\mu^* \leq M \leq 0$ . This implies that (2.20) holds. If  $x(t)$  tends to  $\infty$ , so does  $y$  because  $y(t) \geq x^p(t) - \frac{2|K|}{p-1}x(t) - |M| \left(\frac{2x(t)}{p-1}\right)^{\frac{2p}{p+1}} \rightarrow \infty$  as  $t \rightarrow \infty$ .

Therefore  $y_t \geq Cy^p$  for some  $C > 0$  which would imply that  $t \mapsto y^{1-p}(t) + C(p-1)t$  is increasing, which is impossible.

Next we suppose that the trajectory leaves  $\mathbf{C}$  by crossing the line  $\mathcal{L}$ . Since it cannot enter  $\mathbf{B}$  through  $\mathcal{C}$  (in the case  $K < 0$ ,  $M \leq -\mu^*$ ), it leaves  $\mathbf{C}$  by intersecting  $\mathcal{L}$  and we denote by  $t_1$  the first time where  $\mathbf{T}_{reg}$  intersects  $\mathcal{L}$ . Then  $x_t(t_1) = 0$  and  $x_{tt}(t_1) = -y_t(t_1) < 0$ . Therefore  $t_1$  is a local maximum. Now the trajectory cannot cross again the half line  $\left\{ (x, y) : x = x(t_1), y > \frac{2x(t_1)}{p-1} \right\}$  because on it there holds  $x_t = \frac{2x(t_1)}{p-1} - y < 0$ . Hence  $x(t) \leq x(t_1)$  for any  $t \geq t_1$ .

In the same way, either  $y$  is increasing on  $\mathbb{R}$  and since  $x_t = \frac{2}{p-1}x - y$  and  $x$  is bounded,  $y$  cannot tend to infinity when  $t \rightarrow \infty$ , thus  $y(t) \rightarrow y_0 > 0$ , or  $y$  is not monotone and  $\mathbf{T}_{reg}$  crosses  $\mathcal{C}$  at a first value  $t_2$ , necessarily larger than  $t_1$  and where  $x_t(t_2) < 0$ . Then  $y_{tt}(t_2) = p|x(t_2)|^{p-1}x_t(t_2) < 0$  and  $t_2$  is a local maximum of  $y$ . Therefore  $\cup_{t \geq t_2} (x(t), y(t))$  remains in the subset of  $\mathbf{Q}$  bordered by the portion of trajectory of  $\mathbf{T}_{reg}$  for  $t \leq t_2$  and  $\{(x, y) : 0 < x \leq x(t_2), y = y(t_2)\}$ . This implies that  $y(t) \leq y(t_2)$  for all  $t \in \mathbb{R}$ . Noticing that  $u(r) \leq 1$  since  $u$  is decreasing, we get the conclusion.  $\square$

*Remark.* The above method does not give an explicit estimate of the upper bounds of  $x$  and  $y$  and such a bound can be estimated in some cases. If  $M \geq 0$  it follows from Proposition 2.3 that for any  $p > 1$  there holds

$$u(r) \leq \left( \frac{2N}{p-1} \right)^{\frac{1}{p-1}} r^{-\frac{2}{p-1}},$$

thus this estimate is independent of  $M$ . Here a new critical value is involved in all dimension  $N \geq 2$ , namely

$$\mu^*(2) = (p+1) \left( \frac{1}{p} \right)^{\frac{p}{p+1}}, \quad (2.21)$$

corresponding to the definition (1.20). If  $-\mu^*(2) < M < 0$ , it is easy to check that the function  $v = \ln u$  satisfies

$$-\Delta v \geq ae^{(p-1)v} \quad (2.22)$$

with  $a = 1 - \left( \frac{|M|}{\mu^*(2)} \right)^{p+1} > 0$ . We derive that the function  $w(r) = -(r^{N-1}v_r(r))$  is increasing, with limit  $\ell \in (0, \infty]$ . Hence

$$e^{(1-p)v(r)} \geq e^{(1-p)v(0)} + \frac{a(p-1)r^2}{4N}$$

This implies

$$u(r) \leq \min \left\{ 1, c_0 a^{-\frac{1}{p-1}} r^{-\frac{2}{p-1}} \right\}. \quad (2.23)$$

## 2.4 Linearization of the system (1.17) near equilibria

### 2.4.1 Linearization at $(0, 0)$

The linearized system at  $(0, 0)$  is

$$\begin{aligned} x_t &= \frac{2x}{p-1} - y \\ y_t &= -Ky, \end{aligned} \quad (2.24)$$

which admits the eigenvalues

$$\lambda_1 = -K \quad \lambda_2 = \frac{2}{p-1}. \quad (2.25)$$

Note that  $\lambda_2 - \lambda_1 = N - 2$ .

(a) Assume that  $N \geq 3$  and  $p > \frac{N}{N-2}$ , or equivalently  $K > 0$ . Then  $(0, 0)$  is a saddle point. There exists a unique *unstable trajectory*  $\mathbf{T}_{unst}$  such that

$$\lim_{t \rightarrow -\infty} (x(t), y(t)) = (0, 0) \quad \text{and} \quad \lim_{t \rightarrow -\infty} \frac{y(t)}{x(t)} = 0, \quad (2.26)$$

and more precisely, from (2.8),

$$\begin{aligned} x(t) &= e^{\frac{2t}{p-1}} \left( 1 - \frac{e^{2t}}{2N} - \frac{M(p+1)^2 e^{\frac{4p+2}{p+1}t}}{2(N(p+1)+2p)(2p+1)} (1 + o(1)) \right) \\ y(t) &= e^{\frac{2pt}{p-1}} \left( \frac{1}{N} + \frac{M(p+1)}{N(p+1)+2p} e^{\frac{(p-1)t}{p+1}} (1 + o(1)) \right) \quad \text{as } t \rightarrow -\infty. \end{aligned} \quad (2.27)$$

From Definition 2.1 and the lines which follow, the unstable trajectory  $\mathbf{T}_{unst}$  coincides with the regular trajectory  $\mathbf{T}_{reg}$ . This is included in the region  $\mathbf{C}$  for  $t < T_0$  for some  $-\infty < T_0 \leq \infty$ . There exists also a unique *stable trajectory*  $\mathbf{T}_{st}$  such that

$$\lim_{t \rightarrow \infty} (x(t), y(t)) = (0, 0) \quad \text{and} \quad \lim_{t \rightarrow -\infty} \frac{y(t)}{x(t)} = \frac{2}{p-1} + K = N - 2. \quad (2.28)$$

Since  $N - 2 > \frac{2}{p-1}$ ,  $\mathbf{T}_{st}$  belongs to the region  $\mathbf{A}$  for  $t > T_1$  for some  $T_1 < \infty$ .

*Remark.* If  $\mathbf{T}_{reg} \subset \mathbf{Q}$  satisfies  $\lim_{t \rightarrow \infty} (x(t), y(t)) = (0, 0)$ , then  $\mathbf{T}_{reg} = \mathbf{T}_{st}$  and the corresponding solution is a ground state. The same conclusion holds, if  $\mathbf{T}_{st} \subset \mathbf{Q}$  satisfies  $\lim_{t \rightarrow -\infty} (x(t), y(t)) = (0, 0)$ . Such a solution is called a *homoclinic orbit at  $(0, 0)$* . Because of the uniqueness of the stable and unstable trajectories of a saddle point, it is unique in the class of solutions satisfying (2.28). Equivalently the class of ground states  $u$  of (1.13) satisfying  $u(r) \sim cr^{2-N}$  for some  $c > 0$  is then a one parameter family characterized by  $u(0) = u_0$ .

(b) Assume that  $N \geq 3$  and  $1 < p < \frac{N}{N-2}$ . Then  $K < 0$  and  $0 < \lambda_1 < \lambda_2$ . Hence  $(0, 0)$  is a source and all the trajectories of (1.17) in some neighbourhood of  $(0, 0)$  converge to  $(0, 0)$  when  $t \rightarrow -\infty$ . Among those trajectories there exists *one fast trajectory* which satisfies

$$\lim_{t \rightarrow -\infty} \frac{y(t)}{x(t)} = 0.$$

It is actually the regular trajectory  $\mathbf{T}_{reg}$ . There exist also infinitely many *slow trajectories* which satisfy

$$\lim_{t \rightarrow -\infty} \frac{y(t)}{x(t)} = N - 2.$$

(c) If  $p = \frac{N}{N-2}$ , then  $K = 0$  and  $\lambda_1 = 0 < \lambda_2 = N - 2$ . We still find the regular trajectory  $\mathbf{T}_{reg}$  associated to  $\lambda_2$  and the corresponding eigenvector  $(1, 0)$ . By the central manifold theorem corresponding to  $\lambda_1$  there exists an invariant curve passing through  $(0, 0)$  with slope  $N - 2$ . Using the matched asymptotic expansion method, one finds that if  $M < 0$  there exists a solution  $x$  of (2.1) such that  $x(t) \sim C_N |M|^{-\frac{1}{N-1}} t^{1-N}$  when  $t \rightarrow \infty$ , i.e.  $u(r) \sim C_N |M|^{-\frac{1}{N-1}} r^{2-N} (\ln r)^{1-N}$  when  $r \rightarrow \infty$ , and if  $M > 0$  there exists a solution  $x$  of (2.1) such that  $x(t) \sim C_N M^{-\frac{1}{N-1}} (-t)^{1-N}$  when  $t \rightarrow -\infty$ , equivalently  $u(r) \sim C_N |M|^{-\frac{1}{N-1}} r^{2-N} (\ln(\frac{1}{r}))^{1-N}$  when  $r \rightarrow 0$ .

(d) If  $N = 2$ ,  $(0, 0)$  is a source with  $\lambda_1 = \lambda_2 = \frac{2}{p-1}$ , with corresponding eigenspace  $(1, 0)$ . The linearized problem is equivalent to equation

$$x_{tt} - \frac{4}{q-1} x_t + \frac{4}{(q-1)^2} x = 0$$

with general solutions  $x(t) = ae^{\frac{2t}{q-1}} + be^{-\frac{2t}{q-1}}$  for some real parameters  $a, b$ . Hence there exists infinitely trajectories of (2.1) tending to 0 when  $t \rightarrow -\infty$  and they are tangent to  $(1, 0)$  at  $(0, 0)$ . The regular trajectory  $\mathbf{T}_{reg}$  corresponds to  $b = 0$  and the other trajectories correspond to singular solutions  $u$  of (1.1). They satisfy  $u(r) \sim b \ln r$  as  $r \rightarrow 0$  and there holds

$$-\Delta u = u^p + M |\nabla u|^{\frac{2p}{p+1}} - 2\pi b \delta_0,$$

in the sense of distributions in  $B_\epsilon$  for some  $\epsilon > 0$ .

Next we give a general result in case the system admits only one equilibrium in  $\mathbf{Q}$ .

**Lemma 2.5** *Let  $N \geq 3$ ,  $p > \frac{N}{N-2}$  and  $M \in \mathbb{R}$ . If  $u$  is a regular solution the following trichotomy occurs:*

- (i) either  $\lim_{r \rightarrow \infty} r^{N-2} u(r) = c$  for some  $c > 0$ ,
- (ii) or  $u(r) \sim U_M(r)$  as  $r \rightarrow \infty$ ,
- (iii) or  $u(r)$  has an  $\omega$ -limit cycle surrounding  $P_M$ ,
- (iv) or  $u(r)$  changes sign for some  $r > 0$ .

*Proof.* By assumption  $P_M$  is the unique equilibrium. The trajectory  $\mathbf{T}_{reg}$  starts from  $(0, 0)$  and remains in the region  $\mathbf{C}$  where  $x_t, y_t > 0$  for  $t \leq t_0 \leq \infty$ . If  $t_0 = \infty$ ,  $u$  is a ground state, hence it is bounded from Proposition 2.4. Its  $\omega$ -limit set is non-empty. Because  $x$  and  $y$  are monotone, it converges when  $t \rightarrow \infty$  to some point which is necessarily  $P_M$ . If  $t_0 < \infty$ , then at  $t = t_0$  the trajectory leaves  $\mathbf{C}$  through  $\mathcal{L}$  since it cannot enter in  $\mathbf{B}$ , and it enters the region  $\mathbf{D}$  where  $x_t < 0$ ,  $y_t > 0$ . Moreover  $x_t(t_0) = 0$  and  $x(t_0) > X_M$ . Then three possibilities occur:

- ( $\alpha$ ) either  $x(t) \rightarrow X_M$  monotonically when  $t \rightarrow \infty$ ; thus the trajectory converges to  $P_M$ .
- ( $\beta$ ) either  $x(t) \rightarrow 0$  monotonically when  $t \rightarrow \infty$ . Since  $(0, 0)$  is a saddle point, then  $\mathbf{T}_{reg} = \mathbf{T}_{st}$ . This implies that  $\mathbf{T}_{reg}$  is a homoclinic trajectory at  $(0, 0)$ .

( $\gamma$ ) or there exists  $t_1 > t_0$  such that  $x_t(t_1) = 0$ . Then  $x(t_1) < X_M$ . Hence  $\mathbf{T}_{reg}$  enters the region  $\mathbf{B}$  and by continuity there exists  $t' < t_0$  such that  $x(t') = x(t_1)$  and  $y(t') < y(t_1)$ . Therefore the bounded region of  $\mathbb{R}^2$  bordered by the segment  $I = \{(x, y) : x = x(t_1), y(t') < y < y(t_1)\}$  and the portion of  $\mathbf{T}_{reg}$  defined by  $\{(x(t), y(t)) \in \mathbf{T}_{reg} : t' \leq t \leq t_1\}$  is positively invariant (notice that  $x_t > 0$  on  $I$ ) and it contains  $P_M$  and no other equilibrium. Therefore either the trajectory converges to  $P_M$  or it admits an  $\omega$ -limit cycle which is a closed orbit surrounding  $P_M$ .  $\square$

#### 2.4.2 Linearization at a fixed point $P_M := (X_M, Y_M)$

Suppose that  $P_M$  (or  $P_{j,M}$ ) exists. Then setting  $(x, y) = (X_M, Y_M) + (\bar{x}, \bar{y})$ , the linearized system at this point is

$$\begin{aligned}\bar{x}_t &= \frac{2\bar{x}}{p-1} - \bar{y} \\ \bar{y}_t &= pX_M^{p-1}\bar{x} + \left(\frac{2p}{p+1}MY_M^{\frac{p-1}{p+1}} - K\right)\bar{y},\end{aligned}\tag{2.29}$$

Using equation (2.15), the eigenvalues of its matrix are the roots of the trinomial

$$T(\lambda) = \lambda^2 - \left(\frac{2p}{p+1}MY_M^{\frac{p-1}{p+1}} - L\right)\lambda + 2K - \frac{2p}{p+1}MY_M^{\frac{p-1}{p+1}}.\tag{2.30}$$

If  $M$  is such that  $\frac{2p}{p+1}MY_M^{\frac{p-1}{p+1}} = L$ , then  $2K - \frac{2p}{p+1}MY_M^{\frac{p-1}{p+1}} = 2K - L = K + \frac{2}{p-1} = N - 2$ , and we denote by  $\bar{M}$  such a value of  $M$  which is characterized by

$$\frac{2p}{p+1}\bar{M}Y_M^{\frac{p-1}{p+1}} = L.\tag{2.31}$$

Since  $Y_M$  is a positive root of (2.15), we get

$$\left(\frac{2}{p-1}\right)^p Y_M^{p-1} = A := K - \frac{p+1}{2p}L = \frac{(N-2)(p-1)^2 + 4}{2p(p-1)},$$

hence  $\bar{M}$ , well defined for  $N \geq 2$ , is given by

$$\bar{M} = \frac{(p+1)L}{2p} \left(\frac{1}{A} \left(\frac{p-1}{2}\right)^p\right)^{\frac{1}{p+1}} = \frac{(p+1)((N-2)p - N - 2)}{(4p)^{\frac{p}{p+1}} ((N-2)(p-1)^2 + 4)^{\frac{1}{p+1}}}.\tag{2.32}$$

That is (1.21).

*Remark.* We see that  $\bar{M} > 0$  (resp.  $\bar{M} < 0$ ) if and only if  $N \geq 3$  and  $p > \frac{N+2}{N-2}$  (resp.  $1 < p < \frac{N+2}{N-2}$  if  $N \geq 3$  and any  $p > 1$  if  $N = 2$ ). If  $M = \bar{M}$  and  $N \geq 3$ , the eigenvalues of the linearized system are purely imaginary. If  $p = \frac{N+2}{N-2}$ , then  $\bar{M} = 0$ . It is known that in that case the point  $P_0 = \left(\left(\frac{N-2}{2}\right)^{\frac{N-2}{2}}, \left(\frac{N-2}{2}\right)^{\frac{N}{2}}\right)$  is a center for the system (1.17) associated to  $-\Delta u = u^{\frac{N+2}{N-2}}$  and that there exist infinitely many cycles turning around  $P_0$  with equation

$$\frac{1}{2}y^2 + \frac{N-2}{2N}x^{\frac{2N}{N-2}} - \frac{N-2}{2}xy = E.\tag{2.33}$$

It can be verified that  $E < 0$ , in particular using the function  $F$  defined in (2.36). For  $N \geq 2$  and  $1 < p < \frac{N}{N-2}$ , there always holds  $\bar{M} \leq -\mu^*$  and more precisely

**Lemma 2.6** *If  $N \geq 3$  and  $1 < p < \frac{N}{N-2}$ , then  $\bar{M} < -\mu^*$ . If  $N = 2$  and  $p > 1$ , then  $\bar{M} = -\mu^*$ .*

*Proof.* From Proposition 2.2 and identities (2.17) and (2.30), there holds,

$$\mu^* < \bar{\mu} \iff \mu^* Y_{-\mu^*}^{\frac{p-1}{p+1}} < \bar{\mu} Y_{\bar{M}}^{\frac{p-1}{p+1}} \iff 2|K| < |L|.$$

But  $|L| - 2|K| = \frac{N-2}{p-1}$  and the conclusion follows. If  $N = 2$  we just replace  $<$  by  $=$  in the above series of equivalences.  $\square$

**Lemma 2.7** *Assume  $N \geq 3$ . If  $p > \frac{N+2}{N-2}$  and  $M = \bar{M} > 0$  then  $P_M$  is a weak sink and a Andronov-Hopf bifurcation point. If  $p < \frac{N+2}{N-2}$  and  $M = \bar{M} < 0$  then  $P_M$  is a weak source.*

*Proof.* We recall that a weak sink is an asymptotically stable equilibrium which attracts the nearby points as  $t \rightarrow \infty$  at a rate slower than the usual exponential rate. A weak source is a weak sink of the system obtained by changing  $t$  into  $-t$  (see [17, Chap. 9]). We write  $\bar{x} = x - X_M$ ,  $\bar{y} = y - Y_M$  and obtain the new nonlinear system

$$\begin{aligned} \bar{x}_t &= \frac{2\bar{x}}{p-1} - \bar{y} \\ \bar{y}_t &= pX_M^{p-1}\bar{x} + \frac{2p}{p+1}\bar{M}Y_M^{\frac{p-1}{p+1}}\bar{y} - K\bar{y} + h(\bar{x}, \bar{y}), \end{aligned} \tag{2.34}$$

where

$$h(\bar{x}, \bar{y}) = c_1\bar{x}^2 + c_2\bar{y}^2 + c_3\bar{x}^3 + c_4\bar{y}^3 + \dots,$$

with

$$\begin{aligned} c_1 &= \frac{p(p-1)}{2}X_M^{p-2}, \quad c_2 = \frac{p(p-1)}{(p+1)^2}\bar{M}Y_M^{-\frac{2}{p+1}} \\ c_3 &= \frac{p(p-1)(p-2)}{6}X_M^{p-3}, \quad c_4 = -\frac{2p(p-1)}{3(p+1)^3}\bar{M}Y_M^{-\frac{p+3}{p+1}}. \end{aligned}$$

Setting  $\alpha^2 = \frac{4}{(p-1)^2} + N - 2$ , we have  $pX_M^{p-1} = \alpha^2$ , since  $X_M = \frac{p-1}{2}Y_M$  and  $\bar{M}$  satisfies (2.31), thus

$$c_1 = \frac{\alpha^2}{Y_M}, \quad c_2 = \frac{(p-1)L}{2(p+1)Y_M}, \quad c_3 = -\frac{(p-1)L}{3(p+1)^2Y_M^2}. \tag{2.35}$$

In order to compute the Lyapunov coefficients we transform the system by setting

$$\gamma = \sqrt{N-2}, \quad s = \gamma t, \quad \gamma w = \frac{2\bar{x}}{p-1} - \bar{y}.$$

The new system is

$$\begin{aligned} \bar{x}_s &= w \\ w_s &= -\bar{x} - \frac{1}{\gamma^2}h\left(\bar{x}, \frac{2\bar{x}}{p-1} - \gamma w\right). \end{aligned}$$

By integrating the first line and using the expansion of  $h$ , we obtain

$$w_s = -\bar{x} - \frac{1}{\gamma^2} \left[ c_1 \bar{x}^2 + c_2 \left( \frac{2\bar{x}}{p-1} - \gamma w \right)^2 + c_3 \bar{x}^3 + c_4 \left( \frac{2\bar{x}}{p-1} - \gamma w \right)^3 + \dots \right]$$

This can be written in the following way

$$w_s = -\bar{x} + \nu_{2,0} \bar{x}^2 + \nu_{1,1} \bar{x} w + \nu_{0,2} w^2 + \nu_{3,0} \bar{x}^3 + \nu_{2,1} \bar{x}^2 w + \nu_{1,2} \bar{x} w^2 + \nu_{0,3} w^3 + \dots$$

By [17, Th. 9.2.3], the Lyapunov coefficient is given by  $\Lambda = \nu_{2,0} + 3\nu_{0,3} + \nu_{1,1}(\nu_{2,0} + \nu_{0,2})$  which yields by computation

$$\gamma \Lambda = \alpha^2 \left( 3c_4 - \frac{4c_2}{p-1} (c_2 \alpha + c_1) \right) = -\alpha^2 \frac{(p-1)(N+1)}{(p+1)^2 Y_M^2} L.$$

If  $L > 0$  (resp.  $L < 0$ )  $P_M$  is a weak sink (resp. a weak source).  $\square$

### 2.4.3 Energy and Lyapunov functionals for system (1.17)

If  $x(t)$  is a solution of (2.1) we set

$$F(t) := \frac{x_t^2}{2} + \frac{|x|^{p+1}}{p+1} - \frac{Kx^2}{p-1}, \quad (2.36)$$

Then, if  $(x(t), y(t)) \in \mathbf{Q}$ , we have

$$\begin{aligned} F_t(t) &= -Lx_t^2 - M \left( \frac{2x}{p-1} - x_t \right)^{\frac{2p}{p+1}} x_t = -(Lx_t + My^q) x_t \\ &= - \left( L \left( \frac{2x}{p-1} - y \right) + My^q \right) \left( \frac{2x}{p-1} - y \right). \end{aligned}$$

Hence, if  $LM > 0$ ,  $F$  is monotone in the region  $\left\{ (x, y) \in \mathbf{Q} : \frac{2}{p-1}x - y > 0 \right\}$ , located under  $\mathcal{L}$ .

*Remark.* a) This function was introduced classically in the case  $M = 0$ , leading in particular to (2.33) when  $p = \frac{N+2}{N-2}$ .

b) Using this function we can deduce an upper estimate for regular solutions, completing, Proposition 2.3 and Proposition 2.4, namely if  $M > 0$  and  $p \geq \frac{N+2}{N-2}$  any ground state satisfies

$$u(r) \leq \left( \frac{p+1}{p-1} K \right)^{\frac{1}{p-1}} r^{-\frac{2}{p-1}} \quad \text{for all } r > 0. \quad (2.37)$$

Next we construct a Lyapunov functional adapting the method initiated by [2] and already used in [3] and [4].

**Lemma 2.8** *We define on  $\mathbb{R}^2$*

$$\mathcal{J}(x, y) = \frac{Kx^2}{p-1} - \frac{|x|^{p+1}}{p+1} - M \left( \frac{2}{p-1} \right)^{\frac{2p}{p+1}} \frac{(p+1)|x|^{\frac{3p+1}{p+1}}}{3p+1} - \frac{1}{2} \left( \frac{2x}{p-1} - y \right)^2, \quad (2.38)$$

and, if  $(x(t), y(t))$  is a solution of (2.39),  $\mathcal{V}(t) = \mathcal{J}(x(t), y(t))$ . If  $(x(t), y(t)) = (x, y) \in \mathbf{Q}$  we have

$$\begin{aligned} \mathcal{V}_t(t) &= \left( \frac{2x}{p-1} - y \right) \left[ L \left( \frac{2x}{p-1} - y \right) - M \left( \left( \frac{2x}{p-1} \right)^{\frac{2p}{p+1}} - y^{\frac{2p}{p+1}} \right) \right] \\ &= L \left( \frac{2x}{p-1} - y \right)^2 - M \left( \left( \frac{2x}{p-1} \right)^{\frac{2p}{p+1}} - y^{\frac{2p}{p+1}} \right) \left( \frac{2x}{p-1} - y \right). \end{aligned} \quad (2.39)$$

Consequently, the function  $t \mapsto \mathcal{V}(t)$  is decreasing if  $M > 0$  and  $1 < p \leq \frac{N+2}{N-2}$ , and increasing if  $M < 0$  and  $p \geq \frac{N+2}{N-2}$ .

*Proof.* We recall the ansatz introduced in [2] for finding a Lyapunov function for a system of the form

$$\begin{aligned} x_t &= f(x, y) \\ y_t &= g(x, y). \end{aligned} \quad (2.40)$$

If  $f(x, y) = 0 \iff y = h(x)$ , then consider the function

$$\mathbf{L}(x, y) = \int_{h(x)}^y f(x, t) dt - \int_0^x g(t, h(t)) dt. \quad (2.41)$$

In the case of system (1.17),  $h(x) = \frac{2x}{p-1}$ , and we find  $\mathbf{L}(x, y) = \mathcal{J}(x, y)$ . Then (2.39) and the conclusion follow.  $\square$

*Remark.* If  $LM > 0$  we set

$$\mathcal{H} = \left\{ (x, y) \in \mathbf{Q} : \frac{\left( \frac{2x}{p-1} \right)^{\frac{2p}{p+1}} - y^{\frac{2p}{p+1}}}{\frac{2x}{p-1} - y} = \frac{L}{M}, \text{ or } \frac{2x}{p-1} = y = \left( \frac{L}{Mq} \right)^{\frac{1}{q-1}} \right\}.$$

Then

$$\mathcal{V}_t(t) = 0 \iff (x(t), y(t)) \in \mathcal{H} \cup \left\{ (x, y) \in \mathbf{Q} : \frac{2x}{p-1} = y \right\}.$$

Moreover  $\mathcal{H}$  is starshaped with respect to 0 and we set

$$\mathcal{R} := \bigcup_{0 \leq \theta \leq 1} \theta \mathcal{H} = \left\{ (x, y) \in \mathbf{Q} : \frac{\left( \frac{2x}{p-1} \right)^{\frac{2p}{p+1}} - y^{\frac{2p}{p+1}}}{\frac{2x}{p-1} - y} \leq \frac{L}{M}, \text{ or } \frac{2x}{p-1} = y \leq \left( \frac{L}{Mq} \right)^{\frac{1}{q-1}} \right\}.$$

If  $M > 0$  we have  $\mathcal{V}_t(t) \geq 0$  if  $(x(t), y(t)) \in \mathcal{R}$  and  $\mathcal{V}_t(t) \leq 0$  if  $(x(t), y(t)) \in \mathcal{R}^c \cap \mathbf{Q}$ . If  $M < 0$ , the signs of  $\mathcal{V}_t(t)$  in the same regions are opposite.

In [7] we also used a function introduced in [19] for equation (1.2). When  $q = \frac{2p}{p+1}$  it reduces to

$$\mathcal{Z}(r) = r^a \left( \frac{p+1}{2} u_r^2 + u^{p+1} + a \frac{u u_r}{r} + M u |u_r|^{\frac{2p}{p+1}} \right), \quad (2.42)$$



where  $a = \frac{2(p+1)(N-1)}{p+3}$ . Since  $r = e^t$ , we find that in  $\mathbf{Q}$ ,

$$\mathcal{Z}(r) = e^{\frac{2(p+1)L}{p+3}t} \left( \frac{p+1}{2}y^2 + x^{p+1} - axy + Mxy^{\frac{2p}{p+1}} \right). \quad (2.43)$$

The function satisfies the relation

$$\mathcal{Z}_r - \frac{2p}{p+1}Mu_r^{\frac{p-1}{p+1}}\mathcal{Z} = r^{a-1}\mathcal{U}, \quad (2.44)$$

where

$$\begin{aligned} \mathcal{U} &= \frac{2(N-1)(p^2-1)}{(p+3)^2} \frac{uu_r}{r} \left( -L + \frac{p(p+3)}{(p+1)^2}Mr u_r^{\frac{p-1}{p+1}} \right) \\ &= \frac{2(N-1)(p^2-1)}{(p+3)^2} xy \left( -L + \frac{p(p+3)}{(p+1)^2}My^{\frac{p-1}{p+1}} \right) r^{-\frac{p+3}{p-1}} \end{aligned} \quad (2.45)$$

Note that  $\mathcal{U}$  has a constant sign in  $\mathbf{Q}$  if  $LM < 0$ .

#### 2.4.4 Comparison results

**Lemma 2.9** *Let  $N \geq 1$ ,  $p > 1$  and  $M, M' \in \mathbb{R}$  such that  $M < M'$ . Then, as long as they lie below the line  $\mathcal{L}$ , i.e.  $x_t > 0$ , the regular trajectories  $\mathbf{T}_{reg}^M$  and  $\mathbf{T}_{reg}^{M'}$  associated to  $M$  and  $M'$  respectively do not intersect. Furthermore  $\mathbf{T}_{reg}^M$  is below  $\mathbf{T}_{reg}^{M'}$ .*

*Proof.* We use the expansion (2.27) and deduce, for  $M_i = M$  or  $M'$ , that

$$\frac{y^{M_i}(t)}{x^{M_i}(t)} = e^{2t} \left( \frac{1}{N} + M_i \frac{e^{\frac{p-1}{p+1}t}}{N(N + \frac{2p}{p+1})} (1 + o(1)) \right) \quad (2.46)$$

as  $t \rightarrow -\infty$ . Hence  $\mathbf{T}_{reg}^M$  is below  $\mathbf{T}_{reg}^{M'}$  for  $t \leq t^*$  for some  $t^* \in \mathbb{R}$ . Suppose that the trajectories intersect for a first time at some point  $(x_0, y_0)$  below  $\mathcal{L}$ . Since the system is autonomous, there will exist two solutions of the systems relative to  $M$  and  $M'$  satisfying at the same time  $t_0$ ,  $x^M(t_0) = x^{M'}(t_0) = x_0$  and  $y^M(t_0) = y^{M'}(t_0) = y_0$ . From (2.2),  $x_t^M(t_0) = x_t^{M'}(t_0) > 0$  and  $y_t^M(t_0) = y_t^{M'}(t_0) + (M - M')y_0^{\frac{2p}{p+1}} < y_t^{M'}(t_0)$ . Hence the intersection of the two trajectories is transverse and the slope of  $\mathbf{T}_{reg}^M$  at this intersection point is smaller than the one of  $\mathbf{T}_{reg}^{M'}$  which is impossible. The second assertion follows immediately.  $\square$

### 3 Study of ground states of (1.17) when $M > 0$

When  $M \geq 0$  and either  $N \geq 3$  and  $1 < p \leq \frac{N}{N-2}$ , or  $N = 1, 2$  and  $p > 1$ , there exists no ground state by Proposition 2.3; in this section we assume  $N \geq 3$  and  $p > \frac{N}{N-2}$ .

### 3.1 Behaviour near the equilibrium

Since  $M \geq 0$  and  $p > \frac{N}{N-2}$  there exists a unique equilibrium  $P_M \in \mathbf{Q}$ . The sign of the real part of the characteristic roots of the linearization of the system (2.2) at  $P_M$  depends on the position of  $M$  with respect to  $\overline{M}$  defined in (2.32).

**Lemma 3.1** *Assume  $M > 0$  and  $N \geq 3$ .*

1) *If  $\frac{N}{N-2} < p \leq \frac{N+2}{N-2}$ , then  $P_M$  is a source.*

2) *If  $p > \frac{N+2}{N-2}$ ; then  $P_M$  is a sink when  $M < \overline{M}$  and it is a source when  $M > \overline{M}$ .*

3) *If  $p > \frac{N+2}{N-2}$ , then  $P_{\overline{M}}$  a weak sink. Furthermore, if  $0 < M - \overline{M} < \epsilon$ , for  $\epsilon$  small enough, there exists a periodic trajectory in  $\mathbf{Q}$  surrounding  $P_M$ .*

*Proof. Step 1:* Assume  $\frac{N}{N-2} < p \leq \frac{N+2}{N-2}$ . The linearized system is given in (2.29). Because  $M > 0$ , the product of the characteristic roots given by equation (2.30) is positive since it is given by

$$2K - \frac{2p}{p+1}MY_M^{\frac{p-1}{p+1}} = \frac{2}{p+1}K + \frac{2p}{p+1}\left(\frac{p-1}{2}\right)^p Y_M^{p-1}.$$

The sum (or the real part) of the characteristic roots is equal to  $\frac{2p}{p+1}MY_M^{\frac{p-1}{p+1}} - L$  which is positive, as  $p \leq \frac{N+2}{N-2}$  implies  $L < 0$ . Hence  $P_M$  is a source.

*Step 2:* Assume  $p > \frac{N+2}{N-2}$ , hence  $L > 0$ . As in Step 1, the product of the characteristic roots is positive. By Proposition 2.2-(1),  $MY_M^{\frac{p-1}{p+1}}$  is an increasing function of  $M$ , then the sum (or the real part of) of the characteristic roots, given by  $\frac{2p}{p+1}MY_M^{\frac{p-1}{p+1}} - L$  is also increasing and vanishes if  $M = \overline{M}$ . It is negative if  $0 < M < \overline{M}$  and positive if  $M > \overline{M}$ . It implies assertion 2.

*Step 3:* If  $M = \overline{M}$ , and  $p > \frac{N+2}{N-2}$ , then  $P_{\overline{M}}$  is a weak sink by Lemma 2.7. The appearance of the limit cycle, which is called the Andronov-Hopf bifurcation, occurs for  $M > \overline{M}$  when  $M - \overline{M}$  is small enough (see [17, Chap. 9]). This implies assertion 2.  $\square$

*Remark.* The product of the characteristic roots is also expressed by

$$2K - \frac{2p}{p+1}MY_M^{\frac{p-1}{p+1}} = \frac{2M}{p+1}Y_M^{\frac{p-1}{p+1}} + 2\left(\frac{p-1}{2}\right)^2 Y_M^{p-1}.$$

Hence it is positive for any  $M \geq 0$ ,  $p > 1$ .

Next we give some sufficient conditions for nonexistence of a periodic solution or a homoclinic orbit at  $(0, 0)$ .

**Lemma 3.2** *Assume  $N \geq 3$ . If  $M > 0$  and  $1 < p \leq \frac{N+2}{N-2}$  the system (1.17) admits no closed orbit in  $\overline{\mathbf{Q}}$ . If  $0 < M \leq \overline{M}$  and  $p > \frac{N+2}{N-2}$ , the system (1.17) admits no cycle in  $\mathbf{Q}$  surrounding  $P_M$ .*

*Proof.* If  $\gamma$  is a non-trivial closed orbit it corresponds either to a  $T$ -periodic solution or a solution such that

$$\lim_{t \rightarrow -\infty} (x(t), y(t)) = \lim_{t \rightarrow \infty} (x(t), y(t)).$$

The function  $\mathcal{V}$  defined in (2.38) is monotone and it satisfies either  $\mathcal{V}(0) = \mathcal{V}(T)$  or  $\lim_{t \rightarrow -\infty} \mathcal{V}(t) = \lim_{t \rightarrow \infty} \mathcal{V}(t)$ . Hence it is constant and by (2.39) it implies  $y(t) = \frac{2}{p-1}x(t)$  for all  $t$ , a contradiction.

Next we suppose  $p > \frac{N+2}{N-2}$ ,  $0 < M \leq \bar{M}$  and that there exists a  $T$ -periodic solution  $(x(t), y(t))$  with trajectory  $\gamma \subset \mathbf{Q}$  surrounding  $P_M$ , hence  $P_M$  belongs to the bounded connected component  $\Gamma$  of  $\mathbb{R}^2 \setminus \{\gamma\}$  bordered by  $\gamma$ . Since  $P_M$  is a sink or a weak sink by Lemma 3.1, there exists a neighbourhood  $\mathcal{O}$  of  $P_M$  such that all the trajectories issued from  $\mathcal{O}$  converge to  $P_M$  as  $t \rightarrow \infty$ . Hence any trajectory issued from  $\mathcal{O}$ , necessarily contained in  $\Gamma$ , has an  $\alpha$ -limit set in  $\bar{\Gamma}$  which is either a stationary point different from  $P_M$ , which is excluded, or a limit cycle  $\{(x(t), y(t))\}_{t \in [0, \tau)} := \gamma' \subset \bar{\Gamma}$  ( $\tau$  is its period). This limit cycle is not stable, hence, by Floquet's theory

$$\int_0^\tau \left( \frac{\partial H_1(x(t), y(t))}{\partial x} + \frac{\partial H_2(x(t), y(t))}{\partial y} \right) dt = \int_0^\tau \left( \frac{2pM}{p+1} y^{\frac{p-1}{p+1}}(t) - L \right) dt \geq 0 \quad (3.1)$$

We perform the change of unknowns  $\bar{x} = x - X_M$ ,  $\bar{y} = y - Y_M$  used in Lemma 2.7 which leads to the system (2.34). The explicit value of the remaining term is

$$\begin{aligned} h(\bar{x}, \bar{y}) &= (\bar{x} + X_M)^p - X_M^p - pX_M^{p-1}\bar{x} + M(\bar{y} + Y_M)^{\frac{2p}{p+1}} - MY_M^{\frac{2p}{p+1}} - \frac{2p}{p+1}MY_M^{\frac{p-1}{p+1}}\bar{y} \\ &= \Phi(\bar{x}) + M\Psi(\bar{y}), \end{aligned} \quad (3.2)$$

where  $\Phi$  and  $\Psi$  are defined accordingly. It is positive by convexity because  $M > 0$ . Since from (2.34),

$$\frac{2}{p-1} \int_0^\tau \bar{x}(t) dt - \int_0^\tau \bar{y}(t) dt = 0,$$

and

$$0 = pX_M^{p-1} \int_0^\tau \bar{x}(t) dt + \frac{2p}{p+1}MY_M^{\frac{p-1}{p+1}} \int_0^\tau \bar{y}(t) dt - K \int_0^\tau \bar{y}(t) dt + \int_0^\tau h(\bar{x}, \bar{y}) dt$$

we derive

$$\left( \frac{p(p-1)^p}{2^p} Y_M^{p-1} + \frac{2p}{p+1} M Y_M^{\frac{p-1}{p+1}} - K \right) \int_0^\tau \bar{y}(t) dt < 0.$$

Using the equation (2.15) satisfied by  $Y_M$  it yields

$$(p-1) \left( K - \frac{M}{p+1} Y_M^{\frac{p-1}{p+1}} \right) \int_0^\tau \bar{y}(t) dt < 0.$$

By (2.11)-(ii),  $K - \frac{M}{p+1} Y_M^{\frac{p-1}{p+1}} > 0$ , hence  $\int_0^\tau \bar{y}(t) dt < 0$ , therefore  $\tau^{-1} \int_0^\tau y(t) dt < Y_M$  and by concavity,

$$\frac{1}{\tau} \int_0^\tau (y(t))^{\frac{p-1}{p+1}} dt \leq \left( \frac{1}{\tau} \int_0^\tau y(t) dt \right)^{\frac{p-1}{p+1}} < Y_M^{\frac{p-1}{p+1}}. \quad (3.3)$$

Combining (3.1) and (3.3), we obtain

$$L < \frac{2pM}{p+1} Y_M^{\frac{p-1}{p+1}}. \quad (3.4)$$

Since  $M \mapsto MY_M^{\frac{p-1}{p+1}}$  is decreasing by Proposition 2.2-(1-ii) we have for  $0 \leq M \leq \bar{M}$

$$0 = \frac{2p\bar{M}}{p+1} Y_{\bar{M}}^{\frac{p-1}{p+1}} - L \geq \frac{2pM}{p+1} Y_M^{\frac{p-1}{p+1}} - L,$$

which contradicts (3.4).  $\square$

*Remark.* Up to changing the sense of variation of  $\mathcal{V}(t)$ , the proof of the first assertion shows that there exists no closed orbit in  $\bar{\mathbf{Q}}$  if  $M < 0$  and  $p \geq \frac{N+2}{N-2}$ . However the proof of the second assertion is not valid when  $M < 0$ .

The nonexistence of any periodic solution can also be proved when the equilibrium is a node (i.e. the two characteristic values are real with the same sign).

**Lemma 3.3** 1- Let  $N \geq 3$  and  $p > \frac{N+2}{N-2}$ . There exists a unique and explicit  $M_0 > \bar{M}$  such that for any  $M \geq M_0$ ,  $P_M$  is a repelling node, degenerate if  $M = M_0$ . If  $\bar{M} < M < M_0$ ,  $P_M$  is a repelling spiraling point. If  $3 \leq N \leq 10$  and  $0 < M < \bar{M}$ ,  $P_M$  is an attracting spiraling point.  
 2- If  $N \geq 11$  and  $\frac{N+2}{N-2} < p < \frac{N-2\sqrt{N-1}}{N-4-2\sqrt{N-1}}$  there exists a unique and explicit  $0 < M_1 < \bar{M}$  such that if  $P_M$  is an attracting node if  $0 < M < M_1$  and an attracting spiraling point if  $M_1 < M < \bar{M}$ .  
 3- If  $M > M_0$  or if  $0 < M < M_1$ , there exist no periodic trajectory in  $\mathbf{Q}$  around  $P_M$ , neither no homoclinic trajectory in  $\mathbf{Q}$  at  $(0, 0)$  surrounding  $P_M$ .

*Proof.* The characteristic values of the trinomial  $T(\lambda)$  defined in (2.30) are real if and only if its discriminant  $D$  is nonnegative. By computation we find

$$\begin{aligned} D &= \left( \frac{2p}{p+1} MY_M^{\frac{p-1}{p+1}} - L \right)^2 - 8 \left( K - \frac{p}{p+1} MY_M^{\frac{p-1}{p+1}} \right) \\ &= \left( \frac{2p}{p+1} MY_M^{\frac{p-1}{p+1}} \right)^2 + (2-L) \frac{4p}{p+1} MY_M^{\frac{p-1}{p+1}} + L^2 - 8K \\ &= \left( \frac{2p}{p+1} MY_M^{\frac{p-1}{p+1}} - L + 2 \right)^2 - 4(1-L+2K). \end{aligned} \quad (3.5)$$

Observing that  $1-L+2K = N-1$  by (1.16), we deduce

$$D = \left( \frac{2p}{p+1} MY_M^{\frac{p-1}{p+1}} - L + 2 + 2\sqrt{N-1} \right) \left( \frac{2p}{p+1} MY_M^{\frac{p-1}{p+1}} - L + 2 - 2\sqrt{N-1} \right).$$

Hence (3.5) is satisfied if one of the following conditions holds:

$$\begin{aligned} (i) \quad & MY_M^{\frac{p-1}{p+1}} \leq \frac{p+1}{2p} (L - 2 - 2\sqrt{N-1}), \\ (ii) \quad & MY_M^{\frac{p-1}{p+1}} \geq \frac{p+1}{2p} (L - 2 + 2\sqrt{N-1}). \end{aligned} \quad (3.6)$$

It is easy to check that for  $p > \frac{N+2}{N-2}$ , one has  $0 < L - 2 + 2\sqrt{N-1} < qK$ . Since  $M > 0$ , by Proposition 2.2-(1), the mapping  $M \mapsto MY_M^{\frac{p-1}{p+1}}$  is continuous, increasing and from  $[0, \infty)$  into  $[0, \infty)$ . Therefore there exists a unique  $M_0$  such that

$$M_0 Y_{M_0}^{\frac{p-1}{p+1}} = \frac{p+1}{2p} (L - 2 + 2\sqrt{N-1}). \quad (3.7)$$

Using (2.15) we find

$$M_0 = \frac{p+1}{2p} \left( \frac{p-1}{2} \right)^{\frac{p}{p+1}} \frac{K - \frac{2p}{p-1} + 2\sqrt{N-1}}{\left( \frac{p-1}{2p} K + \frac{p+1}{p-1} - \frac{(p+1)\sqrt{N-1}}{p} \right)^{\frac{1}{p+1}}}. \quad (3.8)$$

Concerning the upper bound in (i), there holds

$$L - 2 - 2\sqrt{N-1} > 0 \iff p > \frac{N - 2\sqrt{N-1}}{N - 2\sqrt{N-1} - 4} \quad \text{and} \quad N \geq 11. \quad (3.9)$$

Then we can define  $M_1$  by

$$M_1 Y_{M_1}^{\frac{p-1}{p+1}} = \frac{p+1}{2p} (L - 2 - 2\sqrt{N-1}). \quad (3.10)$$

which leads to

$$M_1 = \frac{p+1}{2p} \left( \frac{p-1}{2} \right)^{\frac{p}{p+1}} \frac{K - \frac{2p}{p-1} - 2\sqrt{N-1}}{\left( \frac{p-1}{2p} K + \frac{p+1}{p-1} + \frac{(p+1)\sqrt{N-1}}{p} \right)^{\frac{1}{p+1}}}. \quad (3.11)$$

Note that

$$\frac{2p}{p+1} M_1 Y_{M_1}^{\frac{p-1}{p+1}} = L - 2 - 2\sqrt{N-1} < L = \frac{2p}{p+1} \overline{M} Y_{\overline{M}}^{\frac{p-1}{p+1}} \leq L - 2 + 2\sqrt{N-1} = \frac{2p}{p+1} M_0 Y_{M_0}^{\frac{p-1}{p+1}}.$$

Hence

$$M_1 < \overline{M} < M_0. \quad (3.12)$$

Next we prove 3) by adapting an argument introduced in [12] for quadratic systems. We return to system (2.34) that we write under the form

$$\begin{aligned} \bar{x}_t &= a\bar{x} - \bar{y} \\ \bar{y}_t &= c\bar{x} + d\bar{y} + \Phi(\bar{x}) + M\Psi(\bar{y}) \end{aligned} \quad (3.13)$$

which defines  $a, c, d$  with  $\Phi$  and  $\Psi$  given by (3.13), and the trinomial (2.30) for characteristic values endows the form

$$T(\lambda) = \lambda^2 - (d+a)\lambda + ad + c.$$

In the range of values of  $M$ , the discriminant  $D = (d+a)^2 - 4(ad+c) = (d-a)^2 - 4c$  is positive. We consider the intersection of a straight line  $\ell$  passing through  $P_M$  with equation  $\bar{y} = A\bar{x}$  with a trajectory  $(\bar{x}(t), \bar{y}(t))$ . Then

$$\begin{aligned} U &= \bar{y}_t - A\bar{x}_t = (A^2 + (d-a)A + c)\bar{x} + \Phi(\bar{x}) + M\Psi(\bar{y}) \\ &= (A^2 + (d-a)A + c)\bar{x} + \Phi(\bar{x}) + M\Psi(A\bar{x}). \end{aligned}$$

We can choose  $A \neq 0$  such that  $A^2 + (d - a)A + c = 0$  since  $D > 0$ . Since  $\Phi$  and  $\Psi$  achieve positive values, we derive from the expression of  $h$  that  $U > 0$  for  $\bar{x} \neq 0$ . This proves that any closed orbit around  $P_M$  or passing by  $P_M$  can intersect  $\ell$  only one time which is a contradiction.  $\square$

### 3.2 Existence or nonexistence of ground states

**Proposition 3.4** *Let  $N \geq 3$ ,  $\frac{N}{N-2} < p \leq \frac{N+2}{N-2}$  and  $M > 0$ . Then there exists no radial ground state, and there exists a singular solutions  $u(r)$  such that  $\lim_{r \rightarrow 0} r^{\frac{2}{p-1}} u(r) = X_M$  and  $\lim_{r \rightarrow \infty} r^{N-2} u(r) = c$  for some  $c > 0$ .*

*Proof.* Assume that  $\mathbf{T}_{reg}$  remains in  $\mathbf{Q}$ , by Lemma 2.5 we have three possibilities:

( $\alpha$ ) either  $\mathbf{T}_{reg}$  converges to  $P_M$  when  $t \rightarrow \infty$ , which is impossible since  $P_M$  is a source,  
 ( $\beta$ ) or  $\mathbf{T}_{reg}$  has a limit cycle at  $\infty$ , and this is impossible by Lemma 3.2,  
 ( $\gamma$ ) or  $\mathbf{T}_{reg}$  converges to  $(0, 0)$  when  $t \rightarrow \infty$ , hence it is a homoclinic orbit. The function  $\mathcal{V}$  defined in (2.38) is decreasing by Lemma 2.8. Since  $\mathcal{V}(-\infty) = \mathcal{V}(\infty) = 0$ , it is identically 0 and so is  $\mathcal{V}_t$ . This implies that  $\frac{2x(t)}{p-1} - y(t) = 0$  for all  $t$  which is a contradiction.  
 Hence  $\mathbf{T}_{reg}$  does not remain in  $\mathbf{Q}$ .

We denote by  $\Xi$  the connected region of  $\mathbf{Q}$  bordered by the semi-axis  $\{(x, y) : x = 0, y > 0\}$  and  $\mathbf{T}_{reg}$ . Since  $\mathbf{H}$  is outward on the semi-axis,  $\Xi$  is negatively invariant. By Section 3.4.1,  $(0, 0)$  is a saddle point, hence the stable trajectory  $\mathbf{T}_{st} = \{(x(t), y(t))\}$  satisfies  $\lim_{t \rightarrow \infty} \frac{y(t)}{x(t)} = N - 2$  which implies  $u(r) \sim cr^{2-N}$  for some  $c > 0$ . Its  $\alpha$ -limit set  $\alpha(\mathbf{T}_{st})$  cannot be a limit cycle as we have seen it above. If it contains  $(0, 0)$  it implies again that  $\mathcal{V}$ , which is monotone, is equal to 0, hence  $\mathcal{V}_t \equiv 0$  and  $\frac{2x(t)}{p-1} - y(t) \equiv 0$ , which is impossible. Hence  $\alpha(\mathbf{T}_{st})$  contains  $P_M$ . Since  $P_M$  is a source it implies that  $\mathbf{T}_{st}$  converges to  $P_M$  when  $t \rightarrow -\infty$ .  $\square$

**Proposition 3.5** *Let  $N \geq 3$  and  $p > \frac{N+2}{N-2}$ . Then for any  $0 < M \leq \bar{M}$  there exists a ground state  $u$  which satisfies  $u \sim U_M$  at  $\infty$ .*

*Proof.* If  $0 < M < \bar{M}$  (resp.  $M = \bar{M}$ ),  $P_M$  is a sink (resp. a weak sink). Suppose first that the trajectory  $\mathbf{T}_{reg}$  does not stay in  $\mathbf{Q}$ , then it leaves  $\mathbf{Q}$  at some point  $(0, y_s)$  with  $y_s > 0$ . As a consequence, the stable trajectory  $\mathbf{T}_{st}$  at  $(0, 0)$  remains in the negatively invariant region  $\Xi$  defined in the proof of Proposition 3.4. Since it cannot converge to  $P_M$  when  $t \rightarrow -\infty$  it admits a limit cycle surrounding  $P_M$  which contradicts Lemma 3.2. Therefore  $\mathbf{T}_{reg} \subset \mathbf{Q}$  and, again using Lemma 3.2, either it converges to  $P_M$  when  $t \rightarrow \infty$  and the proof is complete, or to  $(0, 0)$  and  $\mathbf{T}_{reg} = \mathbf{T}_{st}$  is a homoclinic trajectory. The trace of the linearized system (2.24) at  $(0, 0)$  is equal to  $\frac{2}{p-1} - K = -L < 0$ . Therefore, from [17, Th. 9.3.3] the connection is attracting and the trajectories inside the bounded region  $\mathcal{T}$  bordered by the homoclinic trajectory  $\mathbf{T}_{reg}$  spiral towards it when  $t \rightarrow \infty$ . Hence any such trajectory inside  $\mathcal{T}$  either has a limit cycle when  $t \rightarrow -\infty$  which is impossible by Lemma 3.2 or converges to  $P_M$  which is also impossible. Consequently there exists no homoclinic trajectory at  $(0, 0)$  which ends the proof.  $\square$

Next we study the case where  $M$  is large enough. We have already proved in [7] that for any  $p > 1$ , there exists  $M_{\dagger} = M_{\dagger}(N, p) > 0$  (see introduction) such that if  $M > M_{\dagger}$  there exists no ground state, radial or non-radial. In the radial case we have a more precise result.

**Proposition 3.6** *Let  $N \geq 3$  and  $p > \frac{N+2}{N-2}$ . Then for any  $M \geq M_0$  there exists no ground state, but there exist singular solutions  $u$  which satisfy  $\lim_{r \rightarrow 0} r^{\frac{2}{p-1}} u(r) = X_M$  and  $\lim_{r \rightarrow \infty} r^{N-2} u(r) = c > 0$ .*

*Proof.* Since  $M \geq M_0 > \bar{M}$ ,  $P_M$  is a source by Lemma 3.1, and there exist no periodic orbit neither a homoclinic trajectory at  $(0, 0)$  by Lemma 3.3. Thus  $\mathbf{T}_{reg}$  leaves  $\mathbf{Q}$  through the semi-axis  $\{(0, y) : y > 0\}$  by Lemma 2.5. As in the proof of Proposition 3.5 the stable trajectory  $\mathbf{T}_{st}$  at  $(0, 0)$  remains in the negatively invariant region  $\Xi$  already defined. Then it converges necessarily to  $P_M$  when  $t \rightarrow -\infty$ .  $\square$

Next we study the case  $\bar{M} < M < M_0$ .

**Theorem 3.7** *Let  $N \geq 3$ ,  $p > \frac{N+2}{N-2}$ . There exist two positive real numbers  $\tilde{M}_{min}$  and  $\tilde{M}_{max}$  such that  $\bar{M} < \tilde{M}_{min} \leq \tilde{M}_{max} < M_0$  such that,*

1- *For  $\bar{M} < M < \tilde{M}_{min}$  there exist ground states undulating around  $U_M$  when  $r \rightarrow \infty$  and positive singular solutions undulating around  $U_M$  on  $[0, \infty)$ .*

2- *For  $M = \tilde{M}_{min}$  and for  $M = \tilde{M}_{max}$  there exist ground states  $u$  such that  $\lim_{r \rightarrow \infty} r^{N-2} u(r) = c > 0$ .*

3- *For  $\tilde{M}_{max} < M < M_0$  there exists no ground state and there exist singular solutions such that  $\lim_{r \rightarrow 0} r^{\frac{2}{p-1}} u(r) = X_M$  or turning around  $P_M$  when  $r \rightarrow 0$ , and  $\lim_{r \rightarrow \infty} r^{N-2} u(r) = c > 0$ .*

*Proof.* Since  $M$  is subject to vary, we put it in exponent in the different specific trajectories of the system. For  $\bar{M} < M < M_0$ ,  $P_M$  is a source and the trajectories converging toward this point when  $t \rightarrow -\infty$  are spiralling. We have three possibilities:

- (i) either  $\mathbf{T}_{reg}^M$  leaves  $\mathbf{Q}$  at some point  $(0, y_s)$ ,  $y_s > 0$ ,
- (ii) or  $\mathbf{T}_{reg}^M$  has a  $\omega$ -limit set which is a periodic orbit surrounding  $P_M$ ,
- (iii) or  $\mathbf{T}_{reg}^M$  converges to  $(0, 0)$  when  $t \rightarrow \infty$ .

If (i) holds, then  $\mathbf{T}_{st}^M$  remains in the region  $\Xi := \Xi^M$  of  $\mathbf{Q}$  bordered by  $\mathbf{T}_{reg}^M$  and the semi-axis  $\{(0, y) : y > 0\}$ . Then, either it converges to  $P_M$  when  $t \rightarrow -\infty$ , or it admits an unstable (from outside)  $\alpha$ -limit cycle. We denote by  $(x_{st}^M, y_{st}^M)$  the first backward intersection of  $\mathbf{T}_{st}^M$  with the straight line  $\mathcal{L}$ . Furthermore  $\mathbf{T}_{reg}^M$  intersects  $\mathcal{L}$  at some point (necessarily unique)  $(x_{reg}^M, y_{reg}^M)$ . Because of the relative position of  $\mathbf{T}_{st}^M$  and  $\mathbf{T}_{reg}^M$ , there holds

$$g(M) = x_{reg}^M - x_{st}^M > 0.$$

If (ii) holds, then we claim that  $\mathbf{T}_{st}^M$  leaves  $\mathbf{Q}$  at some point  $(x_s, 0)$  with  $x_s > 0$ . Indeed, if  $\mathbf{T}_{st}^M \subset \mathbf{Q}$ , it is bounded by Proposition 2.3. Hence the  $\alpha$ -limit set is non-empty. It cannot be  $(0, 0)$  since  $\mathbf{T}_{st}^M \neq \mathbf{T}_{reg}^M$  and  $\mathbf{T}_{st}^M$  cannot converge to  $P_M$  or have a limit cycle around  $P_M$  since it would imply that  $\mathbf{T}_{st}^M \cap \mathbf{T}_{reg}^M \neq \{\emptyset\}$ . Hence  $\mathbf{T}_{st}^M$  intersects  $\mathcal{L}$  at some point  $(x_{st}^M, y_{st}^M)$  and  $\mathbf{T}_{reg}^M$

intersect the first time  $\mathcal{L}$  and  $(x_{reg}^M, y_{reg}^M)$ . Because  $\mathbf{T}_{reg}^M$  lies in the region of  $\mathbf{Q}$  bordered by  $\mathbf{T}_{st}^M$  and the semi-axis  $\{(x, 0) : x > 0\}$ , there holds

$$g(M) = x_{reg}^M - x_{st}^M < 0.$$

If (iii) holds then  $g(M) = 0$ .

The function  $g$  defined for any  $M \in (\bar{M}, M_0)$  is continuous. We know that there exists  $\epsilon > 0$  such that for any  $M \in (\bar{M}, \bar{M} + \epsilon)$ ,  $\mathbf{T}_{reg}^M$  has a  $\omega$ -limit cycle surrounding  $P_M$ , hence  $g(M) < 0$ . If  $M = M_0$ ,  $\mathbf{T}_{reg}^M$  leaves  $\mathbf{Q}$  at some point  $(0, y_s)$  with  $y_s > 0$  from Proposition 3.6 and the intersection of  $\mathbf{T}_{reg}^M$  with the semi-axis  $\{(0, y) : y > 0\}$  is transverse at  $(0, y_s)$ . By continuity with respect to the parameter  $M$ , for any  $M \in (M_0 - \epsilon', M_0]$ , for  $\epsilon' > 0$  small enough,  $\mathbf{T}_{reg}^M$  leaves  $\mathbf{Q}$  at some point of the semi-axis. So we are in situation (i) and  $g(M) > 0$ .

Then, by continuity of  $g$  there exist  $\tilde{M}_{min}$  and  $\tilde{M}_{max}$  such that  $\bar{M} < \tilde{M}_{min} \leq \tilde{M}_{max} < M_0$  such that  $g(\tilde{M}_{min}) = g(\tilde{M}_{max}) = 0$ . If  $M = \tilde{M}_{min}$  or  $M = \tilde{M}_{max}$ ,  $\mathbf{T}_{st}^M = \mathbf{T}_{reg}^M$  and the trajectory  $\mathbf{T}_{reg}^M$  is homoclinic at  $(0, 0)$ . For  $\bar{M} < M < \tilde{M}_{min}$  we are in situation (ii), which proves 1. For  $M > \tilde{M}_{max}$  we are in situation (i) and either  $\mathbf{T}_{st}^M$  converges to  $P_M$  or has a  $\alpha$ -limit cycle surrounding  $P_M$  when  $t \rightarrow -\infty$ .  $\square$

Theorems A and A' follow from the previous results.

*Remark.* It is a challenging question to prove that there is a unique  $M$  such that there is a homoclinic trajectory at  $(0, 0)$ . Up to now all we can prove is that if there exist two parameters  $0 < M_1 < M_2$  such that for each of them there exists a homoclinic trajectory of  $(0, 0)$  in  $\mathbf{Q}$   $\mathbf{T}^{M_i} := \mathbf{T}_{st}^{M_i} = \mathbf{T}_{unst}^{M_i}$  ( $i=1, 2$ ), then  $\mathbf{T}^{M_2}$  is a subset of the domain of  $\mathbf{Q}$  limited by  $\mathbf{T}^{M_1}$ .

## 4 Study of ground states of (1.17) when $M < 0$

We will distinguish the cases  $p \geq \frac{N}{N-2}$  where system (1.17) admits a unique non-trivial equilibrium and  $1 < p < \frac{N}{N-2}$  where the existence of zero, one or two equilibria depends on the value of  $M$  with respect to  $-\mu^*$  defined in (1.20). In order to avoid confusion we set

$$\bar{\mu} = |\bar{M}| = \frac{(p+1)|p(N-2) - (N+2)|}{(4p)^{\frac{p}{p+1}}((N-2)(p-1)^2 + 4)^{\frac{1}{p+1}}}. \quad (4.1)$$

Then  $\bar{\mu} = \bar{M}$  if  $p > \frac{N+2}{N-2}$  and  $\bar{\mu} = -\bar{M}$  if  $p < \frac{N+2}{N-2}$ .

**Proposition 4.1** *let  $N \geq 3$ ,  $p \geq \frac{N}{N-2}$  and  $M < 0$ .*

1- *If  $p \geq \frac{N+2}{N-2}$ ,  $P_M$  is a sink.*

2- *If  $\frac{N}{N-2} \leq p < \frac{N+2}{N-2}$ , then  $P_M$  is a sink if  $M < \bar{M}$ , it is a source if  $\bar{M} < M < 0$ , and  $P_{\bar{M}}$  is a weak source. Moreover there exists  $M_1 < \bar{M}$  such that  $P_M$  is a node if  $M \leq M_1$ . If  $\frac{N}{N-2} \leq p < \frac{N+2\sqrt{N-1}}{N-4+2\sqrt{N-1}}$ , there exists  $M_0 \in (\bar{M}, 0)$  such that  $P_M$  is a node for  $M \geq M_0$  and a spiraling equilibrium if  $M_1 < M < M_0$ .*



*Proof.* The equation (2.30) satisfied by the characteristic roots can be written under the form

$$T(\lambda) = \lambda^2 + \left( \frac{2p}{p+1} |M| Y_M^{\frac{p-1}{p+1}} + L \right) \lambda + 2K + \frac{2p}{p+1} |M| Y_M^{\frac{p-1}{p+1}} = 0. \quad (4.2)$$

Since  $K \geq 0$  the product of the roots is positive; the real part of the roots is positive if and only if  $\frac{2p}{p+1} |M| Y_M^{\frac{p-1}{p+1}} + L < 0$ . If  $p \geq \frac{N+2}{N-2}$ , then  $L \geq 0$  and  $P_M$  is a sink. If  $\frac{N}{N-2} \leq p < \frac{N+2}{N-2}$ , then  $\bar{M}$  is characterized by

$$\frac{2p}{p+1} \bar{M} Y_{\bar{M}}^{\frac{p-1}{p+1}} = L < 0. \quad (4.3)$$

By Proposition 2.2 the function  $M \mapsto M Y_M^{\frac{p-1}{p+1}}$  is increasing and onto on  $(-\infty, 0)$ . Therefore,  $P_M$  is a sink if  $M < \bar{M}$  and a source if  $\bar{M} < M < 0$ . Finally, if  $M = \bar{M}$ , the two roots are imaginary and by Lemma 2.7  $P_M$  is a weak source. Finally, from (3.6), the characteristic roots are real if and only

$$\frac{2p}{p+1} |M| Y_M^{\frac{p-1}{p+1}} \leq |L| + 2 - 2\sqrt{N-1} \quad \text{or} \quad \frac{2p}{p+1} |M| Y_M^{\frac{p-1}{p+1}} \geq |L| + 2 + 2\sqrt{N-1}. \quad (4.4)$$

Notice that there always hold  $|L| + 2 - 2\sqrt{N-1} < |L|$  since  $N \geq 3$ . The first condition in (4.4) requires  $|L| + 2 - 2\sqrt{N-1} > 0$ , equivalently  $p < \frac{N+2\sqrt{N-1}}{N-4+2\sqrt{N-1}}$ . Since there holds  $\frac{N}{N-2} < \frac{N+2\sqrt{N-1}}{N-4+2\sqrt{N-1}} < \frac{N+2}{N-2}$ , the conclusion follows and  $M_0$  and  $M_1$  are given by (3.8) and (3.11).  $\square$

The most intricate case corresponds to  $1 < p < \frac{N}{N-2}$  or  $N = 1, 2$  where there may exist 0, 1 or 2 equilibria.

**Proposition 4.2** *Assume  $N = 1, 2$  and  $p > 1$  or  $N \geq 3$  and  $1 < p < \frac{N}{N-2}$ ,  $M < -\mu^*$  and let  $P_{j,M}$ ,  $j=1$  or  $2$ , be the two equilibria of (1.17).*

1- *Then  $P_{1,M}$  is a saddle point.*

2- *Let  $N \geq 3$ . If  $\bar{M} < M < -\mu^*$ , then  $P_{2,M}$  is a source; if  $M < \bar{M}$ , then  $P_{2,M}$  is a sink;  $P_{2,\bar{M}}$  is a weak source. Moreover there exist  $M_1 < \bar{M}$  such that  $P_{2,M}$  is a node for  $M \leq M_1$ ; there exists also  $M_0 \in (\bar{M}, -\mu^*)$  such that  $P_{2,M}$  is a node for  $M \leq M_1$  or for  $M_0 \leq M < -\mu^*$ , and it is a spiraling equilibrium if  $M_1 < M < M_0$ .*

3- *Let  $N = 2$ . Then  $P_{2,M}$  is a sink. There exists  $M_1 < -\mu^*$  such that  $P_{2,M}$  is a node if and only if  $M \leq M_1$ .*

4- *If  $N = 1$ , then  $P_{2,M}$  is a sink and a node.*

*Proof.* Recall that, from Proposition 2.2,  $M \mapsto M Y_{1,M}^{\frac{p-1}{p+1}}$  is decreasing and  $M \mapsto M Y_{2,M}^{\frac{p-1}{p+1}}$  is increasing. We first consider the linearized operator at  $P_{1,M}$ . The product of the roots of (4.2) is equal to  $2K - \frac{2p}{p+1} M Y_{1,M}^{\frac{p-1}{p+1}}$ . Since  $M < -\mu^*$ ,

$$2K - \frac{2p}{p+1} M Y_{1,M}^{\frac{p-1}{p+1}} < 2K + \frac{2p}{p+1} \mu^* Y_{-\mu^*}^{\frac{p-1}{p+1}} = 0.$$

Hence  $P_{1,M}$  is a saddle point.

Next we consider  $P_{2,M}$ . If  $\bar{M} < M < -\mu^*$ , since  $M \mapsto MY_{2,M}^{\frac{p-1}{p+1}}$  is increasing,

$$MY_{2,M}^{\frac{p-1}{p+1}} > \bar{M}Y_{2,\bar{M}}^{\frac{p-1}{p+1}} = \frac{p+1}{2p}L, \quad (4.5)$$

and

$$2K - \frac{2p}{p+1}MY_{2,M}^{\frac{p-1}{p+1}} > 2K + \frac{2p}{p+1}\mu^*Y_{-\mu^*}^{\frac{p-1}{p+1}} = 0. \quad (4.6)$$

Hence  $P_{2,M}$  is a source. If  $M < \bar{M}$ , then the sign in (4.5) is reversed and

$$2K - \frac{2p}{p+1}MY_{2,M}^{\frac{p-1}{p+1}} > 2K - \frac{2p}{p+1}\bar{\mu}Y_{\bar{\mu}}^{\frac{p-1}{p+1}} > 2K + \frac{2p}{p+1}\mu^*Y_{-\mu^*}^{\frac{p-1}{p+1}} = 0. \quad (4.7)$$

Thus  $P_{2,M}$  is a sink. By Lemma 2.7  $P_{2,\bar{M}}$  is a weak source, and assertion 2 is proved. Finally, if  $N = 2$  and  $M < -\mu^* = \bar{M}$  from Lemma 2.6, therefore  $P_{2,M}$  is a sink.

Next we look for conditions which insure that  $P_{2,M}$  is a node. The characteristic roots are real if and only if one of the two conditions (3.6) where  $Y_M$  is replaced by  $Y_{2,M}$  holds:

$$\begin{aligned} (i) \quad & \frac{2p}{p+1}MY_{2,M}^{\frac{p-1}{p+1}} \geq L - 2 + 2\sqrt{N-1}, \\ (ii) \quad & \frac{2p}{p+1}MY_{2,M}^{\frac{p-1}{p+1}} \leq L - 2 - 2\sqrt{N-1}. \end{aligned} \quad (4.8)$$

For  $1 < p < \frac{N}{N-2}$ ,  $M \mapsto \Phi(M) := \frac{2p}{p+1}MY_{2,M}^{\frac{p-1}{p+1}}$  is an increasing diffeomorphism from  $(-\infty, -\mu^*]$  to  $(-\infty, 2K]$ . Since  $L - 2\sqrt{N-1} - 2 < 2K$  there exists a unique  $M_1 < -\mu^*$  such that  $\Phi(M_1) = L - 2 - 2\sqrt{N-1}$  and (4.8)-(ii) holds when  $M \leq M_1$ . Since  $L - 2 + 2\sqrt{N-1} > 2K$  is equivalent to  $(N-2)^2 < 0$ , there exists no  $M \leq -\mu^*$  such that (4.8)-(i) holds. When  $N = 2$  only  $M = -\mu^*$  satisfies (4.8)-(i) with equality. If  $N \geq 3$  there exists a unique  $M_0 < -\mu^*$  such that (4.8)-(i) holds. Hence for any  $M_0 \leq M < -\mu^*$ , inequality (4.8)-(i) holds with  $M_0$  and  $M_1$  defined by (3.8) and (3.11).

At end assume  $N = 1$ . For any  $M \leq -\mu^*(1) = -(p+1)\left(\frac{p+1}{2p}\right)^{\frac{p+1}{p}}$ , there holds

$$\frac{2p}{p+1}MY_{2,M}^{\frac{p-1}{p+1}} - L \leq -\frac{2p}{p+1}\mu^*Y_{2,-\mu^*}^{\frac{p-1}{p+1}} = 2K - L = -1. \quad (4.9)$$

Therefore  $P_{2,M}$  is always a sink and the discriminant of (2.30) is equal to  $\left(\frac{2p}{p+1}MY_{2,M}^{\frac{p-1}{p+1}} - L + 2\right)^2$ , hence the characteristic roots are negative.  $\square$

*Notations.* Under the assumptions of Proposition 4.2 there exist two stable trajectories  $\mathbf{T}_{st}^{1,j}$ ,  $j=1,2$ , converging to  $P_{1,M}$  when  $t \rightarrow \infty$ , associated to a negative characteristic value  $\lambda$ ; the common slope at  $P_{1,M}$  is  $\frac{2}{p-1} + |\lambda|$ . We assume that  $\mathbf{T}_{st}^{1,1}$  is *locally below*  $\mathcal{L}$  and  $\mathbf{T}_{st}^{1,2}$  *locally above*  $\mathcal{L}$ , thus these trajectories are coming from the regions  $\mathbf{C}$  and  $\mathbf{A}$  defined in the proof of Proposition 2.4. There exist also two unstable trajectories  $\mathbf{T}_{unst}^{1,j}$ ,  $j=3,4$ , converging to  $P_{1,M}$

when  $t \rightarrow -\infty$ , associated to a positive characteristic value  $\lambda'$  and the common slope at  $P_{1,M}$  is  $\frac{2}{p-1} - \lambda'$ . We assume that  $\mathbf{T}_{st}^{1,3}$  is *locally below*  $\mathcal{L}$  and  $\mathbf{T}_{st}^{1,4}$  *locally above*  $\mathcal{L}$ , thus, in a neighbourhood of  $P_{1,M}$ , these trajectories belong also to the regions **C** and **A**. In particular the trajectory  $\mathbf{T}_{st}^{1,1}$  cannot cross  $\mathcal{L}$ , neither the segment  $\{(x, y) : x = X_{1M}, 0 \leq y \leq Y_{1M}\}$ . Hence either it converges to  $(0, 0)$  when  $t \rightarrow -\infty$ , or it crosses the axis  $\{y = 0\}$  at some point with positive  $x$ -coordinate less than  $X_{1,M}$ .

*Remark.* In the critical case  $M = -\mu^*$  there holds  $-\frac{2p}{p+1}\mu^*Y_{-\mu^*}^{\frac{p-1}{p+1}} = 2K = 2 - N - L$ , hence the characteristic polynomial is  $T(\lambda) = \lambda(\lambda + 2 - N)$ . If  $N \geq 3$  the characteristic values are 0 and  $N - 2$ , with respective corresponding eigenvectors  $(1, \frac{2}{p-1})$  and  $(1, \frac{2}{p-1} + 2 - N)$ . There exists an invariant curve  $\Gamma$  passing through  $P_{1,M} = P_{2,M}$ , tangent to  $(1, \frac{2}{p-1})$  by the center manifold theorem. If  $N = 2$  the two characteristic values are 0 with the eigenspace generated by  $(1, \frac{2}{p-1})$  which is tangent to the central manifold (a curve) at  $(0, 0)$ .

Next we look for the existence of limit cycles. Since  $M < 0$ , we cannot argue using the convexity argument used in Lemma 3.2. We use system (2.6) which also has a convexity property, as we will see it in the proof below.

**Lemma 4.3** *1- If  $N \geq 3$  and  $p \geq \frac{N+2}{N-2}$  and  $M < 0$ , there is no cycle surrounding  $P_M$ .  
 2- If  $N \geq 3$ ,  $\frac{N}{N-2} < p < \frac{N+2}{N-2}$  and  $\bar{M} \leq M < 0$ , there is no cycle surrounding  $P_M$ .  
 3- If  $N \geq 3$  and  $1 < p < \frac{N}{N-2}$  or  $N = 2$  and  $\bar{M} \leq M \leq -\mu^*$ , there is no cycle surrounding  $P_{2,M}$ .*

*Proof.* For 1, assume that there exists a periodic trajectory  $(x(t), y(t))$  surrounding  $P_M$ . By Green's formula,

$$\begin{aligned} 0 &= \oint_{\gamma} (-H_2(x, y)dx + H_1(x, y)dy) = \iint_{\Gamma} \left( \frac{\partial H_1(x, y)}{\partial x} + \frac{\partial H_2(x, y)}{\partial y} \right) dx dy \\ &= \iint_{\Gamma} \left( \frac{2pM}{p+1} |y|^{\frac{p-1}{p+1}} - L \right) dx dy. \end{aligned} \quad (4.10)$$

Since  $L \geq 0$  and  $M < 0$  we obtain a contradiction. This proves the first assertion.

For 2 and 3 we assume that  $\gamma_0 \subset \mathbf{Q}$  is a cycle surrounding  $P_M$  or  $P_{2,M}$  and we denote by  $\Gamma_0$  the bounded domain bordered by  $\gamma_0$ . By Proposition 4.1 and Proposition 4.2,  $P_M$  and  $P_{2,M}$  are sources. We use the system (2.6). Setting  $J(\sigma, z) = (\sigma^p z)^{\frac{1}{p+1}}$ , it becomes

$$\begin{aligned} \sigma_t &= \sigma(\sigma + 2 - N + z + MJ(\sigma, z)) := F(\sigma, z) \\ z_t &= z(N - p\sigma - z - MJ(\sigma, z)) := G(\sigma, z). \end{aligned} \quad (4.11)$$

In the phase plane  $(\sigma, z)$  the equilibrium  $P_M$  becomes

$$\mathcal{P}_M = (\sigma_M, z_M) = \left( \frac{Y_M}{X_M}, \frac{X_M^p}{Y_M} \right) = \left( \frac{2}{p-1}, \left( \frac{p-1}{2} \right)^p Y_M^{p-1} \right).$$

Similarly  $P_{2,M}$  becomes  $\mathcal{P}_{2,M} = (\sigma_{2,M}, z_{2,M}) = \left( \frac{Y_{2,M}}{X_{2,M}}, \frac{X_{2,M}^p}{Y_{2,M}} \right) = \left( \frac{2}{p-1}, \left( \frac{p-1}{2} \right)^p Y_{2,M}^{p-1} \right)$  and  $\mathcal{P}_M$  and  $\mathcal{P}_{2,M}$  are sources. Hence, any trajectory converging to this point  $\mathcal{P}_M$  (or  $\mathcal{P}_{2,M}$ ) admits an

omega-limit cycle  $\gamma \subset \bar{\Gamma}_0$ . Consequently this cycle is not unstable which implies that the Floquet integral relative to this  $T$ -periodic solution is nonpositive:

$$\mathcal{I} := \int_0^T \left( \frac{\partial F(\sigma, z)}{\partial \sigma} + \frac{\partial G(\sigma, z)}{\partial z} \right) (t) dt \leq 0. \quad (4.12)$$

The computation gives

$$\begin{aligned} \frac{\partial F}{\partial \sigma} + \frac{\partial G}{\partial z} &= (2-p)\sigma + 2 - z + M \frac{2p+1}{p+1} (\sigma^p z)^{\frac{1}{p+1}} - M \frac{p+2}{p+1} (\sigma^p z)^{\frac{1}{p+1}} \\ &= (2-p)\sigma + 2 - z + M \frac{p-1}{p+1} J(\sigma, z). \end{aligned}$$

We set  $\sigma = \sigma_M + \bar{\sigma}$  and  $z = z_M + \bar{z}$  (the computation would be the same with  $(\sigma_M, z_M)$  replaced by  $(\sigma_{2,M}, z_{2,M})$ ), then

$$\begin{aligned} \frac{\mathcal{I}}{T} &= \frac{1}{T} \int_0^T \left( \frac{\partial F}{\partial \sigma} + \frac{\partial G}{\partial z} \right) (\sigma(t), z(t)) dt \\ &= (2-p) \left( \sigma_M + \frac{1}{T} \int_0^T \bar{\sigma}(t) dt \right) + 2 - z_M - \frac{1}{T} \int_0^T \bar{z}(t) dt + \frac{p-1}{p+1} \frac{M}{T} \int_0^T J(\sigma(t), z(t)) dt. \end{aligned}$$

By addition

$$\begin{aligned} \frac{\sigma_t}{\sigma} + \frac{z_t}{z} &= \sigma - (N-2) + z + N - p\sigma - z = 2 - (p-1)\sigma \\ &= 2 - (p-1)\sigma_M - (p-1)\bar{\sigma} = -(p-1)\bar{\sigma}. \end{aligned}$$

Integrating on a period, we get  $\int_0^T \bar{\sigma}(t) dt = 0$ . We also derive from (4.11)

$$\begin{aligned} 0 &= \sigma_M + \frac{1}{T} \int_0^T \bar{\sigma}(t) dt + 2 - N + z_M + \frac{1}{T} \int_0^T \bar{z}(t) dt + \frac{M}{T} \int_0^T J(\sigma, z) dt \\ &= \frac{2}{p-1} + 2 - N + z_M + \frac{1}{T} \int_0^T \bar{z}(t) dt + \frac{M}{T} \int_0^T J(\sigma, z) dt \\ &= -K + z_M + \frac{1}{T} \int_0^T \bar{z}(t) dt + \frac{M}{T} \int_0^T J(\sigma, z) dt. \end{aligned}$$

Furthermore,  $\sigma_M + 2 - N + z_M + MJ(\sigma_M, z_M) = 0$ . Indeed

$$\frac{1}{T} \int_0^T \bar{z}(t) dt = -\frac{M}{T} \int_0^T (J(\sigma, z) - J(\sigma_M, z_M)) dt. \quad (4.13)$$

Next we show that the function  $J$  is *concave* on  $\mathcal{Q} := \{(\sigma, z) : \sigma > 0, z > 0\}$ : indeed

$$J(\sigma, z) - J(\sigma_M, z_M) = \frac{\partial J}{\partial \sigma}(\sigma_M, z_M) \bar{\sigma} + \frac{\partial J}{\partial z}(\sigma_M, z_M) \bar{z} + \frac{1}{2} (a\bar{\sigma}^2 + 2b\bar{\sigma}\bar{z} + c\bar{z}^2),$$

where  $a$ ,  $b$  and  $c$  depend on  $\sigma$ ,  $\sigma_M$ ,  $z$  and  $z_M$  and:

$$\frac{\partial J}{\partial \sigma}(\sigma, z) = \frac{p}{p+1} \sigma^{-\frac{1}{p+1}} z^{\frac{1}{p+1}} \quad \text{and} \quad \frac{\partial J}{\partial z}(\sigma, z) = \frac{1}{p+1} \sigma^{\frac{p}{p+1}} z^{-\frac{p}{p+1}}.$$

$$a = -\frac{p}{(p+1)^2} \sigma_\theta^{-\frac{p+2}{p+1}} z_\theta^{\frac{1}{p+1}}, \quad b = \frac{p}{(p+1)^2} \sigma_\theta^{-\frac{1}{p+1}} z_\theta^{\frac{p}{p+1}} \quad \text{and} \quad c = -\frac{p}{(p+1)^2} \sigma_\theta^{\frac{p}{p+1}} z_\theta^{-\frac{2p+1}{p+1}},$$

where  $(\sigma_\theta, z_\theta) = (\theta\sigma + (1-\theta)\sigma_M, \theta z + (1-\theta)z_M)$  for some  $\theta \in (0, 1)$ . Since

$$b^2 - ac = \frac{p^2}{(p+1)^4} \left( \sigma_\theta^{-\frac{2}{p+1}} z_\theta^{-\frac{2}{p+1}} - \sigma_\theta^{-\frac{2}{p+1}} z_\theta^{-\frac{2p}{p+1}} \right) = 0,$$

then  $a\bar{\sigma}^2 + 2b\bar{\sigma}\bar{z} + c\bar{z}^2 = a(\bar{\sigma} + \frac{b}{a}\bar{z})^2 = -2R^2(t) \leq 0$  and from (4.13),

$$\begin{aligned} \frac{1}{T} \int_0^T \bar{z}(t) dt &= -\frac{M}{T} \frac{\partial J}{\partial \sigma}(\sigma_M, z_M) \int_0^T \bar{\sigma}(t) dt - \frac{M}{T} \frac{\partial J}{\partial z}(\sigma_M, z_M) \int_0^T \bar{z}(t) dt + \frac{M}{T} \int_0^T R^2(t) dt \\ &= -\frac{M}{T(p+1)} \sigma_M^{\frac{p}{p+1}} z_M^{-\frac{p}{p+1}} \int_0^T \bar{z}(t) dt + \frac{M}{T} \int_0^T R^2(t) dt, \end{aligned}$$

thus

$$\left( 1 + \frac{M \sigma_M^{\frac{p}{p+1}} z_M^{-\frac{p}{p+1}}}{p+1} \right) \int_0^T \bar{z}(t) dt = M \int_0^T R^2(t) dt. \quad (4.14)$$

When  $M < 0$  we have already seen that  $Y_M > y_{0,M} := \left( \frac{2}{p-1} \left( \frac{-M}{p+1} \right)^{\frac{1}{p}} \right)^{\frac{p+1}{p-1}}$ , defined in (2.16) (resp.  $Y_{2,M} > y_{0,M}$ ), which yields

$$1 + \frac{M \sigma_M^{\frac{p}{p+1}} z_M^{-\frac{p}{p+1}}}{p+1} = \left( Y_M^{\frac{p(p-1)}{p+1}} - y_{0,M}^{\frac{p(p-1)}{p+1}} \right) Y_M^{-\frac{p(p-1)}{p+1}} > 0 \implies \int_0^T \bar{z}(t) dt < 0. \quad (4.15)$$

Therefore

$$\begin{aligned} \frac{\mathcal{I}}{T} &= (2-p)\sigma_M + 2 - z_M - \frac{1}{T} \int_0^T \bar{z}(t) dt + \frac{M(p-1)}{T(p+1)} \int_0^T J(\sigma, z) dt \\ &= (2-p)\sigma_M + 2 - z_M - \frac{1}{T} \int_0^T \bar{z}(t) dt + \frac{M(p-1)}{p+1} J(\sigma_M, z_M) \\ &\quad + \frac{M(p-1)}{T(p+1)} \int_0^T (J(\sigma, z) - J(\sigma_M, z_M)) dt. \end{aligned}$$

Hence from (4.13)

$$\begin{aligned} \frac{\mathcal{I}}{T} &= (2-p)\sigma_M + 2 - z_M - \frac{2p}{T(p+1)} \int_0^T \bar{z}(t) dt + \frac{M(p-1)}{p+1} J(\sigma_M, z_M) \\ &= \frac{2}{p-1} - \left( \frac{p-1}{2} \right)^p Y_M^{p-1} - \frac{2p}{T(p+1)} \int_0^T \bar{z}(t) dt + \frac{M(p-1)}{p+1} Y_M^{\frac{p-1}{p+1}} \\ &= \frac{2p}{p+1} M Y_M^{\frac{p-1}{p+1}} - L - \frac{2p}{T(p+1)} \int_0^T \bar{z}(t) dt. \end{aligned} \quad (4.16)$$

But from Proposition 2.2,

$$\overline{M} < M < 0 \implies \frac{2p}{p+1}MY_M^{\frac{p-1}{p+1}} - L = \frac{2p}{p+1} \left( MY_M^{\frac{p-1}{p+1}} - \overline{M}Y_{\overline{M}}^{\frac{p-1}{p+1}} \right) > 0. \quad (4.17)$$

Combining (4.15) and (4.17) we obtain  $\mathcal{I} > 0$  which contradicts the nonpositivity of the Floquet integral in the case  $M > \overline{M}$ . Finally, if  $\overline{M} = M$ , then  $\int_0^T \overline{z}(t)dt = 0$ , which implies by (4.14) that  $R \equiv 0$  on the omega-limit cycle  $\gamma$ . Using the expression of  $R(t)$  we infer

$$\overline{\sigma} + \frac{b}{a}\overline{z} \equiv 0 \quad \text{where} \quad \frac{a}{b} = \frac{1}{z_\theta \sigma_\theta^{\frac{p-1}{p+1}}} \quad \text{and} \quad (\sigma_\theta, z_\theta) = (\theta\sigma + (1-\theta)\sigma_M, \theta z + (1-\theta)z_M),$$

for  $\theta = \theta(t) \in (0, 1)$ . This is clearly impossible if one considers points at the intersection of  $\gamma$  and the straight line  $\mathcal{L}$ . The same argument holds when  $N = 2$  where  $\overline{M} = -\mu^*$ .  $\square$

*Remark.* The above proof can easily be adapted to recover the second statement of Lemma 3.2. Indeed, if  $p > \frac{N+2}{N-2}$  and  $0 < M < \overline{M}$ ,  $P_M$  is a sink. Hence if there is a cycle surrounding  $P_M$ , we can assume that it is an  $\alpha$ -limit cycle, say  $\gamma$ , and the integral  $\mathcal{I}$  given by (4.12) is nonnegative

by Floquet's theory. The inequality (4.14) yields  $\int_0^T \overline{z}(t)dt > 0$  since  $1 + \frac{M\sigma_M^{\frac{p}{p+1}}z_M^{-\frac{p}{p+1}}}{p+1} > 1$ . For  $M < \overline{M}$  there holds  $\frac{2p}{p+1}MY_M^{\frac{p-1}{p+1}} < L$  from the monotonicity of  $M \mapsto MY_M^{\frac{p-1}{p+1}}$ , which contradicts the sign of  $\mathcal{I}$ .

In the next statement we extend [14, Prop. 5.6], which was proved in the case  $1 < p < \frac{N}{N-2}$ , but valid actually for any  $p > 1$ , and give precision on the behavior of the solutions. The constant

$$\mu^*(1) = \left( \frac{(p+1)^{2p+1}}{(2p)^p} \right)^{\frac{1}{p+1}}$$

plays an important role. Recall also that  $\mu^*(1) > \mu^* := \mu^*(N)$  for  $N \geq 2$ .

**Lemma 4.4** *Let  $p > 1$ . If  $N \geq 2$  and  $M \leq -\mu^*(1)$  or  $N = 1$  and  $M < -\mu^*(1)$ , then there exists a ground state  $u$ . Furthermore there holds*

$$u_r^2(r) < \frac{2p}{p+1}u^{p+1}(r) \quad \text{for all } r > 0. \quad (4.18)$$

*As a consequence, the corresponding trajectory  $\mathbf{T}_{reg} = \{(x_{reg}(t), y_{reg}(t))\}_{t \in \mathbb{R}}$  does not converge to  $(0, 0)$  when  $t \rightarrow \infty$ . If  $1 < p < \frac{N}{N-2}$  and  $N \geq 3$  or  $p > 1$  and  $N = 1, 2$ ,  $\mathbf{T}_{reg}$  does not converge to  $P_{1,M}$  when  $t \rightarrow \infty$ .*

*Proof.* Let  $u$  be a regular solution of (1.13) with  $u(0) = 1$  and  $u_r(0) = 0$ . As in [14] we set

$$G(r) = \frac{1}{2} \left( u_r^2(r) - \frac{2p}{p+1}|u|^{p+1}(r) \right).$$

Then  $G(0) < 0$  and  $u > 0$  on some maximal interval  $[0, r_1)$  with  $r_1 \leq \infty$ . If there exists a minimal  $r_0 \leq r_1$  such that  $G(r_0) = 0$ , then  $G_r(r_0) \geq 0$ . From (1.13) we have

$$\begin{aligned} G_r(r_0) &= u_r(r_0) (u_{rr}(r_0) - pu^p(r_0)) \\ &= -\frac{N-1}{r_0} u_r^2(r_0) - u_r(r_0) \left( (p+1)u^p(r_0) + M |u_r(r_0)|^{\frac{2p}{p+1}} \right). \end{aligned}$$

Since  $G(r_0) = 0$ ,  $u^p(r_0) = \left( \frac{p+1}{2p} \right)^{\frac{p}{p+1}} |u_r(r_0)|^{\frac{2p}{p+1}}$ , hence

$$G_r(r_0) = -\frac{N-1}{r_0} u_r^2(r_0) - u_r(r_0) |u_r(r_0)|^{\frac{2p}{p+1}} (\mu^*(1) + M).$$

Since  $u_r(r_0) < 0$  we obtain a contradiction. Therefore  $G(r) < 0$  for any  $r \in (0, r_0)$ . By continuity this implies (4.18) and in particular  $u(r) > 0$ , hence  $u$  is a ground state. Inequality (4.18) implies

$$u(r) > \frac{1}{\left(1 + \frac{p-1}{2} \sqrt{\frac{2p}{p+1}} r\right)^{\frac{2}{p-1}}} \quad \text{for all } r > 0, \quad (4.19)$$

from which follows

$$\liminf_{r \rightarrow \infty} r^{\frac{2}{p-1}} u(r) \geq \left( \frac{2(p+1)}{p(p-1)^2} \right)^{\frac{1}{p-1}}, \quad (4.20)$$

This implies that the trajectory  $\mathbf{T}_{reg}$  does not converge to  $(0, 0)$  at infinity (a result which was clear when  $1 < p < \frac{N}{N-2}$  in which case  $(0, 0)$  is a source). For the last statement we have with the equation (2.14) satisfied by the equilibrium  $X_M$  and the fact that  $M \leq -\mu^*(1)$ ,

$$\begin{aligned} \tilde{f}_M \left( \left( \frac{2(p+1)}{p(p-1)^2} \right)^{\frac{1}{p-1}} \right) &\leq \frac{2(p+1)}{p(p-1)^2} - \mu^*(1) \left( \frac{2}{p-1} \right)^{\frac{2p}{p+1}} \left( \frac{2(p+1)}{p(p-1)^2} \right)^{\frac{1}{p+1}} - \frac{2K}{p-1} \\ &\leq -\frac{2(N-1)}{p-1}. \end{aligned}$$

Since  $\tilde{f}_M$  has two roots  $0 < X_{1,M} < X_{2,M}$  by Proposition 2.2 and  $M \leq -\mu^*(1) < -\mu^*(N)$ , and we have  $\left( \frac{2(p+1)}{p(p-1)^2} \right)^{\frac{1}{p-1}} > X_{1,M}$ , the result follows.  $\square$

Next we give an alternative proof of a result of [16].

**Lemma 4.5** *Let  $N = 1, 2$  and  $p > 1$  or  $N \geq 3$  and  $1 < p < \frac{N}{N-2}$ . If  $-\mu^* < M \leq 0$  there exists no ground state.*

*Proof.* If  $-\mu^* < M < 0$ , the only equilibrium of (1.17) is  $(0, 0)$  and it is a source by Section 3.4.1. If there exists a ground state then the trajectory  $\mathbf{T}_{reg}$  remains in  $\mathbf{Q}$ . It is bounded by Proposition 2.4 hence it admits an  $\omega$ -limit set which either contains an equilibrium or is a periodic orbit. The two possibilities are excluded.  $\square$

Next we study the case  $M \leq -\mu^*$ . In particular we cover the case  $M = -\mu^*$  studied in [22] with a different proof, using the energy function  $\mathcal{Z}$  defined in (2.42).

**Lemma 4.6** *Let  $N \geq 3$  and  $1 < p < \frac{N}{N-2}$ , or  $N = 2$  and  $p > 1$ . If  $M \leq -\mu^*$  then,*

- (i)  $\mathbf{T}_{\text{reg}}$  cannot converge to  $P_{1,M}$  as  $t \rightarrow \infty$  with  $t \mapsto x(t)$  increasing,
- (ii)  $\mathbf{T}_{\text{reg}}$  cannot intersect  $\mathcal{L}$  at a point between  $(0,0)$  and  $P_{1,M}$ .

*Proof.* The function  $\mathcal{Z}$  defined by (2.42) satisfies (2.44) and (2.45). Then  $\mathcal{U} > 0$  as soon as

$$|M| y^{\frac{p-1}{p+1}} \leq |L| \frac{(p+1)^2}{p+3}.$$

For  $M \leq -\mu^*$ , there holds from Proposition 2.2 and (2.17),

$$|M| Y_{1,M}^{\frac{p-1}{p+1}} \leq \mu^* Y_{-\mu^*}^{\frac{p-1}{p+1}} = \frac{p+1}{p} |K| < |L| \frac{(p+1)^2}{p(p+3)}.$$

Indeed we check that

$$(p+1)|L| - (p+3)|K| = 2(N-1).$$

Hence  $\mathcal{U} > 0$  for any  $y \in [0, Y_{1,M}]$ .

(i) Suppose that  $\mathbf{T}_{\text{reg}}$  converges to  $P_{1,M}$  with  $x(t)$  increasing, hence  $(x(t), y(t))$  is below  $\mathcal{L}$  and  $y(t)$  is also increasing. Then from (2.44)  $r \mapsto e^{\frac{2p|M|}{p+1} \int_0^r |u_r|^{\frac{p-1}{p+1}} ds} \mathcal{Z}(r)$  is increasing. But  $\mathcal{Z}(0) = 0$  and  $\lim_{r \rightarrow \infty} \mathcal{Z}(r) = 0$  from (2.43) since  $L < 0$  and  $x$  and  $y$  are bounded. This is a contradiction.

(ii) Suppose that  $\mathbf{T}_{\text{reg}}$  intersects  $\mathcal{L}$  at a point  $(\tilde{x}_M, \tilde{y}_M)$  between  $(0,0)$  and  $P_{1,M}$ , i.e.  $\tilde{y}_M < Y_{1,M}$ .

(a) If  $M < -\mu^*$ , then consider the stable trajectory  $\mathbf{T}_{st}^{1,1}$  at  $P_{1,M}$  which is below  $\mathcal{L}$ :  $\mathbf{T}_{st}^{1,1}$  cannot converge to  $(0,0)$  when  $t \rightarrow -\infty$ ; indeed it would be a unstable trajectory at the source point  $(0,0)$  and since  $\mathbf{T}_{reg}$  is the unique fast unstable trajectory at this point (see Section 2.4.1), it is below  $\mathbf{T}_{st}^{1,1}$  near zero and the two curve would intersect. Therefore  $\mathbf{T}_{st}^{1,1}$  leaves  $\mathbf{Q}$  through the semi-axis  $\{(x,0) : x > 0\}$  at some at some  $x = x(\tau)$  and  $(x(t), y(t)) \in \mathbf{Q}$  for  $t > \tau$ . Setting  $\bar{r} = e^\tau$ , there holds  $\mathcal{Z}(\bar{r}) = \bar{r}^a u^{p+1}(\bar{r}) > 0$  and again  $r \mapsto e^{\frac{2p|M|}{p+1} \int_r^\infty |u_r|^{\frac{p-1}{p+1}} ds} \mathcal{Z}(r)$  is increasing with limit 0 as  $r \rightarrow \infty$ , a contradiction.

(b) If  $M = -\mu^*$  and  $\mathbf{T}_{\text{reg}} := \mathbf{T}_{\text{reg}}^{-\mu^*}$  intersects  $\mathcal{L}$  at some point between  $(0,0)$  and  $P_{-\mu^*}$ , then by continuity and transversality,  $\mathbf{T}_{\text{reg}}^M$  intersects  $\mathcal{L}$  at some point between  $(0,0)$  and  $P_{1,M}$  if  $-\mu^* - \epsilon \leq M < -\mu^*$  provided  $\epsilon > 0$  is small enough. This contradicts (a).  $\square$

**Proposition 4.7** *Let  $N \geq 3$  and  $1 < p < \frac{N}{N-2}$  or  $N = 2$  and  $p > 1$ .*

(i) *If  $M < -\mu^*$  and  $\mathbf{T}_{reg}$  does not converge to  $P_{2,M}$ , then it intersects the line  $\mathcal{L}$  at some point  $(x, y)$  with  $x > X_{2,M}$ .*

(ii) *If  $M = -\mu^*$  then  $\mathbf{T}_{reg}$  intersects  $\mathcal{L}$  at some point  $(x, y)$  with  $x > X_{-\mu^*}$  and leaves  $\mathbf{Q}$ ; there is no ground state.*

*Proof.* (i) Suppose  $M < -\mu^*$ . If  $\mathbf{T}_{reg}$  remains below  $\mathcal{L}$ , then  $x_t > 0$ ,  $y_t > 0$  and  $\mathbf{T}_{reg}$  converges to  $P_{1,M}$  or  $P_{2,M}$ . The first limit is excluded by Lemma 4.6 and the second by assumption. Hence  $\mathbf{T}_{reg}$  intersects  $\mathcal{L}$ . This intersection cannot occur between  $(0,0)$  and  $P_{1,M}$  and between  $P_{1,M}$  and  $P_{2,M}$  since the vector field  $\mathbf{H}$  is inward in the region  $\mathbf{B}$  on this segment, so it occurs at some point  $(x, y)$  with  $x > X_{2,M}$ .



(ii) Suppose  $M = -\mu^*$ . By Lemma 4.6,  $\mathbf{T}_{reg}$  intersects  $\mathcal{L}$  at some point  $(x, y)$  with  $x > X_{-\mu^*}$  and it enters in the region  $\mathbf{D}$  (note that the region  $\mathbf{B}$  is empty). Consider the slope  $\sigma = \frac{y}{x}$  of the trajectory  $\mathbf{T}_{reg}$ . As long as  $\mathbf{T}_{reg}$  stays under  $\mathcal{L}$ ,  $\sigma > \frac{2}{p-1}$ . Now we introduce the system (2.5), (2.6) and set  $\kappa = \frac{z}{\sigma} = \frac{x^{p+1}}{y^2}$ . Hence

$$\frac{\sigma_t}{\sigma} = \sigma + z - \mu^* \sigma^{\frac{p}{p+1}} z^{\frac{1}{p+1}} + 2 - N = \sigma \phi(\kappa) + 2 - N,$$

with

$$\phi(\kappa) = 1 + \kappa - \mu^* \kappa^{\frac{1}{p+1}}.$$

The function  $\phi$  achieves its minimum at  $\kappa^* = \left(\frac{\mu^*}{p+1}\right)^{\frac{p+1}{p}}$  and  $\phi(\kappa^*) = \frac{(N-2)(p-1)}{2}$ . Thus

$$\frac{\sigma_t}{\sigma} \geq (N-2) \left( \sigma - \frac{p-1}{2} \right) \geq 0.$$

Thus  $\sigma$  is nondecreasing. Therefore, after crossing  $\mathcal{L}$ ,  $\sigma > \frac{2}{p-1}$  and  $\mathbf{T}_{reg}$  cannot converge to  $P_{-\mu^*}$ .  $\square$

#### 4.1 Ground states and large solutions when $M < 0$

The following result extends [7, Theorem B'] to a larger class of parameters  $M$  in the radial case. We recall that  $\mu^*(1)$  and  $\mu^*(2)$  are defined in (1.20).

**Proposition 4.8** *Assume  $N \geq 2$ ,  $p > 1$  and  $M > -\mu^*(1)$ . Then there exists no positive solution of (1.13) in  $(a, b)$  for  $a < b$  tending to infinity at  $r = a$ .*

*Proof.* Without loss of generality we can assume  $a = 1$ . If  $M \geq -\mu^*(2)$  the result follows from [7, Theorem B'], hence we can assume  $-\mu^*(1) < M < -\mu^*(2)$ . We put  $m = |M| > 0$  and

$$\alpha = \alpha(m) = \left( p^p \left( \frac{m}{p+1} \right)^{p+1} - 1 \right) > 0.$$

If  $u$  satisfies (1.13) and blows-up at  $r = 1$ , then  $v = \ln u$  satisfies

$$\begin{aligned} -v_{rr} - \frac{N-1}{r} v_r + \alpha e^{(p-1)v} &\geq 0 && \text{in } (1, b) \\ \lim_{r \rightarrow 1} v(r) &= \infty. \end{aligned} \tag{4.21}$$

Up to changing  $b$ , we can assume that  $u(b) \geq 1$ , it follows that  $v$  is bounded from below on  $(1, b)$  by the solution of

$$\begin{aligned} -w_{rr} - \frac{N-1}{r} w_r + \alpha e^{(p-1)w} &\geq 0 && \text{in } (1, b) \\ \lim_{r \rightarrow 1} w(r) = \infty & && v(b) = 0. \end{aligned} \tag{4.22}$$

It is classical, (see e.g. [21]) that  $\left| w(r) - \frac{2}{p-1} \ln \left( \frac{1}{r-1} \right) \right|$  remains bounded on  $(1, b)$ . Returning to the variable  $(x(t), y(t))$  solutions of (1.17), then  $x(t) \geq ct^{-\frac{2}{p-1}}$  on  $(0, \ln b)$ .

(i) We first observe that  $t^{\frac{2}{p-1}}x(t)$  and  $t^{\frac{p+1}{p-1}}y(t)$  remain bounded on  $(0, \ln b)$ . Indeed, by the equation satisfied by  $y$  we get

$$(e^{Kt}y(t))_t \geq -me^{Kt}|y(t)|^{\frac{2p}{p+1}} \geq -c|e^{Kt}y(t)|^{\frac{2p}{p+1}},$$

and there exists a sequence  $\{t_n\}$  converging to 0 such that  $y(t_n) \rightarrow \infty$ . Furthermore  $z(t) = e^{Kt}y(t)$  satisfies

$$-\left(\frac{p+1}{p-1}|z|^{-\frac{p-1}{p+1}}\operatorname{sgn}(z)\right)' \geq c_1 \implies |z(t)|^{-\frac{p-1}{p+1}}\operatorname{sgn}(z(t)) \leq c_2t \implies z(t) \geq c_3t^{-\frac{p+1}{p-1}}.$$

Therefore  $y(t) \geq c_3t^{-\frac{p+1}{p-1}}$ . Using the equation (2.1) we first observe that  $x_t$  remains negative in a right neighbourhood of 0 otherwise there would exist a sequence  $\{t_n\}$  decreasing to 0 where  $x_t(t_n) = 0$  and  $x_{tt}(t_n) \geq 0$  yielding

$$(x(t_n))^p - \frac{K}{p-1}x(t_n) - m\left(\frac{2}{p-1}\right)^{\frac{2p}{p+1}}(x(t_n))^{\frac{2}{p-1}} \leq 0,$$

which is impossible since  $x(t_n) \rightarrow \infty$ . Similarly  $y_t$  remains negative on some interval  $(0, \tau_1)$  otherwise there would exist a sequence  $\{t'_n\}$  decreasing to 0 such that  $y_t(t'_n) = 0$  and  $y_{tt}(t'_n) \geq 0$ . Since

$$y_{tt} = -Ky_t + px^{p-1}x_t - \frac{2mp}{p+1}y^{\frac{p-1}{p+1}}y_t,$$

and  $x_t \leq 0$  we derive a contradiction. Therefore  $y(t) \rightarrow \infty$  as  $t \rightarrow 0$ ,  $y_t \leq 0$  and

$$-Ky = x^p - my^{\frac{2p}{p+1}} \leq 0 \implies y(t) \geq c_4x(t)^{\frac{p+1}{2}},$$

which implies

$$x_t \leq \frac{2x}{p-1} - c_4x^{\frac{p+1}{2}} \leq c_5x^{\frac{p+1}{2}} \implies x(t) \leq c_6ct^{-\frac{2}{p-1}},$$

near  $t = 0$ . Using again (1.17) and the monotonicity of  $x(t)$  and  $y(t)$ ,

$$e^{-\frac{t}{p-1}}x\left(\frac{t}{2}\right) \geq \frac{(p-1)\left(e^{-\frac{t}{p-1}} - e^{-\frac{2t}{p-1}}\right)}{2}y(t) \geq c_6ty(t),$$

which yields  $0 \leq y(t) \leq c_7t^{-\frac{p+1}{p-1}}$  near  $t = 0$ .

(ii) Next we set  $\kappa(\tau) = (r-1)^{\frac{2}{p-1}}u(r)$  with  $\tau = \ln(r-1)$ . Then  $\kappa$  satisfies on  $(-\infty, \beta]$  for some  $\beta \in \mathbb{R}$

$$\begin{aligned} -\kappa_{\tau\tau} + \frac{p+3}{p-1}\kappa_{\tau} - \frac{2(p+1)}{(p-1)^2}\kappa + \frac{(N-1)e^{\tau}}{1+e^{\tau}}\left(\frac{2}{p-1}\kappa - \kappa_{\tau}\right) \\ - \kappa^p + m\left|\frac{2}{p-1}\kappa - \kappa_{\tau}\right|^{\frac{2p}{p-1}} = 0. \end{aligned} \tag{4.23}$$

and  $\kappa$  and  $\kappa_\tau$  remain bounded from above and from below on  $(-\infty, a]$ . Therefore the limit set  $\Sigma$  of the corresponding trajectory at  $-\infty$  is not empty and it is included in the limit set  $\tilde{\Sigma}$  at  $-\infty$ , of some trajectory of a nonnegative function satisfying the autonomous system

$$-\phi_{\tau\tau} + \frac{p+3}{p-1}\phi_\tau - \frac{2(p+1)}{(p-1)^2}\phi - \phi^p + m \left| \frac{2}{p-1}\phi - \phi_\tau \right|^{\frac{2p}{p-1}} = 0, \quad (4.24)$$

Which is precisely equation (2.1) in dimension 1. By the Poincaré-Bendixon theorem,  $\tilde{\Sigma}$  either contains an equilibrium or it is a limit cycle. If the limit set contains an equilibrium, say  $W$ , it is positive and satisfies

$$-\frac{2(p+1)}{(p-1)^2}W - W^p + m \left( \frac{2}{p-1}W \right)^{\frac{2p}{p-1}} = 0.$$

Since  $M > -\mu^*(1)$  the only nonnegative root is zero, which yields a contradiction. If the limit set is a cycle  $\gamma$ , it is a subset of  $\mathbf{Q}$ . This imply that there would exist an equilibrium in the region bordered by  $\gamma$ , contradiction. This ends the proof.  $\square$

**Corollary 4.9** *Assume  $N \geq 2$ ,  $p > 1$  and  $-\mu^*(1) < M \leq -\mu^*$ . Then any solution  $(x(t), y(t))$  of system (1.17) issued from a point  $(x_0, y_0) \in \mathbf{Q}$  and staying in  $\mathbf{Q}$  for  $t \in I_{(x_0, y_0)} \cap \mathbb{R}_-$  where  $I_{(x_0, y_0)}$  is the maximal interval of existence of this solution is defined on  $\mathbb{R}_-$  and is bounded therein.*

*Proof.* Consider any solution such that  $(x(0), y(0)) = (x_0, y_0) \in \mathbf{Q}$  and suppose its negative trajectory  $\mathbf{T}_-$  is defined on some maximal interval  $(\theta, 0]$  with  $\theta \in (-\infty, 0)$  and thus unbounded. We first suppose that  $t \mapsto x(t)$  is not monotone when  $t \rightarrow \theta$ . Then there exists a sequence  $\{t_n\}$  decreasing to  $\theta$  such that  $x_t(t_n) = 0$  and thus  $a_n = (x(t_n), y(t_n)) \in \mathcal{L}$ ,  $x_{tt}(t_n) = y_t(t_n) < 0$  and  $\lim_{t_n \rightarrow \theta} x(t_n) = \infty$ .

Consider now the regular trajectory  $\mathbf{T}_{reg}$ . If  $p \geq \frac{N}{N-2}$  then, from Lemma 2.5, either  $\mathbf{T}_{reg}$  converges to  $P_M$  when  $t \rightarrow \infty$ , or it crosses  $\mathcal{L}$  at a point  $(x^*, \frac{2}{p-1}x^*)$  with  $x^* > X_M$ . If  $n$  is such that  $x^* < x(t_n)$  we get a contradiction: indeed for  $t < t_{n-1}$ ,  $\mathbf{T}_-$  stays in the region bordered from above by  $\mathcal{L}$  and  $\mathbf{T}_{reg}$ , so it cannot intersect  $\mathcal{L}$  at  $a_n$ .

If  $1 < p < \frac{N}{N-2}$ , we infer the same contradiction using Proposition 4.7. Therefore  $x(t)$  decreases monotonically to  $\infty$  when  $t \rightarrow \theta$ . Since  $M > -\mu(1)$  we derive a contradiction.

Hence  $\inf I_{(x_0, y_0)} = -\infty$ . By the same reasons as above,  $t \mapsto x(t)$  is monotone decreasing and  $x(t) \rightarrow \infty$  when  $t \rightarrow -\infty$ , and  $y(t) > \frac{2}{p-1}x(t) \rightarrow \infty$ . Moreover there exists  $\bar{t} < 0$  such that  $y_t(t) \leq 0$ : indeed, if for some  $t_0 < \bar{t}$ ,  $y_t(t_0) > 0$ , then  $(x(t_0), y(t_0)) \in \mathbf{D}$  and necessarily  $(x(t), y(t))$  remains in  $\mathbf{D}$  for  $t \leq t_0$  because  $x_t(t) < 0$  implies that  $(x(t), y(t)) \in \mathbf{D} \cup \mathbf{A}$  and the backward trajectory cannot cross the curve  $\mathcal{C}$  where  $y_t = 0$ ; now this implies that  $y_t(t) > 0$  for  $t \leq t_0$ , a contradiction. Therefore  $y_t(t) \leq 0$  for  $t \leq \bar{t}$ . Then

$$|M| y^{\frac{2p}{p+1}} \geq x^p - |K| y \geq x^p - \frac{|M|}{2} y^{\frac{2p}{p+1}} - c_M.$$

Since  $x(t) \rightarrow \infty$ , we deduce for large  $|t|$ ,

$$\frac{|M|}{2} y^{\frac{2p}{p+1}} \geq \frac{1}{2} x^p.$$

Returning to the system (1.17), we get for large  $|t|$

$$x_t = \frac{2}{p-1}x - y \leq \frac{2}{p-1}x - |M|^{-\frac{2p}{p+1}} x^{\frac{p+1}{2}} \leq -c^2 x^{\frac{p+1}{2}}.$$

Since  $\frac{p+1}{2} > 1$ , it is straightforward to check by integration that a positive function  $x$  satisfying the above differential inequality cannot be defined on an interval unbounded from below, which ends the proof.  $\square$

**Proposition 4.10** *Assume  $p > 1$  and  $-\mu^*(1) < M < -\mu^*(1) + \epsilon$  for  $\epsilon > 0$  small enough. Then there exists a ground state and (4.18) still holds.*

*Proof.* Assume  $-\mu^*(1) < M < -\mu^*(1) + \epsilon$  and the regular solution  $u$  is not a ground state. Then using the notation of Lemma 4.4 there exists  $r_0 > 0$  such that  $G(r_0) = 0$  and  $G'(r_0) \geq 0$ . Hence  $u_r^2(r_0) = \frac{2p}{p+1}u_r^{p+1}(r_0) \neq 0$ , thus

$$\frac{N-1}{r_0} \leq |u_r(r_0)|^{\frac{p-1}{p+1}} (\mu^*(1) + M).$$

Put  $t_0 = \ln r_0$ , then  $y^2(t_0) = \frac{2p}{p+1}x^{p+1}(t_0)$  and  $N-1 \leq |y(t_0)|^{\frac{p-1}{p+1}} (\mu^*(1) + M)$ . Equivalently

$$y(t_0) \geq \left(\frac{N-1}{\epsilon}\right)^{\frac{p+1}{p-1}} \quad \text{hence} \quad x(t_0) \geq \left(\frac{2p}{p+1}\right)^{\frac{1}{p+1}} \left(\frac{N-1}{\epsilon}\right)^{\frac{2}{p-1}}. \quad (4.25)$$

The curve  $\{(x, y) : y^2 = \frac{2p}{p+1}x^{p+1}\}$  cuts  $\mathcal{L}$  at a unique  $S_0 = (x_0, y_0) \in \mathbf{Q}$  and  $y_0^{p-1} = \frac{p+1}{2p} \left(\frac{2}{p-1}\right)^{p+1}$ . If  $M = -\mu^*(1)$  and  $p \geq \frac{N}{N-2}$  (resp.  $1 < p < \frac{N}{N-2}$ ), then  $X_{-\mu^*(1)} > x_0$  (resp.  $X_{2, -\mu^*(1)} > x_0$ ). Indeed this follows from (2.16). In the same way, if  $1 < p < \frac{N}{N-2}$ , then  $X_{1, -\mu^*(1)} < x_0$ . These configurations still hold if  $\epsilon$  is small enough, i.e.  $X_{-\mu^*(1)+\epsilon} > x_0$  if  $p \geq \frac{N}{N-2}$  and  $X_{1, -\mu^*(1)+\epsilon} < x_0 < X_{2, -\mu^*(1)+\epsilon}$  if  $1 < p < \frac{N}{N-2}$ . However, the regular trajectory associated to  $M$ ,  $\mathbf{T}_{reg}^M$  has a unique intersection with  $\mathcal{L}$ , at a point  $(x_{reg}(t_1), y_{reg}(t_1))$  where  $x_{reg}(t_1)$  is maximal, and either  $p \geq \frac{N}{N-2}$  and  $x_{reg}(t_1) > X_{-\mu^*(1)+\epsilon}$ , or  $1 < p < \frac{N}{N-2}$  and  $x_{reg}(t_1) > X_{1, -\mu^*(1)+\epsilon}$ .

In both cases  $t_1 < t_0$ ,  $y_{reg}(t_1) = \frac{2}{p-1}x_{reg}(t_1)$  and  $x_{reg}(t_1) > \left(\frac{p+1}{2p}\right)^{\frac{1}{p+1}} \left(\frac{N-1}{\epsilon}\right)^{\frac{2}{p-1}}$ . Now, for  $M > -\mu^*(1)$  the trajectory  $\mathbf{T}_{reg}^M$  remains above  $\mathbf{T}_{reg}^{-\mu^*(1)}$  as long as they remain below  $\mathcal{L}$  by Lemma 2.9. Then we encounter two possibilities: either  $\mathbf{T}_{reg}^{-\mu^*(1)}$  converges to  $P_{-\mu^*(1)}$  (or  $P_{2, -\mu^*(1)}$  if  $1 < p < \frac{N}{N-2}$ ) with  $x(t)$  and  $y(t)$  increasing, or  $\mathbf{T}_{reg}^{-\mu^*(1)}$  crosses  $\mathcal{L}$  at some point  $(\tilde{x}, \tilde{y})$  depending only on  $N$  and  $p$ . Both possibilities are ruled out if  $\epsilon$  is small enough. Hence  $\mathbf{T}_{reg}^M$  is a ground state if  $M \in (-\mu^*(1), -\mu^*(1) + \epsilon]$  for  $\epsilon > 0$  small enough and  $G$  remains negative.  $\square$

## 4.2 The case $M < 0$ , $N \geq 3$ and $p \geq \frac{N+2}{N-2}$

**Theorem 4.11** *Assume  $M < 0$ ,  $N \geq 3$  and  $p \geq \frac{N+2}{N-2}$ . Then there exist ground states  $u$ . Moreover they satisfy  $u(r) \sim U_M(r)$  as  $r \rightarrow \infty$ .*

*Proof.* Let  $(x_{reg}(t), y_{reg}(t))$  be the regular solution issued from  $(0, 0)$ . By Lemma 2.8 the function  $\mathcal{V}$  defined in (2.38) is increasing. Since it vanishes at  $t = -\infty$  it is positive. If there exists some  $t_0$  such that  $x_{reg}(t_0) = 0$  then  $\mathcal{V}(t_0) < 0$ , which is impossible, hence  $x_{reg}(t) > 0$  for all  $t$ . Thus  $\mathbf{T}_{reg}$  is a ground state; it is bounded by Proposition 2.4, and it cannot converge to  $(0, 0)$  since  $\mathcal{V}(t) > \mathcal{V}(-\infty) = 0$ . From Lemma 4.3 there is no cycle, hence  $\mathbf{T}_{reg}$  converges to  $P_M$  which is a sink.  $\square$

*Remark.* The existence of a ground state was already obtained in [19] with the use of the function  $\mathcal{Z}$  defined in (2.42).

### 4.3 The case $M < 0$ , $N \geq 3$ and $\frac{N}{N-2} < p < \frac{N+2}{N-2}$

In what follows we give an improvement of [22, Theorem A] in which it is shown that in this range of exponent there exists some  $\omega > 0$  such that  $-\omega \leq -M < 0$  there exists no ground state. The expression of  $\omega$  is explicit (and not simple).

**Theorem 4.12** *Assume  $N \geq 3$  and  $\frac{N}{N-2} < p < \frac{N+2}{N-2}$ . If  $\overline{M} \leq M < 0$  there exists no ground state. Moreover there exists a unique, up to similarity transformation, positive solution  $u$  satisfying  $u(r) \sim U_M(r)$  as  $r \rightarrow 0$  such that  $u(r) \sim cr^{2-N}$  ( $c > 0$ ) as  $r \rightarrow \infty$ .*

*Proof.* Suppose that there exists such a ground state, then  $\mathbf{T}_{reg}$  remains in  $\overline{\mathbf{Q}}$ . By Proposition 4.1,  $P_M$  is either a source if  $\overline{M} < M$  or a weak source if  $\overline{M} = M$ , and by Lemma 4.3 there exists no cycle surrounding  $P_M$ . By Lemma 2.5,  $(x_{reg}(t), y_{reg}(t))$  converges to  $(0, 0)$  when  $t \rightarrow \infty$ , hence  $\mathbf{T}_{reg}$  is a homoclinic orbit equivalently  $\mathbf{T}_{reg} = \mathbf{T}_{st} = \mathbf{T}_{unst}$ . Now

$$\text{Trace } D\mathbf{H}(0, 0) = \frac{2}{p-1} - K = -L > 0.$$

Hence, by [17, Th 9.3.3] the homoclinic orbit is repelling. Since  $P_M$  is also repelling, we derive a contradiction because any trajectory issue from  $\mathcal{B}$  must converge to  $\mathbf{T}_{reg}$ . Hence  $\mathbf{T}_{reg}$  intersects the axis  $\{x = 0\}$  for some positive  $y_1 > 0$ , and there exists no ground state. We denote by  $\mathcal{O}$  the region of  $\mathbf{Q}$  delimited by the regular trajectory  $\mathbf{T}_{reg}$  and the segment  $(0, y) : 0 < y < y_1$ . It is negatively invariant. The stable trajectory  $\mathbf{T}_{st} = \{(x_{st}, y_{st})\}$  of  $(0, 0)$  satisfies  $x_{st}(t) = ce^{-Kt}(1 + o(1))$  and  $y_{st}(t) = c(N-2)e^{-Kt}(1 + o(1))$  when  $t \rightarrow -\infty$ , thus it remains in  $\mathcal{O}$ . Because there are no cycle in  $\mathcal{O}$ , it must converge to  $P_M$ , hence the corresponding  $u_{st}$  is equivalent to  $U_M$  near  $r = 0$ , which ends to proof.  $\square$

*Remark.* In the previous theorem the positive solution  $u$  satisfying  $u(r) \sim U_M(r)$  as  $r \rightarrow 0$  such that  $u(r) \sim cr^{2-N}$  ( $c > 0$ ) as  $r \rightarrow \infty$  is the stable trajectory  $\mathbf{T}_{st}$ . It is a heteroclinic orbit connecting  $(0, 0)$  to  $P_M$ . We conjecture that in the case  $p = \frac{N}{N-2}$  the non existence of a ground state still holds and that there exists a unique solution  $u$  such that satisfying  $u(r) \sim U_M(r)$  as  $r \rightarrow 0$  such that  $u(r) \sim cr^{2-N}(\ln r)^{\frac{2-N}{2}}$  ( $c > 0$ ) as  $r \rightarrow \infty$ .

The expression of the result presents some similarity with Theorem 3.7 in the case  $M > 0$ . However a new type of difficulty appears: in order to define properly an intersection function expressing the distance between some trajectories as in [4], we need to find some values of the parameter  $M$  for which there exists a ground state, and all the trajectories in  $\mathbf{Q}$  are bounded.

It is not the case when  $M < -\mu^*(1)$  even if there exists a ground state by Lemma 4.4 but we can easily prove that there exist large solutions. So we need to prove that for  $M = -\mu^*(1) + \epsilon$  there exist a ground state and all the trajectories in  $\mathbf{Q}$  are bounded. This is the object of Proposition 4.10 and Proposition 4.8.

**Theorem 4.13** *Let  $N \geq 3$  and  $\frac{N}{N-2} < p < \frac{N+2}{N-2}$ . There exist positive constants  $\tilde{\mu}_{min} \leq \tilde{\mu}_{max}$ , verifying  $\bar{\mu} < \tilde{\mu}_{min} \leq \tilde{\mu}_{max} < \mu^*(1)$  such that*

(i) *If  $M < -\tilde{\mu}_{max}$  there exist ground states  $u$  such that  $u(r) \sim U_M(r)$  or undulating around  $U_M(r)$  when  $r \rightarrow \infty$ .*

(ii) *If  $M = -\tilde{\mu}_{max}$  or If  $M = -\tilde{\mu}_{min}$  there exist ground states  $u$  such that  $u(r) \sim cr^{2-N}$ ,  $c > 0$ , when  $r \rightarrow \infty$ .*

(iii) *If  $-\tilde{\mu}_{min} < M < 0$  there exists no ground state. Furthermore there exist singular solutions  $u$  undulating around  $U_M(r)$  when  $r > 0$  and singular solutions  $u$  undulating around  $U_M(r)$  when  $r \rightarrow 0$  and such that  $u(r) \sim cr^{2-N}$ ,  $c > 0$ , when  $r \rightarrow \infty$ .*

*Proof.* Recall that  $\bar{M} = -\bar{\mu}$ . First we show that if  $-\mu^*(1) < M < \bar{M}$ , the stable trajectory  $\mathbf{T}_{st} := \mathbf{T}_{st}^M$  either has a limit cycle around  $P_M$  or does not stay in  $\mathbf{Q}$ . If we assume that it stays in  $\mathbf{Q}$ , then it is bounded by Corollary 4.9. Since at  $-\infty$  it cannot converge to  $P_M$  which is a sink by Proposition 4.1, it admits a alpha-limit cycle which is a closed orbit around  $P_M$ .

For  $-\mu^*(1) < M < \bar{M}$  we denote by  $P_{st}^M = (x_{st}^M, x_{st}^M)$  the farthest point of the closure  $\overline{\mathbf{T}_{st}}$  of the trajectory  $\mathbf{T}_{st}$  belonging to the line  $\mathcal{L}$ , i.e. the points with the largest  $x$  (and  $y$ )-coordinate. We also denote by  $P_{reg}^M = (x_{reg}^M, x_{reg}^M)$  the farthest point of the intersection of  $\mathcal{L}$  with the closure  $\overline{\mathbf{T}_{reg}}$  of  $\mathbf{T}_{reg} := \mathbf{T}_{reg}^M$ . More precisely since either  $\mathbf{T}_{st}$  leaves  $\mathbf{Q}$  or has an alpha-limit cycle around  $P_M$ , in that case  $P_{st}^M$  corresponds to the last intersection of  $\mathbf{T}_{st}$  and  $\mathcal{L}$ . If  $\mathbf{T}_{reg}$  converges to  $P_M$  monotonically, then  $P_{reg}^M = P_M \in \overline{\mathbf{T}_{reg}}$ , while if this convergence is not monotone, or if  $\mathbf{T}_{reg}$  admits a omega-limit cycle around  $P_M$ , or if  $\mathbf{T}_{reg}$  leaves  $\mathbf{Q}$ ,  $P_{reg}^M$  is the first intersection of  $\mathbf{T}_{reg}$  with  $\mathcal{L}$ . Both the functions  $M \mapsto x_{st}^M$  and  $M \mapsto x_{reg}^M$  are continuous, either by transversality argument or by the continuity of  $M \mapsto X_M$ . Hence the function  $M \mapsto g(M) = x_{reg}^M - x_{st}^M$  is continuous. For  $M < \bar{M}$  we encounter three possibilities:

(i)  $x_{reg}^M = X_M$  or  $\mathbf{T}_{st}$  converges to  $X_M$  non-monotonically, or  $\mathbf{T}_{reg}$  has a omega-limit cycle around  $P_M$ . In such a case  $\mathbf{T}_{st}$  does not stay in  $\mathbf{Q}$ , thus  $g(M) < 0$ .

(ii)  $\mathbf{T}_{reg}$  does not stay in  $\mathbf{Q}$ , then  $\mathbf{T}_{st}$  belongs to the region of  $\mathbf{Q}$  bordered by  $\mathbf{T}_{reg}$  and the axis  $\{x = 0\}$ . Then thus  $g(M) > 0$ .

(iii)  $g(M) = 0$ , then  $\mathbf{T}_{st} = \mathbf{T}_{reg}$  is a homoclinic orbit.

If  $M = \bar{M}$  there exists no ground state by Theorem 4.12 hence, by continuity, this still holds for  $\bar{M} - \epsilon < M < \bar{M}$  for  $\epsilon > 0$  small enough and then  $g(\bar{M}) > 0$ . By Proposition 4.10, if  $-\mu^*(1) < M < -\mu^*(1) + \epsilon$ , there exists a ground state, hence  $g(M) < 0$ . Since  $g$  is continuous there exists  $M \in (-\mu^*(1), \bar{M})$  such that  $g(M) = 0$ .

If we define

$$\begin{aligned} \tilde{\mu}_{min} &= \min\{|M| \in (\bar{\mu}, \mu^*(1)) : g(M) > 0\} \\ \tilde{\mu}_{max} &= \max\{|M| \in (\bar{\mu}, \mu^*(1)) : g(M) < 0\} \end{aligned} \quad (4.26)$$

then the trajectories  $\mathbf{T}_{reg}$  corresponding to  $M = -\tilde{\mu}_{min}$  and  $M = -\tilde{\mu}_{max}$  are homoclinic and they satisfy the statements (ii) of Theorem 4.13 and the conclusion follows.  $\square$

The proof of Theorem B, B' follows from Theorem 4.12 and Theorem 4.13.

#### 4.4 The case $M < 0$ and $1 < p < \frac{N}{N-2}$

We present first a general existence result of singular solutions.

**Proposition 4.14** *If  $N \geq 3$ ,  $1 < p < \frac{N}{N-2}$  and  $M < -\mu^*$  there exists positive singular solutions  $u$  such that  $r^{N-2}u(r) \rightarrow c$  for some  $c > 0$  when  $r \rightarrow 0$  and  $r^{\frac{2}{p-1}}u(r) \rightarrow X_{1,M}$  when  $r \rightarrow \infty$ .*

*Proof.* By Proposition 4.2  $P_{1,M}$  is a saddle point of system (1.17). By the remark after this proposition there exist two stable trajectories  $\mathbf{T}_{st}^{1,j}$ ,  $j = 1, 2$  converging to  $P_{1,M}$  as  $t \rightarrow \infty$ ; the trajectory  $\mathbf{T}_{st}^{1,1}$  is locally below the line  $\mathcal{L}$  (see Proposition 4.7), hence it belongs to the region  $\mathbf{C}$  for  $t > t_0$ . By Proposition 4.7 either the regular trajectory  $\mathbf{T}_{reg}$  converges to  $P_{2,M}$  or it crosses the line  $\mathcal{L}$  beyond  $P_{2,M}$ . Hence  $\mathbf{T}_{st}^{1,1}$  cannot intersect  $\mathbf{T}_{reg}$ , and is trapped when  $t$  decreases in the region  $\mathbf{C}$  and the curve  $\mathbf{T}_{reg}$ . Thus it converges to  $(0, 0)$  when  $t \rightarrow -\infty$ . Because  $(0, 0)$  is a source (see Section 3.4.1) with one fast trajectory  $\mathbf{T}_{reg}$ , which satisfies  $\lim_{t \rightarrow -\infty} e^{-\frac{2}{p-1}t} x_{reg}(t) = u(0)$ , the trajectory  $\mathbf{T}_{st}^{1,1}$  is a slow one and it satisfies  $\lim_{t \rightarrow -\infty} e^{(N-2-\frac{2}{p-1})t} x_{reg}(t) = c$ .  $\square$

*Remark.* Under the assumptions of Proposition 4.14 the trajectory  $\mathbf{T}_{unst}^{1,4}$  leaves  $\mathbf{Q}$  since in this region it stays in the sector  $\{(x, y) : H_j(x, y) < 0\}$  for  $j = 1, 2$ , and this sector contains no stable equilibrium. The result holds also if  $M = -\mu^*$ .

**Theorem 4.15** *Let  $N \geq 3$  and  $1 < p < \frac{N}{N-2}$ . Then*

- (i) *if  $\bar{M} \leq M \leq 0$  there exists no ground state.*
- (ii) *if  $\bar{M} \leq M < -\mu^*$  there exists a positive singular solution  $u$ , unique up to scaling, such that  $u(r) \sim U_{2,M}(r)$  as  $r \rightarrow 0$ ,  $u(r) \sim U_{1,M}(r)$  as  $r \rightarrow \infty$  and  $u(r) > U_{1,M}(r)$  for any  $r > 0$ .*

*Proof.* (i) Let  $\bar{M} \leq M \leq 0$ . Suppose that there exists a ground state  $\mathbf{T}_{reg}$ , then  $\bar{M} \leq M < -\mu^*$  by Lemma 4.5, and by Proposition 4.7. Furthermore this trajectory is bounded by Proposition 2.4. Moreover  $P_{2,M}$  is a source or a weak source and there exists no cycle surrounding it by Proposition 4.2 and Lemma 4.3. Hence  $\mathbf{T}_{reg}$  converges to an equilibrium which cannot be  $(0, 0)$  neither  $P_{2,M}$ . So, from Proposition 4.7, it converges to  $P_{1,M}$  from above  $\mathcal{L}$  as  $t \rightarrow \infty$ . Since  $P_{1,M}$  is a saddle point  $\mathbf{T}_{reg}$  must coincide with the stable trajectory  $\mathbf{T}_{st}^{1,2}$  of this point. Therefore the region bordered by  $\mathbf{T}_{reg}$ ,  $\mathbf{T}_{st}^{1,1}$  and  $(0, 0)$  is invariant and it contains only one source equilibrium and no cycle around  $P_{2,M}$  by Lemma 4.3. Any trajectory starting from this region must converge to  $P_{1,M}$  which is impossible. Hence  $\mathbf{T}_{reg}$  is not a ground state. Since the trajectory  $\mathbf{T}_{reg}$  intersects the axis  $\{x = 0\}$ , the trajectory  $\mathbf{T}_{st}^{1,2}$  which converges to  $P_{1,M}$  at infinity from above  $\mathcal{L}$  is trapped in the region bordered by  $\mathbf{T}_{reg}$  and the semi-axis  $\{(0, y) : y > 0\}$  which is negatively invariant. As there is no cycle in this region, it converges to  $P_{2,M}$  when  $t \rightarrow \infty$ . To this trajectory corresponds a solution  $u$  of (1.13) which satisfies  $u(r) \sim U_{2,M}(r)$  when  $r \rightarrow \infty$  and  $u(r) \sim U_{1,M}(r)$  when  $r \rightarrow 0$ . Furthermore  $u(r) > U_{1,M}(r)$  for all  $r > 0$ .  $\square$

The next result extends Proposition 4.14 to the case  $M = -\mu^*$ .

**Proposition 4.16** *Let  $N \geq 3$  and  $1 < p < \frac{N}{N-2}$ . If  $M = -\mu^*$  there exists positive solutions  $u$  satisfying  $u(r) \sim cr^{2-N}$  when  $r \rightarrow 0$  for some  $c > 0$  and  $u(r) \sim U_{-\mu^*}(r)$  when  $r \rightarrow \infty$ .*

*Proof.* Since  $N \geq 3$ , there still exist two unstable trajectories  $\mathbf{T}_{unst}^{1,j}$  ( $j=3,4$ ) starting from  $P_{1,M}$  with slope  $N - 2$ ;  $\mathbf{T}_{unst}^{1,4}$ , which is above  $\mathcal{L}$ , leaves  $\mathbf{Q}$  through the semi-axis  $\{(0, y) : y > 0\}$  (see the remark above). From Proposition 4.7 the trajectory  $\mathbf{T}_{reg}$  crosses  $\mathcal{L}$  at a unique point  $A_{reg} = (x_{reg}, \frac{2}{p-1}x_{reg})$  with  $x_{reg} > X_{-\mu^*}$  and leaves  $\mathbf{Q}$  through the semi-axis  $\{(0, y) : y > 0\}$  and its exit point is above the exit point of  $\mathbf{T}_{unst}^{1,4}$ . Hence the other unstable trajectory  $\mathbf{T}_{unst}^{1,3}$  which is trapped in the region of  $\mathbf{Q}$  bordered by  $\mathbf{T}_{unst}^{1,4}$ ,  $\mathbf{T}_{reg}$  and the semi-axis  $\{(0, y) : y > 0\}$ , either converges to  $P_{-\mu^*}$  or leaves  $\mathbf{Q}$  crossing the semi-axis  $\{(0, y) : y > 0\}$ . Necessarily it intersects the line  $\mathcal{L}$  at some point  $\bar{A} = (\bar{x}, \frac{2}{p-1}\bar{x})$  with  $\bar{x} > X_{-\mu^*}$  and enters the region  $\mathbf{D}$ . As long as it stays above  $\mathcal{L}$ , as in the Proposition 4.7, there holds  $\sigma_t > 0$  (with the notations from this proposition). Thus it cannot converges to  $P_{-\mu^*}$ . Therefore  $\mathbf{T}_{unst}^{1,3}$  crosses the semi-axis  $\{(0, y) : y > 0\}$  between the exit points of  $\mathbf{T}_{unst}^{1,4}$  and  $\mathbf{T}_{reg}$ .

Let  $\mathcal{R}$  be the open connected region of  $\mathbf{Q}$  below the line  $\mathcal{L}$  and bordered by  $\mathbf{T}_{reg}$  and  $\mathbf{T}_{unst}^{1,3}$ . If  $(\tilde{x}, \tilde{y}) \in \mathcal{R}$  and  $\tilde{\mathbf{T}}$  is the trajectory through this point, we have three possibilities:

- (i) Either  $\tilde{\mathbf{T}}$  leaves  $\mathcal{R}$  crossing  $\mathcal{L}$  between  $(0, 0)$  and  $P_{-\mu^*}$ ,
- (ii) Or  $\tilde{\mathbf{T}}$  leaves  $\mathcal{R}$  crossing  $\mathcal{L}$  between  $P_{-\mu^*}$  and  $\bar{A}$ ,
- (iii) Or  $\tilde{\mathbf{T}}$  converges to  $P_{-\mu^*}$ .

The set of points satisfying (i) or (ii) are non-empty, disjoint and open. Therefore the set of points satisfying (iii) is non-empty and the corresponding trajectory  $\tilde{\mathbf{T}}$  converges to  $P_{-\mu^*}$  when  $t \rightarrow \infty$ . The backward trajectory remains in  $\mathcal{R}$  which is negatively invariant. Since there is no fixed point in this region, it converges to  $(0, 0)$  when  $t \rightarrow -\infty$  and it is a slow trajectory of this point, which ends the proof.  $\square$

Next we describe the behaviour of the positive solutions for  $M < -\bar{\mu}$ . However, in order to use the method introduced in the proof of Theorem 4.13 we are confronted to another difficulty namely that there can exist homoclinic trajectories at  $P_{1,M}$ .

**Theorem 4.17** *Let  $N \geq 3$ ,  $M < 0$  and  $1 < p < \frac{N}{N-2}$ . Then there exist positive real numbers  $\bar{\mu} < \hat{\mu}_{min} \leq \hat{\mu}_{max} < \tilde{\mu}_{min} \leq \tilde{\mu}_{max} < \mu^*(1)$  with the following properties*

- 1- for  $\bar{\mu} < |M| < \tilde{\mu}_{min}$  there is no radial ground state;
- 2- for  $|M| = \tilde{\mu}_{min}$  or  $|M| = \tilde{\mu}_{max}$  there exist ground states  $u$  satisfying  $u(r) \sim U_{1,M}(r)$  when  $r \rightarrow \infty$ ;
- 3- for  $|M| > \tilde{\mu}_{max}$  there exist ground states either such that  $u(r) \sim U_{2,M}(r)$  when  $r \rightarrow \infty$  or undulating around  $U_{2,M}(r)$  when  $r \rightarrow \infty$ .

Moreover,

- 4- For  $\bar{\mu} < |M| < \hat{\mu}_{min}$  there exist solutions  $u$ , necessarily singular, undulating around  $U_{2,M}$  as  $r \rightarrow 0$  and  $u(r) \sim U_{1,M}(r)$  when  $r \rightarrow \infty$  and solutions  $u$  undulating around  $U_{2,M}$  on  $(0, \infty)$ ;
- 5- for  $|M| = \hat{\mu}_{min}$  or  $|M| = \hat{\mu}_{max}$  there exists a solution  $u \neq U_{1,M}$  such that  $u(r) \sim U_{1,M}(r)$  both when  $r \rightarrow 0$  and  $r \rightarrow \infty$ ;
- 6- for  $\hat{\mu}_{max} < |M| < \tilde{\mu}_{min}$  there exists a solution  $u$  such that  $u(r) \sim cr^{2-N}$  as  $r \rightarrow 0$  and either  $u(r) \sim U_{2,M}(r)$  or undulating around  $U_{2,M}$  when  $r \rightarrow \infty$ .

*Proof.* Step 1. For  $M = \bar{M}$  we know the behaviour of the solutions from Theorem 4.15. The trajectories  $\mathbf{T}_{reg}$ ,  $\mathbf{T}_{unst}^{1,3}$  and  $\mathbf{T}_{unst}^{1,4}$  leave  $\mathbf{Q}$  on  $\{(0, y) : y > 0\}$  with transverse intersections, and



$\mathbf{T}_{unst}^{1,2}$  connects  $P_{2,M}$  to  $P_{1,M}$ . This transversality property is also true for the corresponding trajectories with parameter  $\overline{M} - \epsilon_0 \leq M \leq \overline{M}$  for  $\epsilon_0 > 0$  small enough. Let  $\overline{M}_\epsilon = \overline{M} - \epsilon$  where  $0 < \epsilon \leq \epsilon_0$ . Then  $P_{2,\overline{M}_\epsilon}$  is a sink by Proposition 4.2. By Proposition 4.7, the points  $P_{1,\overline{M}_\epsilon}$  and  $P_{2,\overline{M}_\epsilon}$  belong to the region  $\mathcal{R}_\epsilon$  bordered by  $\mathbf{T}_{reg}$ , and the semi-axis  $\{(0, y) : y > 0\}$ . The trajectory  $\mathbf{T}_{st}^{1,2}$  which converges to  $P_{1,\overline{M}_\epsilon}$  at  $\infty$ , cannot converge to  $P_{2,\overline{M}_\epsilon}$  at  $-\infty$ . Hence it has a limit cycle around  $P_{2,\overline{M}_\epsilon}$ . Note that  $\mathbf{T}_{st}^{1,2}$  is included in the region bordered by  $\mathbf{T}_{unst}^{1,3}$  and  $\mathbf{T}_{unst}^{1,4}$ .

For  $\alpha > 0$  small enough set  $M^\alpha = -\mu^*(1) + \alpha$ . By Proposition 4.10 the corresponding regular trajectory  $\mathbf{T}_{reg}$  is a ground state. Since  $(0, 0)$  is a source and (4.18) holds, it cannot converge neither to  $(0, 0)$  nor to  $P_{1,M^\alpha}$  as in Lemma 4.4. Hence either it converges to  $P_{2,M^\alpha}$  at  $\infty$  or it admits a limit cycle around. Next,  $\mathbf{T}_{unst}^{1,3}$  is trapped in the positively invariant region bordered by  $\mathbf{T}_{st}^{1,1}$  which connects  $(0, 0)$  to  $P_{1,M}$ , the portion of the curve  $\mathcal{C}$  below  $\mathcal{L}$  (hence between  $P_{1,M}$  and  $P_{2,M}$ ) and  $\mathbf{T}_{reg}$  between  $(0, 0)$  and its second intersection with  $\mathcal{C}$ ; therefore, it converges to  $P_{2,M^\alpha}$  at  $\infty$  or it admits a limit cycle too. Finally consider the trajectory  $\mathbf{T}_{st}^{1,2}$  which tends to  $P_{1,M^\alpha}$  at  $\infty$ . It stays in the negatively invariant region  $\{(x, y) : y > \frac{2}{p-1}x \text{ or } x > X_{1,M}\}$ . If it stays in  $\mathbf{Q}$ , then the solution is defined on  $\mathbb{R}$  and the trajectory is bounded from Corollary 4.9. So, either it converges to a fixed point or it has a limit cycle around  $P_{2,M^\alpha}$  when  $t \rightarrow -\infty$ . This is impossible since it would intersect  $\mathbf{T}_{reg}$ . Hence for  $M = M^\alpha$ ,  $\mathbf{T}_{st}^{1,2}$  leaves  $\mathbf{Q}$  in finite time at  $(x(\bar{t}), 0)$  for some  $x(\bar{t}) > 0$ . Then there exist  $t_0 > t_1 > \bar{t}$  such that  $y_t(t_0) = 0$  and  $y(t_0)$  is the maximum of  $y$  on  $(\bar{t}, \infty)$  and  $(x(t_1), y(t_1)) \in \mathcal{L}$ . Hence  $x(t) \leq x(t_1)$  for  $\bar{t} \leq t \leq t_1$ .

*Step 2.* Next consider any  $M \in (-\mu^*(1), \overline{M})$ . In any case the trajectories  $\mathbf{T}_{st}^{1,2}$  and  $\mathbf{T}_{reg}$  are bounded as long as they stay in  $\mathbf{Q}$  from Corollary 4.9. Then we define  $(x_{reg}^M, y_{reg}^M)$  as the farthest point on  $\mathcal{L}$  belonging to the closure  $\overline{\mathbf{T}_{reg}}$  of  $\mathbf{T}_{reg}$ , and  $(x_{st}^M, y_{st}^M)$  as the farthest point on  $\mathcal{L}$  belonging to  $\overline{\mathbf{T}_{st}}$ .

Let  $A$  be the set of  $M \in (-\mu^*(1), \overline{M})$  such that there is no ground state, let  $B_1$  be the set of  $M \in (-\mu^*(1), \overline{M})$  such that there exists a ground state converging to  $P_{1,M}$  at  $\infty$  and let  $B_2$  be the set of  $M \in (-\mu^*(1), \overline{M})$  such that there exists a ground state converging to  $P_{2,M}$  or having a limit cycle at  $\infty$ . Then  $(-\mu^*(1), \overline{M}) = A \cup B_1 \cup B_2$ . Clearly  $\overline{M} \in A$  and  $M^\alpha \in B_2$ , furthermore if  $M \in A \cup B_1 \cup B_2$  we have three possibilities.

- Any  $M \in B_2$  has the same properties as  $M^\alpha$ , hence  $\mathbf{T}_{reg}$  is a ground state which converges to  $P_{2,M}$  or has a limit cycle around  $P_{2,M}$ ;  $\mathbf{T}_{unst}^{1,3}$  either converges to  $P_{2,M}$  or has a limit cycle around  $P_{2,M}$ ; and  $\mathbf{T}_{st}^{1,2}$  intersects  $\mathcal{L}$  at a last value  $t_1$ .

- If  $M \in A$ ,  $\mathbf{T}_{reg}$  is not a ground state and it leaves  $\mathbf{Q}$  through the semi-axis  $\{x = 0, y > 0\}$ ;  $\mathbf{T}_{st}^{1,2}$  and  $\mathbf{T}_{unst}^{1,3}$  are included in the region of  $\mathbf{Q}$  bordered by  $\mathbf{T}_{reg}$  and three configurations are possible:

A-(i) either  $\mathbf{T}_{st}^{1,2}$  has a limit cycle around  $P_{2,M}$  and  $\mathbf{T}_{unst}^{1,3}$  leaves  $\mathbf{Q}$

A-(ii) or  $\mathbf{T}_{unst}^{1,3}$  converges to  $P_{2,M}$  or has a limit cycle around.

A-(iii) or  $\mathbf{T}_{st}^{1,2} = \mathbf{T}_{unst}^{1,3}$  which means this trajectory is homoclinic with respect to  $P_{1,M}$ .

Note that  $\overline{M}_\epsilon$  satisfies A-(i).

As in the proof of Theorem 4.13 the mappings  $M \mapsto x_{reg}^M$  and  $M \mapsto x_{1,2}^M$  are continuous. We set  $g(M) = x_{reg}^M - x_{1,2}^M$ . If  $M \in A$ , then  $g(M) < 0$  and if  $M \in B_2$ , then  $g(M) > 0$ . Since  $g$  is continuous there exists  $M \in (-\mu^*(1), -\overline{\mu})$  such that  $g(M) = 0$ . Hence  $\mathbf{T}_{reg} = \mathbf{T}_{st}^{1,2}$  and  $B_1 \neq \emptyset$ .

More precisely we can define  $\bar{\mu} < \tilde{\mu}_{min} < \tilde{\mu}_{max} < \mu^*(1)$  such that  $\tilde{\mu}_{min} \in B_1$ ,  $\tilde{\mu}_{max} \in B_1$ . If  $|M| < \tilde{\mu}_{min}$ ,  $g(M) > 0$  and there is no ground state. If  $|M| > \tilde{\mu}_{max}$ ,  $g(M) < 0$  and there exists a ground state  $u$  such that  $r^{\frac{2}{p-1}}u(r) \rightarrow X_{2,M}$  or such that  $r^{\frac{2}{p-1}}u(r)$  is turning around  $X_{2,M}$  as  $r \rightarrow \infty$ .

Finally we consider the relative position of  $\mathbf{T}_{st}^{1,2}$  and  $\mathbf{T}_{unst}^{1,3}$  when  $M \in [-\tilde{\mu}_{min}, -\bar{\mu}]$ . In any of the three situations A-(i), A-(ii) and A-(iii),  $\mathcal{L}$  intersects  $\mathbf{T}_{unst}^{1,3}$  at a first point of x-coordinate  $x_{1,3}^M$  and  $\mathbf{T}_{st}^{1,2}$  at a last point of x-coordinate  $x_{1,2}^M$ . We define the continuous function  $M \mapsto h(M) = x_{1,3}^M - x_{1,2}^M$ . Then  $h(\bar{M}_\epsilon) > 0$  and  $h(-\tilde{\mu}_{min}) = x_{1,3}^M - x_{reg}^M < 0$ . Hence there exists at least one  $M \in (-\tilde{\mu}_{min}, -\bar{\mu})$  where  $h(M) = 0$  and for such a  $M$ , A-(iii) holds. Then we define  $\hat{\mu}_{min}$  and  $\hat{\mu}_{max}$  in  $(\bar{\mu}, \tilde{\mu}_{min})$  such that A-(iii) holds if  $|M| = -\hat{\mu}_{min}$  or if  $|M| = -\hat{\mu}_{max}$ . Hence, if  $-\hat{\mu}_{min} < M < -\bar{\mu}$ ,  $h(M) > 0$  and the trajectory  $\mathbf{T}_{unst}^{1,3}$  starts from  $P_{1,M}$  and converges to  $P_{2,M}$  or has a limit cycle around  $P_{2,M}$ . If  $-\hat{\mu}_{max} < M < -\tilde{\mu}_{min}$ ,  $h(M) < 0$  and the trajectory  $\mathbf{T}_{st}^{1,2}$  starts from  $(0, 0)$  with the slope  $N - 2$  and converges to  $P_{1,M}$  when  $t \rightarrow \infty$ .  $\square$

*Proofs of Theorem C and C'.* They are a consequence of Proposition 4.14, Theorem 4.15, Proposition 4.16 and Theorem 4.17.

*Remark.* It is an open problem whether the cycles which may exist for some  $M$  are unique or not. It is a numerical evidence that it holds if  $M > 0$ , but unclear if  $M < 0$ .

#### 4.5 The case $M < 0$ , $N = 2$ and $p > 1$

A first difficulty in this case comes from the fact that there exist singular solutions  $u$  with a logarithmic blow-up. The main difficulty comes from the equality of  $\mu^*(2)$  and  $\bar{\mu}$ . Hence  $-\bar{\mu}$  is no longer a weak source as in the case  $N > 2$ .

**Theorem 4.18** *Assume  $N = 2$ ,  $p > 1$ . There exist positive numbers  $\tilde{\mu}_{min}$  and  $\tilde{\mu}_{max}$  such that  $-\mu^*(2) < \tilde{\mu}_{min} \leq \tilde{\mu}_{max} < \mu^*(1)$  with the following properties:*

- 1- *For  $-\tilde{\mu}_{min} < M < -\mu^*(2)$  there exists a ground state.*
- 2- *for  $M < -\tilde{\mu}_{max}$  there exists a ground state  $u$  either such that  $u(r) \sim U_{2,M}(r)$  or undulating around  $U_{2,M}(r)$  when  $r \rightarrow \infty$ .*
- 3-  *$M = -\tilde{\mu}_{min}$  there exists a ground state  $u$  such that  $u(r) \sim U_{1,M}(r)$  when  $r \rightarrow \infty$ .*

*Proof.* If  $M = \bar{M} = -\mu^*(2)$  there exists no ground state from Proposition 4.7. By continuity this property is still valid for  $\bar{M}_\epsilon = \bar{M} - \epsilon$  for  $\epsilon > 0$  small enough. As in the proof of Theorem 4.17 with  $N \geq 3$  we still denote by  $A$  the set of  $M \in (-\mu^*(1), -\mu^*(2))$  such that there is no ground state. We define in a similar way the set  $B_1$  and  $B_2$ . The previous situation is still valid with the only difference that  $\bar{M}_\epsilon$  does not satisfies A-(i) but A-(ii): indeed from [18, Th. 8.2, Lemma 8.7], see Appendix, for  $\epsilon < \epsilon_0$  small enough there is no cycle around  $P_{2,M}$  which is a sink by Proposition 4.2. Thus  $\mathbf{T}_{unst}^{1,3}$  converges to  $P_{2,M}$  when  $t \rightarrow \infty$  and  $\mathbf{T}_{st}^{1,2}$  converges to 0 when  $t \rightarrow -\infty$ , and since there is no ground state it satisfies  $x(t) \sim ce^{\frac{2t}{p-1}}|t|$  when  $t \rightarrow -\infty$  for some  $c > 0$ . Hence the function  $g : M \mapsto g(M) = x_{reg}^M - x_{1,2}^M$  defined as in the proof of Theorem 4.17 shares the same properties and the conclusion follows.  $\square$

*Remark.* We conjecture that there is no cycle when  $N = 2$ . If it is true, then for any  $-\tilde{\mu} < M < -\mu^*$ ,  $\mathbf{T}_{st}^{1,2}$  converges to  $(0, 0)$  as  $t \rightarrow -\infty$ . Equivalently there exists a positive solution  $u$  of

(1.13) with a logarithmic blow-up at  $r = 0$  and such that  $u(r) \sim U_{1,M}(r)$  as  $r \rightarrow \infty$ . Hence there exist also a positive solution  $u$  of (1.13) such that  $u(r) \sim U_{1,M}(r)$  as  $r \rightarrow 0$  and  $u(r) \sim U_{2,M}(r)$  as  $r \rightarrow \infty$ .

#### 4.6 The case $M < 0$ , $N = 1$ and $p > 1$

In the case  $N = 1$ , the equation is invariant under the translation group  $\mathcal{T}_\alpha[u](\cdot) = u(\cdot + \alpha)$  for  $\alpha = 0$  and any ground state is symmetric with respect to its vertex.

**Theorem 4.19** *Let  $N = 1$ . Then there exists a ground state  $u$  if and only if  $M \leq -\mu^*(1)$ . It satisfies  $u(r) \sim U_{2,M}(r)$  as  $r \rightarrow \infty$ . Furthermore, if  $M < -\mu^*(1)$  there exists a positive singular solution  $u$  which satisfies  $u(r) \sim U_{1,M}(r)$  as  $r \rightarrow 0$  and  $u(r) \sim U_{2,M}(r)$  as  $r \rightarrow \infty$ .*

*Proof.* The existence when  $M < -\mu^*(1)$  is proved in Lemma 4.4 but the proof therein is not valid when  $M = -\mu^*(1)$  in which case a second beautiful construction due to Chipot and Weissler [14] applies: if  $M \leq -\mu^*(1)$  there exist singular solutions  $U_{1,M}$  and  $U_{2,M}$ . If  $\mathbf{T}_{reg}$  is not a ground state, the corresponding solution  $u$  vanishes at  $r = r_0 > 0$ . Hence there exists a translation of  $u$ , say  $r \mapsto u(r - c)$  which is tangent to  $U_{1,M}$  which is impossible. If  $M < -\mu^*(1)$  estimate (4.18) implies (4.20) which in turn implies that  $r^{\frac{2}{p-1}}u(r)$  cannot converge to  $X_{1,M}$ . Notice that there exists no cycle in the phase plane  $(x, y)$  otherwise the corresponding solution  $u$  would be singular and undulating hence a translation of it say  $x \mapsto u(x + c)$  which is now singular at  $x = -c$  and defined for  $x > -c$  could be made tangent somewhere to  $U_{1,M}$  (or  $U_{2,M}$ ) which is impossible. Therefore  $r^{\frac{2}{p-1}}u(r)$  converge to  $X_{2,M}$  as  $r \rightarrow \infty$ .

In order to prove that there exists a heteroclinic connecting  $P_{1,M}$  to  $P_{2,M}$  and since  $\mathbf{T}_{reg}$  converges to  $P_{1,M}$ , there exists a smallest  $\tau$  such that  $x_{reg}(\tau) = X_{1,M}$  and the vector field  $\mathbf{H}$  is directed to the right on the segment  $J = \{(x, y) : x = X_{1,M}, y_{reg}(\tau) \leq y \leq Y_{1,M}\}$ , we have three possibilities:

(i) either  $x_{reg}(t) \rightarrow X_{1,M}$  monotonically. In that case the region bordered by the segment  $J$ , the portion of  $\mathcal{L}$  between  $P_{1,M}$  and  $P_{2,M}$  and the portion of trajectory  $\mathbf{T}_{reg}$  for  $t > \tau$  is positively invariant. Since  $\mathbf{T}_{unst}^{1,3}$  belongs to this region, it converges to  $P_{2,M}$  when  $t \rightarrow \infty$ .

(ii) either  $\mathbf{T}_{reg}$  has a first intersection with  $\mathcal{L}$  at a point  $(x(t_1), y(t_1))$  with  $x(t_1) > X_{2,M}$ . Then it enters successively the region  $\mathbf{D}$  where  $x_t < 0$  and  $y_t > 0$  and the region the region  $\mathbf{A}$  where  $x_t < 0$  and  $y_t < 0$  and finally intersects  $\mathcal{L}$  between  $P_{1,M}$  and  $P_{2,M}$  at some point  $(x(t_2), y(t_2))$ , or converges to  $P_{2,M}$  monotonically, in which case we set  $t_2 = \infty$ . In that case the region bordered by the segment  $J$ , the portion of  $\mathcal{L}$  between  $P_{1,M}$  and  $(x(t_2), y(t_2))$  and the portion of trajectory  $\mathbf{T}_{reg}$  for  $\tau < t < t_2$  is positively invariant. Since  $\mathbf{T}_{unst}^{1,3}$  belongs to this region we conclude as in case (i).  $\square$

#### 4.7 Appendix: Non-existence of cycle in the case $N = 2$

The difficulty comes from the fact that when  $M = \bar{M} = -\bar{\mu} = -\mu^*(2)$ , the linearized system at  $P_{2,\bar{M}}$  has zero as a double eigenvalue. Following Kuznetsov's notations [18, Lemma 8.7] we consider the system associated to (2.1), with two extra parameters  $\vec{\alpha} = (\alpha_1, \alpha_2)$  called the

bifurcation parameters,

$$\begin{aligned} x_t &= v \\ v_t &= \alpha_2 + \frac{4}{p-1}v - \frac{4}{(p-1)^2}x - x^p + (\bar{\mu} + \alpha_1) \left( \frac{2}{p-1}x - v \right)^{\frac{2p}{p+1}} := g(x, v, \alpha_1, \alpha_2). \end{aligned} \quad (4.27)$$

We recall that  $\bar{\mu} = (p+1)p^{-\frac{p}{p+1}}$  and set  $\vec{x} = (x, v)$ ,  $g(x, v, \alpha_1, \alpha_2) = g(\vec{x}, \vec{\alpha})$  and

$$y_0 = Y_{\bar{M},2} = \frac{1}{p^{\frac{1}{p-1}}} \left( \frac{2}{p-1} \right)^{\frac{p+1}{p-1}} =, \quad x_0 = X_{\bar{M},2} = \frac{p-1}{2}y_0$$

linearize (4.27) at  $\tilde{P}_0 = (x_0, 0)$ , with  $\vec{\alpha}$  fixed, we obtain the new system

$$\begin{aligned} \bar{x}_t &= v \\ v_t &= \alpha_2 + \alpha_1 y_0^{\frac{2p}{p+1}} + \frac{2\alpha_1 p}{p^2-1} y_0^{\frac{p-1}{p+1}} \bar{x} - \frac{2\alpha_1 p}{p+1} y_0^{\frac{p-1}{p+1}} v - \alpha_1 y_0^{\frac{2p}{p+1}} + \alpha_2 \\ &\quad + \frac{1}{2} (g_{xx}(P_0) \bar{x}^2 + 2g_{xv}(P_0) \bar{x}v + g_{vv}(P_0) v^2) + R(\vec{x}, \vec{\alpha}). \end{aligned} \quad (4.28)$$

In order to agree with Kuznetsov's notations, we write (4.28) under the form

$$\begin{aligned} y_{1t} &= y_2 \\ y_{2t} &= g_{00} + g_{10}y_1 + g_{01}y_2 + \frac{1}{2} (g_{20}y_1^2 + 2g_{11}y_1y_2 + g_{02}y_2^2) + R(\vec{y}, \vec{\alpha}) \end{aligned} \quad (4.29)$$

where  $R(\vec{y}, \vec{\alpha}) = O(|\vec{y}|^3)$  and

$$\begin{aligned} g_{00} &= \alpha_2 + \alpha_1 y_0^{\frac{2p}{p+1}}, \quad g_{1,0} = \frac{4p\alpha_1}{p^2-1} y_0^{\frac{p-1}{p+1}}, \quad g_{01} = -\frac{2p\alpha_1}{p+1} y_0^{\frac{p-1}{p+1}}, \quad g_{11} = -\frac{4p}{(p+1)^2} \left( \frac{p+1}{p^{\frac{p}{p+1}}} + \alpha_1 \right) y_0^{-\frac{2}{p+1}} \\ g_{20} &= -\frac{4p}{(p+1)^2} \left( \frac{p+1}{p^{\frac{p}{p+1}}} - \frac{2\alpha_1}{p-1} \right) y_0^{-\frac{2}{p+1}}, \quad g_{02} = -\frac{2p(p-1)}{(p+1)^2} \left( \frac{p+1}{p^{\frac{p}{p+1}}} + \alpha_1 \right) y_0^{-\frac{2}{p+1}}. \end{aligned}$$

Note that, if  $\alpha_1 = 0$ , the three coefficients  $g_{11}$ ,  $g_{20}$  and  $g_{02}$  are negative.

Following Kuznetsov proof, we perform several changes of variables:

1- Setting  $y_1 = v_1 + \delta$ ,  $v_2 = y_2$  where  $\delta = \delta(\vec{\alpha})$ , we can get rid of the coefficient of  $v_2$  in the second equation and obtain

$$\begin{aligned} v_{1t} &= v_2 \\ v_{2t} &= h_{00} + h_{10}y_1 + \frac{1}{2} (h_{20}v_1^2 + 2h_{11}v_1v_2 + h_{02}v_2^2) + Q(\vec{v}, \vec{\alpha}) \end{aligned} \quad (4.30)$$

where  $Q(\vec{v}, \vec{\alpha}) = O(|\vec{v}|^3)$  and

$$\delta = -\frac{g_{01}}{g_{11}(0)}(1 + o(1)) \quad \text{as } \vec{\alpha} \rightarrow 0,$$

with  $g_{01}$  (resp.  $g_{00}$  and  $g_{10}$ ) stands for  $g_{01}(\vec{\alpha})$  (resp.  $g_{00}(\vec{\alpha})$  and  $g_{10}(\vec{\alpha})$ ) and

$$h_{00} = g_{00} - \frac{g_{01}g_{10}}{g_{11}(0)} + o(1), \quad h_{10} = g_{10} - \frac{g_{01}g_{20}}{g_{11}(0)} + o(1) \quad \text{and} \quad h_{20} = g_{20}, \quad h_{11} = g_{11}, \quad h_{02} = g_{02}.$$

2- Time scaling,  $\frac{dt}{d\tau} = 1 + \theta v_1(t)$  where  $\theta = \theta(\vec{\alpha})$ , and  $\xi_1 = v_1$ ,  $\xi_2 = (1 + \theta v_1)v_2$  in order to get rid of the coefficients of  $v_2^2$  in the equations for  $\xi_2$ , then

$$\begin{aligned} \xi_{1\tau} &= \xi_2 \\ \xi_{2\tau} &= f_{00} + f_{10}\xi_1 + \frac{1}{2}hf_{20}\xi_1^2 + f_{11}\xi_1\xi_2 + h_{02}v_2^2 + P(\vec{\xi}, \vec{\alpha}) \end{aligned} \quad (4.31)$$

where  $P(\vec{\xi}, \vec{\alpha}) = O(|\vec{\xi}|^3)$ ,

$$f_{00} = h_{00}(1 + o(1)), \quad f_{10} = \left( h_{10} - \frac{h_{00}h_{02}}{2} \right) (1 + o(1)), \quad f_{20} = (h_{20} - h_{10}h_{02})(1 + o(1)),$$

and  $f_{11} = h_{11}(1 + o(1))$ . We rewrite the equation of  $\xi_{2t}$  under the form

$$\xi_{2t} = \mu_1(\vec{\alpha}) + \mu_2(\vec{\alpha})\xi_1 + A(\vec{\alpha})\xi_1^2 + B(\vec{\alpha})\xi_1\xi_2 + O(|\vec{\xi}|^3), \quad (4.32)$$

where, for  $|\vec{\alpha}|$  small enough,

$$\begin{aligned} \mu_1 &= h_{00} = \left( g_{00} - \frac{g_{10}g_{01}}{g_{11}(0)} \right) (1 + o(1)) = \alpha_1 y_0^{\frac{2p}{p+1}} (1 + o(1)), \\ \mu_2 &= \left( h_{10} - \frac{h_{02}h_{00}}{2} \right) (1 + o(1)) = \frac{2(p^2 + 2p - 1)}{p^2 - 1} y_0^{\frac{p-1}{p+1}} (1 + o(1)), \\ A(\vec{\alpha}) &= -\frac{2p}{p+1} \left( \frac{1}{p} \right)^{\frac{p}{p+1}} y_0^{-\frac{2}{p+1}} (1 + o(1)), \end{aligned}$$

and

$$B(\vec{\alpha}) = -\frac{4}{p+1} \left( \frac{1}{p} \right)^{\frac{p}{p+1}} y_0^{-\frac{2}{p+1}} (1 + o(1)).$$

At end we change again time and put

$$t = \left| \frac{B}{A} \right| \tau, \quad \eta_1 = \frac{B^2}{A} \xi_1 \quad \text{and} \quad \eta_2 = \text{sgn} \left( \frac{B}{A} \right) \frac{B^3}{A^2} \xi_2,$$

in order to see that  $\vec{\eta} = (\eta_1, \eta_2)$  verifies

$$\begin{aligned} \eta_{1t} &= \eta_2 \\ \eta_{2t} &= \beta_1(\vec{\alpha}) + \beta_2(\vec{\alpha})\eta_1 + \eta_1^2 + \text{sgn} \left( \frac{B(0)}{A(0)} \right) \eta_1\eta_2 + O(|\vec{\eta}|^3). \end{aligned} \quad (4.33)$$

In our situation

$$\text{sgn} \left( \frac{B(0)}{A(0)} \right) = 1.$$

After further computations and simplifications we obtain, with  $\alpha_2 = 0$  and eliminating the terms which contain  $\alpha_1^2$ ,

$$\beta_1 = -\frac{64}{p^2 - 1} \alpha_1 y_0^{\frac{p-1}{p+1}} (1 + o(1)) \quad \text{and} \quad \beta_2 = \frac{8p^2 + 2p - 1}{p^2 - 1} \alpha_1 y_0^{\frac{p-1}{p+1}} (1 + o(1)). \quad (4.34)$$

Therefore, the discriminant  $\beta_2^2 - 4\beta_1$  of the polynomial  $\mathcal{P}(\eta_1) = \beta_1 + \beta_2\eta_1 + \eta_1^2$  is positive for  $\alpha_1 > 0$ .

By [18, Lemma 8.7] there is no cycle in the region of the plane  $(\beta_1, \beta_2)$  located in the second quadrant, which is our case since  $\alpha_1 > 0$ . Furthermore, the equilibrium  $(\beta_1, \beta_2) = (0, 0)$ , which has a double zero eigenvalue has one stable trajectory converging when  $t \rightarrow \infty$  and one unstable trajectory converging when  $t \rightarrow -\infty$ . Hence it is a saddle point.

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