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# Product of two staircase rank one transformations that is not loosely Bernoulli

Adam Kanigowski and Thierry de la Rue

9th May 2019

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## Abstract

We construct two staircase rank one transformations whose Cartesian product is not loosely Bernoulli.

## 1 Introduction

In this paper we study the loosely Bernoulli property for zero entropy, measure preserving transformations. Loose Bernoullicity was introduced by J. Feldman, [3], and A. Katok, [6]<sup>1</sup>. Recall that a zero entropy measure preserving transformation is loosely Bernoulli (LB for short) if it is isomorphic to a transformation induced from an irrational rotation of the circle. It follows from [8] that the class of loosely Bernoulli systems is broad: it is closed under taking factors, compact extensions and inverse limits. Moreover it contains all *finite rank systems*. First non-loosely Bernoulli examples were constructed by J. Feldman [3] by a cutting and stacking method. D. Ornstein, D. Rudolph and B. Weiss [8] constructed a *rank*

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<sup>1</sup>In [6], zero entropy loosely Bernoulli is called *standard*.

one transformation  $T$  such that  $T \times T$  is not loosely Bernoulli. The example in [8] is based on Ornstein's construction in [7] of a class of almost surely mixing rank one transformation with random spacers. On the other hand, M. Gerber constructed in [4] a mixing rank one transformation whose Cartesian square is loosely Bernoulli. An algebraic example of a zero entropy non-loosely Bernoulli transformation comes from the work of M. Ratner, [10], where it was shown that the Cartesian square of the horocycle flow (on compact quotients) is not loosely Bernoulli, although horocycle flows are loosely Bernoulli [9]. It was recently shown in [5] that there exist smooth flows on  $\mathbb{T}^2$  (Kochergin flows) whose Cartesian product is not loosely Bernoulli (provided that the exponents are different).

In this note we study the loosely Bernoulli property for a natural class of mixing rank one transformations: *staircase* rank one systems (mixing for this class was proved by Adams, [1]). Loose Bernoullicity for products of staircase rank one transformations is not known. Our main result implies in particular that there exist two staircase rank one transformations whose product is not loosely Bernoulli.

Our method uses some ideas from [5], but the main mechanism is different: systems considered in [5] have superlinear (in fact polynomial with degree close to 2) growth, which is the main source of non-loose Bernoullicity. In our case, the orbit growth is linear and instead we take crucial advantage from the  $\delta$ -alternation property (see Definition 1.1), which ensures that the heights of the towers of the two rank one transformations are never of the same order. Since our argument is essentially based on this assumption, it certainly does not apply to the case of the Cartesian square of a single rank-one transformation. In particular, new ideas will be required to answer a famous question by Jean-Paul Thouvenot whether the Cartesian square of Chacon's transformation is loosely Bernoulli or not.

## 1.1 Statement of the main result

For a rank one system  $T$ , let  $(p_n^T)_{n \in \mathbb{N}}$ ,  $(a_{n,i}^T)_{i=1}^{p_n^T}$  and  $(h_n^T)_{n \in \mathbb{N}}$  denote the sequence of *cuts*, *spacers* and *heights* respectively (see Section 2.1). For  $0 < \gamma < \gamma' < 1$  define

$$\mathcal{C}_{\gamma, \gamma'} := \left\{ T \in \text{Rank}(1) : \text{there exists } n'_T \in \mathbb{N} \text{ such that for every } n \geq n'_T, \right. \\ \left. p_n^T \in \left[ (h_n^T)^\gamma, (h_n^T)^{\gamma'} \right], a_{n,i}^T > a_{n,i-1}^T \text{ for } i = 2, \dots, p_n^T, \text{ and } a_{n,p_n}^T \leq (h_n^T)^{\gamma'} \right\}. \quad (1)$$

Notice that if  $T$  is a *staircase* rank one transformation, i.e.  $a_{n,i}^T = i$  for  $n \in \mathbb{N}$  and  $i \in \{1, \dots, p_n^T\}$ , and if  $p_n^T \in [(h_n^T)^\gamma, (h_n^T)^{\gamma'}]$ , then  $T \in \mathcal{C}_{\gamma, \gamma'}$ .

**Definition 1.1** ( $\delta$ -alternating sequences). *Fix  $\delta > 0$  and let  $(a_n)$  and  $(b_n)$  be two increasing sequences of positive integers. We say that  $(b_n)$  is  $(a_n)$ -alternating with exponent  $\delta$  if the following holds: there exists  $n_0 \in \mathbb{N}$  such that for every  $n \geq n_0$ , if  $m(n)$  is unique such that  $b_{m(n)} \leq a_n < b_{m(n)+1}$ , then  $b_{m(n)}^{1+\delta} < a_n$  and  $a_n^{1+\delta} < b_{m(n)+1}$ . Moreover,  $(a_n)$  and  $(b_n)$  are called  $\delta$ -alternating if  $(a_n)$  is  $(b_n)$ -alternating with exponent  $\delta$  and  $(b_n)$  is  $(a_n)$ -alternating with exponent  $\delta$ .*

For  $0 < \gamma < \gamma' < 1$  let

$$\mathcal{L}^{\gamma, \gamma', \delta} := \left\{ (T, S) \in \mathcal{C}_{\gamma, \gamma'} \times \mathcal{C}_{\gamma, \gamma'} : (h_n^T) \text{ and } (h_n^S) \text{ are } \delta\text{-alternating} \right\}. \quad (2)$$

**Theorem 1.** *If  $T$  is weakly mixing and  $(T, S) \in \mathcal{L}^{\gamma, \gamma', \delta}$  with  $\delta < \gamma$  then  $T \times S$  is not loosely Bernoulli.*

In Subsection 3.2 we will also show the following:

**Lemma 1.2.** *For every  $T \in \mathcal{C}_{\gamma, \gamma'}$ ,  $0 < \gamma < \gamma' < 1/3$  there exists a staircase rank one  $S \in \mathcal{C}_{\gamma/3, 3\gamma'}$  such that  $(T, S) \in \mathcal{L}^{\gamma/3, 3\gamma', \gamma/4}$ .*

It follows from [2] that staircase rank one transformations are mixing. Therefore, Theorem 1 together with Lemma 1.2 has the following consequence:

**Corollary 1.3.** *There exist two staircase rank one transformations whose product is not loosely Bernoulli.*

### Outline of the proof

A rank one transformation  $T$  is defined inductively (see Section 2.1) by a sequence of nested towers  $(\mathcal{T}_m)_{m \in \mathbb{N}}$ . Therefore, there is a natural notion of distance: two points  $x, y$  are at distance  $\frac{1}{h_m}$ , if  $m \in \mathbb{N}$  is the largest such that  $x, y$  are in the same level of  $\mathcal{T}_m$  (see (7)). To show that a system is not loosely Bernoulli, one needs to show that “typical” points are not close in the  $\bar{f}$  metric (see Section 2.2), i.e. that for typical points there is no good matching. For two rank one transformations  $T$  and  $S$  and typical points  $(x, y), (x', y')$  in the product space, consider some matching of pieces of orbits of length  $N$ . One can then split such a matching into sets  $A_k$  according to which interval of the form  $[2^{-k-1}, 2^{-k})$  the maximum over the distances (for  $T$  and  $S$ ) of a given iterate of  $T \times S$  belongs (see (31)). The main goal is to show that the cardinality of  $A_k$  is bounded above by  $Nk^{-2}$  (see Proposition 4.1). Then the fact that the matching is of small cardinality follows by summing over  $k$ .

The main result towards this is Lemma 3.2 which describes how the horizontal distance behaves under iterates of  $T$  and  $S$ . First, due to the staircase nature of  $T$  and  $S$ , it is shown that if two points in the same level of  $\mathcal{T}_n$  but not in the same level of  $\mathcal{T}_{n+1}$  reach the top of  $\mathcal{T}_n$  (which happens before  $h_n$ ), then their orbits will not be in the same level of  $\mathcal{T}_n$  for time  $\geq h_n^{1+2\xi}$  (for some  $\xi > 0$ ) (see (25)). This is a consequence of the fact that each time the orbits hit the top they split by a positive shear (due to the staircase nature), but the only way they could end up in the same level of  $\mathcal{T}_n$  is if they split by  $\geq h_n$ , which does not happen before  $h_n^{1+2\xi}$  (the total amount of shear is too small). The second part of Lemma 3.2 (see (26)) shows that if  $x, x'$  are in the same level of  $\mathcal{T}_n$  (not too close to the top or bottom of  $\mathcal{T}_n$ ), the only way that  $T^i x$  and  $T^j x'$  lie in the same level of  $\mathcal{T}_n$  for  $i, j \leq \frac{h_n}{n^3}$  is when  $i = j$ .

We will use Lemma 3.2 for both  $T$  and  $S$  to show that  $A_k$  has cardinality  $\leq Nk^{-2}$ , together with the fact that the sequences of heights are  $\delta$ -alternating. Namely, if  $x, x'$  are at distance  $\frac{1}{h_n^T}$  and  $(y, y')$  are at distance  $\frac{1}{h_m^S}$ , (assume without loss of generality that  $\frac{1}{h_n^T} < \frac{1}{h_m^S}$ ), then by (25) (for  $S$ ), the “vertical” matching (that is, matching  $(T^i x, S^i y)$  with  $(T^j x', S^j y')$ ) does not work on  $A_k$  for  $i \in [h_m^S, (h_m^S)^{1+2\xi}]$  (it only can work for  $i \in [0, h_m^S]$ ). On the other hand, using  $(h_m^S)^{1+2\xi} \leq \frac{h_n^T}{n^3}$  (by  $\delta$ -alternation) and (26) for  $T$ , the only way to match  $(T^i x, S^i y)$  with  $(T^j x', S^j y')$  in  $A_k$  for  $i, j \in [h_m^S, (h_m^S)^{1+2\xi}]$  is that  $i = j$ , i.e. the

matching *has to be* vertical. The inconsistency of the matching on  $[h_m^S, (h_m^S)^{1+2\xi}]$  gives the bound  $|A_k| \leq Nk^{-2}$ .

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## 2 Basic definitions

### 2.1 Rank one maps

Recall that a rank one system is constructed by cutting and stacking: fix a sequence of *cuts*  $(p_n)_{n \in \mathbb{N}}$  and a sequence of *spacers*  $(a_{n,i})_{i=1}^{p_n}$ ,  $n \in \mathbb{N}$ . We define  $h_1 = 1$  and inductively

$$h_n = p_{n-1}h_{n-1} + \sum_{i=1}^{p_{n-1}} a_{n-1,i}, \quad \text{for } n \geq 2. \quad (3)$$

The sequence  $(h_n)_{n \in \mathbb{N}}$  is the sequence of *heights*. The rank one system  $T$  is constructed from the sequences  $(p_n)_{n \in \mathbb{N}}$  and  $(a_{n,i})_{i=1}^{p_n}$ ,  $n \in \mathbb{N}$  in the following way: we start with the interval  $[0, 1]$  which we cut into  $p_1$  equal intervals  $(I_i^1)_{i=1}^{p_1}$ . For every  $i \in \{1, \dots, p_1\}$  we put  $a_{1,i}$  spacers (extra intervals with the same length as  $I_i^1$ ) over  $I_i^1$ , getting the  $i$ th subcolumn. Then we *stack* everything over  $I_1^1$ , putting the  $(i+1)$ th subcolumn over the  $i$ th subcolumn for each  $i = 1, \dots, p_1 - 1$ , and call this new tower  $\mathcal{T}_2$  (with base  $I_1^1$ ). The transformation  $T$  on  $\mathcal{T}_2$  just moves one level up except the last level, where it is not yet defined. Next, inductively at stage  $n$ , we cut the tower  $\mathcal{T}_n$  with base  $I_1^n$  into  $p_n$  equal subtowers, over the  $i$ th subtower we put  $a_{n,i}$  spacers and then we stack to get tower  $\mathcal{T}_{n+1}$  with base  $I_1^n$  (see Figure 1). Finally, the rank one map  $T$  is defined almost everywhere, i.e. it is defined on  $\mathcal{T}_\infty := \bigcup_{n \in \mathbb{N}} \mathcal{T}_n$ . Often, if there are more transformations involved, we will write a superscript  $T$  in the sequences  $(p_n)$ ,  $(a_{n,i})$ ,  $(h_n)$  and  $\mathcal{T}_n$ . In what follows we will always assume that  $p_n > 1$  for every  $n$ . This implies that

$$h_n \geq 2^{n-1}. \quad (4)$$

We also assume below that the total measure of all the spacers is finite:

$$\sum_{n=1}^{+\infty} \frac{1}{\prod_{k=1}^n p_k} \left( \sum_{i=1}^{p_n} a_{n,i} \right) < +\infty. \quad (5)$$

Under this condition,  $T$  preserves a probability measure  $\mu_T$  given by the normalized Lebesgue measure on  $\mathcal{T}_\infty$ . It is a classical fact, easily seen by a density point argument, that rank one systems are ergodic with respect to the Lebesgue measure. Hence  $T$  acts on  $(\mathcal{T}_\infty, \mathcal{B}, \mu_T)$ . Moreover, (5) implies the following bound on the sequence of heights: there exists  $1 \leq K < \infty$  such that for every  $n \in \mathbb{N}$

$$\prod_{i=1}^{n-1} p_i \leq h_n \leq K \prod_{i=1}^{n-1} p_i. \quad (6)$$

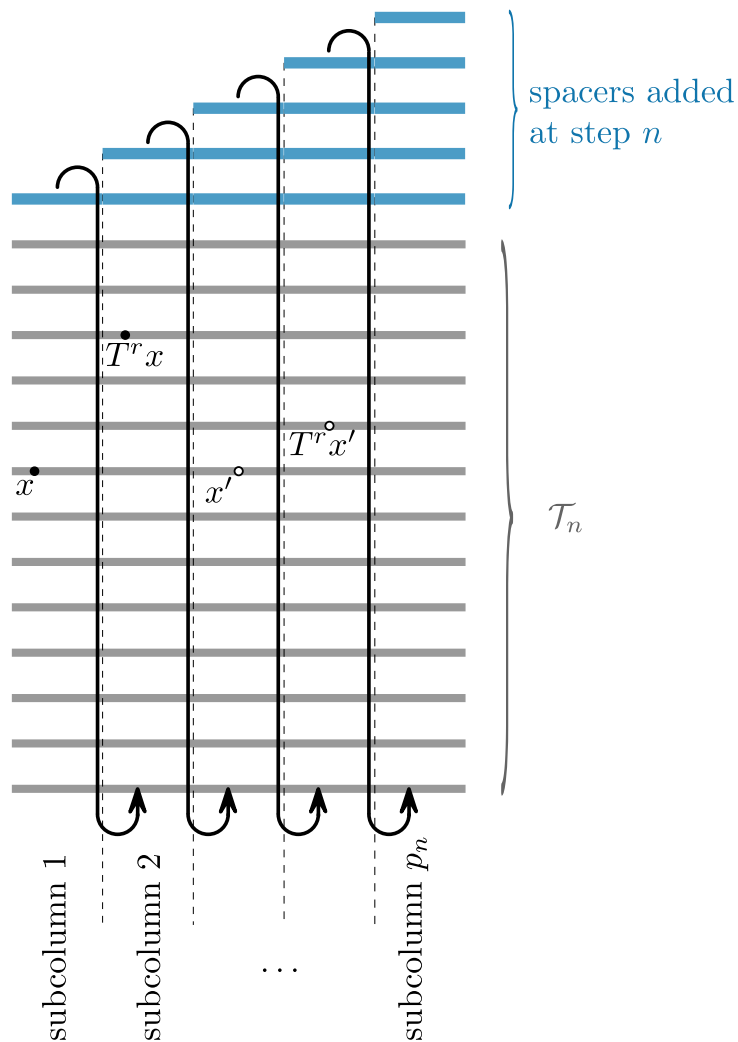


Figure 1: Step  $n$  in the construction of a staircase rank-one transformation  $T$ . Each point not on the top of the figure is mapped to the point directly above him. The arrows show how the subcolumns are stacked together to form  $\mathcal{T}_{n+1}$ . The figure also shows two points  $x, x'$  in the same level of  $\mathcal{T}_n$ , but not in the same level of  $\mathcal{T}_{n+1}$ , and their images  $T^r x$  and  $T^r x'$  for some  $h_n \leq r \leq (h_n)^{1+2\xi}$  as in Lemma 3.2. The points  $T^r x$  and  $T^r x'$  are not any more in the same level of  $\mathcal{T}_n$  because the corresponding orbits went through different numbers of spacers.

Indeed, the LHS is immediate from (3). The RHS follows by (5) and (3) since for  $\epsilon_n = \frac{1}{\prod_{i=1}^n p_i} (\sum_{i=1}^{p_n} a_{n,i})$ , we have

$$\sum_{i=1}^{p_{n-1}} a_{n-1,i} = \epsilon_{n-1} \prod_{i=1}^{n-1} p_i \leq \epsilon_{n-1} p_{n-1} h_{n-1}$$

and so by (3)

$$h_n \leq p_{n-1} h_{n-1} (1 + \epsilon_{n-1})$$

and recursively

$$h_n \leq \left( \prod_{i=1}^{n-1} (1 + \epsilon_i) \right) \prod_{i=1}^{n-1} p_i \leq K \prod_{i=1}^{n-1} p_i,$$

where  $K := \prod_{i=1}^{\infty} (1 + \epsilon_i) < \infty$  by (5). This shows (6).

The set of rank one transformations (satisfying (5)) will be denoted by  $Rank(1)$ .

For  $n \in \mathbb{N}$  and  $x, y$  in the same level of  $\mathcal{T}_n$  we define the *horizontal* distance of  $x, y$ , setting

$$d_H(x, y) := \frac{1}{h_{m(x,y)}}, \quad (7)$$

where  $m = m(x, y) \geq n$  is the largest number such that  $x, y$  are in one level of  $\mathcal{T}_m$  (i.e. they are in different levels of  $\mathcal{T}_{m+1}$ ). If  $x, y \in \mathcal{T}_{\infty}$  and there is no  $n$  such that  $x, y$  are in the same level of  $\mathcal{T}_n$ , we set

$$d_H(x, y) := 1.$$

Observe that if  $T \in C_{\gamma, \gamma'}$ , for  $n$  large enough we have

$$\mu(\mathcal{T}_{n+1}^T) \leq \mu(\mathcal{T}_n^T) \left( 1 + (h_n^T)^{\gamma' - 1} \right),$$

and using (4), we get

$$\mu(\mathcal{T}_n^T) \geq 1 - O(n^{-2}). \quad (8)$$

## 2.2 Loosely Bernoulli transformations

We recall the definition of  $\bar{f}$  metric introduced in [3]. For two finite words (over a finite alphabet)  $A = a_1 \dots a_k$  and  $B = b_1 \dots b_k$ , a *matching* between  $A$  and  $B$  is any pair of strictly increasing sequences  $(i_s, j_s)_{s=1}^r$  such that  $a_{i_s} = b_{j_s}$  for  $s = 1, \dots, r$ . The  $\bar{f}$  distance between  $A$  and  $B$  is defined by

$$\bar{f}(A, B) = 1 - \frac{r}{k},$$

where  $r$  is the maximal cardinality over all matchings between  $A$  and  $B$ .

Let  $T : (X, \mathcal{B}, \mu) \rightarrow (X, \mathcal{B}, \mu)$  be a measure preserving automorphism. For a finite partition  $\mathcal{P} = (P_1, \dots, P_r)$  of  $X$  and an integer  $N \geq 1$  we denote  $\mathcal{P}_0^N(x) = x_0 \dots x_{N-1}$ , where  $x_i \in \{1, \dots, r\}$  is such that  $T^i(x) \in P_{x_i}$  for  $i = 0, \dots, N-1$ .

**Definition 2.1.** A zero entropy process  $(T, \mathcal{P})$  is said to be loosely Bernoulli if for every  $\varepsilon > 0$  there exists  $N_\varepsilon \in \mathbb{N}$  and a set  $A_\varepsilon \in \mathcal{B}$ ,  $\mu(A_\varepsilon) > 1 - \varepsilon$ , such that for every  $x, y \in A_\varepsilon$  and every  $N \geq N_\varepsilon$

$$\bar{f}(\mathcal{P}_0^N(x), \mathcal{P}_0^N(y)) < \varepsilon. \quad (9)$$

A zero entropy automorphism  $T$  is loosely Bernoulli (LB for short) if for every finite measurable partition  $\mathcal{P}$ , the process  $(T, \mathcal{P})$  is loosely Bernoulli.

To prove Theorem 1, in view of Definition 2.1 it is enough to show that there exists a finite partition  $\mathcal{P}$  such that the process  $(T \times S, \mathcal{P})$  is not LB. Recall that we have an exhaustive sequence of towers  $(\mathcal{T}_n^T)$  and  $(\mathcal{T}_n^S)$  for  $T$  and  $S$  respectively. For  $n \in \mathbb{N}$  let  $\mathcal{Q}_n$  be the partition of  $\mathcal{T}_\infty^T$  into levels of  $\mathcal{T}_n^T$  and  $\mathcal{T}_\infty^T \setminus \mathcal{T}_n^T$  ( $\mathcal{T}_\infty^T \setminus \mathcal{T}_n^T$  is the union of all spacers at stages  $\geq n$ ). Let  $\mathcal{R}_n$  be the analogous partition of  $\mathcal{T}_\infty^S$  and define  $\mathcal{P}_n := \mathcal{Q}_n \times \mathcal{R}_n$ . Then  $\mathcal{P}_n$  is a partition of  $\mathcal{T}_\infty^T \times \mathcal{T}_\infty^S$  (in fact, the sequence  $(\mathcal{P}_n)$  converges to the partition into points). Theorem 1 is a consequence of the following theorem:

**Theorem 2.** Let  $T$  be weakly mixing and  $(T, S) \in \mathcal{L}^{\gamma, \gamma', \delta}$  with  $\delta < \gamma$ . There exists  $m_0$  such that the process  $(T \times S, \mathcal{P}_{m_0})$  is not LB.

In the following subsections we will state some lemmas which will help us in proving Theorem 2.

### 3 Preliminary lemmas

#### 3.1 A combinatorial lemma: lower bound on $\bar{f}$

In this section we state a general combinatorial lemma which allows to give a lower bound on the  $\bar{f}$  distance. Let  $(i_s)_{s=1}^r$  and  $(j_s)_{s=1}^r$  be two increasing sequences of positive integers in  $[0, N]$ .

For  $M \leq N$  and  $1 \leq w < r$  define

$$I(M, w) = \{s \in \{1, \dots, r\} : i_s \in [i_w, i_w + M]\} \subset [0, N]$$

and

$$J(M, w) = \{s \in \{1, \dots, r\} : j_s \in [j_w, j_w + M]\} \subset [0, N].$$

The following property of  $I(M, w)$  and  $J(M, w)$  is straightforward and will be useful in the proof of the lemma below: If  $s > w$  and  $s \notin I(M, w)$ , then

$$I(M, w) \cap I(M, s) = \emptyset, \quad (10)$$

and analogously for  $J(M, w)$ . We have the following lemma:

**Lemma 3.1.** Let  $\xi \in (0, 1)$ . Let  $(i_s)_{s=1}^r \in [0, N]$  and  $(j_s)_{s=1}^r \in [0, N]$  be two increasing sequences of integers for which there exists a number  $K \in \mathbb{N}$ ,  $8K^{1+\xi} \leq N$  such that for every  $s = 1, \dots, r$

$$\min\left(|I(K^{1+\xi}, s)|, |J(K^{1+\xi}, s)|\right) \leq 2K, \quad (11)$$

then  $r < \frac{4N}{K^\xi}$ .



*Proof.* Set  $s_1 := 1$ . If  $|I(K^{1+\xi}, s_1)| \geq |J(K^{1+\xi}, s_1)|$ , we set  $B_1 := J(K^{1+\xi}, s_1)$  otherwise, we set  $B_1 := I(K^{1+\xi}, s_1)$ . By (11), we have

$$|B_1| \leq 2K.$$

We then proceed inductively: assume that for some  $\ell \geq 1$  we have constructed  $1 = s_1 < s_2 < \dots < s_\ell \leq r$  and subsets  $B_1, \dots, B_\ell$  of  $\{1, \dots, r\}$ . If  $B_1 \cup \dots \cup B_\ell = \{1, \dots, r\}$ , we stop here and set  $v := \ell$ . Otherwise, we define  $s_{\ell+1}$  as the smallest element of  $\{1, \dots, r\} \setminus B_1 \cup \dots \cup B_\ell$ , and  $B_{\ell+1}$  as either  $I(K^{1+\xi}, s_{\ell+1})$  or  $J(K^{1+\xi}, s_{\ell+1})$  (by choosing the set with the smallest cardinal). By (11), we always have

$$|B_{\ell+1}| \leq 2K.$$

We go on in this way until we obtain a finite sequence of sets  $\{B_i\}_{i=1}^v$ , such that

$$B_1 \cup \dots \cup B_v = \{1, \dots, r\}, \quad (12)$$

and such that

$$|B_\ell| = \min(|I(K^{1+\xi}, s_\ell)|, |J(K^{1+\xi}, s_\ell)|) \leq 2K \text{ for every } \ell \in \{1, \dots, v\}. \quad (13)$$

Moreover, if  $1 \leq \ell_1 < \ell_2 \leq v$ , then  $s_{\ell_2} \notin B_{\ell_1}$ . If we further assume that  $B_{\ell_h} = I(K^{1+\xi}, s_{\ell_h})$  for  $h \in \{1, 2\}$  (or  $B_{\ell_h} = J(K^{1+\xi}, s_{\ell_h})$  for  $h \in \{1, 2\}$ ), then by construction (see also (10))

$$B_{\ell_1} \cap B_{\ell_2} = \emptyset. \quad (14)$$

By pigeonhole principle, there exists a set  $A \subset \{1, \dots, v\}$ , with  $|A| \geq \frac{v}{2}$  and such that either for every  $\ell \in A$ ,  $B_\ell = I(K^{1+\xi}, s_\ell)$ , or for every  $\ell \in A$ ,  $B_\ell = J(K^{1+\xi}, s_\ell)$ . By (14), the subsets  $B_\ell$ ,  $\ell \in A$  correspond to disjoint subintervals of  $[0, N]$ , each of them of length  $K^{1+\xi}$ . We thus get

$$\frac{v}{2} \leq \frac{N}{K^{1+\xi}}. \quad (15)$$

Now, (12), (13) and (15) yield

$$r \leq |B_1| + \dots + |B_v| \leq 2Kv \leq \frac{4N}{K^\xi}.$$

□

### 3.2 Proof of Lemma 1.2

*Proof of Lemma 1.2:* Let  $(h_n^T)$  denote the sequence of heights for  $T$ . We will construct a staircase  $S \in C_{\gamma/3, 3\gamma'}$  such that  $(T, S) \in \mathcal{L}^{\gamma/3, 3\gamma', \gamma/4}$ , i.e. the sequences  $(h_n^T)$  and  $(h_n^S)$  are  $\gamma/4$ -alternating. For this, we will show that there exists  $n_0$  such that for every  $n \geq n_0$ , we have

$$(h_{n-1}^S)^{1+\gamma/4} < h_n^T \quad \text{and} \quad (h_n^T)^{1+\gamma/4} < h_n^S. \quad (16)$$

Fix  $\eta < 1/100$ . Let  $K_T$  denote the constant  $K$  for  $T$  in (6). We will first construct  $(p_i^S)_{i=1}^{+\infty}$  so that for  $n \geq n_0$ , we have

$$\left( \prod_{i=1}^{n-2} p_i^S \right)^{1+(\frac{1}{4}+\eta)\gamma} < \prod_{i=1}^{n-1} p_i^T \quad \text{and} \quad \left( K_T \prod_{i=1}^{n-1} p_i^T \right)^{1+\gamma/4} < \prod_{i=1}^{n-1} p_i^S. \quad (17)$$

Then we will show how to derive (16) from (17).

Since  $T \in \mathcal{C}_{\gamma, \gamma'}$  it follows that for  $n > n'_T$ , by (6), we have

$$K_T \prod_{i=1}^n p_i^T \geq h_{n+1}^T \geq (h_n^T)^{1+\gamma} \geq \left( \prod_{i=1}^{n-1} p_i^T \right)^{1+\gamma}. \quad (18)$$

Set  $p_n^S = 2$  for every  $n \in \{1, \dots, n'_T + 1\}$ . Then (by enlarging  $n'_T$  if necessary), we have

$$\left( \prod_{i=1}^{n'_T+1} p_i^S \right)^{1+(\frac{1}{4}+\eta)\gamma} = (2^{n'_T+1})^{1+(\frac{1}{4}+\eta)\gamma} \leq \left( \frac{1}{K_T} 2^{n'_T+1} \right)^{1+\gamma} \leq \prod_{i=1}^{n'_T+2} p_i^T.$$

(The last inequality by (18) and since  $p_n^T \geq 2$ .) So the left inequality in (17) holds for  $n = n'_T + 3$ . We then proceed inductively: having defined  $(p_i^S)_{i=1}^w$  so that the left inequality in (17) holds for  $n = w + 2$ , we choose  $p_{w+1}^S \geq 2$  so that

$$\left( \prod_{i=1}^w p_i^S \right) p_{w+1}^S \in \left[ \left( K_T \prod_{i=1}^{w+1} p_i^T \right)^{1+\gamma/4}, \left( \prod_{i=1}^{w+2} p_i^T \right)^{\frac{1}{1+(\frac{1}{4}+\eta)\gamma}} \right]. \quad (19)$$

We explain below why such a choice is always possible. Notice that by (18) for  $n = w + 2$ , we have

$$\begin{aligned} \left( \prod_{i=1}^{w+2} p_i^T \right)^{\frac{1}{1+(\frac{1}{4}+\eta)\gamma}} - \left( K_T \prod_{i=1}^{w+1} p_i^T \right)^{1+\gamma} &\geq \\ &\left( \frac{1}{K_T} \prod_{i=1}^{w+1} p_i^T \right)^{\frac{1+\gamma}{1+(\frac{1}{4}+\eta)\gamma}} - \left( K_T \prod_{i=1}^{w+1} p_i^T \right)^{1+\gamma/4} \geq 4 \prod_{i=1}^{w+1} p_i^T, \end{aligned}$$

the last inequality by  $\frac{1+\gamma}{1+(\frac{1}{4}+\eta)\gamma} > 1 + \gamma/4$  (recall also that  $K_T$  is a fixed constant and  $w \geq n'_T + 3$  and  $n'_T$  can be made sufficiently large with respect to  $K_T$ .) Moreover, since we assume that the left inequality in (17) holds for  $w+2$ , it follows that  $4 \prod_{i=1}^{w+1} p_i^T \geq 4 \prod_{i=1}^w p_i^S$ . Therefore the length of the interval on the right of (19) is at least  $4 \prod_{i=1}^w p_i^S$ , and so such  $p_{w+1}^S \geq 2$  always exists.

By (19), the first inequality in (17) is satisfied for  $n = w + 3$ , and the second inequality in (17) holds for  $n = w + 2$ . Recursively, it follows that (17) holds for  $n \geq n_0 := n'_T + 3$ .

To guarantee that  $S$  is a staircase rank one system we set  $a_{n,i}^S = i$  for  $n \in \mathbb{N}$  and  $i \in \{1, \dots, p_n\}$ . It remains to show that  $S \in \mathcal{C}_{\gamma/3, 3\gamma'}$  and, using (17), that (16) holds. Notice that by (17), for  $w \geq n_0$ , we have

$$\frac{p_w^S}{\prod_{i=1}^{w-1} p_i^S} \leq \frac{\prod_{i=1}^{w+1} p_i^T}{\left( \prod_{i=1}^{w-1} p_i^S \right)^2} \leq \frac{\prod_{i=1}^{w+1} p_i^T}{\left( \prod_{i=1}^{w-1} p_i^T \right)^2} = \frac{p_w^T p_{w+1}^T}{\prod_{i=1}^{w-1} p_i^T}. \quad (20)$$

Since  $T \in \mathcal{C}_{\gamma, \gamma'}$  with  $0 < \gamma < \gamma' < 1/3$  and by (6), we get

$$\frac{p_w^T p_{w+1}^T}{\prod_{i=1}^{w-1} p_i^T} \leq \frac{h_w^{\gamma'} h_{w+1}^{\gamma'}}{\prod_{i=1}^{w-1} p_i^T} \leq \frac{K_T^2 \left( \prod_{i=1}^{w-1} p_i^T \right)^{2\gamma'} (p_w^T)^{\gamma'}}{\prod_{i=1}^{w-1} p_i^T} \leq \frac{K_T^3 \left( \prod_{i=1}^{w-1} p_i^T \right)^{3\gamma'}}{\prod_{i=1}^{w-1} p_i^T}. \quad (21)$$

Notice that (20) and (21) together with the definition of spacers for  $S$  and  $\gamma' < 1/3$  imply that

$$\begin{aligned} \sum_{n=1}^{+\infty} \frac{1}{\prod_{i=1}^n p_i^S} \left( \sum_{i=1}^{p_n} a_{n,i}^S \right) &\leq \sum_{n=1}^{+\infty} \frac{(p_n^S)^2}{\prod_{i=1}^n p_i^S} \\ &\leq \sum_{n=1}^{+\infty} \frac{p_n^S}{\prod_{i=1}^{n-1} p_i^S} < +\infty, \end{aligned}$$

and hence (6) holds for  $S$  (with constant  $K_S$ ). Then, (17) and (6) (for  $T$  and for  $S$ ) ensure the validity of (16) for  $n$  large enough. This proves that  $(h_n^T)$  and  $(h_n^S)$  are  $\gamma/4$ -alternating.

Now it remains to check that  $S \in \mathcal{C}_{\gamma/3, 3\gamma'}$ . For this, we will use the following inequality: for  $n$  large enough, using several times (6) and the fact that  $T \in \mathcal{C}_{\gamma, \gamma'}$ , we get

$$\begin{aligned} h_{n+2}^T &\leq K_T \prod_{i=1}^{n+1} p_i^T \\ &= K_T p_{n+1}^T p_n^T \prod_{i=1}^{n-1} p_i^T \\ &\leq K_T p_{n+1}^T p_n^T h_n^T \\ &\leq K_T (h_{n+1}^T)^{\gamma'} (h_n^T)^{1+\gamma'} \\ &\leq K_T (K_T p_n^T h_n^T)^{\gamma'} (h_n^T)^{1+\gamma'} \\ &\leq K_T^2 ((h_n^T)^{1+\gamma'})^{\gamma'} (h_n^T)^{1+\gamma'} \\ &= K_T^2 (h_n^T)^{1+2\gamma'+\gamma'^2} \\ &\leq (h_n^T)^{1+3\gamma'}. \end{aligned}$$

By (16), it follows that for  $n$  large enough, we have

$$\frac{K_S h_{n+1}^S}{h_n^S} \leq \frac{K_S (h_{n+2}^T)^{\frac{1}{1+\gamma/4}}}{(h_n^T)^{1+\gamma/4}} \leq \frac{K_S h_{n+2}^T}{h_n^T} \leq K_S (h_n^T)^{3\gamma'} \leq (h_n^S)^{3\gamma'}.$$

On the other hand, again by (16)

$$\frac{h_{n+1}^S}{h_n^S} \geq \frac{(h_{n+1}^T)^{1+\gamma/4}}{(h_{n+1}^T)^{\frac{1}{1+\gamma/4}}} \geq (h_{n+1}^T)^{\frac{(1+\gamma/4)^2-1}{1+\gamma/4}} \geq (h_n^S)^{\gamma/2}.$$

Therefore and by (6), for  $n$  large enough, we have

$$p_n^S \in \left[ \frac{h_{n+1}^S}{h_n^S}, \frac{K_S h_{n+1}^S}{h_n^S} \right] \subset \left[ (h_n^S)^{\gamma/3}, (h_n^S)^{3\gamma'} \right]. \quad (22)$$

Using (22) and the definition of spacers for  $S$  ( $S$  is a staircase transformation) we get that  $S \in \mathcal{C}_{\gamma/3, 3\gamma'}$ . This finishes the proof.  $\square$

### 3.3 Distribution of points for maps from $\mathcal{C}_{\gamma,\gamma'}$

In this section we will do quantitative estimates on recurrence of points. Fix  $G \in \mathcal{C}(\gamma, \gamma')$ . Since  $G$  will be fixed throughout this section, we drop the superscript  $G$  and denote by  $\mathcal{T}_n, B_n, h_n$  and  $p_n$  the tower, base, height and the number of cuts at stage  $n$  (recall that  $p_n \in [h_n^\gamma, h_n^{\gamma'}]$  for  $n \geq n'_G$ ). For  $i \in \{1, \dots, p_n\}$ , let  $\mathcal{T}_{n,i} \subset \mathcal{T}_n$  denote the column over the  $i$ th cut.

We will construct some subsets of  $\mathcal{T}_\infty$ , on which we control the dynamics well. First, we cut off the ‘‘boundaries’’ of  $\mathcal{T}_n$ : let

$$F_n^G := \left( \bigcup_{i=\lfloor \frac{h_n}{n^2} \rfloor}^{h_n - \lfloor \frac{h_n}{n^2} \rfloor} G^i(B_n) \right) \cap \left( \bigcup_{i=\lfloor h_n^{\gamma/4} \rfloor}^{p_n - \lfloor h_n^{\gamma/4} \rfloor} \mathcal{T}_{n,i} \right). \quad (23)$$

Notice that since  $p_n \geq h_n^\gamma$  for  $n \geq n'_G$  and every column has equal measure, we have  $\mu_G(F_n^G) \geq 1 - \frac{4}{n^2}$  for  $n$  large enough. Let

$$F^G := \bigcap_{i \geq n_1} F_i^G, \quad (24)$$

where  $n_1$  is such that  $\mu(F^G) \geq 1 - 10^{-5}$ . In particular, if  $x \in F^G$ , then  $x \in \mathcal{T}_{n_1}$ . This will be used in the statement of the lemma below.

**Lemma 3.2.** *Fix  $0 < \xi \leq 100^{-1} \min(\gamma, 1 - \gamma')$ . There exists  $m_1 \in \mathbb{N}$  such that for every  $n \geq m_1$  the following holds: for every  $x, x' \in F^G$  such that  $d_H(x, x') = \frac{1}{h_n}$  (see (7)), for every  $r \in \{h_n, h_n + 1, \dots, (h_n)^{1+2\xi}\}$  for which  $G^r x, G^r x' \in F^G$ , we have*

$$d_H(G^r(x), G^r(x')) \geq \frac{1}{h_{n-1}}. \quad (25)$$

Moreover, for every  $x, x' \in F^G$  such that  $d_H(x, x') \leq \frac{1}{h_n}$  and for every  $i, j \in \{0, \dots, \frac{h_n}{n^3}\}$ ,  $i \neq j$ , we have

$$d_H(G^i(x), G^j(x')) \geq \frac{1}{h_{n-1}}. \quad (26)$$

*Proof.* Let  $n \geq n_1$  be large enough so that  $h_n^{\gamma'+2\xi} < h_n/n^2$ , and let  $x, x' \in F^G$  be such that  $d_H(x, x') = \frac{1}{h_n}$  (see Figure 1). Fix  $r \in \{h_n, h_n + 1, \dots, h_n^{1+2\xi}\}$  for which  $G^r x, G^r x' \in F^G$ . Let  $0 \leq \ell_1, \ell_2 \leq p_n$  be such that  $x \in \mathcal{T}_{n,\ell_1}$  and  $x' \in \mathcal{T}_{n,\ell_2}$ . Since  $x, x' \in F^G \subset F_n^G$  (see (23)),  $\ell_1, \ell_2 \in [h_n^{\gamma/4}, p_n - h_n^{\gamma/4}]$ . Notice that  $r \leq h_n^{1+2\xi}$  and the height of  $\mathcal{T}_n$  is  $h_n$ . By the definition of  $F_n^G$  it follows that

$$G^r x, G^r x' \notin \bigcup_{i=p_n - \frac{1}{2}\lfloor h_n^{\gamma/4} \rfloor}^{p_n} \mathcal{T}_{n,i}. \quad (27)$$

Since  $d_H(x, x') = \frac{1}{h_n}$ , we have  $\ell_1 \neq \ell_2$  (otherwise  $d_H(x, x') \leq \frac{1}{h_{n+1}}$ ). Assume without loss of generality that  $\ell_1 < \ell_2$ . Let  $\ell_r \geq \ell_1$  be such that  $G^r x \in \mathcal{T}_{n,\ell_r}$ . Notice that since  $r \geq h_n$  and  $G^r x \in F^G \subset \mathcal{T}_n$ , it follows that in fact  $\ell_r > \ell_1$ . Define

$$w_r := \sum_{i=\ell_1}^{\ell_r-1} (a_{n,\ell_2-\ell_1+i} - a_{n,i}).$$

By  $G \in C_{\gamma, \gamma'}$ , using (27) we get that  $a_{n, \ell_2 - \ell_1 + i} - a_{n, i} > 0$  (the spacers are monotonically placed). Since  $\ell_r > \ell_1$  it follows that  $w_r > 0$ . Moreover,  $a_{n, i} \leq h_n^{\gamma'}$  for  $i \in \{1, \dots, p_n\}$ , hence

$$w_r \leq (\ell_r - \ell_1) h_n^{\gamma'} \leq \frac{h_n^{1+2\xi}}{h_n} h_n^{\gamma'} = h_n^{\gamma'+2\xi} < h_n/n^2.$$

Let  $s$  be such that  $G^r x \in G^s B_n$ . Since  $G^r x \in F^G \subset F_n^G$ , we have  $s \geq h_n/n^2$ , and it follows by definition of  $w_r$  and the fact that  $w_r < s$  that  $G^r x' \in G^{s-w_r} B_n$ . Hence  $G^r x$  and  $G^r x'$  are not in the same level of  $\mathcal{T}_n$  and (25) follows.

For (26), notice that  $x, x' \in F^G \subset F_n^G$  (see (23)) and by assumptions  $x, x'$  are in one level of  $\mathcal{T}_n$ , i.e. for some  $s \in [\frac{h_n}{n^2}, h_n - \frac{h_n}{n^2}]$ ,  $x, x' \in G^s(B_n)$ . But by the bounds on  $s$  it follows that for  $u \in \{i, j\}$ , we have  $G^u(x) \in G^{s+u}(B_n)$  (and  $0 \leq s+u \leq h_n$ ). Since  $i \neq j$ , we have  $s+i \neq s+j$  and hence  $G^i(x)$  and  $G^j(x')$  are in different levels of  $\mathcal{T}_n$ . This gives (26).  $\square$

For  $n \in \mathbb{N}$  and  $x \in \mathcal{T}_n$  let  $i_{n,x} \in \{0, \dots, h_n - 1\}$  be such that  $x \in G^{i_{n,x}}(B_n)$ . Define

$$D_G^{x,n} := \mathcal{T}_n \setminus \bigcup_{k=-\lfloor \frac{h_n}{n^2} \rfloor}^{\lfloor \frac{h_n}{n^2} \rfloor} G^{k+i_{n,x}}(B_n) \quad (28)$$

Notice that, since  $G \in C_{\gamma, \gamma'}$ , by (8) we have  $\mu(D_G^{x,n}) \geq 1 - O(n^{-2})$ . We define

$$D_x^G := \bigcap_{n \geq n_3} D_G^{x,n}, \quad (29)$$

where  $n_3$  is such that for all  $x$ ,  $\mu(D_x^G) \geq 1 - 10^{-5}$ .

**Lemma 3.3.** *There exists  $m_2 \in \mathbb{N}$  such that for every  $x \in F^G$  (see (24)), every  $x' \in D_x^G$  every  $n \geq m_2$  and every  $i, j \in \left\{0, \dots, \left\lfloor \frac{h_{n+1}}{(n+1)^2} \right\rfloor\right\}$ , we have*

$$d_H(G^i(x), G^j(x')) \geq \frac{1}{h_n}. \quad (30)$$

*Proof.* Let  $m_2 := 2 \max(n_1, n_3)$  (where  $n_1$  and  $n_2$  are defined just after (24) and (29) respectively), and fix  $n \geq m_2$ . Let  $x, x'$  and  $i, j$  be as in the assumptions of the lemma. Assume by contradiction that  $d_H(G^i(x), G^j(x')) < \frac{1}{h_n}$ . By (7) this implies that  $G^i(x)$  and  $G^j(x')$  are in the same level of  $\mathcal{T}_{n+1}$ . Since  $x \in F^G \subset F_{n+1}^G$  (see (23)) and  $0 \leq i \leq \frac{h_{n+1}}{(n+1)^2}$ , this implies that  $G^i x$  and  $G^j(x')$  are both located  $i$  levels above  $x$  in  $\mathcal{T}_{n+1}$ . Therefore  $x$  and  $G^{j-i} x'$  are in the same level of  $\mathcal{T}_{n+1}$ . But  $|j-i| \leq \frac{h_{n+1}}{(n+1)^2}$ , which contradicts the fact that  $x' \in D_x^G \subset D_G^{x, n+1}$  (see (28)). The contradiction finishes the proof.  $\square$

## 4 A proposition which implies Theorem 2

For the rest of the paper  $(T, S) \in \mathcal{L}^{\gamma, \gamma', \delta}$  with  $0 < \delta < \gamma < \gamma' < 1$  are fixed. Since we deal with two transformations  $T$  and  $S$ , we will denote the horizontal distance (see (7)) for  $T$  and  $S$  respectively by  $d_1$  and  $d_2$ .

Recall from (24) the sets  $F^T$  and  $F^S$ . Let  $n_0 \geq 4$ , and set  $\mathcal{P} := \mathcal{P}_{n_0}$ . For  $(x, y), (x', y') \in \mathcal{T}_\infty^T \times \mathcal{T}_\infty^S$ , and  $N \in \mathbb{N}$ , consider a matching  $\theta = (i_s, j_s)_{s=1}^r$  of  $\mathcal{P}_0^N(x, y)$  and  $\mathcal{P}_0^N(x', y')$  (see Section 2.2). Then for each  $k \in \mathbb{N}$  we define

$$A_\theta^k((x, y)(x', y')) := \left\{ s \in \{1, \dots, r\} : \begin{aligned} &(T^{i_s} \times S^{i_s})(x, y), (T^{j_s} \times S^{j_s})(x', y') \in (F^T \cap \mathcal{T}_{n_0}^T) \times (F^S \cap \mathcal{T}_{n_0}^S) \text{ and} \\ &2^{-k-1} \leq \max(d_1(T^{i_s}x, T^{j_s}x'), d_2(S^{i_s}y, S^{j_s}y')) < 2^{-k} \end{aligned} \right\}. \quad (31)$$

(The dependence on  $n_0$  is only implicit to avoid cumbersome notation.) Notice that by definition of the partition  $\mathcal{P}_{n_0}$  (every product of a level in  $\mathcal{T}_{n_0}^T$  with a level in  $\mathcal{T}_{n_0}^S$  is a different atom), the definition of the horizontal distance and (4), we have for each  $s \in \{1, \dots, r\}$

$$\max(d_1(T^{i_s}x, T^{j_s}x'), d_2(S^{i_s}y, S^{j_s}y')) \leq \max\left(\frac{1}{h_{n_0}^S}, \frac{1}{h_{n_0}^T}\right) \leq \frac{1}{2^{n_0-1}}.$$

As  $n_0 \geq 4$ , for  $k < \frac{n_0}{2}$  we have  $2^{-k-1} > 2^{-n_0+1}$  and hence

$$A_\theta^k((x, y)(x', y')) = \emptyset. \quad (32)$$

The following proposition implies Theorem 2:

**Proposition 4.1.** *There exist  $n_0 \geq 100$ ,  $N_0 \in \mathbb{N}$  and a set  $B \times C \subset \mathcal{T}_\infty^T \times \mathcal{T}_\infty^S$ ,  $(\mu_T \times \mu_S)(B \times C) \geq 99/100$ , for which the following holds: for every  $(x, y) \in B \times C$  there exists a set  $D_x \times D_y \in \mathcal{T}_\infty^T \times \mathcal{T}_\infty^S$ ,  $(\mu_T \times \mu_S)(D_x \times D_y) \geq 99/100$  such that for every  $(x, y) \in B \times C$ ,  $(x', y') \in D_x \times D_y$  and every  $N \geq N_0$*

(P1) for each  $(z, w) \in \{(x, y), (x', y')\}$ ,

$$|\{i \in [0, N] : (T^i z, S^i w) \in (\mathcal{T}_{n_0}^T \cap F^T) \times (\mathcal{T}_{n_0}^S \cap F^S)\}| \geq \frac{9}{10}N;$$

(P2) for every  $N > N_0$ , every  $k \in \mathbb{N}$  and every matching  $\theta = (i_s, j_s)_{s=1}^r$  of  $(\mathcal{P}_{n_0})_0^N(x, y)$  and  $(\mathcal{P}_{n_0})_0^N(x', y')$ , we have

$$\left| A_\theta^k((x, y)(x', y')) \right| \leq \frac{N}{k^2}.$$

The proof of Proposition 4.1 is the most technical part of the paper. We will devote a separate section to its proof. Let us first show how Proposition 4.1 implies Theorem 2.

*Proof of Theorem 2.* We will prove the following:

**Claim:** For every  $(x, y) \in B \times C$ ,  $(x', y') \in D_x \times D_y$  and every  $N \geq N_0$ , we have

$$\bar{f}(\mathcal{P}_0^N(x, y), \mathcal{P}_0^N(x', y')) \geq 1/100. \quad (33)$$

(Here  $\mathcal{P}$  is the partition  $\mathcal{P}_{n_0}$ , where  $n_0$  is given by Proposition 4.1.) Before we show the **Claim**, let us show how it implies the result. Assume by contradiction that the process  $(T \times S, \mathcal{P})$  is LB. Let  $\epsilon := \frac{1}{200}$  and let  $N_\epsilon \in \mathbb{N}$ ,  $A_\epsilon \subset \mathcal{T}_\infty^T \times \mathcal{T}_\infty^S$ ,  $(\mu_T \times \mu_S)(A_\epsilon) > 1 - \epsilon$  be

from Definition 2.1. Take  $N \geq \max(N_0, N_\epsilon)$ . Notice that for every  $(x, y) \in (B \times C) \cap A_\epsilon$ , we have

$$(D_x \times D_y) \cap A_\epsilon = \emptyset. \quad (34)$$

Indeed, if the set above is non-empty, then there exists  $(x, y) \in (B \times C) \cap A_\epsilon$  and  $(x', y') \in (D_x \times D_y) \cap A_\epsilon$ , but then by (33) and (9)

$$1/100 \leq \bar{f}(\mathcal{P}_0^N(x, y), \mathcal{P}_0^N(x', y')) \leq \epsilon$$

and this is a contradiction. So (34) holds. This in turn contradicts  $\mu_T \times \mu_S(B \times C), \mu_T \times \mu_S(D_x \times D_y) \geq 99/100$  and  $\mu_T \times \mu_S(A_\epsilon) \geq 1 - \epsilon$ . So the proof is finished up to proving the **Claim**.

To prove the **Claim**, we will show that for every matching  $\theta = (i_s, j_s)_{s=1}^r$  of  $\mathcal{P}_0^N(x, y)$  and  $\mathcal{P}_0^N(x', y')$ , we have  $r \leq \frac{9N}{10}$ . Fix any such matching  $\theta = (i_s, j_s)_{s=1}^r$  and let

$$H_N := \left\{ s \in \{1, \dots, r\} : (T^{i_s}x, S^{i_s}y), (T^{j_s}x', S^{j_s}y') \in (\mathcal{T}_{n_0}^T \cap F^T) \times (\mathcal{T}_{n_0}^S \cap F^S) \right\}.$$

By (P1) we have  $|\{1, \dots, r\} \setminus H_N| \leq \frac{1}{10}N$ , hence the **Claim** follows by showing that

$$|H_N| \leq \frac{7}{10}N. \quad (35)$$

Notice that by definition of  $A_\theta^k((x, y)(x', y'))$ , we have

$$H_N = \bigcup_{k=0}^{+\infty} A_\theta^k((x, y)(x', y')).$$

Therefore, using (32) and remembering that  $n_0 \geq 100$ , we get

$$|H_N| \leq \sum_{k \geq n_0/2} \left| A_\theta^k((x, y)(x', y')) \right| \leq \sum_{k \geq 50} \left| A_\theta^k((x, y)(x', y')) \right|.$$

By (P2), this implies that

$$|H_N| \leq N \sum_{k \geq 50} \frac{1}{k^2} \leq \frac{7}{10}N.$$

This shows (35), which concludes the proof of the **Claim** and the proof of Theorem 2.  $\square$

## 5 Proof of Proposition 4.1

This section will be devoted to the proof of Proposition 4.1. We will divide the proof into several steps.

### Choice of $n_0$

Recall that  $(T, S) \in \mathcal{L}^{\gamma, \gamma', \delta}$ . Set  $\xi := \min(\gamma, 1 - \gamma', \delta)/100$ . Recall the definitions of  $m_1$  from Lemma 3.2,  $m_2$  from Lemma 3.3,  $n_1$  from (24) and  $n_3$  from (29)<sup>2</sup>. We choose  $n_0 \geq \max(m_1, m_2, n_1, n_2)$  large enough so that

$$\mu_G(\mathcal{T}_{n_0}^G) \geq 1 - 2^{-100} \quad \text{for } G \in \{T, S\}.$$

<sup>2</sup>All the constants come in two copies: for  $T$  and  $S$ , for instance we have  $m_1^T$  and  $m_1^S$  and we define  $m_1 := \max(m_1^T, m_1^S)$ . We define all other constants analogously.

We also assume  $n_0$  large enough so that some inequalities depending only on  $\xi$  and  $\delta$  hold for each  $k \geq n_0/2$  (see (42), (44), (45), (46) and (47)).

### Construction of $B$ and $C$

Since the construction follows similar lines for  $T$  and  $S$  we will do it simultaneously for  $G \in \{T, S\}$ . Recall the definition of  $F^G$  from (24). By the ergodic theorem, there exists a set  $F_{erg}^G$ ,  $\mu(F_{erg}^G) > 1 - 10^{-4}$ , and a number  $m_3^G \in \mathbb{N}$  such that for every  $x \in F_{erg}^G$  and every  $N \geq m_3^G$ ,

$$\left| \{j \in [0, N-1] : G^j x \in F^G \cap \mathcal{T}_{n_0}^G\} \right| \geq (1 - 10^{-4})N. \quad (36)$$

We then define  $B := F^T \cap F_{erg}^T$  and  $C := F^S \cap F_{erg}^S$ . We will write this in the product form:

$$B \times C = (F^T \cap F_{erg}^T) \times (F^S \cap F_{erg}^S). \quad (37)$$

Notice in particular that  $(\mu_T \times \mu_S)(B \times C) \geq 99/100$  as required in Proposition 4.1.

### Construction of $D_x$ and $D_y$ .

We define

$$D_x \times D_y := (D_x^T \cap B) \times (D_y^S \cap C), \quad (38)$$

where  $D_x^T$  and  $D_y^S$  come from (29) for  $G = T$  and  $G = S$  respectively (see also (28)). Notice that  $(\mu_T \times \mu_S)(D_x \times D_y) \geq 99/100$  as in the statement of Proposition 4.1.

### Definition of $N_0$ and proof of (P1) in Proposition 4.1

Let  $N_0 := \max(m_3^S, m_3^T)$ , where  $m_3^G$  comes from (36). Notice now that for  $(x, y) \in B \times C$  and  $(x', y') \in D_x \times D_y \subset B \times C$  (see (38)) and for  $N \geq N_0$ , (P1) holds since by (36), for any  $(z, w) \in B \times C$ ,

$$\left| \{i \in [0, N] : (T^i z, S^i w) \in (F^T \cap \mathcal{T}_{n_0}^T) \times (F^S \cap \mathcal{T}_{n_0}^S)\} \right| \geq \left(1 - \frac{2}{10^4}\right) N.$$

### 5.1 Proof of (P2) in Proposition 4.1

*Proof.* We will use Lemma 3.2, Lemma 3.3 for  $G = T$  and  $G = S$  and then Lemma 3.1. Fix  $N > N_0$ ,  $(x, y) \in B \times C$  and  $x', y' \in D_x \times D_y$ . Consider any matching  $\theta = (i_s, j_s)_{s=1}^r$  between  $(\mathcal{P}_{n_0})_0^N(x, y)$  and  $(\mathcal{P}_{n_0})_0^N(x', y')$ , and let  $k \in \mathbb{N}$ . If  $k < n_0/2$ , then  $A_\theta^k((x, y), (x', y')) = \emptyset$  (see (32)) and (P2) follows trivially. We therefore assume that  $k \geq n_0/2$ .

Take any  $s \in \{1, \dots, r\} \cap A_\theta^k((x, y), (x', y'))$ . To simplify notation, denote  $x_s = T^{i_s} x$ ,  $x'_s = T^{j_s} x'$  and  $y_s = S^{i_s} y$ ,  $y'_s = S^{j_s} y'$ . By definition of  $A_\theta^k((x, y), (x', y'))$ , we have

$$2^{-k-1} \leq \max(d_1(x_s, x'_s), d_2(y_s, y'_s)) < 2^{-k}. \quad (39)$$



Let  $n, m \in \mathbb{N}$  be unique such that  $(h_n^T)^{-1} = d_1(x_s, x'_s)$  and  $(h_m^S)^{-1} = d_2(y_s, y'_s)$ . We assume without loss of generality that  $h_n^T < h_m^S$  (notice that, by the  $\delta$ -alternation, we cannot have  $h_n^T = h_m^S$ , and the case  $h_n^T > h_m^S$  is analogous). Then by (39), we have

$$2^k < h_n^T \leq 2^{k+1}, \quad (40)$$

Then, again by the  $\delta$ -alternation and (40) we know that

$$h_m^S \geq 2^{k(1+\delta)}. \quad (41)$$

We will first show that  $2^{k(1+2\xi)+3} < N$ . Indeed, otherwise we would have  $N \leq 2^{k(1+\delta/50)+3}$  (recall that  $\xi \leq \delta/100$ ). Then by (41) there would exist some  $\eta = \eta(\delta) > 0$  such that  $h_m^S \geq N^{1+\eta}$ . We assume here that  $n_0$  is sufficiently large so that, for  $k \geq n_0/2$  and any  $h \geq 2^{k(1+\delta)}$ , we have

$$\frac{h}{(1 + \log_2 h)^2} \geq h^{\frac{1}{1+\eta}}. \quad (42)$$

Since  $m \leq 1 + \log_2 h_m$  by (4), we would have  $\frac{h_m^S}{m^2} \geq N$ . By an application of Lemma 3.3 with  $y \in F^S$ ,  $y' \in D_y^S$  and  $S^{i_s}y \in F^S \subset \mathcal{T}_m^S$ , and since  $0 \leq i_s, j_s \leq N \leq \frac{h_m^S}{m^2}$ , we would get

$$\frac{1}{h_m^S} = d_2(y_s, y'_s) = d_2(S^{i_s}y, S^{j_s}y') \geq \frac{1}{h_{m-1}^S},$$

a contradiction. Hence

$$2^{k(1+2\xi)+3} < N. \quad (43)$$

Let  $v = v(n)$  be unique such that

$$h_v^S \leq h_n^T < h_{v+1}^S.$$

By the  $\delta$ -alternation, since  $n$  is large enough and  $\xi \leq \delta/100$ , we know that

$$(h_n^T)^{1+2\xi} \leq \frac{h_{v+1}^S}{(v+1)^3} \quad \text{and} \quad (h_v^S)^{1+2\xi} \leq h_n^T. \quad (44)$$

Notice that by definition of  $v$  and since  $h_n^T < h_m^S$ , we have  $m \geq v+1$  and therefore  $d_2(y_s, y'_s) = (h_m^S)^{-1} \leq (h_{v+1}^S)^{-1}$ . Moreover,  $y_s, y'_s \in F^S$  (see (31)). Then, by (26) (for  $n = v+1$ ,  $y_s, y'_s$  and  $G = S$ ), for every  $0 \leq i, j \leq (h_n^T)^{1+2\xi} < \frac{h_{v+1}^S}{(v+1)^3}$ ,  $i \neq j$ , we get

$$d_2(S^i(y_s), S^j(y'_s)) \geq (h_v^S)^{-1} \stackrel{(44)}{\geq} (h_n^T)^{-\frac{1}{1+2\xi}} \stackrel{(40)}{\geq} 2^{-k+1}. \quad (45)$$

(For the last inequality, we also used the fact that  $k \geq n_0/2$  is large enough.) Hence, if for some  $w > s$ , we have

$$(i_w, j_w) \in [i_s, i_s + (h_n^T)^{1+2\xi}] \times [j_s, j_s + (h_n^T)^{1+2\xi}] \cap A_\theta^k((x, y), (x', y')),$$

then  $i_w - i_s = j_w - j_s$ . Indeed, if not then by (45) for  $i = i_w - i_s$  and  $j = j_w - j_s$

$$d_2(y_w, y'_w) = d_2(S^{i_w - i_s}(y_s), S^{j_w - j_s}(y'_s)) \geq 2^{-k+1}.$$

This would contradict the definition of  $A_\theta^k((x, y), (x', y'))$ . So  $i_w - i_s = j_w - j_s$ . Let  $w > s$  be such that  $i_w - i_s = j_w - j_s \in [h_n^T, (h_n^T)^{1+2\xi}]$ . By (25) (for  $G = T$  and  $x_s, x'_s \in F^T$ ) and since  $x_w, x'_w \in F^T$  (see (31)) we get for  $r_w := i_w - i_s = j_w - j_s$ ,

$$d_1(x_w, x'_w) = d_1(T^{r_w}(x_s), T^{r_w}(x'_s)) \geq (h_{n-1}^T)^{-1} \stackrel{(40)}{\geq} 2^{-k+1}. \quad (46)$$

The last inequality holds because  $T \in C_{\gamma, \gamma'}$ , hence  $(h_{n-1}^T)^{-1} \geq (h_n^T)^{-\frac{1}{1+\gamma}} \geq (h_n^T)^{-\frac{1}{1+2\xi}}$  and we remember the last inequality in (45). Therefore, if  $w > s$  is such that

$$(i_w, j_w) \in [i_s + h_n^T, i_s + (h_n^T)^{1+2\xi}] \times [j_s + h_n^T, j_s + (h_n^T)^{1+2\xi}]$$

then by (46),  $w \notin A_\theta^k((x, y), (x', y'))$ .

By (40) it follows that if

$$(i_w, j_w) \in [i_s, i_s + 2^{j(1+2\xi)}] \times [j_s, j_s + 2^{j(1+2\xi)}] \subset [i_s, i_s + (h_n^T)^{1+2\xi}] \times [j_s, j_s + (h_n^T)^{1+2\xi}]$$

is such that  $w \in A_\theta^k((x, y), (x', y'))$ , then

$$i_w - i_s = j_w - j_s < h_n^T \leq 2^{k+1}.$$

Let  $\{w_s\}_{s=1}^{r'} := A_\theta^k((x, y), (x', y')) \cap \{1, \dots, r\}$  and consider the matching subsequence given by  $\{i_{w_s}, j_{w_s}\}_{s=1}^{r'}$ .

For  $0 \leq M \leq N$  and  $w \in \{1, \dots, r'\}$ , consider the sets  $I(M, w) \subset \{1, \dots, r'\}$  and  $J(M, w) \subset \{1, \dots, r'\}$  defined as in Section 3.1. Then by the above reasoning, it follows that for  $K = 2^k$ , we have

$$\min(|I(K^{1+\xi}, s)|, |J(K^{1+\xi}, s)|) \leq 2K.$$

Since (43) holds, the assumptions of Lemma 3.1 are satisfied, and we get

$$\left| A_\theta^k((x, y), (x', y')) \right| \leq \frac{4N}{2^{k\xi}} \leq \frac{N}{k^2}. \quad (47)$$

(The last inequality since  $k \geq n_0/2$  is sufficiently large.) This finishes the proof of (P2) and therefore also the proof of Proposition 4.1.  $\square$

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