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Product of two staircase rank one transformations that is not loosely Bernoulli

Adam Kanigowski and Thierry de la Rue

18th December 2018

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Abstract

We construct two staircase rank one transformations whose Cartesian product is not loosely Bernoulli.

1 Introduction

In this note we study the loosely Bernoulli property for zero entropy, measure preserving transformations. Loose Bernoulliness was introduced by J. Feldman, [4], and A. Katok, [6]¹. Recall that a zero entropy measure preserving transformation is loosely Bernoulli (LB for short) if it is isomorphic to a transformation induced from an irrational rotation of the circle. It follows from [3] that the class of loosely Bernoulli systems is broad: it is closed under taking factors, compact extensions and inverse limits. Moreover it contains all *finite rank systems*. First non-loosely Bernoulli examples were constructed by J. Feldman [4] by

¹In [6] loosely Bernoulli is called standard.

a cutting and stacking method. D. Ornstein, D. Rudolph and B. Weiss, [3], constructed a *rank one* transformation T such that $T \times T$ is not loosely Bernoulli. The example in [3] is based on Ornstein's construction in [7] of a class of almost surely mixing rank one transformation with random spacers. On the other hand, M. Gerber constructed in [5] a mixing rank one transformation whose Cartesian square is loosely Bernoulli. An algebraic example of a zero entropy non-loosely Bernoulli transformation comes from the work of M. Ratner, [9], where it was shown that the Cartesian square of the horocycle flow (on compact quotients) is not loosely Bernoulli, although horocycle flows are loosely Bernoulli, [8].

In this note we study the loosely Bernoulli property for a natural class of mixing rank one transformations: *staircase* rank one systems (mixing for this class was proved by Adams, [1]). Loose Bernoulliness for products of staircase rank one transformations is not known. Our main result implies in particular that there exist two staircase rank one transformations whose product is not loosely Bernoulli.

1.1 Statement of the main result

For a rank one system T , let $(p_n^T)_{n \in \mathbb{N}}$, $(a_{n,i})_{i=1}^{p_n^T}$ and $(h_n^T)_{n \in \mathbb{N}}$ denote the sequence of *cuts*, *spacers* and *heights* respectively (see Section 2.1). For $0 < \gamma < \gamma' < 1$ define

$$\mathcal{C}_{\gamma, \gamma'} := \{T \in \text{Rank}(1) : \text{there exists } n'_T \in \mathbb{N} \text{ such that for every } n \geq n'_T, \\ p_n^T \in [(h_n^T)^\gamma, (h_n^T)^{\gamma'}], a_{n,i}^T > a_{n,i-1}^T \text{ for } i = 2, \dots, p_n^T, \text{ and } a_{n,p_n}^T \leq (h_n^T)^{\gamma'}\}. \quad (1)$$

Notice that if T is a *staircase* rank one transformation, i.e. $a_{n,i}^T = i$ for $n \in \mathbb{N}$ and $i \in \{1, \dots, p_n^T\}$, and if $p_n^T \in [(h_n^T)^\gamma, (h_n^T)^{\gamma'}]$, then $T \in \mathcal{C}_{\gamma, \gamma'}$.

Definition 1.1 (δ -alternating sequences). *Fix $\delta > 0$ and let (a_n) and (b_n) be two increasing sequences of positive integers. We say that (b_n) is (a_n) -alternating with exponent δ if the following holds: there exists $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$, if $m(n)$ is unique such that $b_{m(n)} \leq a_n < b_{m(n)+1}$, then $b_{m(n)}^{1+\delta} < a_n$ and $a_n^{1+\delta} < b_{m(n)+1}$. Moreover, (a_n) and (b_n) are called δ -alternating if (a_n) is (b_n) -alternating with exponent δ and (b_n) is (a_n) -alternating with exponent δ .*

For $0 < \gamma < \gamma' < 1$ let

$$\mathcal{L}^{\gamma, \gamma', \delta} := \{(T, S) \in \mathcal{C}_{\gamma, \gamma'} \times \mathcal{C}_{\gamma, \gamma'} : (h_n^T) \text{ and } (h_n^S) \text{ are } \delta\text{-alternating}\}. \quad (2)$$

Theorem 1. *If T is weakly mixing and $(T, S) \in \mathcal{L}^{\gamma, \gamma', \delta}$ with $\delta < \gamma$ then $T \times S$ is not loosely Bernoulli.*

In Subection 3.2 we will also show the following:

Lemma 1.2. *For every $T \in \mathcal{C}_{\gamma, \gamma'}$, $0 < \gamma < \gamma' < 1/3$ there exists a staircase rank one $S \in \mathcal{C}_{\gamma/3, 3\gamma'}$ such that $(T, S) \in \mathcal{L}^{\gamma/3, 3\gamma', \gamma/4}$.*

It follows from [2] that staircase rank one transformations are mixing. Therefore, Theorem 1 together with Lemma 1.2 has the following consequence:

Corollary 1.3. *There exists two staircase rank one transformations whose product is not loosely Bernoulli.*

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2 Basic definitions

2.1 Rank one maps

Recall that a rank one system is constructed by cutting and stacking: fix a sequence of *cuts* $(p_n)_{n \in \mathbb{N}}$ and a sequence of *spacers* $(a_{n,i})_{i=1}^{p_n}$, $n \in \mathbb{N}$. We define $h_1 = 1$ and inductively

$$h_n = p_{n-1}h_{n-1} + \sum_{i=1}^{p_{n-1}} a_{n-1,i}. \quad (3)$$

The sequence $(h_n)_{n \in \mathbb{N}}$ is the sequence of *heights*. The rank one system T is constructed from the sequences $(p_n)_{n \in \mathbb{N}}$ and $(a_{n,i})_{i=1}^{p_n}$, $n \in \mathbb{N}$ in the following way: we start with the interval $[0, 1]$ which we cut into p_n equal intervals $(I_i^1)_{i=1}^{p_n}$. For every $i \in \{1, \dots, p_n\}$ we put $a_{1,i}$ spacers over I_i^1 . Then we *stack* everything over I_1^1 and call this tower \mathcal{T}_2 with base I_1^1 . The transformation T on \mathcal{T}_2 just moves one level up except the last level, where it is not defined. Next, inductively at stage n , we cut the tower \mathcal{T}_n with base I_1^n into p_n equal subtowers, over i 'th subtower we put $a_{n,i}$ spacers and then we stack to get tower \mathcal{T}_{n+1} with base I_1^n . Finally, the rank one map T is defined almost everywhere, i.e. it is defined on $\mathcal{T}_\infty := \bigcup_{n \in \mathbb{N}} \mathcal{T}_n$. Often, if there are more transformations involved, we will write a superscript T in the sequences (p_n) , $(a_{n,i})$, (h_n) and \mathcal{T}_n . In what follows we will always assume that $p_n > 1$ for every n . This implies that

$$h_n \geq 2^{n-1}. \quad (4)$$

We also assume below that the total measure of all the spacers is finite:

$$\sum_{n=1}^{+\infty} \frac{1}{\prod_{k=1}^n p_k} \left(\sum_{i=1}^{p_n} a_{n,i} \right) < +\infty. \quad (5)$$

Under this condition, T preserves a probability measure μ_T given by the normalized Lebesgue measure on \mathcal{T}_∞ . It is a classical fact that rank one systems are uniquely ergodic. Hence T acts on $(\mathcal{T}_\infty, \mathcal{B}, \mu_T)$. Moreover, (5) implies the following bounds on the sequence of heights: there exists $1 \leq K < \infty$ such that for every $n \in \mathbb{N}$

$$\prod_{i=1}^{n-1} p_i \leq h_n \leq K \prod_{i=1}^{n-1} p_i. \quad (6)$$

Indeed, the LHS is immediate from (3). The RHS follows by (5) and (3) since for $\epsilon_n = \frac{1}{\prod_{i=1}^n p_i} (\sum_{i=1}^{p_n} a_{n,i})$, we have

$$\sum_{i=1}^{p_{n-1}} a_{n-1,i} = \epsilon_{n-1} \prod_{i=1}^{n-1} p_i \leq \epsilon_{n-1} p_{n-1} h_{n-1}$$

and so by (3)

$$h_n \leq p_{n-1} h_{n-1} (1 + \epsilon_{n-1})$$

and recursively

$$h_n \leq \left(\prod_{i=1}^{n-1} (1 + \epsilon_i) \right) \prod_{i=1}^{n-1} p_i \leq K \prod_{i=1}^{n-1} p_i,$$

where $K := \prod_1^\infty (1 + \epsilon_i) < \infty$ by (5). This shows (6).

For $n \in \mathbb{N}$ and x, y in one level of \mathcal{T}_n we define the *horizontal distance* of x, y , setting

$$d_H(x, y) := \frac{1}{h_{m(x, y)}}, \quad (7)$$

where $m = m(x, y) \geq n$ is the largest number such that x, y are in one level of \mathcal{T}_m (i.e. they are in different levels of \mathcal{T}_{m+1}). The set of rank one transformations (satisfying (5)) will be denoted $Rank(1)$.

2.2 Loosely Bernoulli transformations

We recall the definition of \bar{f} metric introduced in [4]. For two finite words (over a finite alphabet) $A = a_1 \dots a_k$ and $B = b_1 \dots b_k$ a *matching* between A and B is

any pair of strictly increasing sequences $(i_s, j_s)_{s=1}^r$ such that $a_{i_s} = b_{j_s}$ for $s = 1, \dots, r$. The \bar{f} distance between A and B is defined by

$$\bar{f}(A, B) = 1 - \frac{r}{k},$$

where r is the maximal cardinality over all matchings between A and B .

Let $T : (X, \mathcal{B}, \mu) \rightarrow (X, \mathcal{B}, \mu)$ be a measure preserving automorphism. For a finite partition $\mathcal{P} = (P_1, \dots, P_r)$ of X and an integer $N \geq 1$ we denote $\mathcal{P}_0^N(x) = x_0 \dots x_{N-1}$, where $x_i \in \{1, \dots, r\}$ is such that $T^i(x) \in P_{x_i}$ for $i = 0, \dots, N-1$.

Definition 2.1. *The process (T, \mathcal{P}) is said to be loosely Bernoulli if for every $\varepsilon > 0$ there exists $N_\varepsilon \in \mathbb{N}$ and a set $A_\varepsilon \in \mathcal{B}$, $\mu(A_\varepsilon) > 1 - \varepsilon$ such that for every $x, y \in A_\varepsilon$ and every $N \geq N_\varepsilon$*

$$\bar{f}(\mathcal{P}_0^N(x), \mathcal{P}_0^N(y)) < \varepsilon. \quad (8)$$

The automorphism T is loosely Bernoulli (LB for short) if for every finite measurable partition \mathcal{P} , the process (T, \mathcal{P}) is loosely Bernoulli.

To prove Theorem 1, in view of Definition 2.1 it is enough to show that there exists a finite partition \mathcal{P} such that the process $(T \times S, \mathcal{P})$ is not LB. Recall that we have an exhaustive sequence of towers (\mathcal{T}_n^T) and (\mathcal{T}_n^S) for T and S respectively. For $n \in \mathbb{N}$ let \mathcal{Q}_n be the partition of \mathcal{T}_∞^T into levels of \mathcal{T}_n^T and $\mathcal{T}_\infty^T \setminus \mathcal{T}_n^T$ ($\mathcal{T}_\infty^T \setminus \mathcal{T}_n^T$ is the union of all spacers at stages $\geq n$). Let \mathcal{R}_n be the analogous partition of \mathcal{T}_∞^S and define $\mathcal{P}_n := \mathcal{Q}_n \times \mathcal{R}_n$. Then \mathcal{P}_n is a partition of $\mathcal{T}_\infty^T \times \mathcal{T}_\infty^S$ (in fact, the sequence (\mathcal{P}_n) converges to partition into points). Theorem 1 follows by the following theorem:

Theorem 2. *Let T be weakly mixing and $(T, S) \in \mathcal{L}^{\gamma, \gamma', \delta}$ with $\delta < \gamma$. There exists m_0 such that the process $(T \times S, \mathcal{P}_{m_0})$ is not LB.*

In the following subsections we will state some lemmas which will help us in proving Theorem 2.

3 Preliminary lemmas

3.1 A combinatorial lemma: lower bound on \bar{f}

In this section we state a general combinatorial lemma which allows to give a lower bound on the \bar{f} distance. Let $(i_s)_{s=1}^r$ and $(j_s)_{s=1}^r$ be two increasing sequences of positive integers in $[0, N]$.

For $M \leq N$ and $1 \leq w < r$ define

$$I(M, w) = \{s \in \{1, \dots, r\} : i_s \in [i_w, i_w + M]\} \subset [0, N]$$

and

$$J(M, w) = \{s \in \{1, \dots, r\} : j_s \in [j_w, j_w + M]\} \subset [0, N].$$

The following property of $I(M, w)$ and $J(M, w)$ is straightforward and will be useful in the proof of the lemma below: If $s > w$ and $s \notin I(M, w)$, then

$$I(M, w) \cap I(M, s) = \emptyset, \quad (9)$$

and analogously for $J(M, w)$. We have the following lemma:

Lemma 3.1. *Let $\xi \in (0, 1)$. Let $(i_s)_{s=1}^r \in [0, N]^r$ and $(j_s)_{s=1}^r \in [0, N]^r$ be two increasing sequences of integers for which there exists a number $K \in \mathbb{N}$, $8K^{1+\xi} \leq N$ such that for every $s = 1, \dots, r$*

$$\min\left(|I(K^{1+\xi}, s)|, |J(K^{1+\xi}, s)|\right) \leq 2K, \quad (10)$$

then $r < \frac{4N}{K^\xi}$.

Proof. Set $s_1 := 1$. If $|I(K^{1+\xi}, s_1)| \geq |J(K^{1+\xi}, s_1)|$, we set $B_1 := J(K^{1+\xi}, s_1)$ otherwise, we set $B_1 := I(K^{1+\xi}, s_1)$. By (10), we have

$$|B_1| \leq 2K.$$

We then proceed inductively: assume that for some $\ell \geq 1$ we have constructed $1 = s_1 < s_2 < \dots < s_\ell \leq r$ and subsets B_1, \dots, B_ℓ of $\{1, \dots, r\}$. If $B_1 \cup \dots \cup B_\ell = \{1, \dots, r\}$, we stop here and set $v := \ell$. Otherwise, we define $s_{\ell+1}$ as the smallest element of $\{1, \dots, r\} \setminus B_1 \cup \dots \cup B_\ell$, and $B_{\ell+1}$ as either $I(K^{1+\xi}, s_{\ell+1})$ or $J(K^{1+\xi}, s_{\ell+1})$ (by choosing the set with the smallest cardinal). By (10), we always have

$$|B_{\ell+1}| \leq 2K.$$

We go on in this way until we obtain a finite sequence of set $\{B_i\}_{i=1}^v$, such that

$$B_1 \cup \dots \cup B_v = \{1, \dots, r\}, \quad (11)$$

and such that

$$|B_\ell| = \min(|I(K^{1+\xi}, s_\ell)|, |J(K^{1+\xi}, s_\ell)|) \leq 2K \text{ for every } \ell \in \{1, \dots, v\}. \quad (12)$$

Moreover, if $1 \leq \ell_1 < \ell_2 \leq v$, then $s_{\ell_2} \notin B_{\ell_1}$. If we further assume that $B_{\ell_h} = I(K^{1+\xi}, s_{\ell_h})$ for $h \in \{1, 2\}$ (or $B_{\ell_h} = J(K^{1+\xi}, s_{\ell_h})$ for $h \in \{1, 2\}$), then by construction (see also (9))

$$B_{\ell_1} \cap B_{\ell_2} = \emptyset. \quad (13)$$

By pigeonhole principle, there exists a set $A \subset \{1, \dots, v\}$, with $|A| \geq \frac{v}{2}$ and such that either for every $\ell \in A$, $B_\ell = I(K^{1+\xi}, s_\ell)$, or for every $\ell \in A$, $B_\ell = J(K^{1+\xi}, s_\ell)$. By (13), the subsets B_ℓ , $\ell \in A$ correspond to disjoint subintervals of $[0, N]$, each of them of length $K^{1+\xi}$. We thus get

$$\frac{v}{2} \leq \frac{N}{K^{1+\xi}}. \quad (14)$$

Now, (11), (12) and (14) yield

$$r \leq |B_1| + \dots + |B_v| \leq 2Kv \leq \frac{4N}{K^\xi}.$$

□

3.2 Proof of Lemma 1.2

Proof of Lemma 1.2: Let (h_n^T) denote the sequence of heights for T . We will construct a staircase $S \in \mathcal{C}_{\gamma/3, 3\gamma'}$ such that $(T, S) \in \mathcal{L}^{\gamma/3, 3\gamma', \gamma/4}$, *i.e.* the sequences (h_n^T) and (h_n^S) are $\gamma/4$ -alternating. For this, we will show that there exists n_0 such that for every $n \geq n_0$, we have

$$(h_{n-1}^S)^{1+\gamma/4} < h_n^T \quad \text{and} \quad (h_n^T)^{1+\gamma/4} < h_n^S. \quad (15)$$

Fix $\eta < 1/100$. Let K_T denote the constant K for T in (6). We will first construct $(p_i^S)_{i=1}^{+\infty}$ so that for $n \geq n_0$, we have

$$\left(\prod_{i=1}^{n-2} p_i^S \right)^{1+(\frac{1}{4}+\eta)\gamma} < \prod_{i=1}^{n-1} p_i^T \quad \text{and} \quad \left(K_T \prod_{i=1}^{n-1} p_i^T \right)^{1+\gamma/4} < \prod_{i=1}^{n-1} p_i^S. \quad (16)$$

Then we will show how to derive (15) from (16).

Since $T \in \mathcal{C}_{\gamma, \gamma'}$ it follows that for $n > n'_T$, by (6), we have

$$K_T \prod_{i=1}^n p_i^T \geq h_n^T \geq (h_{n-1}^T)^{1+\gamma} \geq \left(\prod_{i=1}^{n-1} p_i^T \right)^{1+\gamma}. \quad (17)$$

Set $p_n^S = 2$ for every $n \in \{1, \dots, n'_T + 1\}$ and $a_{n,i}^S = 0$ for every $i \in \{1, 2\}$ and $n \in \{1, \dots, n'_T + 1\}$. Then (by enlarging n'_T if necessary), we have

$$\left(\prod_{i=1}^{n'_T+1} p_i^S \right)^{1+(\frac{1}{4}+\eta)\gamma} = (2^{n'_T+1})^{1+(\frac{1}{4}+\eta)\gamma} \leq \left(\frac{1}{K_T} 2^{n'_T+1} \right)^{1+\gamma} \leq \prod_{i=1}^{n'_T+2} p_i^T.$$

(The last inequality by (17) and since $p_n^T \geq 2$.) So the left inequality in (16) holds for $n = n'_T + 3$. We then proceed inductively: having defined $(p_i^S)_{i=1}^w$ so that the left inequality in (16) holds for $n = w + 2$, we choose $p_{w+1}^S \in \mathbb{N}$ so that

$$\left(\prod_{i=1}^w p_i^S \right) p_{w+1}^S \in \left[\left(K_T \prod_{i=1}^{w+1} p_i^T \right)^{1+\gamma/4}, \left(\prod_{i=1}^{w+2} p_i^T \right)^{\frac{1}{1+(\frac{1}{4}+\eta)\gamma}} \right]. \quad (18)$$

We explain below why such a choice is always possible. Notice that by (17) for $n = w + 2$, we have

$$\left| \left(\prod_{i=1}^{w+2} p_i^T \right)^{\frac{1}{1+(\frac{1}{4}+\eta)\gamma/4}} - \left(K_T \prod_{i=1}^{w+1} p_i^T \right)^{1+\gamma/4} \right| \geq \left| \left(\frac{1}{K_T} \prod_{i=1}^{w+1} p_i^T \right)^{\frac{1+\gamma}{1+(\frac{1}{4}+\eta)\gamma}} - \left(K_T \prod_{i=1}^{w+1} p_i^T \right)^{1+\gamma/4} \right| \geq 4 \prod_{i=1}^{w+1} p_i^T,$$

the last inequality by $\frac{1+\gamma}{1+(\frac{1}{4}+\eta)\gamma} \geq 1 + \gamma/4$ (recall also that K_T is a fixed constant and $w \geq n'_T + 3$ and n'_T can be made sufficiently large with respect to K_T .) Moreover, since we assume that the left inequality in (16) holds for $w+2$, it follows that $4 \prod_{i=1}^{w+1} p_i^T \geq 4 \prod_{i=1}^w p_i^S$. Therefore the length of the interval on the right of (18) is at least $4 \prod_{i=1}^w p_i^S$, and so such $p_{w+1}^S \geq 2$ always exists. Recursively, it follows that (16) holds for $n \geq n_0 := n'_T + 3$.

To guarantee that S is a staircase rank one system we set $a_{n,i}^S = i$ for $n \in \mathbb{N}$ and $i \in \{1, \dots, p_n\}$. It remains to show that $S \in \mathcal{C}_{\gamma/3, 3\gamma'}$ and, using (16), that (15) holds. Notice that by (16), for $w \geq n_0$, we have

$$\frac{p_w^S}{\prod_{i=1}^{w-1} p_i^S} \leq \frac{\prod_{i=1}^{w+1} p_i^T}{\left(\prod_{i=1}^{w-1} p_i^S \right)^2} \leq \frac{\prod_{i=1}^{w+1} p_i^T}{\left(\prod_{i=1}^{w-1} p_i^T \right)^2} = \frac{p_w^T p_{w+1}^T}{\prod_{i=1}^{w-1} p_i^T}. \quad (19)$$

Since $T \in \mathcal{C}_{\gamma, \gamma'}$ with $0 < \gamma < \gamma' < 1/3$ and by (6), we get

$$\frac{p_w^T p_{w+1}^T}{\prod_{i=1}^{w-1} p_i^T} \leq \frac{h_w^{\gamma'} h_{w+1}^{\gamma'}}{\prod_{i=1}^{w-1} p_i^T} \leq \frac{K_T^2 \left(\prod_{i=1}^{w-1} p_i^T \right)^{2\gamma'} (p_w^T)^{\gamma'}}{\prod_{i=1}^{w-1} p_i^T} \leq \frac{K_T^3 \left(\prod_{i=1}^{w-1} p_i^T \right)^{3\gamma'}}{\prod_{i=1}^{w-1} p_i^T}. \quad (20)$$

Notice that (19) and (20) together with the definition of spacers for S and $\gamma' < 1/3$ imply that

$$\begin{aligned} \sum_{n=1}^{+\infty} \frac{1}{\prod_{i=1}^n p_i^S} \left(\sum_{i=1}^{p_n} a_{n,i}^S \right) &\leq \sum_{n=1}^{+\infty} \frac{(p_n^S)^2}{\prod_{i=1}^n p_i^S} \\ &\leq \sum_{n=1}^{+\infty} \frac{p_n^S}{\prod_{i=1}^{n-1} p_i^S} < +\infty, \end{aligned}$$

and hence (6) holds for S (with constant K_S). Then, (16) and (6) (for T and for S) ensure the validity of (15) for n large enough. This proves that (h_n^T) and (h_n^S) are $\gamma/4$ -alternating.

Now it remains to check that $S \in \mathcal{C}_{\gamma/3, 3\gamma'}$. For this, we will use the following inequality:

for n large enough, using several times (6) and the fact that $T \in \mathcal{C}_{\gamma, \gamma'}$, we get

$$\begin{aligned}
h_{n+2}^T &\leq K_T \prod_{i=1}^{n+1} p_i^T \\
&= K_T p_{n+1}^T p_n^T \prod_{i=1}^{n-1} p_i^T \\
&\leq K_T p_{n+1}^T p_n^T h_n^T \\
&\leq K_T (h_{n+1}^T)^{\gamma'} (h_n^T)^{1+\gamma'} \\
&\leq K_T (K_T p_n^T h_n^T)^{\gamma'} (h_n^T)^{1+\gamma'} \\
&\leq K_T^2 ((h_n^T)^{1+\gamma'})^{\gamma'} (h_n^T)^{1+\gamma'} \\
&= K_T^2 (h_n^T)^{1+2\gamma'+\gamma'^2} \\
&\leq (h_n^T)^{1+3\gamma'}.
\end{aligned}$$

By (15), it follows that for n large enough, we have

$$\frac{h_{n+1}^S}{h_n^S} \leq \frac{(h_{n+2}^T)^{\frac{1}{1+\gamma/4}}}{(h_n^T)^{1+\gamma/4}} \leq \frac{h_{n+2}^T}{h_n^T} \leq (h_n^T)^{3\gamma'} \leq (h_n^S)^{3\gamma'}.$$

On the other hand, again by (15)

$$\frac{h_{n+1}^S}{h_n^S} \geq \frac{(h_{n+1}^T)^{1+\gamma/4}}{(h_{n+1}^T)^{\frac{1}{1+\gamma/4}}} \geq (h_{n+1}^T)^{\frac{(1+\gamma/4)^2-1}{1+\gamma/4}} \geq (h_n^S)^{\gamma/2}.$$

Therefore and by (6), for n large enough, we have

$$p_n^S \in \left[\frac{h_{n+1}^S}{h_n^S}, \frac{K_S h_{n+1}^S}{h_n^S} \right] \subset \left[(h_n^S)^{\gamma/3}, (h_n^S)^{3\gamma'} \right]. \quad (21)$$

Using (21) and the definition of spacers for S (S is a staircase transformation) we get that $S \in \mathcal{C}_{\gamma/3, 3\gamma'}$. This finishes the proof. \square

3.3 Distribution of points for maps from $\mathcal{C}_{\gamma, \gamma'}$

In this section we will do quantitative estimates on recurrence of points. Fix $G \in \mathcal{C}(\gamma, \gamma')$. Since G will be fixed throughout this section, we drop the superscript G and denote by \mathcal{T}_n , B_n , h_n and p_n the tower, base, height and the number of cuts at stage n (recall that $p_n \in [h_n^{\gamma'}, h_n^{\gamma'}]$ for $n \geq n'_G$). For $i \in \{1, \dots, p_n\}$, let $\mathcal{T}_{n,i} \subset \mathcal{T}_n$ denote the column over the i -th cut.

We will construct some subsets of \mathcal{T}_∞ , on which we control the dynamics well. First, we cut off the ‘‘boundaries’’ of \mathcal{T}_n : let

$$F_n^G := \left(\bigcup_{i=\lfloor \frac{h_n}{n^2} \rfloor}^{\lfloor h_n - \frac{h_n}{n^2} \rfloor} G^i(B_n) \right) \cap \left(\bigcup_{i=h_n^{\gamma/4}}^{p_n - h_n^{\gamma/4}} \mathcal{T}_{n,i} \right). \quad (22)$$

Notice that since $p_n \geq h_n^\gamma$ for $n \geq n'_G$ and every column has equal measure, we have $\mu_G(F_n^G) \geq 1 - \frac{4}{n^2}$ for n large enough. Let

$$F^G := \bigcap_{i \geq n_1} F_i^G, \quad (23)$$

where n_1 is such that $\mu(F^G) \geq 1 - 10^{-5}$. In particular, if $x \in F^G$, then $x \in \mathcal{T}_{n_1}$. This will be used in the statement of the lemma below.

Lemma 3.2. *Fix $0 < \xi \leq 100^{-1} \min(\gamma, 1 - \gamma')$. There exists $m_1 \in \mathbb{N}$ such that for every $n \geq m_1$ the following holds: for every $x, x' \in F^G$ such that $d_H(x, x') = \frac{1}{h_n}$ (see (7)), for every $r \in \{h_n, h_n + 1, \dots, (h_n)^{1+2\xi}\}$ for which $G^r x, G^r x' \in F^G$, we have*

$$d_H(G^r(x), G^r(x')) \geq \frac{1}{h_{n-1}}. \quad (24)$$

Moreover, for every $x, x' \in F^G$ such that $d_H(x, x') \leq \frac{1}{h_n}$ and for every $i, j \in \{0, \dots, \frac{h_n}{n^3}\}$, $i \neq j$, we have

$$d_H(G^i(x), G^j(x')) \geq \frac{1}{h_{n-1}}. \quad (25)$$

Proof. Let $n \geq n_1$ be large enough so that $h_n^{\gamma'+2\xi} < h_n/n^2$, and let $x, x' \in F^G$ be such that $d_H(x, x') = \frac{1}{h_n}$. Fix $r \in \{h_n, h_n + 1, \dots, h_n^{1+2\xi}\}$ for which $G^r x, G^r x' \in F^G$. Let $0 \leq \ell_1, \ell_2 \leq p_n$ be such that $x \in \mathcal{T}_{n, \ell_1}$ and $x' \in \mathcal{T}_{n, \ell_2}$. Since $x, x' \in F^G \subset F_n^G$ (see (22)), $\ell_1, \ell_2 \in [h_n^{\gamma/4}, p_n - h_n^{\gamma/4}]$. Notice that $r \leq h_n^{1+2\xi}$ and the height of \mathcal{T}_n is h_n . By the choice of ξ it follows that

$$G^r x, G^r x' \notin \bigcup_{i=p_n - \frac{1}{2}h_n^{\gamma/4}}^{p_n} \mathcal{T}_{n, i}. \quad (26)$$

Since $d_H(x, x') = \frac{1}{h_n}$, we have $\ell_1 \neq \ell_2$ (otherwise $d_H(x, x') \leq \frac{1}{h_{n+1}}$). Assume WLOG that $\ell_1 < \ell_2$. Let $\ell_r \geq \ell_1$ be such that $G^r x \in \mathcal{T}_{n, \ell_r}$. Notice that since $r \geq h_n$ and $G^r x \in F^G \subset \mathcal{T}_n$, it follows that in fact $\ell_r > \ell_1$. Define

$$w_r := \sum_{i=\ell_1}^{\ell_r-1} (a_{n, \ell_2 - \ell_1 + i} - a_{n, i}).$$

By $G \in C_{\gamma, \gamma'}$, using (26) we get that $a_{n, \ell_2 - \ell_1 + i} - a_{n, i} > 0$ (the spacers are monotonically placed). Since $\ell_r > \ell_1$ it follows that $w_r > 0$. Moreover, $a_{n, i} \leq h_n^{\gamma'}$ for $i \in \{1, \dots, p_n\}$, hence

$$w_r \leq (\ell_r - \ell_1) h_n^{\gamma'} \leq \frac{h_n^{1+2\xi}}{h_n} h_n^{\gamma'} = h_n^{\gamma'+2\xi} < h_n/n^2.$$

Let s be such that $G^r x \in G^s B_n$. Since $G^r x \in F^G \subset F_n^G$, we have $s \geq h_n/n^2$, and it follows by definition of w_r and the fact that $w_r < s$ that $G^r x' \in G^{s-w_r} B_n$. Hence $G^r x$ and $G^r x'$ are not in the same level of \mathcal{T}_n and (24) follows.

For (25), notice that $x, x' \in F^G \subset F_n^G$ (see (22)) and by assumptions x, x' are in one level of \mathcal{T}_n , i.e. for some $s \in [\frac{h_n}{n^2}, h_n - \frac{h_n}{n^2}]$, $x, x' \in G^s(B_n)$. But by the bounds on s it follows that for $u \in \{i, j\}$, we have $G^u(x) \in G^{s+u}(B_n)$ (and $0 \leq s+u \leq h_n$). Since $i \neq j$, we have $s+i \neq s+j$ and hence $G^i(x)$ and $G^j(x')$ are in different levels of \mathcal{T}_n . This gives (25). \square

For $n \in \mathbb{N}$ and $x \in \mathcal{T}_n$ let $i_{n,x} \in \{0, \dots, h_n - 1\}$ be such that $x \in G^{i_{n,x}}(B_n)$. Define

$$D_G^{x,n} := \mathcal{T}_n \setminus \bigcup_{k=-\lfloor \frac{h_n}{n^2} \rfloor}^{\lfloor \frac{h_n}{n^2} \rfloor} G^{k+i_{n,x}}(B_n) \quad (27)$$

Notice that, since $G \in C_{\gamma,\gamma'}$, we have $\mu(\mathcal{T}_n) \geq 1 - O(n^{-2})$, hence $\mu(D_G^{x,n}) \geq 1 - O(n^{-2})$. We define

$$D_x^G := \bigcap_{n \geq n_3} D_G^{x,n}, \quad (28)$$

where n_3 is such that for all x , $\mu(D_x^G) \geq 1 - 10^{-5}$.

Lemma 3.3. *There exists $m_2 \in \mathbb{N}$ such that for every $x \in F^G$ (see (23)), every $x' \in D_x^G$ every $n \geq m_2$ and every $i, j \in \left\{0, \dots, \left\lfloor \frac{h_{n+1}}{(n+1)^2} \right\rfloor\right\}$ for which $G^i x \in \mathcal{T}_n$, we have*

$$d_H(G^i(x), G^j(x')) \geq \frac{1}{h_n}. \quad (29)$$

Proof. Let $m_2 := 2 \max(n_1, n_3)$ and fix $n \geq m_2$. Let x, x' and i, j be as in the assumptions of the lemma. Assume by contradiction that $d_H(G^i(x), G^j(x')) < \frac{1}{h_n}$. By (7) this implies that $G^i(x)$ and $G^j(x')$ are in the same level of \mathcal{T}_{n+1} . Since $x \in F^G \subset F_{n+1}^G$ (see (22)) and $0 \leq i \leq \frac{h_{n+1}}{(n+1)^2}$, this implies that $G^i x$ and $G^j(x')$ are both located i levels above x in \mathcal{T}_{n+1} . Therefore x and $G^{j-i}x'$ are in the same level of \mathcal{T}_{n+1} . But $|j - i| \leq \frac{h_{n+1}}{(n+1)^2}$, which contradicts the fact that $x' \in D_x^G \subset D_G^{x,n+1}$ (see (27)). The contradiction finishes the proof. \square

4 A proposition which implies Theorem 2

For the rest of the paper $(T, S) \in \mathcal{L}^{\gamma,\gamma',\delta}$ with $0 < \delta < \gamma < \gamma' < 1$ are fixed. Since we deal with two transformations T and S , we will denote the horizontal distance (see (7)) for T and S respectively by d_1 and d_2 .

Recall from (23) the sets F^T and F^S . Let $\mathcal{P} = \mathcal{P}_n$ for some $n \geq 3$. For $(x, y), (x', y') \in \mathcal{T}_\infty^T \times \mathcal{T}_\infty^S$, and $N \in \mathbb{N}$, consider a matching $\theta = (i_s, j_s)_{s=1}^r$ of $\mathcal{P}_0^N(x, y)$ and $\mathcal{P}_0^N(x', y')$. Then for each $k \in \mathbb{N}$ we define

$$\begin{aligned} A_\theta^k((x, y)(x', y')) &:= \left\{ s \in \{1, \dots, r\} : \right. \\ &(T^{i_s} \times S^{i_s})(x, y), (T^{j_s} \times S^{j_s})(x', y') \in (F^T \cap \mathcal{T}_n^T) \times (F^S \cap \mathcal{T}_n^S) \text{ and} \\ &\left. 2^{-k-1} \leq \max(d_1(T^{i_s}x, T^{j_s}x'), d_2(S^{i_s}y, S^{j_s}y')) < 2^{-k} \right\}. \quad (30) \end{aligned}$$

Notice that by definition of the partition \mathcal{P}_n (every level of $\mathcal{T}_n^T \times \mathcal{T}_n^S$ is a different atom), the definition of the horizontal distance and (4), we have for each $s \in \{1, \dots, r\}$

$$\max(d_1(T^{i_s}x, T^{j_s}x'), d_2(S^{i_s}y, S^{j_s}y')) \leq \frac{1}{h_n} \leq \frac{1}{2^{n-1}} < \frac{1}{2^{n/2}}$$

Hence for $k < \frac{n}{2}$, we have

$$A_\theta^k((x, y)(x', y')) = \emptyset. \quad (31)$$

The following proposition implies Theorem 2:

Proposition 4.1. *There exist $n_0 \geq 100$, $N_0 \in \mathbb{N}$ and a set $B \times C \subset \mathcal{T}_\infty^T \times \mathcal{T}_\infty^S$, $(\mu_T \times \mu_S)(B \times C) \geq 99/100$, for which the following holds: for every $(x, y) \in B \times C$ there exists a set $D_x \times D_y \in \mathcal{T}_\infty^T \times \mathcal{T}_\infty^S$, $(\mu_T \times \mu_S)(D_x \times D_y) \geq 99/100$ such that for every $(x, y) \in B \times C$, $(x', y') \in D_x \times D_y$ and every $N \geq N_0$*

(P1) for each $(z, w) \in \{(x, y), (x', y')\}$,

$$|\{i \in [0, N] : (T^i z, S^i w) \in (\mathcal{T}_{n_0}^T \cap F^T) \times (\mathcal{T}_{n_0}^S \cap F^S)\}| \geq \frac{9}{10}N;$$

(P2) for every $N > N_0$, every $k \in \mathbb{N}$ and every matching $\theta = (i_s, j_s)_{s=1}^r$ of $(\mathcal{P}_{n_0})_0^N(x, y)$ and $(\mathcal{P}_{n_0})_0^N(x', y')$, we have

$$\left| A_\theta^k((x, y)(x', y')) \right| \leq \frac{N}{k^2}.$$

The proof of Proposition 4.1 is the most technical part of the paper. We will devote a separate section to its proof. Let us first show how Proposition 4.1 implies Theorem 2.

Proof of Theorem 2. To simplify the notation, set $\mathcal{P} := \mathcal{P}_{n_0}$. We will prove the following:

Claim: For every $(x, y) \in B \times C$, $(x', y') \in D_x \times D_y$ and every $N \geq N_0$, we have

$$\bar{f}(\mathcal{P}_0^N(x, y), \mathcal{P}_0^N(x', y')) \geq 1/100. \quad (32)$$

Before we show the **Claim**, let us show how it implies the result. Assume by contradiction that the process $(T \times S, \mathcal{P})$ is LB. Let $\epsilon := \frac{1}{200}$ and let $N_\epsilon \in \mathbb{N}$, $A_\epsilon \subset \mathcal{T}_\infty^T \times \mathcal{T}_\infty^S$, $(\mu_T \times \mu_S)(A_\epsilon) > 1 - \epsilon$ be from Definition 2.1. Take $N \geq \max(N_0, N_\epsilon)$. Notice that for every $(x, y) \in (B \times C) \cap A_\epsilon$, we have

$$(D_x \times D_y) \cap A_\epsilon = \emptyset. \quad (33)$$

Indeed, if the set above is non-empty, then there exists $(x, y) \in (B \times C) \cap A_\epsilon$ and $(x', y') \in (D_x \times D_y) \cap A_\epsilon$, but then by (32) and (8)

$$1/100 \leq \bar{f}(\mathcal{P}_0^N(x, y), \mathcal{P}_0^N(x', y')) \leq \epsilon$$

and this is a contradiction. So (33) holds. This in turn contradicts $\mu_T \times \mu_S(B \times C), \mu_T \times \mu_S(D_x \times D_y) \geq 99/100$ and $\mu_T \times \mu_S(A_\epsilon) \geq 1 - \epsilon$. So the proof is finished up to proving the **Claim**.

To prove the **Claim**, we will show that for every matching $\theta = (i_s, j_s)_{s=1}^r$ of $\mathcal{P}_0^N(x, y)$ and $\mathcal{P}_0^N(x', y')$, we have $r \leq \frac{9N}{10}$. Fix any such matching $\theta = (i_s, j_s)_{s=1}^r$ and let

$$H_N := \left\{ s \in \{1, \dots, r\} : (T^{i_s} x, S^{i_s} y), (T^{j_s} x', S^{j_s} y') \in (\mathcal{T}_{n_0}^T \cap F^T) \times (\mathcal{T}_{n_0}^S \cap F^S) \right\}.$$

By (P1) we have $|\{1, \dots, r\} \setminus H_n| \leq \frac{1}{10}N$, hence the **Claim** follows by showing that

$$|H_N| \leq \frac{7}{10}N. \quad (34)$$

Notice that by definition of $A_\theta^k((x, y)(x', y'))$, we have

$$H_N = \bigcup_{k=0}^{+\infty} A_\theta^k((x, y)(x', y')).$$

Therefore, using (31) and remembering that $n_0 \geq 100$, we get

$$|H_N| \leq \sum_{k \geq n_0/2} \left| A_\theta^k((x, y)(x', y')) \right| \leq \sum_{k \geq 50} \left| A_\theta^k((x, y)(x', y')) \right|.$$

By (P2), this implies that

$$|H_N| \leq N \sum_{k \geq 50} \frac{1}{k^2} \leq \frac{7}{10}N.$$

This shows (34), which concludes the proof of the **Claim** and the proof of Theorem 2. \square

5 Proof of Proposition 4.1

This section will be devoted to the proof of Proposition 4.1. We will divide the proof into several steps.

5.1 Construction of B and C in Proposition 4.1

Recall that $(T, S) \in \mathcal{L}^{\gamma, \gamma', \delta}$. Set $\xi := \min(\gamma, 1 - \gamma', \delta)/100$. To define the sets B and C , we will use some notation from Section 3.3. First let n_0 be large enough so that

$$\mu_G(\mathcal{T}_{n_0}^G) \geq 1 - 2^{-100} \quad \text{for } G \in \{T, S\}.$$

Since the construction follows similar lines for T and S we will do it simultaneously for $G \in \{T, S\}$. First recall the definition of F^G from (23). By the ergodic theorem, there exists a set F_{erg}^G , $\mu(F_{erg}^G) > 1 - 10^{-4}$ and a number $m_3 \in \mathbb{N}$ such that for every $x \in F_{erg}^G$ and every $N \geq m_3$,

$$\left| \{j \in [0, N - 1] : G^j x \in F^G \cap \mathcal{T}_{n_0}^G\} \right| \geq (1 - 10^{-4})N. \quad (35)$$

We then define $B := F^T \cap F_{erg}^T$ and $C := F^S \cap F_{erg}^S$. We will write this in the product form:

$$B \times C = (F^T \cap F_{erg}^T) \times (F^S \cap F_{erg}^S). \quad (36)$$

Notice in particular that $(\mu_T \times \mu_S)(B \times C) \geq 99/100$ as required in Proposition 4.1.

Construction of D_x and D_y . We define

$$D_x \times D_y := (D_x^T \cap B) \times (D_y^S \cap C), \quad (37)$$

where D_x^T and D_y^S come from (28) for $G = T$ and $G = S$ respectively (see also (27)). Notice that $(\mu_T \times \mu_S)(D_x \times D_y) \geq 99/100$ as in the statement of Proposition 4.1

Definition of N_0 and proof of (P1) in Proposition 4.1 Let $N_0 := \max\{m_3, m_2, m_1, n_3, n_1\}$, where m_3 comes from (35), m_2 is from Lemma 3.3, m_1 is from Lemma 3.2, n_1 from (23) and n_3 from (28)². Notice now that for $(x, y) \in B \times C$ and $(x', y') \in D_x \times D_y \subset B \times C$ (see (37)) and for $N \geq N_0 \geq m_2$, (P1) holds since by (35), for any $(z, w) \in B \times C$,

$$|\{i \in [0, N] : (T^i z, S^i w) \in (F^T \cap \mathcal{T}_{n_0}^T) \times (F^S \cap \mathcal{T}_{n_0}^S)\}| \geq \left(1 - \frac{2}{10^4}\right) N.$$

5.2 Proof of (P2) in Proposition 4.1

Proof. We will use Lemma 3.2, Lemma 3.3 for $G = T$ and $G = S$ and then Lemma 3.1. Fix $N > N_0$, $(x, y) \in B \times C$ and $x', y' \in D_x \times D_y$. Consider any matching $\theta = (i_s, j_s)_{s=1}^r$ between $(\mathcal{P}_{n_0})_0^N(x, y)$ and $(\mathcal{P}_{n_0})_0^N(x', y')$, and let $k \in \mathbb{N}$. If $k < n_0/2$, then $A_\theta^k((x, y), (x', y')) = \emptyset$ (see (31)) and (P2) follows trivially. We therefore assume that $k \geq n_0/2$.

Take any $s \in \{1, \dots, r\} \cap A_\theta^k((x, y), (x', y'))$. To simplify notation, denote $x_s = T^{i_s} x$, $x'_s = T^{j_s} x'$ and $y_s = S^{i_s} y$, $y'_s = S^{j_s} y'$. By definition of $A_\theta^k((x, y), (x', y'))$, we have

$$2^{-k-1} \leq \max(d_1(x_s, x'_s), d_2(y_s, y'_s)) < 2^{-k}. \quad (38)$$

Let $n, m \in \mathbb{N}$ be unique such that $(h_n^T)^{-1} = d_1(x_s, x'_s)$ and $(h_m^S)^{-1} = d_2(y_s, y'_s)$. We assume WLOG that $h_n^T < h_m^S$ (notice that, by the δ -alternation, we cannot have $h_n^T = h_m^S$, and the case $h_n^T > h_m^S$ is analogous). Then by (38), we have

$$2^k < h_n^T \leq 2^{k+1}, \quad (39)$$

Then, again by the δ -alternation and (38) we know that

$$h_m^S \geq 2^{k(1+\frac{\delta}{2})}. \quad (40)$$

We will first show that $2^{k(1+2\xi)+3} < N$. Indeed, otherwise we would have $N \leq 2^{k(1+\delta/50)+3}$ (recall that $\xi \leq \delta/100$). Then by (40) there would exist some $\eta = \eta(\delta) > 0$ such that $h_m^S \geq N^{1+\eta}$; in particular, $\frac{h_m^S}{m^2} \geq N$ (m is sufficiently large). By an application of Lemma 3.3 with $y \in F^S$, $y' \in D_y^S$ and $S^{i_s} y \in F^G \subset \mathcal{T}_m$, and since $0 \leq i_s, j_s \leq N \leq \frac{h_m^S}{m^2}$, we would get

$$\frac{1}{h_m^S} = d_2(y_s, y'_s) = d_2(S^{i_s} y, S^{j_s} y') \geq \frac{1}{h_{m-1}^S},$$

a contradiction. Hence

$$2^{k(1+2\xi)+3} < N. \quad (41)$$

Let $v = v(n)$ be unique such that

$$h_v^S \leq h_n^T < h_{v+1}^S.$$

By the δ -alternation, since n is large enough and $\xi \leq \delta/100$, we know that

$$(h_n^T)^{1+2\xi} \leq \frac{h_{v+1}^S}{(v+1)^3} \quad \text{and} \quad (h_v^S)^{1+2\xi} \leq h_n^T. \quad (42)$$

²All the constants come in two copies: for T and S , for instance we have m_3^T and m_3^S and we define $m_3 := \max(m_3^T, m_3^S)$. We define all other constants analogously.

Notice that by definition of v and since $h_n^T < h_m^S$, we have $m \geq v + 1$ and therefore $d_2(y_s, y'_s) = (h_m^S)^{-1} \leq (h_{v+1}^S)^{-1}$. Moreover, $y_s, y'_s \in F^S$ (see (30)). Then, by (25) (for $n = v + 1$, y_s, y'_s and $G = S$), for every $0 \leq i, j \leq (h_n^T)^{1+2\xi} < \frac{h_{v+1}^S}{(v+1)^3}$, $i \neq j$, we get

$$d_2(S^i(y_s), S^j(y'_s)) \geq (h_v^S)^{-1} \stackrel{(42)}{\geq} (h_n^T)^{-\frac{1}{1+2\xi}} \stackrel{(39)}{\geq} 2^{-k+1}. \quad (43)$$

(For the last inequality, we also used the fact that $k \geq n_0/2$ is large enough.) Hence, if for some $w > s$, we have

$$(i_w, j_w) \in [i_s, i_s + (h_n^T)^{1+2\xi}] \times [j_s, j_s + (h_n^T)^{1+2\xi}] \cap A_\theta^k((x, y), (x', y')),$$

then $i_w - i_s = j_w - j_s$. Indeed, if not then by (43) for $i = i_w - i_s$ and $j = j_w - j_s$

$$d_2(y_w, y'_w) = d_2(S^{i_w - i_s}(y_s), S^{j_w - j_s}(y'_s)) \geq 2^{-k+1}.$$

This would contradict the definition of $A_\theta^k((x, y), (x', y'))$. So $i_w - i_s = j_w - j_s$. Let $w > s$ be such that $i_w - i_s = j_w - j_s \in [h_n^T, (h_n^T)^{1+2\xi}]$. By (24) (for $G = T$ and $x_s, x'_s \in F^T$) and since $x_w, x'_w \in F^T$ (see (30)) we get for $r_w := i_w - i_s = j_w - j_s$,

$$d_1(x_w, x'_w) = d_1(T^{r_w}(x_s), T^{r_w}(x'_s)) \geq (h_{n-1}^T)^{-1} \stackrel{(39)}{\geq} 2^{-k+1}. \quad (44)$$

The last inequality holds because $T \in C_{\gamma, \gamma'}$, hence $(h_{n-1}^T)^{-1} \geq (h_n^T)^{-\frac{1}{1+\gamma}} \geq (h_n^T)^{-\frac{1}{1+2\xi}}$ and we remember the last inequality in (43). Therefore, if $w > s$ is such that

$$(i_w, j_w) \in [i_s + h_n^T, i_s + (h_n^T)^{1+2\xi}] \times [j_s + h_n^T, j_s + (h_n^T)^{1+2\xi}]$$

then by (44), $w \notin A_\theta^k((x, y), (x', y'))$.

By (39) it follows that if

$$(i_w, j_w) \in [i_s, i_s + 2^{j(1+2\xi)}] \times [j_s, j_s + 2^{j(1+2\xi)}] \subset [i_s, i_s + (h_n^T)^{1+2\xi}] \times [j_s, j_s + (h_n^T)^{1+2\xi}]$$

is such that $w \in A_\theta^k((x, y), (x', y'))$, then

$$i_w - i_s = j_w - j_s < h_n^T \leq 2^{k+1}.$$

Let $\{w_s\}_{s=1}^{r'} := A_\theta^k((x, y), (x', y')) \cap \{1, \dots, r\}$ and consider the matching subsequence given by $\{i_{w_s}, j_{w_s}\}_{s=1}^{r'}$.

For $0 \leq M \leq N$ and $w \in \{1, \dots, r'\}$, consider the sets $I(M, w) \subset \{1, \dots, r'\}$ and $J(M, w) \subset \{1, \dots, r'\}$ defined as in Section 3.1. Then by the above reasoning, it follows that for $K = 2^k$, we have

$$\min \left(|I(K^{1+\xi}, s)|, |J(K^{1+\xi}, s)| \right) \leq 2K.$$

Since (41) holds, the assumptions of Lemma 3.1 are satisfied, and we get

$$\left| A_\theta^k((x, y), (x', y')) \right| \leq \frac{4N}{2^{k\xi}} \leq \frac{N}{k^2}.$$

(The last inequality since $k \geq n_0/2$ is sufficiently large.) This finishes the proof of (P2) and therefore also the proof of Proposition 4.1 \square

References

- [1] T. Adams, *Smorodinsky's conjecture on rank-one mixing*, Proceedings of the American Mathematical Society **126** (1998), no. 3, 739–744.
- [2] C. Silva D. Creutz, *Mixing on rank-one transformations*, Studia Math. **199** (2010), no. 1, 43–72.
- [3] B. Weiss D. Ornstein, D. Rudolph, *Equivalence of measure preserving transformations*, Mem. Amer. Math. Soc. **37** (1982), no. 262, xii+116. MR 653094
- [4] J. Feldman, *New K -automorphisms and a problem of Kakutani*, Israel J. Math. **24** (1976), no. 1, 16–38.
- [5] M. Gerber, *A zero-entropy mixing transformation whose product with itself is loosely bernoulli*, Israel Journal of Mathematics **38** (1981), 1–22.
- [6] A. B. Katok, *Monotone equivalence in ergodic theory*, Izv. Akad. Nauk SSSR Ser. Mat. **41** (1977), no. 1, 104–157, 231.
- [7] D. Ornstein, *On the root problem in ergodic theory.*, In Proceedings of the Sixth Berkeley Symposium on Mathematical Statistics and Probability (Univ. California, Berkeley, Calif., 1970/1971), Vol. II: Probability theory **II** (1972), 347–356.
- [8] M. Ratner, *Horocycle flows are loosely Bernoulli*, Israel J. Math. **31** (1978), no. 2, 122–132.
- [9] ———, *The Cartesian square of the horocycle flow is not loosely Bernoulli*, Israel J. Math. **34** (1979), no. 1-2, 72–96 (1980).