

Common mathematical framework for stochastic reverberation models

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1 In the field of room acoustics, it is well known that reverberation can be character-
2 ized statistically in a particular region of the time-frequency domain (after the tran-
3 sition time and above Schroeder's frequency). Since the 1950s, various formulas have
4 been established, focusing on particular aspects of reverberation: exponential decay
5 over time, correlations between frequencies, correlations between sensors at each fre-
6 quency, and time-frequency distribution. In this paper, we introduce a stochastic
7 reverberation model, that permits us to retrieve all these well-known results within
8 a common mathematical framework. To the best of our knowledge, this is the first
9 time that such a unification work is presented. The benefits are multiple: several
10 formulas generalizing the classical results are established, that jointly characterize
11 the spatial, temporal and spectral properties of late reverberation.

Keywords: Reverberation; Diffusion; Room impulse response; Stochastic models

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12 **I. INTRODUCTION**

13

14 When a microphone records a sound produced by an audio source in a room, the received
15 signal is made of several contributions ([Kuttruff, 2014](#)): firstly, the *direct sound*, that cor-
16 responds to the direct propagation of the sound wave from the source to the microphone,
17 then a few *early reflections*, that are due to the sound wave reflections on the various room
18 surfaces (walls, floor, ceiling...), and finally the *late reverberation*: after a time called *tran-*
19 *sition time* ([Polack, 1992](#), [1993](#)), reflections are so frequent that they form a continuum and,
20 because the sound is partially *absorbed* by the room surfaces at every reflection, the sound
21 level decays exponentially over time. This phenomenon is called *reverberation*, and it can be
22 modeled as the convolution between the source signal and a causal *room impulse response*
23 (RIR), made of a few isolated impulses before the transition time, and of a continuous,
24 exponentially decaying, random process in late reverberation. The Fourier transform of the
25 RIR is called *room frequency response* (RFR), and the modal theory ([Balian and Bloch,](#)
26 [1970](#); [Maa, 1939](#); [Polack, 2007](#)) shows that its profile is qualitatively similar to that of the
27 RIR: below a frequency called *Schroeder's frequency*, the RFR is made of a few isolated
28 modes, and above this frequency the modes become so dense that they can be represented
29 as a continuous random process ([Schroeder, 1962](#), [1987](#); [Schroeder and Kuttruff, 1962](#)).

30 To sum up, reverberation can be modeled as a stochastic process in a rectangular region
31 of the time-frequency domain ([Jot et al., 1997](#)), as depicted in Fig. 1. If in addition the
32 source and the microphones are located at least a half-wavelength away from the walls,

33 then in this time-frequency region, the sound field can generally be approximated as *diffuse*
 34 (Cremer *et al.*, 1982; Joyce, 1975; Schultz, 1971). Diffusion is a consequence of the reflections
 35 on the room surfaces not being *specular* (*i.e.* mirror-like), but rather *scattered* in various
 36 directions, as represented in Fig. 2. After many reflections, the sound field can be considered
 37 as *isotropic*: the sound waves come uniformly from all directions.

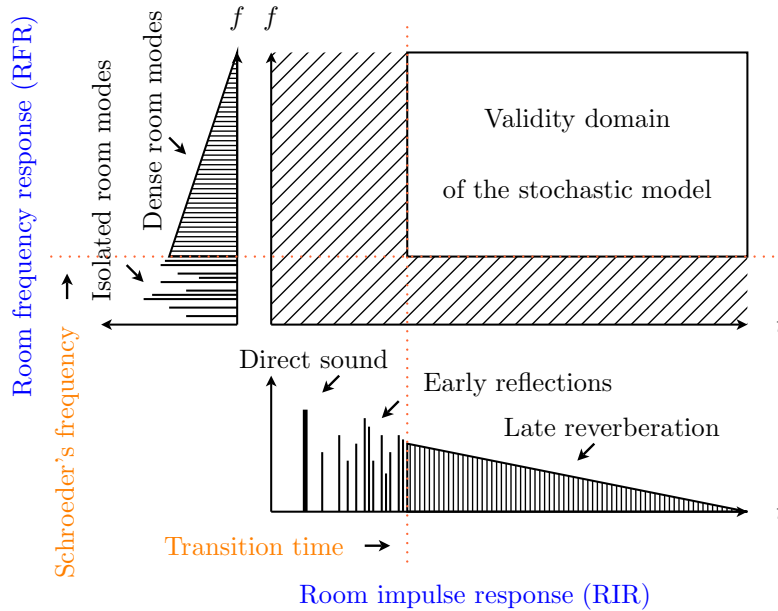


FIG. 1. Time-frequency profile of reverberation (adapted from (Jot *et al.*, 1997, p. 30) and (Baskind, 2003, p. 20)).

38 Historically, the first stochastic reverberation model is due to Schroeder (1962) and Moorer
 39 (1979): the RIR at microphone i is

$$h_i(t) = b_i(t)e^{-\alpha t}\mathbf{1}_{t \geq 0} \quad (1)$$

40 where $\alpha > 0$ and $b_i(t)$ is a centered white Gaussian process. Parameter α is related to the
 41 reverberation time T_r in seconds by the equation $T_r = \frac{3 \ln(10)}{\alpha}$. The Gaussian distribution of

42 $b_i(t)$ arises from the central limit theorem: in late reverberation, $h_i(t)$ is the sum of many
 43 independent contributions.

44 [Schroeder \(1962, 1987\)](#); [Schroeder and Kuttruff \(1962\)](#) also noticed that the independency
 45 of the samples $h_i(t)$ implies that the RFR, defined as their Fourier transform¹ $\mathcal{F}_{h_i}(f)$, is a
 46 stationary random process. From (1), he derived several formulas that can be summarized
 47 by expressing the complex autocorrelation function² of $\mathcal{F}_{h_i}(f)$:

$$\text{corr} [\mathcal{F}_{h_i}(f_1), \mathcal{F}_{h_i}(f_2)] = \frac{1}{1 + i\pi \frac{f_1 - f_2}{\alpha}}. \quad (2)$$

48 Following a similar approach in the spectral domain, under the diffuse field assumption, [Cook](#)
 49 *et al.* (1955) computed the correlation at frequency f between two sensors at distance D
 50 (with $c > 0$ the speed of sound in the air):

$$\text{corr} [\mathcal{F}_{h_1}(f), \mathcal{F}_{h_2}(f)] = \text{sinc} \left(\frac{2\pi f D}{c} \right). \quad (3)$$

51 Equation (3) was later generalized to combinations of pressure and velocity sensors ([Jacobsen](#)
 52 *and Roisin, 2000*) and to differential microphones ([Elko, 2001\).](#)

53 Finally, [Polack \(1988\)](#) generalized model (1) by assuming that $b_i(t)$ is a centered station-
 54 ary Gaussian process, whose power spectral density (PSD) $B(f)$ has slow variations³. Then
 55 he showed that the Wigner distribution⁴ ([Cohen, 1989](#)) of the RIR is

$$\mathcal{W}_{h_i, h_i}(t, f) = B(f)e^{-2\alpha t} \mathbf{1}_{t \geq 0}. \quad (4)$$

56 In order to account for the fact that the attenuation coefficient α actually depends on
 57 the frequency f , he also proposed an empirical generalization of (4):

$$\mathcal{W}_{h_i, h_i}(t, f) = B(f)e^{-2\alpha(f)t} \mathbf{1}_{t \geq 0}. \quad (5)$$

58 In other respects, based on the billiard theory, Polack (1992, 1993) also showed that
 59 the durations of the various trajectories in a room, from a given source position to the
 60 microphone, are distributed according to a Poisson process (Chiu *et al.*, 2013).

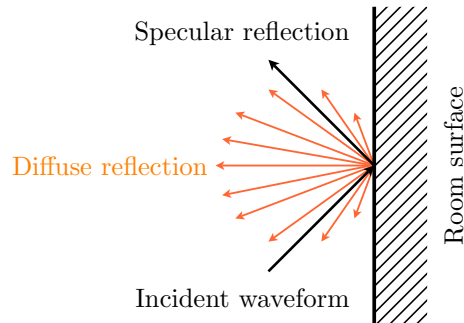


FIG. 2. Specular vs. diffuse reflection.

61 In this paper, we propose a unified stochastic model of reverberation, that will permit
 62 us to retrieve all formulas (1) to (4) in a common mathematical framework⁵, to establish
 63 a link with the Poisson distribution proposed by Polack, and to show how the probability
 64 distribution of the RIR, which is impulsive in early reverberation, converges to the Gaussian
 65 distribution in late reverberation. In addition, this model will also permit us to go deeper
 66 into the description of the statistical properties of the RIR over the space, time and frequency
 67 domains, and to prove several new results.

68 This paper is structured as follows: Section II presents important mathematical defini-
 69 tions and notation that will be used throughout the paper. Then our general stochastic
 70 reverberation model is introduced in Section III. The statistical properties of this model at
 71 one sensor are investigated in Section IV. The statistical relationships between two sensors
 72 are then analyzed in Section V. The new model is validated experimentally in Section VI,

73 by comparing the predicted second order statistics with their estimates from a database of
 74 synthetic RIRs. Finally, some conclusions and perspectives are presented in Section VII. In
 75 order to make the main discussion as clear as possible, all mathematical proofs were moved
 76 to Appendices A to G.

77 II. MATHEMATICAL DEFINITIONS

- 78 • \mathbb{N} : set of whole numbers
- 79 • \mathbb{R}, \mathbb{C} : sets of real and complex numbers, respectively
- 80 • \mathbb{R}_+ : set of nonnegative real numbers
- 81 • $i = \sqrt{-1}$: imaginary unit
- 82 • \mathbf{x} (bold font), z (regular): vector and scalar, respectively
- 83 • $[a, b]$: closed interval, including a and $b \in \mathbb{R}$
- 84 • $]a, b[$: open interval, excluding a and $b \in \mathbb{R}$
- 85 • $L^p(V)$, where V is a Borel set and $p \in \mathbb{N} \setminus \{0\}$: Lebesgue space of measurable functions
 86 f of support V , such that $\|f\|_p = (\int_V |f(\mathbf{x})|^p d\mathbf{x})^{\frac{1}{p}} < +\infty$
- 87 • $L^\infty(V)$, where V is a Borel set: Lebesgue space of essentially bounded functions f of
 88 support V (i.e. such that $\|f\|_\infty = \text{ess sup}_V |f| < +\infty$)
- 89 • δ : Dirac delta function
- 90 • $\|\cdot\|_2$: Euclidean/Hermitian norm of a vector or a function

- 91 • \bar{z} : complex conjugate of $z \in \mathbb{C}$
- 92 • \mathbf{x}^\top : transpose of vector \mathbf{x}
- 93 • \mathcal{S}^2 : unit sphere in \mathbb{R}^3 ($\mathcal{S}^2 = \{\mathbf{x} \in \mathbb{R}^3; \|\mathbf{x}\|_2 = 1\}$)
- 94 • $\mathbb{E}[X]$: expected value of a random variable X
- 95 • $\phi_X(\theta) = \mathbb{E}[e^{i\theta X}]$: characteristic function of a real random variable X
- 96 • Covariance of two complex random variables X and Y :

$$\text{cov}[X, Y] = \mathbb{E}[(X - \mathbb{E}[X])(\overline{Y - \mathbb{E}[Y]})] \quad (6)$$

- 97 • $\text{var}[X] = \text{cov}[X, X]$: variance of a random variable X
- 98 • Correlation of two complex random variables X and Y :

$$\text{corr}[X, Y] = \frac{\text{cov}[X, Y]}{\sqrt{\text{var}[X] \text{var}[Y]}} \quad (7)$$

- 99 • $\mathcal{P}(\lambda)$: Poisson distribution of parameter $\lambda > 0$:

$$N \sim \mathcal{P}(\lambda) \Leftrightarrow P(N = n) = e^{-\lambda} \frac{\lambda^n}{n!} \Leftrightarrow \phi_N(\theta) = e^{\lambda(e^{i\theta} - 1)} \quad (8)$$

- 100 • $\text{sinc}(x) = \frac{\sin(x)}{x}$: cardinal sine function
- 101 • $\mathbf{1}_A$: indicator function of a set A ($\mathbf{1}_A(x)$ is 1 if $x \in A$ or 0 if $x \notin A$)
- 102 • $\tilde{\psi}(t) = \overline{\psi(-t)}$: conjugate and time-reverse of $\psi : \mathbb{R} \rightarrow \mathbb{C}$
- Convolution of two functions ψ_1 and $\psi_2 : \mathbb{R} \rightarrow \mathbb{C}$:

$$(\psi_1 * \psi_2)(t) = \int_{u \in \mathbb{R}} \psi_1(u) \psi_2(t - u) du$$

- 103 • Fourier transform of a function $\psi : \mathbb{R} \rightarrow \mathbb{C}$:

$$\mathcal{F}_\psi(f) = \int_{t \in \mathbb{R}} \psi(t) e^{-2i\pi f t} dt \quad (f \in \mathbb{R}) \quad (9)$$

- 104 • Two-sided Laplace transform of a function $\psi : \mathbb{R} \rightarrow \mathbb{C}$:

$$\mathcal{L}_\psi(s) = \int_{t \in \mathbb{R}} \psi(t) e^{-st} dt \quad (s \in \mathbb{C}) \quad (10)$$

- 105 • Wigner distribution (*a.k.a.* Wigner-Ville distribution) of two second-order random
 106 processes $\psi_1(t)$ and $\psi_2(t)$:

$$\mathcal{W}_{\psi_1, \psi_2}(t, f) = \int_{\mathbb{R}} \text{cov}[\psi_1(t + \frac{u}{2}), \psi_2(t - \frac{u}{2})] e^{-2i\pi f u} du. \quad (11)$$

107 **III. DEFINITION OF THE STOCHASTIC MODEL**

108 The model that we present in this section is based on the source image principle ([Allen](#)
 109 [and Berkley, 1979](#); [Kuttruff, 2014](#)). As illustrated in Fig. 3 in the case of specular reflections
 110 in a rectangular room⁶, the trajectory inside the room from the real source to the microphone
 111 is equivalent to a virtual straight trajectory from a so-called *source image* which is outside
 112 the room. A remarkable property of this principle is that, regardless of the room dimen-
 113 sions, the density of the source images is uniform in the whole space: the number of source
 114 images contained in a given disk, of radius sufficiently larger than the room dimensions, is
 116 approximately invariant under any translation of this disk.

117 Since we aim to define a general stochastic reverberation model that holds for any room
 118 geometry, based on the billiard theory ([Polack, 1992](#); [1993](#)), we will consider that the posi-
 119 tions of the source images are random and uniformly distributed (note that this assumption

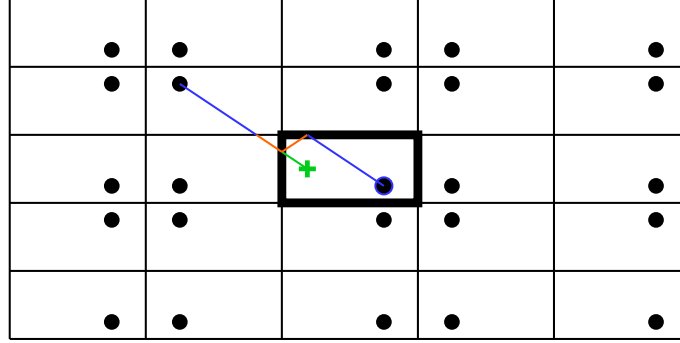


FIG. 3. Positions of microphone (plus sign), source (thick point) and source images (other points). The original room walls are drawn with thick lines. A virtual straight trajectory from one source image to the microphone is drawn, along with the real trajectory in the original room.

120 is *a fortiori* valid in the case of a diffuse sound field, which is uniform). More precisely, given
 121 a Borel set $V \subset \mathbb{R}^3$ of finite volume $|V|$, we will assume that the number $N(V)$ of source
 122 images contained in V follows a Poisson distribution of parameter $\lambda|V|$: $N(V) \sim \mathcal{P}(\lambda|V|)$.
 123 Mathematically, this is formalized through the concept of Poisson random measures with
 124 independent increments: given a non-negative, locally integrable function $\Lambda(\mathbf{x})$ on \mathbb{R}^p , the
 125 Poisson random increment $dN(\mathbf{x}) \sim \mathcal{P}(\Lambda(\mathbf{x})d\mathbf{x})$ corresponds to an infinitesimal volume
 126 $|V| = d\mathbf{x}$. Then for any Borel set $V \subset \mathbb{R}^p$ of finite Lebesgue measure, the number
 127 $N(V) = \int_V dN(\mathbf{x})$ of points contained in V follows a Poisson distribution of parameter
 128 $\int_V \Lambda(\mathbf{x})d\mathbf{x}$: $N(V) \sim \mathcal{P}(\int_V \Lambda(\mathbf{x})d\mathbf{x})$. In the proposed stochastic reverberation model, we
 129 will consider a spatially uniform distribution of the source images in the 3D-space, so that
 130 $p = 3$ and $\Lambda(\mathbf{x}) = \lambda > 0$ is constant.

131 In other respects, we will assume that the sound waves undergo an exponential attenu-
 132 ation along their trajectories, that is due to the multiple reflections on the room surfaces

133 and to the propagation in the air. In this paper we focus on the case of omnidirectional
 134 microphones, so we will further assume that this attenuation is isotropic (in accordance with
 135 the diffuse field approximation) and independent of the frequency. It will thus only depend
 136 on the length of the trajectory, as in (Polack, 1988).

137 Finally, we suppose that several microphones⁷ indexed by an integer i are placed at
 138 arbitrary positions \mathbf{x}_i in the room.

139 We end up with the following model:

$$h_i(t) = \int_{\mathbf{x} \in \mathbb{R}^3} h_i(t, \mathbf{x}) e^{-\frac{\alpha}{c} \|\mathbf{x} - \mathbf{x}_i\|_2} dN(\mathbf{x}), \quad (12)$$

140 where $h_i(t)$ is the RIR at microphone i , $\alpha > 0$ is the attenuation coefficient (in Hz), and
 141 $c > 0$ is the speed of sound in the air (approximately 340 m/s in usual conditions). The
 142 impulse $h_i(t, \mathbf{x})$, propagated from the source image at position \mathbf{x} , is modeled as a coherent
 143 sum of monochromatic spherical waves:

$$h_i(t, \mathbf{x}) = \int_{f \in \mathbb{R}} A(f) \frac{e^{2i\pi f \left(t - \frac{\|\mathbf{x} - \mathbf{x}_i\|_2}{c} \right)}}{\|\mathbf{x} - \mathbf{x}_i\|_2} df, \quad (13)$$

144 where $A(f)$ is a linear-phase frequency response (in order to ensure coherence). In Ap-
 145 pendix A, we show that (12) and (13) are equivalent to the following model:

146 **Definition 1** (Unified stochastic reverberation model).

147 *Let $\lambda > 0$, $T > 0$, $\alpha > 0$, and $c > 0$. Let $dN(\mathbf{x})$ be a uniform Poisson random measure on
 148 \mathbb{R}^3 with independent increments:*

$$dN(\mathbf{x}) \sim \mathcal{P}(\lambda d\mathbf{x}). \quad (14)$$

149 Let $g(t) \in L^\infty([-T, T])$, such that

$$\mathcal{F}_g(0) = \frac{d\mathcal{F}_g}{df}(0) = 0, \quad (15)$$

$$\forall f \in \mathbb{R}, \mathcal{L}_g(\alpha + 2\iota\pi f) \geq 0. \quad (16)$$

150 At any sensor position $\mathbf{x}_i \in \mathbb{R}^3$, $h_i(t)$ is defined as

$$\forall t \in \mathbb{R}, h_i(t) = e^{-\alpha(t-T)} b_i(t), \quad (17)$$

151 where

$$b_i(t) = \int_{\mathbf{x} \in \mathbb{R}^3} g\left(t - T - \frac{\|\mathbf{x} - \mathbf{x}_i\|_2}{c}\right) \frac{dN(\mathbf{x})}{\|\mathbf{x} - \mathbf{x}_i\|_2}. \quad (18)$$

152 Equivalently, the Fourier transform of $h_i(t)$ is

$$\mathcal{F}_{h_i}(f) = \mathcal{L}_g(\alpha + 2\iota\pi f) e^{-2\iota\pi f T} \int_{\mathbb{R}^3} \frac{e^{-\frac{\alpha+2\iota\pi f}{c}\|\mathbf{x}-\mathbf{x}_i\|_2}}{\|\mathbf{x} - \mathbf{x}_i\|_2} dN(\mathbf{x}). \quad (19)$$

153 This definition calls for comments. Firstly, the linear-phase frequency response $A(f)$

154 in (13) was parameterized as

$$A(f) = \mathcal{L}_g(\alpha + 2\iota\pi f) e^{-2\iota\pi f T}. \quad (20)$$

155 This technical definition aims to simplify the mathematical developments in the next sec-

156 tions. Secondly, any function $g \in L^\infty([-T, T])$ is such that $\mathcal{F}_g(f)$ and $f \mapsto \mathcal{L}_g(\alpha + 2\iota\pi f)$ are

157 smooth, so $\mathcal{F}_g(0)$, $\frac{d\mathcal{F}_g}{df}(0)$ and $\mathcal{L}_g(\alpha + 2\iota\pi f)$ are well-defined. The constraints (15) and (16)

158 imposed to g are required to prove Lemma 1 and Proposition 2 in Section IV⁸. Moreover, the

159 support of g is chosen so that $h_i(t)$ in (17) and $b_i(t)$ in (18) are causal. Thirdly, the existence

160 of functions g that satisfy these constraints is guaranteed by Lemma 3 in Appendix A.

161 Now it is time to investigate the properties of this model. In Section IV, we will focus on
 162 one sensor at spatial position \mathbf{x}_i . Then in Section V, we will analyze the spatial relationships
 163 between two sensors at different positions \mathbf{x}_i and \mathbf{x}_j .

164 **IV. STATISTICAL PROPERTIES AT ONE SENSOR**

165 Let us first introduce an equivalent model definition:

166 **Proposition 1** (Equivalent model definition at one sensor). *With the same notation as in*
 167 *Definition 1, we have:*

$$b_i(t) = \int_{r \in \mathbb{R}_+} g\left(t - T - \frac{r}{c}\right) \frac{dN(r)}{r}, \quad (21)$$

$$\mathcal{F}_{h_i}(f) = \mathcal{L}_g(\alpha + 2i\pi f) e^{-2i\pi f T} \int_{r \in \mathbb{R}_+} \frac{e^{-\frac{\alpha + 2i\pi f}{c} r} dN(r)}{r} \quad (22)$$

168 where $dN(r)$ are independent Poisson increments on \mathbb{R}_+ :

$$dN(r) \sim \mathcal{P}(4\pi\lambda r^2 dr). \quad (23)$$

169 Proposition 1 is proved in Appendix A. Let us now investigate the statistical properties
 170 of this model:

171 **Lemma 1.** *The random process $b_i(t)$ introduced in Definition 1 is such that $\forall t \geq 2T$,*

172 1. $b_i(t)$ is a centered wide sense stationary (WSS) process⁹, of autocovariance function

$$\forall \tau \in \mathbb{R}, \Gamma(\tau) = \text{cov}[b_i(t + \tau), b_i(t)] = 4\pi\lambda c (\tilde{g} * g)(\tau), \quad (24)$$

173 autocorrelation function

$$\forall \tau \in \mathbb{R}, \gamma(\tau) = \text{corr}[b_i(t + \tau), b_i(t)] = \frac{(\tilde{g} * g)(\tau)}{\|g\|_2^2}, \quad (25)$$

174 *and power spectral density*

$$\forall f \in \mathbb{R}, B(f) = \mathcal{F}_\Gamma(f) = 4\pi\lambda c |\mathcal{F}_g(f)|^2. \quad (26)$$

175 2. When $t \rightarrow +\infty$ (i.e. $t \gg T$), $b_i(t)$ converges in law to a stationary Gaussian process.

176 Lemma 1 is proved in Appendix C.

177 **Proposition 2** (Statistical properties at one sensor position). *With the notation introduced*
 178 *in Lemma 1, the reverberation model in Definition 1 has the following properties:*

179 1. *First order moments:*

180 • *in the spectral domain:*

$$\forall f \in \mathbb{R}, \mathbb{E}[\mathcal{F}_{h_i}(f)] = \frac{4\pi\lambda c^2 \mathcal{L}_g(\alpha + 2i\pi f) e^{-2i\pi f T}}{(\alpha + 2i\pi f)^2} \quad (27)$$

181 • *in the time domain:*

$$\forall t \geq 2T, \mathbb{E}[h_i(t)] = 0. \quad (28)$$

182 2. *Second order moments:*

183 • *in the spectral domain:*

$$\forall f \in \mathbb{R}, \text{var}[\mathcal{F}_{h_i}(f)] = 2\pi\lambda c \mathcal{L}_g(\alpha + 2i\pi f)^2 / \alpha \quad (29)$$

$$\forall f_1, f_2 \in \mathbb{R}, \text{corr}[\mathcal{F}_{h_i}(f_1), \mathcal{F}_{h_i}(f_2)] = \frac{e^{-2i\pi(f_1-f_2)T}}{1 + i\pi \frac{f_1-f_2}{\alpha}} \quad (30)$$

184 • *in time domain:*

$$\forall t \geq 2T, \text{var}[h_i(t)] = 4\pi\lambda c \|g\|_2^2 e^{-2\alpha(t-T)} \quad (31)$$

$$\forall t_1, t_2 \geq 2T, \text{corr}[h_i(t_1), h_i(t_2)] = \gamma(t_1 - t_2) \quad (32)$$

185

- *in the time-frequency domain:*

$$\forall f \in \mathbb{R}, \forall t \geq 2T, \mathcal{W}_{h_i, h_i}(t, f) = B(f) e^{-2\alpha(t-T)}. \quad (33)$$

186

3. *Asymptotic normality: when $t \rightarrow +\infty$, $h_i(t)$ converges in law to a Gaussian process.*

187

Proposition 2 is proved in Appendix D. Note that (26) and (15) show that $B(f)$ is very

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flat at $f = 0$: $B(0) = \frac{dB}{df}(0) = \frac{d^2B}{df^2}(0) = \frac{d^3B}{df^3}(0) = 0$. The asymptotic normality is related to

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the central limit theorem: when r becomes large, the volume contained between the spheres

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of radius r and $r + dr$ increases as $r^2 dr$, and so does the number of source images included

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in this volume as shown in (23), which leads to the addition of an increasing number of

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independent and identically distributed (i.i.d.) random increments $dN(\mathbf{x})$.

193

This proposition permits us to retrieve most of the classical results listed in the intro-

194

duction. Firstly, $h_i(t)$ is centered for $t \geq 2T$ (the fact that it is *not* centered for $t \in [0, 2T]$

195

explains why the expected value of the frequency response $\mathbb{E}[\mathcal{F}_{h_i}(f)]$ in (27) is not zero).

196

Secondly, (17) corresponds to Schroeder and Moorer's model defined in (1) when $T \rightarrow 0$ (in

197

this case the process $b_i(t)$ becomes white, and it is Gaussian when $t \gg T$), and in the general

198

case it is equivalent to Polack's model (Polack, 1988, chap. 1) ($b_i(t)$ is a centered stationary

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Gaussian process when $t \gg T$). When $T \rightarrow 0$, (30) reduces to Schroeder's formula (2),

200

which was indeed established by assuming that $b_i(t)$ is white. Finally, (33) is equivalent

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to Polack's time-frequency model defined in (4). To the best of our knowledge, the other

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formulas in Proposition 2 are novel.

203 **V. STATISTICAL PROPERTIES BETWEEN TWO SENSORS**

204 Let us now focus on the relationships between two sensors:

205 **Lemma 2.** *Let us consider the model in Definition 1 at two positions \mathbf{x}_i and $\mathbf{x}_j \in \mathbb{R}^3$. Let*
 206 *us define the rectangular window*

$$\forall t \in \mathbb{R}, w(t) = \frac{c}{2D} \mathbf{1}_{[-\frac{D}{c}, \frac{D}{c}]}(t), \quad (34)$$

207 where $D = \|\mathbf{x}_i - \mathbf{x}_j\|_2$. Then, in addition to the properties listed in Lemma 1, the random
 208 process $\mathbf{b}(t) = [b_i(t), b_j(t)]^\top$ is also such that

- 209 • $\forall t \geq 2T + \frac{D}{c}$, $\mathbf{b}(t)$ is a centered WSS process⁹, of cross-autocovariance function

$$\forall \tau \in \mathbb{R}, \Gamma_{i,j}(\tau) = \text{cov}[b_i(t + \tau), b_j(t)] = w * \Gamma(\tau), \quad (35)$$

210 cross-autocorrelation function

$$\forall \tau \in \mathbb{R}, \gamma_{i,j}(\tau) = \text{corr}[b_i(t + \tau), b_j(t)] = w * \gamma(\tau), \quad (36)$$

211 and cross-power spectral density

$$\forall f \in \mathbb{R}, B_{i,j}(f) = \mathcal{F}_{\Gamma_{i,j}}(f) = B(f) \text{sinc}\left(\frac{2\pi f D}{c}\right). \quad (37)$$

- 212 • When $t \rightarrow +\infty$ (i.e. $t \gg T$), $\mathbf{b}(t)$ converges in law to a stationary Gaussian process.

213 Lemma 2 is proved in Appendix F.

214 **Proposition 3** (Statistical properties between two sensors).

215 *Let us consider the reverberation model in Definition 1 at two positions \mathbf{x}_i and $\mathbf{x}_j \in \mathbb{R}^3$, and*
 216 *the rectangular window $w(t)$ defined in (34) with $D = \|\mathbf{x}_i - \mathbf{x}_j\|_2$. Then, with the notation*

217 introduced in Lemma 1 and in addition to the properties listed in Proposition 2, we also
 218 have:

219 • in the spectral domain:

$$\forall f_1, f_2 \in \mathbb{R}, \text{corr}[\mathcal{F}_{h_i}(f_1), \mathcal{F}_{h_j}(f_2)] = \frac{e^{-\frac{\alpha D}{c} - 2i\pi(f_1 - f_2)(T + \frac{D}{2c})} \text{sinc}(\frac{\pi(f_1 + f_2)D}{c})}{1 + i\pi \frac{f_1 - f_2}{\alpha}} \quad (38)$$

220 • in the time domain:

$$\forall t_1, t_2 \geq 2T + \frac{D}{c}, \text{corr}[h_i(t_1), h_j(t_2)] = w * \gamma(t_1 - t_2) \quad (39)$$

221 • in the time-frequency domain:

$$\forall f \in \mathbb{R}, \forall t \geq 2T + \frac{D}{2c}, \mathcal{W}_{h_i, h_j}(t, f) = B(f)e^{-2\alpha(t-T)} \text{sinc}(\frac{2\pi f D}{c}). \quad (40)$$

222 • Asymptotic normality: when $t \rightarrow +\infty$, $\mathbf{h}(t)$ converges in law to a Gaussian process.

223 Proposition 3 is proved in Appendix G. Applying equation (38) to $f_1 = f_2 = f$, we get
 224 Cook's formula (3) when $\alpha \rightarrow 0$ (no exponential decay). This formula was indeed originally
 225 proved by considering plane waves under a far-field assumption (Cook *et al.*, 1955). Besides,
 226 equation (39) shows that $h_i(t)$ and $h_j(t)$ are correlated on a time interval that corresponds
 227 to the wave propagation from one sensor to the other. To the best of our knowledge, all
 228 formulas in Proposition 3 are novel.

229 VI. SIMULATION RESULTS

230 In Propositions 2 and 3, we have listed several statistical properties of the reverberation
 231 model introduced in Definition 1. Some of these properties are already-known experimental

232 facts, such as the zero mean and asymptotic Gaussianity of late reverberation. In this
 233 section, we aim to experimentally validate the second-order statistics specific to this new
 234 stochastic reverberation model, that were not predicted by the existing models listed in
 235 Section I. Firstly, we note that at the second order, the spectral domain, time domain, and
 236 time-frequency domain formulas in both Propositions 2 and 3 are mutually equivalent; we
 237 can thus focus on the time-frequency domain only. Secondly, from (33) and (40), we get the
 238 closed-form expression of the time-frequency correlation: $\forall f \in \mathbb{R}, \forall t \geq 2T + \frac{D}{2c}$,

$$\frac{\mathcal{W}_{h_i, h_j}(t, f)}{\mathcal{W}_{h_i, h_i}(t, f)} = \text{sinc}\left(\frac{2\pi f D}{c}\right). \quad (41)$$

239 Since (33) was already known (Polack, 1988) and since (40) can actually be deduced from (33)
 240 and (41), the novelty regarding the second order statistics of late reverberation in Propo-
 241 sitions 2 and 3 lies in the single equation (41). Therefore we just need to experimentally
 242 validate (41).

243 For this purpose, we need an experimental setup with the greatest spatial diversity possi-
 244 ble: a high number (typically a thousand) of RIRs, with random source position \mathbf{x} , random
 245 middle position of the sensors $\frac{\mathbf{x}_i + \mathbf{x}_j}{2}$, random orientation of the sensor axis $\frac{\mathbf{x}_j - \mathbf{x}_i}{D}$, for a fixed
 246 distance $\|\mathbf{x}_j - \mathbf{x}_i\|_2 = D$. Because we do not have access to a suitable database of mea-
 247 sured RIRs, we used synthetic RIRs instead. These synthetic RIRs were generated with
 248 the Roomsimove toolbox (Vincent and Campbell, 2008), which is a state-of-the-art RIR
 249 generator based on the source image principle. Roomsimove is dedicated to parallelepipedic
 250 ("shoebox") rooms and applies high-pass filtering above 20 Hz. We used it with the default
 251 physical parameters (humidity: 40%, temperature: 20 °C, speed of sound: $c = 343$ m/s), to
 252 simulate a shoebox room matching the two properties of the classroom described in (Kut-

253 [truff, 2014](#), p. 84): a volume of 200 m³ (the room dimensions are 7.4 m × 9 m × 3 m), and
 254 a reverberation time of 1 s (the absorption coefficient is set to 0.13 for all room surfaces).
 255 For this setup, Schroeder’s frequency is about 140 Hz. In accordance with our stochastic
 256 model, we considered omnidirectional sources and omnidirectional microphones. We thus
 257 generated $M = 1000$ RIRs sampled at 16 kHz, with random source positions and random
 258 middle positions of the sensors (both uniformly distributed inside the room volume), ran-
 259 dom orientation of the sensor axis (uniformly distributed on the unit sphere), for a fixed
 260 microphone distance $D = 20$ cm.

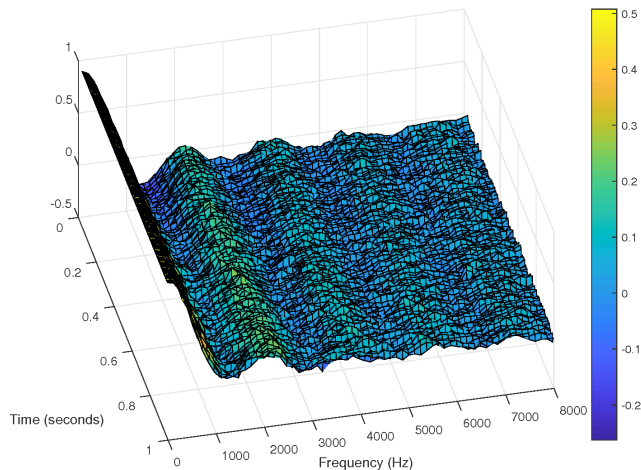


FIG. 4. Time-frequency correlation between sensors of the late RIR

261 Fig. 4 represents the estimate of the time-frequency correlation (41) that we obtained with
 262 this experimental setup. The Wigner distributions $\mathcal{W}_{h_i, h_i}(t, f)$ and $\mathcal{W}_{h_i, h_j}(t, f)$ were esti-
 263 mated as $\widehat{\mathcal{W}}_{h_i, h_i}(t, f) = \frac{1}{M} \sum_{m=1}^M |S_{h_i}^{(m)}(t, f)|^2$ and $\widehat{\mathcal{W}}_{h_i, h_j}(t, f) = \frac{1}{M} \sum_{m=1}^M S_{h_i}^{(m)}(t, f) \overline{S_{h_j}^{(m)}(t, f)}$,
 264 where $S_{h_i}^{(m)}(t, f)$ (resp. $S_{h_j}^{(m)}(t, f)$) is the short time Fourier transform (STFT) of the m^{th}
 265 generated RIR $h_i^{(m)}(t)$ (resp. $h_j^{(m)}(t)$), computed with a 128-sample long Hann window and

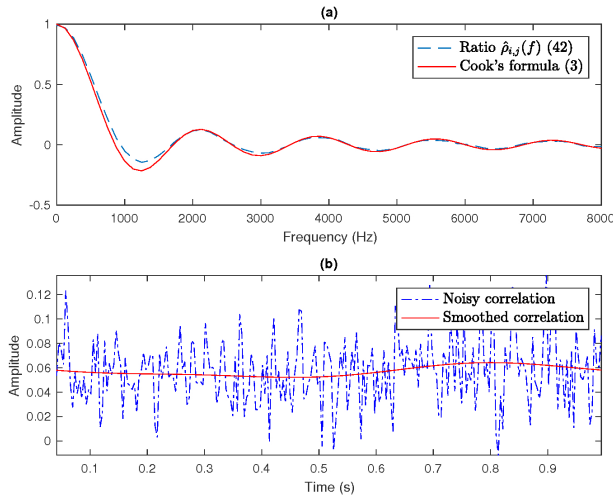


FIG. 5. Correlation between sensors of the late RFR over frequency (a) and over time (b)

266 an overlap of 64 samples in the time domain. The distributions $\widehat{\mathcal{W}}_{h_i, h_i}(t, f)$ and $\widehat{\mathcal{W}}_{h_i, h_j}(t, f)$
 267 obtained in this way are smoothed estimates of $\mathcal{W}_{h_i, h_i}(t, f)$ and $\mathcal{W}_{h_i, h_j}(t, f)$ in the time-
 268 frequency domain (Cohen, 1989). We observe in Fig. 4 that the estimated time-frequency
 269 correlation is noisy but, as expected, it seems approximately constant as a function of time,
 270 and looks like a cardinal sine function as a function of frequency.

271 In order to accurately compare the frequency variation with Cook's formula (3), we
 272 computed the ratio $\hat{\rho}_{i,j}(f)$ of the projections of the estimated distributions onto the frequency
 273 axis, i.e.

$$\hat{\rho}_{i,j}(f) = \frac{\int_{\mathbb{R}} \widehat{\mathcal{W}}_{h_i, h_j}(t, f) dt}{\int_{\mathbb{R}} \widehat{\mathcal{W}}_{h_i, h_i}(t, f) dt} \quad (42)$$

274 (represented by the dashed curve in Fig. 5-(a)), which is an estimate of (3) (the cardinal
 275 sine function is represented by the solid curve in Fig. 5-(a)). We can observe a good match
 276 between the two curves¹⁰. Finally, the dash-dotted curve in Fig. 5-(b) represents, as a
 277 function of time, the same noisy estimate of the time-frequency correlation as in Fig. 4, at

278 a fixed frequency $f = 3860$ Hz (which corresponds to a local maximum of the cardinal sine
 279 function in Fig. 5-(a)), and the solid curve represents a smoothed (i.e. low-pass filtered)
 280 version of this noisy signal. We can observe that the average time-frequency correlation is
 281 indeed approximately independent of time.

282 VII. CONCLUSION AND PERSPECTIVES

283 In this paper, we proposed a new stochastic model of reverberation, that permitted us
 284 to retrieve various well-known results within a common framework. This unification work
 285 resulted in several new results, that jointly characterize the properties of late reverberation in
 286 the space, time, and frequency domains. The most noticeable result in our opinion is (40),
 287 which very simply makes the connection between Polack’s time-frequency model (4) and
 288 Cook’s formula (3).

289 Although this model was motivated by physical assumptions that only hold in a partic-
 290 ular region of the time-frequency domain (after the transition time and above Schroeder’s
 291 frequency), from a signal processing perspective however, one of its most interesting features
 292 is that it is also applicable *before* the transition time and *below* Schroeder’s frequency. In-
 293 deed, since the parameter of the Poisson distribution $dN(r)$ in (23) increases quadratically
 294 with the distance r , the model permits us to describe both the impulsiveness of the RIR
 295 before the transition time, and its asymptotic normality in late reverberation. However, be-
 296 cause it does not take the exact geometry of the acoustic scene into account (i.e. the room
 297 geometry and the source and sensors positions), it is able to capture neither the accurate
 298 temporal localization in the RIR of the isolated peaks related to early reflections before the

299 transition time, nor the spectral localization of the isolated room modes below Schroeder's
300 frequency. In practice, this blindness to such fine details does not prevent us to estimate
301 the model parameters from the full RIRs, in the whole time-frequency domain. The missing
302 geometrical information can still be recovered afterwards in an ad-hoc manner, for instance
303 by estimating the Poisson random measure $dN(\mathbf{x})$ given the observed RIRs. Indeed, the
304 knowledge of $dN(\mathbf{x})$ permits us to identify the spatial positions of the source images, which
305 carry the geometrical information related to both the early reflections and the room modes.
306 Finally, we end up with a stochastic model involving very few parameters (α , λ , filter g ,
307 and the distances between microphones), that is able to describe reverberation in the whole
308 time-frequency domain. We thus believe that this model has an interesting potential in a
309 variety of signal processing applications.

310 However this reverberation model, as it is presented in this paper, is not yet suitable for
311 modeling real RIRs. Indeed, one assumption has to be relaxed: the attenuation coefficient
312 α is not constant but rather depends on frequency in practice, as in Polack's generalized
313 time-frequency model (5). In a future paper, we will thus present a generalization of the
314 proposed model where we will introduce a frequency-varying attenuation coefficient. A
315 second generalization of this model would be to represent acoustic fields that are not perfectly
316 diffuse. Finally, the generalization to directional microphones is straightforward, by using
317 the same approach as presented in (Elko, 2001).

318 Our future contributions will also focus on the signal processing aspects of this work: we
319 will propose a fast algorithm for estimating the parameters of the model in discrete time.

320 Finally, our ultimate goal is to investigate the potential of this model in applications such
 321 as source separation, dereverberation, and synthetic reverberation.

322 **ACKNOWLEDGMENTS**

323 Ahead of this work, I had some very interesting and inspiring discussions about rever-
 324 beration with several persons that I would like to thank here, including my former PhD
 325 students Simon Leglaive and Arthur Belhomme, and my colleagues Gaël Richard, Yves
 326 Grenier, Laurent Girin, Antoine Liutkus, Philippe Depalle and Xavier Boutillon.

327 **APPENDIX A: PROOFS FOR THE DEFINITION OF THE STOCHASTIC MODEL**

328 The following lemma aims to prove the existence of functions g that satisfy conditions (15)
 329 and (16) in Definition 1.

330 **Lemma 3.** *Let $\psi(t) \in L^2([0, T])$ with $T > 0$, such that $\int_{t=0}^T \psi(t)e^{\alpha t} dt = \int_{t=0}^T \psi(t)e^{-\alpha t} dt = 0$,
 331 where $\alpha \in \mathbb{R}$ ¹¹. Let $g(t) = (\tilde{\psi} * \psi)(t) e^{\alpha(t-T)}$. Then function g satisfies conditions (15)
 332 and (16) in Definition 1.*

333 *Proof of Lemma 3.* Since $\psi(t) \in L^2([0, T])$, function g is continuous and bounded. Firstly,
 334 $\mathcal{F}_g(f) = e^{-\alpha T} \overline{\mathcal{L}_\psi(\alpha + 2i\pi f)} \mathcal{L}_\psi(-\alpha + 2i\pi f)$. Moreover, since $\psi(t)$ has finite support, both
 335 functions $f \mapsto \mathcal{L}_\psi(-\alpha + 2i\pi f)$ and $f \mapsto \mathcal{L}_\psi(\alpha + 2i\pi f)$ are infinitely differentiable. In addition,
 336 $\mathcal{L}_\psi(-\alpha) = \mathcal{L}_\psi(\alpha) = 0$, which finally proves (15). Secondly, $\mathcal{L}_g(\alpha + 2i\pi f) = e^{-\alpha T} |\mathcal{F}_\psi(f)|^2$,
 337 which proves (16). □

We can now derive equations (17), (18) and (19) in Definition 1. By substituting (13) and (20) into (12), we get

$$h_i(t) = \int_{f \in \mathbb{R}} \mathcal{L}_g(\alpha + 2i\pi f) e^{2i\pi f(t-T)} \int_{\mathbf{x} \in \mathbb{R}^3} \frac{e^{-\frac{\alpha + 2i\pi f}{c} \|\mathbf{x} - \mathbf{x}_i\|_2}}{\|\mathbf{x} - \mathbf{x}_i\|_2} dN(\mathbf{x}) df$$

338 which shows that $h_i(t)$ as defined in (12) is the inverse Fourier transform of (19) and therefore
 339 proves (19). Besides, applying the Fourier transform (9) to (17) also leads to (19), which
 340 proves the equivalence between (17)-(18), and (12)-(13)-(20).

341 Finally, let us prove Proposition 1.

342 *Proof of Proposition 1.* Let $dN(r) = \int_{\mathbf{u} \in \mathcal{S}^2} dN(r\mathbf{u})$. Then (14) implies (23). The change of
 343 variables $\mathbf{x} = \mathbf{x}_i + r\mathbf{u}$ with $r \in \mathbb{R}_+$ and $\mathbf{u} \in \mathcal{S}^2$ in (18) and (19) leads to (21) and (22). \square

344 APPENDIX B: PROPERTIES OF THE POISSON RANDOM MEASURE

345 The following Proposition 4 characterizes the distribution of the Poisson random mea-
 346 sure introduced in Section III, by providing the closed-form expression of the characteristic
 347 functional of this random measure:

348 **Proposition 4.** *Let $dN(\mathbf{x})$ be a Poisson random measure on \mathbb{R}^p with independent incre-*
 349 *ments: $dN(\mathbf{x}) \sim \mathcal{P}(\Lambda(\mathbf{x})d\mathbf{x})$, where Λ is a non-negative, locally integrable function. We*
 350 *consider the stochastic integral $I = \int \psi(\mathbf{x})dN(\mathbf{x})$, where ψ is an essentially bounded real-*
 351 *valued function with compact support, and its characteristic function $\phi_I(\theta) = \mathbb{E}[e^{i\theta I}]$. Then*
 352 *we have*

$$\forall \theta \in \mathbb{R}, \phi_I(\theta) = \exp \left(\int (e^{i\theta\psi(\mathbf{x})} - 1) \Lambda(\mathbf{x}) d\mathbf{x} \right), \quad (\text{B1})$$

353 *which yields*

$$\forall \theta \in \mathbb{R}, \ln \phi_I(\theta) = \sum_{n=1}^{+\infty} \frac{\iota^n}{n!} \theta^n \kappa_n \quad (\text{B2})$$

354 *where* $\forall n \in \mathbb{N} \setminus \{0\}$,

$$\kappa_n = \int \psi(\mathbf{x})^n \Lambda(\mathbf{x}) d\mathbf{x} \quad (\text{B3})$$

355 *is the n -th order cumulant of I . In particular, we get*

$$\mathbb{E}[I] = \int \psi(\mathbf{x}) \Lambda(\mathbf{x}) d\mathbf{x}. \quad (\text{B4})$$

356 *Proof.* In the particular case where ψ is the indicator function of a Borel set V of finite
 357 Lebesgue measure, we have $I = N(V) \sim \mathcal{P}(\int_V \Lambda(\mathbf{x}) d\mathbf{x})$. Therefore (8) yields (B1), because
 358 both members of (B1) are equal to $\exp((e^{\iota\theta} - 1) \int_V \Lambda(\mathbf{x}) d\mathbf{x})$. Consequently, since (B1) holds
 359 for any indicator function and since the increments $dN(\mathbf{x})$ are assumed independent, it is
 360 easy to show that (B1) also holds for *simple functions*¹². Following the construction of the
 361 Lebesgue integral, we conclude that (B1) also holds for any essentially bounded function ψ
 362 with compact support.

363 Then the Taylor series expansion of the second exponential function in (B1) yields $\forall \theta \in \mathbb{R}$,

364

$$\ln \phi_I(\theta) = \sum_{n=1}^{+\infty} \frac{\iota^n}{n!} \theta^n \int \psi(\mathbf{x})^n \Lambda(\mathbf{x}) d\mathbf{x}, \quad (\text{B5})$$

365 which proves (B2). Note that the permutation of the integral and the sum in (B5) is allowed

366 thanks to Fubini's theorem, because $\sum_{n=1}^{+\infty} \frac{1}{n!} |\theta|^n \int |\psi(\mathbf{x})|^n \Lambda(\mathbf{x}) d\mathbf{x} \leq e^{|\theta| \|\psi\|_\infty} \int_V \Lambda(\mathbf{x}) d\mathbf{x} <$

367 $+\infty$, since ψ is essentially bounded, $V = \text{support}(\psi)$ is compact, and Λ is locally integrable.

368 In particular, for $n = 1$, we get $\mathbb{E}[I] = \kappa_1$ thus (B3) yields (B4). □

369 The following Proposition 5 provides the closed-form expression of the joint cumulants
 370 of two stochastic integrals involving the Poisson random measure:

371 **Proposition 5.** *Let $dN(\mathbf{x})$ be a Poisson random measure on \mathbb{R}^p with independent incre-*
 372 *ments: $dN(\mathbf{x}) \sim \mathcal{P}(\Lambda(\mathbf{x})d\mathbf{x})$, where Λ is a non-negative, locally integrable function. We*
 373 *consider two stochastic integrals $I_1 = \int \psi_1(\mathbf{x})dN(\mathbf{x})$ and $I_2 = \int \psi_2(\mathbf{x})dN(\mathbf{x})$, where ψ_1*
 374 *and ψ_2 are essentially bounded real-valued functions with compact support, and their joint*
 375 *characteristic function $\phi_{I_1, I_2}(\theta_1, \theta_2) = \mathbb{E}[e^{i(\theta_1 I_1 + \theta_2 I_2)}]$. Then we have*

$$\forall \theta_1, \theta_2 \in \mathbb{R}, \ln(\phi_{I_1, I_2}(\theta_1, \theta_2)) = \sum_{n=1}^{+\infty} \frac{i^n}{n!} \sum_{k=0}^n \binom{n}{k} \theta_1^k \theta_2^{n-k} \kappa_{k, n-k}, \quad (\text{B6})$$

376 where $\forall n_1, n_2 \in \mathbb{N}$ such that $n_1 + n_2 \geq 1$,

$$\kappa_{n_1, n_2} = \int \psi_1(\mathbf{x})^{n_1} \psi_2(\mathbf{x})^{n_2} \Lambda(\mathbf{x}) d\mathbf{x} \quad (\text{B7})$$

377 is the (n_1, n_2) -th order joint cumulant of (I_1, I_2) . In particular, we get

$$\text{cov}[I_1, I_2] = \int \psi_1(\mathbf{x}) \psi_2(\mathbf{x}) \Lambda(\mathbf{x}) d\mathbf{x}. \quad (\text{B8})$$

378 *Proof.* By applying (B1) to $\theta = 1$ and to $\psi(\mathbf{x}) = \theta_1 \psi_1(\mathbf{x}) + \theta_2 \psi_2(\mathbf{x})$, we get $\forall \theta_1, \theta_2 \in \mathbb{R}$,

$$\begin{aligned} \ln \phi_{I_1, I_2}(\theta_1, \theta_2) &= \int (e^{i(\theta_1 \psi_1(\mathbf{x}) + \theta_2 \psi_2(\mathbf{x}))} - 1) \Lambda(\mathbf{x}) d\mathbf{x} \\ &= \sum_{n=1}^{+\infty} \frac{i^n}{n!} \int (\theta_1 \psi_1(\mathbf{x}) + \theta_2 \psi_2(\mathbf{x}))^n \Lambda(\mathbf{x}) d\mathbf{x} \\ &= \sum_{n=1}^{+\infty} \frac{i^n}{n!} \sum_{k=0}^n \binom{n}{k} \theta_1^k \theta_2^{n-k} \int \psi_1(\mathbf{x})^k \psi_2(\mathbf{x})^{n-k} \Lambda(\mathbf{x}) d\mathbf{x}, \end{aligned} \quad (\text{B9})$$

which proves (B6). Note that the permutation of the integral and the two sums in (B9) is allowed thanks to Fubini's theorem, because

$$\sum_{n=1}^{+\infty} \frac{1}{n!} \sum_{k=0}^n \binom{n}{k} |\theta_1|^k |\theta_2|^{n-k} \int |\psi_1(\mathbf{x})|^k |\psi_2(\mathbf{x})|^{n-k} \Lambda(\mathbf{x}) d\mathbf{x} \leq e^{|\theta_1| \|\psi_1\|_\infty + |\theta_2| \|\psi_2\|_\infty} \int_V \Lambda(\mathbf{x}) d\mathbf{x} < +\infty$$

379 since ψ_1 and ψ_2 are essentially bounded, $V = \text{support}(\psi_1) \cup \text{support}(\psi_2)$ is compact, and Λ
 380 is locally integrable. In particular, for $n_1 = 1$ and $n_2 = 1$, we get $\text{cov}[I_1, I_2] = \kappa_{1,1}$, thus (B7)
 381 yields (B8). □

382 APPENDIX C: PROOF OF LEMMA 1

383 1. Wide sense stationarity

384 a. First order moments

385 Considering the Poisson random measure $dN(r)$ defined in (23), equation (B4) in Propo-
 386 sition 4, applied to $p = 1$ and to function $\Lambda(r) = 4\pi\lambda r^2$ defined on \mathbb{R}_+ , shows that for
 387 any essentially bounded (possibly complex) function $r \mapsto \psi(r)$ with compact support in
 388 \mathbb{R}_+ , $\mathbb{E}[\int_{r \in \mathbb{R}_+} \psi(r) dN(r)] = 4\pi\lambda \int_{r \in \mathbb{R}_+} \psi(r) r^2 dr$, which we will rewrite in the symbolic, more
 389 compact form:

$$\mathbb{E}[dN(r)] = 4\pi\lambda r^2 dr. \tag{C1}$$

390 From (C1), since $\int_{u=-T}^{\min(t-T, T)} |g(u)|(t - T - u) du < +\infty$ (with the change of variable
 391 $r \mapsto u = t - T - \frac{r}{c}$), Fubini's theorem shows that we can permute the mathematical
 392 expectation with the integral over r in (21), leading to:

$$\mathbb{E}[b_i(t)] = 4\pi\lambda c^2 \left(\int_{u=-T}^{\min(t-T, T)} (t - T - u) g(u) du \right). \tag{C2}$$

393 Finally, substituting (15) into (C2) shows that $\forall t \geq 2T$, $\mathbb{E}[b_i(t)] = 0$.

394 **b. Second order moments**

395 Considering the Poisson random measure $dN(r)$ defined in (23), equation (B8) in Propo-
 396 sition 5, applied to $p = 1$ and to function $\Lambda(r) = 4\pi\lambda r^2$ defined on \mathbb{R}_+ , shows that for any
 397 essentially bounded (possibly complex) functions $r \mapsto \psi_1(r)$ and $r \mapsto \psi_2(r)$ with compact
 398 supports in \mathbb{R}_+ , $\text{cov} \left[\int_{r_1 \in \mathbb{R}_+} \psi_1(r_1) dN(r_1), \int_{r_2 \in \mathbb{R}_+} \psi_2(r_2) dN(r_2) \right] = 4\pi\lambda \int_{r \in \mathbb{R}_+} \psi_1(r) \overline{\psi_2(r)} r^2 dr$,
 399 which we will rewrite in the symbolic, more compact form:

$$\text{cov}[dN(r_1), dN(r_2)] = 4\pi\lambda \delta(r_2 - r_1) r_1 r_2 dr_1 dr_2. \quad (\text{C3})$$

400 From (C3), since $\int_{u \in \mathbb{R}_+} |g(u + \tau) g(u)| du < +\infty$ (with the change of variable $r \mapsto u =$
 401 $t - T - \frac{r}{c}$), Fubini's theorem shows that we can permute the mathematical expectation with
 402 the integral over r in (21), leading to (24), which jointly proves both (25) and (26).

403 **2. Asymptotic normality**

404 From (21), $b_i(t)$ can be written as the stochastic integral I in Proposition 4, applied to
 405 $p = 1$, to function $\Lambda(r) = 4\pi\lambda r^2$ defined on \mathbb{R}_+ , and to $\psi(r) = \frac{g(t-T-\frac{r}{c})}{r}$. Then (B2) yields

$$\forall \theta \in \mathbb{R}, \ln(\phi_{b_i(t)}(\theta)) = \sum_{n=1}^{+\infty} \frac{(i\theta)^n}{n!} \kappa_n(t) \quad (\text{C4})$$

406 where $\kappa_n(t)$ is the n -th order cumulant of $b_i(t)$, whose expression is given by (B3):

$$\kappa_n(t) = 4\pi\lambda \int_{r \in \mathbb{R}_+} g\left(t - T - \frac{r}{c}\right)^n r^{2-n} dr. \quad (\text{C5})$$

The change of variables $r = c(t - T - u)$ in (C5) implies

$$\forall t \geq 2T, \kappa_n(t) = 4\pi\lambda c^{3-n} \int_{u=-T}^T g(u)^n (t - T - u)^{2-n} du.$$

407 Therefore, $\forall n \geq 2, \forall t > 2T$,

$$|\kappa_n(t)| \leq \frac{4\pi\lambda c^{3-n} \int_{u=-T}^T |g(u)|^n du}{(t-2T)^{n-2}}. \quad (\text{C6})$$

Let $\varepsilon > 0$. Substituting (C6) into (C4), we get: $\forall t \geq (2 + \varepsilon)T$,

$$\left| \ln(\phi_{b_i(t)}(\theta)) - \left(i\theta\kappa_1(t) - \frac{\theta^2}{2}\kappa_2(t) \right) \right| \leq \frac{\psi(\theta)}{t-2T} \xrightarrow{t \rightarrow +\infty} 0,$$

408 where $\psi(\theta) = 4\pi\lambda(\varepsilon cT)^3 \int_{u=-T}^T \left(e^{|\frac{\theta g(u)}{\varepsilon cT}|} - \sum_{n=0}^2 \frac{|\frac{\theta g(u)}{\varepsilon cT}|^n}{n!} \right) du$.

409 Therefore the characteristic function of $b_i(t)$ converges pointwise to that of the normal
410 distribution when $t \rightarrow +\infty$, which proves that $b_i(t)$ is asymptotically normally distributed.

411 In the same way, it can be proved¹³ that the random variables $b_i(t+t_1) \dots b_i(t+t_K)$ for all
412 $K \in \mathbb{N}$ and $t, t_1 \dots t_K \in \mathbb{R}$ are jointly normally distributed when $t \rightarrow +\infty$ and $t_1 \dots t_K$ are

413 fixed, which shows that $b_i(t)$ converges in law to a stationary Gaussian process.

414 APPENDIX D: PROOF OF PROPOSITION 2

415 1. First order moments

416 From (C1), since $\int_{r \in \mathbb{R}_+} e^{-\frac{\alpha}{\varepsilon} r} r dr < +\infty$, Fubini's theorem shows that we can permute the

417 mathematical expectation with the integral over r in (22), leading to (27). Besides, since

418 $\mathbb{E}[b_i(t)] = 0$ as shown in Lemma 1, equation (17) implies (28).

419 **2. Second order moments**

420 *a. In the spectral domain*

421 From (C3), since $\int_{r \in \mathbb{R}_+} e^{-\frac{2\alpha}{c}r} dr < +\infty$, Fubini's theorem shows that we can permute the
 422 mathematical expectation with the integral over r in (22), leading to:

$$\text{cov}[\mathcal{F}_{h_i}(f_1), \mathcal{F}_{h_i}(f_2)] = 2\pi\lambda c \mathcal{L}_g(\alpha + 2i\pi f_1) \mathcal{L}_g(\alpha + 2i\pi f_2) \frac{e^{-2i\pi(f_1-f_2)T}}{\alpha + i\pi(f_1-f_2)}. \quad (\text{D1})$$

423 Equation (D1) implies both (29) and (30).

424 *b. In the time domain*

425 Equations (24) and (17) imply both (31) and (32).

426 *c. In the time-frequency domain*

Finally, from (C3) and (17), since

$$\int_{u \in \mathbb{R}} \int_{r \in \mathbb{R}_+} \left| g\left(t + \frac{u}{2} - T - \frac{r}{c}\right) g\left(t - \frac{u}{2} - T - \frac{r}{c}\right) \right| dr du < +\infty,$$

427 Fubini's theorem shows that we can permute the mathematical expectation with the integrals
 428 over r in (21) and over u in (11) (with $\psi_1 = \psi_2 = h_i$), leading to:

$$\mathcal{W}_{h_i, h_i}(t, f) = 4\pi\lambda e^{-2\alpha(t-T)} \int_{r \in \mathbb{R}_+} I\left(t - T - \frac{r}{c}, f\right) dr \quad (\text{D2})$$

429 where

$$I(t, f) = \int_{u \in \mathbb{R}} g\left(t + \frac{u}{2}\right) g\left(t - \frac{u}{2}\right) e^{-2i\pi f u} du = \mathcal{W}_{g, g}(t, f). \quad (\text{D3})$$

430 In order to conclude, we will use two properties of the Wigner distribution. Firstly, the
 431 time support of $\mathcal{W}_{g,g}$ is the same as that of g . Secondly, the *projection property* shows
 432 that $\int_{t \in \mathbb{R}} \mathcal{W}_{g,g}(t, f) dt = |\mathcal{F}_g(f)|^2$. Therefore substituting (D3) and (26) into (D2) finally
 433 implies (33).

434 3. Asymptotic normality

435 Since $b_i(t)$ converges in law to a stationary Gaussian process as shown in Lemma 1,
 436 equation (17) shows that $h_i(t)$ converges in law to a Gaussian process.

437 APPENDIX E: GEOMETRY WITH TWO MICROPHONES

438 Let $\mathbf{x}_i, \mathbf{x}_j \in \mathbb{R}^3$. Let $\xi_1, \xi_2 : \mathbb{R}_+ \rightarrow \mathbb{C}$. In the next sections, we will have to compute
 439 several integrals of the form:

$$J_{\xi_1, \xi_2} = \int_{\mathbf{x} \in \mathbb{R}^3} \frac{\xi_1(\|\mathbf{x} - \mathbf{x}_i\|_2)}{\|\mathbf{x} - \mathbf{x}_i\|_2} \frac{\overline{\xi_2(\|\mathbf{x} - \mathbf{x}_j\|_2)}}{\|\mathbf{x} - \mathbf{x}_j\|_2} d\mathbf{x}. \quad (\text{E1})$$

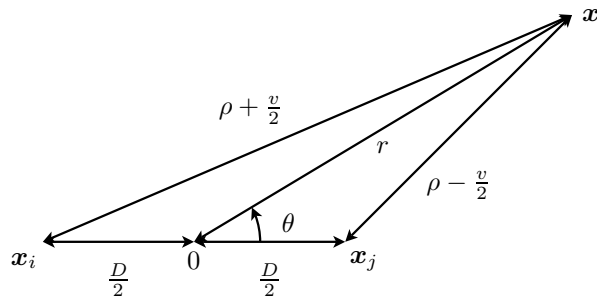


FIG. 6. Geometry with two microphones at $\mathbf{x}_i, \mathbf{x}_j$ and a source image at \mathbf{x} .

440 To compute such an integral, we will use the spherical coordinates (r, θ, φ) , as illustrated
 441 in Fig. 6, where $\theta = 0$ corresponds to the direction of vector $\mathbf{x}_j - \mathbf{x}_i$, and the origin of the

442 coordinates is the middle of the line segment $[\mathbf{x}_i, \mathbf{x}_j]$ (so that $\frac{\mathbf{x}_i + \mathbf{x}_j}{2} = 0$ as represented in

443 Fig. 6). We thus get $\mathbf{x} = [r \sin(\theta) \cos(\varphi), r \sin(\theta) \sin(\varphi), r \cos(\theta)]^\top$ and $d\mathbf{x} = r^2 dr \sin(\theta) d\theta d\varphi$,

444 with $r \in \mathbb{R}_+$, $\theta \in [0, \pi]$ and $\varphi \in [0, 2\pi]$. Moreover, as can be seen in Fig. 6, we have

445 • $\|\mathbf{x} - \mathbf{x}_i\|_2 = \sqrt{r^2 + \frac{D^2}{4} + rD \cos(\theta)},$

446 • $\|\mathbf{x} - \mathbf{x}_j\|_2 = \sqrt{r^2 + \frac{D^2}{4} - rD \cos(\theta)}.$

447 By substitution into (E1), we get

$$J_{\xi_1, \xi_2} = 2\pi \int_{r=0}^{+\infty} \int_{\theta=0}^{\pi} \frac{\xi_1 \left(\sqrt{r^2 + \frac{D^2}{4} + rD \cos(\theta)} \right) \xi_2 \left(\sqrt{r^2 + \frac{D^2}{4} - rD \cos(\theta)} \right)}{\sqrt{r^2 + \frac{D^2}{4} + rD \cos(\theta)} \sqrt{r^2 + \frac{D^2}{4} - rD \cos(\theta)}} r^2 dr \sin(\theta) d\theta. \quad (\text{E2})$$

Finally, we make a last change of variables, that is also illustrated in Fig. 6:

$$\rho = \frac{\|\mathbf{x} - \mathbf{x}_i\|_2 + \|\mathbf{x} - \mathbf{x}_j\|_2}{2} = \frac{\sqrt{r^2 + \frac{D^2}{4} + rD \cos(\theta)} + \sqrt{r^2 + \frac{D^2}{4} - rD \cos(\theta)}}{2},$$

$$v = \|\mathbf{x} - \mathbf{x}_i\|_2 - \|\mathbf{x} - \mathbf{x}_j\|_2 = \sqrt{r^2 + \frac{D^2}{4} + rD \cos(\theta)} - \sqrt{r^2 + \frac{D^2}{4} - rD \cos(\theta)},$$

which is such that $\rho \in [\frac{D}{2}, +\infty[$, $v \in [-D, D]$, and

$$\frac{r^2 dr \sin(\theta) d\theta}{\sqrt{r^2 + \frac{D^2}{4} + rD \cos(\theta)} \sqrt{r^2 + \frac{D^2}{4} - rD \cos(\theta)}} = \frac{d\rho dv}{D}.$$

448 Equation (E2) thus becomes

$$J_{\xi_1, \xi_2} = \frac{2\pi}{D} \int_{\rho=\frac{D}{2}}^{+\infty} \int_{v=-D}^D \xi_1 \left(\rho + \frac{v}{2} \right) \xi_2 \left(\rho - \frac{v}{2} \right) d\rho dv. \quad (\text{E3})$$

449 APPENDIX F: PROOF OF LEMMA 2

450 1. Wide sense stationarity

451 Considering the Poisson random measure $dN(\mathbf{x})$ defined in (14), equation (B8) in Propo-

452 sition 5, applied to $p = 3$ and to the constant function $\Lambda(\mathbf{x}) = \lambda$, shows that for any

453 essentially bounded (possibly complex) functions $\mathbf{x} \mapsto \psi_1(\mathbf{x})$ and $\mathbf{x} \mapsto \psi_2(\mathbf{x})$ with compact
 454 supports in \mathbb{R}^3 , $\text{cov} \left[\int_{\mathbf{x}_1 \in \mathbb{R}^3} \psi_1(\mathbf{x}_1) dN(\mathbf{x}_1), \int_{\mathbf{x}_2 \in \mathbb{R}^3} \psi_2(\mathbf{x}_2) dN(\mathbf{x}_2) \right] = \lambda \int_{\mathbf{x} \in \mathbb{R}^3} \psi_1(\mathbf{x}) \overline{\psi_2(\mathbf{x})} d\mathbf{x}$,
 455 which we will rewrite in the symbolic, more compact form:

$$\text{cov}[dN(\mathbf{x}_1), dN(\mathbf{x}_2)] = \lambda \delta(\mathbf{x}_2 - \mathbf{x}_1) d\mathbf{x}_1 d\mathbf{x}_2. \quad (\text{F1})$$

From (F1), since

$$\int_{\mathbf{x} \in \mathbb{R}^3} \frac{\left| g \left(t_1 - T - \frac{\|\mathbf{x} - \mathbf{x}_i\|_2}{c} \right) g \left(t_2 - T - \frac{\|\mathbf{x} - \mathbf{x}_j\|_2}{c} \right) \right|}{\|\mathbf{x} - \mathbf{x}_i\|_2 \|\mathbf{x} - \mathbf{x}_j\|_2} d\mathbf{x} < +\infty,$$

456 Fubini's theorem shows that we can permute the mathematical expectation with the integral
 457 over \mathbf{x} in (18), leading to:

$$\text{cov}[b_i(t_1), b_j(t_2)] = \lambda J_{\xi_1, \xi_2} \quad (\text{F2})$$

458 where $\xi_1(r) = g \left(t_1 - T - \frac{r}{c} \right)$, $\xi_2(r) = g \left(t_2 - T - \frac{r}{c} \right)$, and J_{ξ_1, ξ_2} was defined in (E1). Then
 459 substituting (E3) into (F2) shows that

$$\text{cov}[b_i(t_1), b_j(t_2)] = 4\pi\lambda \int_{\rho = \frac{D}{2}}^{+\infty} I(t_1 - T - \frac{\rho}{c}, t_2 - T - \frac{\rho}{c}) d\rho \quad (\text{F3})$$

460 where

$$I(t_1, t_2) = \frac{1}{2D} \int_{v=-D}^D g \left(t_1 - \frac{v}{2c} \right) g \left(t_2 + \frac{v}{2c} \right) dv. \quad (\text{F4})$$

461 With $v = cu$, substituting (34) into (F4) proves that

$$I(t_1, t_2) = \int_{u \in \mathbb{R}} g \left(t_1 - \frac{u}{2} \right) g \left(t_2 + \frac{u}{2} \right) w(u) du. \quad (\text{F5})$$

462 Substituting (F5) into (F3) proves (35), which with (25) implies (36). Moreover, applying
 463 the Fourier transform (9) to (35) and substituting (26) leads to (37).

464 **2. Asymptotic normality**

465 Let $\mathbf{b}(t) = [b_i(t), b_j(t)]^\top$. From (18), $b_i(t)$ (resp. $b_j(t)$) can be written as the stochastic
 466 integral I_1 (resp. I_2) in Proposition 5, applied to $p = 3$, to the constant function $\Lambda(\mathbf{x}) = \lambda$,
 467 and to $\psi_1(\mathbf{x}) = \frac{g(t-T-\frac{\|\mathbf{x}-\mathbf{x}_i\|_2}{c})}{\|\mathbf{x}-\mathbf{x}_i\|_2}$ and $\psi_2(\mathbf{x}) = \frac{g(t-T-\frac{\|\mathbf{x}-\mathbf{x}_j\|_2}{c})}{\|\mathbf{x}-\mathbf{x}_j\|_2}$. Then (B6) yields

$$\forall \theta_1, \theta_2 \in \mathbb{R}, \ln(\phi_{\mathbf{b}(t)}(\theta_1, \theta_2)) = \sum_{n=1}^{+\infty} \frac{i^n}{n!} \sum_{k=0}^n \binom{n}{k} \theta_1^k \theta_2^{n-k} \kappa_{k, n-k}(t) \quad (\text{F6})$$

468 where $\kappa_{n_1, n_2}(t)$ is the (n_1, n_2) -th order cumulant of $\mathbf{b}(t)$, whose expression is given by (B7):

469

$$\kappa_{n_1, n_2}(t) = \lambda J_{\xi_1, \xi_2}, \quad (\text{F7})$$

470 where $\xi_1(r) = \frac{(g(t-T-\frac{r}{c}))^{n_1}}{r^{n_1-1}}$, $\xi_2(r) = \frac{(g(t-T-\frac{r}{c}))^{n_2}}{r^{n_2-1}}$, and J_{ξ_1, ξ_2} was defined in (E1).

471 Then substituting (E3) into (F7) shows that

$$\kappa_{n_1, n_2}(t) = \frac{2\pi\lambda}{D} \int_{\rho=\frac{D}{2}}^{+\infty} \int_{v=-D}^D \frac{\left(g\left(t-T-\frac{\rho+\frac{v}{2}}{c}\right)\right)^{n_1}}{\left(\rho+\frac{v}{2}\right)^{n_1-1}} \frac{\left(g\left(t-T-\frac{\rho-\frac{v}{2}}{c}\right)\right)^{n_2}}{\left(\rho-\frac{v}{2}\right)^{n_2-1}} d\rho dv. \quad (\text{F8})$$

The change of variables $\rho = c(t - T - u)$ and $v = cw$ in (F8) implies that $\forall t \geq 2T + \frac{D}{c}$,

$$\kappa_{n_1, n_2}(t) = \frac{2\pi\lambda c^{4-n_1-n_2}}{D} \int_{u=-T-\frac{D}{2c}}^{T+\frac{D}{2c}} \int_{w=-\frac{D}{c}}^{\frac{D}{c}} \frac{\left(g\left(u-\frac{w}{2}\right)\right)^{n_1}}{\left(t-T-u+\frac{w}{2}\right)^{n_1-1}} \frac{\left(g\left(u+\frac{w}{2}\right)\right)^{n_2}}{\left(t-T-u-\frac{w}{2}\right)^{n_2-1}} dudw.$$

472 Therefore $\forall n_1 + n_2 \geq 2, \forall t \geq 2T + \frac{D}{c}$,

$$|\kappa_{n_1, n_2}(t)| \leq \frac{2\pi\lambda c^{4-n_1-n_2}}{D} \frac{\int_{u=-T-\frac{D}{2c}}^{T+\frac{D}{2c}} \int_{w=-\frac{D}{c}}^{\frac{D}{c}} |g\left(u-\frac{w}{2}\right)|^{n_1} |g\left(u+\frac{w}{2}\right)|^{n_2} dudw}{(t-2T)^{n_1+n_2-2}}. \quad (\text{F9})$$

473 Substituting (F9) into (F6), we get $\forall t \geq 2T + \frac{D}{c}$,

$$\left| \ln(\phi_{\mathbf{b}(t)}(\theta)) - \sum_{n=1}^2 \frac{i^n}{n!} \sum_{k=0}^n \binom{n}{k} \theta_1^k \theta_2^{n-k} \kappa_{k, n-k}(t) \right| \leq \frac{1}{(t-2T)} \psi(\theta_1, \theta_2) \xrightarrow{t \rightarrow +\infty} 0,$$

474 where

$$475 \quad \psi(\theta_1, \theta_2) = 2\pi\lambda c D^2 \int_{u=-T-\frac{D}{2c}}^{T+\frac{D}{2c}} \int_{w=-\frac{D}{c}}^{\frac{D}{c}} e^{\frac{|\theta_1 g(u-\frac{w}{2})| + |\theta_2 g(u+\frac{w}{2})|}{D}} - \sum_{n=0}^2 \frac{1}{n!} \left(\frac{|\theta_1 g(u-\frac{w}{2})| + |\theta_2 g(u+\frac{w}{2})|}{D} \right)^n dudw.$$

476 Therefore the characteristic function of $\mathbf{b}(t)$ converges pointwise to that of the normal
477 distribution when $t \rightarrow +\infty$, which proves that $\mathbf{b}(t)$ is asymptotically normally distributed.

478 In the same way, it can be proved¹³ that the random variables $\mathbf{b}(t+t_1) \dots \mathbf{b}(t+t_K)$ for all
479 $K \in \mathbb{N}$ and $t, t_1 \dots t_K \in \mathbb{R}$ are jointly normally distributed when $t \rightarrow +\infty$ and $t_1 \dots t_K$ are
480 fixed, which proves that $\mathbf{b}(t)$ converges in law to a stationary Gaussian process.

481 APPENDIX G: PROOF OF PROPOSITION 3

482 1. In the spectral domain

From (F1), since

$$\int_{\mathbb{R}^3} \frac{e^{-\frac{\alpha}{c}\|\mathbf{x}-\mathbf{x}_i\|_2} e^{-\frac{\alpha}{c}\|\mathbf{x}-\mathbf{x}_j\|_2}}{\|\mathbf{x}-\mathbf{x}_i\|_2 \|\mathbf{x}-\mathbf{x}_j\|_2} d\mathbf{x} < +\infty,$$

483 Fubini's theorem shows that we can permute the mathematical expectation with the integral
484 over \mathbf{x} in (19), leading to:

$$\text{cov}[\mathcal{F}_{h_i}(f_1), \mathcal{F}_{h_j}(f_2)] = \lambda \mathcal{L}_g(\alpha + 2i\pi f_1) \mathcal{L}_g(\alpha + 2i\pi f_2) e^{-2i\pi(f_1-f_2)T} J_{\xi_1, \xi_2}, \quad (\text{G1})$$

485 where $\xi_1(r) = e^{-\frac{\alpha+2i\pi f_1}{c}r}$, $\xi_2(r) = e^{-\frac{\alpha+2i\pi f_2}{c}r}$, and J_{ξ_1, ξ_2} was defined in (E1). Then equa-

486 tion (E3) shows that

$$J_{\xi_1, \xi_2} = \frac{2\pi}{D} \int_{\rho=\frac{D}{2}}^{+\infty} e^{-2\frac{\alpha+i\pi(f_1-f_2)}{c}\rho} \rho d\rho \int_{v=-D}^D e^{-2i\pi\frac{f_1+f_2}{2c}v} dv = 2\pi c \frac{e^{-\frac{\alpha+i\pi(f_1-f_2)}{c}D}}{\alpha + i\pi(f_1 - f_2)} \text{sinc}\left(2\pi\frac{f_1+f_2}{2c}D\right). \quad (\text{G2})$$

487 By substituting (G2) and (29) into (G1), we finally get (38).

488 **2. In the time domain**

489 Equations (17) and (35) prove that

$$\text{cov}[h_i(t_1), h_j(t_2)] = 4\pi\lambda c e^{-2\alpha(\frac{t_1+t_2}{2}-T)} w * \tilde{g} * g(t_1 - t_2). \quad (\text{G3})$$

490 Finally, substituting (31) and (25) into (G3) implies (39).

491 **3. In the time-frequency domain**

Finally, from (F1) and (17), since

$$\int_{u \in \mathbb{R}} \int_{\mathbf{x} \in \mathbb{R}^3} \frac{\left| g\left(t + \frac{u}{2} - T - \frac{\|\mathbf{x}-\mathbf{x}_i\|_2}{c}\right) g\left(t - \frac{u}{2} - T - \frac{\|\mathbf{x}-\mathbf{x}_j\|_2}{c}\right) \right|}{\|\mathbf{x} - \mathbf{x}_i\|_2 \|\mathbf{x} - \mathbf{x}_j\|_2} d\mathbf{x} du < +\infty,$$

492 Fubini's theorem shows that we can permute the mathematical expectation with the integrals

493 over \mathbf{x} in (18) and over u in (11) (with $\psi_1 = h_i$ and $\psi_2 = h_j$), leading to:

$$\mathcal{W}_{h_i, h_j}(t, f) = \lambda e^{-2\alpha(t-T)} \int_{\mathbb{R}} J_{\xi_1, \xi_2} e^{-2i\pi f u} du. \quad (\text{G4})$$

494 where $\xi_1(r) = g\left(t + \frac{u}{2} - T - \frac{r}{c}\right)$, $\xi_2(r) = g\left(t - \frac{u}{2} - T - \frac{r}{c}\right)$, and J_{ξ_1, ξ_2} was defined in (E1).

495 Then substituting (E3) into (G4) shows that

$$\mathcal{W}_{h_i, h_j}(t, f) = \frac{4\pi\lambda e^{-2\alpha(t-T)}}{2D} \int_{\rho=\frac{D}{2}}^{+\infty} \int_{v=-D}^D I\left(t - T - \frac{\rho}{c}, f, v\right) d\rho dv \quad (\text{G5})$$

496 with

$$I(t, f, v) = \int_{u \in \mathbb{R}} g\left(t + \frac{u}{2} - \frac{v}{2c}\right) g\left(t - \frac{u}{2} + \frac{v}{2c}\right) e^{-2i\pi f u} du = e^{-2i\pi f \frac{v}{c}} \mathcal{W}_{g, g}(t, f), \quad (\text{G6})$$

497 where $\mathcal{W}_{g, g}(t, f)$ was expressed in (D3), and we have used the change of variable $u' = u - \frac{v}{c}$.

498 In order to conclude, we will use the two properties of the Wigner distribution that
 499 we already used in Appendix D 2c. Firstly, the time support of $\mathcal{W}_{g,g}$ is the same as that
 500 of g . Secondly, $\int_{t \in \mathbb{R}} \mathcal{W}_{g,g}(t, f) dt = |\mathcal{F}_g(f)|^2$. Substituting (G6) and (26) into (G5) finally
 501 implies (40).

502 4. Asymptotic normality

503 Since $\mathbf{b}(t)$ converges in law to a stationary Gaussian process as shown in Lemma 2,
 504 equation (17) shows that $\mathbf{h}(t)$ converges in law to a Gaussian process.

505 ¹The Fourier transform is defined in equation (9).

506 ²The correlation of two complex random variables is defined in equation (7).

507 ³Note that in this case, $\mathcal{F}_{h_i}(f)$ is no longer a stationary process.

508 ⁴The Wigner distribution is defined in equation (11).

509 ⁵A straightforward generalization of this model will also permit us to prove (5), that will be presented in a
 510 future paper .

511 ⁶Fig. 3 illustrates the source image principle in 2D-space for convenience, but of course our model will be
 512 defined in the 3D-space.

513 ⁷For the sake of simplicity, we will focus on the case of two microphones; the generalization to an arbitrary
 514 number of microphones is straightforward.

515 ⁸More precisely, (15) is required to prove that the random processes $b_i(t)$ and $h_i(t)$ are centered after a
 516 given time (*cf.* Appendix C 1 a), and (16) enforces the nonnegativity of $\mathcal{L}_g(\alpha + 2i\pi f)$ in (20), which makes

517 the filter $A(f)$ linear-phase, and which is required to get an expression (30) of $\text{corr}[\mathcal{F}_{h_i}(f_1), \mathcal{F}_{h_i}(f_2)]$ in
 518 Proposition 2 that simplifies to Schroeder’s equation (2) when $T \rightarrow 0$.

519 ⁹A WSS process is a random process with finite second order moments, whose mean and covariances are
 520 invariant with respect to any translation of time.

521 ¹⁰Actually, the reader may notice that the magnitudes of the extrema of the dashed curve provided by (42)
 522 are slightly lower than those of the solid curve given by Cook’s formula (3). This difference is due to the way
 523 Wigner distributions are estimated: the STFT produces estimated time-frequency distributions $\widehat{\mathcal{W}}_{h_i, h_i}(t, f)$
 524 and $\widehat{\mathcal{W}}_{h_i, h_j}(t, f)$ that are smoothed both in time and frequency. Therefore the estimate (42) is the ratio
 525 between two spectra that are both smoothed in frequency, which explains the deviations between the two
 526 curves in Fig. 5-(a).

¹¹An example of function $\psi(t)$ satisfying the conditions in Lemma 3 is

$$\psi(t) = \left(\left(\alpha^2 + \frac{4\pi^2}{T^2} \right) \cos \left(\frac{2\pi t}{T} \right) - \alpha^2 \right) \mathbf{1}_{[0, T]}(t).$$

527 ¹²A *simple function* is defined as a linear combination of indicator functions of measurable sets.

528 ¹³The proof is the same and it is omitted here for the sake of conciseness.

529

530 Allen, J. B., and Berkley, D. A. (1979). “Image method for efficiently simulating small-room
 531 acoustics,” *Journal of the Acoustical Society of America* **65**(4), 943–950.

532 Balian, R., and Bloch, C. (1970). “Distribution of eigenfrequencies for the wave equation
 533 in a finite domain: I. three-dimensional problem with smooth boundary surface,” *Annals*
 534 *of Physics* **60**(2), 401–447.

535 Baskind, A. (2003). “Modèles et méthodes de description spatiale de scènes sonores : appli-
 536 cation aux enregistrements binauraux (*Models and methods of spatial description of sound*)

- 537 *scenes: application to binaural recordings*),” Ph.D. thesis, Université Pierre et Marie Curie
 538 (UPMC), Paris, France.
- 539 Chiu, S. N., Stoyan, D., Kendall, W. S., and Mecke, J. (2013). *Stochastic Geometry and*
 540 *Its Applications*, Chap. 2 ”Point processes I - The Poisson point process”, 3rd edition ed.
 541 (Wiley, Hoboken, NJ, USA).
- 542 Cohen, L. (1989). “Time-frequency distributions-a review,” *Proc. IEEE* **77**(7), 941–981.
- 543 Cook, R. K., Waterhouse, R. V., Berendt, R. D., Edelman, S., and Thompson Jr., M. C.
 544 (1955). “Measurement of correlation coefficients in reverberant sound fields,” *The Journal*
 545 *of the Acoustical Society of America* **27**(6), 1072–1077.
- 546 Cremer, L., Müller, H. A., and Schultz, T. J. (1982). *Principles and applications of room*
 547 *acoustics*, **1**, Chap. (ii) ”Statistical Room Acoustics” (Applied Science Publishers, London,
 548 UK).
- 549 Elko, G. W. (2001). “Spatial coherence functions for differential microphones in isotropic
 550 noise fields,” *Microphone Arrays* 61–85.
- 551 Jacobsen, F., and Roisin, T. (2000). “The coherence of reverberant sound fields,” *The*
 552 *Journal of the Acoustical Society of America* **108**(1), 204–210.
- 553 Jot, J.-M., Cerveau, L., and Warusfel, O. (1997). “Analysis and synthesis of room rever-
 554 beration based on a statistical time-frequency model,” in *AES Convention 103*, Audio
 555 Engineering Society, New, NY, USA, pp. 4629–4658.
- 556 Joyce, W. B. (1975). “Sabine’s reverberation time and ergodic auditoriums,” *The Journal*
 557 *of the Acoustical Society of America* **58**(3), 643–655.

- 558 Kuttruff, H. (2014). *Room Acoustics, Fifth Edition* (CRC Press, Boca Raton, FL, USA),
 559 pp. 1–374.
- 560 Maa, D.-Y. (1939). “Distribution of eigentones in a rectangular chamber at low frequency
 561 range,” *The Journal of the Acoustical Society of America* **10**(3), 235–238.
- 562 Moorer, J. A. (1979). “About this reverberation business,” *Computer Music Journal* **3**(2),
 563 13–28.
- 564 Polack, J. D. (1988). “La transmission de l’énergie sonore dans les salles (*The transmission
 565 of sound energy in rooms*),” Ph.D. thesis, Université du Maine, Le Mans, France.
- 566 Polack, J.-D. (1992). “Modifying chambers to play billiards: the foundations of reverbera-
 567 tion theory,” *Acta Acustica united with Acustica* **76**(6), 256–272(17).
- 568 Polack, J.-D. (1993). “Playing billiards in the concert hall: The mathematical foundations
 569 of geometrical room acoustics,” *Applied Acoustics* **38**(2), 235–244.
- 570 Polack, J.-D. (2007). “The relationship between eigenfrequency and image source distribu-
 571 tions in rectangular rooms,” *Acta Acustica united with Acustica* **93**, 1000–1011.
- 572 Schroeder, M. R. (1962). “Frequency-correlation functions of frequency responses in rooms,”
 573 *The Journal of the Acoustical Society of America* **34**(12), 1819–1823.
- 574 Schroeder, M. R. (1987). “Statistical parameters of the frequency response curves of large
 575 rooms,” *The Journal of the Audio Engineering Society* **35**(5), 299–306.
- 576 Schroeder, M. R., and Kuttruff, K. H. (1962). “On frequency response curves in rooms.
 577 Comparison of experimental, theoretical, and Monte Carlo results for the average frequency
 578 spacing between maxima,” *The Journal of the Acoustical Society of America* **34**(1), 76–80.

579 Schultz, T. (1971). “Diffusion in reverberation rooms,” Journal of Sound and Vibration
580 **16**(1), 17–28.

581 Vincent, E., and Campbell, D. R. (2008). “Roomsimove” GNU Public License, [http://](http://homepages.loria.fr/evincent/software/Roomsimove_1.4.zip)
582 homepages.loria.fr/evincent/software/Roomsimove_1.4.zip.