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# $\mathcal{H}_\infty$ control under asymptotic tracking constraint and nonproper output weights

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## ABSTRACT

This paper presents a complete solution to the measurement output feedback  $\mathcal{H}_\infty$  control problem subject to asymptotic tracking of an infinite energy reference and the presence of nonproper output weights. Relying on generalized Sylvester-type equations, a set of all stabilizing controllers is characterized, explicitly exhibiting the impacts of the reference signal and the output weight on the resulting controllers. Based on this specific parameterization, additional disturbance attenuation is tackled, and both full-order and reduced-order controller syntheses are discussed. It is observed that the multiobjective problem is equivalent to a standard problem for an auxiliary system constructed from the original weighted system with the control input matrix and measurement output matrix being augmented by solutions to generalized Sylvester-type equations. An example is included to show the effectiveness of the present results.

## KEYWORDS

Asymptotic tracking; controller parameterization;  $\mathcal{H}_\infty$  performance; nonproper weights; reduced-order controller; robust control

## 1. Introduction

One of the fundamental issues in systems and control theory is to track (reject) infinite energy references (disturbances) asymptotically, which is also referred to as the regulation problem [Stoorvogel, Saberi, and Sannuti \(2000\)](#). Such a problem commonly appears in practical engineering systems, such as helicopters [Cai, Chen, Peng, Lee, and Dong \(2008\)](#), disk drive servo system [Lan, Thum, and Chen \(2010\)](#), lateral vehicle control [Fauvel, Claveau, and Chevrel \(2013\)](#), spacecraft [Fadakar, Fidan, and Huissoon \(2017\)](#), robotic manipulator [Andreev and Peregudova \(2017\)](#), etc. Various solutions have been provided to guarantee tracking performance ; for example, repetitive control [Pipeleers, Demeulenaere, Schutter, and Swevers \(2008\)](#); [Tomizuka \(2008\)](#), adaptive control [Zhang and Xie \(2014\)](#), and nonlinear control [Chen, Lee, Peng, and Venkataramanan \(2003\)](#); [Feng and Ho \(2014\)](#). Among these different techniques, one seminal result, known as the internal model principle [Francis \(1977\)](#); [Wonham \(1985\)](#), achieves exact asymptotic tracking by a structured controller containing a copy of

the dynamics of the so-called exo-system. Extensions of this scheme have been widely developed by considering other performance objectives and such multiobjective problems have been extensively investigated in the literature; e.g., see [Köroğlu and Scherer \(2011\)](#); [Meng, Yang, and Dimarogonas \(2015\)](#); [Su and Huang \(2014\)](#); [Wieland, Sepulchre, and Allgöwer \(2011\)](#); [Yu, Zhao, and Wu \(2014\)](#) and the references therein.

An alternative approach to tackling the aforementioned problems is the use of weighting filters [Feng and Yagoubi \(2016\)](#); [Meinsma \(1995\)](#); [Mita, Xin, and Anderson \(2000\)](#). It is well known that, in many problems, weights are often required for the definition of a standard model consisting of a physical plant, disturbances and reference signals, and control objectives. A well recognized example of adopting weights is the  $\mathcal{H}_\infty$  control in which it is often desirable to choose a weight with a pole at the origin since the closed-loop system is finite only if the sensitivity has a zero at the origin. This choice indicates that the underlying stabilizing controller achieves either the perfect rejection of constant disturbances or the tracking of constant references. Moreover, in order to avoid undesirable high-frequency noise sensitivity and limited robustness, it is also often advisable to select a nonproper weight, whose  $\mathcal{H}_\infty$  norm is generally outside the desirable closed-loop bandwidth [Meinsma \(1995\)](#). However, the use of weights potentially introduces some unstabilizable or undetectable finite dynamics, even uncontrollable or unobservable impulsive elements, if nonproper weights are involved, into the standard model. Under these circumstances, the stabilization issue yields a nonstandard problem which traditional methods fail to solve, since the conventional internal stability can not usually be achieved. These undesirable elements can be treated approximately ; for example, by slight perturbation to render the problem standard [Chiang and Safonov \(1992\)](#). However, this approach is vulnerable to the issues related to lightly-damped poles and may lead to higher order and not strictly proper controllers. To solve such problems, Kwakernaak [Kwakernaak \(1993\)](#) uses the so-called polynomial method, which copes naturally with nonproper weights, but can not be applied to the case in which weights have imaginary poles. Plant augmentation is rather similar to the “plant state tapping” technique which is also used to address unstable and nonproper weights [Meinsma \(1995\)](#). Recently, through the framework of descriptor systems [Dai \(1989\)](#); [Feng and Yagoubi \(2017\)](#); [Xu and Lam \(2006\)](#), which can easily model nonproper elements, the concept of comprehensive admissibility [Feng, Yagoubi, and Chevrel \(2012a, 2012b\)](#), which expands the results in [Liu, Zhang, and Mita \(1997\)](#); [Mita et al. \(2000\)](#) into the nonproper case, has been introduced to handle such nonstandard problems, and  $\mathcal{H}_2$  and  $\mathcal{H}_\infty$  performance control are also solved via Riccati-type solutions.

In this paper, we consider the measurement output feedback  $\mathcal{H}_\infty$  control with exact asymptotic tracking constraint and nonproper output weighting filter, and we present a complete solution to such a multiobjective problem. The infinite energy reference to be tracked is described as an autonomous weight in order to address the problem within a uniform framework. A set of all stabilizing controllers is first explicitly characterized in terms of weights’ dynamics and solutions to generalized Sylvester-type equations. Based on the structured controller, the nonstandard problem is transformed into a standard control problem with respect to an auxiliary system, which is constructed from the original weighted system with the control input matrix and measurement output matrix being augmented by solutions to generalized Sylvester-type equations, and additional  $\mathcal{H}_\infty$  performance is further tackled through LMI-based solutions. Moreover, for the sake of easy implementation, reduced-order  $\mathcal{H}_\infty$  controller synthesis is included. Unlike the Riccati-type solution reported in [Feng et al. \(2012a, 2012b\)](#), the current results do not require assumptions of invariant zeros, and the  $\mathcal{H}_\infty$  performance design

is independent of solutions to generalized Sylvester-type equations, which implies a convex optimization. The remainder of this paper is organized as follows. The multi-objective problem under consideration is defined in Section 2. Section 3 presents the parameterization of all stabilizing controllers under asymptotic tracking and potentially nonproper output weights. Based on this specific structure, both full-order and reduced-order  $\mathcal{H}_\infty$  controller design are further discussed in Section 4. An example is included to illustrate the effectiveness of the present results in Section 5, and concluding remarks are given in Section 6. Some fundamental concepts of descriptor systems are recalled in Appendix A.

*Notation:* The notation used in this paper is fairly standard. For a real square matrix  $X$  the notation  $X \geq 0$  (respectively  $X > 0$ ) means that the matrix  $X$  is symmetric and positive-semi-definite (respectively positive definite). The superscripts ‘ $T$ ’ and ‘ $-1$ ’ represent the transpose and inverse, respectively.  $\text{Tr}(\cdot)$  and  $\lambda(\cdot)$  are the trace and eigenvalues of the matrix, while the notation  $\mathbf{He}\{P\}$  stands for  $P^\top + P$ . For a symmetric matrix represented blockwise, off-diagonal blocks are abbreviated by ‘ $\bullet$ ’.  $\mathcal{F}_l(\cdot, \cdot)$  stands for the lower linear fractional transformation. Moreover, for a matrix  $\Lambda \in \mathbb{R}^{m \times n}$  with  $m > n$ , let  $\Lambda^\perp \in \mathbb{R}^{(m-n) \times m}$  be any matrix satisfying  $\Lambda^\perp \Lambda = 0$  and  $\Lambda^\perp (\Lambda^\perp)^\top > 0$ . Note that  $\Lambda^\perp$  exists if and only if  $\Lambda$  has linearly dependent rows. For  $m < n$ , let  $\Lambda^\perp \triangleq ((\Lambda^\top)^\perp)^\top$ . For simplicity, the arguments of a function will sometimes be omitted when no confusion can arise. Besides, unless stated otherwise, all matrices used in this paper are assumed to have appropriate dimensions.

## 2. Problem formulation

The problem setup is depicted in Fig. 1, where  $\tilde{G}$  stands for the physical plant described by the following continuous-time linear time-invariant (LTI) system

$$\tilde{G} : \begin{cases} \dot{x} &= A_g x + B_r r + B_w w + B_u u, \\ v &= C_v x + D_{vr} r + D_{vw} w + D_{vu} u, \\ y &= C_g x + D_r r + D_w w, \\ e &= C_e x + D_{er} r + D_{ew} w + D_{eu} u, \end{cases} \quad (1)$$

where  $x \in \mathbb{R}^n$ ,  $y \in \mathbb{R}^p$ ,  $v \in \mathbb{R}^l$ ,  $e \in \mathbb{R}^q$ ,  $r \in \mathbb{R}^{p_i}$ ,  $w \in \mathbb{R}^w$ ,  $z \in \mathbb{R}^{p_o}$  and  $u \in \mathbb{R}^m$  are the plant state, measurement output, controlled output, tracking error, exogenous reference, external disturbance, weighted output and control input, respectively. The matrices  $A_g \in \mathbb{R}^{n \times n}$ ,  $B_r \in \mathbb{R}^{n \times p_i}$ ,  $B_w \in \mathbb{R}^{n \times w}$ ,  $B_u \in \mathbb{R}^{n \times m}$ ,  $C_v \in \mathbb{R}^{l \times n}$ ,  $C_g \in \mathbb{R}^{p \times n}$ ,  $C_e \in \mathbb{R}^{q \times n}$ ,  $D_{vr} \in \mathbb{R}^{l \times p_i}$ ,  $D_{vw} \in \mathbb{R}^{l \times w}$ ,  $D_{vu} \in \mathbb{R}^{l \times m}$ ,  $D_r \in \mathbb{R}^{p \times p_i}$ ,  $D_w \in \mathbb{R}^{p \times w}$ ,  $D_{er} \in \mathbb{R}^{q \times p_i}$ ,  $D_{ew} \in \mathbb{R}^{q \times w}$ , and  $D_{eu} \in \mathbb{R}^{q \times m}$  are known constant matrices. The direct feedthrough matrix from  $u$  to  $y$  is left out to make the arguments simpler and can be handled via standard methods.

The exogenous reference  $r$  to follow is assumed to be generated by the linear autonomous system  $W_i$ , known as the exo-system,

$$W_i : \begin{cases} \dot{x}_i &= A_i x_i, \\ r &= C_i x_i, \end{cases} \quad (2)$$

with  $A_i \in \mathbb{R}^{n_i \times n_i}$  and  $C_i \in \mathbb{R}^{p_i \times n_i}$ . The weighted output  $z$  is related to the controlled

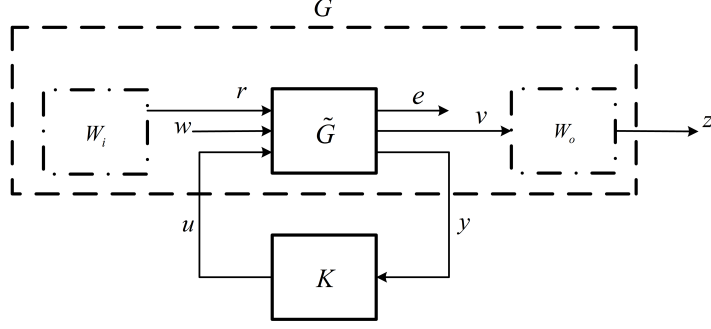


Figure 1. Problem setup

output  $v$  by the filter  $W_o$  of the form

$$W_o : \begin{cases} E_o \dot{x}_o &= A_o x_o + B_o v, \\ z &= C_o x_o + D_o v, \end{cases} \quad (3)$$

where  $E_o \in \mathbb{R}^{n_o \times n_o}$ ,  $A_o \in \mathbb{R}^{n_o \times n_o}$ ,  $B_o \in \mathbb{R}^{n_o \times l}$ ,  $C_o \in \mathbb{R}^{p_o \times n_o}$ , and  $D_o \in \mathbb{R}^{p_o \times l}$  are known real constant matrices. The matrix  $E_o$  may be singular, i.e.  $\text{rank}(E_o) = r_o \leq n_o$ . Note that systems of the form (3) are called descriptor (singular) systems and are known for their capacity to preserve the structure of physical systems and to describe non-dynamic constraints and impulsive behaviors Dai (1989); Feng and Yagoubi (2017); Xu and Lam (2006). The purpose of using descriptor representation here is to introduce nonproper output weights. Since the descriptor system itself is not the main focus of this paper, we do not give too much detail here, but include some basic notations in Appendix A to be self-contained.

The generalized system, incorporating the exo-system  $W_i$ , output weight  $W_o$  and physical plant  $\tilde{G}$ , is thus written by

$$G : \begin{cases} E \dot{\zeta} &= A \zeta + B_1 w + B_2 u, \\ z &= C_1 \zeta + D_{11} w + D_{12} u, \\ y &= C_2 \zeta + D_{21} w, \\ e &= C_3 \zeta + D_{31} w + D_{32} u, \end{cases} := \left[ \begin{array}{c|cc} A - sE & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & 0 \\ C_3 & D_{31} & D_{32} \end{array} \right], \quad (4)$$

where

$$\begin{aligned} \zeta &= \begin{bmatrix} x_o \\ x \\ x_i \end{bmatrix}, \quad E = \begin{bmatrix} E_o & 0 & 0 \\ 0 & I_n & 0 \\ 0 & 0 & I_{n_i} \end{bmatrix}, \quad A = \begin{bmatrix} A_o & B_o C_v & B_o D_{vr} C_i \\ 0 & A_g & B_r C_i \\ 0 & 0 & A_i \end{bmatrix}, \\ B_1 &= \begin{bmatrix} B_o D_{vw} \\ B_w \\ 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} B_o D_{vu} \\ B_u \\ 0 \end{bmatrix}, \quad C_1 = [C_o \quad D_o C_v \quad D_o D_{vr} C_i], \\ C_2 &= [0 \quad C_g \quad D_r C_i], \quad C_3 = [0 \quad C_e \quad D_{er} C_i], \quad D_{11} = D_o D_{vw}, \\ D_{12} &= D_o D_{vu}, \quad D_{21} = D_w, \quad D_{31} = D_{ew}, \quad D_{32} = D_{eu}. \end{aligned}$$

The measurement feedback controller  $K$  is assumed to be

$$K := \left[ \begin{array}{c|c} \frac{A_K - sE_K}{C_K} & \frac{B_K}{D_K} \end{array} \right], \quad (5)$$

where  $E_K \in \mathbb{R}^{n_k \times n_k}$ ,  $A_K \in \mathbb{R}^{n_k \times n_k}$ ,  $B_K \in \mathbb{R}^{n_k \times p}$ ,  $C_K \in \mathbb{R}^{m \times n_k}$  and  $D_K \in \mathbb{R}^{m \times p}$ . The matrix  $E_K$  may be singular, i.e.  $\text{rank}(E_K) = r_k \leq n_k$ .

The closed-loop system formed by  $G$  and  $K$  is denoted by  $\mathcal{F}_l(G, K)$ , and the transfer function from the external disturbance  $w$  to weighted output  $z$  of  $\mathcal{F}_l(G, K)$  is denoted by  $T_{zw}(s)$ .

**Problem 2.1** (Asymptotic Tracking with Weighted  $\mathcal{H}_\infty$  Performance). The problem under consideration is to find a measurement feedback controller such that the following conditions are satisfied

- (C1) (Asymptotic tracking) The tracking error  $e(t)$  satisfies  $\lim_{t \rightarrow \infty} e(t) = 0$  for any  $w(t) \in \mathcal{L}_2$ , and for all  $x(0) \in \mathbb{R}^n$ , and  $x_i(0) \in \mathbb{R}^{n_i}$ ;  
(C2) ( $\mathcal{H}_\infty$  performance) The  $\mathcal{H}_\infty$  norm of  $T_{zw}(s)$  is bounded by  $\gamma$  for some  $\gamma > 0$ .

In order to solve this problem, we introduce here two partitions of  $G$  with regard to  $W_i$  and  $W_o$ , denoted as  $G^{W_i}$  and  $G^{W_o}$ , respectively,

$$G^{W_i} := \left[ \begin{array}{c|c} \left[ \begin{array}{cc} \bar{A}_{11} & \bar{A}_{12} \\ 0 & A_i \end{array} \right] - s \left[ \begin{array}{cc} \bar{E} & 0 \\ 0 & I_{n_i} \end{array} \right] & \begin{array}{cc} \bar{B}_{11} & \bar{B}_{12} \\ 0 & 0 \end{array} \\ \hline \begin{array}{cc} C_{11} & C_{12} \\ \bar{C}_{21} & \bar{C}_{22} \\ \bar{C}_{31} & \bar{C}_{32} \end{array} & \begin{array}{cc} D_{11} & D_{12} \\ D_{21} & 0 \\ D_{31} & D_{32} \end{array} \end{array} \right], \quad (6)$$

where

$$\begin{aligned} \bar{E} &= \begin{bmatrix} E_o & 0 \\ 0 & I_n \end{bmatrix}, \quad \bar{A}_{11} = \begin{bmatrix} A_o & B_o C_v \\ 0 & A_g \end{bmatrix}, \quad \bar{A}_{12} = \begin{bmatrix} B_o D_{vr} C_i \\ B_r C_i \end{bmatrix}, \\ \bar{B}_{11} &= \begin{bmatrix} B_o D_{vr} \\ B_w \end{bmatrix}, \quad \bar{B}_{12} = \begin{bmatrix} B_o D_{vu} \\ B_u \end{bmatrix}, \quad \bar{C}_{11} = [C_o \quad D_o C_v], \\ \bar{C}_{12} &= D_o D_{vr} C_i, \quad \bar{C}_{21} = [0 \quad C_g], \quad \bar{C}_{22} = D_r C_i, \quad \bar{C}_{31} = [0 \quad C_e], \\ \bar{C}_{32} &= D_{er} C_i, \quad D_{11} = D_o D_{vw}, \quad D_{12} = D_o D_{vu}, \quad D_{21} = D_w, \\ D_{31} &= D_{ew}, \quad D_{32} = D_{eu}; \end{aligned}$$

$$G^{W_o} := \left[ \begin{array}{c|c} \left[ \begin{array}{cc} A_o & \hat{A}_{12} \\ 0 & \hat{A}_{22} \end{array} \right] - s \left[ \begin{array}{cc} E_o & 0 \\ 0 & I_{n+n_i} \end{array} \right] & \begin{array}{cc} \hat{B}_{11} & \hat{B}_{12} \\ \hat{B}_{21} & \hat{B}_{22} \end{array} \\ \hline \begin{array}{cc} C_o & \hat{C}_{12} \\ 0 & \hat{C}_{22} \\ 0 & \hat{C}_{32} \end{array} & \begin{array}{cc} D_{11} & D_{12} \\ D_{21} & 0 \\ D_{31} & D_{32} \end{array} \end{array} \right], \quad (7)$$

where

$$\begin{aligned}\hat{A}_{12} &= [B_o C_v \quad B_o D_{vr} C_i], \quad \hat{A}_{22} = \begin{bmatrix} A_g & B_r C_i \\ 0 & A_i \end{bmatrix}, \quad \hat{B}_{11} = B_o D_{vw}, \\ \hat{B}_{12} &= B_o D_{vu}, \quad \hat{B}_{21} = \begin{bmatrix} B_w \\ 0 \end{bmatrix}, \quad \hat{B}_{22} = \begin{bmatrix} B_u \\ 0 \end{bmatrix}, \quad \hat{C}_{12} = [D_o C_v \quad D_o D_{vr} C_i], \\ \hat{C}_{22} &= [C_g \quad D_r C_i], \quad \hat{C}_{32} = [C_e \quad D_{er} C_i].\end{aligned}$$

Before ending this section, the following assumptions are made.

- (A1)  $W_i$  and  $W_o$  only possess unstable and impulsive modes;
- (A2)  $(\bar{E}, \bar{A}_{11}, \bar{B}_{12})$  is finite dynamics stabilizable and impulse controllable;
- (A3)  $(\hat{A}_{22}, \hat{C}_{22})$  is detectable.

Assumption (A1) is made without loss of generality to simplify the argument, since stable modes, if they exist, of  $W_i$  and  $W_o$  do not affect the internal stability of the underlying closed-loop system and can always be incorporated into the physical plant  $\tilde{G}$ . Assumptions (A2) and (A3) are necessary for the existence of a stabilizing measurement feedback controller to the weighted system  $G$ . Moreover, it is worth noting that due to the use of unstable and nonproper weights, the multiobjective problem under consideration results in a design problem with unstabilizable and undetectable dynamics, which standard methods cannot solve. In this paper, we present a complete solution to this nonstandard multiobjective problem through a structured controller together with LMI-based conditions to both full-order and reduced-order controller synthesis.

### 3. Controller parameterization

In this section, we show a specific parameterization for stabilizing controller design to fulfil the asymptotic tracking and stability of the transfer function  $T_{zw}(s)$ . To this end, we mildly modify Condition (C2) in Problem 2.1 to

(C'2) The transfer function  $T_{zw}(s)$  is stable.

The following theorem gives the necessary and sufficient solution to Conditions (C1) and (C'2).

**Theorem 3.1.** *Consider the partitions (6) and (7). Conditions (C1) and (C'2) are satisfied, if and only if there exist matrices  $X_i \in \mathbb{R}^{(n+n_o) \times n_i}$ ,  $Y_i \in \mathbb{R}^{(n+n_o) \times n_i}$ ,  $\Pi_i \in \mathbb{R}^{m \times n_i}$ ,  $X_o \in \mathbb{R}^{n_o \times (n+n_i)}$ ,  $Y_o \in \mathbb{R}^{n_o \times (n+n_i)}$  and  $\Pi_o \in \mathbb{R}^{n_o \times p}$  such that the following generalized Sylvester-type equations hold.*

$$\begin{cases} \bar{A}_{12} &= \bar{A}_{11} Y_i + \bar{B}_{12} \Pi_i + X_i A_i, \\ \bar{C}_{32} &= \bar{C}_{31} Y_i + D_{32} \Pi_i, \\ 0 &= \bar{E} Y_i + X_i. \end{cases} \quad (8)$$

$$\begin{cases} \hat{A}_{12} &= \Pi_o \hat{C}_{22} - X_o \hat{A}_{22} + A_o Y_o, \\ \hat{B}_{11} &= \Pi_o D_{21} - X_o \hat{B}_{21}, \\ 0 &= E_o Y_o - X_o. \end{cases} \quad (9)$$

Moreover, the measurement feedback controller  $K$  is illustrated in Fig. 2 and

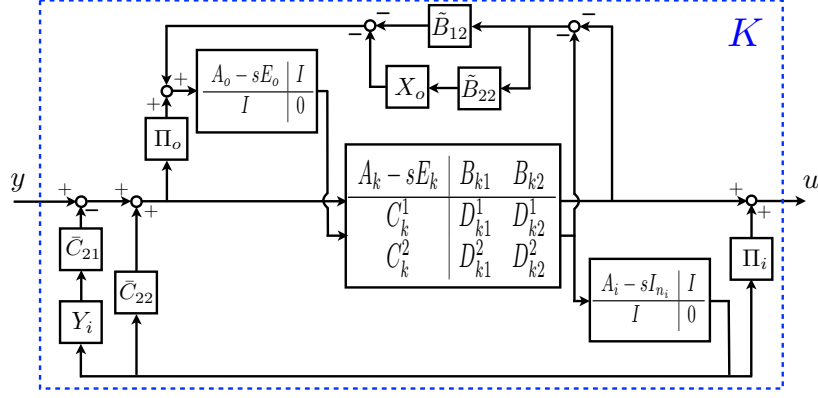


Figure 2. Structured controller

parametrized by

$$K := \left[ \begin{array}{c|c} A_K - sE_K & B_K \\ \hline C_K & D_{k1}^1 \end{array} \right], \quad (10)$$

where

$$E_K = \begin{bmatrix} I_{n_i} & 0 & 0 \\ 0 & E_k & 0 \\ 0 & 0 & E_o \end{bmatrix}, \quad A_K = \begin{bmatrix} A_i + D_{k1}^2 \Gamma_i & C_k^2 & D_{k2}^2 \\ B_{k1} \Gamma_i & A_k & B_{k2} \\ \Omega_o \Gamma_i & -\Gamma_o C_k & A_o - \Gamma_o D_{k2} \end{bmatrix},$$

$$B_K = \begin{bmatrix} D_{k1}^2 \\ B_{k1} \\ \Omega_o \end{bmatrix}, \quad C_K = [\Omega_i \quad C_k^1 \quad D_{k2}^1], \quad \Gamma_i = \bar{C}_{22} - \bar{C}_{21} Y_i,$$

$$\Gamma_o = -X_o \tilde{B}_{22} - \tilde{B}_{12}, \quad \tilde{B}_{12} = [\hat{B}_{12} \quad X_i^1], \quad \tilde{B}_{22} = \left[ \begin{array}{c|c} \hat{B}_{22} & X_i^2 \\ \hline & I_{n_i} \end{array} \right],$$

$$\Omega_i = D_{k1}^1 \Gamma_i + \Pi_i, \quad \Omega_o = \Pi_o - \Gamma_o D_{k1}, \quad X_i^1 = [I_{n_o} \quad 0] X_i,$$

$$X_i^2 = [0 \quad I_n] X_i, \quad C_k = \begin{bmatrix} C_k^1 \\ C_k^2 \end{bmatrix}, \quad D_{k1} = \begin{bmatrix} D_{k1}^1 \\ D_{k1}^2 \end{bmatrix}, \quad D_{k2} = \begin{bmatrix} D_{k2}^1 \\ D_{k2}^2 \end{bmatrix},$$

and the matrices  $E_k, A_k, B_{k1}, B_{k2}, C_k^1, C_k^2, D_{k1}^1, D_{k1}^2, D_{k2}^1$  and  $D_{k2}^2$  are parameters of the controller  $\mathcal{K}$  given by

$$\mathcal{K} := \left[ \begin{array}{c|cc} A_k - sE_k & B_{k1} & B_{k2} \\ \hline C_k^1 & D_{k1}^1 & D_{k2}^1 \\ C_k^2 & D_{k1}^2 & D_{k2}^2 \end{array} \right], \quad (11)$$

which internally stabilizes the auxiliary system  $\Sigma$  of the form

$$\Sigma := \left[ \begin{array}{c|c} A - sE & \left[ \begin{array}{c} B_2 \\ X_i \\ I_{n_i} \end{array} \right] \\ \hline \left[ \begin{array}{c} C_2 \\ I_{n_o} \quad Y_o \end{array} \right] & \left[ \begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array} \right] \end{array} \right]. \quad (12)$$



**Proof.** We show that the two generalized Sylvester-type equations (8) and (9) are necessary for the existence of a solution to Conditions (C1) and (C'2). For the partition (6), by Lemma 5.3 (page 71) of Feng and Yagoubi (2017), there exist matrices  $X_i \in \mathbb{R}^{(n+n_o) \times n_i}$ ,  $Y_i \in \mathbb{R}^{(n+n_o) \times n_i}$  and  $\Upsilon \in \mathbb{R}^{n_k \times n_i}$  such that

$$\begin{aligned} 0 &= D_{32} (D_k \bar{C}_{22} - D_k \bar{C}_{21} Y_i - C_k \Upsilon) - \bar{C}_{31} Y_i + \bar{C}_{32}, \\ -X_i A_i &= \bar{A}_{11} Y_i + \bar{B}_{12} (D_k \bar{C}_{21} Y_i + C_k \Upsilon - D_k \bar{C}_{22}) - \bar{A}_{12}, \\ -X_i &= \bar{E} Y_i. \end{aligned}$$

Defining  $\Pi_i = D_k \bar{C}_{21} Y_i + C_k \Upsilon - D_k \bar{C}_{22}$ , the generalized Sylvester-type equation (8) holds. (9) can be proven by the same token for the partition (7).

Sufficiency is achieved in such a way: through the two generalized Sylvester-type equations (8) and (9), together with certain transformation matrices, we show that the structured controller (10) eliminates the unstabilizable and undetectable dynamics introduced by the exo-system  $W_i$  and output weight  $W_o$  in the closed-loop system. The multiobjective problem is equivalent to a standard problem for an auxiliary system constructed from the data of the weighted system.

To this end, we first reform the controller (10) as

$$K := \left[ \begin{array}{cc|c} A_i + D_{k1}^2 \Gamma_i - sI_{n_i} & \bar{C}_{k2} & D_{k1}^2 \\ \bar{B}_k \Gamma_i & \bar{A}_k - s\bar{E}_k & \bar{B}_k \\ \hline \Omega_i & \bar{C}_{k1} & D_{k1}^1 \end{array} \right],$$

where

$$\begin{aligned} \bar{E}_k &= \begin{bmatrix} E_k & 0 \\ 0 & E_o \end{bmatrix}, \quad \bar{A}_k = \begin{bmatrix} A_k & B_{k2} \\ -\Gamma_o C_k & A_o - \Gamma_o D_{k2} \end{bmatrix}, \quad \bar{B}_k = \begin{bmatrix} B_{k1} \\ \Omega_o \end{bmatrix}, \\ \bar{C}_{k1} &= [C_k^1 \quad D_{k2}^1], \quad \bar{C}_{k2} = [C_k^2 \quad D_{k2}^2]. \end{aligned}$$

The closed-loop system  $G_{CL}$  formed by  $G$  and  $K$  is thus given as follows

$$G_{CL} := \left[ \begin{array}{c|c} AC - sEC & BC \\ \hline CC_1 & DC_1 \\ CC_3 & DC_3 \end{array} \right],$$

where

$$\begin{aligned} A_C &= \begin{bmatrix} \bar{A}_{11} + \bar{B}_{12} D_{k1}^1 \bar{C}_{21} & \bar{A}_{12} + \bar{B}_{12} D_{k1}^1 \bar{C}_{22} & \bar{B}_{12} \Omega_i & \bar{B}_{12} \bar{C}_{k1} \\ 0 & A_i & 0 & 0 \\ D_{k1}^2 \bar{C}_{21} & D_{k1}^2 \bar{C}_{22} & A_i + D_{k1}^2 \Gamma_i & \bar{C}_{k2} \\ \bar{B}_k \bar{C}_{21} & \bar{B}_k \bar{C}_{22} & \bar{B}_k \Gamma_i & \bar{A}_k \end{bmatrix}, \\ E_C &= \begin{bmatrix} \bar{E} & 0 & 0 & 0 \\ 0 & I_{n_i} & 0 & 0 \\ 0 & 0 & I_{n_i} & 0 \\ 0 & 0 & 0 & \bar{E}_k \end{bmatrix}, \quad B_C = \begin{bmatrix} \bar{B}_{11} + \bar{B}_{12} D_{k1}^1 D_{21} \\ 0 \\ D_{k1}^2 D_{21} \\ \bar{B}_k D_{21} \end{bmatrix}, \\ C_{C1} &= [\bar{C}_{11} + D_{12} D_{k1}^1 \bar{C}_{21} \quad \bar{C}_{12} + D_{12} D_{k1}^1 \bar{C}_{22} \quad D_{12} \Omega_i \quad D_{12} \bar{C}_{k1}], \\ C_{C3} &= [\bar{C}_{31} + D_{32} D_{k1}^1 \bar{C}_{21} \quad \bar{C}_{32} + D_{32} D_{k1}^1 \bar{C}_{22} \quad D_{32} \Omega_i \quad D_{32} \bar{C}_{k1}], \\ D_{C1} &= D_{11} + D_{12} D_{k1}^1 D_{21}, \quad D_{C3} = D_{31} + D_{32} D_{k1}^1 D_{21}. \end{aligned}$$

With the solution to (8), we define the following transformation matrices

$$M_1 = \begin{bmatrix} I_{n+n_o} & -X_i & 0 & 0 \\ 0 & I_{n_i} & 0 & 0 \\ 0 & I_{n_i} & I_{n_i} & 0 \\ 0 & 0 & 0 & I_{n_k} \end{bmatrix}, \quad N_1 = \begin{bmatrix} I_{n+n_o} & -Y_i & 0 & 0 \\ 0 & I_{n_i} & 0 & 0 \\ 0 & -I_{n_i} & I_{n_i} & 0 \\ 0 & 0 & 0 & I_{n_k} \end{bmatrix}.$$

Hence, we have

$$M_1 A_C N_1 = \begin{bmatrix} \bar{A}_{11} + \bar{B}_{12} D_{k1}^1 \bar{C}_{21} & \phi_1 & \bar{B}_{12} \Omega_i & \bar{B}_{12} \bar{C}_{k1} \\ 0 & A_i & 0 & 0 \\ D_{k1}^2 \bar{C}_{21} & \phi_2 & A_i + D_{k1}^2 \Gamma_i & \bar{C}_{k2} \\ \bar{B}_k \bar{C}_{21} & \phi_3 & \bar{B}_k \Gamma_i & \bar{A}_k \end{bmatrix},$$

$$M_1 E_C N_1 = E_C, \quad M_1 B_C = \begin{bmatrix} \bar{B}_{11} + \bar{B}_{12} D_{k1}^1 D_{21} \\ 0 \\ D_{k1}^2 D_{21} \\ \bar{B}_k D_{21} \end{bmatrix},$$

$$C_{C3} N_1 = [\bar{C}_{31} + D_{32} D_{k1}^1 \bar{C}_{21} \quad \phi_4 \quad D_{32} \Omega_i \quad D_{32} \bar{C}_{k1}],$$

$$C_{C1} N_1 = [\bar{C}_{11} + D_{12} D_{k1}^1 \bar{C}_{21} \quad \phi_5 \quad D_{12} \Omega_i \quad D_{12} \bar{C}_{k1}],$$

where

$$\begin{aligned} \phi_1 &= -\bar{A}_{11} Y_i + \bar{A}_{12} - X_i A_i + \bar{B}_{12} (D_{k1}^1 \bar{C}_{22} - D_{k1}^1 \bar{C}_{21} Y_i - \Omega_i) = 0, \\ \phi_2 &= D_{k1}^2 (\bar{C}_{22} - \bar{C}_{21} Y_i - \Gamma_i) = 0, \quad \phi_3 = \bar{B}_k (\bar{C}_{22} - \bar{C}_{21} Y_i - \Gamma_i) = 0, \\ \phi_4 &= -\bar{C}_{31} Y_i + \bar{C}_{32} + D_{32} (D_{k1}^1 \bar{C}_{22} - D_{k1}^1 \bar{C}_{21} Y_i - \Omega_i) = 0, \\ \phi_5 &= -\bar{C}_{11} Y_i + \bar{C}_{12} - D_{12} \Pi_i, \end{aligned}$$

in the light of (8). Therefore, the closed-loop system  $G_{CL}$  is restricted system equivalent to  $\bar{G}_{CL}$  of the form

$$\bar{G}_{CL} := \left[ \begin{array}{c|c} \bar{A}_C - s \bar{E}_C & \bar{B}_C \\ \hline \bar{C}_{C1} & D_{C1} \\ \bar{C}_{C3} & D_{C3} \end{array} \right],$$

where

$$\bar{A}_C = \begin{bmatrix} \bar{A}_{11} + \bar{B}_{12} D_{k1}^1 \bar{C}_{21} & \bar{B}_{12} \Omega_i & \bar{B}_{12} \bar{C}_{k1} & 0 \\ D_{k1}^2 \bar{C}_{21} & A_i + D_{k1}^2 \Gamma_i & \bar{C}_{k2} & 0 \\ \bar{B}_k \bar{C}_{21} & \bar{B}_k \Gamma_i & \bar{A}_k & 0 \\ 0 & 0 & 0 & A_i \end{bmatrix},$$

$$\bar{E}_C = \begin{bmatrix} \bar{E} & 0 & 0 & 0 \\ 0 & I_{n_i} & 0 & 0 \\ 0 & 0 & \bar{E}_k & 0 \\ 0 & 0 & 0 & I_{n_i} \end{bmatrix}, \quad \bar{B}_C = \begin{bmatrix} \bar{B}_{11} + \bar{B}_{12} D_{k1}^1 D_{21} \\ D_{k1}^2 D_{21} \\ \bar{B}_k D_{21} \\ 0 \end{bmatrix},$$

$$\bar{C}_{C1} = [\bar{C}_{11} + D_{12} D_{k1}^1 \bar{C}_{21} \quad D_{12} \Omega_i \quad D_{12} \bar{C}_{k1} \quad \phi_5],$$

$$\bar{C}_{C3} = [\bar{C}_{31} + D_{32} D_{k1}^1 \bar{C}_{21} \quad D_{32} \Omega_i \quad D_{32} \bar{C}_{k1} \quad 0].$$

It is observed that this system can be rewritten by  $\mathcal{F}_l(\bar{G}, \bar{K})$ , where

$$\bar{G} := \left[ \begin{array}{ccc|cc} \bar{A}_{11} - s\bar{E} & \bar{B}_{12}\Pi_i & 0 & \bar{B}_{11} & \begin{bmatrix} \bar{B}_{12} & 0 \\ 0 & I_{n_i} \\ 0 & 0 \end{bmatrix} \\ 0 & A_i - sI_{n_i} & 0 & 0 & \\ 0 & 0 & A_i - sI_{n_i} & 0 & \\ \hline \bar{C}_{11} & D_{12}\Pi_i & \phi_5 & D_{11} & \begin{bmatrix} D_{12} & 0 \\ 0 & 0 \end{bmatrix} \\ \bar{C}_{21} & \Gamma_i & 0 & D_{21} & \\ \bar{C}_{31} & D_{32}\Pi_i & 0 & D_{31} & \begin{bmatrix} D_{32} & 0 \end{bmatrix} \end{array} \right],$$

$$\bar{K} := \left[ \begin{array}{c|c} \bar{A}_k - s\bar{E}_k & \bar{B}_k \\ \hline \bar{C}_{k1} & \begin{bmatrix} D_{k1}^1 \\ D_{k1}^2 \end{bmatrix} \\ \bar{C}_{k2} & \end{array} \right],$$

and the second output of  $\bar{G}$  is the measurement. Then, using the two transformation matrices as follows

$$M_2 = \begin{bmatrix} I_{n+n_o} & X_i & 0 \\ 0 & I_{n_i} & 0 \\ 0 & 0 & I_{n_i} \end{bmatrix}, \quad N_2 = \begin{bmatrix} I_{n+n_o} & Y_i & 0 \\ 0 & I_{n_i} & 0 \\ 0 & 0 & I_{n_i} \end{bmatrix},$$

the system  $\bar{G}$  is restricted system equivalent to  $\hat{G}$  of the form

$$\hat{G} := \left[ \begin{array}{ccc|cc} \bar{A}_{11} - s\bar{E} & \bar{A}_{12} & 0 & \bar{B}_{11} & \begin{bmatrix} \bar{B}_{12} & X_i \\ 0 & I_{n_i} \\ 0 & 0 \end{bmatrix} \\ 0 & A_i - sI_{n_i} & 0 & 0 & \\ 0 & 0 & A_i - sI_{n_i} & 0 & \\ \hline \bar{C}_{11} & \phi_6 & \phi_5 & D_{11} & \begin{bmatrix} D_{12} & 0 \\ 0 & 0 \end{bmatrix} \\ \bar{C}_{21} & \bar{C}_{22} & 0 & D_{21} & \\ \bar{C}_{31} & \bar{D}_{32} & 0 & D_{31} & \begin{bmatrix} D_{32} & 0 \end{bmatrix} \end{array} \right],$$

$$= \left[ \begin{array}{ccc|cc} A_o - sE_o & \hat{A}_{12} & 0 & \hat{B}_{11} & \hat{B}_{12} \\ 0 & \hat{A}_{22} - sI_{n+n_i} & 0 & \hat{B}_{21} & \hat{B}_{22} \\ 0 & 0 & A_i - sI_{n_i} & 0 & 0 \\ \hline \bar{C}_o & \tilde{C}_{12} & \phi_5 & D_{11} & \tilde{D}_{12} \\ 0 & \hat{C}_{22} & 0 & D_{21} & 0 \\ 0 & \hat{C}_{32} & 0 & D_{31} & \tilde{D}_{32} \end{array} \right],$$

where  $\tilde{C}_{12} = [D_o C_v \ \phi_6]$ ,  $\phi_6 = \bar{C}_{11}Y_i + D_{12}\Pi_i$ ,  $\tilde{D}_{12} = [D_{12} \ 0]$ ,  $\tilde{D}_{32} = [D_{32} \ 0]$ , and other data are given in (7) and (10). Then, the closed-loop system  $\bar{G}_{CL}$  is further restricted system equivalent to  $\hat{G}_{CL} = \mathcal{F}_l(\hat{G}, \bar{K})$  given by

$$\hat{G}_{CL} := \left[ \begin{array}{c|c} \hat{A}_C - s\hat{E}_C & \hat{B}_C \\ \hline \hat{C}_{C1} & \hat{D}_{C1} \\ \hat{C}_{C3} & \hat{D}_{C3} \end{array} \right],$$

where

$$\begin{aligned} \hat{A}_C &= \begin{bmatrix} A_o & \hat{A}_{12} + \tilde{B}_{12}D_{k1}\hat{C}_{22} & \tilde{B}_{12}C_k & \tilde{B}_{12}D_{k2} & 0 \\ 0 & \hat{A}_{22} + \tilde{B}_{22}D_{k1}\hat{C}_{22} & \tilde{B}_{22}C_k & \tilde{B}_{22}D_{k2} & 0 \\ 0 & B_{k1}\hat{C}_{22} & A_k & B_{k2} & 0 \\ 0 & \Omega_o\hat{C}_{22} & -\Gamma_oC_k & A_o - \Gamma_oD_{k2} & 0 \\ 0 & 0 & 0 & 0 & A_i \end{bmatrix}, \\ \hat{E}_C &= \begin{bmatrix} E_o & 0 & 0 & 0 & 0 \\ 0 & I_{n+n_o} & 0 & 0 & 0 \\ 0 & 0 & E_k & 0 & 0 \\ 0 & 0 & 0 & E_o & 0 \\ 0 & 0 & 0 & 0 & I_{n_i} \end{bmatrix}, \quad \hat{B}_C = \begin{bmatrix} \hat{B}_{11} + \tilde{B}_{12}D_{k1}D_{21} \\ \hat{B}_{21} + \tilde{B}_{22}D_{k1}D_{21} \\ B_{k1}D_{21} \\ \Omega_oD_{21} \\ 0 \end{bmatrix}, \\ \hat{C}_{C1} &= [C_o \quad \tilde{C}_{12} + \tilde{D}_{12}D_{k1}\hat{C}_{22} \quad \tilde{D}_{12}C_k \quad \tilde{D}_{12}D_{k2} \quad \phi_5], \\ \hat{C}_{C3} &= [0 \quad \hat{C}_{32} + \tilde{D}_{32}D_{k1}\hat{C}_{22} \quad \tilde{D}_{32}C_k \quad \tilde{D}_{32}D_{k2} \quad 0], \\ \hat{D}_{C1} &= D_{11} + \tilde{D}_{12}D_{k1}D_{21}, \quad \hat{D}_{C3} = D_{31} + \tilde{D}_{32}D_{k1}D_{21}. \end{aligned}$$

Similarly, with the solution to (9), we define the following transformation matrices

$$M_3 = \begin{bmatrix} I_{n_o} & X_o & 0 & -I_{n_o} & 0 \\ 0 & I_{n+n_i} & 0 & 0 & 0 \\ 0 & 0 & I_{n_k} & 0 & 0 \\ 0 & 0 & 0 & I_{n_o} & 0 \\ 0 & 0 & 0 & 0 & I_{n_i} \end{bmatrix}, \quad N_3 = \begin{bmatrix} I_{n_o} & -Y_o & 0 & I_{n_o} & 0 \\ 0 & I_{n+n_i} & 0 & 0 & 0 \\ 0 & 0 & I_{n_k} & 0 & 0 \\ 0 & 0 & 0 & I_{n_o} & 0 \\ 0 & 0 & 0 & 0 & I_{n_i} \end{bmatrix}.$$

Hence, there holds

$$\begin{aligned} M_3\hat{A}_CN_3 &= \begin{bmatrix} A_o & \psi_1 & \psi_2 & \psi_3 & 0 \\ 0 & \hat{A}_{22} + \tilde{B}_{22}D_{k1}\hat{C}_{22} & \tilde{B}_{22}C_k & \tilde{B}_{22}D_{k2} & 0 \\ 0 & B_{k1}\hat{C}_{22} & A_k & B_{k2} & 0 \\ 0 & \Omega_o\hat{C}_{22} & -\Gamma_oC_k & A_o - \Gamma_oD_{k2} & 0 \\ 0 & 0 & 0 & 0 & A_i \end{bmatrix}, \\ M_3\hat{E}_CN_3 &= \hat{E}_C, \quad M_3\hat{B}_C = \begin{bmatrix} \psi_4 \\ \hat{B}_{21} + \tilde{B}_{22}D_{k1}D_{21} \\ B_{k1}D_{21} \\ \Omega_oD_{21} \\ 0 \end{bmatrix}, \\ \hat{C}_{C1}N_3 &= [C_o \quad \tilde{C}_{12} + \tilde{D}_{12}D_{k1}\hat{C}_{22} - C_oY_o \quad \tilde{D}_{12}C_k \quad \tilde{D}_{12}D_{k2} - C_o \quad \phi_5], \\ \hat{C}_{C3}N_3 &= [0 \quad \hat{C}_{32} + \tilde{D}_{32}D_{k1}\hat{C}_{22} \quad \tilde{D}_{32}C_k \quad \tilde{D}_{32}D_{k2} \quad 0]. \end{aligned}$$

where

$$\begin{aligned}
\psi_1 &= \hat{A}_{12} + \tilde{B}_{12}D_{k1}\hat{C}_{22} - A_oY_o + X_o \left( \hat{A}_{22} + \tilde{B}_{22}D_{k1}\hat{C}_{22} \right) - \Omega_o\hat{C}_{22} = 0, \\
\psi_2 &= \left( \tilde{B}_{12} + X_o\tilde{B}_{22} + \Gamma_o \right) C_k = 0, \\
\psi_3 &= -A_o + \left( \tilde{B}_{12} + X_o\tilde{B}_{22} + \Gamma_o \right) C_k + A_o = 0, \\
\psi_4 &= \hat{B}_{11} + X_o\hat{B}_{21} + \left( \tilde{B}_{12}D_{k1} + X_o\tilde{B}_{22}D_{k1} - \Omega_o \right) D_{21} = 0,
\end{aligned}$$

in the light of (9). Hence, the closed-loop system can be written as

$$\left\{ \begin{array}{l} \begin{bmatrix} I & 0 & 0 \\ 0 & E_k & 0 \\ 0 & 0 & E_o \end{bmatrix} \dot{x}_{cl} = \begin{bmatrix} \hat{A}_{22} + \tilde{B}_{22}D_{k1}\hat{C}_{22} & \tilde{B}_{22}C_k & \tilde{B}_{22}D_{k2} \\ B_{k1}\hat{C}_{22} & A_k & B_{k2} \\ \Omega_o\hat{C}_{22} & -\Gamma_oC_k & A_o - \Gamma_oD_{k2} \end{bmatrix} x_{cl} \\ \quad + \begin{bmatrix} \hat{B}_{21} + \tilde{B}_{22}D_{k1}D_{21} \\ B_{k1}D_{21} \\ \Omega_oD_{21} \end{bmatrix} w, \\ \dot{x}_i = A_ix_i, \\ z = [\tilde{C}_{12} + \tilde{D}_{12}D_{k1}\hat{C}_{22} - C_oY_o \quad \tilde{D}_{12}C_k \quad \tilde{D}_{12}D_{k2} + C_o] x_{cl} + \phi_5x_i + \hat{D}_{11}w, \\ e = [\hat{C}_{32} + \tilde{D}_{32}D_{k1}\hat{C}_{22} \quad \tilde{D}_{32}C_k \quad \tilde{D}_{32}D_{k2}] x_{cl} + \hat{D}_{31}w. \end{array} \right.$$

It is observed that the exo-system  $W_i$  neither affects the state  $x_{cl}$  and the tracking error  $e$ , nor appears in the the transfer function  $T_{zw}$ . Hence, we can simply consider the following system instead

$$\left\{ \begin{array}{l} \begin{bmatrix} I & 0 & 0 \\ 0 & E_k & 0 \\ 0 & 0 & E_o \end{bmatrix} \dot{x}_{cl} = \begin{bmatrix} \hat{A}_{22} + \tilde{B}_{22}D_{k1}\hat{C}_{22} & \tilde{B}_{22}C_k & \tilde{B}_{22}D_{k2} \\ B_{k1}\hat{C}_{22} & A_k & B_{k2} \\ \Omega_o\hat{C}_{22} & -\Gamma_oC_k & A_o - \Gamma_oD_{k2} \end{bmatrix} x_{cl} \\ \quad + \begin{bmatrix} \hat{B}_{21} + \tilde{B}_{22}D_{k1}D_{21} \\ B_{k1}D_{21} \\ \Omega_oD_{21} \end{bmatrix} w, \\ z = [\tilde{C}_{12} + \tilde{D}_{12}D_{k1}\hat{C}_{22} - C_oY_o \quad \tilde{D}_{12}C_k \quad \tilde{D}_{12}D_{k2} + C_o] x_{cl} + \hat{D}_{11}w, \\ e = [\hat{C}_{32} + \tilde{D}_{32}D_{k1}\hat{C}_{22} \quad \tilde{D}_{32}C_k \quad \tilde{D}_{32}D_{k2}] x_{cl} + \hat{D}_{31}w, \end{array} \right.$$

which can be further written by  $\mathcal{F}_l(\bar{\Xi}, \mathcal{K})$ , where  $\mathcal{K}$  is given in (11) and

$$\bar{\Xi} := \left[ \begin{array}{cc|cc} \hat{A}_{22} - sI_{n+n_i} & 0 & \hat{B}_{21} & \tilde{B}_{22} \\ \Pi_o\hat{C}_{22} & A_o - sE_o & \Pi_oD_{21} & -\Gamma_o \\ \hline \tilde{C}_{12} - C_oY_o & C_o & D_{11} & \tilde{D}_{12} \\ \begin{bmatrix} \hat{C}_{22} & 0 \\ 0 & I_{n_o} \end{bmatrix} & & \begin{bmatrix} D_{21} \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \hat{C}_{32} & 0 & D_{31} & \tilde{D}_{32} \end{array} \right].$$

Using the two transformation matrices

$$M_4 = \begin{bmatrix} I_{n+n_i} & 0 \\ -X_o & I_{n_o} \end{bmatrix}, \quad N_4 = \begin{bmatrix} I_{n+n_i} & 0 \\ Y_o & I_{n_o} \end{bmatrix},$$

one can show that  $\bar{\Xi}$  is restricted system equivalent to  $\Xi$  given by

$$\Xi := \left[ \begin{array}{c|cc} A - sE & B_1 & \left[ \begin{array}{c|c} B_2 & X_i \\ \hline & I_{n_i} \end{array} \right] \\ \hline C_1 \left[ \begin{array}{cc} I & Y_i \\ 0 & 0 \end{array} \right] + [0 \quad D_{12}\Pi_i] & D_{11} & [D_{12} \quad 0] \\ \left[ \begin{array}{cc} C_2 & \\ \hline I_{n_o} & Y_o \end{array} \right] & \left[ \begin{array}{c} D_{21} \\ 0 \end{array} \right] & \left[ \begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array} \right] \\ C_3 & D_{31} & [D_{32} \quad 0] \end{array} \right],$$

where the second output is the measurement. It has been seen that  $T_{zw} = \mathcal{F}_l(\bar{\Xi}, \mathcal{K}) = \mathcal{F}_l(\Xi, \mathcal{K})$  with the third output being removed. Note that  $\mathcal{K}$  internally stabilizes the system  $\Sigma$  in (12), which is indeed the system  $\Xi_{22}$  induced from  $\Xi$ . Therefore,  $T_{zw}$  is stable. Moreover, the tracking error  $e$  is a linear combination of the state  $x_{cl}$  and the external disturbance  $w(t) \in \mathcal{L}_2$ , hence, there holds  $\lim_{t \rightarrow \infty} e(t) = 0$ .  $\square$

In order to tackle the asymptotic tracking and nonproper output weighting filter, a structured measurement feedback controller is conducted and Fig. 2 exhibits explicitly the impact of the exo-system and output weight on this controller. An extra degree of freedom is also fully shown for the performance objective control, which will be discussed in the next section. Hence, Theorem 3.1 characterizes the set of all stabilizing controllers addressing asymptotic tracking and nonproper weights. The nonstandard problem is thus equivalent to a conventional control problem for the auxiliary system  $\Xi$ , which is constructed from the weighted system  $G$  with the control input matrix and measurement output matrix being augmented by solutions to the two generalized Sylvester-type equations.

#### 4. Constrained $\mathcal{H}_\infty$ control

In this section, relying on the parameterization of all stabilizing controllers given previously, the  $\mathcal{H}_\infty$  performance control is tackled, and both full-order and reduced-order  $\mathcal{H}_\infty$  controller syntheses are studied in terms of LMI-based conditions. It has been shown that the structured controller (10) ensures asymptotic tracking and leads to a stable transfer function  $T_{zw}$ . Hence, the additional performance objective, i.e. **Condition (C2)** can be further approached with the benefit of the free controller  $\mathcal{K}$  in (11). We assume that the two generalized Sylvester-type equations (8) and (9) admit solutions. As stated previously, Problem 2.1 is equivalent to an  $\mathcal{H}_\infty$  control problem with respect to the auxiliary system  $\Xi$  specified as follows

$$\begin{aligned} \Xi &:= \left[ \begin{array}{c|cc} A - sE & B_1 & \left[ \begin{array}{c|c} B_2 & X_i \\ \hline & I_{n_i} \end{array} \right] \\ \hline C_1 \left[ \begin{array}{cc} I & Y_i \\ 0 & 0 \end{array} \right] + [0 \quad D_{12}\Pi_i] & D_{11} & [D_{12} \quad 0] \\ \left[ \begin{array}{cc} C_2 & \\ \hline I_{n_o} & Y_o \end{array} \right] & \left[ \begin{array}{c} D_{21} \\ 0 \end{array} \right] & \left[ \begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array} \right] \end{array} \right] \\ &= \left[ \begin{array}{c|cc} A - sE & B_1 & \mathcal{B}_2(X_i) \\ \hline \mathcal{C}_1(Y_i, \Pi_i) & D_{11} & \mathcal{D}_{12} \\ C_2(Y_o) & D_{21} & 0 \end{array} \right], \end{aligned} \quad (13)$$

with the third output being removed, since the tracking issue is guaranteed through the structured controller. Note that this system has invariant zeros at infinity, which is why the Riccati-based solution [Wang, Yung, and Chang \(2006\)](#) fails. Hence, we adopt the LMI-based approach to solve the underlying descriptor  $\mathcal{H}_\infty$  control problem. Moreover, it is observed that the auxiliary system depends on the generalized Sylvester-type equations solutions, which are generally not unique. This fact indicates that the underlying controller design may result in a nonlinear matrix inequality problem, and proceeding with certain specific solution by directly applying existing methods may lead to a conservative synthesis. In this section, we fully explore the structure of the auxiliary system and present a design route independent of the solutions to generalized Sylvester-type equations.

Before presenting the controller synthesis, we assume without loss of generality that the matrix  $E$  is of the form  $E = \text{diag}(I_{n+n_i+r_o}, 0_{(n_o-r_o) \times (n_o-r_o)})$ . Note that if  $E$  is not of this form, then a pair of transformation matrices can easily be obtained due to a singular value decomposition (SVD) to make  $E$  satisfy this condition.

#### 4.1. Full-Order $\mathcal{H}_\infty$ Controller

**Theorem 4.1.** *Recall the partitions  $G^{W_i}$  and  $G^{W_o}$  in (6) and (7). Given  $\gamma > 0$ , there exists a measurement output controller (11) with  $\text{rank}(E_k) = n + n_i + r_o$  such that the  $\mathcal{H}_\infty$  norm of the closed-loop system formed by  $\Xi$  in (13) and  $\mathcal{K}$  in (11) is bounded by  $\gamma$ , if and only if there exist matrices  $X_i \in \mathbb{R}^{(n+n_o) \times n_i}$ ,  $Y_i \in \mathbb{R}^{(n+n_o) \times n_i}$ ,  $\Pi_i \in \mathbb{R}^{m \times n_i}$ ,  $X_o \in \mathbb{R}^{n_o \times (n+n_i)}$ ,  $Y_o \in \mathbb{R}^{n_o \times (n+n_i)}$ ,  $\Pi_o \in \mathbb{R}^{n_o \times p}$ ,  $\mathcal{X} \in \mathbb{R}^{(n+n_i+n_o) \times (n+n_i+n_o)}$ ,  $\mathcal{Y} \in \mathbb{R}^{(n+n_i+n_o) \times (n+n_i+n_o)}$ ,  $\mathcal{W} \in \mathbb{R}^{(n+n_i+n_o) \times m}$ , and  $\mathcal{Z} \in \mathbb{R}^{m \times (n+n_i+n_o)}$  satisfying (8), (9), and*

$$\begin{bmatrix} E^T \mathcal{X} & \bullet \\ E & E \mathcal{Y}^T \end{bmatrix} = \begin{bmatrix} \mathcal{X}^T E & \bullet \\ E & \mathcal{Y} E^T \end{bmatrix} \geq 0, \quad (14)$$

$$E^T \mathcal{W} = 0, \quad E \mathcal{Z}^T = 0, \quad (15)$$

$$L_c(\bar{Y}, \bar{Z}) < 0, \quad (16)$$

$$L_o(\bar{X}, \bar{W}) < 0, \quad (17)$$

where

$$\begin{aligned} L_c(\bar{Y}, \bar{Z}) &= \begin{bmatrix} N_c & 0 \\ 0 & I \end{bmatrix}^T \begin{bmatrix} \mathbf{He}\{\bar{A}_{11} \bar{Y}^T\} & \bullet & \bullet \\ \bar{C}_{11} \bar{Y}^T & -\gamma I & \bullet \\ \bar{Z} \bar{A}_{11}^T + \bar{B}_{11}^T & D_{11}^T + \bar{Z} \bar{C}_{11}^T & -\gamma I \end{bmatrix} \begin{bmatrix} N_c & 0 \\ 0 & I \end{bmatrix}, \\ L_o(\bar{X}, \bar{W}) &= \begin{bmatrix} N_o & 0 \\ 0 & I \end{bmatrix}^T \begin{bmatrix} \mathbf{He}\{\hat{A}_{22}^T \bar{X}\} & \bullet & \bullet \\ \bar{W}^T \hat{A}_{22} + \hat{B}_{21}^T \bar{X} & \mathbf{He}\{\hat{B}_{21}^T \bar{W}\} - \gamma I & \bullet \\ \tilde{C}_{12} - C_o Y_o & D_{11} & -\gamma I \end{bmatrix} \begin{bmatrix} N_o & 0 \\ 0 & I \end{bmatrix}, \\ N_c &= \begin{bmatrix} \bar{B}_{12} \\ D_{12} \end{bmatrix}^\perp, \quad N_o = \begin{bmatrix} \hat{B}_{22} & D_{21} \end{bmatrix}^\perp, \quad \bar{X} = \begin{bmatrix} 0 & I_{n+n_i} \end{bmatrix} \mathcal{X} \begin{bmatrix} 0 \\ I_{n+n_i} \end{bmatrix}, \\ \bar{Y} &= \begin{bmatrix} I_{n+n_o} & 0 \end{bmatrix} \mathcal{Y} \begin{bmatrix} I_{n+n_o} \\ 0 \end{bmatrix}, \quad \bar{W} = \begin{bmatrix} 0 & I_{n+n_i} \end{bmatrix} \mathcal{W}, \quad \bar{Z} = \mathcal{Z} \begin{bmatrix} I_{n+n_o} \\ 0 \end{bmatrix}, \\ \tilde{C}_{12} &= \begin{bmatrix} D_o C_v & \bar{C}_{11} Y_i + D_{12} \Pi_i \end{bmatrix}. \end{aligned}$$

**Proof.** By Theorem 1 in Masubuchi (2007), the  $\mathcal{H}_\infty$  controller exists if and only if there exist matrices  $\mathcal{X}$ ,  $\mathcal{Y}$ ,  $\mathcal{W}$  and  $\mathcal{Z}$  with appropriate dimensions satisfying (14), (15), and

$$\begin{bmatrix} \mathcal{N}_c & 0 \\ 0 & I \end{bmatrix}^T \Theta_c \begin{bmatrix} \mathcal{N}_c & 0 \\ 0 & I \end{bmatrix} < 0, \quad (18)$$

$$\begin{bmatrix} \mathcal{N}_o & 0 \\ 0 & I \end{bmatrix}^T \Theta_o \begin{bmatrix} \mathcal{N}_o & 0 \\ 0 & I \end{bmatrix} < 0, \quad (19)$$

where

$$\mathcal{N}_c = \begin{bmatrix} \mathcal{B}_2(X_i) \\ \mathcal{D}_{12} \end{bmatrix}^\perp, \quad \Theta_c = \begin{bmatrix} \mathbf{He}\{A\mathcal{Y}^T\} & \bullet & \bullet \\ \mathcal{C}_1(Y_i, \Pi_i)\mathcal{Y}^T & -\gamma I & \bullet \\ B_1^T + \mathcal{Z}A^T & D_{11}^T + \mathcal{Z}\mathcal{C}_1(Y_i, \Pi_i) & -\gamma I \end{bmatrix},$$

$$\mathcal{N}_o = \begin{bmatrix} \mathcal{C}_2(Y_o) & \mathcal{D}_{21} \end{bmatrix}^\perp, \quad \Theta_o = \begin{bmatrix} \mathbf{He}\{A^T\mathcal{X}\} & \bullet & \bullet \\ B_1^T\mathcal{X} + \mathcal{W}^T A & \mathbf{He}\{B_1^T\mathcal{W}\} - \gamma I & \bullet \\ \mathcal{C}_1(Y_i, \Pi_i) & D_{11} & -\gamma I \end{bmatrix}.$$

Note that  $\Theta_c$  contains a multiplication between the decision variable  $\mathcal{Y}$  and the auxiliary system data  $\mathcal{C}_1(Y_i, \Pi_i)$  depending on both  $Y_i$  and  $\Pi_i$ . Hence, this condition is obviously not an LMI. Same observations hold for  $\mathcal{N}_c$  and  $\mathcal{N}_o$ . In order to obtain linear conditions, let us set

$$M_i = \begin{bmatrix} I_{n+n_o} & X_i \\ 0 & I_{n_i} \end{bmatrix}, \quad N_i = \begin{bmatrix} I_{n+n_o} & Y_i \\ 0 & I_{n_i} \end{bmatrix}.$$

Note that

$$\begin{bmatrix} \mathcal{B}_2(X_i) \\ \mathcal{D}_{12} \end{bmatrix} = \begin{bmatrix} M_i & 0 \\ 0 & I_{p_o} \end{bmatrix} \begin{bmatrix} \bar{B}_{12} & 0 \\ 0 & I_{n_i} \\ D_{12} & 0 \end{bmatrix},$$

which implies

$$\mathcal{N}_c = \begin{bmatrix} M_i^{-T} & 0 \\ 0 & I_{p_o} \end{bmatrix} \begin{bmatrix} \bar{B}_{12} & 0 \\ 0 & I_{n_i} \\ D_{12} & 0 \end{bmatrix}^\perp = \begin{bmatrix} M_i^{-T} & 0 \\ 0 & I_{p_o} \end{bmatrix} \begin{bmatrix} N_{c1} \\ 0 \\ N_{c2} \end{bmatrix},$$

where  $\begin{bmatrix} N_{c1} \\ N_{c2} \end{bmatrix} = \begin{bmatrix} \bar{B}_{12} \\ D_{12} \end{bmatrix}^\perp$ . Then, the condition (18) is equivalent to

$$\begin{bmatrix} N_{c1} & 0 \\ 0 & 0 \\ N_{c2} & 0 \\ 0 & I \end{bmatrix}^T \begin{bmatrix} M_i^{-1} & 0 \\ 0 & I_{p_o} \end{bmatrix} \Theta_c \begin{bmatrix} M_i^{-T} & 0 \\ 0 & I_{p_o} \end{bmatrix} \begin{bmatrix} N_{c1} & 0 \\ 0 & 0 \\ N_{c2} & 0 \\ 0 & I \end{bmatrix} < 0.$$



Set  $Y = M_i^{-1} \mathcal{Y} N_i^T$  and  $Z = \mathcal{Z} N_i^T$ . Denote the partitions as follows

$$Y = \begin{bmatrix} \bar{Y} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix}, \quad Z = [\bar{Z} \quad Z_2].$$

Then, the condition (18) can be further rewritten by

$$\begin{aligned} & \begin{bmatrix} N_{c1} & 0 \\ 0 & 0 \\ N_{c2} & 0 \\ 0 & I \end{bmatrix}^T \begin{bmatrix} \mathbf{He}\{\bar{A}_{11}\bar{Y} + \bar{B}_{12}\Pi_i Y_{12}^T\} & \bullet & \bullet & \bullet \\ & * & \bullet & \bullet \\ D_{12}\Pi_i Y_{12}^T + \bar{C}_{11}\bar{Y}^T & * & -\gamma I & \bullet \\ \bar{Z}\bar{A}_{11}^T + Z_2\Pi_i^T \bar{B}_{12}^T + \bar{B}_{11}^T & * & D_{11}^T + \bar{Z}\bar{C}_{11}^T + Z_2\Pi_i^T D_{12}^T & -\gamma I \end{bmatrix} \begin{bmatrix} N_{c1} & 0 \\ 0 & 0 \\ N_{c2} & 0 \\ 0 & I \end{bmatrix} \\ = & \begin{bmatrix} N_{c1} & 0 \\ N_{c2} & 0 \\ 0 & I \end{bmatrix}^T \begin{bmatrix} \mathbf{He}\{\bar{A}_{11}\bar{Y} + \bar{B}_{12}\Pi_i Y_{12}^T\} & \bullet & \bullet \\ D_{12}\Pi_i Y_{12}^T + \bar{C}_{11}\bar{Y}^T & -\gamma I & \bullet \\ \bar{Z}\bar{A}_{11}^T + Z_2\Pi_i^T \bar{B}_{12}^T - \bar{B}_{11}^T & D_{11}^T + \bar{Z}\bar{C}_{11}^T + Z_2\Pi_i^T D_{12}^T & -\gamma I \end{bmatrix} \begin{bmatrix} N_{c1} & 0 \\ N_{c2} & 0 \\ 0 & I \end{bmatrix} \\ = & L_c(\bar{Y}, \bar{Z}) + \begin{bmatrix} N_{c1} & 0 \\ N_{c2} & 0 \\ 0 & I \end{bmatrix}^T \begin{bmatrix} \mathbf{He}\{\bar{B}_{12}\Pi_i Y_{12}^T\} & \bullet & \bullet \\ D_{12}\Pi_i Y_{12}^T & 0 & \bullet \\ Z_2\Pi_i^T \bar{B}_{12}^T & Z_2\Pi_i^T D_{12}^T & 0 \end{bmatrix} \begin{bmatrix} N_{c1} & 0 \\ N_{c2} & 0 \\ 0 & I \end{bmatrix} < 0, \end{aligned}$$

where ‘\*’ stands for the elements irrelevant to the discussion. Since  $N_{c1}^T \bar{B}_{12} + N_{c2}^T D_{12} = 0$ , the second term of the last inequality is zero. Hence, the condition (16) holds. Similarly, let us set

$$M_o = \begin{bmatrix} I_{n_o} & -X_o \\ 0 & I_{n+n_i} \end{bmatrix}, \quad N_o = \begin{bmatrix} I_{n_o} & Y_o \\ 0 & I_{n+n_i} \end{bmatrix}.$$

Note that

$$[C_2(Y_o) \quad \mathcal{D}_{21}] = \begin{bmatrix} 0 & \hat{C}_{22} & D_{21} \\ I_{n_o} & 0 & 0 \end{bmatrix} \begin{bmatrix} N_o & 0 \\ 0 & I_p \end{bmatrix},$$

which implies

$$\mathcal{N}_o = \begin{bmatrix} N_o^{-1} & 0 \\ 0 & I_p \end{bmatrix} \begin{bmatrix} 0 & I_{n_o} \\ \hat{C}_{22}^T & 0 \\ D_{21}^T & 0 \end{bmatrix}^\perp = \begin{bmatrix} N_o^{-1} & 0 \\ 0 & I_p \end{bmatrix} \begin{bmatrix} 0 \\ N_{o1} \\ N_{o2} \end{bmatrix},$$

where  $\begin{bmatrix} N_{o1} \\ N_{o2} \end{bmatrix} = \begin{bmatrix} \hat{C}_{22}^T \\ D_{21}^T \end{bmatrix}^\perp$ . Then, the condition (19) is equivalent to

$$\begin{bmatrix} 0 & 0 \\ N_{o1} & 0 \\ N_{o2} & 0 \\ 0 & I \end{bmatrix}^T \begin{bmatrix} N_o^{-T} & 0 \\ 0 & I_p \end{bmatrix} \Theta_o \begin{bmatrix} N_o^{-1} & 0 \\ 0 & I_p \end{bmatrix} \begin{bmatrix} 0 & 0 \\ N_{o1} & 0 \\ N_{o2} & 0 \\ 0 & I \end{bmatrix} < 0.$$

Set  $X = M_o^T \mathcal{X} N_o^{-1}$  and  $W = M_o^T \mathcal{W}$ . Denote the partitions as follows

$$X = \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & \bar{X} \end{bmatrix}, \quad W = \begin{bmatrix} W_1 \\ \bar{W} \end{bmatrix}.$$

Then, the condition (19) can be further rewritten by

$$\begin{aligned}
& \begin{bmatrix} 0 & 0 \\ N_{o1} & 0 \\ N_{o2} & 0 \\ 0 & I \end{bmatrix}^T \begin{bmatrix} * & \bullet & \bullet \\ * & \mathbf{He}\{\hat{A}_{22}\bar{X} + X_{12}^T \Pi_o \hat{C}_{22}\} & \bullet \\ * & \hat{B}_{21}^T \bar{X} + D_{21}^T \Pi_o^T X_{12} + W_1^T \Pi_o \hat{C}_{22} & \mathbf{He}\{W_1^T \Pi_o D_{21} + \hat{B}_{21} \bar{W}\} - \gamma I \\ * & \hat{C}_{12} - C_o Y_o & D_{11} \\ & & & -\gamma I \end{bmatrix} \begin{bmatrix} 0 & 0 \\ N_{o1} & 0 \\ N_{o2} & 0 \\ 0 & I \end{bmatrix} \\
& \begin{bmatrix} N_{o1} & 0 \\ N_{o2} & 0 \\ 0 & I \end{bmatrix}^T \begin{bmatrix} \bullet & \bullet & \bullet \\ \mathbf{He}\{\hat{A}_{22}\bar{X} + X_{12}^T \Pi_o \hat{C}_{22}\} & \bullet & \bullet \\ \hat{B}_{21}^T \bar{X} + D_{21}^T \Pi_o^T X_{12} + W_1^T \Pi_o \hat{C}_{22} & \mathbf{He}\{W_1^T \Pi_o D_{21} + \hat{B}_{21} \bar{W}\} - \gamma I & \bullet \\ \hat{C}_{12} - C_o Y_o & D_{11} & -\gamma I \end{bmatrix} \begin{bmatrix} N_{o1} & 0 \\ N_{o2} & 0 \\ 0 & I \end{bmatrix} \\
& = L_o(\bar{X}, \bar{W}) + \begin{bmatrix} N_{o1} & 0 \\ N_{o2} & 0 \\ 0 & I \end{bmatrix}^T \begin{bmatrix} \mathbf{He}\{X_{12}^T \Pi_o \hat{C}_{22}\} & \bullet & \bullet \\ D_{21}^T \Pi_o^T X_{12} + W_1^T \Pi_o \hat{C}_{22} & \mathbf{He}\{W_1^T \Pi_o D_{21}\} & \bullet \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} N_{o1} & 0 \\ N_{o2} & 0 \\ 0 & I \end{bmatrix} < 0.
\end{aligned}$$

Since  $N_{o1}^T \hat{C}_{22} + N_{o2}^T D_{21}^T = 0$ , the second term is zero. Hence, the condition (17) holds.  $\square$

The LMI (14) implies that solutions  $\mathcal{X}$  and  $\mathcal{Y}$  are in the form of

$$\mathcal{X} = \begin{bmatrix} \mathcal{X}_{p11} & 0 \\ \mathcal{X}_{p21} & \mathcal{X}_{p22} \end{bmatrix}, \quad \mathcal{Y} = \begin{bmatrix} \mathcal{Y}_{p11} & \mathcal{Y}_{p12} \\ 0 & \mathcal{Y}_{p22} \end{bmatrix}.$$

Furthermore, we define the following two matrices

$$\mathcal{X}_{11} = \begin{bmatrix} \mathcal{X}_{p11} & \mathcal{X}_{pc11} \\ \mathcal{X}_{cp11} & \mathcal{X}_{c11} \end{bmatrix}, \quad \mathcal{X}_{11}^{-1} = \mathcal{Y}_{11} = \begin{bmatrix} \mathcal{Y}_{p11} & \mathcal{Y}_{pc11} \\ \mathcal{Y}_{cp11} & \mathcal{Y}_{c11} \end{bmatrix}.$$

By the standard way for state-space  $\mathcal{H}_\infty$  control [Gahinet and Apkarian \(1994\)](#), we have  $\mathcal{X}_{pc11} \mathcal{X}_{c11}^{-1} \mathcal{X}_{cp11} = \mathcal{X}_{p11} - \mathcal{Y}_{p11}^{-1}$ . Then, we can obtain the closed-loop Lyapunov matrix as

$$\mathcal{X}_{CL} = \left[ \begin{array}{cc|cc} \mathcal{X}_{p11} & 0 & \mathcal{X}_{pc11} & 0 \\ \mathcal{X}_{p21} & \mathcal{X}_{p22} & 0 & I_{n_o - r_o} \\ \hline \mathcal{X}_{cp11} & 0 & \mathcal{X}_{c11} & 0 \\ \mathcal{X}_{cp21} & \mathcal{X}_{cp22} & \mathcal{X}_{c21} & -\mathcal{Y}_{p22}^T \end{array} \right], \quad (20)$$

where  $\mathcal{X}_{cp21} = -(\mathcal{Y}_{p22}^T \mathcal{X}_{p21} + \mathcal{Y}_{p12} \Delta^{-T})$ ,  $\mathcal{X}_{cp22} = I - \mathcal{Y}_{p22}^T \mathcal{X}_{p22}$ ,  $\mathcal{X}_{c21} = \mathcal{Y}_{p12}^T \Delta^{-T} \mathcal{Y}_{cp11}^T \mathcal{Y}_{c11}^{-T}$ ,  $\Delta = \mathcal{Y}_{p11} - \mathcal{Y}_{pc11} \mathcal{Y}_{c11}^{-1} \mathcal{Y}_{cp11}$ .

Once the closed-loop Lyapunov matrix is done, we form the following matrices

$$\begin{aligned}
A_0 &= \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}, \quad B_{01} = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}, \quad B_{02} = \begin{bmatrix} 0 & \mathcal{B}_2(X_i) \\ I & 0 \end{bmatrix}, \quad C_{01} = [\mathcal{C}_1(Y_i, \Pi_i) \quad 0], \\
C_{02} &= \begin{bmatrix} 0 & I \\ \mathcal{C}_2(Y_o) & 0 \end{bmatrix}, \quad D_{012} = [0 \quad \mathcal{D}_{12}], \quad D_{021} = \begin{bmatrix} 0 \\ \mathcal{D}_{21} \end{bmatrix}.
\end{aligned}$$

Then, the resulting  $\mathcal{H}_\infty$  controller  $\mathcal{K}$  can be obtained through solving the following

LMI

$$\begin{bmatrix} A_0^T \mathcal{X}_{CL} & \bullet & \bullet \\ B_0^T \mathcal{X}_{CL} & -\gamma I & \bullet \\ C_{01} & D_{11} & -\gamma I \end{bmatrix} + \text{He} \left\{ \mathcal{P}^T \left[ \begin{array}{c|cc} A_k & B_{k1} & B_{k2} \\ \hline C_k^1 & D_{k1}^1 & D_{k2}^1 \\ C_k^2 & D_{k1}^2 & D_{k2}^2 \end{array} \right] \mathcal{Q} \right\} < 0, \quad (21)$$

where  $\mathcal{P} = [B_{02}^T \mathcal{X}_{CL} \ 0 \ D_{012}^T]$  and  $\mathcal{Q} = [C_{02} \ D_{021} \ 0]$ . Moreover,  $E_k$  is set by  $E_k = E = \text{diag}(I_{n+n_i+r_o}, 0_{(n_o-r_o) \times (n_o-r_o)})$ .

Now, we are in a position to give a complete procedure to form a full-order controller solving Problem 2.1.

**Algorithm 4.2** (Full-order Solution).

- (1) Simultaneously solve the generalized Sylvester-type equations (8) and (9), and LMI conditions (14)-(17). Get solutions  $(X_i, Y_i, \Pi_i, X_o, Y_o, \Pi_o)$  and  $(\mathcal{X}, \mathcal{Y}, \mathcal{W}, \mathcal{Z})$ .
- (2) Based on  $\mathcal{X}$  and  $\mathcal{Y}$ , form the Lyapunov matrix  $\mathcal{X}_{CL}$  (20), and obtain the  $\mathcal{H}_\infty$  controller  $\mathcal{K}$  (11) by solving (21).
- (3) Construct the controller  $K$  by (10) with  $(X_i, Y_i, \Pi_i, X_o, Y_o, \Pi_o)$  and  $\mathcal{K}$ .

#### 4.2. Reduced-Order $\mathcal{H}_\infty$ Controller

It is worthy of noting that the auxiliary system  $\Xi$  given in (13) contains  $n + n_i + r_o$  finite dynamic modes and  $n_o - r_o$  infinite modes. In the parts to follow, for the sake of simplicity, let us abuse a bit the use of notation by calling the sum of finite and infinite modes as the order of system. Since the  $\mathcal{H}_\infty$  controller obtained previously has the same order of the the auxiliary system  $\Xi$ , the overall controller  $K$  given in (10) has the order  $n + 2n_i + 2n_o$ , which exceeds the order of the original generalized system  $G$ , and is obviously not desirable, especially for the case where higher order exo-systems or output weighting filters are involved. In order to conduct a solution that is easy to implement, we explore the issue of reduced-order controller synthesis in this subsection.

Consider the two systems, denoted respectively as  $\Xi_{12}$  and  $\Xi_{21}$ , induced from the auxiliary system  $\Xi$  as below

$$\Xi_{12} = \left[ \begin{array}{c|c} A - sE & B_2(X_i) \\ \hline C_1(Y_i, \Pi_i) & [D_{12} \ 0] \end{array} \right], \quad \Xi_{21} = \left[ \begin{array}{c|c} A - sE & B_1 \\ \hline C_2(Y_o) & \left[ \begin{array}{c} D_{21} \\ 0 \end{array} \right] \end{array} \right].$$

It is observed that both of them contain zeros at infinity, since the D-matrix of  $\Xi_{12}$  is not full row rank, and the D-matrix of  $\Xi_{21}$  is not full column rank. According to [Xin, Hara, and Kaneda \(2008\)](#), there exists a reduced-order  $\mathcal{H}_\infty$  controller, with which the overall controller in (10) has an order less than  $n + n_i + n_o$ . The following theorem states a way to reduced-order controller synthesis.

**Theorem 4.3.** *Given  $\gamma > 0$ , there exists a measurement output feedback controller with an order less than  $n$ , such that the  $\mathcal{H}_\infty$  norm of the closed-loop system formed by  $\Xi$  and the controller is bounded by  $\gamma$ , if there exist matrices  $X_i \in \mathbb{R}^{(n+n_o) \times n_i}$ ,  $Y_i \in \mathbb{R}^{(n+n_o) \times n_i}$ ,  $\Pi_i \in \mathbb{R}^{m \times n_i}$ ,  $X_o \in \mathbb{R}^{n_o \times (n+n_i)}$ ,  $Y_o \in \mathbb{R}^{n_o \times (n+n_i)}$ ,  $\Pi_o \in \mathbb{R}^{n_o \times p}$ ,  $\mathcal{X} \in \mathbb{R}^{(n+n_i+n_o) \times (n+n_i+n_o)}$ ,  $\mathcal{Y} \in \mathbb{R}^{(n+n_i+n_o) \times (n+n_i+n_o)}$ ,  $\mathcal{W} \in \mathbb{R}^{(n+n_i+n_o) \times m}$ ,  $\mathcal{Z} \in$*

$\mathbb{R}^{m \times (n+n_i+n_o)}$ , and two positive scalars  $\epsilon_1, \epsilon_2$  satisfying (8), (9), and

$$\begin{bmatrix} E^T \mathcal{X} & \bullet \\ E & E\mathcal{Y}^T \end{bmatrix} = \begin{bmatrix} \mathcal{X}^T E & \bullet \\ E & \mathcal{Y}E^T \end{bmatrix} \geq 0, \quad (22)$$

$$E^T \mathcal{W} = 0, \quad E\mathcal{Z}^T = 0, \quad (23)$$

$$L_c(\bar{Y}, \bar{Z}) + \epsilon_1 I < 0, \quad (24)$$

$$L_c(\bar{X}, \bar{W}) + \epsilon_2 I < 0, \quad (25)$$

where

$$\begin{aligned} L_c(\bar{Y}, \bar{Z}) &= \begin{bmatrix} N_c & 0 \\ 0 & I \end{bmatrix}^T \begin{bmatrix} \mathbf{He}\{\bar{A}_{11}\bar{Y}^T\} & \bullet & \bullet \\ \bar{C}_{11}\bar{Y}^T & -\gamma I & \bullet \\ \bar{Z}\bar{A}_{11}^T + \bar{B}_{11}^T & D_{11}^T + \bar{Z}\bar{C}_{11}^T & -\gamma I \end{bmatrix} \begin{bmatrix} N_c & 0 \\ 0 & I \end{bmatrix}, \\ L_o(\bar{X}, \bar{W}) &= \begin{bmatrix} N_o & 0 \\ 0 & I \end{bmatrix}^T \begin{bmatrix} \mathbf{He}\{\hat{A}_{22}^T \bar{X}\} & \bullet & \bullet \\ \bar{W}^T \hat{A}_{22} + \hat{B}_{21}^T \bar{X} & \mathbf{He}\{\hat{B}_{21}^T \bar{W}\} - \gamma I & \bullet \\ \tilde{C}_{12} - C_o Y_o & D_{11} & -\gamma I \end{bmatrix} \begin{bmatrix} N_o & 0 \\ 0 & I \end{bmatrix}, \\ N_c &= \begin{bmatrix} \bar{B}_{12} \\ D_{12} \end{bmatrix}^\perp, \quad N_o = \begin{bmatrix} \hat{B}_{22} & D_{21} \end{bmatrix}^\perp, \quad \bar{X} = \begin{bmatrix} 0 & I_{n+n_i} \end{bmatrix} \mathcal{X} \begin{bmatrix} 0 \\ I_{n+n_i} \end{bmatrix}, \\ \bar{Y} &= \begin{bmatrix} I_{n+n_o} & 0 \end{bmatrix} \mathcal{Y} \begin{bmatrix} I_{n+n_o} \\ 0 \end{bmatrix}, \quad \bar{W} = \begin{bmatrix} 0 & I_{n+n_i} \end{bmatrix} \mathcal{W}, \quad \bar{Z} = \mathcal{Z} \begin{bmatrix} I_{n+n_o} \\ 0 \end{bmatrix}, \\ \tilde{C}_{12} &= [D_o C_v \quad \bar{C}_{11} Y_i + D_{12} \Pi_i]. \end{aligned}$$

**Proof.** Since the D-matrices of systems  $\Xi_{12}$  and  $\Xi_{21}$  are not full row rank and not full column rank, respectively, a reduced-order  $\mathcal{H}_\infty$  controller exists. Then, the proof is achieved by applying the results of Xin et al. (2008) to Theorem 4.1.  $\square$

To construct the reduced-order control, let  $(\mathcal{X}_m, \mathcal{Y}_m, \mathcal{W}_m, \mathcal{Z}_m)$  be solutions to  $\min_{\mathcal{X}, \mathcal{Y}, \mathcal{W}, \mathcal{Z}} \text{Tr}(E^T \mathcal{X} + E\mathcal{Y}^T)$ , subject to the conditions (8), (9), and (22)-(25). Then, there holds

$$\mathcal{X}_m = \begin{bmatrix} \mathcal{X}_{m11} & 0 \\ \mathcal{X}_{p21} & \mathcal{X}_{p22} \end{bmatrix}, \quad \mathcal{Y}_m = \begin{bmatrix} \mathcal{Y}_{m11} & \mathcal{Y}_{p12} \\ 0 & \mathcal{Y}_{p22} \end{bmatrix}.$$

Let an SVD of  $\mathcal{Y}_{m11} - \mathcal{X}_{m11}^{-1}$  be

$$\mathcal{Y}_{m11} - \mathcal{X}_{m11}^{-1} = U \text{Diag}(\lambda_1, \lambda_2, \dots, \lambda_\mu) V,$$

where  $U, V$  are unitary matrices, and  $\lambda_i, i = 1, 2, \dots, \mu$  are singular values of  $\mathcal{Y}_{m11} - \mathcal{X}_{m11}^{-1}$  listed in a descending manner. Assume that there exist  $\nu$  singular values negligible with respect to others. Then, we set

$$\hat{\mathcal{Y}}_{m11} = \mathcal{X}_{m11}^{-1} + U \text{Diag}(\lambda_1, \dots, \lambda_{\mu-\nu}, 0 \dots, 0) V.$$

Similarly, we define matrices  $\mathcal{X}_{11}$  and  $\mathcal{Y}_{11}$  as

$$\mathcal{X}_{11} = \begin{bmatrix} \mathcal{X}_{m11} & \mathcal{X}_{pc11} \\ \mathcal{X}_{cp11} & \mathcal{X}_{c11} \end{bmatrix}, \quad \mathcal{X}_{11}^{-1} = \mathcal{Y}_{11} = \begin{bmatrix} \hat{\mathcal{Y}}_{m11} & \mathcal{Y}_{pc11} \\ \mathcal{Y}_{cp11} & \mathcal{Y}_{c11} \end{bmatrix}.$$

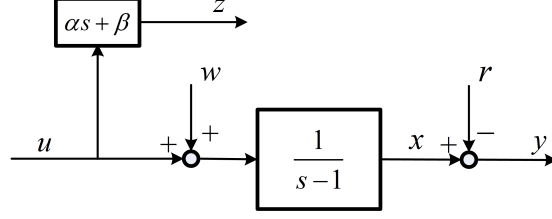


Figure 3. Example

with  $\mathcal{X}_{pc11}\mathcal{X}_{c11}^{-1}\mathcal{X}_{cp11} = \mathcal{X}_{m11} - \hat{\mathcal{Y}}_{m11}^{-1}$ . Then, the underlying closed-loop Lyapunov matrix is constructed by (20). The reduced-order  $\mathcal{H}_\infty$  controller  $\mathcal{K}$  can be also obtained by solving the LMI condition (21). The following algorithm gives a way to form a reduced-order controller for solving Problem 2.1.

**Algorithm 4.4** (Reduced-order Solution).

- (1) Simultaneously solve the generalized Sylvester-type equations (8) and (9), and LMI conditions (14)-(17). Get solutions  $(X_i, Y_i, \Pi_i, X_o, Y_o, \Pi_o)$  and  $(\mathcal{X}, \mathcal{Y}, \mathcal{W}, \mathcal{Z})$ .
- (2) Choose  $\epsilon_1$  and  $\epsilon_2$  such that  $0 < \epsilon_1 < \lambda_{\min}(-L_c(\bar{Y}, \bar{Z}))$  and  $0 < \epsilon_2 < \lambda_{\min}(-L_o(\bar{X}, \bar{W}))$ .
- (3) Solve  $\min_{\mathcal{X}, \mathcal{Y}, \mathcal{W}, \mathcal{Z}} \text{Tr}(E^T \mathcal{X} + E \mathcal{Y}^T)$ , subject to the conditions (22)-(25). Get solutions  $(\mathcal{X}_m, \mathcal{Y}_m, \mathcal{W}_m, \mathcal{Z}_m)$ .
- (4) Based on  $\mathcal{X}_m$  and  $\mathcal{Y}_m$ , form the Lyapunov matrix  $\mathcal{X}_{CL}$  (20), and obtain a reduced-order  $\mathcal{H}_\infty$  controller  $\mathcal{K}$  by solving (21).
- (5) Obtain the controller  $K$  by (10) with  $(X_i, Y_i, \Pi_i, X_o, Y_o, \Pi_o)$  and  $\mathcal{K}$ .

## 5. Simulation illustration

In this section, we use an unstable first order system to demonstrate the effectiveness of previous results. Consider the system depicted in Fig. 3. Here, we attempt to make the state  $x(t)$  follow a unit step  $r(t) = 1, t \geq 0$ . Hence, the tracking error is defined as  $e(t) = x(t) - r(t)$ . Meanwhile, it is assume that  $y(t) = e(t)$ , that is an error feedback controller is appealed. Moreover, we penalize the control law  $u(t)$  with a nonproper weight specified by  $\alpha s + \beta$ , where  $\alpha = 0.5$  and  $\beta = 1$ . Therefore, the system data are as follows

$$\begin{aligned}
 A_g &= 1, & B_w &= B_u = 1, & B_r &= 0, & C_g &= 1, & D_r &= -1, & D_w &= 0, & C_v &= 0, \\
 D_{vr} &= D_{vw} = 0, & D_{vu} &= 1, & C_e &= 1, & D_{er} &= -1, & D_{eu} &= D_{ew} = 0, & A_i &= 0, \\
 C_i &= 1, & E_o &= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, & A_o &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, & B_o &= \begin{bmatrix} 0 \\ -1 \end{bmatrix}, & C_o &= [0 \ 0.5], & D_o &= 1.
 \end{aligned}$$

First we design a full-order  $\mathcal{H}_\infty$  controller with  $\gamma = 1.5$ . Simultaneously solving generalized Sylvester-type equations (8), (9) and the LMIs (14)-(17) yields

$$\begin{aligned}
X_i &= \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \quad Y_i = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad \Pi_i = 1, \quad X_o = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad Y_o = \begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix}, \quad \Pi_o = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \\
\mathcal{X} &= \begin{bmatrix} 4.6340 & -6.0760 & 0.4202 & 0 \\ -6.0760 & 14.6977 & -0.4036 & 0 \\ 0.4202 & -0.4036 & 6.1263 & 0 \\ 0.4016 & -0.4023 & 0.4201 & 0 \end{bmatrix}, \quad \mathcal{W}^T = [0 \ 0 \ 0 \ 0], \\
\mathcal{Y} &= \begin{bmatrix} 4.0267 & 4.2060 & -0.3858 & 2.4954 \\ 4.2060 & 6.6941 & 2.1276 & 0 \\ -0.3858 & 2.1276 & 6.0479 & -2.8476 \\ 0 & 0 & 0 & -6.5825 \end{bmatrix}, \quad \mathcal{Z} = [0 \ 0 \ 0 \ -0.8696].
\end{aligned}$$

Then, by Algorithm 4.2, the following controller, denoted as  $K_1$ , is obtained

$$K_1 = \frac{1.9556(s - 135.1)(s + 27.64)(s + 2.502)(s + 0.1712)}{s(s + 47.04)(s + 15.19)(s + 5.563)(s + 2.437)}.$$

Second, we use Algorithm 4.4 to construct a reduced-order  $\mathcal{H}_\infty$  controller with the same level of disturbance attenuation. With  $\epsilon_1 = \epsilon_2 = 0.0008$ , we obtain

$$\begin{aligned}
\mathcal{X}_m &= \begin{bmatrix} 2.2907 & -2.4659 & 0.7677 & 0 \\ -2.4659 & 4.2337 & -0.9921 & 0 \\ 0.7677 & -0.9921 & 1.1222 & 0 \\ 0.7316 & -0.8123 & 0.9301 & 0 \end{bmatrix}, \quad \mathcal{W}^T = [0 \ 0 \ 0 \ 0], \\
\mathcal{Y}_m &= \begin{bmatrix} 1.3078 & 0.9072 & -0.0319 & 1.9593 \\ 0.9072 & 1.4542 & 0.8591 & -0.1130 \\ -0.0319 & 0.8591 & 1.8491 & -3.8675 \\ 0 & 0 & 0 & -3.3741 \end{bmatrix}, \quad \mathcal{Z} = [0 \ 0 \ 0 \ -0.8810].
\end{aligned}$$

Hence, there holds

$$\mathcal{Y}_{m11} - \mathcal{X}_{m11}^{-1} = U \text{diag}(1.5564, 8.9535 \times 10^{-9}, 3.4332 \times 10^{-9})V,$$

which indicates that the last two singular values are negligible. Hence

$$\hat{\mathcal{Y}}_{m11} = \mathcal{X}_{m11}^{-1} + U \text{diag}(1.5564, 0, 0)V = \begin{bmatrix} 1.3078 & 0.9071 & -0.0319 \\ 0.9072 & 1.4542 & 0.8591 \\ -0.0319 & 0.8591 & 1.8491 \end{bmatrix}.$$

Therefore, the reduced-order  $\mathcal{H}_\infty$  controller denoted as  $K_2$  is given as follows

$$K_2 = \frac{-6.1685(s + 0.8958)(s + 0.226)}{s(s + 1.001)(s + 3.005)}$$

Finally, based on the same solutions to the generalized Sylvester-type equations, a stabilizing controller denoted as  $K_3$  is also designed by making (11) merely stabilize

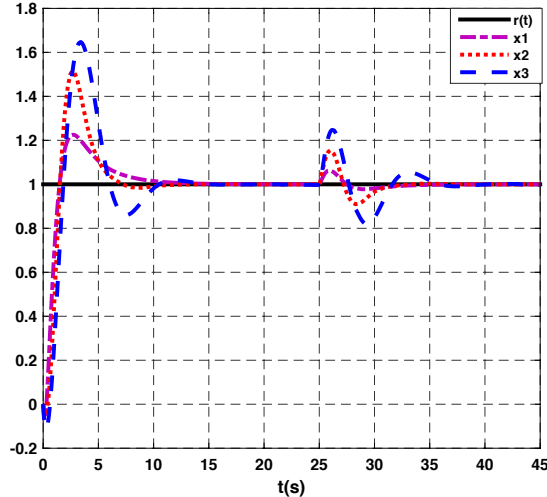


Figure 4. Asymptotic tracking with different controllers

the auxiliary system (13), and is given by

$$K_3 = \frac{9.3669(s - 3332)(s + 1.726)(s + 0.4105)(s + 0.1903)}{s(s + 370.7)(s + 66.19)(s + 1.256)(s + 0.4684)}.$$

Fig. 4 exhibits the tracking performance obtained by using these three controllers. The purple curve ( $x_1$ ), the red curve ( $x_2$ ), and the blue curve ( $x_3$ ) present the state trajectories by using the full-order  $\mathcal{H}_\infty$  controller, reduced-order  $\mathcal{H}_\infty$  controller, and stabilizing controller, respectively. Note that all controllers contain an integrator for achieving asymptotic tracking. This specific structure is also explicitly shown in (10). Moreover, when  $t = 25s$ , a disturbance signal  $w(t) = 0.3te^{-2t}$  is further added. The curve obtained with the stabilizing controller  $K_3$  is affected much more significantly compared with the two  $\mathcal{H}_\infty$  controllers. From this example, it is observed that the full-order  $\mathcal{H}_\infty$  controller leads no surprise to the best transient performance, the stabilizing controller gives relatively bad transient performance, and a tradeoff between transient performance and design complexity is reached by the reduced-order  $\mathcal{H}_\infty$  controller.

Note that the semidefinite programming solver SeDuMi Sturm (1999) in MATLAB® is used to construct controllers, and the running times are 0.718s, 0.863s, and 0.689s for the full-order  $H_\infty$  controller, reduced-order  $H_\infty$  controller, and stabilizing controller, respectively.

## 6. Conclusion

A multiobjective control problem is addressed in this paper, where the worst-case energy gain from a finite energy disturbance to a certain performance output weighted by a nonproper filter is assumed to be bounded, together with an exact asymptotic tracking requirement. A structured controller, simultaneously achieving asymptotic tracking and stabilization under nonproper weights, is conducted by solving two generalized Sylvester-type equations. Using this parameterization, the defined multiobjective problem is transformed into a standard robust disturbance attenuation problem

for an auxiliary system, and non conservative LMI-based solutions, for both full-order and reduced-order controllers synthesis problems, are given. One of our future research topics is to expand the current results to the case in which both the plant and the weights contain parametric dependence uncertainties.

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## Appendix A. Ddescriptor systems

In this appendix, we recall some fundamental notations for continuous-time descriptor systems. Consider the following LTI descriptor system

$$\begin{cases} E\dot{x}(t) = Ax(t) + Bu(t), \\ y(t) = Cx(t) + Du(t), \end{cases} \quad (\text{A1})$$

where  $x \in \mathbb{R}^n$ ,  $y \in \mathbb{R}^p$  and  $u \in \mathbb{R}^m$  are the descriptor variable, measurement and control input vector, respectively. The matrix  $E \in \mathbb{R}^{n \times n}$  may be singular, *i.e.*  $\text{rank}(E) = r \leq n$ ; and  $A, B, C$  and  $D$  are known real constant matrices with appropriate dimensions. The finite eigenvalues of  $sE - A$  are called the finite dynamic modes. The infinite eigenvalues of  $sE - A$  with the eigenvector  $v$  satisfying  $Ev = 0$  determine the static modes; while the infinite eigenvalues with the generalized eigenvectors  $v_k$  such that  $Ev_1 = 0$  and  $Ev_k = Av_{k-1}$  ( $k \geq 2$ ) are the impulsive modes.

**Definition A.1.** Dai (1989) The descriptor system (A1) is said to be

- a) regular if  $\det(sE - A)$  is not identically null;
- b) impulse-free if  $\deg(\det(sE - A)) = \text{rank}(E)$ ;
- c) stable if all the roots of  $\det(sE - A) = 0$  have negative real parts;
- d) admissible if it is regular, impulse-free and stable;
- e) finite dynamics stabilizable (respectively finite dynamics detectable) if there exists  $F$  such that  $(E, A + BF)$  is regular and stable (respectively if there exists  $L$  such that  $(E, A + LC)$  is regular and stable);
- f) impulse controllable (respectively impulse observable) if there exists  $F$  such that  $(E, A + BF)$  is regular and impulse-free (respectively if there exists  $L$  such that  $(E, A + LC)$  is regular and impulse-free).

If the descriptor system is regular, then its transfer function can be defined as

$$T(s) = C(sE - A)^{-1}B + D.$$

**Definition A.2** (Restricted System Equivalence). Rosenbrock (1974) Consider the descriptor system (A1) and  $\bar{T}$  given by

$$\bar{T} : \begin{cases} \bar{E}\dot{\bar{x}}(t) &= \bar{A}\bar{x}(t) + \bar{B}\bar{u}(t), \\ \bar{y}(t) &= \bar{C}\bar{x}(t) + \bar{D}\bar{u}(t). \end{cases}$$

The two systems are termed restricted system equivalent, if there exist nonsingular matrices  $M$  and  $N$  such that

$$\begin{bmatrix} M & 0 \\ 0 & I \end{bmatrix} \left[ \begin{array}{c|c} A - sE & B \\ \hline C & D \end{array} \right] \begin{bmatrix} N & 0 \\ 0 & I \end{bmatrix} = \left[ \begin{array}{c|c} \bar{A} - s\bar{E} & \bar{B} \\ \hline \bar{C} & \bar{D} \end{array} \right].$$