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# State Leakage and Coordination with Causal State Knowledge at the Encoder

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## Abstract

We revisit the problems of state masking and state amplification through the lens of empirical coordination and show that empirical coordination provides a natural framework in which to jointly study the trade-offs between reliable communication, state masking, and state amplification. Specifically, we characterize the rate-equivocation-coordination trade-offs regions of a state-dependent channel in which the encoder has causal and strictly causal state knowledge. We also extend this characterization to the cases of two-sided state information and noisy channel feedback. Our approach is based on the notion of “core of the receiver’s knowledge,” which we introduce to capture what the decoder can infer about all the signals involved in the model. Finally, we exploit the aforementioned results to solve a channel state estimation zero-sum game in which the encoder prevents the decoder to estimate the channel state accurately.

## Index Terms

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Shannon theory, state-dependent channel, state leakage, empirical coordination, state masking, state amplification, causal encoding, two-sided state information, noisy channel feedback.

## I. INTRODUCTION

The study of state-dependent channels can be traced back to the early works of Shannon [2] and Gelf'and and Pinsker [3], which identified optimal coding strategies to transmit reliably in the presence of a state known at the encoder causally or non-causally, respectively. The insights derived from the models have since proved central to the study of diverse topics including wireless communications [4], information-hiding and watermarking [5], and information transmission in repeated games [6]. The present work relates to the latter application and studies state-dependent channels with causal state knowledge from the perspective of *empirical coordination* [7].

Previous studies that have explored the problem of not only decoding messages at the receiver but also estimating the channel state are particularly relevant to the present work. The *state masking* formulation of the problem [8] aims at characterizing the trade-off between the rate of reliable communication and the minimal leakage about the channel state. The rate-leakage capacity region of state masking has been successfully characterized for both causal and non-causal state knowledge. The *state amplification* formulation [9], in which the state is conveyed to the receiver instead of being masked, aims at characterizing the trade-off between the rate of reliable communication and the reduction of uncertainty about the state. The rate-uncertainty reduction capacity region of state amplification has also been successfully characterized for causal and non-causal state knowledge. The state amplification formulation was subsequently extended in the causal case by replacing the reduction of uncertainty about the state by an average distortion function [10] (this model was dubbed causal *state communication*). Note that, in such a scenario, the channel output feedback at the encoder increases the region of achievable rate-distortion pairs [11]. The rate-distortion capacity region of state communication has been successfully characterized for causal and strictly causal state knowledge, and has been characterized for noiseless and noisy non-causal state knowledge in the case of Gaussian channels with a quadratic distortion [12], [13]. Both formulations have been combined in [14] to study the trade-off between amplification and leakage rates in a channel with two receivers having opposite objectives. The amplification-leakage capacity region has been investigated for non-causal state knowledge, via generally non-matching inner and outer bounds. As a perhaps more concrete example, [15] has studied the trade-off between amplification and leakage in the context of an energy harvesting scenario.

We revisit here, the problems of state masking and state amplification with causal and strictly causal state knowledge through the lens of *empirical coordination* [7], [16]. Empirical coordination refers to the

control of the joint histograms of the various sequences such as states, codewords, that appear in channel models, and is related to the coordination of autonomous decision makers in game theory [6]. Specifically, the study of empirical coordination over state-dependent channels is a proxy for characterizing the utility of autonomous decision makers playing a repeated game in the presence of an environment variable (the state), random [6], [17] or adversarial [18], [19], [20], and of an observation structure (the channel) describing how agents observe each other’s actions. The characterization of the empirical coordination capacity requires the design of coding schemes in which the actions of the decision makers are sequences that embed coordination information. The empirical coordination capacity has been studied for state-dependent channels under different constraints including strictly causal and causal encoding [21], for perfect channel [22], for strictly causal and causal decoding [23], with source feedforward [24], for lossless decoding [25], with secrecy constraint [26], with two-sided state information [27] and with channel feedback [28]. Interestingly, empirical coordination is also a powerful tool for controlling the Bayesian posterior beliefs of the decoder, e.g. in the problems of “Bayesian persuasion” [29] and “strategic communication” [30].

The main contribution of the present work is to show that empirical coordination provides a natural framework in which to jointly study the problems of reliable communication, state masking, and state amplification. This connection highlights some of the benefits of empirical coordination beyond those already highlighted in earlier works [21]–[28]. In particular, we obtain the following.

- We introduce and characterize the notion of *core of the receiver’s knowledge*, which captures what the decoder can exactly know about the other variables in the system. For instance, this allows us to characterize the rate-leakage-coordination region for the causal state-dependent channel (Theorem II.3). Our definition of leakage refines previous work by exactly characterizing the leakage rate instead of only providing a single-sided bound. When specialized, our result (Theorem II.6) simultaneously recovers the constraints already established both in [8, Section V] and [9, Theorem 2].
- We revisit the problem of causal state communication and characterize the normalized Kullback-Leibler (KL)-divergence between the decoder’s posterior beliefs and a target belief induced by coordination (Theorem III.1). This allows us to characterize the rate-distortion trade-off for a zero-sum game, in which the decoder attempts to estimate the state while the encoder tries to mask it (Theorem III.3).
- We extend the results to other models, including two-sided state information (Theorem IV.2), noisy feedback (Theorem IV.4), and strictly causal encoding (Theorem V.2).

The rest of the paper is organized as follows. In Section II, we formally introduce the model, along with necessary definitions and notation, and we state our main results. In Section III, we investigate the channel state estimation problem by introducing the KL-divergence and the decoder's posterior beliefs. In Section IV and Section V, we present some extensions of our results to different scenarios. The proofs of most results are provided in Appendices A-E, with some details relegated to Supplementary Materials.

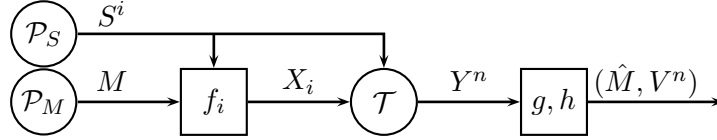


Fig. 1. The memoryless channel  $\mathcal{T}_{Y|X,S}$  depends on the state drawn i.i.d. according to  $\mathcal{P}_S$ . The encoding function is causal  $f_i : \mathcal{M} \times \mathcal{S}^i \rightarrow \mathcal{X}$ , for all  $i \in \{1, \dots, n\}$  and the decoding functions  $g : \mathcal{Y}^n \rightarrow \mathcal{M}$  and  $h : \mathcal{Y}^n \rightarrow \Delta(\mathcal{V}^n)$  are non-causal.

## II. PROBLEM FORMULATION AND MAIN RESULT

### A. Notation

Throughout the paper, capital letters, e.g.,  $S$ , denote random variables while lowercase letters, e.g.,  $s$ , denote their realizations and calligraphic fonts, e.g.,  $\mathcal{S}$ , denote the alphabets in which the realizations take values. All alphabets considered in the paper are assumed finite, i.e.,  $|\mathcal{S}| < \infty$ . Sequences of random variables and realizations are denoted by  $S^n = (S_1, \dots, S_n)$  and  $s^n = (s_1, \dots, s_n)$ , respectively. We denote the set of probability distributions over  $\mathcal{S}$  by  $\Delta(\mathcal{S})$ . For a distribution  $\mathcal{Q}_S \in \Delta(\mathcal{S})$ , we drop the subscript and simply write  $\mathcal{Q}(s)$  in place of  $\mathcal{Q}_S(s)$  for the probability mass assigned to realization  $s \in \mathcal{S}$ . The notation  $\mathcal{Q}_X(\cdot|y) \in \Delta(\mathcal{X})$  denotes the conditional distribution of  $X \in \mathcal{X}$ , given the realization  $y \in \mathcal{Y}$ . For two distributions  $\mathcal{Q}_X, \mathcal{P}_X \in \Delta(\mathcal{X})$ ,  $\|\mathcal{Q}_X - \mathcal{P}_X\|_1 = \sum_{x \in \mathcal{X}} |\mathcal{Q}(x) - \mathcal{P}(x)|$  stands for the  $\ell_1$ -distance between the vectors representing the distributions, see also [31, pp. 370] and [32, pp. 19]. We write  $Y \ominus X \ominus W$  when  $Y$ ,  $X$ , and  $W$  form a Markov chain in that order. The notation  $\mathbb{1}(v = s)$  denotes the indicator function, which is equal to 1 if  $v = s$  and 0 otherwise.

For a sequence  $s^n \in \mathcal{S}^n$ ,  $\mathbf{N}(\tilde{s}|s^n)$  denotes the occurrence number of symbol  $\tilde{s} \in \mathcal{S}$  in the sequence  $s^n$ . The empirical distribution  $Q_S^n \in \Delta(\mathcal{S})$  of sequence  $s^n \in \mathcal{S}^n$  is then defined as

$$\forall \tilde{s} \in \mathcal{S} \quad Q^n(\tilde{s}) = \frac{\mathbf{N}(\tilde{s}|s^n)}{n}. \quad (1)$$

Given  $\delta > 0$  and a distribution  $\mathcal{Q}_{SX} \in \Delta(\mathcal{S} \times \mathcal{X})$ ,  $T_\delta(\mathcal{Q}_{SX})$  stands for the set of sequences  $(s^n, x^n)$  that are jointly typical with tolerance  $\delta > 0$  with respect to the distribution  $\mathcal{Q}_{SX}$ , i.e., such that

$$\left\| Q_{SX}^n - \mathcal{Q}_{SX} \right\|_1 = \sum_{s,x} \left| Q^n(s,x) - \mathcal{Q}(s,x) \right| \leq \delta. \quad (2)$$

We denote by  $\mathbb{P}(S^n \in T_\delta(\mathcal{P}_S))$  the probability value assigned to the event  $\{S^n \in T_\delta(\mathcal{P}_S)\}$ , according to the distribution of  $S^n$ .

### B. System model

The problem under investigation is illustrated in Figure 1. A uniformly distributed message represented by the random variable  $M \in \mathcal{M}$  is to be transmitted over a state dependent memoryless channel characterized by the conditional distribution  $\mathcal{T}_{Y|XS}$  and a channel state  $S \in \mathcal{S}$  drawn according to the i.i.d. distribution  $\mathcal{P}_S$ . For  $n \in \mathbb{N}^*$ , the message  $M$  and the state sequence  $S^n$  are encoded into a codeword  $X^n \in \mathcal{X}^n$  using an encoder, subject to causal constraints to be precised later. Upon observing the output  $Y^n \in \mathcal{Y}^n$  of the noisy channel, the receiver uses a decoder to form an estimate  $\hat{M} \in \mathcal{M}$  of  $M$  and to generate actions  $V^n \in \mathcal{V}^n$ , whose exact role will be precised shortly. For now,  $V^n$  can be thought of as an estimate of the state sequence  $S^n$  but more generally captures the ability of the receiver to coordinate with the transmitter and the channel state. Both  $\mathcal{T}_{Y|XS}$  and  $\mathcal{P}_S$  are assumed known to all parties. We are specifically interested in causal encoders formally defined as follows.

**Definition II.1** *A code with causal encoding consists of stochastic encoding functions  $f_i : \mathcal{M} \times \mathcal{S}^i \rightarrow \Delta(\mathcal{X}) \forall i \in \{1, \dots, n\}$ , a deterministic decoding function  $g : \mathcal{Y}^n \rightarrow \mathcal{M}$ , and a stochastic receiver action function  $h : \mathcal{Y}^n \rightarrow \Delta(\mathcal{V}^n)$ . The set of codes with causal encoding with length  $n$  and message set  $\mathcal{M}$  is denoted  $\mathcal{C}_c(n, \mathcal{M})$ .*

A code  $c \in \mathcal{C}_c(n, \mathcal{M})$ , the uniform distribution of the messages  $\mathcal{P}_M$ , the source  $\mathcal{P}_S$  and the channel  $\mathcal{T}_{Y|XS}$ , induce a distribution on  $(M, S^n, X^n, Y^n, V^n, \hat{M})$  given by

$$\mathcal{P}_M \prod_{i=1}^n \left[ \mathcal{P}_{S_i} f_{X_i|S^i M} \mathcal{T}_{Y_i|X_i S_i} \right] h_{V^n|Y^n} \mathbb{1}(\hat{M} = g(Y^n)). \quad (3)$$

Since the sequences  $(S^n, X^n, Y^n, V^n)$  are random, the empirical distribution  $Q_{SXYV}^n$  is also a random variable. The performance of codes is measured as follows.

**Definition II.2** *Fix a target rate  $R \geq 0$ , a target state leakage  $E \geq 0$  and a target distribution  $\mathcal{Q}_{SXYV}$ . The triple  $(R, E, \mathcal{Q}_{SXYV})$  is achievable if for all  $\varepsilon > 0$ , there exist  $\bar{n} \in \mathbb{N}$ , for all  $n \geq \bar{n}$ , there exist a code  $c \in \mathcal{C}_c(n, \mathcal{M})$  that satisfies*

$$\begin{aligned} \frac{\log_2 |\mathcal{M}|}{n} &\geq R - \varepsilon, \\ \left| \mathcal{L}_e(c) - E \right| &\leq \varepsilon, \quad \text{with} \quad \mathcal{L}_e(c) = \frac{1}{n} \cdot I(S^n; Y^n), \\ \mathcal{P}_e(c) &= \mathbb{P}(M \neq \hat{M}) + \mathbb{P}\left(\left\| Q_{SXYV}^n - \mathcal{Q}_{SXYV} \right\|_1 > \varepsilon\right) \leq \varepsilon. \end{aligned}$$

We denote by  $\mathcal{A}_c$  the set of achievable triples  $(R, E, \mathcal{Q}_{SXYV})$ .

In layman's term, performance is captured along three metrics: i) the rate at which the message  $M$  can be reliably transmitted; ii) the information leakage rate about the state sequence  $S^n$  at the receiver; and iii) the ability of the encoder to coordinate with the receiver, captured by the empirical coordination with respect to  $\mathcal{Q}_{SXYV}$ . The need to coordinate with receiver action  $V$  is motivated by problems in which the terminals represent decision makers that choose actions  $(X, V)$  as a function of the system state  $S$ , as in [6].

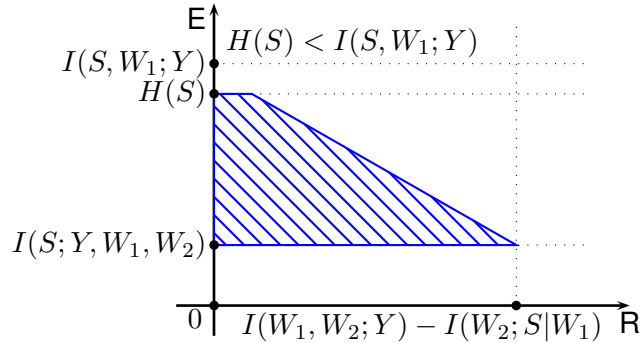


Fig. 2. The region of achievable  $(R, E) \in \mathcal{A}_c$  for a given distribution  $\mathcal{Q}_{SW_1W_2XYV}$  for which  $H(S) < I(S, W_1; Y)$ .

### C. Main result

**Theorem II.3** Consider a target distribution  $\mathcal{Q}_{SXYV}$  that decomposes as  $\mathcal{Q}_{SXYV} = \mathcal{P}_S \mathcal{Q}_{X|S} \mathcal{T}_{Y|XS} \mathcal{Q}_{V|SXY}$ . Then,  $(R, E, \mathcal{Q}_{SXYV}) \in \mathcal{A}_c$  if and only if there exist two auxiliary random variables  $(W_1, W_2)$  with distribution  $\mathcal{Q}_{SW_1W_2XYV} \in \mathcal{Q}_c$  satisfying

$$I(S; W_1, W_2, Y) \leq E \leq H(S), \quad (4)$$

$$R + E \leq I(W_1, S; Y), \quad (5)$$

where  $\mathcal{Q}_c$  is the set of distributions  $\mathcal{Q}_{SW_1W_2XYV}$  with marginal  $\mathcal{Q}_{SXYV}$  that decompose as

$$\mathcal{P}_S \mathcal{Q}_{W_1} \mathcal{Q}_{W_2|SW_1} \mathcal{Q}_{X|SW_1} \mathcal{T}_{Y|XS} \mathcal{Q}_{V|YW_1W_2}, \quad (6)$$

and such that  $\max(|\mathcal{W}_1|, |\mathcal{W}_2|) \leq |\mathcal{S} \times \mathcal{X} \times \mathcal{Y} \times \mathcal{V}| + 1$ .

The achievability and converse proofs are provided in Appendices A and B, respectively, with the cardinality bounds established in the Supplementary Materials. The key idea behind the achievability proof is the following. The encoder operates in a Block-Markov fashion to ensure that the transmitted signals, the state, the received sequence, and the receiver actions are coordinated subject to the causal

constraint at the encoder. This requires the use of two auxiliary codebooks, captured by the auxiliary random variables  $W_1$  and  $W_2$ , where the first codebook is used for reliable communication while the second one is used to coordinate with the state. Simultaneously, the encoder quantizes the channel state and transmits carefully chosen bin indices on top of its messages to finely control how much the receiver can infer about the channel state. The region of achievable pairs  $(R, E)$  is depicted in Fig. 2 for a given distribution  $\mathcal{Q}_{SW_1W_2XYV}$ , assuming  $H(S) < I(S, W_1; Y)$ .

**Remark II.4** Equation (5) and the first inequality of (4) imply the information constraints of [10, Theorem 3] for causal state communication and of [21, Theorem 2] for empirical coordination.

$$R \leq I(W_1, W_2; Y) - I(W_2; S|W_1). \quad (7)$$

Indeed, both Markov chains  $X \circlearrowleft (S, W_1) \circlearrowleft W_2$  and  $Y \circlearrowleft (X, S) \circlearrowleft (W_1, W_2)$  imply  $Y \circlearrowleft (W_1, S) \circlearrowleft W_2$ .

Theorem II.3 has several important consequences. First, the coordination of both encoder and decoder actions according to  $\mathcal{P}_S \mathcal{Q}_{X|S} \mathcal{T}_{Y|XS} \mathcal{Q}_{V|SXY}$  is compatible with the reliable transmission of additional information at rate  $R \geq 0$ . Second, the case of equality in the right-hand-side inequality of (4) corresponds to the full disclosure of the channel state  $S$  to the decoder. Third, for any  $(R, \mathcal{Q}_{SXYV})$ , the minimal state leakage  $E^*(R, \mathcal{Q}_{SXYV})$  such that  $(R, E^*(R, \mathcal{Q}_{SXYV}), \mathcal{Q}_{SXYV}) \in \mathcal{A}_c$ , if it exists, is given by

$$E^*(R, \mathcal{Q}_{SXYV}) = \min_{\substack{\mathcal{Q}_{SW_1W_2XYV} \in \mathcal{Q}_c, \\ \text{s.t. } R \leq I(W_1, W_2; Y) - I(W_2; S|W_1)}} I(S; W_1, W_2, Y). \quad (8)$$

The reliable transmission of information requires the decoder to know the encoding function, from which it can estimate the channel state  $S$ . In Section III, we investigate the relationship between the state leakage  $\mathcal{L}_e(c)$  and the decoder's posterior belief  $\mathcal{P}_{S^n|Y^n}$  induced by the coding process.

#### D. Special case without receiver actions

We now assume that the decoder does not return an action  $V$  coordinated with the other symbols  $(S, X, Y)$  to compare our setting with the problems of “state masking” [8, Section V] and “state amplification” [9, Section IV]. Note that these earlier works involve slightly different notions of achievable state leakage. In [8], the state leakage is upper bounded by  $\mathcal{L}_e(c) = \frac{1}{n}I(S^n; Y^n) \leq E + \varepsilon$ . In [9], the decoder forms a list  $L_n(Y^n) \subseteq S^n$  with cardinality  $\log_2 |L_n(Y^n)| = H(S) - E$  such that the list decoding error probability  $\mathbb{P}(S^n \notin L_n(Y^n)) \leq \varepsilon$  is small, hence reducing the uncertainty about the state. Here, we require the leakage  $\mathcal{L}_e(c) = \frac{1}{n}I(S^n; Y^n)$  induced by the code to be controlled by  $|\mathcal{L}_e(c) - E| \leq \varepsilon$ . Nevertheless, we shall see that our definition allows us to obtain the rate constraints of [8], [9] as extreme cases.



**Definition II.5** A code without receiver actions consists of stochastic encoding functions  $f_i : \mathcal{M} \times \mathcal{S}^i \rightarrow \Delta(\mathcal{X}) \forall i \in \{1, \dots, n\}$  and a deterministic decoding function  $g : \mathcal{Y}^n \rightarrow \mathcal{M}$ . The set of such codes with length  $n$  and message set  $\mathcal{M}$  is denoted  $\mathcal{C}_d(n, \mathcal{M})$ . The corresponding set of achievable triples  $(R, E, \mathcal{Q}_{SXY})$  is defined as in Definition II.2 and is denoted  $\mathcal{A}_d$ .

Note that the target distribution is here restricted to  $\mathcal{Q}_{SXY} \in \Delta(\mathcal{S} \times \mathcal{X} \times \mathcal{Y})$  since the receiver does not take an action.

**Theorem II.6** Consider a target distribution  $\mathcal{Q}_{SXY}$  that decomposes as  $\mathcal{Q}_{SXY} = \mathcal{P}_S \mathcal{Q}_{X|S} \mathcal{T}_{Y|XS}$ . Then,  $(R, E, \mathcal{Q}_{SXY}) \in \mathcal{A}_d$  if and only if there exist an auxiliary random variable  $W_1$  with distribution  $\mathcal{Q}_{SW_1XY} \in \mathcal{Q}_d$  that satisfies

$$I(S; W_1, Y) \leq E \leq H(S), \quad (9)$$

$$R + E \leq I(W_1, S; Y), \quad (10)$$

where  $\mathcal{Q}_d$  is the set of distributions  $\mathcal{Q}_{SW_1XY}$  with marginal  $\mathcal{Q}_{SXY}$  that decompose as

$$\mathcal{P}_S \mathcal{Q}_{W_1} \mathcal{Q}_{X|SW_1} \mathcal{T}_{Y|XS}, \quad (11)$$

and such that  $|W_1| \leq |\mathcal{S} \times \mathcal{Y}| + 1$ .

The achievability proof is obtained from Theorem II.3 by setting  $W_2 = \emptyset$  and by considering a single block coding instead of block-Markov coding. The converse proof is similar to the converse of Theorem II.3 and is provided in the Supplementary Materials.

**Remark II.7** When setting  $W_2 = \emptyset$ , (7) in Remark II.4 simplifies to

$$R \leq I(W_1; Y), \quad (12)$$

which, together with the first inequality in (9), coincides with the information constraints of [8, pp. 2260]. Furthermore, (12), (10) and the second inequality of (9) correspond to the region  $\mathcal{R}_0$  stated in [9, Lemma 3]. Formally, the region characterized by Theorem II.6 is the intersection of the regions identified in [8, pp. 2260] and [9, Lemma 3].

### III. CHANNEL STATE ESTIMATION VIA DISTORTION FUNCTION

#### A. Decoder posterior belief

In this section, we provide an upper bound on the KL-divergence between the decoder posterior belief  $\mathcal{P}_{S^n|Y^n}$  induced by the encoding defined in (3), and the target conditional distribution  $\mathcal{Q}_{S|YW_1W_2}$ .

**Theorem III.1 (Channel state estimation)** *Assume that the distribution  $\mathcal{Q} = \mathcal{Q}_{SW_1W_2XY}$  has full support. For any conditional distribution  $\mathcal{P}_{W_1^n W_2^n X^n | S^n}$ , we have*

$$\begin{aligned} & \frac{1}{n} D \left( \mathcal{P}_{S^n | Y^n} \left\| \prod_{i=1}^n \mathcal{Q}_{S_i | Y_i W_{1,i} W_{2,i}} \right. \right) \\ & \leq \mathcal{L}_e(c) - I(S; W_1, W_2, Y) + \alpha_1 \delta + \alpha_2 \mathbb{P} \left( (S^n, W_1^n, W_2^n, Y^n) \notin T_\delta(\mathcal{Q}) \right), \end{aligned} \quad (13)$$

where  $\delta > 0$  denotes the tolerance of the set of typical sequences  $T_\delta(\mathcal{Q})$  and the constants  $\alpha_1 = \sum_{\substack{s, w_1, \\ w_2, y}} \log_2 \frac{1}{\mathcal{Q}(s|w_1, w_2, y)}$  and  $\alpha_2 = \log_2 \frac{1}{\min_{s, y, w_1, w_2} \mathcal{Q}(s|y, w_1, w_2)}$  are strictly positive.

The proof of Theorem III.1 is given in Appendix C. Consider a target leakage  $\mathbf{E} = I(S; W_1, W_2, Y)$  and a pair  $(\mathbf{R}, \mathcal{Q}_{SXYV})$ , and assume there exist a distribution  $\mathcal{Q}_{SW_1W_2XYV} \in \mathbb{Q}_c$  with full support, satisfying (4) and (5). By Theorem II.3, for all  $\varepsilon > 0$  and for all  $\delta > 0$ , there exist a  $\bar{n}$ , for all  $n \geq \bar{n}$  there exist a code  $c \in \mathcal{C}(n, \mathcal{M})$  with two auxiliary sequences  $(W_1^n, W_2^n)$ , such that:

$$\left| \mathcal{L}_e(c) - I(S; W_1, W_2, Y) \right| \leq \varepsilon \quad \text{and} \quad \mathbb{P} \left( \left\| \mathcal{Q}_{SW_1W_2Y}^n - \mathcal{Q}_{SW_1W_2Y} \right\|_1 > \delta \right) \leq \varepsilon. \quad (14)$$

Hence, by Theorem III.1 we have

$$\frac{1}{n} D \left( \mathcal{P}_{S^n | Y^n} \left\| \prod_{i=1}^n \mathcal{Q}_{S_i | Y_i W_{1,i} W_{2,i}} \right. \right) \leq \varepsilon + \alpha_1 \delta + \alpha_2 \varepsilon, \quad (15)$$

where  $\varepsilon$  and  $\delta$  may go to zero when  $n$  goes to infinity. The control of the leakage  $\mathcal{L}_e(c)$  and the joint typicality of the sequences  $(S^n, W_1^n, W_2^n, Y^n) \in T_\delta(\mathcal{Q})$  implies that the decoder posterior belief  $\mathcal{P}_{S^n | Y^n}$  approaches the single-letter distribution  $\mathcal{Q}_{S|Y W_1 W_2}$ . Based on the triple of symbols  $(Y, W_1, W_2)$ , the decoder generates action  $V$  using the conditional distribution  $\mathcal{Q}_{V|Y W_1 W_2}$  and infers the channel state  $S$  with the conditional distribution  $\mathcal{Q}_{S|Y W_1 W_2}$ . In that regard, the random variables  $(Y, W_1, W_2)$  capture the "core of the receiver's knowledge," regarding other random variables  $S$  and  $V$ . The bound on the KL-divergence in (13) relates to the notion of "strategic distance" [17, Section 5.2], later used in several articles on repeated game [18], [19], [20], on "Bayesian persuasion" [29] and on "strategic communication" [30].

### B. Channel state estimation zero-sum game

We now introduce a channel state estimation zero-sum game, in which the encoder and decoder are opponents choosing an encoding and a decoding strategy, respectively. Although the encoder and the decoder cooperate in transmitting reliably at rate  $\mathbf{R}$ , the encoder seeks to prevent the decoder from returning a good estimate  $v \in \mathcal{V}$  of the channel state  $s \in \mathcal{S}$  by maximizing the expected long-run distortion, while the decoder attempts to minimize it.

**Definition III.2** A target rate  $R \geq 0$  and a target distortion  $D \geq 0$  are achievable if for all  $\varepsilon > 0$ , there exist a  $\bar{n} \in \mathbb{N}$ , for all  $n \geq \bar{n}$ , there exist a code in  $\mathcal{C}_d(n, \mathcal{M})$  such that

$$\frac{\log_2 |\mathcal{M}|}{n} \geq R - \varepsilon, \quad (16)$$

$$\mathbb{P}\left(M \neq \hat{M}\right) \leq \varepsilon, \quad (17)$$

$$\left| \min_{h_{V^n|Y^n}} \frac{1}{n} \sum_{i=1}^n \mathbb{E}\left[d(S_i, V_i)\right] - D \right| \leq \varepsilon. \quad (18)$$

We denote by  $\mathcal{A}_g$  the set of achievable pairs  $(R, D) \in \mathcal{A}_g$ .

**Theorem III.3 (Zero-sum game)** A pair of rate and distortion  $(R, D) \in \mathcal{A}_g$  is achievable if and only if there exist an auxiliary random variable  $W_1$  with distribution  $\mathcal{Q}_{SW_1XY} \in \mathbb{Q}_d$  that satisfies

$$R \leq I(W_1; Y), \quad (19)$$

$$D = \min_{\mathcal{P}_{V|W_1Y}} \mathbb{E}\left[d(S, V)\right], \quad (20)$$

where the set  $\mathbb{Q}_d$  is defined in Theorem II.6.

The achievability proof of Theorem III.3 is provided in Appendix D and is a consequence of Theorem II.6 and Theorem III.1, and of [29, Lemma A.8, Lemma A.21]. The converse proof of Theorem is provided in Appendix E.

**Remark III.4 (Maximin-minimax result)** The optimal distortion-rate function  $D^*(R)$  reformulates as a maximin problem:

$$D^*(R) = \max_{\substack{\mathcal{Q}_{W_1}, \mathcal{Q}_{X|SW_1} \\ R \leq I(W_1; Y)}} \min_{\mathcal{P}_{V|W_1Y}} \mathbb{E}\left[d(S, V)\right] = \min_{\mathcal{P}_{V|W_1Y}} \max_{\substack{\mathcal{Q}_{W_1}, \mathcal{Q}_{X|SW_1} \\ R \leq I(W_1; Y)}} \mathbb{E}\left[d(S, V)\right]. \quad (21)$$

The maximum and the minimum are taken over compact and convex sets and the distortion function is linear. Hence by Sion's Theorem [33] the maximin equal the minimax and the value of this channel state estimation zero-sum game is  $D^*(R)$ .

**Remark III.5 (One auxiliary random variable)** The formulation of Theorem III.3 is based on the set of distributions  $\mathbb{Q}_d$  with only one auxiliary random variable  $W_1$ , instead of the two random variables

$(W_1, W_2)$  of the set  $\mathbb{Q}_c$ . When the encoder tries to mask the channel state, it does not require the auxiliary random variable  $W_2$  anymore, since

$$D^\circ = \max_{\substack{\mathcal{Q}_{W_1}, \mathcal{Q}_{X|SW_1}, \mathcal{Q}_{W_2|SW_1} \\ R \leq I(W_1, W_2; Y) - I(W_2; S|W_1)}} \min_{\mathcal{P}_{V|W_1 W_2 Y}} \mathbb{E} [d(S, V)] \quad (22)$$

$$\leq \max_{\substack{\mathcal{Q}_{W_1}, \mathcal{Q}_{X|SW_1}, \mathcal{Q}_{W_2|SW_1} \\ R \leq I(W_1, W_2; Y) - I(W_2; S|W_1)}} \min_{\mathcal{P}_{V|W_1 Y}} \mathbb{E} [d(S, V)] \quad (23)$$

$$\leq \max_{\substack{\mathcal{Q}_{W_1}, \mathcal{Q}_{X|SW_1} \\ R \leq I(W_1; Y)}} \min_{\mathcal{P}_{V|W_1 Y}} \mathbb{E} [d(S, V)] = D^*, \quad (24)$$

where (23) comes from taking the minimum over  $\mathcal{P}_{V|W_1 Y}$  instead of  $\mathcal{P}_{V|W_1 W_2 Y}$ ; (24) comes from the Markov chain  $Y \text{---} (S, W_1) \text{---} W_2$  stated in (6), that ensures:  $I(W_1, W_2; Y) - I(W_2; S|W_1) \leq I(W_1; Y)$ . Hence, the information constraint  $R \leq I(W_1, W_2; Y) - I(W_2; S|W_1)$  is more restrictive than  $R \leq I(W_1; Y)$ .

**Remark III.6 (Zero rate case)** In the special case  $R = 0$ , which corresponds to a channel estimation game without communication, the encoding functions reduce to  $f_{X_i|S^i}$  instead of  $f_{X_i|S^i M}$ . The channel state estimation zero-sum game becomes the maximin problem

$$\max_{\{f_{X_i|S^i}\}_{i \in \{1, \dots, n\}}} \min_{h_{V^n|Y^n}} \mathbb{E} \left[ \frac{1}{n} \sum_{i=1}^n d(S_i, V_i) \right], \quad (25)$$

in which the encoder chooses  $\{f_{X_i|S^i}\}_{i \in \{1, \dots, n\}}$  and the decoder chooses  $h_{V^n|Y^n}$ . Theorem III.3 shows that the single-letter solution is  $\max_{\mathcal{Q}_{W_1}, \mathcal{Q}_{X|SW_1}} \min_{\mathcal{P}_{V|W_1 Y}} \mathbb{E} [d(S, V)]$ .

If the objectives of encoder and decoder were aligned, i.e., they would both try to minimize the long term average distortion

$$\min_{\substack{\{f_{X_i|S^i}\}_{i \in \{1, \dots, n\}}, \\ h_{V^n|Y^n}}} \mathbb{E} \left[ \frac{1}{n} \sum_{i=1}^n d(S_i, V_i) \right], \quad (26)$$

the problem would become the causal channel state communication studied in [10].

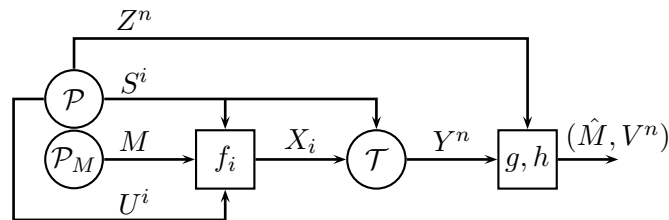


Fig. 3. The causal encoding function writes  $f_i : \mathcal{M} \times \mathcal{U}^i \times \mathcal{S}^i \rightarrow \mathcal{X}$ , for all  $i \in \{1, \dots, n\}$  and the non-causal decoding functions write  $g : \mathcal{Y}^n \times \mathcal{Z}^n \rightarrow \mathcal{M}$  and  $h : \mathcal{Y}^n \times \mathcal{Z}^n \rightarrow \Delta(\mathcal{V}^n)$ .

#### IV. EXTENSIONS TO MORE GENERAL SCENARIOS

##### A. Two-sided state information

The case of two-sided state information is illustrated in Fig. 3. The channel state  $S^n$ , information source  $U^n$  and decoder state information  $Z^n$  are jointly distributed according to the i.i.d. distribution  $\mathcal{P}_{USZ} \in \Delta(\mathcal{U} \times \mathcal{S} \times \mathcal{Z})$ .

**Definition IV.1** *A code with two-sided state information consists of stochastic functions  $f_i : \mathcal{M} \times \mathcal{U}^i \times \mathcal{S}^i \rightarrow \Delta(\mathcal{X}) \forall i \in \{1, \dots, n\}$ , a deterministic decoding function  $g : \mathcal{Y}^n \times \mathcal{Z}^n \rightarrow \mathcal{M}$ , and a stochastic receiver action function  $h : \mathcal{Y}^n \times \mathcal{Z}^n \rightarrow \Delta(\mathcal{V}^n)$ . The set of codes with causal encoding with length  $n$  and message set  $\mathcal{M}$  is denoted  $\mathcal{C}_s(n, \mathcal{M})$ .*

A code  $c \in \mathcal{C}_s(n, \mathcal{M})$ , the uniform distribution of the messages  $\mathcal{P}_M$ , the source  $\mathcal{P}_{USZ}$  and the channel  $\mathcal{T}_{Y|XS}$  induce a distribution on  $(M, U^n, S^n, Z^n, X^n, Y^n, V^n, \hat{M})$  given by

$$\mathcal{P}_M \prod_{i=1}^n \left[ \mathcal{P}_{U_i S_i Z_i} f_{X_i|U_i S_i M} \mathcal{T}_{Y_i|X_i S_i} \right] h_{V^n|Y^n Z^n} \mathbf{1}(\hat{M} = g(Y^n, Z^n)). \quad (27)$$

We denote by  $\mathcal{A}_s$  the set of achievable triples  $(R, E, \mathcal{Q}_{USZXYV})$ , defined similarly as in Definition II.2.

**Theorem IV.2 (Two-sided state information)** *Consider a target distribution  $\mathcal{Q}_{USZXYV}$  that decomposes as  $\mathcal{Q}_{USZXYV} = \mathcal{P}_{USZ} \mathcal{Q}_{X|US} \mathcal{T}_{Y|XS} \mathcal{Q}_{V|USZXY}$ . Then,  $(R, E, \mathcal{Q}_{USZXYV}) \in \mathcal{A}_s$  if and only if there exist two auxiliary random variables  $(W_1, W_2)$  with distribution  $\mathcal{Q}_{USZW_1 W_2 XYV} \in \mathbb{Q}_s$  satisfying*

$$I(U, S; W_1, W_2, Y, Z) \leq E \leq H(U, S), \quad (28)$$

$$R + E \leq I(W_1, U, S; Y, Z), \quad (29)$$

where  $\mathbb{Q}_s$  is the set of distributions  $\mathcal{Q}_{USZW_1 W_2 XYV}$  that decompose as

$$\mathcal{P}_{USZ} \mathcal{Q}_{W_1} \mathcal{Q}_{W_2|USW_1} \mathcal{Q}_{X|USW_1} \mathcal{T}_{Y|XS} \mathcal{Q}_{V|YZW_1 W_2}, \quad (30)$$

and such that  $\max(|\mathcal{W}_1|, |\mathcal{W}_2|) \leq d + 1$  with  $d = |\mathcal{U} \times \mathcal{S} \times \mathcal{Z} \times \mathcal{X} \times \mathcal{Y} \times \mathcal{V}|$ .

The achievability proof follows directly from the proof of Theorem II.3, by replacing the random variable of the channel state  $S$  by the pair  $(U, S)$  and the random variable of the channel output  $Y$  by the pair  $(Y, Z)$ . The converse proof is provided in the Supplementary Materials.

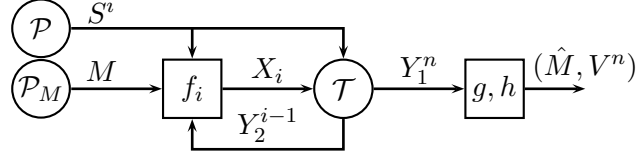


Fig. 4. The noisy feedback sequence  $Y_2^{i-1}$  is drawn i.i.d. according to  $\mathcal{T}_{Y_1 Y_2 | X S}$ . The encoding writes  $f_i : \mathcal{M} \times \mathcal{S}^i \times \mathcal{Y}_2^{i-1} \rightarrow \mathcal{X}$ ,  $\forall i \in \{1, \dots, n\}$ .

**Remark IV.3** *The Markov chains  $X \text{---} (U, S, W_1) \text{---} W_2$ ,  $Y \text{---} (X, S) \text{---} (U, Z, W_1, W_2)$  and  $Z \text{---} (U, S) \text{---} (X, Y, W_1, W_2)$  imply another Markov chain property  $(Y, Z) \text{---} (W_1, U, S) \text{---} W_2$ . Indeed, for all  $(u, s, z, w_1, w_2, x, y)$  we have*

$$\begin{aligned} \mathcal{P}(y, z | w_1, w_2, u, s) &= \sum_{x \in \mathcal{X}} \mathcal{P}(x | w_1, w_2, u, s) \mathcal{P}(y | x, w_1, w_2, u, s) \mathcal{P}(z | x, w_1, w_2, u, s, y) \\ &= \sum_{x \in \mathcal{X}} \mathcal{Q}(x | u, s, w_1) \mathcal{T}(y | x, s) \mathcal{P}(z | u, s) = \mathcal{P}(y, z | w_1, u, s). \end{aligned}$$

By combining (28) and (29) with the Markov chain  $(Y, Z) \text{---} (W_1, U, S) \text{---} W_2$ , we recover the information constraint of [27, Theorem V.1]:

$$R \leq I(W_1, W_2; Y, Z) - I(W_2; U, S | W_1). \quad (31)$$

### B. Noisy channel feedback observed by the encoder

In this section, we consider that the encoder has noisy feedback  $Y_2$  from the state-dependent channel  $\mathcal{T}_{Y_1 Y_2 | X S}$ , as depicted in Fig. 4. The encoding function becomes  $f_i : \mathcal{M} \times \mathcal{S}^i \times \mathcal{Y}_2^{i-1} \rightarrow \mathcal{X}$ ,  $\forall i \in \{1, \dots, n\}$  while the decoding functions and the definition of the state leakage remain unchanged. The corresponding set of achievable triples  $(R, E, \mathcal{Q}_{S X Y_1 Y_2 V})$  is denoted  $\mathcal{A}_f$ .

**Theorem IV.4 (Noisy channel feedback)** *We consider a target distribution  $\mathcal{Q}_{S X Y_1 Y_2 V}$  that decomposes as  $\mathcal{Q}_{S X Y_1 Y_2 V} = \mathcal{P}_S \mathcal{Q}_{X | S} \mathcal{T}_{Y_1 Y_2 | X S} \mathcal{Q}_{V | S X Y_1 Y_2}$ . Then,  $(R, E, \mathcal{Q}_{S X Y_1 Y_2 V}) \in \mathcal{A}_f$  if and only if there exist two auxiliary random variables  $(W_1, W_2)$  with distribution  $\mathcal{Q}_{S W_1 W_2 X Y_1 Y_2 V} \in \mathcal{Q}_f$  that satisfy*

$$R \leq I(W_1, W_2; Y_1) - I(W_2; S, Y_2 | W_1), \quad (32)$$

$$I(S; W_1, W_2, Y_1) \leq E \leq H(S), \quad (33)$$

$$R + E \leq I(W_1, S; Y_1), \quad (34)$$

where  $\mathcal{Q}_f$  is the set of distributions with marginal  $\mathcal{Q}_{S W_1 W_2 X Y_1 Y_2 V}$  that decompose as

$$\mathcal{P}_S \mathcal{Q}_{W_1} \mathcal{Q}_{X | S W_1} \mathcal{T}_{Y_1 Y_2 | X S} \mathcal{Q}_{W_2 | S W_1 Y_2} \mathcal{Q}_{V | Y_1 W_1 W_2},$$

and such that  $\max(|\mathcal{W}_1|, |\mathcal{W}_2|) \leq d + 1$  with  $d = |\mathcal{S} \times \mathcal{X} \times \mathcal{Y}_1 \times \mathcal{Y}_2 \times \mathcal{V}|$ .

The achievability proof of Theorem IV.4 follows directly from the proof of Theorem II.3, by replacing the pair  $(S^n, W_1^n)$  by the triple  $(S^n, W_1^n, Y_2^n)$  in order to select  $W_2^n$ . The decoding functions and the leakage analysis remain unchanged. The converse proof is stated in the Supplementary Materials.

**Remark IV.5 (Noisy feedback improves coordination)** *The channel feedback increases the set of achievable triples, i.e.  $\mathcal{A}_c \subset \mathcal{A}_f$ , since the conditional distribution  $\mathcal{Q}_{W_2|SW_1Y_2}$  depends on channel outputs  $Y_2$ . The information constraints of Theorem IV.4 are reduced to that of Theorem II.3 since  $\mathcal{Q}_{W_2|SW_1Y_2} = \mathcal{Q}_{W_2|SW_1} \iff W_2 \perp (S, W_1) \perp Y_2 \iff I(W_2; Y_2|S, W_1) = 0$ . This was already pointed out for the coordination problem in [28], and for the rate-and-state capacity problem in [11].*

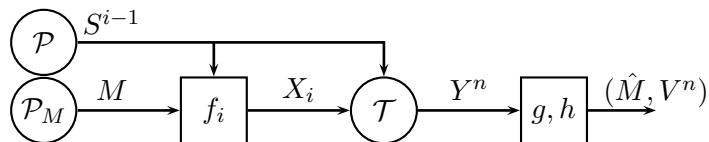


Fig. 5. The strictly causal encoding function writes  $f_i : \mathcal{M} \times \mathcal{S}^{i-1} \rightarrow \Delta(\mathcal{X})$ , for all  $i \in \{1, \dots, n\}$  and the non-causal decoding functions write  $g : \mathcal{Y}^n \rightarrow \mathcal{M}$  and  $h : \mathcal{Y}^n \rightarrow \Delta(\mathcal{V}^n)$ .

## V. STRICTLY CAUSAL ENCODING

**Definition V.1** *A code with strictly causal encoding consists of stochastic encoding functions  $f_i : \mathcal{M} \times \mathcal{S}^{i-1} \rightarrow \Delta(\mathcal{X}) \forall i \in \{1, \dots, n\}$ , a deterministic decoding function  $g : \mathcal{Y}^n \rightarrow \mathcal{M}$ , and a stochastic receiver action function  $h : \mathcal{Y}^n \rightarrow \Delta(\mathcal{V}^n)$ . The set of codes with strictly causal encoding with length  $n$  and message set  $\mathcal{M}$  is denoted  $\mathcal{C}_{\text{sc}}(n, \mathcal{M})$ . The corresponding set of achievable triples  $(R, E, \mathcal{Q}_{SXYV})$  is defined similarly as in Definition II.2 and is denoted  $\mathcal{A}_{\text{sc}}$ .*

**Theorem V.2 (Strictly causal encoding)** *Consider a target distribution  $\mathcal{Q}_{SXYV}$  that decomposes as  $\mathcal{Q}_{SXYV} = \mathcal{P}_S \mathcal{Q}_X \mathcal{T}_{Y|XS} \mathcal{Q}_{V|SXY}$ . Then,  $(R, E, \mathcal{Q}_{SXYV}) \in \mathcal{A}_{\text{sc}}$  if and only if there exist an auxiliary random variable  $W_2$  with distribution  $\mathcal{Q}_{SW_2XYV} \in \mathcal{Q}_{\text{sc}}$  that satisfy*

$$I(S; X, W_2, Y) \leq E \leq H(S), \quad (35)$$

$$R + E \leq I(X, S; Y), \quad (36)$$

where  $\mathcal{Q}_{\text{sc}}$  is the set of distributions  $\mathcal{Q}_{SW_2XYV}$  with marginal  $\mathcal{Q}_{SW_2XYV}$  that decompose as

$$\mathcal{Q}_{SW_2XYV} = \mathcal{P}_S \mathcal{Q}_X \mathcal{Q}_{W_2|SX} \mathcal{T}_{Y|XS} \mathcal{Q}_{V|XYW_2} \quad (37)$$

and such that  $|\mathcal{W}_2| \leq |\mathcal{S} \times \mathcal{X} \times \mathcal{Y}| + 1$ .

The achievability proof is obtained from Theorem II.3 by replacing the auxiliary random variable  $W_1$  by the channel input  $X$ . The converse proof is provided in the Supplementary Materials.

**Remark V.3** Equation (36), the first inequality of (35), the Markov chain  $Y \dashv\vdash (X, S) \dashv\vdash W_2$ , and the independence between  $S$  and  $X$  imply

$$R \leq I(X, W_2; Y) - I(W_2; S|X). \quad (38)$$

**Corollary V.4 (Strictly causal encoding without receiver's outputs)** A pair of rate and state leakage  $(R, E)$  is achievable if and only if there exist a distribution  $Q_X$  that satisfies

$$I(S; Y|X) \leq E \leq H(S), \quad (39)$$

$$R + E \leq I(X, S; Y). \quad (40)$$

The achievability proof of Corollary V.4 comes from the achievability proof of Theorem V.2. The converse proof is based on standard arguments. Equations (39) and (40) imply  $R \leq I(X; Y)$ .

## APPENDIX A

### ACHIEVABILITY PROOF OF THEOREM II.3

#### A. Random coding

The case  $H(S) = 0$  can be handled using a standard channel coding scheme and is detailed in the Supplementary Materials. In the remaining of the proof, we assume that  $H(S) > 0$  and we fix a rate, a state leakage, and a distribution  $(R, E, Q_{SXYV})$  for which there exist a distribution  $Q_{SW_1W_2XYV} \in \mathcal{Q}_c$  with marginal  $Q_{SXYV}$ , that satisfies

$$I(S; W_1, W_2, Y) < E \leq H(S), \quad (41)$$

$$R + E \leq I(W_1, S; Y), \quad (42)$$

where the inequality in (41) is strict. The case of equality has to be treated with care because the channel capacity might be zero, a detailed analysis is available in the Supplementary Materials. We show that  $(R, E, Q_{SXYV})$  is achievable by introducing the rate parameters  $R_L, R_J, R_K$  and by considering a block-Markov random code  $c \in \mathcal{C}(nB, \mathcal{M})$  defined over  $B \in \mathbb{N}$  blocks of length  $n \in \mathbb{N}$ . The codebook is defined over one block of length  $n \in \mathbb{N}$  and the total length of the code is denoted by  $N = nB \in \mathbb{N}$ . In the following, the notation  $T_\delta(\mathcal{Q})$  stands for the set of typical sequences with respect to the distribution  $\mathcal{Q} = Q_{SW_1W_2XYV}$ .



**Random Codebook.**

- 1) We draw  $2^{n(H(S)+\varepsilon)}$  sequences  $S^n(l, j)$  according to the i.i.d. distribution  $\mathcal{P}_S$ , with indices  $(l, j) \in \mathcal{M}_L \times \mathcal{M}_J$  such that  $|\mathcal{M}_L| = 2^{nR_L}$  and  $|\mathcal{M}_J| = 2^{nR_J}$ .
- 2) We draw  $2^{n(R+R_L+R_K)}$  sequences  $W_1^n(m, l, k)$  according to the i.i.d. distribution  $\mathcal{Q}_{W_1}$  with indices  $(m, l, k) \in \mathcal{M} \times \mathcal{M}_L \times \mathcal{M}_K$ .
- 3) For each triple of indices  $(m, l, k) \in \mathcal{M} \times \mathcal{M}_L \times \mathcal{M}_K$ , we draw the same number  $2^{n(R+R_L+R_K)}$  of sequences  $W_2^n(m, l, k, \hat{m}, \hat{l}, \hat{k})$  with indices  $(\hat{m}, \hat{l}, \hat{k}) \in \mathcal{M} \times \mathcal{M}_L \times \mathcal{M}_K$  according to the i.i.d. conditional distribution  $\mathcal{Q}_{W_2|W_1}$  depending on  $W_1^n(m, l, k)$ .

**Encoding function at the beginning of block  $b \in \{2, \dots, B-1\}$ .**

- 1) The encoder observes the sequence of channel states  $S_{b-1}^n$  corresponding to the block  $b-1$  and finds the indices  $(l_{b-1}, j_{b-1}) \in \mathcal{M}_L \times \mathcal{M}_J$  such that  $(S^n(l_{b-1}, j_{b-1}), S_{b-1}^n) \in T_\delta(\mathcal{P})$  for the distribution  $\mathcal{P}(s, \hat{s}) = \mathcal{P}(s)\mathbb{1}(\hat{s} = s) \forall (s, \hat{s}) \in S \times S$ .
- 2) The encoder observes the message  $m_b$  and the index  $l_{b-1}$  and recalls  $W_1^n(m_{b-1}, l_{b-2}, k_{b-1})$  corresponding to the block  $b-1$ . It finds the index  $k_b \in \mathcal{M}_K$  such that  $(S_{b-1}^n, W_1^n(m_{b-1}, l_{b-2}, k_{b-1}), W_2^n(m_{b-1}, l_{b-2}, k_{b-1}, m_b, l_{b-1}, k_b)) \in T_\delta(\mathcal{Q})$ .
- 3) The encoder sends  $X_b^n$  drawn from the i.i.d. conditional distribution  $\mathcal{Q}_{X|S_{W_1}}$  depending on  $W_1^n(m_b, l_{b-1}, k_b)$  and  $S_b^n$  observed causally on the current block  $b \in \{2, \dots, B-1\}$ .

**Decoding function at the end of block  $b \in \{2, \dots, B-1\}$ .**

- 1) The receiver recalls  $Y_{b-1}^n$  and the indices  $(m_{b-1}, l_{b-2}, k_{b-1})$  corresponding to  $W_1^n(m_{b-1}, l_{b-2}, k_{b-1})$  decoded at the end of the block  $b-1$ .
- 2) The receiver observes  $Y_b^n$  and finds the triple of indices  $(m_b, l_{b-1}, k_b)$  such that  $(Y_b^n, W_1^n(m_b, l_{b-1}, k_b)) \in T_\delta(\mathcal{Q})$  and  $(Y_{b-1}^n, W_1^n(m_{b-1}, l_{b-2}, k_{b-1}), W_2^n(m_{b-1}, l_{b-2}, k_{b-1}, m_b, l_{b-1}, k_b)) \in T_\delta(\mathcal{Q})$ .
- 3) The receiver returns the message  $m_b$  corresponding to block  $b$ .
- 4) The receiver returns  $V_{b-1}^n$  drawn from the conditional distribution  $\mathcal{Q}_{V|Y_{W_1}W_2}$  depending on  $(Y_{b-1}^n, W_1^n(m_{b-1}, l_{b-2}, k_{b-1}), W_2^n(m_{b-1}, l_{b-2}, k_{b-1}, m_b, l_{b-1}, k_b))$ .
- 5) The receiver knows that over block  $b-1$ , the sequences  $(S_{b-1}^n, W_1^n(m_{b-1}, l_{b-2}, k_{b-1}), W_2^n(m_{b-1}, l_{b-2}, k_{b-1}, m_b, l_{b-1}, k_b), X_{b-1}^n, Y_{b-1}^n, V_{b-1}^n) \in T_\delta(\mathcal{Q})$  and the sequence of states  $S_{b-1}^n$  belongs to the bin with index  $l_{b-1} \in \mathcal{M}_L$ .

**Initialization of the encoder.** Arbitrary indices  $(m_1, l_0, k_1)$  are given to both encoder and decoder. The encoder sends the sequence  $X_{b_1}^n$  drawn according to the conditional distribution  $\mathcal{Q}_{X|S_{W_1}}$  depending on  $(S_{b_1}^n, W_1^n(m_1, l_0, k_1))$ . At the beginning of the second block  $b_2$ , encoder recalls  $W_1^n(m_1, l_0, k_1)$ , observes the message  $m_2$ , finds the index  $l_1$  such that  $(S_{b_1}^n, S^n(l_1, j_1)) \in T_\delta(\mathcal{P})$  and finds the index  $k_2$  such that  $(S_{b_1}^n, W_1^n(m_1, l_0, k_1), W_2^n(m_1, l_0, k_1, m_2, l_1, k_2)) \in T_\delta(\mathcal{Q})$ . The encoder sends  $X_{b_2}^n$  drawn

from the conditional distribution  $\mathcal{Q}_{X|SW_1}$  depending on  $(S_{b_2}^n, W_1^n(m_2, l_1, k_2))$ . The index  $m_1$  does not correspond to an informational message.

**Initialization of the decoder.** At the end of second block  $b_2$ , the decoder finds the triple of indices  $(m_2, l_1, k_2)$  such that  $(Y_{b_2}^n, W_1^n(m_2, l_1, k_2)) \in T_\delta(\mathcal{Q})$  and  $(Y_{b_1}^n, W_1^n(m_1, l_0, k_1), W_2^n(m_1, l_0, k_1, m_2, l_1, k_2)) \in T_\delta(\mathcal{Q})$ . The decoder returns the message  $m_2$  corresponding to the block  $b_2$  and the sequence  $V_{b_1}^n \in \mathcal{V}^n$  drawn from the conditional probability  $\mathcal{Q}_{V|YW_1W_2}$  depending on  $(Y_{b_1}^n, W_1^n(m_1, l_0, k_1), W_2^n(m_1, l_0, k_1, m_2, l_1, k_2))$ . The decoder knows that over the first block  $b_1$ , the sequence  $S_{b_1}^n$  belongs to the bin  $l_1$ . The sequences  $(S_{b_1}^n, W_1^n(m_1, l_0, k_1), W_2^n(m_1, l_0, k_1, m_2, l_1, k_2), X_{b_1}^n, Y_{b_1}^n, V_{b_1}^n) \in T_\delta(\mathcal{Q})$  and  $(S_{b_2}^n, W_1^n(m_2, l_1, k_2), W_2^n(m_2, l_1, k_2, m_3, l_2, k_3), X_{b_2}^n, Y_{b_2}^n, V_{b_2}^n) \in T_\delta(\mathcal{Q})$ .

**Last block at the encoder.** At the beginning of the last block  $B$ , the encoder recalls  $W_1^n(m_{B-1}, l_{B-1}, k_{B-1})$ , observes message  $m_B$ , finds the index  $l_{B-1}$  such that  $(S_{B-1}^n, W_1^n(m_{B-1}, l_{B-1}, k_{B-1})) \in T_\delta(\mathcal{P})$ . It finds the index  $k_B$  such that  $(S_{B-1}^n, W_1^n(m_{B-1}, l_{B-2}, k_{B-1}), W_2^n(m_{B-1}, l_{B-2}, k_{B-1}, m_B, l_{B-1}, k_B)) \in T_\delta(\mathcal{Q})$ . The encoder sends the sequence  $X_B^n$  drawn from the conditional distribution  $\mathcal{Q}_{X|SW_1}$  depending on  $(S_B^n, W_1^n(m_B, l_{B-1}, k_B))$ .

**Last block at the decoder.** At the end of the last block  $B$ , the decoder finds the triple of indices  $(m_B, l_{B-1}, k_B)$  such that  $(Y_B^n, W_1^n(m_B, l_{B-1}, k_B)) \in T_\delta(\mathcal{Q})$  and  $(Y_{B-1}^n, W_1^n(m_{B-1}, l_{B-2}, k_{B-1}), W_2^n(m_{B-1}, l_{B-2}, k_{B-1}, m_B, l_{B-1}, k_B)) \in T_\delta(\mathcal{Q})$ . The decoder returns the message  $m_B$  corresponding to the last block  $B$  and the sequence  $V_{B-1}^n \in \mathcal{V}^n$  drawn from the conditional probability  $\mathcal{Q}_{V|YW_1W_2}$  depending on  $(Y_{B-1}^n, W_1^n(m_{B-1}, l_{B-2}, k_{B-1}), W_2^n(m_{B-1}, l_{B-2}, k_{B-1}, m_B, l_{B-1}, k_B))$ . The decoder knows that over the block  $B - 1$ , the sequence  $S_{B-1}^n$  belongs to the bin  $l_{B-1}$ . Then  $(S_{B-1}^n, W_1^n(m_{B-1}, l_{B-2}, k_{B-1}), W_2^n(m_{B-1}, l_{B-2}, k_{B-1}, m_B, l_{B-1}, k_B), X_{B-1}^n, Y_{B-1}^n, V_{B-1}^n) \in T_\delta(\mathcal{Q})$  but  $(S_B^n, W_1^n(m_B, l_{B-1}, k_B), W_2^n(m_B, l_{B-1}, k_B, m_{B+1}, l_B, k_{B+1}), X_B^n, Y_B^n, V_B^n) \notin T_\delta(\mathcal{Q})$  on the last block  $B$ . The decoder does not know the index  $l_B$  of the bin corresponding to sequence  $S_B^n$ .

In the following, we introduce the notation  $W_{1,b}^n = W_1^n(m_b, l_{b-1}, k_b)$  and  $W_{2,b}^n = W_2^n(m_b, l_{b-1}, k_b, m_{b+1}, l_b, k_{b+1})$ , with  $b \in \{1, \dots, B-1\}$ . If there is no error in the coding scheme, the messages  $(m_2, \dots, m_B)$  are correctly decoded and the decoder knows the bin indices  $(l_1, \dots, l_{B-1})$  of the sequences  $(S_1^n, \dots, S_{B-1}^n)$ , and  $(S_b^n, W_{1,b}^n, W_{2,b}^n, X_b^n, Y_b^n, V_b^n) \in T_\delta(\mathcal{Q})$ , for each blocks  $b \in \{1, \dots, B-1\}$ .

### B. Rate parameters $R$ , $R_L$ and $R_K$

1) At the end of block  $b \in \{2, \dots, B\}$ , the decoder observes  $(Y_{b-1}^n, Y_b^n)$  and decodes  $(W_{1,b-1}^n, W_{2,b-1}^n)$  corresponding to the block  $b - 1$ . Intuitively, the observation of  $(W_{1,b-1}^n, W_{2,b-1}^n, Y_{b-1}^n)$  leaks  $nI(S; W_1, W_2, Y) = nI(S; W_2, Y|W_1)$  bits of information regarding  $S_{b-1}^n$ . By fixing the rate parameter

$R_L = E - I(S; W_1, W_2, Y)$ , the encoder will transmit  $nR_L = n(E - I(S; W_1, W_2, Y))$  additional bits of information corresponding to the state sequence  $S_{b-1}^n$ . As it will be proven in the Section A-E, the leakage rate  $I(S_{b-1}^n; Y_{b-1}^n)$  over block  $b-1$  is close to  $n(I(S; W_1, W_2, Y) + E - I(S; W_1, W_2, Y)) = nE$ . We fix the rate parameter  $R_L$  equal to:

$$R_L = E - I(S; W_1, W_2, Y) - 2\varepsilon \geq 0. \quad (43)$$

The first inequality  $I(S; W_1, W_2, Y) \leq E$  in (41) implies there exist a positive rate parameter  $R_L$ . In case of equality  $E = I(S; W_1, W_2, Y)$ , then the rate  $R_L = 0$  and no index  $l \in \mathcal{M}_L$  is transmitted to the decoder.

2) The rates parameters  $R_L, R_J$  corresponding to the indices  $(l_{b-1}, j_{b-1})$ , guarantee that almost every sequences  $S_{b-1}^n$  appear in the codebook.

$$R_L + R_J = H(S) + \varepsilon, \quad (44)$$

$$\implies R_J = H(S) - E + I(S; W_1, W_2, Y) + 3\varepsilon. \quad (45)$$

The second inequality  $E \leq H(S)$  in (41) implies there exist a positive rate parameter  $R_J$ .

3) The rates parameter  $R_K$  corresponding to the index  $k_b$ , is used by the encoder in order to correlate  $(W_1^n(m_{b-1}, l_{b-2}, k_{b-1}), W_2^n(m_{b-1}, l_{b-2}, k_{b-1}, m_b, l_{b-1}, k_b))$  with  $S_{b-1}^n$ .

$$R_K = I(W_2; S|W_1) + \varepsilon, \quad (46)$$

Since the random variables  $W_1$  and  $S$  are independent, we have  $I(W_1, W_2; S) = I(W_2; S|W_1)$ .

4) The rate parameters  $R, R_L, R_K$  are correctly decoded if

$$R + R_L + R_K \leq I(W_1; Y) + I(W_2; Y|W_1) - \varepsilon \quad (47)$$

$$\iff R + E - I(S; W_1, W_2, Y) - 2\varepsilon + I(W_2; S|W_1) + \varepsilon \leq I(W_1, W_2; Y) - \varepsilon \quad (48)$$

$$\iff R + E \leq I(W_1, W_2, S; Y) \quad (49)$$

$$\iff R + E \leq I(W_1, S; Y), \quad (50)$$

where (49) comes from the independence between  $W_1$  and  $S$ ; (50) comes from the Markov chain  $Y \circlearrowleft (W_1, S) \circlearrowleft W_2$  stated in Remark II.4.

Equation (42) implies that for each block  $b \in \{2, \dots, B\}$ , the indices with rates  $R, R_L, R_K$  are recovered by the decoder, with large probability. Hence the rate of the total code of length  $N = nB$  is given by

$$\frac{1}{nB} \sum_{b=2}^B \log_2 |\mathcal{M}| = \frac{B-1}{B} R = R - \frac{1}{B} R \geq R - \frac{1}{B} \log_2 |\mathcal{Y}| \geq R - \varepsilon. \quad (51)$$

The last equation is satisfied when the number of blocks is sufficiently large such that  $\frac{1}{B} \log_2 |\mathcal{Y}| \leq \varepsilon$ .

C. *Expected error probability by block*

For each block  $b \in \{2, \dots, B\}$ , we consider the expected probability of the following error events. The properties of the typical sequences, see [34, pp. 27], implies that for all  $\varepsilon > 0$  there exist  $n_1 \in \mathbb{N}$  such that for all  $n \geq n_1$ , the expected probability of the error event is bounded by

$$\mathbb{E}_c \left[ \mathbb{P} \left( S_{b-1}^n \notin T_\delta(\mathcal{Q}) \right) \right] \leq \varepsilon. \quad (52)$$

From the covering Lemma, see [34, pp. 62],

$$\mathbf{R}_L + \mathbf{R}_J \geq H(S) + \varepsilon$$

implies that  $\forall \varepsilon > 0, \exists n_2 \in \mathbb{N}$  such that  $\forall n \geq n_2$ ,

$$\mathbb{E}_c \left[ \mathbb{P} \left( \forall (L_{b-1}, J_{b-1}) \in \mathcal{M}_L \times \mathcal{M}_J, (S^n(L_{b-1}, J_{b-1}), S_{b-1}^n) \notin T_\delta(\mathcal{Q}) \right) \right] \leq \varepsilon. \quad (53)$$

From the covering Lemma, see [34, pp. 62],

$$\mathbf{R}_K \geq I(W_2; S|W_1) + \varepsilon$$

implies that  $\forall \varepsilon > 0, \exists n_3 \in \mathbb{N}$  such that  $\forall n \geq n_3$ ,

$$\mathbb{E}_c \left[ \mathbb{P} \left( \forall K_b \in \mathcal{M}_K, (S_{b-1}^n, W_1^n(M_{b-1}, L_{b-2}, K_{b-1}), W_2^n(M_{b-1}, L_{b-2}, K_{b-1}, M_b, L_{b-1}, K_b)) \notin T_\delta(\mathcal{Q}) \right) \right] \leq \varepsilon. \quad (54)$$

From the packing Lemma, see [34, pp. 46],

$$\mathbf{R} + \mathbf{R}_L + \mathbf{R}_K \leq I(W_1; Y) + I(W_2; Y|W_1) - \varepsilon$$

implies that  $\forall \varepsilon > 0, \exists n_4 \in \mathbb{N}$  such that  $\forall n \geq n_4$ ,

$$\mathbb{E}_c \left[ \mathbb{P} \left( \exists (M_b, L_{b-1}, K_b) \neq (M'_b, L'_{b-1}, K'_b), \text{ s.t. } \left\{ (Y_b^n, W_1^n(M'_b, L'_{b-1}, K'_b)) \in T_\delta(\mathcal{Q}) \right\} \cap \right. \right. \quad (55)$$

$$\left. \left. \left\{ (Y_{b-1}^n, W_1^n(M_{b-1}, L_{b-2}, K_{b-1}), W_2^n(M_{b-1}, L_{b-2}, K_{b-1}, M'_b, L'_{b-1}, K'_b)) \in T_\delta(\mathcal{Q}) \right\} \right) \right] \leq \varepsilon. \quad (56)$$

Thus for each block  $b \in \{2, \dots, B\}$  and for all  $n \geq \bar{n} \geq \max(n_1, n_2, n_3, n_4)$ , the expected probability of non-decoding the indices  $(M_b, L_{b-1}, K_b)$  is bounded by:

$$\mathbb{E}_c \left[ \mathbb{P} \left( (M_b, L_{b-1}, K_b) \neq (\hat{M}_b, \hat{L}_{b-1}, \hat{M}_b) \right) \right] \leq 4\varepsilon. \quad (57)$$

#### D. Expected error probability of the block-Markov code

We evaluate the expected probability of error for the random indices  $(M_b, L_{b-1}, K_b)$ , for  $b \in \{2, \dots, B\}$  of the block-Markov random code:

$$\mathbb{E}_c \left[ \mathbb{P} \left( \left( M_2, L_1, K_2, \dots, M_B, L_{B-1}, K_B \right) \neq \left( \hat{M}_2, \hat{L}_1, \hat{K}_2, \dots, \hat{M}_B, \hat{L}_{B-1}, \hat{K}_B \right) \right) \right] \quad (58)$$

$$= 1 - \mathbb{E}_c \left[ \mathbb{P} \left( (M_2, L_1, K_2) = (\hat{M}_2, \hat{L}_1, \hat{K}_2) \right) \right] \times \dots$$

$$\times \mathbb{E}_c \left[ \mathbb{P} \left( (M_B, L_{B-1}, K_B) = (\hat{M}_B, \hat{L}_{B-1}, \hat{K}_B) \mid \left\{ (M_2, L_1, K_2) = (\hat{M}_2, \hat{L}_1, \hat{K}_2) \right\} \right) \right] \quad (59)$$

$$\cap \dots \cap \left\{ (M_{B-1}, L_{B-2}, K_{B-1}) = (\hat{M}_{B-1}, \hat{L}_{B-2}, \hat{K}_{B-1}) \right\} \right] \leq 1 - \left( 1 - 4\varepsilon \right)^{B-1}. \quad (60)$$

We denote by  $\tilde{Q}^N \in \Delta(\mathcal{S} \times \mathcal{X} \times \mathcal{Y} \times \mathcal{V})$ , the empirical distribution of symbols over every blocks  $b \in \{1, \dots, B-1\}$  removing the last block. We show  $\tilde{Q}^N$  is close to the empirical distribution  $Q^N$  over all the  $B$  blocks, for a number of blocks  $B \in \mathbb{N}$  sufficiently large, *i.e.* for which  $\frac{2}{B}|\mathcal{S} \times \mathcal{X} \times \mathcal{Y} \times \mathcal{V}| \leq \varepsilon$ . We denote by  $Q_B$ , the empirical distribution of symbols over the last block.

$$\left\| Q^N - \tilde{Q}^N \right\|_1 = \left\| \frac{1}{B} \left( (B-1)\tilde{Q}^N + Q_B \right) - \tilde{Q}^N \right\|_1 \quad (61)$$

$$= \frac{1}{B} \left\| Q_B - \tilde{Q}^N \right\|_1 \leq \frac{2}{B} |\mathcal{S} \times \mathcal{X} \times \mathcal{Y} \times \mathcal{V}| \leq \varepsilon. \quad (62)$$

Then, the expected probability that  $(S^N, W_1^N, W_2^N, X^N, Y^N, V^N) \notin T_\delta(\mathcal{Q})$ , is upper bounded by:

$$\begin{aligned} & \mathbb{E}_c \left[ \mathbb{P} \left( \left\| Q^N - \mathcal{Q} \right\|_1 > 2\varepsilon \right) \right] = \mathbb{E}_c \left[ \mathbb{P} \left( \left\| Q^N - \tilde{Q}^N + \tilde{Q}^N - \mathcal{Q} \right\|_1 > 2\varepsilon \right) \right] \\ & \leq \mathbb{E}_c \left[ \mathbb{P} \left( \left\| Q^N - \tilde{Q}^N \right\| + \left\| \tilde{Q}^N - \mathcal{Q} \right\|_1 > 2\varepsilon \right) \right] \leq \mathbb{E}_c \left[ \mathbb{P} \left( \left\| \tilde{Q}^N - \mathcal{Q} \right\|_1 > 2\varepsilon - \frac{2}{B} |\mathcal{S} \times \mathcal{X} \times \mathcal{Y} \times \mathcal{V}| \right) \right] \\ & \leq \mathbb{E}_c \left[ \mathbb{P} \left( \left\| \tilde{Q}^N - \mathcal{Q} \right\|_1 > \varepsilon \right) \right] \leq 1 - \left( 1 - 4\varepsilon \right)^{B-1}. \end{aligned} \quad (63)$$

Hence, we obtain the following bound on the expected error probability:

$$\mathbb{E}_c \left[ \mathcal{P}_e(c) \right] = \mathbb{E}_c \left[ \mathbb{P} \left( M \neq \hat{M} \right) + \mathbb{P} \left( \left\| Q^N - \mathcal{Q} \right\|_1 > \varepsilon \right) \right] \leq 2 - 2 \left( 1 - 4\varepsilon \right)^{B-1}. \quad (64)$$

This implies that for all  $\varepsilon > 0$ , there exist a code  $c^* \in \mathcal{C}(N)$  with  $N \geq B\bar{n}$  such that  $\mathcal{P}_e(c^*) \leq 2 - 2 \left( 1 - 4\varepsilon \right)^{B-1}$ .

#### E. Expected state leakage rate

In this section, we provide upper and lower bounds on the expected state leakage rate, depending on the parameters  $\varepsilon_1$  and  $\varepsilon_2$  given by equations (105) and (106).

$$\mathbf{E} - \varepsilon_1 - \varepsilon_2 \leq \mathbb{E}_c \left[ \mathcal{L}_e(c) \right] = \mathbb{E}_c \left[ \frac{1}{nB} I(S^{nB}; Y^{nB} | C = c) \right] = \frac{1}{nB} I(S^{nB}; Y^{nB} | C) \leq \mathbf{E} + \varepsilon_1 + \varepsilon_2. \quad (65)$$

The notation  $S^{nB}$  denotes the sequence of random variables of channel states of length  $N = nB$ , whereas  $S_b^n$  denotes the sub-sequence of length  $n \in \mathbb{N}$  over the block  $b \in \{1, \dots, B\}$ .

**Upper bound.** We provide an upper bound on the expected state leakage rate by using the chain rule from one block to another.

$$I(S^{nB}; Y^{nB} | C) \leq \sum_{b=b_1}^{B-1} I(S_b^n; Y^{nB} | S_{b+1}^n, \dots, S_B^n, C) + n \log_2 |\mathcal{S}| \quad (66)$$

$$\leq \sum_{b=b_1}^{B-1} I(S_b^n; Y^{nB}, W_{1,b}^n, W_{2,b}^n, L_b, M_{b+1}, S_{b+1}^n, \dots, S_B^n | C) + n \log_2 |\mathcal{S}| \quad (67)$$

$$\begin{aligned} &= \sum_{b=b_1}^{B-1} I(S_b^n; W_{1,b}^n, W_{2,b}^n, Y_b^n, L_b, M_{b+1} | C) + n \log_2 |\mathcal{S}| \\ &+ \sum_{b=b_1}^{B-1} I(S_b^n; S_{b+1}^n, \dots, S_B^n | W_{1,b}^n, W_{2,b}^n, Y_b^n, L_b, M_{b+1}, C) \end{aligned} \quad (68)$$

$$+ \sum_{b=b_1}^{B-1} I(S_b^n; Y^{nB} | W_{1,b}^n, W_{2,b}^n, Y_b^n, L_b, M_{b+1}, S_{b+1}^n, \dots, S_B^n, C) \quad (69)$$

$$= \sum_{b=b_1}^{B-1} I(S_b^n; W_{1,b}^n, W_{2,b}^n, L_b, M_{b+1} | C) + \sum_{b=b_1}^{B-1} I(S_b^n; Y_b^n | W_{1,b}^n, W_{2,b}^n, L_b, M_{b+1}, C) + n \log_2 |\mathcal{S}|. \quad (70)$$

The term at line (68) is equal to zero since the causal encoding and the i.i.d. property of the channel state, imply that the random variables  $(S_{b+1}^n, \dots, S_B^n)$  are independent of the message  $M_{b+1}$  and of the random variables  $(S_b^n, W_{1,b}^n, W_{2,b}^n, Y_b^n, L_b)$  of the block  $b$ , and the term at line (69) is equal to zero since in the encoding process, the sequence  $S_b^n$  only affects the choice of the bin index  $L_b$  and of the sequences  $(W_{1,b}^n, W_{2,b}^n, Y_b^n)$  of the current block  $b$ . This induces the following Markov chain:  $S_b^n \text{---} (W_{1,b}^n, W_{2,b}^n, Y_b^n, L_b, C) \text{---} (Y^{nB}, M_{b+1}, S_{b+1}^n, \dots, S_B^n)$ , that is valid for each block  $b \in \{1, \dots, B-1\}$ . The sequence  $S_b^n$  is correlated with the random variables of the other blocks  $b' \neq b$  only through  $(W_{1,b}^n, W_{2,b}^n, Y_b^n, L_b, C)$ .

For each block  $b \in \{1, \dots, B-1\}$ , the first term in equation (70) satisfies:

$$I(S_b^n; W_{1,b}^n, W_{2,b}^n, L_b, M_{b+1} | C) = I(S_b^n; W_{2,b}^n, L_b | W_{1,b}^n, M_{b+1}, C) \quad (71)$$

$$\leq H(W_{2,b}^n, L_b | W_{1,b}^n, M_{b+1}, C) \leq \log_2 |\mathcal{M}_L| + H(W_{2,b}^n | W_{1,b}^n, L_b, M_{b+1}, C) \quad (72)$$

$$\leq \log_2 |\mathcal{M}_L| + \log_2 |\mathcal{M}_K| = n \left( \mathbf{E} - I(S; W_1, W_2, Y) - 2\varepsilon + I(S; W_2 | W_1) + \varepsilon \right) \quad (73)$$

$$= n \left( \mathbf{E} - I(S; Y | W_1, W_2) - \varepsilon \right), \quad (74)$$

where (71) comes from the causal encoding that induces the independence between the sequence auxiliary random variables  $W_{1,b}^n$  and the sequence of channel states  $S_b^n$ , in block  $b \in \{1, \dots, B-1\}$ . Hence  $S_b^n$

is independent of  $(W_{1,b}^n, M_{b+1}, C)$ ; (72) comes from the cardinality of the set of indices  $\mathcal{M}_L$ ; (73) comes from the coding scheme described in Appendix A. By considering a fixed sequence  $W_{1,b}^n$  and fixed indices  $(L_b, M_{b+1})$ , the encoder chooses an index  $K_{b+1} \in \mathcal{M}_K$  corresponding to the sequence  $W_{2,b}^n$ . Hence, the sequence  $W_{2,b}^n$  belongs to the bin of cardinality  $|\mathcal{M}_K|$ . The rate parameters are given by  $\mathbf{R}_L = \mathbf{E} - I(S; W_1, W_2, Y) - 2\varepsilon$  and  $\mathbf{R}_K = I(W_2; S|W_1) + \varepsilon$ ; (74) comes from the independence between  $W_1$  and  $S$  that induces  $I(S; W_2|W_1) = I(S; W_1, W_2)$ .

For each block  $b \in \{1, \dots, B-1\}$ , we introduce the random event of error  $E_b \in \{0, 1\}$  defined with respect to the achievable distribution  $\mathcal{Q}_{SXW_1W_2YV}$ , as follows:

$$E_b = \begin{cases} 0 & \text{if } (S_b^n, X_b^n, W_{1,b}^n, W_{2,b}^n, Y_b^n, V_b^n) \in T_\delta(\mathcal{Q}) \text{ and } (\hat{M}_{b+1}, \hat{L}_b, \hat{K}_{b+1}) = (M_{b+1}, L_b, K_{b+1}), \\ 1 & \text{if } (S_b^n, X_b^n, W_{1,b}^n, W_{2,b}^n, Y_b^n, V_b^n) \notin T_\delta(\mathcal{Q}) \text{ or } (\hat{M}_{b+1}, \hat{L}_b, \hat{K}_{b+1}) \neq (M_{b+1}, L_b, K_{b+1}). \end{cases} \quad (75)$$

The second term in equation (70) satisfies:

$$\begin{aligned} & I(S_b^n; Y_b^n | W_{1,b}^n, W_{2,b}^n, L_b, M_{b+1}, C) \\ &= H(Y_b^n | W_{1,b}^n, W_{2,b}^n, L_b, M_{b+1}, C) - H(Y_b^n | S_b^n, W_{1,b}^n, W_{2,b}^n, L_b, M_{b+1}, C) \end{aligned} \quad (76)$$

$$= H(Y_b^n | W_{1,b}^n, W_{2,b}^n, L_b, M_{b+1}, C) - nH(Y | W_1, W_2, S) \quad (77)$$

$$\leq H(Y_b^n | W_{1,b}^n, W_{2,b}^n, L_b, M_{b+1}, C, E_b = 0) + 1 + \mathbb{P}(E_b = 1)n \log_2 |\mathcal{Y}| - nH(Y | W_1, W_2, S) \quad (78)$$

$$\leq n \left( H(Y | W_1, W_2) + \varepsilon \right) + 1 + \mathbb{P}(E_b = 1)n \log_2 |\mathcal{Y}| - nH(Y | W_1, W_2, S) \quad (79)$$

$$= n \left( I(S; Y | W_1, W_2) + \varepsilon + \frac{1}{n} + \mathbb{P}(E_b = 1) \log_2 |\mathcal{Y}| \right), \quad (80)$$

where (76) comes from the properties of the mutual information; (77) comes from the cascade of memoryless channels  $\mathcal{Q}_{X|W_1S} \mathcal{T}_{Y|XS}$  of the coding scheme  $C$  described in Appendix A, that implies  $H(Y_b^n | S_b^n, W_{1,b}^n, W_{2,b}^n, L_b, M_{b+1}, C) = nH(Y | W_1, S) = nH(Y | W_1, W_2, S)$ ; (78) is inspired by the proof of Fano's inequality, see [34, pp. 19]; (79) comes from the cardinality bound for the set of sequences  $y^n$  such that  $(w_1^n, w_2^n, y^n) \in T_\delta(\mathcal{Q})$ , see [34, pp. 26]; (80) comes from the properties of the mutual information.

Hence we have the following upper bound on the leakage rate:

$$\begin{aligned} nB\mathbb{E}_c \left[ \mathcal{L}_e(c) \right] &\leq \sum_{b=b_1}^{B-1} I(S_b^n; W_{1,b}^n, W_{2,b}^n, L_b, M_{b+1} | C) + \sum_{b=b_1}^{B-1} I(S_b^n; Y_b^n | W_{1,b}^n, W_{2,b}^n, L_b, M_{b+1}, C) + n \log_2 |\mathcal{S}| \\ &\leq nB \left( \mathbf{E} - I(S; Y | W_1, W_2) - \varepsilon + I(S; Y | W_1, W_2) + \varepsilon + \frac{1}{n} + \mathbb{P}(E_b = 1) \log_2 |\mathcal{Y}| + \frac{1}{B} \log_2 |\mathcal{S}| \right) \\ &\leq nB \left( \mathbf{E} + \frac{1}{n} + \mathbb{P}(E_b = 1) \log_2 |\mathcal{Y}| + \frac{1}{B} \log_2 |\mathcal{S}| \right). \end{aligned}$$

**Lower bound.** We provide a lower bound on the expected state leakage rate.

$$nB\mathbb{E}_c\left[\mathcal{L}_e(c)\right] = I(S^{nB}; Y^{nB}|C) = nBH(S) - H(S^{nB}|Y^{nB}, C) \quad (81)$$

$$\geq nBH(S) - \mathbb{P}(E_b = 1)nB\log_2|\mathcal{S}| - 1 - H(S^{nB}|Y^{nB}, C, E_b = 0) \quad (82)$$

$$\geq nBH(S) - \mathbb{P}(E_b = 1)nB\log_2|\mathcal{S}| - 1 - \sum_{b=b_1}^{B-1} H(S_b^n|Y^{nB}, S_{b+1}^n, \dots, S_B^n, C, E_b = 0) - n\log_2|\mathcal{S}| \quad (83)$$

$$\begin{aligned} &= nBH(S) - \left(\mathbb{P}(E_b = 1)B + 1\right)n\log_2|\mathcal{S}| - 1 \\ &\quad - \sum_{b=b_1}^{B-1} H(S_b^n|W_{1,b}^n, W_{2,b}^n, L_b, Y^{nB}, S_{b+1}^n, \dots, S_B^n, C, E_b = 0) \\ &\quad - \sum_{b=b_1}^{B-1} I(S_b^n; W_{1,b}^n, W_{2,b}^n, L_b|Y^{nB}, S_{b+1}^n, \dots, S_B^n, C, E_b = 0) \end{aligned} \quad (84)$$

$$= nBH(S) - \left(\mathbb{P}(E_b = 1)B + 1\right)n\log_2|\mathcal{S}| - 1 - \sum_{b=b_1}^{B-1} H(S_b^n|W_{1,b}^n, W_{2,b}^n, L_b, Y^{nB}, S_{b+1}^n, \dots, S_B^n, C, E_b = 0) \quad (85)$$

$$\geq nBH(S) - \left(\mathbb{P}(E_b = 1)B + 1\right)n\log_2|\mathcal{S}| - 1 - \sum_{b=b_1}^{B-1} H(S_b^n|W_{1,b}^n, W_{2,b}^n, L_b, Y_b^n, E_b = 0), \quad (86)$$

where (81) comes from the i.i.d. property of the channel states  $S$ ; (82) is inspired by the proof of Fano's inequality, see [34, pp. 19]; (85) comes from the non-error event  $E_b = 0$ , that implies for all block  $b \in \{1, \dots, B-1\}$ , the sequences  $(W_{1,b}^n, W_{2,b}^n, L_b)$  are correctly decoded based on the observation of  $Y^{nB}$ , hence  $I(S_b^n; W_{1,b}^n, W_{2,b}^n, L_b|Y^{nB}, S_{b+1}^n, \dots, S_B^n, C, E_b = 0) = 0$ ; (86) comes from removing the conditioning over the sequences  $(Y_{b_1}^n, \dots, Y_{b-1}^n, Y_{b+1}^n, \dots, Y_B^n, S_{b+1}^n, \dots, S_B^n, C)$  and the random code in the conditional entropy  $H(S_b^n|W_{1,b}^n, W_{2,b}^n, L_b, Y_b^n, E_b = 0)$ .

In order to provide an upper bound on  $H(S_b^n|W_{1,b}^n, W_{2,b}^n, L_b, Y_b^n, E_b = 0)$ , we fix an index  $l \in \mathcal{M}_L$ , some typical sequences  $(w_1^n, w_2^n, y^n) \in T_\delta(\mathcal{Q})$ . We define the set

$$\mathcal{S}^*(w_1^n, w_2^n, y^n, l) = \left\{ s^n \in \mathcal{S}^n, \text{ s.t. } \left\{ (s^n, w_1^n, w_2^n, y^n) \in T_\delta(\mathcal{Q}) \right\} \cap \left\{ s^n \in \mathcal{B}(l) \right\} \right\} \quad (87)$$

Since the code  $C$  is random, the above set  $\mathcal{S}^*(w_1^n, w_2^n, y^n, l)$  is also random and we have

$$\begin{aligned} &\mathbb{E}_c \left[ \left| \mathcal{S}^*(W_1^n, W_2^n, Y^n, L) \right| \right] \\ &= \mathbb{E}_c \left[ \left| \left\{ s^n \in \mathcal{S}^n, \text{ s.t. } \left\{ (s^n, W_1^n, W_2^n, Y^n) \in T_\delta(\mathcal{Q}) \right\} \cap \left\{ s^n \in \mathcal{B}(L) \right\} \right\} \right| \right] \end{aligned} \quad (88)$$

$$= \sum_{S^n \in T_\delta(\mathcal{Q})(W_1^n, W_2^n, Y^n)} \mathbb{E}_c \left[ \mathbf{1}(S^n \in \mathcal{B}(L)) \right] \quad (89)$$



$$\leq \sum_{S^n \in T_\delta(\mathcal{Q})(W_1^n, W_2^n, Y^n)} 2^{-n \cdot R_L} \leq 2^{n \cdot (H(S|W_1, W_2, Y) - R_L + \varepsilon)}, \quad (90)$$

where (90) comes from the definition of the random code that induces a uniform distribution over the bins  $\mathcal{B}(l)$  and the properties of the typical sequences, see [34, pp. 27].

By Markov's inequality, for all  $l \in \mathcal{M}_L$  and for all  $(w_1^n, w_2^n, y^n) \in T_\delta(\mathcal{Q})$  we have

$$\mathbb{P} \left[ \left| \mathcal{S}^*(W_1^n, W_2^n, Y^n, L) \right| \geq 2^{n \cdot (H(S|W_1, W_2, Y) - R_L + 2\varepsilon)} \right] \leq \frac{\mathbb{E}_c \left[ \left| \mathcal{S}^*(W_1^n, W_2^n, Y^n, L) \right| \right]}{2^{n \cdot (H(S|W_1, W_2, Y) - R_L + 2\varepsilon)}} \quad (91)$$

$$\leq \frac{2^{n \cdot (H(S|W_1, W_2, Y) - R_L + \varepsilon)}}{2^{n \cdot (H(S|W_1, W_2, Y) - R_L + 2\varepsilon)}} \leq 2^{-n\varepsilon} \leq \varepsilon. \quad (92)$$

The last equation is valid for  $n \geq n_5 = \frac{1}{\varepsilon} \log_2 \frac{1}{\varepsilon}$ . For each block  $b \in \{1, \dots, B-1\}$  we define the event

$$F_b = \begin{cases} 1 & \text{if } \left| \mathcal{S}^*(W_{1,b}^n, W_{2,b}^n, Y_b^n, L_b) \right| \geq 2^{n \cdot (H(S|W_1, W_2, Y) - R_L + 2\varepsilon)}, \\ 0 & \text{if } \left| \mathcal{S}^*(W_{1,b}^n, W_{2,b}^n, Y_b^n, L_b) \right| < 2^{n \cdot (H(S|W_1, W_2, Y) - R_L + 2\varepsilon)}. \end{cases} \quad (93)$$

Then for each block  $b \in \{b_1, \dots, B-1\}$  we have

$$\begin{aligned} & H(S_b^n | W_{1,b}^n, W_{2,b}^n, Y_b^n, L_b, E_b = 0) \\ & \leq 1 + \mathbb{P}(F_b = 1) n \log_2 |\mathcal{S}| + H(S_b^n | W_{1,b}^n, W_{2,b}^n, Y_b^n, L_b, E_b = 0, F_b = 0) \end{aligned} \quad (94)$$

$$\leq 1 + \varepsilon \log_2 |\mathcal{S}| + H(S_b^n | W_{1,b}^n, W_{2,b}^n, Y_b^n, L_b, E_b = 0, F_b = 0) \quad (95)$$

$$\leq 1 + \varepsilon \log_2 |\mathcal{S}| + \log_2 2^{n \cdot (H(S|W_1, W_2, Y) - R_L + 2\varepsilon)} \quad (96)$$

$$= n \cdot \left( H(S|W_1, W_2, Y) - R_L + \frac{1}{n} + \varepsilon \log_2 |\mathcal{S}| + 2\varepsilon \right) \quad (97)$$

$$= n \cdot \left( H(S|W_1, W_2, Y) - \mathbf{E} + I(S; W_1, W_2, Y) + 2\varepsilon + \frac{1}{n} + \varepsilon \log_2 |\mathcal{S}| + 2\varepsilon \right) \quad (98)$$

$$= n \cdot \left( H(S) - \mathbf{E} + \frac{1}{n} + \varepsilon \log_2 |\mathcal{S}| + 4\varepsilon \right), \quad (99)$$

where (94) is inspired by the proof of Fano's inequality, see [34, pp. 19]; (95) comes from equation (92), that corresponds to  $\mathbb{P}(F_b = 1) \leq \varepsilon$ ; (96) comes from the definition of event  $F_b = 0$ , that implies the set  $\mathcal{S}^*(w_1^n, w_2^n, y^n, l)$  has cardinality bounded by  $2^{n \cdot (H(S|W_1, W_2, Y) - R_L + 2\varepsilon)}$ ; (98) comes from the definition of the rate  $R_L = \mathbf{E} - I(S; W_1, W_2, Y) - 2\varepsilon$  stated in equation (43).

This provides the following lower bound:

$$\begin{aligned} & nB \mathbb{E}_c \left[ \mathcal{L}_e(c) \right] \\ & \geq nBH(S) - \left( \mathbb{P}(E_b = 1)B + 1 \right) n \log_2 |\mathcal{S}| - 1 - \sum_{b=b_1}^{B-1} H(S_b^n | W_{1,b}^n, W_{2,b}^n, Y_b^n, E_b = 0) \end{aligned} \quad (100)$$

$$\geq nBH(S) - \left( \mathbb{P}(E_b = 1)B + 1 \right) n \log_2 |\mathcal{S}| - 1 - (B-1)n \left( H(S) - \mathbf{E} + \frac{1}{n} + \varepsilon \log_2 |\mathcal{S}| + 4\varepsilon \right) \quad (101)$$

$$\geq nBH(S) - \left(\mathbb{P}(E_b = 1)B + 1\right)n \log_2 |\mathcal{S}| - 1 - Bn \left(H(S) - \mathbf{E} + \frac{1}{n} + \varepsilon \log_2 |\mathcal{S}| + 4\varepsilon\right) - n \log_2 |\mathcal{S}| \quad (102)$$

$$\geq nB \left(\mathbf{E} - \left(\mathbb{P}(E_b = 1) + \varepsilon + \frac{2}{B}\right) \log_2 |\mathcal{S}| - \frac{2}{nB} - 4\varepsilon\right), \quad (103)$$

where (100) comes from equation (86); (101) comes from equation (99); (102) comes the lower bound:

$$n \left(H(S) - \mathbf{E} + \frac{1}{n} + \varepsilon \log_2 |\mathcal{S}| + 4\varepsilon\right) \geq -n\mathbf{E} \geq -n \log_2 |\mathcal{S}|. \quad (104)$$

Equation (103) provides the lower bound on the expected state leakage rate.

**Conclusion for Exact state leakage Rate:**

$$\mathbb{E}_c \left[ \mathcal{L}_e(c) \right] \leq \mathbf{E} + \frac{1}{n} + \max_b \mathbb{P}(E_b = 1) \log_2 |\mathcal{Y}| + \frac{1}{B} \log_2 |\mathcal{S}| = \mathbf{E} + \varepsilon_2, \quad (105)$$

$$\mathbb{E}_c \left[ \mathcal{L}_e(c) \right] \geq \mathbf{E} - \left( \max_b \mathbb{P}(E_b = 1) + \varepsilon + \frac{2}{B} \right) \log_2 |\mathcal{S}| - \frac{2}{nB} - 4\varepsilon = \mathbf{E} - \varepsilon_1. \quad (106)$$

By defining  $\varepsilon_1$  and  $\varepsilon_2$  as in (106) and (105), we have

$$\left| \mathbb{E}_c \left[ \mathcal{L}_e(c) \right] - \mathbf{E} \right| \leq \varepsilon_1 + \varepsilon_2. \quad (107)$$

*F. Conclusion of the achievability proof of Theorem II.3*

By choosing appropriately the number of blocks  $B$ , the error probability  $\max_b \mathbb{P}(E_b = 1)$ , the block-length  $n$  and the tolerance for the typical sequences  $\delta$ , we show the existence of a code  $c^* \in \mathcal{C}(nB, \mathcal{M})$  such that

$$\frac{1}{nB} \sum_{b=2}^B \log_2 |\mathcal{M}| \geq \mathbf{R} - \varepsilon_3, \quad (108)$$

$$\mathcal{P}_e(c^*) \leq \varepsilon_3, \quad (109)$$

$$\left| \mathcal{L}_e(c^*) - \mathbf{E} \right| \leq \varepsilon_3, \quad (110)$$

with  $\varepsilon_3 = \frac{1}{B} \log_2 |\mathcal{Y}| + 2 - 2 \left(1 - 4\varepsilon\right)^{B-1} + \varepsilon_1 + \varepsilon_2$ . This concludes the achievability proof of Theorem II.3.

APPENDIX B  
CONVERSE PROOF OF THEOREM II.3

A. *Information Constraints*

Consider that the triple of rate, state leakage and distribution  $(\mathbf{R}, \mathbf{E}, \mathcal{Q}_{SXYV})$  is achievable by using a sequence of codes with causal encoding. We simplify the notation by using  $\mathcal{Q}$  in place of  $\mathcal{Q}_{SXYV}$  and we introduce the random event of error  $E \in \{0, 1\}$  defined with respect to  $\mathcal{Q}$  as follows:

$$E = \begin{cases} 0 & \text{if } \|Q^n - \mathcal{Q}\|_1 \leq \varepsilon \iff (S^n, X^n, Y^n, V^n) \in T_\delta(\mathcal{Q}), \\ 1 & \text{if } \|Q^n - \mathcal{Q}\|_1 > \varepsilon \iff (S^n, X^n, Y^n, V^n) \notin T_\delta(\mathcal{Q}). \end{cases} \quad (111)$$

The event  $E = 1$  occurs if the sequences  $(S^n, X^n, Y^n, V^n) \notin T_\delta(\mathcal{Q})$  for the target distribution  $\mathcal{Q}$ . By definition II.1, for all  $\varepsilon > 0$ , there exist a  $\bar{n}$  such that for all  $n \geq \bar{n}$ , there exist a code  $c^* \in \mathcal{C}(n, \mathcal{M})$  that satisfies the three following equations:

$$\frac{\log_2 |\mathcal{M}|}{n} \geq \mathbf{R} - \varepsilon, \quad (112)$$

$$\left| \mathcal{L}_e(c^*) - \mathbf{E} \right| = \left| \frac{1}{n} I(S^n; Y^n) - \mathbf{E} \right| \leq \varepsilon, \quad (113)$$

$$\mathcal{P}_e(c^*) = \mathbb{P}(M \neq \hat{M}) + \mathbb{P}\left(\|Q^n - \mathcal{Q}\|_1 > \varepsilon\right) \leq \varepsilon. \quad (114)$$

We introduce the auxiliary random variables  $W_{1,i} = (M, S^{i-1})$  and  $W_{2,i} = Y_{i+1}^n$  that satisfy the Markov chains of the set of distribution  $\mathbb{Q}_c$  for all  $i \in \{1, \dots, n\}$ :

$$S_i \text{ independent of } W_{1,i}, \quad (115)$$

$$X_i \text{ --- } (S_i, W_{1,i}) \text{ --- } W_{2,i}, \quad (116)$$

$$Y_i \text{ --- } (X_i, S_i) \text{ --- } (W_{1,i}, W_{2,i}), \quad (117)$$

$$V_i \text{ --- } (Y_i, W_{1,i}, W_{2,i}) \text{ --- } (S_i, X_i), \quad (118)$$

where (115) comes from the i.i.d. property of the source that induces the independence between  $S_i$  and  $(M, S^{i-1}) = W_{1,i}$ ; (116) comes from Lemma 1 since the encoding is causal and the channel is memoryless; (117) also comes from the memoryless property of the channel  $\mathcal{T}_{Y|XS}$ ; (118) comes from Lemma 2 since the encoding is causal, the decoding is non-causal and the channel is memoryless.

We introduce the random variable  $T$  that is uniformly distributed over the indices  $\{1, \dots, n\}$  and we denote by  $W_{1,T}, W_{2,T}, S_T, X_T, Y_T, V_T$  the corresponding averaged random variables. The auxiliary

random variables  $W_1 = (W_{1,T}, T)$  and  $W_2 = W_{2,T}$  belong to the set of distributions  $\mathbb{Q}_{\mathcal{C}}$  and satisfy the three information constraints of Theorem II.3:

$$I(S; W_1, W_2, Y) \leq \mathbf{E} \leq H(S), \quad (119)$$

$$\mathbf{R} + \mathbf{E} \leq I(W_1, S; Y). \quad (120)$$

**First Constraint:**

$$n\mathbf{E} \geq I(S^n; Y^n) - n\varepsilon = I(S^n; Y^n, M) - I(S^n; M|Y^n) - n\varepsilon \quad (121)$$

$$\geq \sum_{i=1}^n I(S_i; Y^n, M|S^{i-1}) - H(M|Y^n) - n\varepsilon \quad (122)$$

$$\geq \sum_{i=1}^n I(S_i; Y^n, M|S^{i-1}) - n2\varepsilon \quad (123)$$

$$= \sum_{i=1}^n I(S_i; Y^n, M, S^{i-1}) - n2\varepsilon \geq \sum_{i=1}^n I(S_i; Y_{i+1}^n, M, S^{i-1}, Y_i) - n2\varepsilon \quad (124)$$

$$= \sum_{i=1}^n I(S_i; W_{1,i}, W_{2,i}, Y_i) - n2\varepsilon \quad (125)$$

$$= nI(S_T; W_{1,T}, W_{2,T}, Y_T|T) - n2\varepsilon \quad (126)$$

$$= nI(S_T; W_{1,T}, W_{2,T}, Y_T, T) - n2\varepsilon \quad (127)$$

$$= n \left( I(S_T; W_1, W_2, Y_T) - 2\varepsilon \right) \quad (128)$$

$$\geq n \left( I(S_T; W_1, W_2, Y_T|E=0) - 3\varepsilon \right) \quad (129)$$

$$\geq n \left( I(S; W_1, W_2, Y) - 4\varepsilon \right), \quad (130)$$

where (122) comes from the definition of achievable state leakage rate  $\mathbf{E}$  in (113); (123) comes from equation (114) and Fano's inequality, see [34, pp. 19]; (124) comes from the i.i.d. property of the channel states that implies  $S_i$  is independent of  $S^{i-1}$ ; (125) comes from the introduction of the auxiliary random variables  $W_{1,i} = (M, S^{i-1})$  and  $W_{2,i} = Y_{i+1}^n$ , for all  $i \in \{1, \dots, n\}$ ; (126) comes from the introduction of the uniform random variable  $T$  over  $\{1, \dots, n\}$  and the corresponding mean random variables  $S_T, W_{1,T}, W_{2,T}, Y_T$ ; (127) comes from the independence between  $T$  and  $S_T$ ; (128) comes from the identification of the auxiliary random variables  $W_1 = (W_{1,T}, T)$  and  $W_2 = W_{2,T}$ ; (129) comes from the empirical coordination requirement as stated in Lemma 4. The sequences of symbols  $(S^n, X^n, Y^n, V^n)$  are not jointly typical with small error probability  $\mathbb{P}(E = 1)$ ; (130) comes from Lemma 6 since the distribution  $\mathcal{P}_{S_T X_T Y_T V_T|E=0}$  is close to the target distribution  $\mathcal{Q}_{SXYV}$ . The result of [32, Lemma 2.7, pp. 19] concludes.

**Second Constraint:**

$$n\mathbf{E} \leq I(S^n; Y^n) + n\varepsilon \leq H(S^n) + n\varepsilon \quad (131)$$

$$= n \left( H(S) + \varepsilon \right), \quad (132)$$

where (131) comes from the definition of the achievable state leakage rate  $\mathbf{E}$ , stated in equation (113); (132) comes from the i.i.d. property of the channel states  $S$ .

**Third Constraint:**

$$n(\mathbf{E} + \mathbf{R}) \leq I(S^n; Y^n) + H(M) + n2\varepsilon \quad (133)$$

$$= I(S^n; Y^n) + I(M; Y^n) + H(M|Y^n) + n2\varepsilon \quad (134)$$

$$\leq I(S^n; Y^n) + I(M; Y^n) + n3\varepsilon \quad (135)$$

$$\leq I(S^n; Y^n) + I(M; Y^n | S^n) + n3\varepsilon = I(S^n, M; Y^n) + n3\varepsilon \quad (136)$$

$$\leq \sum_{i=1}^n I(S_i, M, S^{i-1}; Y_i) + \sum_{i=1}^n I(S_{i+1}^n, Y_{i+1}^n; Y_i | S_i, M, S^{i-1}) + n3\varepsilon \quad (137)$$

$$= \sum_{i=1}^n I(S_i, M, S^{i-1}; Y_i) + n3\varepsilon \quad (138)$$

$$= \sum_{i=1}^n I(S_i, W_{1,i}; Y_i) + n \cdot 3\varepsilon \quad (139)$$

$$= nI(S_T, W_{1,T}; Y_T | T) + n \cdot 3\varepsilon \quad (140)$$

$$\leq nI(S_T, W_{1,T}, T; Y_T) + n \cdot 3\varepsilon \quad (141)$$

$$\leq n \left( I(S_T, W_1; Y_T) + 3\varepsilon \right) \quad (142)$$

$$\leq n \left( I(S_T, W_1; Y_T | E = 0) + 4\varepsilon \right) \quad (143)$$

$$\leq n \left( I(S, W_1; Y) + 5\varepsilon \right), \quad (144)$$

where (133) comes from the definition of achievable rate and information leakage  $(\mathbf{R}, \mathbf{E})$ , stated in equations (112) and (113); (135) comes from equation (114) and Fano's inequality, stated in [34, pp. 19]; (136) comes from the independence between the message  $M$  and the channel states  $S^n$ , hence  $I(M; Y^n) \leq I(M; Y^n, S^n) = I(M; Y^n | S^n)$ ; (137) comes from the properties of the mutual information; (138) comes from the Markov chain  $Y_i \text{---} (S_i, M, S^{i-1}) \text{---} (S_{i+1}^n, Y_{i+1}^n)$ , stated in Lemma 3; (139) comes from the identification of the auxiliary random variable  $W_{1,i} = (M, S^{i-1})$ ,  $\forall i \in \{1, \dots, n\}$ ; (140) comes from the introduction of  $T, S_T, W_{1,T}, Y_T$ ; (142) comes from the identification  $W_1 = (W_{1,T}, T)$

and  $W_2 = W_{2,T}$ ; (143) comes from the empirical coordination requirement as stated in Lemma 5; (144) comes from Lemma 6 since  $(S^n, X^n, Y^n, V^n)$  are jointly typical, hence  $\mathcal{P}_{S_T X_T Y_T V_T | E=0}$  is close to  $\mathcal{Q}_{SXYV}$ . The result of [32, Lemma 2.7, pp. 19] concludes.

**Conclusion:** If the triple of rate, state leakage and distribution  $(R, E, \mathcal{Q}_{SXYV})$  is achievable with causal encoding, then the following equations are satisfied for all  $\varepsilon > 0$ :

$$I(S; W_1, W_2, Y) - 4\varepsilon \leq E \leq H(S) + \varepsilon, \quad (145)$$

$$R + E \leq I(S, W_1; Y) + 5\varepsilon. \quad (146)$$

We recover (4) and (5) as  $\varepsilon \rightarrow 0$  and this concludes the converse proof of Theorem II.3.

**Remark B.1** *For the converse proof of Theorem II.3, the code with causal encoding is not necessarily deterministic. The same optimal performances can be obtained by considering a stochastic code with causal encoding.*

#### B. Lemmas for the converse proof with empirical coordination

**Lemma 1** *The causal encoding function and the memoryless property of the channel induce the Markov chain property  $X_i \ominus (S_i, W_{1,i}) \ominus W_{2,i}$ . This Markov chain is satisfied with  $W_{1,i} = (M, S^{i-1})$  and  $W_{2,i} = Y_{i+1}^n$ , for all  $i \in \{1, \dots, n\}$ .*

*Proof.* [Lemma 1] We identify  $w_{1,i} = (m, s^{i-1})$ ,  $w_{2,i} = y_{i+1}^n$  and for all  $(s^n, x^n, w_1^n, w_2^n, y^n, m)$

$$\mathcal{P}(w_{2,i} | s_i, w_{1,i}, x_i) = \mathcal{P}(y_{i+1}^n | s_i, m, s^{i-1}, x_i) \quad (147)$$

$$= \sum_{s_{i+1}^n, x_{i+1}^n} \mathcal{P}(s_{i+1}^n, x_{i+1}^n | s_i, m, s^{i-1}, x_i) \mathcal{P}(y_{i+1}^n | s_{i+1}^n, x_{i+1}^n, s_i, m, s^{i-1}, x_i) \quad (148)$$

$$= \sum_{s_{i+1}^n, x_{i+1}^n} \mathcal{P}(s_{i+1}^n, x_{i+1}^n | s_i, m, s^{i-1}) \mathcal{P}(y_{i+1}^n | s_{i+1}^n, x_{i+1}^n) \quad (149)$$

$$= \sum_{s_{i+1}^n, x_{i+1}^n} \mathcal{P}(s_{i+1}^n, x_{i+1}^n, y_{i+1}^n | s_i, m, s^{i-1}) = \mathcal{P}(y_{i+1}^n | s_i, m, s^{i-1}) = \mathcal{P}(w_{2,i} | s_i, w_{1,i}), \quad (150)$$

where (149) comes from the causal encoding function that induces the Markov chain  $X_i \ominus (S_i, M, S^{i-1}) \ominus (S_{i+1}^n, X_{i+1}^n)$  and the memoryless channel  $Y_{i+1}^n \ominus (S_{i+1}^n, X_{i+1}^n) \ominus (S_i, M, S^{i-1}, X_i)$ .  $\square$

**Lemma 2** *The causal encoding function, the non-causal decoding function and the memoryless channel induce  $V_i \ominus (Y_i, W_{1,i}, W_{2,i}) \ominus (S_i, X_i)$ . This Markov chain is satisfied with  $W_{1,i} = (M, S^{i-1})$  and  $W_{2,i} = Y_{i+1}^n$ , for all  $i \in \{1, \dots, n\}$ .*

*Proof.* [Lemma 2] We identify  $w_{1,i} = (m, s^{i-1})$ ,  $w_{2,i} = y_{i+1}^n$  and for all  $(s^n, x^n, w_1^n, w_2^n, y^n, v^n, m)$

$$\begin{aligned} & \mathcal{P}(v_i | y_i, w_{1,i}, w_{2,i}, s_i, x_i) = \mathcal{P}(v_i | y_i, m, s^{i-1}, y_{i+1}^n, s_i, x_i) \\ &= \sum_{x^{i-1}, y^{i-1}} \mathcal{P}(x^{i-1} | y_i, m, s^{i-1}, y_{i+1}^n, s_i, x_i) \mathcal{P}(y^{i-1} | y_i, m, s^{i-1}, y_{i+1}^n, s_i, x_i, x^{i-1}) \\ & \mathcal{P}(v_i | y_i, m, s^{i-1}, y_{i+1}^n, s_i, x_i, x^{i-1}, y^{i-1}). \end{aligned} \quad (151)$$

We can remove  $(s_i, x_i)$ , in the three conditional distributions

$$\mathcal{P}(x^{i-1} | y_i, m, s^{i-1}, y_{i+1}^n, s_i, x_i) = \mathcal{P}(x^{i-1} | m, s^{i-1}), \quad (152)$$

$$\mathcal{P}(y^{i-1} | y_i, m, s^{i-1}, y_{i+1}^n, s_i, x_i, x^{i-1}) = \mathcal{P}(y^{i-1} | s^{i-1}, x^{i-1}), \quad (153)$$

$$\mathcal{P}(v_i | y_i, m, s^{i-1}, y_{i+1}^n, s_i, x_i, x^{i-1}, y^{i-1}) = \mathcal{P}(v_i | y_i, y_{i+1}^n, y^{i-1}), \quad (154)$$

where (152) comes from the causal encoding that induces  $X^{i-1} \text{---} (M, S^{i-1}) \text{---} (Y_i, Y_{i+1}^n, X_i, S_i)$ ; (153) comes from the memoryless channel; (154) comes from the non-causal decoding that induces  $V_i \text{---} (Y_i, Y_{i+1}^n, Y^{i-1}) \text{---} (M, S^{i-1}, S_i, X_i, X^{i-1})$ . Hence we have for all  $(s^n, x^n, w_1^n, w_2^n, y^n, v^n, m)$ :

$$\mathcal{P}(v_i | y_i, w_{1,i}, w_{2,i}, s_i, x_i) = \sum_{x^{i-1}, y^{i-1}} \mathcal{P}(v_i, x^{i-1}, y^{i-1} | y_i, m, s^{i-1}, y_{i+1}^n) \quad (155)$$

$$= \mathcal{P}(v_i | y_i, m, s^{i-1}, y_{i+1}^n) = \mathcal{P}(v_i | y_i, w_{1,i}, w_{2,i}). \quad (156)$$

This concludes the proof of Lemma 2.  $\square$

**Lemma 3** *The causal encoding function and the memoryless property of the channel induce  $Y_i \text{---} (S_i, M, S^{i-1}) \text{---} (S_{i+1}^n, Y_{i+1}^n)$ ,  $\forall i \in \{1, \dots, n\}$ .*

*Proof.* [Lemma 3] We have the following equations for all  $(s^n, x^n, y^n, m)$ :

$$\mathcal{P}(y_i | s_i, m, s^{i-1}, s_{i+1}^n, y_{i+1}^n) = \sum_{x_i} \mathcal{P}(x_i, y_i | s_i, m, s^{i-1}, s_{i+1}^n, y_{i+1}^n) \quad (157)$$

$$= \sum_{x_i} \mathcal{P}(x_i | s_i, m, s^{i-1}, s_{i+1}^n, y_{i+1}^n) \mathcal{P}(y_i | x_i, s_i, m, s^{i-1}, s_{i+1}^n, y_{i+1}^n) \quad (158)$$

$$= \sum_{x_i} \mathcal{P}(x_i | s_i, m, s^{i-1}) \mathcal{P}(y_i | x_i, s_i) = \sum_{x_i} \mathcal{P}(x_i, y_i | s_i, m, s^{i-1}) = \mathcal{P}(y_i | s_i, m, s^{i-1}), \quad (159)$$

where (159) comes from  $X_i \text{---} (S_i, M, S^{i-1}) \text{---} (S_{i+1}^n, Y_{i+1}^n)$  and  $Y_i \text{---} (X_i, S_i) \text{---} (M, S^{i-1}, S_{i+1}^n, Y_{i+1}^n)$ .

This concludes the proof of Lemma 3.  $\square$

The random error event  $E \in \{0, 1\}$  of the following Lemmas is defined in (111), in which  $E = 1$  if  $(S^n, X^n, Y^n, V^n) \notin T_\delta(\mathcal{Q})$ .

**Lemma 4** Fix a distribution  $\mathcal{Q}_{S_T W_1 W_2 X_T Y_T V_T} \in \mathbb{Q}_c$  and suppose that the error probability  $\mathbb{P}(E = 1)$  is small enough such that  $\mathbb{P}(E = 1) \cdot \log_2 |\mathcal{S}| + h_b(\mathbb{P}(E = 1)) \leq \varepsilon$ , where  $h_b$  denotes the binary entropy function. Then we have

$$I(S_T; W_1, W_2, Y_T) \geq I(S_T; W_1, W_2, Y_T | E = 0) - \varepsilon. \quad (160)$$

*Proof.* [Lemma 4]

$$I(S_T; W_1, W_2, Y_T) = I(S_T; W_1, W_2, Y_T | E) + I(S_T; E) - I(S_T; E | W_1, W_2, Y_T) \quad (161)$$

$$\geq (1 - \mathbb{P}(E = 1)) \cdot I(S_T; W_1, W_2, Y_T | E = 0) + \mathbb{P}(E = 1) \cdot I(S_T; W_1, W_2, Y_T | E = 1) - H(E) \quad (162)$$

$$\geq I(S_T; W_1, W_2, Y_T | E = 0) - \mathbb{P}(E = 1) \cdot \log_2 |\mathcal{S}| - h_b(\mathbb{P}(E = 1)). \quad (163)$$

□

**Lemma 5** Fix a distribution  $\mathcal{Q}_{S_T W_1 W_2 X_T Y_T V_T} \in \mathbb{Q}_c$  and suppose that the error probability  $\mathbb{P}(E = 1)$  is small enough such that  $\mathbb{P}(E = 1) \cdot \log_2 |\mathcal{Y}| + h_b(\mathbb{P}(E = 1)) \leq \varepsilon$ . Then we have

$$I(S_T, W_1; Y_T) \leq I(S_T, W_1; Y_T | E = 0) + \varepsilon. \quad (164)$$

*Proof.* [Lemma 5]

$$I(S_T, W_1; Y_T) = I(S_T, W_1; Y_T | E) + I(E; Y_T) - I(E; Y_T | S_T, W_1) \quad (165)$$

$$\leq \mathbb{P}(E = 0) \cdot I(S_T, W_1; Y_T | E = 0) + \mathbb{P}(E = 1) \cdot I(S_T, W_1; Y_T | E = 1) + H(E) \quad (166)$$

$$\leq I(S_T, W_1; Y_T | E = 0) + \mathbb{P}(E = 1) \cdot \log_2 |\mathcal{Y}| + H(E) \leq \varepsilon. \quad (167)$$

□

We denote by  $\mathcal{P}_{S_T X_T Y_T V_T | E=0}$  the distribution induced by the random variables  $(S_T, X_T, Y_T, V_T)$  knowing the event  $E = 0$  is realized.

**Lemma 6** The distribution defined by  $\mathcal{P}_{S_T X_T Y_T V_T | E=0}$  is close to the target distribution  $\mathcal{Q}_{S X Y V}$ , i.e.

$$\left| \mathcal{P}_{S_T X_T Y_T V_T | E=0}(s, x, y, v) - \mathcal{Q}(s, x, y, v) \right| \leq \varepsilon, \quad \forall (s, x, y, v). \quad (168)$$



*Proof.* [Lemma 6] We evaluate the probability value  $\mathcal{P}_{S_T|E=0}(s)$  and we show that it is close to the probability value  $\mathcal{P}(s)$ , for all  $s \in \mathcal{S}$ .

$$\mathcal{P}_{S_T|E=0}(s) = \sum_{s^n \in T_\delta(\mathcal{P})} \sum_{i=1}^n \mathbb{P}(S^n = s^n, T = i, S_T = s | E = 0) \quad (169)$$

$$= \sum_{s^n \in T_\delta(\mathcal{P})} \sum_{i=1}^n \mathbb{P}(S^n = s^n | E = 0) \mathbb{P}(T = i | S^n = s^n, E = 0) \mathbb{P}(S_T = s | S^n = s^n, T = i, E = 0) \quad (170)$$

$$= \sum_{s^n \in T_\delta(\mathcal{P})} \sum_{i=1}^n \mathbb{P}(S^n = s^n | E = 0) \mathbb{P}(T = i) \mathbf{1}(s_T = s) \quad (171)$$

$$= \sum_{s^n \in T_\delta(\mathcal{P})} \mathbb{P}(S^n = s^n | E = 0) \sum_{i=1}^n \frac{1}{n} \mathbf{1}(s_T = s) = \sum_{s^n \in T_\delta(\mathcal{P})} \mathbb{P}(S^n = s^n | E = 0) \frac{N(s|s^n)}{n}, \quad (172)$$

where (171) comes from the independence of event  $\{T = i\}$  with events  $\{S^n = s^n\}$  and  $\{E = 0\}$ ; (172) comes from the definition of the number of occurrence  $N(s|s^n) = \sum_{i=1}^n \mathbf{1}(s_T = s)$ .

Since the sequences  $s^n \in T_\delta(\mathcal{P})$  are typical, we have for all  $\tilde{s} \in \mathcal{S}$ :

$$\mathcal{P}(\tilde{s}) - \varepsilon \leq \frac{N(\tilde{s}|s^n)}{n} \leq \mathcal{P}(\tilde{s}) + \varepsilon, \quad (173)$$

which provides an upper bound and a lower bound on  $\mathbb{P}(S_T = \tilde{s} | E = 0)$  as

$$\mathcal{P}(\tilde{s}) - \varepsilon = \sum_{s^n \in T_\delta(\mathcal{P})} \mathbb{P}(S^n = s^n | E = 0) (\mathcal{P}(\tilde{s}) - \varepsilon) \leq \mathbb{P}(S_T = \tilde{s} | E = 0) \quad (174)$$

$$\leq \sum_{s^n \in T_\delta(\mathcal{P})} \mathbb{P}(S^n = s^n | E = 0) (\mathcal{P}(\tilde{s}) + \varepsilon) = \mathcal{P}(\tilde{s}) + \varepsilon. \quad (175)$$

Using similar arguments, we show that the distribution  $\mathcal{P}_{S_T X_T Y_T V_T | E=0}(s, x, y, v)$  is close to the target distribution  $\mathcal{Q}(s, x, y, v)$ , i.e.

$$\left| \mathcal{P}_{S_T X_T Y_T V_T | E=0}(s, x, y, v) - \mathcal{Q}(s, x, y, v) \right| \leq \varepsilon, \quad \forall (s, x, y, v). \quad (176)$$

This concludes the proof of Lemma 6.  $\square$

## APPENDIX C

### PROOF OF THEOREM III.1

The conditional distribution  $\mathcal{P}_{W_1^n W_2^n X^n | S^n}$  combined with  $\mathcal{P}_S$  and  $\mathcal{T}_{Y|X_S}$  induces the distribution

$$\mathcal{P}_{S^n W_1^n W_2^n X^n Y^n} = \prod_{i=1}^n \mathcal{P}_{S_i} \mathcal{P}_{W_1^n W_2^n X^n | S^n} \prod_{i=1}^n \mathcal{T}_{Y_i | X_i S_i}. \quad (177)$$

We introduce the random event  $F \in \{0, 1\}$  depending on whether the random sequences  $(S^n, W_1^n, W_2^n, Y^n)$  are jointly typical or not.

$$F = \begin{cases} 0 & \text{if } (S^n, W_1^n, W_2^n, Y^n) \in T_\delta(\mathcal{Q}), \\ 1 & \text{if } (S^n, W_1^n, W_2^n, Y^n) \notin T_\delta(\mathcal{Q}). \end{cases} \quad (178)$$

Since the target distribution  $\mathcal{Q}_{SW_1W_2XY}$  has full support, the distribution  $\mathcal{P}_{S^n|Y^n}$  is absolutely continuous with respect to  $\prod_{i=1}^n \mathcal{Q}_{S_i|Y_iW_{1,i}W_{2,i}}$ , and the conditional KL-divergence writes

$$\begin{aligned} & \frac{1}{n} D\left(\mathcal{P}_{S^n|Y^n} \left\| \prod_{i=1}^n \mathcal{Q}_{S_i|Y_iW_{1,i}W_{2,i}}\right.\right) \\ &= \frac{1}{n} \sum_{w_1^n, w_2^n, y^n} \mathcal{P}(w_1^n, w_2^n, y^n) \sum_{s^n} \mathcal{P}(s^n|y^n) \log_2 \frac{1}{\prod_{i=1}^n \mathcal{Q}(s_i|y_i, w_{1,i}, w_{2,i})} \\ & - \frac{1}{n} \sum_{w_1^n, w_2^n, y^n} \mathcal{P}(w_1^n, w_2^n, y^n) \sum_{s^n} \mathcal{P}(s^n|y^n) \log_2 \frac{1}{\mathcal{P}(s^n|y^n)} \quad (179) \\ &= \frac{1}{n} \sum_{w_1^n, w_2^n, y^n, s^n, F} \mathcal{P}(w_1^n, w_2^n, y^n) \mathcal{P}(s^n|y^n) \mathbb{P}(F|s^n, w_1^n, w_2^n, y^n) \log_2 \frac{1}{\prod_{i=1}^n \mathcal{Q}(s_i|y_i, w_{1,i}, w_{2,i})} - \frac{1}{n} H(S^n|Y^n) \quad (180) \end{aligned}$$

$$\begin{aligned} &= \mathbb{P}(F=0) \frac{1}{n} \sum_{\substack{(w_1^n, w_2^n, y^n, s^n) \\ \in T_\delta(\mathcal{Q})}} \mathbb{P}(s^n, w_1^n, w_2^n, y^n|F=0) \log_2 \frac{1}{\prod_{i=1}^n \mathcal{Q}(s_i|y_i, w_{1,i}, w_{2,i})} \\ & + \mathbb{P}(F=1) \frac{1}{n} \sum_{w_1^n, w_2^n, y^n, s^n} \mathbb{P}(s^n, w_1^n, w_2^n, y^n|F=1) \log_2 \frac{1}{\prod_{i=1}^n \mathcal{Q}(s_i|y_i, w_{1,i}, w_{2,i})} - \frac{1}{n} H(S^n|Y^n) \quad (181) \end{aligned}$$

$$\begin{aligned} &\leq \frac{1}{n} \sum_{\substack{(w_1^n, w_2^n, y^n, s^n) \\ \in T_\delta(\mathcal{Q})}} \mathbb{P}(s^n, w_1^n, w_2^n, y^n|F=0) n \left( H(S|W_1, W_2, Y) + \delta \sum_{\substack{s, w_1, \\ w_2, y}} \log_2 \frac{1}{\mathcal{Q}(s|w_1, w_2, y)} \right) \\ & + \mathbb{P}(F=1) \log_2 \frac{1}{\min_{s, y, w_1, w_2} \mathcal{Q}(s|y, w_1, w_2)} - \frac{1}{n} H(S^n|Y^n) \quad (182) \end{aligned}$$

$$\begin{aligned} &\leq H(S|W_1, W_2, Y) - \frac{1}{n} H(S^n|Y^n) + \delta \sum_{\substack{s, w_1, \\ w_2, y}} \log_2 \frac{1}{\mathcal{Q}(s|w_1, w_2, y)} + \mathbb{P}(F=1) \log_2 \frac{1}{\min_{s, y, w_1, w_2} \mathcal{Q}(s|y, w_1, w_2)} \quad (183) \end{aligned}$$

$$\begin{aligned} &= \frac{1}{n} I(S^n; Y^n) - I(S; W_1, W_2, Y) + \delta \sum_{\substack{s, w_1, \\ w_2, y}} \log_2 \frac{1}{\mathcal{Q}(s|w_1, w_2, y)} + \mathbb{P}(F=1) \log_2 \frac{1}{\min_{s, y, w_1, w_2} \mathcal{Q}(s|y, w_1, w_2)} \quad (184) \end{aligned}$$

$$\begin{aligned} &\leq \mathcal{L}_e(c) - I(S; W_1, W_2, Y) + \delta \sum_{\substack{s, w_1, \\ w_2, y}} \log_2 \frac{1}{\mathcal{Q}(s|w_1, w_2, y)} \\ & + \mathbb{P}\left((s^n, w_1^n, w_2^n, y^n) \notin T_\delta(\mathcal{Q})\right) \log_2 \frac{1}{\min_{s, y, w_1, w_2} \mathcal{Q}(s|y, w_1, w_2)}, \quad (185) \end{aligned}$$

where (179) comes from the definition of the KL-divergence; (180) comes from the introduction of the random event  $F \in \{0, 1\}$ ; (181) is a reformulation of (180) where  $\mathbb{P}(F) \cdot \mathbb{P}(s^n, w_1^n, w_2^n, y^n|F) = \mathcal{P}(w_1^n, w_2^n, y^n) \mathcal{P}(s^n|y^n) \mathbb{P}(F|s^n, w_1^n, w_2^n, y^n)$  and where the event  $F=0$  implies that the sequences  $(w_1^n, w_2^n, y^n, s^n) \in T_\delta(\mathcal{Q})$ ; (182) comes from two properties: 1) the property  $2^{-n \cdot \left( H(S|Y, W_1, W_2) + \delta \sum_{\substack{s, w_1, \\ w_2, y}} \log_2 \frac{1}{\mathcal{Q}(s|w_1, w_2, y)} \right)} \leq \prod_{i=1}^n \mathcal{Q}_{S_i|Y_iW_{1,i}W_{2,i}}$  for typical sequences

$(w_1^n, w_2^n, y^n, s^n) \in T_\delta(\mathcal{Q})$ , stated in [34, pp. 26]; 2) the hypothesis  $\mathcal{Q}_{SW_1W_2XY}$  has full support, which implies that  $\min_{s,y,w_1,w_2} \mathcal{Q}(s|y, w_1, w_2) > 0$ ; (184) comes from the i.i.d. property of  $S$  that implies  $H(S) = \frac{1}{n}H(S^n)$ ; (185) comes from the definition of the state leakage  $\mathcal{L}_e(c) = \frac{1}{n}I(S^n; Y^n)$  and the event  $F = 0$ .

## APPENDIX D

### ACHIEVABILITY PROOF OF THEOREM III.3

We consider a pair of rate and distortion  $(\mathbf{R}, \mathbf{D}) \in \mathcal{A}_g$  that satisfies (19) and (20) with distribution  $\mathcal{P}_S \mathcal{Q}_{W_1} \mathcal{Q}_{X|SW_1} \mathcal{T}_{Y|XS}$  and we introduce the target leakage  $\mathbf{E} = I(S; W_1, Y)$ . Theorem II.6 guarantees that the triple  $(\mathbf{R}, \mathbf{E}, \mathcal{Q}_{SXYV})$  is achievable, i.e. for all  $\varepsilon > 0$ , there exist a  $\bar{n} \in \mathbb{N}$ , for all  $n \geq \bar{n}$ , there exist a code with causal encoding  $c \in \mathcal{C}(n, \mathcal{M})$  that satisfies

$$\frac{\log_2 |\mathcal{M}|}{n} \geq \mathbf{R} - \varepsilon, \quad (186)$$

$$\left| \mathcal{L}_e(c) - \mathbf{E} \right| \leq \varepsilon, \quad \text{with} \quad \mathcal{L}_e(c) = \frac{1}{n}I(S^n; Y^n), \quad (187)$$

$$\mathcal{P}_e(c) = \mathbb{P}\left(M \neq \hat{M}\right) + \mathbb{P}\left(\left\|Q_{SXYV}^n - \mathcal{Q}_{SXYV}\right\|_1 > \varepsilon\right) \leq \varepsilon. \quad (188)$$

**Remark D.1** *The achievability proof of Theorem II.6 guarantees that the sequences  $(S^n, W_1^n, X^n, Y^n) \in T_\delta(\mathcal{Q})$  with large probability and that the decoding of  $W_1^n$  is correct with large probability, i.e.  $\mathbb{P}(\hat{W}_1^n \neq W_1^n) \leq \varepsilon$ .*

We assume that the distribution  $\mathcal{P}_S \mathcal{Q}_{W_1} \mathcal{Q}_{X|SW_1} \mathcal{T}_{Y|XS}$  has full support. Otherwise, we would consider a sequence of distributions with full support, that converges to the target distribution. By replacing the pair  $(W_1, W_2)$  by  $W_1$  in Theorem III.1, we obtain:

$$\begin{aligned} \frac{1}{n}D\left(\mathcal{P}_{S^n|Y^n} \left\| \prod_{i=1}^n \mathcal{Q}_{S_i|Y_i W_{1,i}}\right.\right) &\leq \mathcal{L}_e(c) - I(S; W_1, Y) + \delta \sum_{s,w_1,y} \log_2 \frac{1}{\mathcal{Q}(s|w_1, y)} \\ &+ \mathbb{P}\left((s^n, w_1^n, y^n) \notin T_\delta(\mathcal{Q})\right) \log_2 \frac{1}{\min_{s,y,w_1} \mathcal{Q}(s|y, w_1)}. \end{aligned} \quad (189)$$

The equations (186)–(188) and (189) show that for any  $\varepsilon > 0$ , there exist a  $\bar{n}$ , for all  $n \geq \bar{n}$ , there exist a code with causal encoding  $c \in \mathcal{C}(n, \mathcal{M})$  involving an auxiliary sequence  $W_1^n$ , such that

$$\frac{\log_2 |\mathcal{M}|}{n} \geq \mathbf{R} - \varepsilon, \quad (190)$$

$$\mathbb{P}\left(M \neq \hat{M}\right) \leq \varepsilon, \quad (191)$$

$$\frac{1}{n}D\left(\mathcal{P}_{S^n|Y^n} \left\| \prod_{i=1}^n \mathcal{Q}_{S_i|Y_i W_{1,i}}\right.\right) \leq \varepsilon. \quad (192)$$

Following [29, Notations A.6 and A.7], we define the sets  $J_\alpha(w_1^n, y^n)$  and  $B_{\alpha, \gamma, \delta}$  depending on small parameters  $\alpha > 0$ ,  $\gamma > 0$  and  $\delta > 0$ :

$$J_\alpha(w_1^n, y^n) = \left\{ i \in \{1, \dots, n\}, \quad \text{s.t.} \quad D\left(\mathcal{P}_{S_i|Y^n}(\cdot|y^n) \parallel \mathcal{Q}_{S_i|Y_i W_{1,i}}(\cdot|y_i, w_{1,i})\right) \leq \frac{\alpha^2}{2 \ln 2} \right\}, \quad (193)$$

$$B_{\alpha, \gamma, \delta} = \left\{ (w_1^n, y^n) \quad \text{s.t.} \quad \frac{|J_\alpha(w_1^n, y^n)|}{n} \geq 1 - \gamma \quad \text{and} \quad (w_1^n, y^n) \in T_\delta(\mathcal{Q}) \right\}. \quad (194)$$

The notation  $B_{\alpha, \gamma, \delta}^c$  stands for the complementary set of  $B_{\alpha, \gamma, \delta} \subset \mathcal{W}_1^n \times \mathcal{Y}^n$ . The sequences  $(w_1^n, y^n)$  belong to the set  $B_{\alpha, \gamma, \delta}$  if they are jointly typical and if the decoder's posterior belief  $\mathcal{P}_{S_i|Y^n}(\cdot|y^n)$  is close in K-L divergence to the target belief  $\mathcal{Q}_{S_i|Y_i W_{1,i}}(\cdot|y_i, w_{1,i})$ , for a large fraction of stages  $i \in \{1, \dots, n\}$ .

### Lemma 7

$$\mathbb{P}(B_{\alpha, \gamma, \delta}^c) \leq \frac{2 \ln 2}{\alpha^2 \gamma} \frac{1}{n} D\left(\mathcal{P}_{S^n|Y^n} \parallel \prod_{i=1}^n \mathcal{Q}_{S_i|Y_i W_{1,i}}\right) + \mathbb{P}((w_1^n, y^n) \notin T_\delta(\mathcal{Q})). \quad (195)$$

Lemma 7 is a reformulation of [29, equations (40) - (46)], a detailed proof is stated in Appendix D-A.

Then [29, Lemma A.8] ensures that for each code with causal encoding  $c \in \mathcal{C}(n, \mathcal{M})$  we have

$$\left| \min_{h_{V^n|Y^n}} \frac{1}{n} \sum_{i=1}^n \mathbb{E}[d(S_i, V_i)] - \min_{\mathcal{P}_{V|W_1 Y}} \mathbb{E}[d(S, V)] \right| \\ = \left| \frac{1}{n} \sum_{i=1}^n \sum_{y^n} \mathcal{P}(y^n) \min_{h_{V_i|Y^n}} \sum_{s_i} \mathcal{P}(s_i|y^n) d(s_i, v_i) - \sum_{w_1, y} \mathcal{Q}(w_1, y) \min_{\mathcal{P}_{V|W_1 Y}} \sum_s \mathcal{Q}(s|w_1, y) d(s, v) \right| \quad (196)$$

$$\leq (\alpha + 2\gamma + \delta + \mathbb{P}(B_{\alpha, \gamma, \delta}^c)) \bar{d}, \quad (197)$$

where  $\bar{d} = \max_{s, v} d(s, v)$  is the maximal distortion value. We combine (197) with (195) and (192), and then we choose the parameters  $\alpha > 0$ ,  $\gamma > 0$ ,  $\delta > 0$  small and  $n$  large such as to obtain

$$\left| \min_{h_{V^n|Y^n}} \frac{1}{n} \sum_{i=1}^n \mathbb{E}[d(S_i, V_i)] - \min_{\mathcal{P}_{V|W_1 Y}} \mathbb{E}[d(S, V)] \right| \leq \varepsilon. \quad (198)$$

This concludes the achievability proof of Theorem III.3.

#### A. Proof of Lemma 7

The union bound implies

$$\mathbb{P}(B_{\alpha, \gamma, \delta}^c) = \mathbb{P}\left((W_1^n, Y^n) \quad \text{s.t.} \quad \frac{|J_\alpha(W_1^n, Y^n)|}{n} < 1 - \gamma \quad \text{or} \quad (W_1^n, Y^n) \notin T_\delta(\mathcal{Q})\right) \quad (199)$$

$$\leq \mathbb{P}\left((W_1^n, Y^n) \quad \text{s.t.} \quad \frac{|J_\alpha(W_1^n, Y^n)|}{n} < 1 - \gamma\right) + \mathbb{P}\left((W_1^n, Y^n) \notin T_\delta(\mathcal{Q})\right). \quad (200)$$

Moreover,

$$\mathbb{P}\left((W_1^n, Y^n) \quad \text{s.t.} \quad \frac{|J_\alpha(W_1^n, Y^n)|}{n} < 1 - \gamma\right)$$

$$= \mathbb{P}\left(\frac{1}{n} \left| \left\{ i \in \{1, \dots, n\}, \quad \text{s.t.} \quad D\left(\mathcal{P}_{S_i|Y^n}(\cdot|Y^n) \middle| \mathcal{Q}_{S_i|Y_i W_{1,i}}(\cdot|Y_i, W_{1,i})\right) \leq \frac{\alpha^2}{2 \ln 2} \right\} \right| < 1 - \gamma\right) \quad (201)$$

$$= \mathbb{P}\left(\frac{1}{n} \left| \left\{ i \in \{1, \dots, n\}, \quad \text{s.t.} \quad D\left(\mathcal{P}_{S_i|Y^n}(\cdot|Y^n) \middle| \mathcal{Q}_{S_i|Y_i W_{1,i}}(\cdot|Y_i, W_{1,i})\right) > \frac{\alpha^2}{2 \ln 2} \right\} \right| \geq \gamma\right) \quad (202)$$

$$\leq \frac{2 \ln 2}{\alpha^2 \gamma} \mathbb{E} \left[ \frac{1}{n} \sum_{i=1}^n D\left(\mathcal{P}_{S_i|Y^n}(\cdot|Y^n) \middle| \mathcal{Q}_{S_i|Y_i W_{1,i}}(\cdot|Y_i, W_{1,i})\right) \right] \quad (203)$$

$$\leq \frac{2 \ln 2}{\alpha^2 \gamma} \frac{1}{n} D\left(\mathcal{P}_{S^n|Y^n} \middle| \prod_{i=1}^n \mathcal{Q}_{S_i|Y_i W_{1,i}}\right), \quad (204)$$

where (201)-(202) are reformulations; (203) comes from [29, Lemma A.21]; (204) comes from Lemma 8.

**Lemma 8** *We consider the distributions  $\mathcal{P}_{A_1 A_2 B_1 B_2}$ ,  $\mathcal{Q}_{B_1|A_1}$  and  $\mathcal{Q}_{B_2|A_2}$ . We have*

$$\begin{aligned} & D\left(\mathcal{P}_{B_1 B_2|A_1 A_2} \middle| \mathcal{Q}_{B_1|A_1} \mathcal{Q}_{B_2|A_2}\right) \\ &= D\left(\mathcal{P}_{B_1|A_1 A_2} \middle| \mathcal{Q}_{B_1|A_1}\right) + D\left(\mathcal{P}_{B_2|A_1 A_2} \middle| \mathcal{Q}_{B_2|A_2}\right) + I(B_1; B_2|A_1, A_2), \end{aligned} \quad (205)$$

where the mutual information  $I(B_1; B_2|A_1, A_2)$  is evaluated with respect to  $\mathcal{P}_{A_1 A_2 B_1 B_2}$ . In particular, this implies for all  $n \geq 1$

$$D\left(\mathcal{P}_{B^n|A^n} \middle| \prod_{i=1}^n \mathcal{Q}_{B_i|A_i}\right) \geq \sum_{i=1}^n D\left(\mathcal{P}_{B_i|A_i} \middle| \mathcal{Q}_{B_i|A_i}\right). \quad (206)$$

*Proof.* [Lemma 8]

$$\begin{aligned} & D\left(\mathcal{P}_{B_1 B_2|A_1 A_2} \middle| \mathcal{Q}_{B_1|A_1} \times \mathcal{Q}_{B_2|A_2}\right) \\ &= \sum_{a_1, a_2} \mathcal{P}(a_1, a_2) \sum_{b_1} \mathcal{P}(b_1|a_1, a_2) \log_2 \frac{1}{\mathcal{Q}(b_1|a_1)} + \sum_{a_1, a_2} \mathcal{P}(a_1, a_2) \sum_{b_2} \mathcal{P}(b_2|a_1, a_2) \log_2 \frac{1}{\mathcal{Q}(b_2|a_2)} - H(B_1, B_2|A_1, A_2) \end{aligned} \quad (207)$$

$$= \sum_{a_1, a_2} \mathcal{P}(a_1, a_2) \sum_{b_1} \mathcal{P}(b_1|a_1, a_2) \log_2 \frac{\mathcal{P}(b_1|a_1, a_2)}{\mathcal{Q}(b_1|a_1)} + \sum_{a_1, a_2} \mathcal{P}(a_1, a_2) \sum_{b_2} \mathcal{P}(b_2|a_1, a_2) \log_2 \frac{\mathcal{P}(b_2|a_1, a_2)}{\mathcal{Q}(b_2|a_2)} \quad (208)$$

$$+ H(B_1|A_1, A_2) + H(B_2|A_1, A_2) - H(B_1, B_2|A_1, A_2) \quad (209)$$

$$= D\left(\mathcal{P}_{B_1|A_1 A_2} \middle| \mathcal{Q}_{B_1|A_1}\right) + D\left(\mathcal{P}_{B_2|A_1 A_2} \middle| \mathcal{Q}_{B_2|A_2}\right) + I(B_1; B_2|A_1, A_2). \quad (210)$$

□

## APPENDIX E

## CONVERSE PROOF OF THEOREM III.3

We introduce the random event  $F \in \{0, 1\}$  indicating whether  $M$  is correctly decoded or not. We assume that there exist a code with causal encoding  $c \in \mathcal{C}(n, \mathcal{M})$  that satisfies

$$\frac{\log_2 |\mathcal{M}|}{n} \geq \mathbf{R} - \varepsilon, \quad (211)$$

$$\mathcal{P}_e(c) = \mathbb{P}(M \neq \hat{M}) \leq \varepsilon, \quad (212)$$

$$\left| \min_{h_{\mathcal{V}^n | \mathcal{Y}^n}} \frac{1}{n} \sum_{i=1}^n \mathbb{E} [d(S_i, V_i)] - \mathbf{D} \right| \leq \varepsilon. \quad (213)$$

We first show the constraint (19).

$$\mathbf{R} \leq \frac{\log_2 |\mathcal{M}|}{n} + \varepsilon = \frac{1}{n} H(M) + \varepsilon = \frac{1}{n} I(M; Y^n) + \frac{1}{n} H(M | Y^n) + \varepsilon \quad (214)$$

$$\leq \frac{1}{n} I(M; Y^n) + 2\varepsilon = \frac{1}{n} \sum_{i=1}^n I(M; Y_i | Y^{i-1}) + 2\varepsilon \leq \frac{1}{n} \sum_{i=1}^n I(M, Y^{i-1}, Y_{i+1}^n; Y_i) + 2\varepsilon \quad (215)$$

$$= \frac{1}{n} \sum_{i=1}^n I(W_{1,i}; Y_i) + 2\varepsilon = I(W_{1,T}; Y_T | T) + 2\varepsilon \leq I(W_{1,T}, T; Y_T) + 2\varepsilon \leq I(W_1; Y) + 2\varepsilon, \quad (216)$$

where (214) comes from assumption (211) and the uniform distribution of the message  $M$ ; (215) comes from Fano's inequality [31, Theorem 2.10.1] and assumption (212) and adding the mutual informations  $I(Y^{i-1}; Y_i)$  and  $I(Y_{i+1}^n; Y_i | M, Y^{i-1})$ ; (216) comes from the identification of the auxiliary random variable  $W_{1,i} = (M, Y^{i-1}, Y_{i+1}^n)$  and the introduction of the uniform random variable  $T$  over the indices  $\{1, \dots, n\}$  and  $(W_{1,T}, Y_T)$  and by identifying  $W_1 = (W_{1,T}, T)$  and  $Y_T = Y$ .

We now show the constraint (20). The notation  $y^{-i} = (y^{i-1}, y_{i+1}^n)$  stands for the subsequence of  $y^n$  where  $y_i$  has been removed. We assume that the event  $F = 0$  is realized. Then, the average distortion satisfies

$$\min_{h_{\mathcal{V}^n | \mathcal{Y}^n}} \sum_{s^n, y^n, m, v^n} \mathbb{P}(s^n, y^n, m | F = 0) h(v^n | y^n) \left[ \frac{1}{n} \sum_{i=1}^n d(s_i, v_i) \right] \quad (217)$$

$$= \sum_{y^n, m} \mathbb{P}(y^n, m | F = 0) \min_{\tilde{h}: \mathcal{Y}^n \times \mathcal{M} \rightarrow \mathcal{V}^n} \sum_{s^n} \mathbb{P}(s^n | y^n, m, F = 0) \left[ \frac{1}{n} \sum_{i=1}^n d(s_i, v_i) \right] \quad (218)$$

$$= \sum_{y_i, y^{-i}, m} \mathbb{P}(y^n, m | F = 0) \frac{1}{n} \sum_{i=1}^n \min_{\hat{h}: \mathcal{Y}^n \times \mathcal{M} \rightarrow \mathcal{V}_i} \sum_{s_i} \mathbb{P}(s_i | y_i, y^{-i}, m, F = 0) d(s_i, v_i) \quad (219)$$

$$= \sum_{y_i, w_{1,i}} \mathcal{P}(y_i, w_{1,i}) \frac{1}{n} \sum_{i=1}^n \min_{h: \mathcal{Y}_i \times \mathcal{W}_{1,i} \rightarrow \mathcal{V}_i} \sum_{s_i} \mathcal{P}(s_i | y_i, w_{1,i}) d(s_i, v_i) \quad (220)$$

$$= \sum_{y_T, w_{1,T}, T} \mathcal{P}(y_T, w_{1,T}, T) \min_{h: \mathcal{Y} \times \mathcal{W}_1 \rightarrow \mathcal{V}} \sum_{s_T} \mathcal{P}(s_T | y_T, w_{1,T}, T) d(s_T, v_T) \quad (221)$$

$$= \sum_{y, w_1} \mathcal{P}(y, w_1) \min_{h: \mathcal{Y} \times \mathcal{W}_1 \rightarrow \mathcal{V}} \sum_s \mathcal{P}(s|y, w_1) d(s, v) = \min_{\mathcal{P}_{V|W_1Y}} \mathbb{E} \left[ d(S, V) \right], \quad (222)$$

where (218), comes from the hypothesis  $F = 0$ , which guarantees the correct decoding of the message  $m$  based on the observation of  $y^n$ ; (219) is a reformulation; (220) comes from the identification of the auxiliary random variable  $w_{1,i} = (m, y^{-i})$ ; (221) comes from the introduction of the uniform random variable  $T$ ; (222) comes from the identification of the auxiliary random variables  $w_1 = (w_{1,T}, T)$ . We have

$$\left| \min_{\mathcal{P}_{V|W_1Y}} \mathbb{E} \left[ d(S, V) \right] - \mathbf{D} \right| \quad (223)$$

$$\leq \left| \min_{\mathcal{P}_{V|W_1Y}} \mathbb{E} \left[ d(S, V) \right] - \min_{h_{V^n|Y^n}} \frac{1}{n} \sum_{i=1}^n \mathbb{E} \left[ d(S_i, V_i) \right] \right| + \left| \min_{h_{V^n|Y^n}} \frac{1}{n} \sum_{i=1}^n \mathbb{E} \left[ d(S_i, V_i) \right] - \mathbf{D} \right| \quad (224)$$

$$\leq \left| \min_{\mathcal{P}_{V|W_1Y}} \mathbb{E} \left[ d(S, V) \right] - \min_{h_{V^n|Y^n}} \frac{1}{n} \sum_{i=1}^n \mathbb{E} \left[ d(S_i, V_i) \right] \right| + \varepsilon \quad (225)$$

$$= \left| \min_{\mathcal{P}_{V|W_1Y}} \mathbb{E} \left[ d(S, V) \right] - \sum_{y^n, m, F} \mathbb{P}(y^n, m, F) \times \min_{h_{V^n|Y^n}} \sum_{s^n} \mathbb{P}(s^n|y^n, F) \left[ \frac{1}{n} \sum_{i=1}^n d(s_i, v_i) \right] \right| + \varepsilon \quad (226)$$

$$\leq \left| \min_{\mathcal{P}_{V|W_1Y}} \mathbb{E} \left[ d(S, V) \right] - \sum_{y^n, m} \mathbb{P}(y^n, m|F=0) \times \min_{h_{V^n|Y^n}} \sum_{s^n} \mathbb{P}(s^n|y^n, F=0) \left[ \frac{1}{n} \sum_{i=1}^n d(s_i, v_i) \right] \right| + \mathbb{P}(F=1)2\bar{d} + \varepsilon \quad (227)$$

$$= \left| \min_{\mathcal{P}_{V|W_1Y}} \mathbb{E} \left[ d(S, V) \right] - \min_{\mathcal{P}_{V|W_1Y}} \mathbb{E} \left[ d(S, V) \right] \right| + \mathbb{P}(F=1)2\bar{d} + \varepsilon \quad (228)$$

$$= \mathbb{P}(F=1)2\bar{d} + \varepsilon \leq \varepsilon \cdot (2\bar{d} + 1), \quad (229)$$

where (224) comes from the triangle inequality; (225) comes from assumption (213); (226) is a reformulation that introduces the random event  $F \in \{0, 1\}$ ; (227) comes from removing the term  $\mathbb{P}(F=1)\bar{d}$  from the triangle inequality; (228) comes from (222); (229) comes from assumption (212). This concludes the converse proof of Theorem III.3.

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**STATE LEAKAGE AND COORDINATION  
WITH CAUSAL STATE KNOWLEDGE AT THE ENCODER**

– SUPPLEMENTARY MATERIALS –

Maël Le Treust and Matthieu Bloch

APPENDIX F

CASE OF  $H(S) = 0$  IN THE ACHIEVABILITY PROOF OF THEOREM II.3

In this section, we consider the case where  $H(S) = 0$ . We choose a target distribution  $\mathcal{Q}_{XYV}$  that decomposes as  $\mathcal{Q}_{XYV} = \mathcal{Q}_X \mathcal{T}_{Y|X} \mathcal{Q}_{V|XY}$ . Standard channel coding arguments, see [34, Sec. 3.1], show that a pair  $(\mathbf{R}, \mathcal{Q}_{XYV})$  is achievable if and only if

$$\mathbf{R} \leq I(X; Y). \quad (230)$$

We recover the information constraints of Theorem II.3 by setting  $\mathbf{E} = 0$ ,  $W_1 = X$  and  $|\mathcal{W}_2| = 1$ .

APPENDIX G

CASE OF EQUALITY IN (41)

In this section, we consider  $(\mathbf{R}, \mathbf{E})$  and a distribution  $\mathcal{Q}_{SW_1W_2XYV} \in \mathbb{Q}_c$  that satisfies

$$I(S; W_1, W_2, Y) = \mathbf{E} \leq H(S), \quad (231)$$

$$\mathbf{R} + \mathbf{E} \leq I(W_1, S; Y). \quad (232)$$

- If  $I(S; W_1, W_2, Y) < I(W_1, S; Y)$ , then we select  $\tilde{\mathbf{E}}$  close to  $\mathbf{E}$  such that  $I(S; W_1, W_2, Y) < \tilde{\mathbf{E}} \leq H(S)$  and  $\mathbf{R} + \tilde{\mathbf{E}} \leq I(W_1, S; Y)$ , and thus the achievability proof of Appendix A applies.
- If  $I(S; W_1, W_2, Y) = I(W_1, S; Y)$  and the channel capacity, see [34, pp. 176], is strictly positive

$$\max_{\tilde{\mathcal{P}}_{W_1}, \tilde{\mathcal{P}}_{X|W_1S}} I(W_1; Y) > 0, \quad (233)$$

then we refer to Sec. G-A.

- If  $I(S; W_1, W_2, Y) = I(W_1, S; Y)$  and the channel capacity in (233) is equal to zero, then we refer to Sec. G-B.

### A. Strictly positive channel capacity

In this section, we assume that  $I(S; W_1, W_2, Y) = I(W_1, S; Y)$  and the channel capacity is strictly positive. We denote by  $\mathcal{P}_{W_1}^*$  and  $\mathcal{P}_{X|W_1S}^*$  the distributions that achieve the maximum in (233). We define  $\mathcal{Q}_{SW_1W_2XYV}^* = \mathcal{P}_S \mathcal{P}_{W_1}^* \mathcal{P}_{W_2} \mathcal{P}_{X|W_1S}^* \mathcal{T}_{Y|XS} \mathcal{Q}_V$  where  $V$  is independent of  $(S, W_1, W_2, X, Y)$  and  $W_2$  is independent of  $(S, W_1, X, Y)$ , hence  $I(W_2; S|W_1) = I(W_2; Y|W_1) = 0$ , and therefore

$$I_{\mathcal{Q}^*}(W_1, W_2; Y) - I_{\mathcal{Q}^*}(W_2; S|W_1) = I_{\mathcal{Q}^*}(W_1; Y) > 0. \quad (234)$$

We assume that the distributions  $\mathcal{Q}_{SW_1W_2XYV}$  and  $\mathcal{Q}_{SW_1W_2XYV}^*$  are selected according to the random variable  $T \in \{0, 1\}$ , where  $T = 0$  corresponds to the distribution  $\mathcal{Q}_{SW_1W_2XYV}$  and  $T = 1$  corresponds to the distribution  $\mathcal{Q}_{SW_1W_2XYV}^*$ . By hypothesis, we have

$$0 < \mathbb{P}(T = 1) \cdot I_{\mathcal{Q}^*}(W_1; Y) \quad (235)$$

$$= \mathbb{P}(T = 0) \cdot \left( I_{\mathcal{Q}}(W_1, W_2; Y|T = 0) - I_{\mathcal{Q}}(W_2; S|W_1, T = 0) \right) \quad (236)$$

$$+ \mathbb{P}(T = 1) \cdot \left( I_{\mathcal{Q}^*}(W_1, W_2; Y|T = 1) - I_{\mathcal{Q}^*}(W_2; S|W_1, T = 1) \right) \quad (237)$$

$$= I(W_1, W_2; Y|T) - I(W_2; S|W_1, T) \quad (238)$$

$$\leq I(W_1, T, W_2; Y) - I(W_2; S|W_1, T). \quad (239)$$

This show the existence of a distribution  $\tilde{\mathcal{Q}}_{SW_1TW_2XYV}$  that decomposes as

$$\mathcal{P}_S \tilde{\mathcal{Q}}_{W_1T} \tilde{\mathcal{Q}}_{W_2|SW_1T} \tilde{\mathcal{Q}}_{X|SW_1T} \mathcal{T}_{Y|XS} \tilde{\mathcal{Q}}_{V|YW_1TW_2}, \quad (240)$$

that converges to the distribution  $\mathcal{Q}_{SW_1W_2XYV}$  as  $\mathbb{P}(T = 1)$  goes to zero, and that satisfies the information constraint (239) with strict inequality. Thus the achievability proof of Appendix A applies.

### B. Strictly positive channel capacity

In this section, we assume that  $I(S; W_1, W_2, Y) = I(W_1, S; Y)$  and the channel capacity is equal to zero, hence

$$0 = \max_{\mathcal{P}_{W_1}, \mathcal{P}_{X|W_1S}} I(W_1; Y) \quad (241)$$

$$\geq I(W_1; Y) \quad (242)$$

$$= I(S; W_1, W_2, Y) - I(S; Y|W_1) \quad (243)$$

$$= I(S; W_2|W_1, Y) \geq 0, \quad (244)$$

where (243) comes from the hypothesis  $I(S; W_1, W_2, Y) = I(W_1, S; Y)$  and (244) comes from the independence between  $W_1$  and  $S$ .

Moreover, the two Markov chains  $W_2 \text{---} (W_1, S) \text{---} X$  and  $Y \text{---} (X, S) \text{---} (W_1, W_2)$  induce  $W_2 \text{---} (W_1, S) \text{---} Y$ . Therefore, we have

$$W_2 \text{---} (W_1, S) \text{---} Y \iff I(W_2; Y|W_1, S) = 0, \quad (245)$$

$$W_2 \text{---} (W_1, Y) \text{---} S \iff I(W_2; S|W_1, Y) = 0. \quad (246)$$

As a consequence, for all  $(w_1, s, y) \in \mathcal{W}_1 \times \mathcal{S} \times \mathcal{Y}$  such that  $\mathcal{Q}(w_1, s, y) > 0$ , we have

$$\mathcal{Q}(w_2|w_1, s, y) = \mathcal{Q}(w_2|w_1, s) = \mathcal{Q}(w_2|w_1, y). \quad (247)$$

We design a coding scheme for this special case.

*Codebook.* We choose a typical sequence  $W_1^n$  that is known by both the encoder and the decoder.

*Encoder.* At stage  $i \in \{1, \dots, n\}$ , the encoder observes the symbol  $S_i$ , recalls the sequence  $W_1^n$  and generates  $X^n$  according to the i.i.d. conditional distribution  $\mathcal{Q}_{X|SW_1}$ .

*Decoder.* The decoder observes the sequence of channel output  $Y^n$ , recalls the pre-defined sequence  $W_1^n$  and generates sequence  $W_2^n$  according to the i.i.d. conditional distribution  $\mathcal{Q}_{W_2|YW_1}$  which, according to (247), satisfies  $\mathcal{Q}_{W_2}(\cdot|w_1, y) = \mathcal{Q}_{W_2}(\cdot|w_1, s)$  for all  $(w_1, s, y) \in \mathcal{W}_1 \times \mathcal{S} \times \mathcal{Y}$  such that  $\mathcal{Q}(w_1, s, y) > 0$ . Then, the decoder generates  $V^n$  by using the conditional distribution  $\mathcal{Q}_{V|W_1W_2Y}$ .

This coding scheme achieves the target distribution

$$\mathcal{P}_S \mathcal{Q}_{W_1} \mathcal{Q}_{W_2|SW_1} \mathcal{Q}_{X|SW_1} \mathcal{T}_{Y|XS} \mathcal{Q}_{V|YW_1W_2}. \quad (248)$$

## APPENDIX H

### CARDINALITY BOUND FOR THEOREM II.3

**Lemma 9** *We consider the following information constraints with two auxiliary random variables  $(W_1, W_2)$*

$$I(S; W_1, W_2, Y) \leq E \leq H(S), \quad (249)$$

$$R + E \leq I(W_1, S; Y). \quad (250)$$

*The cardinality of the supports of the auxiliary random variables  $(W_1, W_2)$  are bounded by*

$$\max(|\mathcal{W}_1|, |\mathcal{W}_2|) \leq d + 1, \quad \text{with} \quad d = |\mathcal{S} \times \mathcal{X} \times \mathcal{Y} \times \mathcal{V}|.$$

This result is based on the Lemma of Fenchel-Eggleston-Carathéodory, see [34, pp. 631], and the proof is provided below. From Lemma 9, we deduce the cardinality bounds of Theorems IV.2 and IV.4.

*Proof.* [Lemma 9] We denote by  $d = |\mathcal{S} \times \mathcal{X} \times \mathcal{Y} \times \mathcal{V}|$ , the cardinality of the product of the discrete sets. We consider the family of continuous functions  $h_i : \Delta(\mathcal{S} \times \mathcal{X} \times \mathcal{Y} \times \mathcal{V}) \mapsto \mathbb{R}$ , with  $i \in \{1, \dots, d+1\}$ , defined as follows:

$$h_i(\mathcal{P}_{SXYV|W_1W_2}) = \begin{cases} \mathcal{P}_{SXYV|W_1W_2}(i), & \text{for } i \in \{1, \dots, d-1\}, \\ H(Y|S, W_1 = w_1), & \text{for } i = d, \\ H(S|Y, W_1 = w_1, W_2 = w_2), & \text{for } i = d+1. \end{cases}$$

The support Lemma, see [34, pp. 631], implies that there exist a pair of auxiliary random variables  $(W'_1, W'_2) \sim \mathcal{P}_{W'_1W'_2}$  defined on a set  $\mathcal{W}'_1 \times \mathcal{W}'_2$  with finite cardinality  $\max(|\mathcal{W}'_1|, |\mathcal{W}'_2|) \leq d+1$  such that for all  $i \in \{1, \dots, d+1\}$  we have:

$$\int_{\mathcal{W}_1 \times \mathcal{W}_2} h_i(\mathcal{P}_{SXYV|W_1W_2}) dF(w_1, w_2) = \sum_{(w'_1, w'_2) \in \mathcal{W}'_1 \times \mathcal{W}'_2} h_i(\mathcal{P}_{SXYV|W'_1W'_2}) \mathcal{P}(w'_1, w'_2).$$

This implies that the probability  $\mathcal{P}_{SXYV}$  is preserved and we have:

$$\begin{aligned} \mathcal{P}_{SXYV}(i) &= \int_{\mathcal{W}_1 \times \mathcal{W}_2} \mathcal{P}_{SXYV|W_1W_2}(i) dF(w_1, w_2) \\ &= \sum_{(w'_1, w'_2) \in \mathcal{W}'_1 \times \mathcal{W}'_2} \mathcal{P}_{SXYV|W'_1W'_2}(i) \mathcal{P}(w'_1, w'_2), \quad \text{for } i \in \{1, \dots, d-1\} \\ H(Y|S, W_1) &= \int_{\mathcal{W}_1} H(Y|S, W_1 = w_1) dF(w_1) \\ &= \sum_{(w'_1) \in \mathcal{W}'_1} H(Y|S, W'_1 = w'_1) \mathcal{P}(w'_1) = H(Y|S, W'_1), \\ H(S|Y, W_1, W_2) &= \int_{\mathcal{W}_1 \times \mathcal{W}_2} H(S|Y, W_1 = w_1, W_2 = w_2) dF(w_1, w_2) \\ &= \sum_{(w'_1, w'_2) \in \mathcal{W}'_1 \times \mathcal{W}'_2} H(S|Y, W'_1 = w'_1, W'_2 = w'_2) \mathcal{P}(w'_1, w'_2) = H(S|Y, W'_1, W'_2). \end{aligned}$$

Hence the three information constraints remain equal with  $\max(|\mathcal{W}'_1|, |\mathcal{W}'_2|) \leq d+1$ .

$$I(S; W_1, W_2, Y) = H(S) - H(S|W'_1, W'_2, Y) = I(S; W'_1, W'_2, Y),$$

$$I(W_1, S; Y) = H(Y) - H(Y|W'_1, S) = I(W'_1, S; Y).$$

This concludes the proof of Lemma 9.  $\square$

## APPENDIX I

## CONVERSE PROOF OF THEOREM II.6

We consider that the pair of rate and state leakage  $(\mathbf{R}, \mathbf{E})$  is achievable with a code with causal encoding. By definition II.5, for all  $\varepsilon > 0$ , there exist a  $\bar{n}$ , for all  $n \geq \bar{n}$ , there exist a code  $c^* \in \mathcal{C}(n, \mathcal{M})$  that satisfies

$$\frac{\log_2 |\mathcal{M}|}{n} \geq \mathbf{R} - \varepsilon, \quad (251)$$

$$\left| \mathcal{L}_{\mathbf{e}}(c) - \mathbf{E} \right| = \left| \frac{1}{n} I(S^n; Y^n) - \mathbf{E} \right| \leq \varepsilon, \quad (252)$$

$$\mathcal{P}_{\mathbf{e}}(c) = \mathbb{P}(M \neq \hat{M}) \leq \varepsilon. \quad (253)$$

We introduce the auxiliary random variables  $W_{1,i} = (M, S^{i-1})$  that satisfy the Markov chains of the set of distribution  $\mathbb{Q}_{\mathbf{c}}$  for all  $i \in \{1, \dots, n\}$ :

$$S_i \text{ independent of } W_{1,i}, \quad (254)$$

$$Y_i \text{ } \textcircled{-} \text{ } (X_i, S_i) \text{ } \textcircled{-} \text{ } W_{1,i}, \quad (255)$$

where (254) comes from the i.i.d. property of the source that induces the independence between  $S_i$  and  $(M, S^{i-1}) = W_{1,i}$ ; (255) comes from the memoryless property of the channel  $\mathcal{T}_{Y|XS}$ .

We introduce the random variable  $T$  that is uniformly distributed over the indices  $\{1, \dots, n\}$  and the corresponding mean random variables  $(S_T, X_T, W_{1,T}, Y_T)$ . The auxiliary random variables  $W_1 = (W_{1,T}, T)$  belongs to the set of distributions  $\mathbb{Q}_{\mathbf{c}}$  and satisfies the three information constraints of Theorem II.6

$$I(S; Y | W_1) \leq \mathbf{E} \leq H(S), \quad (256)$$

$$\mathbf{R} + \mathbf{E} \leq I(W_1, S; Y). \quad (257)$$

**First Constraint:**

$$n\mathbf{E} \geq I(S^n; Y^n) - n\varepsilon \quad (258)$$

$$= I(S^n; Y^n, M) - I(S^n; M|Y^n) - n\varepsilon \quad (259)$$

$$\geq \sum_{i=1}^n I(S_i; Y^n, M|S^{i-1}) - H(M|Y^n) - n\varepsilon \quad (260)$$

$$\geq \sum_{i=1}^n I(S_i; Y^n, M|S^{i-1}) - n2\varepsilon \quad (261)$$

$$= \sum_{i=1}^n I(S_i; Y^n|M, S^{i-1}) - n2\varepsilon \quad (262)$$

$$\geq \sum_{i=1}^n I(S_i; Y_i|M, S^{i-1}) - n2\varepsilon \quad (263)$$

$$= \sum_{i=1}^n I(S_i; Y_i|W_{1,i}) - n2\varepsilon \quad (264)$$

$$= nI(S_T; Y_T|W_{1,T}, T) - n2\varepsilon \quad (265)$$

$$= n \left( I(S; Y|W_1) - 2\varepsilon \right), \quad (266)$$

where (258) comes from the definition of achievable state leakage rate  $\mathbf{E}$ , stated in equation (252); (259) and (260) come from the properties of the mutual information; (261) comes from equation (253) and Fano's inequality, see [34, pp. 19]; (262) comes from the independence between the message  $M$  and the channel states  $(S^{i-1}, S_i)$ ; (263) comes from the properties of the mutual information; (264) comes from the introduction of the auxiliary random variable  $W_{1,i} = (M, S^{i-1})$ , for all  $i \in \{1, \dots, n\}$ ; (265) comes from the introduction of the uniform random variable  $T$  over  $\{1, \dots, n\}$  and the corresponding mean random variables  $S_T, W_{1,T}, Y_T$ ; (266) comes from identifying  $W_1 = (W_{1,T}, T)$  and  $S = S_T, Y = Y_T$ .

**Second Constraint:** From (132), we have

$$n\mathbf{E} \leq n \left( H(S) + \varepsilon \right). \quad (267)$$

**Third Constraint:** From (133) - (144), we have

$$n \left( \mathbf{E} + \mathbf{R} \right) \leq n \left( I(S, W_1; Y) + 3\varepsilon \right). \quad (268)$$

**Conclusion:** If the pair of rate and state leakage  $(\mathbf{R}, \mathbf{E})$  is achievable with a code with causal encoding, then the following equations are satisfied for all  $\varepsilon > 0$

$$I(S; Y|W_1) - 2\varepsilon \leq \mathbf{E} \leq H(S) + \varepsilon, \quad (269)$$

$$\mathbf{R} + \mathbf{E} \leq I(S, W_1; Y) + 3\varepsilon. \quad (270)$$

This corresponds to equations (9) and (10) and this concludes the converse proof of Theorem II.6.

**Remark I.1** *For the converse proof of Theorem II.6, the code with causal encoding is not necessarily deterministic. The same optimal performances can be obtained by considering a stochastic code with causal encoding.*

## APPENDIX J

### CONVERSE PROOF OF THEOREM IV.2

Consider that the triple of rate, state leakage and distribution  $(\mathbf{R}, \mathbf{E}, \mathcal{Q}_{USZXYV})$  is achievable with a code with causal encoding. We simplify the notation by using  $\mathcal{Q}$  in place of  $\mathcal{Q}_{USZXYV}$ , and we introduce the random event of error  $E \in \{0, 1\}$  defined with respect to the achievable distribution  $\mathcal{Q}$  by

$$E = \begin{cases} 0 & \text{if } \|Q^n - \mathcal{Q}\|_1 \leq \varepsilon \iff (U^n, S^n, Z^n, X^n, Y^n, V^n) \in T_\delta(\mathcal{Q}), \\ 1 & \text{if } \|Q^n - \mathcal{Q}\|_1 > \varepsilon \iff (U^n, S^n, Z^n, X^n, Y^n, V^n) \notin T_\delta(\mathcal{Q}). \end{cases} \quad (271)$$

The event  $E = 1$  occurs if the sequences  $(U^n, S^n, Z^n, X^n, Y^n, V^n) \notin T_\delta(\mathcal{Q})$  for the target distribution  $\mathcal{Q}$ . By definition IV.1, for all  $\varepsilon > 0$ , there exist a  $\bar{n}$ , for all  $n \geq \bar{n}$ , there exist a code  $c^* \in \mathcal{C}(n, \mathcal{M})$  that satisfies

$$\frac{\log_2 |\mathcal{M}|}{n} \geq \mathbf{R} - \varepsilon, \quad (272)$$

$$\left| \mathcal{L}_e(c^*) - \mathbf{E} \right| = \left| \frac{1}{n} I(U^n, S^n; Y^n, Z^n) - \mathbf{E} \right| \leq \varepsilon, \quad (273)$$

$$\mathcal{P}_e(c^*) = \mathbb{P}(M \neq \hat{M}) + \mathbb{P}\left(\|Q^n - \mathcal{Q}\|_1 > \varepsilon\right) \leq \varepsilon. \quad (274)$$

We introduce the auxiliary random variables  $W_{1,i} = (M, U^{i-1}, S^{i-1})$  and  $W_{2,i} = (Y_{i+1}^n, Z_{i+1}^n)$ , that satisfy the Markov chain of the set of distribution  $\mathbb{Q}_s$  for all  $i \in \{1, \dots, n\}$ :

$$(U_i, S_i) \text{ independent of } W_{1,i}, \quad (275)$$

$$X_i \text{ --- } (U_i, S_i, W_{1,i}) \text{ --- } W_{2,i}, \quad (276)$$

$$Y_i \text{ --- } (X_i, S_i) \text{ --- } (U_i, Z_i, W_{1,i}, W_{2,i}), \quad (277)$$

$$Z_i \text{ --- } (U_i, S_i) \text{ --- } (X_i, Y_i, W_{1,i}, W_{2,i}), \quad (278)$$

$$V_i \text{ --- } (Y_i, Z_i, W_{1,i}, W_{2,i}) \text{ --- } (U_i, S_i, X_i), \quad (279)$$

where (275) comes from the i.i.d. property of the source that induces the independence between  $(U_i, S_i)$  and  $(M, U^{i-1}, S^{i-1}) = W_{1,i}$ ; (276) comes from Lemma 10. It is a direct consequence of the causal encoding function, the memoryless property of the channel and the i.i.d. property of the source; (277) comes from the memoryless property of the channel  $\mathcal{T}_{Y|XS}$ ; (278) comes from the i.i.d. property of the



source  $\mathcal{P}_{USZ}$ ; (279) comes from Lemma 11. It is a direct consequence of the causal encoding function, the non-causal decoding function, the memoryless property of the channel and the i.i.d. property of the source.

We introduce the random variable  $T$  that is uniformly distributed over the indices  $\{1, \dots, n\}$  and the corresponding mean random variables  $W_{1,T}, W_{2,T}, U_T, S_T, Z_T, X_T, Y_T, V_T$ . The auxiliary random variables  $W_1 = (W_{1,T}, T)$  and  $W_2 = W_{2,T}$  belong to the set of distributions  $\mathbb{Q}_s$  and satisfy the three information constraints of Theorem IV.2:

$$I(U, S; W_1, W_2, Y, Z) \leq \mathbf{E} \leq H(U, S), \quad (280)$$

$$\mathbf{R} + \mathbf{E} \leq I(W_1, U, S; Y, Z). \quad (281)$$

### First Constraint:

$$n\mathbf{E} \geq I(U^n, S^n; Y^n, Z^n) - n\varepsilon \quad (282)$$

$$= I(U^n, S^n; Y^n, Z^n, M) - I(U^n, S^n; M|Y^n, Z^n) - n\varepsilon \quad (283)$$

$$\geq \sum_{i=1}^n I(U_i, S_i; Y^n, Z^n, M|U^{i-1}, S^{i-1}) - H(M|Y^n, Z^n) - n\varepsilon \quad (284)$$

$$\geq \sum_{i=1}^n I(U_i, S_i; Y^n, Z^n, M|U^{i-1}, S^{i-1}) - n2\varepsilon \quad (285)$$

$$= \sum_{i=1}^n I(U_i, S_i; Y^n, Z^n, M, U^{i-1}, S^{i-1}) - n2\varepsilon \quad (286)$$

$$\geq \sum_{i=1}^n I(U_i, S_i; Y_{i+1}^n, Z_{i+1}^n, M, U^{i-1}, S^{i-1}, Y_i, Z_i) - n2\varepsilon \quad (287)$$

$$= \sum_{i=1}^n I(U_i, S_i; W_{1,i}, W_{2,i}, Y_i, Z_i) - n2\varepsilon \quad (288)$$

$$= nI(U_T, S_T; W_{1,T}, W_{2,T}, Y_T, Z_T|T) - n2\varepsilon \quad (289)$$

$$= nI(U_T, S_T; W_{1,T}, W_{2,T}, Y_T, Z_T, T) - n2\varepsilon \quad (290)$$

$$= n \left( I(U_T, S_T; W_1, W_2, Y_T, Z_T) - 2\varepsilon \right) \quad (291)$$

$$\geq n \left( I(U_T, S_T; W_1, W_2, Y_T, Z_T|E=0) - 3\varepsilon \right) \quad (292)$$

$$\geq n \left( I(U, S; W_1, W_2, Y, Z) - 4\varepsilon \right), \quad (293)$$

where (282) comes from the definition of achievable state leakage rate  $\mathbf{E}$ , stated in equation (273); (283) and (284) come from the properties of the mutual information; (285) comes from equation (274) and Fano's inequality, see [34, pp. 19]; (286) comes from the i.i.d. property of the channel states that implies

$(U_i, S_i)$  is independent of  $(U^{i-1}, S^{i-1})$ ; (287) comes from the properties of the mutual information; (288) comes from the introduction of the auxiliary random variables  $W_{1,i} = (M, U^{i-1}, S^{i-1})$  and  $W_{2,i} = (Y_{i+1}^n, Z_{i+1}^n)$ , for all  $i \in \{1, \dots, n\}$ ; (289) comes from the introduction of the uniform random variable  $T$  over  $\{1, \dots, n\}$  and the corresponding mean random variables  $U_T, S_T, W_{1,T}, W_{2,T}, Y_T, Z_T$ ; (290) comes from the independence between  $T$  and  $(U_T, S_T)$ ; (291) comes from the identification of the auxiliary random variables  $W_1 = (W_{1,T}, T)$  and  $W_2 = W_{2,T}$ ; (292) comes from the empirical coordination requirement as stated in Lemma 4. The sequences of symbols  $(U^n, S^n, Z^n, X^n, Y^n, V^n)$  are not jointly typical with small error probability  $\mathbb{P}(E = 1)$ ; (293) comes from Lemma 6. The sequences of symbols  $(U^n, S^n, Z^n, X^n, Y^n, V^n)$  are jointly typical, hence the distribution of the mean random variables  $\mathcal{P}_{U_T S_T Z_T X_T Y_T V_T | E=0}$  is close to the target distribution  $\mathcal{Q}_{USZXYV}$ . The result of [32, Lemma 2.7, pp. 19] concludes.

**Second Constraint:**

$$n\mathbf{E} \leq I(U^n, S^n; Y^n, Z^n) + n\varepsilon \quad (294)$$

$$\leq H(U^n, S^n) + n\varepsilon \quad (295)$$

$$= n \left( H(U, S) + \varepsilon \right), \quad (296)$$

where (294) comes from the definition of the achievable state leakage rate  $\mathbf{E}$ , stated in equation (273); (295) comes from the properties of the mutual information; (296) comes from the i.i.d. property of the channel states  $(U, S)$ .

**Third Constraint:**

$$n(\mathbf{E} + \mathbf{R}) \leq I(U^n, S^n; Y^n, Z^n) + H(M) + n2\varepsilon \quad (297)$$

$$= I(U^n, S^n; Y^n, Z^n) + I(M; Y^n, Z^n) + H(M|Y^n, Z^n) + n2\varepsilon \quad (298)$$

$$\leq I(U^n, S^n; Y^n, Z^n) + I(M; Y^n, Z^n) + n3\varepsilon \quad (299)$$

$$\leq I(U^n, S^n; Y^n, Z^n) + I(M; Y^n, Z^n | U^n, S^n) + n3\varepsilon \quad (300)$$

$$= I(U^n, S^n, M; Y^n, Z^n) + n3\varepsilon \quad (301)$$

$$= \sum_{i=1}^n I(U^n, S^n, M; Y_i, Z_i | Y_{i+1}^n, Z_{i+1}^n) + n3\varepsilon \quad (302)$$

$$\leq \sum_{i=1}^n I(U^n, S^n, M, Y_{i+1}^n, Z_{i+1}^n; Y_i, Z_i) + n3\varepsilon \quad (303)$$

$$= \sum_{i=1}^n I(U_i, S_i, M, U^{i-1}, S^{i-1}; Y_i, Z_i)$$

$$\begin{aligned}
& + \sum_{i=1}^n I(U_{i+1}^n, S_{i+1}^n, Y_{i+1}^n, Z_{i+1}^n; Y_i | U_i, S_i, M, U^{i-1}, S^{i-1}) \\
& + \sum_{i=1}^n I(U_{i+1}^n, S_{i+1}^n, Y_{i+1}^n, Z_{i+1}^n; Z_i | U_i, S_i, M, U^{i-1}, S^{i-1}, Y_i) + n3\varepsilon \tag{304}
\end{aligned}$$

$$\begin{aligned}
& = \sum_{i=1}^n I(U_i, S_i, M, U^{i-1}, S^{i-1}; Y_i, Z_i) \\
& + \sum_{i=1}^n I(U_{i+1}^n, S_{i+1}^n, Y_{i+1}^n, Z_{i+1}^n; Z_i | U_i, S_i, M, U^{i-1}, S^{i-1}, Y_i) + n3\varepsilon \tag{305}
\end{aligned}$$

$$= \sum_{i=1}^n I(U_i, S_i, M, U^{i-1}, S^{i-1}; Y_i, Z_i) + n3\varepsilon \tag{306}$$

$$= \sum_{i=1}^n I(U_i, S_i, W_{1,i}; Y_i, Z_i) + n \cdot 3\varepsilon \tag{307}$$

$$= nI(U_T, S_T, W_{1,T}; Y_T, Z_T | T) + n \cdot 3\varepsilon \tag{308}$$

$$\leq nI(U_T, S_T, W_{1,T}, T; Y_T, Z_T) + n \cdot 3\varepsilon \tag{309}$$

$$= n \left( I(U_T, S_T, W_1; Y_T, Z_T) + 3\varepsilon \right) \tag{310}$$

$$\leq n \left( I(U_T, S_T, W_1; Y_T, Z_T | E = 0) + 4\varepsilon \right) \tag{311}$$

$$\leq n \left( I(U, S, W_1; Y, Z) + 5\varepsilon \right), \tag{312}$$

where (297) comes from the definition of achievable rate and information leakage  $(\mathbf{R}, \mathbf{E})$ , stated in equations (272) and (273); (299) comes from equation (274) and Fano's inequality, see [34, pp. 19]; (300) comes from the independence between the message  $M$  and the channel states  $(U^n, S^n)$ , hence  $I(M; Y^n, Z^n) \leq I(M; Y^n, Z^n, U^n, S^n) = I(M; Y^n, Z^n | U^n, S^n)$ ; (305) comes from the Markov chain  $Y_i \text{ --- } (U_i, S_i, M, U^{i-1}, S^{i-1}) \text{ --- } (U_{i+1}^n, S_{i+1}^n, Y_{i+1}^n, Z_{i+1}^n)$  stated in Lemma 12, which is a direct consequence of the causal encoding function and the memoryless property of the channel; (306) comes from the i.i.d. property of the source that induces the Markov chain  $Z_i \text{ --- } (U_i, S_i) \text{ --- } (U_{i+1}^n, S_{i+1}^n, Y_{i+1}^n, Z_{i+1}^n, M, U^{i-1}, S^{i-1}, Y_i)$ , hence we have:  $I(U_{i+1}^n, S_{i+1}^n, Y_{i+1}^n, Z_{i+1}^n; Z_i | U_i, S_i, M, U^{i-1}, S^{i-1}, Y_i) = 0$ ; (307) comes from the introduction of the auxiliary random variable  $W_{1,i} = (M, U^{i-1}, S^{i-1})$ , for all  $i \in \{1, \dots, n\}$ ; (308) comes from the introduction of the uniform random variable  $T$  over  $\{1, \dots, n\}$  and the corresponding mean random variables  $U_T, S_T, W_{1,T}, Y_T, Z_T$ ; (310) comes from the identification of the auxiliary random variable  $W_1 = (W_{1,T}, T)$ ; (311) comes from the empirical coordination requirement as stated in Lemma 5. The sequences of symbols  $(U^n, S^n, Z^n, X^n, Y^n, V^n)$  are not jointly typical with small error probability

$\mathbb{P}(E = 1)$ ; (312) comes from Lemma 6, since the sequences of symbols  $(U^n, S^n, Z^n, X^n, Y^n, V^n)$  are jointly typical, hence the distribution of the mean random variables  $\mathcal{P}_{U_T S_T Z_T X_T Y_T V_T | E=0}$  is close to the target distribution  $\mathcal{Q}_{USZXYV}$ . The result of [32, Lemma 2.7, pp. 19] concludes.

**Conclusion:** If the triple of rate, state leakage and distribution  $(\mathbf{R}, \mathbf{E}, \mathcal{Q}_{USZXYV})$  is achievable with a code with causal encoding, then the following equations are satisfied for all  $\varepsilon > 0$ :

$$I(U, S; W_1, W_2, Y, Z) - 4\varepsilon \leq \mathbf{E} \leq H(S) + \varepsilon, \quad (313)$$

$$\mathbf{R} + \mathbf{E} \leq I(U, S, W_1; Y, Z) + 5\varepsilon. \quad (314)$$

This corresponds to (28) and (29) and this concludes the converse proof of Theorem IV.2.

**Remark J.1** *For the converse proof of Theorem IV.2, the a code with causal encoding is not necessarily deterministic. The same optimal performances can be obtained by considering a stochastic code with causal encoding.*

**Lemma 10** *The causal encoding function, the memoryless property of the channel and the i.i.d. property of the source induce the Markov chain property*

$$X_i \text{ --- } (U_i, S_i, W_{1,i}) \text{ --- } W_{2,i}. \quad (315)$$

*This Markov chain is satisfied with  $W_{1,i} = (M, U^{i-1}, S^{i-1})$  and  $W_{2,i} = (Y_{i+1}^n, Z_{i+1}^n)$ , for all  $i \in \{1, \dots, n\}$ .*

*Proof.* [Lemma 10] The auxiliary random variables  $W_{1,i} = (M, U^{i-1}, S^{i-1})$  and  $W_{2,i} = (Y_{i+1}^n, Z_{i+1}^n)$  satisfy the following equations for all  $(u^n, s^n, z^n, x^n, y^n, v^n, m)$ :

$$\begin{aligned} & \mathcal{P}(w_{2,i} | u_i, s_i, w_{1,i}, x_i) = \mathcal{P}(y_{i+1}^n, z_{i+1}^n | u_i, s_i, m, u^{i-1}, s^{i-1}, x_i) \\ &= \sum_{u_{i+1}^n, s_{i+1}^n, x_{i+1}^n} \mathcal{P}(u_{i+1}^n, s_{i+1}^n, x_{i+1}^n, y_{i+1}^n, z_{i+1}^n | u_i, s_i, m, u^{i-1}, s^{i-1}, x_i) \end{aligned} \quad (316)$$

$$\begin{aligned} &= \sum_{u_{i+1}^n, s_{i+1}^n, x_{i+1}^n} \mathcal{P}(u_{i+1}^n, s_{i+1}^n, x_{i+1}^n | u_i, s_i, m, u^{i-1}, s^{i-1}, x_i) \\ & \mathcal{P}(y_{i+1}^n, z_{i+1}^n | u_{i+1}^n, s_{i+1}^n, x_{i+1}^n, u_i, s_i, m, u^{i-1}, s^{i-1}, x_i) \end{aligned} \quad (317)$$

$$= \sum_{u_{i+1}^n, s_{i+1}^n, x_{i+1}^n} \mathcal{P}(u_{i+1}^n, s_{i+1}^n, x_{i+1}^n | u_i, s_i, m, u^{i-1}, s^{i-1}) \quad (318)$$

$$= \sum_{u_{i+1}^n, s_{i+1}^n, x_{i+1}^n} \mathcal{P}(u_{i+1}^n, s_{i+1}^n, x_{i+1}^n | u_i, s_i, m, u^{i-1}, s^{i-1}) \mathcal{P}(y_{i+1}^n | s_{i+1}^n, x_{i+1}^n) \mathcal{P}(z_{i+1}^n | u_{i+1}^n, s_{i+1}^n) \quad (319)$$

$$= \sum_{u_{i+1}^n, s_{i+1}^n, x_{i+1}^n} \mathcal{P}(u_{i+1}^n, s_{i+1}^n, x_{i+1}^n, y_{i+1}^n, z_{i+1}^n | u_i, s_i, m, u^{i-1}, s^{i-1}) \quad (320)$$

$$= \mathcal{P}(y_{i+1}^n, z_{i+1}^n | u_i, s_i, m, u^{i-1}, s^{i-1}) = \mathcal{P}(w_{2,i} | u_i, s_i, w_{1,i}), \quad (321)$$

where (318) comes from the causal encoding function that induces the Markov chain  $X_i \text{---} (U_i, S_i, M, U^{i-1}, S^{i-1}) \text{---} (U_{i+1}^n, S_{i+1}^n, X_{i+1}^n)$ ; (319) comes from the memoryless property of the channel  $Y_{i+1}^n \text{---} (S_{i+1}^n, X_{i+1}^n) \text{---} (U_{i+1}^n, U_i, S_i, M, U^{i-1}, S^{i-1}, X_i)$  and the i.i.d. property of the source  $Z_{i+1}^n \text{---} (S_{i+1}^n, U_{i+1}^n) \text{---} (U_{i+1}^n, U_i, S_i, M, U^{i-1}, S^{i-1}, X_i, Y_{i+1}^n)$ . This concludes the proof of Lemma 10.  $\square$

**Lemma 11** *The causal encoding function, the non-causal decoding function, the memoryless property of the channel and the i.i.d. property of the source induce the Markov chain property*

$$V_i \text{---} (Y_i, Z_i, W_{1,i}, W_{2,i}) \text{---} (U_i, S_i, X_i). \quad (322)$$

This Markov chain is satisfied with  $W_{1,i} = (M, U^{i-1}, S^{i-1})$  and  $W_{2,i} = (Y_{i+1}^n, Z_{i+1}^n)$ , for all  $i \in \{1, \dots, n\}$ .

*Proof.* [Lemma 11] The auxiliary random variables  $W_{1,i} = (M, U^{i-1}, S^{i-1})$  and  $W_{2,i} = (Y_{i+1}^n, Z_{i+1}^n)$  satisfy the following equations for all  $(u^n, s^n, z^n, w_1^n, w_2^n, x^n, y^n, v^n, m)$ :

$$\begin{aligned} & \mathcal{P}(v_i | y_i, z_i, w_{1,i}, w_{2,i}, u_i, s_i, x_i) \\ &= \mathcal{P}(v_i | y_i, z_i, m, u^{i-1}, s^{i-1}, y_{i+1}^n, z_{i+1}^n, u_i, s_i, x_i) \\ &= \sum_{x^{i-1}, y^{i-1}, z^{i-1}} \mathcal{P}(v_i, x^{i-1}, y^{i-1}, z^{i-1} | y_i, z_i, m, u^{i-1}, s^{i-1}, y_{i+1}^n, z_{i+1}^n, u_i, s_i, x_i) \quad (323) \\ &= \sum_{x^{i-1}, y^{i-1}, z^{i-1}} \mathcal{P}(z^{i-1} | y_i, z_i, m, u^{i-1}, s^{i-1}, y_{i+1}^n, z_{i+1}^n, u_i, s_i, x_i) \end{aligned}$$

$$\begin{aligned} & \mathcal{P}(x^{i-1} | y_i, z_i, m, u^{i-1}, s^{i-1}, y_{i+1}^n, z_{i+1}^n, u_i, s_i, x_i, z^{i-1}) \\ & \mathcal{P}(y^{i-1} | y_i, z_i, m, u^{i-1}, s^{i-1}, y_{i+1}^n, z_{i+1}^n, u_i, s_i, x_i, z^{i-1}, x^{i-1}) \\ & \mathcal{P}(v_i | y_i, z_i, m, u^{i-1}, s^{i-1}, y_{i+1}^n, z_{i+1}^n, u_i, s_i, x_i, z^{i-1}, x^{i-1}, y^{i-1}). \quad (324) \end{aligned}$$

We can remove  $(u_i, s_i, x_i)$ , in the four conditional distributions

$$\mathcal{P}(z^{i-1}|y_i, z_i, m, u^{i-1}, s^{i-1}, y_{i+1}^n, z_{i+1}^n, u_i, s_i, x_i) = \mathcal{P}(z^{i-1}|u^{i-1}, s^{i-1}), \quad (325)$$

$$\mathcal{P}(x^{i-1}|y_i, z_i, m, u^{i-1}, s^{i-1}, y_{i+1}^n, z_{i+1}^n, u_i, s_i, x_i, z^{i-1}) = \mathcal{P}(x^{i-1}|m, u^{i-1}, s^{i-1}), \quad (326)$$

$$\mathcal{P}(y^{i-1}|y_i, z_i, m, u^{i-1}, s^{i-1}, y_{i+1}^n, z_{i+1}^n, u_i, s_i, x_i, z^{i-1}, x^{i-1}) = \mathcal{P}(y^{i-1}|s^{i-1}, x^{i-1}), \quad (327)$$

$$\mathcal{P}(v_i|y_i, z_i, m, u^{i-1}, s^{i-1}, y_{i+1}^n, z_{i+1}^n, u_i, s_i, x_i, z^{i-1}, x^{i-1}, y^{i-1}) = \mathcal{P}(v_i|y_i, z_i, y_{i+1}^n, z_{i+1}^n, z^{i-1}, y^{i-1}), \quad (328)$$

where (325) comes from the i.i.d. property of the information source:  $Z^{i-1}$  only depends on  $(U^{i-1}, S^{i-1})$ ; (326) comes from the causal encoding that induces the Markov chain  $X^{i-1} \text{---} (M, U^{i-1}, S^{i-1}) \text{---} (Y_i, Z_i, Y_{i+1}^n, Z_{i+1}^n, X_i, Z^{i-1}, U_i, S_i)$ ; (327) comes from the memoryless property of the channel:  $Y^{i-1}$  only depends on  $(X^{i-1}, S^{i-1})$ ; (328) comes from the non-causal decoding that induces the Markov chain  $V_i \text{---} (Y_i, Z_i, Y_{i+1}^n, Z_{i+1}^n, Z^{i-1}, Y^{i-1}) \text{---} (M, U^{i-1}, S^{i-1}, U_i, S_i, X_i, X^{i-1})$ . Hence, for all  $(u^n, s^n, z^n, x^n, y^n, v^n, m)$  we have

$$\begin{aligned} & \mathcal{P}(v_i|y_i, z_i, w_{1,i}, w_{2,i}, u_i, s_i, x_i) \\ &= \sum_{x^{i-1}, y^{i-1}, z^{i-1}} \mathcal{P}(v_i, x^{i-1}, y^{i-1}, z^{i-1}|y_i, z_i, m, u^{i-1}, s^{i-1}, y_{i+1}^n, z_{i+1}^n) \end{aligned} \quad (329)$$

$$= \mathcal{P}(v_i|y_i, z_i, m, u^{i-1}, s^{i-1}, y_{i+1}^n, z_{i+1}^n) \quad (330)$$

$$= \mathcal{P}(v_i|y_i, z_i, w_{1,i}, w_{2,i}). \quad (331)$$

The above equation corresponds to the Markov chain  $V_i \text{---} (Y_i, Z_i, W_{1,i}, W_{2,i}) \text{---} (U_i, S_i, X_i)$  and it concludes the proof of Lemma 11.  $\square$

**Lemma 12** *The causal encoding function and the memoryless property of the channel induce the following Markov chain property*

$$Y_i \text{---} (U_i, S_i, M, U^{i-1}, S^{i-1}) \text{---} (U_{i+1}^n, S_{i+1}^n, Y_{i+1}^n, Z_{i+1}^n). \quad (332)$$

*This Markov chain is satisfied for all  $i \in \{1, \dots, n\}$ .*

*Proof.* [Lemma 12] For all  $(u^n, s^n, z^n, x^n, y^n, v^n, m)$ , we have

$$\begin{aligned} & \mathcal{P}(y_i | u_i, s_i, m, u^{i-1}, s^{i-1}, u_{i+1}^n, s_{i+1}^n, y_{i+1}^n, z_{i+1}^n) \\ &= \sum_{x_i} \mathcal{P}(x_i, y_i | u_i, s_i, m, u^{i-1}, s^{i-1}, u_{i+1}^n, s_{i+1}^n, y_{i+1}^n, z_{i+1}^n) \end{aligned} \quad (333)$$

$$\begin{aligned} &= \sum_{x_i} \mathcal{P}(x_i | u_i, s_i, m, u^{i-1}, s^{i-1}, u_{i+1}^n, s_{i+1}^n, y_{i+1}^n, z_{i+1}^n) \\ & \quad \mathcal{P}(y_i | x_i, u_i, s_i, m, u^{i-1}, s^{i-1}, u_{i+1}^n, s_{i+1}^n, y_{i+1}^n, z_{i+1}^n) \end{aligned} \quad (334)$$

$$= \sum_{x_i} \mathcal{P}(x_i | u_i, s_i, m, u^{i-1}, s^{i-1}) \mathcal{P}(y_i | x_i, u_i, s_i, m, u^{i-1}, s^{i-1}, u_{i+1}^n, s_{i+1}^n, y_{i+1}^n, z_{i+1}^n) \quad (335)$$

$$= \sum_{x_i} \mathcal{P}(x_i | u_i, s_i, m, u^{i-1}, s^{i-1}) \cdot \mathcal{P}(y_i | x_i, s_i) \quad (336)$$

$$= \sum_{x_i} \mathcal{P}(x_i, y_i | u_i, s_i, m, u^{i-1}, s^{i-1}) = \mathcal{P}(y_i | u_i, s_i, m, u^{i-1}, s^{i-1}), \quad (337)$$

where (335) comes from the causal encoding function that induces the Markov chain  $X_i \text{---} (U_i, S_i, M, U^{i-1}, S^{i-1}) \text{---} (U_{i+1}^n, S_{i+1}^n, Y_{i+1}^n, Z_{i+1}^n)$ ; (336) comes from the memoryless property of the channel that induces the Markov chain:  $Y_i \text{---} (X_i, S_i) \text{---} (U_i, M, U^{i-1}, S^{i-1}, U_{i+1}^n, S_{i+1}^n, Y_{i+1}^n, Z_{i+1}^n)$ . This concludes the proof of Lemma 12.  $\square$

## APPENDIX K

### CONVERSE PROOF OF THEOREM IV.4

The converse proof for information constraints (33) and (34) follows from similar arguments as for the converse proof of Theorem II.3, in Appendix B. We now prove the converse result for the information constraint (32). We consider the distribution  $\mathcal{Q}_{SXY_1Y_2V}$ , also denoted by  $\mathcal{Q}$ , and we introduce the random event of error  $E \in \{0, 1\}$  defined by

$$E = \begin{cases} 0 & \text{if } \|Q^n - \mathcal{Q}\|_1 \leq \varepsilon \iff (S^n, X^n, Y_1^n, Y_2^n, V^n) \in T_\delta(\mathcal{Q}), \\ 1 & \text{if } \|Q^n - \mathcal{Q}\|_1 > \varepsilon \iff (S^n, X^n, Y_1^n, Y_2^n, V^n) \notin T_\delta(\mathcal{Q}). \end{cases} \quad (338)$$





$$=n \left( I(W_{1,T}, W_{2,T}; Y_{1,T}|T) - I(W_{2,T}; S_T, Y_{2,T}|W_{1,T}, T) + 2\varepsilon \right) \quad (352)$$

$$\leq n \left( I(T, W_{1,T}, W_{2,T}; Y_{1,T}) - I(W_{2,T}; S_T, Y_{2,T}|W_{1,T}, T) + 2\varepsilon \right) \quad (353)$$

$$\leq n \max_{Q \in \mathbb{Q}_f} \left( I(W_1, W_2; Y_{1,T}) - I(W_2; S_T, Y_{2,T}|W_1) + 2\varepsilon \right) \quad (354)$$

$$\leq n \max_{Q \in \mathbb{Q}_f} \left( I(W_1, W_2; Y_{1,T}|E=0) - I(W_2; S_T, Y_{2,T}|W_1, E=0) + 3\varepsilon \right) \quad (355)$$

$$\leq n \max_{Q \in \mathbb{Q}_f} \left( I(W_1, W_2; Y_1) - I(W_2; S, Y_2|W_1) + 4\varepsilon \right), \quad (356)$$

where (352) comes from the introduction of the uniform random variable  $T$  over  $\{1, \dots, n\}$  and the introduction of the corresponding mean random variables  $S_T, W_{1,T}, W_{2,T}, X_T, Y_{1,T}, Y_{2,T}, V_T$ ; (353) and comes from the properties of the mutual information; (354) comes from the identification of  $W_1$  with  $(W_{1,T}, T)$ ,  $W_2$  with  $W_{2,T}$  and taking the maximum over the distributions that belong to the set  $\mathbb{Q}_f$ ; (355) comes from the empirical coordination requirement, as stated in Lemma 5 in Section B-B, since the sequences are not jointly typical with small error probability  $\mathbb{P}(E=1)$ ; (356) comes from Lemma 6 in Appendix B-B, that states that the distribution induced by the coding scheme  $\mathcal{P}_{S_T X_T Y_{1,T} Y_{2,T} V_T|E=0}$  is close to the target distribution  $\mathcal{Q}(s, x, y_1, y_2, v)$ . The result of [32, Lemma 2.7, pp. 19] concludes the converse proof of Theorem IV.4.

*Proof.* [Lemma 13] Equation (348) comes from the i.i.d. property of the source  $S$ , the independence of  $S$  with respect to the message  $M$  and the causal encoding function, hence  $S_i$  is independent of the past channel inputs  $X^{i-1}$ , hence  $S_i$  is independent of  $Y_2^{i-1}$ ; (349) comes from the memoryless property of the channel, hence  $(Y_{1,i}, Y_{2,i})$  is drawn with  $(X_i, S_i)$ ; (350) comes from

$$\mathcal{P}(x_i, y_{1,i}|s_i, y_{2,i}, w_{1,i}, w_{2,i}) \quad (357)$$

$$= \mathcal{P}(x_i|s_i, y_{2,i}, m, s^{i-1}, y_2^{i-1}, y_{1,i+1}^n) \mathcal{P}(y_{1,i}|x_i, s_i, y_{2,i}, m, s^{i-1}, y_2^{i-1}, y_{1,i+1}^n) \quad (358)$$

$$= \mathcal{P}(x_i|s_i, y_{2,i}, m, s^{i-1}, y_2^{i-1}) \mathcal{P}(y_{1,i}|x_i, s_i, y_{2,i}, m, s^{i-1}, y_2^{i-1}, y_{1,i+1}^n) \quad (359)$$

$$= \mathcal{P}(x_i|s_i, y_{2,i}, m, s^{i-1}, y_2^{i-1}) \mathcal{P}(y_{1,i}|x_i, s_i, y_{2,i}, m, s^{i-1}, y_2^{i-1}) \quad (360)$$

$$= \mathcal{P}(x_i, y_{1,i}|s_i, y_{2,i}, w_{1,i}) \quad \forall (s^n, x^n, y_1^n, y_2^n, w_1^n, w_2^n). \quad (361)$$

Equation (358) comes from the choice of the auxiliary random variables  $W_{1,i} = (M, S^{i-1}, Y_2^{i-1})$  and  $W_{2,i} = Y_{1,i+1}^n$ ; (359) comes from the causal encoding that implies  $X_i$  is a function of  $(S_i, M, S^{i-1}, Y_2^{i-1})$  but not of  $Y_{1,i+1}^n$ ; (360) comes from the memoryless property of the channel that implies  $Y_{1,i}$  is drawn depending on  $(X_i, S_i, Y_{2,i})$  and not on  $Y_{1,i+1}^n$ ; (361) concludes that the Markov chain (350) holds.

Equation (351) comes from the following equations, for all  $(u^n, s^n, z^n, x^n, y^n, v^n, m)$

$$\mathcal{P}(v_i | y_{1,i}, w_{1,i}, w_{2,i}, x_i, s_i, y_{2,i}) \quad (362)$$

$$= \sum_{x^{i-1}, y_1^{i-1}} \mathcal{P}(v_i, x^{i-1}, y_1^{i-1} | y_{1,i}, m, s^{i-1}, y_2^{i-1}, y_{1,i+1}^n, x_i, s_i, y_{2,i}) \quad (363)$$

$$= \sum_{x^{i-1}, y_1^{i-1}} \mathcal{P}(x^{i-1} | y_{1,i}, m, s^{i-1}, y_2^{i-1}, y_{1,i+1}^n, x_i, s_i, y_{2,i})$$

$$\mathcal{P}(y_1^{i-1} | x^{i-1}, y_{1,i}, m, s^{i-1}, y_2^{i-1}, y_{1,i+1}^n, x_i, s_i, y_{2,i})$$

$$\mathcal{P}(v_i | x^{i-1}, y_1^{i-1}, y_{1,i}, m, s^{i-1}, y_2^{i-1}, y_{1,i+1}^n, x_i, s_i, y_{2,i}) \quad (364)$$

$$= \sum_{x^{i-1}, y_1^{i-1}} \mathcal{P}(x^{i-1} | y_{1,i}, m, s^{i-1}, y_2^{i-1}, y_{1,i+1}^n) \quad (365)$$

$$\mathcal{P}(y_1^{i-1} | x^{i-1}, y_{1,i}, m, s^{i-1}, y_2^{i-1}, y_{1,i+1}^n) \quad (366)$$

$$\mathcal{P}(v_i | x^{i-1}, y_1^{i-1}, y_{1,i}, m, s^{i-1}, y_2^{i-1}, y_{1,i+1}^n) \quad (367)$$

$$= \mathcal{P}(v_i | y_{1,i}, w_{1,i}, w_{2,i}), \quad (368)$$

where (363) comes from the choice of the auxiliary random variables  $W_{1,i} = (M, S^{i-1}, Y_2^{i-1})$  and  $W_{2,i} = Y_{1,i+1}^n$ ; (364) comes from the decomposition of the probability; (365) comes from the causal encoding that implies  $X^{i-1}$  is a function of  $(M, S^{i-1}, Y_2^{i-2})$  but not of  $(X_i, S_i, Y_{2,i})$ ; (366) comes from the memoryless property of the channel that implies  $Y_1^{i-1}$  depends only on  $(X^{i-1}, S^{i-1}, Y_2^{i-1})$  and not on  $(X_i, S_i, Y_{2,i})$ ; (367) comes from the non-causal decoding that implies  $V_i$  is a function of  $(Y_1^{i-1}, Y_{1,i}, Y_{1,i+1}^n)$  but not of  $(X_i, S_i, Y_{2,i})$ ; (368) concludes that the Markov chain (351) holds. This concludes the proof of Lemma 13.  $\square$

## APPENDIX L

### CONVERSE PROOF OF THEOREM V.2

We consider that the triple of rate, information leakage and distribution  $(\mathbf{R}, \mathbf{E}, \mathcal{Q}_{SXYV})$  is achievable with a code with strictly causal encoding. We introduce the random event of error  $E \in \{0, 1\}$  defined with respect to the achievable distribution  $\mathcal{Q}_{SXYV}$ , also denoted by  $\mathcal{Q}$ , by

$$E = \begin{cases} 0 & \text{if } \|Q^n - \mathcal{Q}\|_{\text{tv}} \leq \varepsilon \iff (S^n, X^n, Y^n, V^n) \in T_\delta(\mathcal{Q}), \\ 1 & \text{if } \|Q^n - \mathcal{Q}\|_{\text{tv}} > \varepsilon \iff (S^n, X^n, Y^n, V^n) \notin T_\delta(\mathcal{Q}). \end{cases} \quad (369)$$

The event  $E = 1$  occurs if the sequences  $(S^n, X^n, Y^n, V^n) \notin T_\delta(\mathcal{Q})$  for the target distribution  $\mathcal{Q}$ . By definition V.1, for all  $\varepsilon > 0$ , there exist a  $\bar{n}$  such that for all  $n \geq \bar{n}$ , there exist a code  $c^* \in \mathcal{C}(n, \mathcal{M})$  that

satisfies

$$\frac{\log_2 |\mathcal{M}|}{n} \geq \mathbf{R} - \varepsilon, \quad (370)$$

$$\left| \mathcal{L}_e(c^*) - \mathbf{E} \right| = \left| \frac{1}{n} I(S^n; Y^n) - \mathbf{E} \right| \leq \varepsilon, \quad (371)$$

$$\mathcal{P}_e(c^*) = \mathbb{P}(M \neq \hat{M}) + \mathbb{P}\left(\|Q^n - \mathcal{Q}\|_{\text{tv}} > \varepsilon\right) \leq \varepsilon. \quad (372)$$

We introduce the auxiliary random variables  $W_{2,i} = (M, S^{i-1}, Y_{i+1}^n)$  that satisfy the Markov chains of the set of distribution  $\mathbb{Q}_{\text{sc}}$ , for all  $i \in \{1, \dots, n\}$ :

$$S_i \text{ independent of } X_i, \quad (373)$$

$$Y_i \text{ --- } (X_i, S_i) \text{ --- } W_{2,i}, \quad (374)$$

$$V_i \text{ --- } (Y_i, X_i, W_{2,i}) \text{ --- } S_i, \quad (375)$$

where (373) follows from the strictly causal encoding function; (374) follows from the memoryless property of the channel  $\mathcal{T}_{Y|XS}$ ; (375) is obtained with a slight modification of Lemma 2 in which only the random variable  $S_i$  is removed from (152), (153), and (154). It is a direct consequence of the strictly causal encoding function, the non-causal decoding function and the memoryless property of the channel  $\mathcal{T}_{Y|XS}$ .

We introduce the random variable  $T$  that is uniformly distributed over the indices  $\{1, \dots, n\}$  and the corresponding mean random variables  $W_{2,T}$ ,  $S_T$ ,  $X_T$ ,  $Y_T$ ,  $V_T$ . The auxiliary random variable  $W_2 = (W_{2,T}, T)$  belongs to the set of distributions  $\mathbb{Q}_{\text{sc}}$  and satisfies the information constraints of Theorem V.2

$$I(S; X, W_2, Y) \leq \mathbf{E} \leq H(S), \quad (376)$$

$$\mathbf{R} + \mathbf{E} \leq I(X, S; Y). \quad (377)$$

### First Constraint:

$$n\mathbf{E} \geq I(S^n; Y^n) - n\varepsilon \quad (378)$$

$$= I(S^n; Y^n, M) - I(S^n; M|Y^n) - n\varepsilon \quad (379)$$

$$\geq nH(S) - H(S^n|Y^n, M) - H(M|Y^n) - n\varepsilon \quad (380)$$

$$\geq nH(S) - H(S^n|Y^n, M) - n2\varepsilon \quad (381)$$

$$= nH(S) - \sum_{i=1}^n H(S_i|Y^n, M, S^{i-1}) - n2\varepsilon \quad (382)$$

$$\geq nH(S) - \sum_{i=1}^n H(S_i|Y_{i+1}^n, Y_i, M, S^{i-1}) - n2\varepsilon \quad (383)$$

$$=nH(S) - \sum_{i=1}^n H(S_i|Y_{i+1}^n, Y_i, M, S^{i-1}, X_i) - n2\varepsilon \quad (384)$$

$$=nH(S) - \sum_{i=1}^n H(S_i|W_{2,i}, X_i, Y_i) - n2\varepsilon \quad (385)$$

$$=nH(S) - nH(S_T|W_{2,T}, X_T, Y_T, T) - n2\varepsilon \quad (386)$$

$$=nH(S) - nH(S_T|W_2, X_T, Y_T) - n2\varepsilon \quad (387)$$

$$\geq nH(S) - nH(S_T|W_2, X_T, Y_T, E=0) - n3\varepsilon \quad (388)$$

$$\geq nH(S) - nH(S|W_2, X, Y) - n4\varepsilon \quad (389)$$

$$=n\left(I(S; W, X, Y) - 4\varepsilon\right), \quad (390)$$

where (378) comes from the definition of achievable information leakage rate  $\mathbf{E}$ , stated in equation (371); (381) comes from equation (372) and Fano's inequality, see [34, pp. 19]; (384) comes from the strictly causal encoding  $X_i = f_i(M, S^{i-1})$  that implies  $I(S_i; X_i|Y_{i+1}^n, Y_i, M, S^{i-1}) = 0$ , for all  $i \in \{1, \dots, n\}$ ; (385) comes from the introduction of the auxiliary random variable  $W_{2,i} = (M, S^{i-1}, Y_{i+1}^n)$ , for all  $i \in \{1, \dots, n\}$ ; (386) comes from the introduction of the uniform random variable  $T$  over  $\{1, \dots, n\}$  and the introduction of the corresponding mean random variables  $S_T, W_{2,T}, X_T, Y_T$ ; (387) comes from identifying  $W_2 = (W_{2,T}, T)$ ; (388) comes from the empirical coordination requirement as stated in Lemma 4, since the sequences of symbols  $(S^n, X^n, Y^n, V^n)$  are not jointly typical with small error probability  $\mathbb{P}(E = 1)$ ; (389) comes from Lemma 6. The sequences of symbols  $(S^n, X^n, Y^n, V^n)$  are jointly typical, hence the distribution of the mean random variables  $\mathcal{P}_{S_T X_T Y_T V_T|E=0}$  is close to the target distribution  $\mathcal{Q}_{SXYV}$ . The result of [32, Lemma 2.7, pp. 19] concludes.

**Second Constraint:** From (132), we have

$$n\mathbf{E} \leq n\left(H(S) + \varepsilon\right). \quad (391)$$

**Third Constraint:**

$$n(\mathbf{E} + \mathbf{R}) \leq I(S^n; Y^n) + H(M) + n2\varepsilon \quad (392)$$

$$= I(S^n; Y^n) + I(M; Y^n) + H(M|Y^n) + n2\varepsilon \quad (393)$$

$$\leq I(S^n; Y^n) + I(M; Y^n) + n3\varepsilon \quad (394)$$

$$\leq I(S^n; Y^n) + I(M; Y^n|S^n) + n3\varepsilon \quad (395)$$

$$= I(S^n, M; Y^n) + n3\varepsilon \quad (396)$$

$$\leq I(S^n, X^n; Y^n) + n3\varepsilon \quad (397)$$

$$\leq n\left(I(S, X; Y) + 3\varepsilon\right), \quad (398)$$

where (392) comes from the definition of achievable rate and information leakage  $(\mathbf{R}, \mathbf{E})$ , stated in equations (370) and (371); (394) comes from equation (372) and Fano's inequality, see [34, pp. 19]; (395) comes from the independence between the message  $M$  and the channel states  $S^n$ , hence  $I(M; Y^n) \leq I(M; Y^n, S^n) = I(M; Y^n|S^n)$ ; (397) comes from the Markov chain  $Y^n \dashv\vdash (X^n, S^n) \dashv\vdash M$ , induced by the channel; (398) comes from the memoryless property of the channel that implies:  $H(Y^n|S^n, X^n) = nH(Y|X, S)$ .

**Conclusion:** If the triple of rate, information leakage and distribution  $(\mathbf{R}, \mathbf{E}, \mathcal{Q}_{SXYV})$  is achievable with a code with strictly causal encoding, then the following equations are satisfied for all  $\varepsilon > 0$ :

$$I(S; X, Y, W_2) - 4\varepsilon \leq \mathbf{E} \leq H(S) + \varepsilon, \quad (399)$$

$$\mathbf{R} + \mathbf{E} \leq I(X, S; Y) + 3\varepsilon. \quad (400)$$

This corresponds to (35) and (36) and this concludes the converse proof of Theorem V.2.