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# Toeplitz operators with analytic symbols

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## Abstract

We develop a new semiclassical calculus in analytic regularity, and we apply these techniques to the study of Berezin-Toeplitz quantization in real-analytic regularity.

We provide asymptotic formulas for the Bergman projector and Berezin-Toeplitz operators on a compact Kähler manifold. These objects depend on an integer  $N$  and we study, in the limit  $N \rightarrow +\infty$ , situations in which one can control them up to an error  $O(e^{-cN})$  for some  $c > 0$ .

We develop a calculus of Toeplitz operators with real-analytic symbols, which applies to Kähler manifolds with real-analytic metrics. In particular, we prove that the Bergman kernel is controlled up to  $O(e^{-cN})$  on any real-analytic Kähler manifold as  $N \rightarrow +\infty$ . We also prove that Toeplitz operators with analytic symbols can be composed and inverted up to  $O(e^{-cN})$ . As an application, we study eigenfunction concentration for Toeplitz operators if both the manifold and the symbol are real-analytic. In this case we prove exponential decay in the classically forbidden region.

## 1 Introduction

Toeplitz quantization associates, to a real-valued function  $f$  on a Kähler manifold  $M$ , a family of *Toeplitz operators*, which are self-adjoint linear operators  $(T_N(f))_{N \geq 1}$  acting on holomorphic sections over  $M$ . Examples of Toeplitz operators are spin operators (where  $M = \mathbb{S}^2$ ), which are indexed

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by the total spin  $\frac{N}{2} \in \frac{1}{2}\mathbb{N}$ . Toeplitz operators also include pseudodifferential operators on  $\mathbb{R}^n$ . In this paper we study *exponential estimates* for these operators, that is, approximate expressions with  $O(e^{-cN})$  remainder for some  $c > 0$ .

The family of holomorphic section spaces in Toeplitz quantization is described by a sequence of *Bergman projectors*  $(S_N)_{N \geq 1}$  (also known as Szegő projectors). The operators  $S_N$  can be written as integral operators (the integral kernels are sections of suitable bundles over  $M \times M$ ), and a first step toward understanding Toeplitz quantization is the asymptotic study, in the limit  $N \rightarrow +\infty$ , of the Bergman kernel.

We show (Theorem A) that the Bergman kernel admits an asymptotic expansion in decreasing powers of  $N$ , up to an error  $O(e^{-cN})$ , as soon as the Kähler manifold is compact and real-analytic. To study the Bergman projector, as well as compositions of Toeplitz operators (Theorem B), it is useful to interpret the  $N \rightarrow +\infty$  limit as a semiclassical limit (with semiclassical parameter  $\hbar = \frac{1}{N}$ ). We build new semiclassical tools in real-analytic regularity (in particular, new analytic symbol classes, see Definition 3.3), which can be of more general use.

This study of the calculus of Toeplitz operators allows us to state results concerning sequences of eigenfunctions of Toeplitz operators  $(T_N(f))_{N \geq 1}$  for a real-analytic  $f$ . We prove the following (Theorem C): if  $(u_N)_{N \geq 1}$  is a sequence of normalised eigenfunctions with energy near  $E \in \mathbb{R}$ , that is,

$$T_N(f)u_N = \lambda_N u_N, \quad \lambda_N \xrightarrow{N \rightarrow +\infty} E, \quad \|u_N\|_{L^2(M, L^{\otimes N})} = 1,$$

and if  $V \subset M$  is an open set at positive distance from  $\{x \in M, f(x) = E\}$ , then

$$\|u_N\|_{L^2(V, L^{\otimes N})} \leq C e^{-cN}$$

for some  $C > 0, c > 0$  independent on  $N$ . We say that  $(u_N)_{N \geq 1}$  has an *exponential decay rate* on  $V$ .

In [15], we provide an asymptotic expansion, with error  $O(e^{-cN})$ , for the ground state of a Toeplitz operator  $T_N(f)$ , for  $f$  real-analytic and Morse.

## 1.1 Bergman kernels and Toeplitz operators

This article is devoted to the study of exponential estimates concerning the Bergman kernel and Toeplitz operators on Kähler manifolds with real-analytic data. In this subsection we quickly recall the framework of Toeplitz operators, introduced in [5, 3]. We refer the reader to more detailed introductions [4, 9, 14].

### Definition 1.1.

- A compact Kähler manifold  $(M, J, \omega)$  is said to be quantizable when the symplectic form  $\omega$  has integer cohomology: there exists a unique Hermitian line bundle  $(L, h)$  over  $M$  such that the curvature of  $h$  is

$-2i\pi\omega$ . This line bundle is called the prequantum line bundle over  $(M, J, \omega)$ . The manifold  $(M, J, \omega)$  is said to be *real-analytic* when  $\omega$  (or, equivalently,  $h$ ) is real-analytic on the complex manifold  $(M, J)$ .

- Let  $(M, J, \omega)$  be a quantizable compact Kähler manifold with  $(L, h)$  its prequantum bundle and let  $N \in \mathbb{N}$ .

- The Hilbert space  $L^2(M, L^{\otimes N})$  is the closure of  $C^\infty(M, L^{\otimes N})$ , the space of smooth sections of the  $N$ -th tensor power of  $L$ , for the scalar product

$$\langle u, v \rangle = \int_M \langle u(x), v(x) \rangle_{L_x^{\otimes N}} \frac{\omega^{\wedge \dim_{\mathbb{C}} M}}{(\dim_{\mathbb{C}} M)!}.$$

- The Hardy space  $H^0(M, L^{\otimes N})$  is the space of holomorphic sections of  $L^{\otimes N}$ . It is a finite-dimensional, closed subspace of  $L^2(M, L^{\otimes N})$ .
- The Bergman projector  $S_N$  is the orthogonal projector from the space  $L^2(M, L^{\otimes N})$  to its subspace  $H^0(M, L^{\otimes N})$ .
- The contravariant Toeplitz operator associated with a symbol  $f \in L^\infty(M, \mathbb{C})$  is defined as

$$\begin{aligned} T_N(f) : H^0(M, L^{\otimes N}) &\rightarrow H^0(M, L^{\otimes N}) \\ u &\mapsto S_N(fu). \end{aligned}$$

In a related way, one can define *covariant* Toeplitz operators, which are kernel operators acting on  $H^0(M, L^{\otimes N})$  (see Definition 4.1). We are interested the Bergman projector and both types of Toeplitz operators in the *semiclassical limit*  $N \rightarrow +\infty$ .

A particular motivation for the study of Toeplitz operators is the quantization, on  $M = (\mathbb{S}^2)^d$ , of polynomials in the coordinates (in the standard immersion of  $\mathbb{S}^2$  into  $\mathbb{R}^3$ ). The operators obtained are spin operators, with total spin  $\frac{N}{2}$ . Tunnelling effects in spin systems, in the large spin limit, are widely studied in the physics literature (see [37] for a review). This article also aims at giving a mathematical ground to this study.

The Bergman kernel plays a role in many aspects of complex geometry and complex algebraic geometry [44, 38] as well as random matrices [1, 26], expanding the range of potential applications for Theorem A. Beyond the statements of our main results, our new microlocal analytic tools, developed in Section 3 may be used again in many different contexts, including transfer operators and quantized symplectomorphisms. As a matter of fact, we only deal with direct summation techniques, and the involved techniques of resummation or resurgence might be used to further broaden the range of applications of our tools.

We will use the following estimate on the operator  $\bar{\partial}$  acting on  $L^2(M, L^{\otimes N})$  and the Bergman projector  $S_N$ .

**Proposition 1.2.** *Let  $(M, \omega, J)$  be a compact quantizable Kähler manifold and  $(S_N)_{N \geq 1}$  be the associated sequence of Bergman projectors. There exists  $C > 0$  such that, for every  $N \geq 1$  and  $u \in L^2(M, L^{\otimes N})$ , one has:*

$$\|\bar{\partial}u\|_{L^2} \geq C\|u - S_N u\|_{L^2}. \quad (1)$$

This estimate initially follows from the work of Kohn [27, 28]; it is widely used in the asymptotic study of the Bergman kernel, where it is sometimes named after Hörmander or Kodaira.

The Bergman projector  $S_N$  admits a kernel, in a sense which we make precise here. The space  $H^0(M, L^{\otimes N})$  is finite-dimensional, so that it is spanned by a Hilbert basis  $s_1, \dots, s_{d_N}$  of holomorphic sections of  $L^{\otimes N}$ . The following section of  $L^{\otimes N} \boxtimes \bar{L}^{\otimes N}$  is the integral kernel of the Bergman projector:

$$S_N(x, y) = \sum_{i=1}^{d_N} s_i(x) \otimes \overline{s_i(y)}.$$

Here  $\bar{L}$  is the complex conjugate bundle of  $L$ , and  $\boxtimes$  stands for pointwise direct product:  $L^{\otimes N} \boxtimes \bar{L}^{\otimes N}$  is a bundle over  $M \times M$ . More generally, any section of  $L^{\otimes N} \boxtimes \bar{L}^{\otimes N}$  gives rise to an operator on  $L^2(M, L^{\otimes N})$ .

## 1.2 Statement of the main results

We begin with the definition of what will be the phase of the Bergman kernel. We use the standard notion of holomorphic extensions of real-analytic functions and manifolds, under a notation convention which is recalled in detail in Section 2.3.

**Definition 1.3.** Let  $M$  be a real-analytic Kähler manifold. Let  $\bar{M} = (M, \omega, -J)$  be the complex conjugate of  $M$ : holomorphic data on  $M$  correspond to anti-holomorphic data on  $\bar{M}$ . The codiagonal  $\{(x, x), x \in M\} \subset M \times \bar{M}$  is a totally real submanifold.

In particular, there exists a neighbourhood  $U$  of the diagonal in  $M \times \bar{M}$  and a unique holomorphic section  $\Psi$  of  $L \boxtimes \bar{L}$  over  $U$  such that  $\Psi = 1$  on the diagonal. Its  $N$ -th tensor power  $\Psi^N$  is the unique holomorphic section of  $L \boxtimes \bar{L}$  over  $U$  such that  $\Psi^N = 1$  on the codiagonal.

It is well-known that the pointwise norm of  $\Psi$  decays away from the codiagonal:

$$|\Psi(x, y)|_h = e^{-\frac{1}{2} \text{dist}(x, y)^2 + O(\text{dist}(x, y)^3)}.$$

In the general setting of a Kähler manifold with real-analytic data, it has been conjectured by S. Zelditch that the Bergman kernel takes the following form: for some  $c > 0, c' > 0$ , for all  $(x, y) \in M^2$ ,

$$S_N(x, y) = \Psi^N(x, y) \sum_{k=0}^{cN} N^{d-k} a_k(x, y) + O(e^{-c'N}),$$

where the  $a_k$  are, in a neighbourhood of the diagonal in  $M \times M$ , holomorphic in the first variable and anti-holomorphic in the second variable, with

$$\|a_k\|_{C^0} \leq CR^k k!.$$

The well-behaviour of such sequences of functions when the sum  $\sum N^{-k} a_k$  is computed up to the rank  $k = cN$  with  $c < e/2R$  is well described in [41] and is the foundation for a theory of analytic pseudodifferential operators and Fourier Integral Operators. Here, we rely on more specific function classes. Without giving a precise definition at this stage let us call “analytic symbols” well-controlled sequences of real-analytic functions. See Definition 3.3 about the analytic symbol spaces  $S_m^{r,R}(X)$  and the associated summation. The introduction of these classes allows us to prove the conjecture.

**Theorem A.** *Let  $M$  be a quantizable compact real-analytic Kähler manifold of complex dimension  $d$ . There exists positive constants  $r, R, m, c, c', C$ , a neighbourhood  $U$  of the diagonal in  $M \times M$ , and an analytic symbol  $a \in S_m^{r,R}(U)$ , holomorphic in the first variable, anti-holomorphic in the second variable, such that the Bergman kernel  $S_N$  on  $M$  satisfies, for each  $x, y \in M \times M$  and  $N \geq 1$ :*

$$\left\| S_N(x, y) - \Psi^N(x, y) \sum_{k=0}^{cN} N^{d-k} a_k(x, y) \right\|_{h^{\otimes N}} \leq C e^{-c'N}.$$

Equivalently, the operator with kernel  $\Psi^N(x, y) \sum_{k=0}^{cN} N^{d-k} a_k(x, y)$  is exponentially close (in the  $L^2 \rightarrow L^2$  operator sense) to the Bergman projector.

Theorem A also appears in recent and independent work [40], where the authors use Local Bergman kernels as developed in [2] to study locally the Bergman kernel as an analytic Fourier Integral Operator. Here, we obtain it as a byproduct of the next theorem about composition and inversion of Toeplitz operators.

In order to study contravariant Toeplitz operators of Definition 1.1, as well as the Bergman kernel itself, it is useful to consider *covariant* Toeplitz operators [9]. With  $\Psi^N$  as above, and  $f : M \times \overline{M} \rightarrow \mathbb{C}$  holomorphic near the diagonal, we let

$$T_N^{\text{cov}}(f)(x, y) = \Psi^N(x, y) \left( \sum_{k=0}^{cN} N^{d-k} f_k(x, y) \right),$$

for some small  $c > 0$ ; see Definition 4.1.

**Theorem B.** *Let  $M$  be a quantizable compact real-analytic Kähler manifold. Let  $f$  and  $g$  be analytic symbols on a neighbourhood  $U$  of the diagonal in  $M \times M$ , which are holomorphic in the first variable and anti-holomorphic in the second variable.*

Then there exists  $c' > 0$  and an analytic symbol  $f\sharp g$  on the same neighbourhood  $U$ , holomorphic in the first variable and anti-holomorphic in the second variable, and such that

$$T_N^{\text{cov}}(f)T_N^{\text{cov}}(g) = T_N^{\text{cov}}(f\sharp g) + O(e^{-c'N}).$$

For any  $r, R, m$  large enough, the product  $\sharp$  is a continuous bilinear map from  $S_m^{r,R}(U) \times S_m^{2r,2R}(U)$  to  $S_m^{2r,2R}(U)$  (see Definition 3.3); the constant  $c'$  depends only on  $r, R, m$ .

If the principal symbol of  $f$  does not vanish on  $M$  then there is an analytic symbol  $f^{\sharp-1}$  such that, for some  $c' > 0$ , one has

$$T_N^{\text{cov}}(f)T_N^{\text{cov}}(f^{\sharp-1}) = S_N + O(e^{-c'N}).$$

Given an analytic symbol  $f \in S_{m_0}^{r_0,R_0}(U)$  with non-vanishing subprincipal symbol, there exists  $C > 0$  such that for every  $r, R, m$  large enough (depending on  $f, r_0, R_0, m_0$ ), one has

$$\|f^{\sharp-1}\|_{S_m^{r,R}(U)} \leq C\|f\|_{S_m^{r,R}(U)}.$$

The stationary phase lemma allows one to prove relatively easily that the product  $(f, g) \mapsto f\sharp g$  is continuous from  $S_m^{r,R} \times S_m^{r,R}$  to  $S_{Cm}^{Cr,CR}$  for some  $C > 0$ . Theorem B is stronger in that respect, since the analytic class for  $f\sharp g$  is the same as the one for  $g$  if  $f$  is in a significantly better class. We conjecture that, as it is the case for pseudodifferential operators on  $\mathbb{R}^d$  [6], the  $\sharp$  product is a Banach algebra product in some analytic space, that is, is actually continuous from  $S_m^{r,R} \times S_m^{r,R}$  to  $S_m^{r,R}$ . This kind of results is subtler than the general techniques of analytic microlocal analysis originating from [41] allow for, and cannot be reached from equivalence of analytic quantizations, for instance. Theorem B relies on the Wick property of Toeplitz covariant quantization (Proposition 4.4): as in the Moyal product of pseudodifferential operators, to compute the  $k$ -th term in  $f\sharp g$  one differentiates  $f$  or  $g$  at most  $k$  times.

The fact that  $f\sharp g$  belongs to the same analytic class as  $g$  in Theorem B is a key point in our proof of Theorem A.

As an application of composition and inversion properties, one can study the concentration rate of eigenfunctions, in the general case (exponential decay in the forbidden region) as well as in the particular case where the principal symbol has a non-degenerate minimum.

**Theorem C.** *Let  $M$  be a quantizable compact real-analytic Kähler manifold. Let  $f$  be a real-analytic, real-valued function on  $M$  and  $E \in \mathbb{R}$ . Let  $(u_N)_{N \geq 1}$  be a normalized sequence of  $(\lambda_N)_{N \geq 1}$ -eigenstates of  $T_N(f)$  with  $\lambda_N \xrightarrow{N \rightarrow +\infty} E$ . Then, for every open set  $V$  at positive distance from  $\{f = E\}$  there exist positive constants  $c, C$  such that, for every  $N \geq 1$ , one has*

$$\int_V \|u_N(x)\|_h^2 \frac{\omega^{\wedge n}}{n!}(dx) \leq Ce^{-cN}.$$

We say informally that, in the forbidden region  $\{f \neq E\}$ , the sequence  $(u_N)_{N \geq 1}$  has an exponential decay rate.

### 1.3 Exponential estimates in semiclassical analysis

Exact or approximate eigenstates of quantum Hamiltonians are often searched for in the form of a Wentzel-Kramers-Brillouin (WKB) ansatz:

$$e^{\frac{\phi(x)}{\hbar}}(a_0(x) + \hbar a_1(x) + \hbar^2 a_2(x) + \dots),$$

where  $\hbar$  is the Planck constant, and is very small at the observer's scale. In the formula above,  $\Re(\phi) \leq 0$  so that this expression is extremely small outside the set  $\{\Re(\phi) = 0\}$  where it concentrates.

From this intuition, an interest developed towards decay rates for solutions of PDEs with small parameters. The most used setting in the mathematical treatment of quantum mechanics is the Weyl calculus of pseudodifferential operators [49]. Typical decay rates in this setting are of order  $O(\hbar^\infty)$ . Indeed, the composition of two pseudodifferential operators (or, more generally, Fourier Integral Operators) associated with smooth symbols can only be expanded in powers of  $\hbar$  up to an error  $O(\hbar^\infty)$ .

In the particular case of a Schrödinger operator  $P_\hbar = -\hbar^2 \Delta + V$  where  $V$  is a smooth function, one can obtain an *Agmon estimate* [18], which is an  $O(e^{\frac{\phi(x)}{\hbar}})$  pointwise control of eigenfunctions of  $P_\hbar$  with eigenvalues close to  $E$ . Here,  $\phi < 0$  on  $\{V > E\}$ . In this setting one can easily conjugate  $P_\hbar$  with multiplication operators of the form  $e^{-\frac{\phi}{\hbar}}$ , which allows to prove the control above. This conjugation property is not true for more general pseudodifferential operators. Moreover, Agmon estimates yield exponential decay in space variables, and give no information about the concentration rate of the semiclassical Fourier transform, which is only known to decay at  $O(\hbar^\infty)$  speed outside zero.

In the setting of pseudodifferential operators on  $\mathbb{R}^d$  with *real-analytic* symbols, following analytic microlocal techniques [41], exponential decay rates in phase space (that is, exponential decay of the FBI or Bargmann transform) were obtained in [31, 32, 33, 35]. Exponential estimates in semiclassical analysis have important applications in physics [12] where they validate the WKB ansatz which, in turn, yields precise results on spectral gaps or dynamics of quantum states (quantum tunnelling). Moreover, on the mathematical level, these techniques can be used to study non-self-adjoint perturbations [23, 24] and resonances [19, 42, 36, 43, 17].

Since exponential decay in phase space for pseudodifferential operators is defined by means of the FBI or Bargmann transform, it seems natural to formulate these questions in terms of Bargmann quantization, which then generalises to Berezin-Toeplitz quantization on Kähler manifolds, where the semiclassical parameter is the inverse of an integer:  $\hbar = N^{-1}$ . Yet, for instance, the validity of the WKB ansatz for a Toeplitz operator, at the bottom of a non-degenerate real-analytic well, was only performed when the underlying manifold is  $\mathbb{C}$  (see [45]), and some results were recently obtained for non-self-adjoint perturbations of Toeplitz operators on complex one-dimensional tori [39].



The analysis of Toeplitz operators depends on the knowledge of the Bergman projector, which encodes the geometrical data of the manifold on which the quantization takes place. The original microlocal techniques for the study of this projector [7, 47, 9] allow for a direct control of the Bergman kernel up to  $O(N^{-\infty})$ , from which one can deduce  $O(N^{-\infty})$  estimates for composition and eigenpairs of Toeplitz operators with smooth symbols [30, 13, 14]. Based on analytic pseudodifferential techniques, local Bergman kernels make it possible to show, under real-analyticity hypothesis, exponential (that is,  $O(e^{-cN})$ ) decay of the coherent states in Toeplitz quantization [2].

There is a recent increase of activity in the topic of exponential estimates in Toeplitz quantization: control of the Bergman kernel in real-analytic or Gevrey regularity [20, 21, 22, 10], but also estimates for the localisation of eigenfunctions of the form  $O(e^{-cN^\alpha})$  for  $C^\infty$  or rougher symbols [48, 11, 29, 16].

**Remark 1.4** (Gevrey case). The methods and symbol classes developed in this paper can be easily applied to the Gevrey case.  $s$ -Gevrey symbol classes are defined, and studied, by putting all factorials to the power  $s > 1$ .  $s$ -Gevrey functions have almost holomorphic extensions with controlled error near the real locus, so that all results in this paper should be valid in the Gevrey case under the two following modifications:

- The summation of  $s$ -Gevrey symbols is performed up to  $k = cN^{\frac{1}{s}}$ .
- All  $O(e^{-c'N})$  controls are replaced with  $O(e^{-c'N^{\frac{1}{s}}})$ .

For instance, we conjecture that the Bergman kernel on a quantizable compact Gevrey Kähler manifold is known up to  $O(e^{-c'N^{\frac{1}{s}}})$ . Its kernel decays at speed  $N^{\dim(M)} e^{-(\frac{1}{2}-\varepsilon)N \operatorname{dist}(x,y)^2}$  as long as  $\operatorname{dist}(x,y) \leq cN^{-\frac{s-1}{2s}}$ . This would improve recent results [21].

## 1.4 Outline

In Section 2 we recall the basic properties of holomorphic extensions of real-analytic functions. Then, in Section 3, we define analytic symbol classes for sequences of functions  $(f_k)_{k \geq 0}$  and we give a meaning to the sum  $\sum N^{-k} f_k$  up to exponential precision. These symbol classes are more precise than the ones appearing in the literature since [41]. In Section 4 we show Theorems A and B: the Bergman kernel on a compact quantizable real-analytic Kähler manifold, and the composition of analytic covariant Toeplitz operators, are known up to  $O(e^{-cN})$  precision, in terms of analytic symbols, from which we deduce, in Subsection 4.5, general exponential decay (Theorem C) in the forbidden region, for covariant as well as contravariant Toeplitz operators with analytic symbols.

In Sections 3 and those that follow, the fundamental tool is a version in real-analytic regularity of the stationary phase lemma (Lemma 3.12).

The various proofs in the second part have a common denominator: the general strategy consists in applying the complex stationary phase lemma and controlling the growth of the derivatives of the successive terms.

## 2 Holomorphic extensions

In this section we provide a general formalism for holomorphic extensions of various real-analytic data, which we use throughout this paper. The constructions of holomorphic extensions of real-analytic functions and manifolds is somewhat standard. We refer to [46] for details on these constructions. In particular, we study in Subsection 2.4 a specific class of analytic function spaces, which is a prerequisite to the Definition 3.3 of analytic symbol classes.

### 2.1 Combinatorial notations and inequalities

In this subsection we recall some basic combinatorial notation. Analytic functions and analytic symbol spaces are defined using sequences which grow as fast as a factorial (see Definitions 2.11 and 3.3) so that we will frequently need to bound expressions involving binomial or multinomial coefficients.

**Definition 2.1.** Let  $0 \leq i \leq j$  be integers. The associated *binomial* coefficient is

$$\binom{j}{i} = \frac{j!}{i!(j-i)!}.$$

Let more generally  $(i_k)_{1 \leq k \leq n}$  be a family of non-negative integers and let  $j \geq \sum_{k=1}^n i_k$ . The associated *multinomial* coefficient is

$$\binom{j}{i_1, \dots, i_n} = \frac{j!}{(j - \sum_{k=1}^n i_k)! \prod_{k=1}^n i_k!}$$

**Remark 2.2.** An alternative definition of multinomial coefficient assumes  $j = i_1 + \dots + i_n$ , in which case one defines  $\binom{j}{i_1, \dots, i_n} = \frac{j!}{i_1! \dots i_n!}$ . The definition we give contains this one, and is more consistent with the notation for binomial coefficients.

**Definition 2.3.**

1. A *polyindex* (plural: polyindices)  $\mu$  is an ordered family  $(\mu_1, \dots, \mu_d)$  of non-negative integers (the set of non-negative integers is denoted by  $\mathbb{N}_0$ ). The cardinal  $d$  of the family is called the *dimension* of the polyindex (we will only consider the case where  $d$  is finite).
2. The norm  $|\mu|$  of the polyindex  $\mu = (\mu_1, \dots, \mu_d)$  is defined as  $\sum_{i=1}^d \mu_i$ .
3. The partial order  $\leq$  on polyindices of same dimension is defined as follows:  $\nu \leq \mu$  when, for every  $1 \leq i \leq d$ , one has  $\nu_i \leq \mu_i$ .

4. The factorial  $\mu!$  is defined as  $\prod_{i=1}^d \mu_i!$ . Together with the partial order, this allows to extend the notation for binomial coefficients. If  $\nu \leq \mu$ , then we define the associated binomial coefficient as

$$\binom{\mu}{\nu} = \frac{\mu!}{\nu!(\mu - \nu)!}$$

We state here a few useful inequalities about binomial coefficients.

**Lemma 2.4.** *Let  $\nu \leq \mu$  be polyindices. Then*

$$\binom{\mu}{\nu} \leq \binom{|\mu|}{|\nu|}.$$

**Lemma 2.5.** *Let  $i \leq j$  and  $1 \leq k \leq l - 1$  be integers. Then*

$$\frac{(i + k - 1)!(j + l - i - k - 1)!}{i!k!(j - i)!(l - k)!} \leq \frac{(j + l - 2)!}{j!(l - 1)!}.$$

*In particular, if  $a_1, \dots, a_n$  are nonnegative integers and  $b_1, \dots, b_n$  are positive integers, with  $\sum_{i=1}^n a_i = j$  and  $\sum_{i=1}^n b_i = l$ , then*

$$\frac{(a_1 + b_1 - 1)! \dots (a_n + b_n - 1)!}{a_1!b_1! \dots a_n!b_n!} \leq \frac{(j + l - n)!}{j!(l - n + 1)!}.$$

*Proof.* For the first part, let  $k' = k - 1$ , then

$$\frac{(i + k - 1)!(j + l - i - k - 1)!}{i!k!(j - i)!(l - k)!} = \frac{1}{k(l - k)} \binom{i + k'}{i} \binom{j + l - 2 - i - k'}{j - i}.$$

Since  $1 \leq k \leq l - 1$  there holds  $\frac{1}{k(l - k)} \leq \frac{1}{l - 1}$ . Moreover, from Lemma 2.4, one has

$$\binom{i + k'}{i} \binom{j + l - 2 - i - k'}{j - i} \leq \binom{j + l - 2}{j} = \frac{(j + l - 1)!}{j!(l - 2)!}.$$

Hence,

$$\frac{(i + k - 1)!(j + l - i - k - 1)!}{i!k!(j - i)!(l - k)!} \leq \frac{(j + l - 2)!}{j!(l - 1)!}.$$

The second part is deduced from the first part by induction. Indeed, we just proved that, denoting  $a'_{n-1} = a_{n-1} + a_n$  and  $b'_{n-1} = b_{n-1} + b_n - 1$ , one has

$$\begin{aligned} & \frac{(a_1 + b_1 - 1)! \dots (a_n + b_n - 1)!}{a_1!b_1! \dots a_n!b_n!} \\ & \leq \frac{(a_1 + b_1 - 1)! \dots (a_{n-2} + b_{n-2} - 1)!(a'_{n-1} + b'_{n-1} - 1)!}{a_1!b_1! \dots a_{n-2}!b_{n-2}!a'_{n-1}!b'_{n-1}!}. \end{aligned}$$

Here, the sum of the  $a_i$ 's has not changed but the sum of the  $b_i$ 's has been reduced by one. By induction,

$$\frac{(a_1 + b_1 - 1)! \dots (a_n + b_n - 1)!}{a_1! b_1! \dots a_n! b_n!} \leq \frac{(j + l - n)!}{j!(l - n + 1)!}.$$

□

**Lemma 2.6.** *Let  $\ell \geq 2$  and  $n \geq 2$  be integers. The set*

$$\left\{ (i_1, \dots, i_n) \in \mathbb{N}_0^n, \sum_{k=1}^n i_k = \ell, \text{ at least two of them are } \geq 1 \right\}.$$

*is contained in the convex hull of all permutations of  $(\ell - 1, 1, 0, \dots, 0)$ .*

*Proof.* Let us call *support* of a tuple  $(i_1, \dots, i_n)$  the number of its elements which are non-zero. We will prove by induction on  $2 \leq k \leq \min(n, \ell)$  that the convex hull  $S$  of the permutations of  $(\ell - 1, 1, 0, \dots, 0)$  contain all tuples of support  $k$  such that the sum of all elements is  $\ell$ .

For  $k = 2$ , we can indeed recover all elements of the form  $(\ell - x, x, 0, \dots, 0)$  for all  $1 \leq x \leq \ell - 1$  by a convex combination of  $(\ell - 1, 1, 0, \dots, 0)$  and  $(1, \ell - 1, 0, \dots, 0)$ .

We now proceed to the induction. Suppose that  $S$  contains all elements of the form  $(i_1, \dots, i_{k-1}, 0, \dots, 0)$  and their permutations. Then, in particular, it contains  $a_0 = (\ell - k + 2, 1, \dots, 1, 0, \dots, 0)$ . For every  $1 \leq j \leq k - 2$ ,  $S$  also contains the image of  $a_0$  by the transposition  $(k, k - j)$ , which we denote by  $a_j$ . Moreover,  $S$  contains  $(\frac{\ell}{k-1}, \dots, \frac{\ell}{k-1}, 0, \dots, 0)$  and its permutations. From the  $(a_j)_{0 \leq j \leq k-2}$  and  $(\frac{\ell}{k-1}, \dots, \frac{\ell}{k-1}, 0, \dots, 0)$ , one can form the convex combination

$$\begin{aligned} \frac{\ell - k + 1}{(\ell - k + 2)(k - 2)} \sum_{j=0}^{k-2} a_j + \frac{1}{\ell - k + 2} \left( 0, \frac{\ell}{k-1}, \dots, \frac{\ell}{k-1}, 0, \dots, 0 \right) \\ = (\ell - k + 1, \underbrace{1, \dots, 1}_{k-1}, 0, \dots, 0). \end{aligned}$$

In particular,  $S$  contains all permutations of  $(\ell - k + 1, 1, \dots, 1, 0, \dots, 0)$ . Thus,  $S$  contains all elements of support  $k$ , since the  $k$ -uple  $(\ell - k, 0, \dots, 0)$  and its permutations are the extremal points of the convex  $\{\sum_{j=1}^k i_j = \ell - k\}$ . This concludes the induction. □

## 2.2 Extensions of real-analytic functions

The fundamental object that one is allowed to extend in a holomorphic way is a real-analytic function.

**Definition 2.7.** Let  $f : U \rightarrow E$  be a real-analytic function on an open set  $U \in \mathbb{R}^n$ , which takes values into a real or complex Banach space  $E$ . A *holomorphic extension* of  $f$  is a couple  $(f, \tilde{U})$ , where  $\tilde{U}$  is an open set of  $\mathbb{C}^n$  and  $\tilde{f} : \tilde{U} \rightarrow E \otimes \mathbb{C}$ , such that

- $\bar{\partial}\tilde{f} = 0$ .
- $U \subset \tilde{U}$ ,
- $\tilde{f}|_U = f$

Naturally, two holomorphic extensions coincide on the connected components of their intersections which intersect  $U$  since, on a connected open set of  $\mathbb{C}^d$ , a holomorphic function which vanishes on a real set vanishes everywhere.

If  $E$  is a real Banach space then  $E \otimes \mathbb{C}$  is the complexification of  $E$ ; if  $E$  is complex to begin with then  $E \otimes \mathbb{C} = E$ .

The local expression of a real-analytic function as a convergent power series gives a natural and non-ambiguous way to define a holomorphic extension.

### 2.3 Extensions of manifolds

**Proposition 2.8** ([46]). *Let  $M$  be a real-analytic manifold. There is a complex manifold  $(\tilde{M}, J_e)$  with boundary, such that  $M$  is a totally real submanifold of  $\tilde{M}$ . Then  $\tilde{M}$  is called a holomorphic extension of  $M$ .*

In this setting, “totally real” means that

$$\forall x \in M, T_x M \cap J_e(T_x M) = \{0\}.$$

The extension of real-analytic manifolds is naturally associated with an extension of their real-analytic functions.

**Proposition 2.9.** *Let  $f$  be a real-analytic function on a real-analytic manifold  $M$ . Then there exists a holomorphic function  $\tilde{f}$  on a holomorphic extension  $\tilde{M}$  of  $M$  such that  $\tilde{f}|_M = f$ .*

In the body of this article we will frequently extend real-analytic functions on holomorphic manifolds. We introduce a convenient notation to this end, which is reminiscent of Definition 1.3. Locally, a real-analytic function  $f$  on a complex manifold of dimension  $d$  can be written as

$$f : z \mapsto \sum_{\nu, \rho \in \mathbb{N}_0^d} c_{\nu, \rho} z^\nu \bar{z}^\rho.$$

As the function  $f$  is not holomorphic, we specifically write  $f(z, \bar{z})$ . There is then a natural notion of an extension

$$\tilde{f} : (z, w) \mapsto \sum_{\nu, \rho \in \mathbb{N}_0^d} c_{\nu, \rho} z^\nu w^\rho.$$

This function is holomorphic on a neighbourhood of 0 in  $\mathbb{C}^{2d}$ . It coincides with  $\tilde{f}$ , but the totally real manifold of interest is not  $\{\Im(z) = 0\}$  anymore but rather  $\{(z, w), w = \bar{z}\}$ .

Let  $M$  be a complex manifold; using the convention above let us treat local charts for  $M$  and its holomorphic extension  $\widetilde{M}$ . A change of charts in  $M$  is a biholomorphism  $\phi$  which, in the convention above, depends only on  $z$  as a function on  $\widetilde{M}$ . The extended biholomorphism  $\tilde{\phi}$  constructed in the previous subsection can be written as

$$(z, w) \mapsto (\phi(z), \overline{\phi(\bar{w})}).$$

Gluing open sets along the charts  $\bar{\phi}$  (defined by  $\bar{\phi}(z) = \overline{\phi(\bar{z})}$ ) yields a manifold  $\overline{M}$ , and there is a natural identification  $M \ni z \mapsto \bar{z} \in \overline{M}$ , so that  $\overline{M}$  is simply  $M$  with reversed complex structure.

The expression of  $\tilde{\phi}$  above yields

$$\widetilde{M} = M \times \overline{M},$$

and  $M$  sits in  $\widetilde{M}$  as the totally real submanifold

$$\{(z, w) \in M \times \overline{M}, \bar{z} = w\}.$$

This copy of  $M$  is said to be the codiagonal of  $M \times \overline{M}$ .

Any real-analytic function on  $M$  can be extended as a holomorphic function in a neighbourhood of the codiagonal of  $\widetilde{M}$ . If the function was holomorphic (on a small open set of  $M$ ) to begin with, then its extension depends only on the first variable (on a small open set of  $M \times \overline{M}$ ).

## 2.4 Analytic functional spaces

In this subsection we derive a few tools about the study of holomorphic functions near a compact totally real set. We first fix a notion of convenient open sets on which our analysis can take place.

**Definition 2.10.** A *domain* of  $\mathbb{R}^d$  is an open, relatively compact set  $U$  with piecewise smooth boundary.

Recall that a holomorphic function  $f$  near zero can be written as

$$f(z) = \sum_{\nu \in \mathbb{N}_0^d} \frac{f_\nu}{\nu!} z^\nu.$$

Then, in particular  $f_\nu = \partial^\nu f(0)$ . Since  $f$  is holomorphic, the sum above converges for  $|z|$  sufficiently small. In other terms, there exists  $r > 0$  and  $C > 0$  such that, for every  $\nu \in \mathbb{N}_0^d$ , one has

$$|f_\nu| \leq C \nu! r^{|\nu|}.$$

**Definition 2.11.** For  $j \in \mathbb{N}_0$  and  $f$  a function on a domain of  $\mathbb{R}^d$  of class  $C^j$ , we denote by  $\nabla^j f$  the function  $(\partial^\alpha f(x))_{|\alpha|=j}$ , which maps  $U$  to  $\mathbb{R}^{\binom{j+d-1}{d-1}}$ . For  $n \in \mathbb{N}$  and  $v \in \mathbb{R}^n$ , we denote  $\|v\|_{\ell^1} = \sum_{j=1}^n |v_j| + \dots + |v_n|$ .

Let  $m \in \mathbb{N}_0$  and  $r > 0$ . Let  $U$  be a domain in  $\mathbb{R}^d$ . The space  $H(m, r, U)$  is defined as the set of real-analytic functions on  $U$  such that there exists a constant  $C$  satisfying, for every  $j \in \mathbb{N}_0$ ,

$$\sup_{x \in U} \|\nabla^j f(x)\|_{\ell^1} \leq \frac{Cr^j j!}{(j+1)^m}.$$

The space  $H(m, r, U)$  is a Banach space for the norm  $\|\cdot\|_{H(m, r, U)}$  defined as the smallest constant  $C$  such that the inequality above is true for every  $j$ .

Such functions can be extended to a neighbourhood of  $U$  in  $\mathbb{C}^d$ , with imaginary part bounded by  $r^{-1}$  (and by the distance to the boundary of  $U$ ). The spaces  $H(m, r, U)$  are compactly embedded in each other for the lexicographic order on  $(r, -m)$ : if either  $r < r'$  or  $r = r', m > m'$ , then

$$H(m, r, U) \subset H(m', r', U).$$

Introducing a parameter  $m$  will allow us to control polynomial quantities which appear when one manipulates these holomorphic function spaces, using Lemmas 2.13 and 3.7. They correspond to a regularity condition at the boundary of a maximal holomorphic extension: for instance, the function  $x \mapsto x \log(x)$  belongs to  $H(1, 1, (1/2, 3/2))$  but not to  $H(m, 1, (1/2, 3/2))$  for  $m > 1$ .

It will be useful in the course of this paper to consider various analytic norms for the same function while maintaining a fixed norm. The definition of the spaces  $H(m, r, U)$  immediately imply the following fact.

**Proposition 2.12.** *Let  $m_0 \in \mathbb{N}_0$  and  $r_0 > 0$ . Let  $U$  be a domain in  $\mathbb{R}^d$ . Let  $f \in H(m_0, r_0, U)$ . Then, for all  $m \geq m_0$ , for all  $r \geq r_0 2^{m-m_0}$ , one has  $f \in H(m, r, U)$  with*

$$\|f\|_{H(m, r, U)} \leq \|f\|_{H(m_0, r_0, U)}.$$

The following lemma will be used several times in what follows.

**Lemma 2.13.** *Let  $d \in \mathbb{N}_0$ . There exists  $C > 0$  such that, for any  $m \geq \max(d+2, 2(d+1))$ , for any  $j \in \mathbb{N}_0$ , one has*

$$\sum_{i=0}^j \frac{\min(i+1, j-i+1)^d (j+1)^m}{(i+1)^m (j-i+1)^m} \leq 2 + C \frac{3^m}{4^m}.$$

We postpone the proof of this lemma until Section 3. More specifically, this is a particular case of Lemma 3.7.

Analytic function classes form an algebra for  $m$  large enough, and non-vanishing functions can be inverted.

**Proposition 2.14.** *There exists  $C > 0$  such that the following is true. Let  $m \geq 2$ . Let  $r > 0$  and let  $U$  be a domain in  $\mathbb{R}^n$ . Let  $f, g \in H(m, r, U)$ . Then  $fg \in H(m, r, U)$ , and*

$$\|fg\|_{H(m, r, U)} \leq C \|f\|_{H(m, r, U)} \|g\|_{H(m, r, U)}.$$

The constant  $C$  is universal.

If  $f$  is bounded away from zero on  $U$ , then  $f^{-1} \in H(m, r, U)$ , with

$$\|f^{-1}\|_{H(m,r,U)} \leq \frac{\|f\|_{H(m,r,U)}}{\inf_U(|f|)^2}.$$

*Proof.* Let  $f, g \in H(m, r, U)$  and  $j \in \mathbb{N}_0$ . Then

$$\sum_{|\alpha|=j} |\partial^\alpha(fg)| \leq \sum_{|\beta+\gamma|=j} \binom{\beta+\gamma}{\beta} |\partial^\beta f| |\partial^\gamma g|$$

By Lemma 2.4, one has, for every  $\beta$  and  $\gamma$  such that  $|\beta+\gamma|=j$ ,

$$\binom{\beta+\gamma}{\beta} \leq \binom{|\beta+\gamma|}{|\beta|} = \binom{j}{|\beta|}.$$

Hence,

$$\sum_{|\alpha|=j} |\partial^\alpha(fg)| \leq \sum_{i=0}^j \binom{j}{i} \|\nabla^i f\|_{\ell^1} \|\nabla^{|\alpha|-i} g\|_{\ell^1},$$

so that, for any  $j \geq 0$ , one has

$$\begin{aligned} & \|\nabla^j(fg)\|_{\ell^1} \\ & \leq \|f\|_{H(m,r,U)} \|g\|_{H(m,r,U)} \frac{r^j j!}{(j+1)^m} \sum_{i=0}^j \binom{j}{i}^{-1} \binom{j}{i} \frac{(j+1)^m}{(i+1)^m (j-i+1)^m}. \end{aligned}$$

Hence,

$$\|\nabla^j(fg)\|_{\ell^1} \leq \|f\|_{H(m,r,U)} \|g\|_{H(m,r,U)} \frac{r^j j!}{(j+1)^m} \sum_{i=0}^j \frac{(j+1)^m}{(i+1)^m (j-i+1)^m}.$$

Let us use Lemma 2.13 with  $d = 0$ . If  $m \geq 2$ , this quantity is bounded independently of  $j$  and  $m$ , so that

$$\|\nabla^j(fg)\|_{\ell^1} \leq C \|f\|_{H(m,r,U)} \|g\|_{H(m,r,U)} \frac{r^j j!}{(j+1)^m}.$$

This concludes the first part of the proof.

Let now  $f \in H(m, r, U)$  be bounded away from zero on  $U$ . We introduce the modified product  $f \cdot g = \frac{fg}{C}$ , for which  $H(m, r, U)$  is a Banach algebra.

First,  $|f|^2$  is real-valued and strictly positive; moreover

$$|f|^2 = f\bar{f} \in H(m, r, U)$$

and, by the property above,

$$\||f|^2\|_{H(m,r,U)} \leq C \|f\|_{H(m,r,U)}^2.$$



Let  $g = \frac{|f|^2}{2\|f\|_{H(m,r,U)}^2}$ . Then

$$\|1 - g\|_{H(m,r,U)} \leq 1 - \frac{\inf_U(|f|^2)}{2\|f\|_{H(m,r,U)}^2} < 1.$$

In particular,  $g = 1 - (1 - g)$  so that, letting  $h$  be such that  $g \cdot h = 1$ , one has

$$h = \sum_{k=0}^{+\infty} (1 - g)^{\cdot k},$$

where the power series converge because the  $\cdot$  product induces a Banach algebra structure on  $H(m, r, U)$ .

Hence, one can control

$$\|h\|_{H(m,r,U)} \leq \frac{2\|f\|_{H(m,r,U)}^2}{\inf_U(|f|^2)}.$$

Now  $|f|^{-2} = \frac{h}{2C\|f\|_{H(m,r,U)}^2}$  so that

$$\||f|^{-2}\|_{H(m,r,U)} \leq \frac{1}{C \inf_U(|f|^2)}.$$

We now turn to  $f^{-1} = \bar{f}|f|^{-2}$ , which is controlled as follows:

$$\|f^{-1}\|_{H(m,r,U)} \leq \frac{\|f\|_{H(m,r,U)}}{\inf_U(|f|^2)}.$$

This concludes the proof.  $\square$

The spaces  $H(r, m, U)$  contain all holomorphic functions.

**Proposition 2.15.** *Let  $d \in \mathbb{N}$ . For every  $T > 0$  we let  $P(0, T)$  be the polydisk of center 0 and of radius  $T$  in  $\mathbb{C}^d$ .*

*Let  $f$  be a holomorphic, bounded function on  $P(0, 2T)$ , continuous up to the boundary. Then*

$$\|f\|_{H(-d, dT^{-1}, P(0, T))} \leq C \sup_{P(0, 2T)} |f|.$$

*Proof.* The proof relies on the Cauchy formula. For all  $z \in P(0, T)$  and  $\nu \in \mathbb{N}_0^d$ , there holds

$$\partial^\nu f(z) = C \int_{|\xi_1|=\dots=|\xi_d|=2T} \frac{\nu! f(\xi)}{(\xi_1 - z_1)^{\nu_1} (\xi_2 - z_2)^{\nu_2} \dots (\xi_d - z_d)^{\nu_d}} d\xi.$$

As  $z \in P(0, r)$  and  $|\xi_1| = \dots = |\xi_d| = 2T$ , for every  $1 \leq i \leq d$  there holds  $|\xi_i - z_i| \geq T$ , so that

$$\sup_{P(0, T)} |\partial^\nu(f)| \leq CT^{-|\nu|} \nu! \sup_{P(0, 2T)} |f|.$$

In particular, since  $\nu! \leq |\nu|!d^{|\nu|}$ , by summing over  $\nu$ 's with same norm we obtain

$$\sup_{x \in P(0,T)} \|\nabla^j f(x)\|_{\ell^1} \leq C(j+1)^d (dT^{-1})^j j!,$$

hence the claim.  $\square$

### 3 Calculus of analytic symbols

In this section we define and study (formal) *analytic symbols*, which we will show to be well suited to the study of stationary phases with complex, real-analytic phases.

#### 3.1 Analytic symbols

We begin with an explicit definition of  $C^j$ -seminorms on compact manifolds.

**Definition 3.1.** Let  $X$  be a compact manifold (with smooth boundary). We fix a finite set  $(\rho_V)_{V \in \mathcal{V}}$  of local charts on open sets  $V$  which cover  $X$ .

Let  $j \geq 0$ . The  $C^j$  seminorm of a function  $f : X \mapsto \mathbb{C}$  which is continuously differentiable  $j$  times is defined as

$$\|f\|_{C^j(X)} = \max_{V \in \mathcal{V}} \sup_{x \in V} \sum_{|\mu|=j} |\partial^\mu (f \circ \rho_V)(x)|.$$

This definition is adapted to the multiplication of two functions. The Leibniz formula yields directly:

**Proposition 3.2.** *Let  $X$  be a compact manifold (with smooth boundary) with fixed local charts, and  $f, g \in C^j(X, \mathbb{R})$ .*

*Then  $fg \in C^j(X, \mathbb{R})$  with*

$$\|fg\|_{C^j(X)} \leq \sum_{i=0}^j \binom{j}{i} \|f\|_{C^i(X)} \|g\|_{C^{j-i}(X)}.$$

Using the convention above, let us generalise Definition 2.11, in order to define analytic symbols.

**Definition 3.3.** Let  $X$  be a compact manifold (with boundary), with a fixed set of covering local charts.

Let  $r, R, m$  be positive real numbers. The space of analytic symbols  $S_m^{r,R}(X)$  consists of sequences  $(a_k)_{k \geq 0}$  of real-analytic functions on  $X$ , such that there exists  $C \geq 0$  such that, for every  $j \geq 0, k \geq 0$ , one has

$$\|a_k\|_{C^j(X)} \leq C \frac{r^j R^k (j+k)!}{(j+k+1)^m}.$$

The norm of an element  $a \in S_m^{r,R}(X)$  is defined as the smallest  $C$  as above; then  $S_m^{r,R}(X)$  is a Banach space.

We are interested in symbols which have an expansion in increasing powers of the semiclassical parameter. We will use the term “symbols” while, in the usual semiclassical vocabulary, we are dealing with formal symbols to which we associate classical symbols by a summation process in Proposition 3.6.

As for the analytic function classes  $H(m, r, U)$  of Definition 2.11, the spaces  $S_m^{r,R}(X)$  are included in each other for a lexicographic order, and the constants of injection are controlled as follows:

**Proposition 3.4.** *Let  $X$  be a compact manifold (with boundary) with a fixed finite set of covering charts. Let  $r_0, R_0, m_0$  positive. Let  $f \in S_{m_0}^{r_0, R_0}(X)$ . For every  $m \geq m_0$ , for every  $r \geq r_0 2^{m-m_0}$  and  $R \geq R_0 2^{m-m_0}$ , one has  $f \in S_m^{r,R}$  with*

$$\|f\|_{S_m^{r,R}(X)} \leq \|f\|_{S_{m_0}^{r_0, R_0}(X)}.$$

The notion of sum of a formal series in  $N^{-1}$  is well-defined up to  $O(N^{-\infty})$ , by a process known as *Borel summation*. In a similar but more explicit way, formal series corresponding to analytic symbols can be summed up to an exponentially small error.

**Definition 3.5.** Let  $X$  be a compact Riemannian manifold (with boundary) and let  $f \in S_m^{r,R}(X)$ . Let  $c_R = \frac{e}{3R}$ . The *summation* of  $f$  is defined as

$$X \times \mathbb{N} \ni (x, N) \mapsto f(N)(x) = \sum_{k=0}^{c_R N} N^{-k} f_k(x).$$

**Proposition 3.6.** *Let  $X$  be a compact Riemannian manifold with boundary and let  $f \in S_m^{r,R}(X)$ . Let  $c_R = \frac{e}{3R}$ . Then*

1. *The function  $f(N)$  is bounded on  $X$  uniformly for  $N \in \mathbb{N}$ .*
2. *For every  $0 < c_1 < c_R$ , there exists  $c_2 > 0$  such that*

$$\sup_{x \in X} \left| \sum_{k=c_1 N}^{c_R N} N^{-k} f_k(x) \right| = O(e^{-c_2 N}).$$

*Proof.*

1. Since

$$\sup_{x \in X} |f_k(x)| \leq \|f\|_{S_m^{r,R}(X)} R^k k!,$$

it remains to control

$$\sum_{k=0}^{c_R N} N^{-k} R^k k!.$$

In this series, the first term is 1, and the ratio between two consecutive terms is

$$\frac{N^{-k} R^k k!}{N^{-k+1} R^{k-1} (k-1)!} = \frac{Rk}{N} \leq Rc_R = \frac{e}{3} < 1.$$

Hence,

$$\sup_{x \in X} |f(x, N)| \leq \|f\|_{S_m^{r,R}(X)} \sum_{k=0}^{c_R N} (e/3)^k \leq \|f\|_{S_m^{r,R}(X)} \frac{3}{3-e}.$$

2. The claim reduces to a control on

$$\sum_{k=c_1 N}^{c_R N} N^{-k} R^k k!.$$

In this series, on which each term is smaller than  $(e/3)^k$ , the first term is controlled by

$$(e/3)^{c_1 N} = \exp(c_1 \log(e/3)N).$$

Hence the claim, with  $c_2 = c_1 \log(e/3)$ . □

From the second point of Proposition 3.6, we see that the constant  $c_R = \frac{e}{3R}$  is quite arbitrary (using the Stirling formula to control factorials, one could in fact consider any constant smaller than  $\frac{e}{R}$ ). We use it in Definition 3.5 to avoid dealing with equivalence classes of sequences whose difference is  $O(e^{-c'N})$  for some  $c'$ , as in [41].

Before studying further the space  $S_m^{r,R}(X)$ , let us prove a generalisation of Lemma 2.13.

**Lemma 3.7.** *Let  $d \in \mathbb{N}$  and  $n \geq 2$ . There exists  $C(n, d) > 0$  such that, for any  $m \geq \max(d+2, 2(d+n-1))$ , for any  $\ell \in \mathbb{N}_0$ , one has*

$$\sum_{\substack{0 \leq i_1 \leq i_2 \leq \dots \leq i_n \\ i_1 + \dots + i_n = \ell}} \frac{(i_{n-1} + 1)^d (\ell + 1)^m}{(i_1 + 1)^m \dots (i_n + 1)^m} \leq 1 + C \frac{3^m}{4^m}.$$

This is indeed, up to a factor 2, a generalisation of Lemma 2.13 which corresponds to the case  $n = 2$ .

*Proof.* The case  $\ell = 1$  is trivial, so we assume  $\ell \geq 2$ . The only term in the sum such that  $i_{n-1} = 0$  is equal to 1; let us control the sum restricted on  $\{i_{n-1} \geq 1\}$ . Let us first show that, if  $i_{n-1} \geq 1$ , then

$$\frac{(i_{n-1} + 1)^d (\ell + 1)^m}{(i_1 + 1)^m \dots (i_n + 1)^m} \leq (\ell + 1)^d \frac{3^m}{4^m}. \quad (2)$$

One has directly  $(i_{n-1} + 1)^d \leq (\ell + 1)^d$ .

We are left with

$$\frac{(\ell + 1)^m}{(i_1 + 1)^m \dots (i_n + 1)^m},$$

which is a symmetric expression of  $(i_1, \dots, i_n)$ , log-convex as soon as  $m \geq 0$ , and which we wish to bound on the symmetrised set

$$\left\{ (i_1, \dots, i_n) \in \mathbb{N}_0^n, \sum_{k=1}^n i_k = \ell, \text{ at least two of them are } \geq 1 \right\}.$$

By Lemma 2.6, it is sufficient to control the quantity above at the permutations of  $(\ell - 1, 1, 0, \dots, 0)$ . At each of those points, since  $\ell \geq 2$ , one has

$$\frac{(\ell + 1)^m}{(i_1 + 1)^m \dots (i_n + 1)^m} = \left( \frac{\ell + 1}{2\ell} \right)^m \leq \frac{3^m}{4^m}.$$

We are now in position to prove the claim. Let us first restrict our attention to  $\{i_1 \geq \frac{\ell+1}{3(n-1)}\}$ . There are less than  $(\ell + 1)^{n-1}$  such terms (since there are less than  $(\ell + 1)^{n-1}$  terms in total), and each of these terms is smaller than

$$\frac{(\ell + 1)^d (\ell + 1)^m}{\left( \frac{\ell+1}{3(n-1)} \right)^{mn}} = \frac{(\ell + 1)^d (3(n-1))^{mn}}{(\ell + 1)^{m(n-1)}}.$$

Hence, this sum is controlled by

$$\frac{(\ell + 1)^{n+d-1} (3(n-1))^{mn}}{(\ell + 1)^{m(n-1)}}$$

We now consider the sum on  $\{i_1 \leq \frac{\ell+1}{3n-1} \leq i_2\}$ . There are again less than  $(\ell + 1)^{n-1}$  such terms, each of them smaller than

$$\frac{(\ell + 1)^d (\ell + 1)^m}{\left( \frac{\ell+1}{3(n-1)} \right)^{m(n-1)}} = \frac{(\ell + 1)^d (3(n-1))^{m(n-1)}}{(\ell + 1)^{m(n-2)}}.$$

Thus, this sum is smaller than

$$\frac{(\ell + 1)^{n+d-1} (3(n-1))^{m(n-1)}}{(\ell + 1)^{m(n-2)}}.$$

Similarly, we are able to control the sum on  $\{i_k \leq \frac{\ell+1}{3(n-1)} \leq i_{k+1}\}$ , for  $k \leq n - 2$ , by

$$\frac{(\ell + 1)^{n+d-1} (3(n-1))^{m(n-k)}}{(\ell + 1)^{m(n-k-1)}}.$$

If  $m \geq 2(d + n - 1)$ , then  $(\ell + 1)^{n+d-1+m} \leq (\ell + 1)^{3m/2}$ , so that, for any  $k \leq n - 2$ , if  $\ell + 1 \geq 3n$ , one has

$$\begin{aligned} \frac{(\ell + 1)^{n+d-1} (3(n-1))^{m(n-k)}}{(\ell + 1)^{m(n-k-1)}} &\leq (\ell + 1)^{\frac{3m}{2}} \left( \frac{3(n-1)}{\ell + 1} \right)^{m(n-k)} \\ &\leq (\ell + 1)^{3m/2} \left( \frac{3(n-1)}{\ell + 1} \right)^{2m} \\ &= \left( \frac{9(n-1)^2}{\sqrt{\ell + 1}} \right)^m. \end{aligned}$$

Thus, for  $\ell$  large enough (depending on  $n$ ), this quantity is smaller than  $\frac{3^m}{4^m}$ ; for  $\ell$  small we have a number of terms bounded by a function of  $n$ , each term being smaller than  $C(n, d)\frac{3^m}{4^m}$  by (2).

It remains to control the sum restricted on  $\{1 \leq i_{n-1} \leq \frac{\ell+1}{3(n-1)}\}$ . In this case,  $i_n + 1 \geq \frac{2(\ell+1)}{3}$ , so that the sum is smaller than

$$\begin{aligned} \frac{3^m}{2^m} \sum_{\substack{0 \leq i_1 \leq \dots \leq i_{n-1} \leq \frac{\ell+1}{3(n-1)} \\ i_{n-1} \geq 1}} \frac{(i_{n-1} + 1)^d}{(i_1 + 1)^m (i_2 + 1)^m \dots (i_{n-1} + 1)^m} \\ \leq \frac{3^m}{2^m} (\zeta(m))^{n-2} (\zeta(m-d) - 1). \end{aligned}$$

The Riemann zeta function is decreasing, and if  $m \geq d + 2$ , then

$$\zeta(m-d) \leq 1 + 3 \cdot 2^{-(m-d)},$$

so that the expression above is controlled by  $C(n, d)\frac{3^m}{4^m}$ . This concludes the proof.  $\square$

Analytic symbols behave well with respect to the Cauchy product, which corresponds to the product of their summations.

**Proposition 3.8.** *There exists  $C_0 \in \mathbb{R}$  and a function  $C : \mathbb{R}^2 \mapsto \mathbb{R}$  such that the following is true.*

*Let  $X$  be a compact Riemannian manifold (with boundary) and with a fixed finite set of covering charts. Let  $r, R \geq 0$  and  $m \geq 4$ . For  $a, b \in S_m^{r,R}(X)$ , let us define the Cauchy product of  $a$  and  $b$  as*

$$(a * b)_k = \sum_{i=0}^k a_i b_{k-i}.$$

1. *The space  $S_m^{r,R}(X)$  is an algebra for this Cauchy product, that is,*

$$\|a * b\|_{S_m^{r,R}} \leq C_0 \|a\|_{S_m^{r,R}} \|b\|_{S_m^{r,R}},$$

*Moreover, there exists  $c > 0$  depending only on  $R$  such that as  $N$  tends to infinity, one has*

$$(a * b)(N) = a(N)b(N) + O(e^{-cN}).$$

2. *Let  $r_0, R_0, m_0$  positive and  $a \in S_{m_0}^{r_0, R_0}(X)$  with  $a_0$  nonvanishing. Then, for every  $m$  large enough depending on  $a$ , for every  $r \geq r_0 2^{m-m_0}$  and  $R \geq R_0 2^{m-m_0}$ , the symbol  $a$  is invertible (for the Cauchy product) in  $S_m^{r,R}(X)$ , and its inverse  $a^{*-1}$  satisfies:*

$$\|a^{*-1}\|_{S_m^{r,R}(X)} \leq 2 \min(|a_0|)^{-4} \|a\|_{S_{m_0}^{r_0, R_0}}^3.$$

*Proof.*

1. From Proposition 3.2, one has, for every  $0 \leq i \leq k$  and  $j \geq 0$ ,

$$\|a_i b_{k-i}\|_{C^j} \leq \sum_{\ell=0}^j \binom{j}{\ell} \|a_i\|_{C^\ell} \|b_{k-i}\|_{C^{j-\ell}}.$$

In particular,

$$\begin{aligned} \|(a * b)_k\|_{C^j} &\leq \|a\|_{S_m^{r,R}} \|b\|_{S_m^{r,R}} \frac{r^j R^k (j+k)!}{(j+k+1)^m} \\ &\times \sum_{i=0}^k \sum_{\ell=0}^j \binom{j+k}{i+\ell}^{-1} \binom{j}{\ell} \frac{(j+k+1)^m}{(i+\ell+1)^m (j+k-i-\ell+1)^m}. \end{aligned}$$

Since,

$$\binom{j}{\ell} \leq \binom{j+i}{\ell+i} \leq \binom{j+k}{\ell+i},$$

one has

$$\begin{aligned} \|(a * b)_k\|_{C^j} &\leq \|a\|_{S_m^{r,R}} \|b\|_{S_m^{r,R}} \frac{r^j R^k (j+k)!}{(j+k+1)^m} \\ &\times \sum_{i=0}^k \sum_{\ell=0}^j \frac{(j+k+1)^m}{(i+\ell+1)^m (j+k-i-\ell+1)^m} \\ &\leq \|a\|_{S_m^{r,R}} \|b\|_{S_m^{r,R}} \frac{r^j R^k (j+k)!}{(j+k+1)^m} \\ &\times \sum_{i'=0}^{k+j} \frac{\min(i'+1, j+k-i'+1) (j+k+1)^m}{(i'+1)^m (j+k-i'+1)^m}, \end{aligned}$$

where  $i' = i + \ell$ . We are reduced to Lemma 2.13 with  $d = 1$ . If  $m \geq 4$ , this sum is smaller than a universal constant  $C$  independently of  $j, k$ , so that

$$\|a * b\|_{S_m^{r,R}} \leq C \|a\|_{S_m^{r,R}} \|b\|_{S_m^{r,R}}.$$

Let us control the product of the associated analytic series. By Proposition 3.6, for some  $c > 0$  depending only on  $R$ , one has

$$a(N) = \sum_{k=0}^{\frac{eN}{12R}} N^{-k} a_k + O(e^{-cN}),$$

and similar controls for  $b(N)$  and  $(a * b)(N)$ .

The first  $\frac{eN}{12R}$  terms of the expansion in decreasing powers of  $(a * b)(N)$  and  $a(N)b(N)$  then coincide by definition of the Cauchy product. It remains to control

$$\sum_{\frac{eN}{12R} \leq i+j \leq \frac{eN}{6R}} N^{-(i+j)} a_i b_j.$$

From

$$\sup(|a_i b_j|) \leq CR^{i+j} i! j! \leq C(2R)^{i+j} (i+j)!,$$

one has, as in Proposition 3.6,

$$\left| \sum_{\substack{\frac{eN}{12R} \leq i+j \leq \frac{eN}{6R}}} N^{-(i+j)} a_i b_j \right| \leq \sum_{\substack{\frac{eN}{12R} \leq i+j \leq \frac{eN}{6R}}} N^{-(i+j)} (2R)^{i+j} (i+j)! \leq e^{-cN},$$

hence the claim.

2. The unit element of the Cauchy product is  $(1, 0, 0, \dots)$ , which belongs to  $S_m^{r,R}(X)$ . Let  $a \in S_{m_0}^{r_0, R_0}(X)$  be such that  $a_0$  does not vanish on  $X$ , and let us try to find  $b$  such that  $(a * b)_0 = 1$  and  $(a * b)_k = 0$  whenever  $k \neq 0$ .

The first condition yields  $b_0 = a_0^{-1}$ , which is a function with real-analytic regularity and same radius as  $a_0$ , by Proposition 2.14, so that

$$\|b_0\|_{C^j} \leq C_0 \frac{r_0^j j!}{(j+1)^{m_0}}.$$

In particular, by Lemma 2.12, for all  $m \geq m_0, r \geq r_0 2^{m-m_0}$ , one has

$$\|b_0\|_{C^j} \leq C_0 \frac{r^j j!}{(j+1)^m}.$$

The coefficients  $b_k$  are then determined by induction:

$$b_k = a_0^{-1} \sum_{i=1}^k a_i b_{k-i} = b_0 \sum_{i=1}^k a_i b_{k-i}.$$

Let us control  $\|b\|_{S_m^{r,R}(X)}$  by  $\|a\|_{S_m^{r,R}(X)}$  by induction, for some  $r, R, m$  which will be chosen later.

We now proceed by induction on  $k$ . Suppose that, for all  $\ell \leq k-1$  and  $j \geq 0$ , one has

$$\|b_\ell\|_{C^j} \leq C_b \frac{r^j R^\ell (j+\ell)!}{(j+\ell+1)^m},$$

We wish to prove the same control for  $\ell = k$ . The constant  $C_b$  will be chosen later.



By induction hypothesis,

$$\begin{aligned}
\|b_k\|_{C^j} &\leq C_0 C_b \|a\|_{S_m^{r,R}} \sum_{j_1=0}^j \sum_{i=1}^k \sum_{j_2=0}^{j-j_1} \binom{j}{j_1, j_2} \frac{r^{j_1} j_1!}{(j_1+1)^m} \\
&\quad \times \frac{r^{j_2} R^i(j_2+i)! r^{j-j_1-j_2} R^{k-i}(j-j_1-j_2+k-i)!}{(i+j_2+1)^m (j-j_1-j_2+k-i+1)^m} \\
&\leq C_b C_0 \|a\|_{S_m^{r,R}} \frac{r^j R^k(j+k)!}{(j+k+1)^m} \sum_{j_1=0}^j \sum_{i=1}^k \sum_{j_2=0}^{j-j_1} \binom{j}{j_1, j_2} \binom{j+k}{j_1, j_2+i}^{-1} \\
&\quad \times \frac{(j+k+1)^m}{(j_1+1)^m (j_2+i+1)^m (j-j_1-j_2+k-i+1)^m}.
\end{aligned}$$

Let us prove that, for every  $i, j, j_1, j_2, k$  in the range above, one has

$$\binom{j+k}{j_1, j_2+i} \geq \binom{j}{j_1, j_2}.$$

There holds

$$\binom{j+1}{j_1, j_2+1} = \binom{j}{j_1, j_2} \frac{j+1}{j-j_1-j_2} \geq \binom{j}{j_1, j_2},$$

so that

$$\binom{j+k}{j_1, j_2+i} \geq \binom{j+i}{j_1, j_2+i} \geq \binom{j}{j_1, j_2}.$$

Hence,

$$\begin{aligned}
\|b_k\|_{C^j} &\leq C_b C_0 \|a\|_{S_m^{r,R}} \frac{r^j R^k(j+k)!}{(j+k+1)^m} \\
&\quad \times \sum_{j_1=0}^j \sum_{i=1}^k \sum_{j_2=0}^{j-j_1} \frac{(j+k+1)^m}{(j_1+1)^m (j_2+i+1)^m (j-j_1-j_2+k-i+1)^m} \\
&\leq C_b C_0 \|a\|_{S_m^{r,R}} \frac{r^j R^k(j+k)!}{(j+k+1)^m} \\
&\quad \times \sum_{\substack{j_1+i_1+i_2=j+k \\ i_1 \geq 1}} \frac{\min(i_1+1, i_2+1)(j+k+1)^m}{(j_1+1)^m (i_1+1)^m (i_2+1)^m}.
\end{aligned}$$

From Lemma 3.7 with  $n = 3$  and  $d = 1$ , the sum

$$\sum_{\substack{j_1+i_1+i_2=j+k \\ i_1 \geq 1}} \frac{\min(i_1+1, i_2+1)(j+k+1)^m}{(j_1+1)^m (i_1+1)^m (i_2+1)^m}$$

is bounded independently of  $j$  and  $k$  for  $m \geq 6$ . However this control is not enough since it yields a constant in front of  $\frac{r^j R^k (j+k)!}{(j+k+1)^m}$  which is a priori  $CC_0 C_b \|a\|_{S_m^{r,R}} \geq C_b$ .

However, the only term in this expansion which contributes as 1 is  $j_1 = 0, i_1 = k+j, i_2 = 0$ , which corresponds to  $j_1 = 0, i = k, j_2 = j$ . One can control this term independently of  $C_b$  since

$$|a_0^{-1}| \|a_k\|_{C^j} |b_0| \leq C_0^2 \|a\|_{S_{m_0}^{r_0, R_0}} \frac{r^j R^k (j+k)!}{(j+k+1)^m}.$$

The sum over all other terms is smaller than  $CC_b C_0 \|a\|_{S_m^{r,R}} (3/4)^m$  for some  $C$ , by Lemma 3.7.

We can conclude: if  $m$  is large with respect to  $\|a\|_{S_m^{r,R}}$  (which can be done using Proposition 3.4 by setting  $r \geq r_0 2^{m-m_0}$  and  $R \geq R_0 2^{m-m_0}$ ) and if  $C_b \geq 2C_0^2 \|a\|_{S_{m_0}^{r_0, R_0}}$ , where we recall from Proposition 2.14 that

$$C_0^2 = \min(|a_0|)^{-4} \|a\|_{S_{m_0}^{r_0, R_0}}^2,$$

one has, by induction,

$$\|b_k\|_{C^j} \leq C_b \frac{r^j R^k (j+k)!}{(j+k+1)^m}.$$

This concludes the proof. □

**Remark 3.9.** The method of proof for Proposition 3.8 will be used again in Section 4. This method consists in an induction, in which quotients of factorials must be bounded; this reduces the control by induction to Lemma 3.7. Constants which appear must be carefully chosen so that the induction can proceed. In particular, given a fixed object in an analytic class, it will be useful to change the parameters (typically  $m, r, R$ ) in its control, while maintaining a fixed norm.

The classes  $H(m, r, V)$  of real-analytic functions introduced in Section 2 contain all holomorphic functions. In a similar manner, the symbol classes  $S_m^{r,R}$  contain all classical analytic symbols in the sense of Sjöstrand [41].

**Proposition 3.10.** *Let  $U$  be an open set of  $\mathbb{C}^n$  and let  $a = (a_k)_{k \geq 0}$  be a sequence of bounded holomorphic functions on  $U$  such that there exists  $C > 0$  and  $R > 0$  satisfying, for all  $k \geq 0$ ,*

$$\sup_U |a_k| \leq CR^k k!.$$

*Then for every  $V \subset\subset U$  there exists  $r > 0$  such that  $a \in S_0^{r,R}(V)$ .*

*In particular, given an analytic symbol  $a$  and an analytic change of variables  $\kappa$ , then  $a \circ \kappa$  is an analytic symbol.*

*Proof.* By Proposition 2.15, there exists  $C_1 > 0$  and  $r > 0$  such that, for every  $k \geq 0$ , one has  $a_k \in H(r, 0, V)$  with

$$\|a_k\|_{H(0,r,V)} \leq C_1 \sup_U |a_k|.$$

In other terms, for every  $k \geq 0, j \geq 0$ , one has

$$\|a_k\|_{C^j(V)} \leq C_1 C r^j R^k j! k! \leq C_1 C r^j R^k (j+k)!.$$

Hence  $a \in S_0^{r,R}(V)$ .

By a power series expansion, an analytic symbol  $a$  satisfies, on some holomorphic extension  $U$  of its domain of definition,

$$\sup_U |\tilde{a}_k| \leq C R^k k!.$$

This control is not affected by application of the biholomorphism  $\tilde{\kappa}$ , so that

$$\sup_{\tilde{\kappa}^{-1}(U)} |\tilde{a}_k \circ \tilde{\kappa}| \leq C R^k k!.$$

By the lines above,  $a \circ \kappa$  is an analytic symbol. □

### 3.2 Complex stationary phase lemma

In this subsection we present the tools of stationary phase in the context of real-analytic regularity, as developed by Sjöstrand [41]. We wish to study integrals of the form

$$\int_{\Omega} e^{N\Phi(x)} a(x) dx,$$

as  $N \rightarrow +\infty$ . If  $\Phi$  is purely imaginary, then by integration by parts, this integral is  $O(N^{-\infty})$  away from the points where  $d\Phi$  vanishes. At such points, if  $\Phi$  is Morse, a change of variables leads to the usual case where  $\Phi$  is quadratic nondegenerate; then there is a full expansion of the integral in decreasing powers of  $N$ . If  $\Phi$  is real-valued, a similar analysis (Laplace method) yields a related expansion.

On one hand, we wish to study such an integral, in the more general case where  $i\Phi$  is complex-valued. On the other hand we want to improve the  $O(N^{-\infty})$  estimates into  $O(e^{-cN})$ . This is done via a complex change of variables; to this end we have to impose real-analytic regularity on  $\Phi$  and  $a$ .

Let us introduce a notion of analytic phase, which generalises positive phase functions as appearing in [41].

**Definition 3.11.** Let  $d, k \in \mathbb{N}$ . Let  $\Omega$  be a domain of  $\mathbb{R}^d$ . Let  $\Phi$  be a real-analytic function on  $\Omega \times \mathbb{R}^k$ . For each  $\lambda \in \mathbb{R}^k$  we let  $\Phi_\lambda = \Phi(\cdot, \lambda)$ . Then  $\Phi$  is said to be an *analytic phase* on  $\Omega$  under the following conditions.

- There exists an open set  $\tilde{\Omega} \subset \mathbb{C}^d$  such that, for every  $\lambda \in \mathbb{R}^k$ , the function  $\Phi_\lambda$  extends to a holomorphic function  $\tilde{\Phi}_\lambda$  on  $\tilde{\Omega}$ .

- For every  $\lambda \in \mathbb{R}^k$ , there exists exactly one point  $\tilde{x}_\lambda \in \tilde{\Omega}$  such that  $d\tilde{\Phi}_\lambda(\tilde{x}_\lambda) = 0$ ; this critical point is non-degenerate, with  $\tilde{\Phi}_\lambda(\tilde{x}_\lambda) = 0$ .
- One has  $\tilde{x}_0 = 0$  and moreover  $\Re\Phi_0 < 0$  on  $\Omega \setminus \{0\}$ .

Under the conditions of Definition 3.11, the function  $\lambda \mapsto \tilde{x}_\lambda$  is real-analytic.

We now recall the stationary phase lemma in analytic regularity.

**Proposition 3.12.** [41, 25] *Let  $\Phi$  be an analytic phase on a domain  $\Omega$ . There exists  $c > 0, c' > 0, C' > 0$ , a neighbourhood  $\Lambda \subset \mathbb{R}^k$  of zero, and a biholomorphism  $\tilde{\kappa}_\lambda$ , with real-analytic dependence<sup>1</sup> on  $\lambda \in \Lambda$ , such that the associated Laplace operator  $\tilde{\Delta}(\lambda) = \kappa_\lambda \circ \Delta \circ \kappa_\lambda^{-1}$  satisfies, for every function  $a_\lambda$  holomorphic on  $\tilde{\Omega}$ :*

$$\int_{\Omega} e^{N\Phi_\lambda} a_\lambda = \sum_{k=0}^{cN} \left(k! N^{\frac{d}{2}+k}\right)^{-1} \tilde{\Delta}(\lambda)^k (\tilde{a}_\lambda J_\lambda^{-1})(\tilde{x}_\lambda) + R_\lambda(N),$$

where, uniformly in  $\lambda \in \Lambda$ ,

$$|R_\lambda(N)| \leq C e^{-c'N} \sup_{\tilde{\Omega}} |\tilde{a}_\lambda|,$$

and  $J_\lambda$  is the Jacobian determinant associated with the change of variables.

## 4 Calculus of covariant Toeplitz operators

In this section we prove our three main theorems.

We begin in Subsection 4.1 with the definition, and the first properties, of covariant Toeplitz operators. Then, in Subsections 4.2 to 4.4, we study them. We prove that they can be composed (Proposition 4.7), and inverted (Propositions 4.8 and 4.9), with a precise control on the analytic classes involved. This allows us to prove Theorem A: see the beginning of Section 4.4 for a detailed proof strategy for Theorems A and B. To conclude, in Subsection 4.5 we prove Theorem C.

Until the end of Section 4,  $M$  is a compact real-analytic quantizable Kähler manifold of dimension  $d$ .

### 4.1 Covariant Toeplitz operators

**Definition 4.1.** Let  $U$  denote a small, smooth neighbourhood of the co-diagonal in  $M \times M$ ; for instance  $U = \{(x, y) \in M \times M, \text{dist}(x, y) < \epsilon\}$  with  $\epsilon$  small enough so that the section  $\Psi^N$  of Definition 1.3 is defined on

<sup>1</sup>By this we mean: a real-analytic function  $\kappa$  on  $U \times \Lambda$ , where  $U$  is a neighbourhood of 0 in  $\tilde{\Omega}$ , holomorphic in the first variable, such that there exists  $\sigma$  with the same properties, satisfying  $\sigma(\kappa(x, \lambda), \lambda) = \kappa(\sigma(x, \lambda), \lambda) = x$  for all  $(x, \lambda) \in U \times \Lambda$ .

a neighbourhood of  $U$ . The space  $T_m^{-,r,R}(U)$  of *covariant analytic Toeplitz operators* consists of operators with kernel

$$T_N^{\text{cov}}(f) : (x, y) \mapsto N^d \mathbb{1}_{(x,y) \in U} \Psi^N(x, y) f(N)(x, y),$$

where  $f(N)$  is the summation of an analytic symbol  $f \in S_m^{r,R}(U)$ , with  $f$  holomorphic in the first variable and anti-holomorphic in the second variable.

**Remark 4.2.** Since  $\Psi^N$  is exponentially small near the boundary of  $U$ , by Proposition 1.2, the image of a covariant Toeplitz operator is exponentially close to its projection on  $H^0(M, L^{\otimes N})$ .

## 4.2 Study of an analytic phase

In this work, covariant Toeplitz operators of Definition 4.1 have the following integral kernels:

$$T_N^{\text{cov}}(f) : (x, y) \mapsto \Psi^N(x, y) \left( \sum_{k=0}^{cN} N^{d-k} f_k(x, y) \right).$$

The integral kernel of the composition of two covariant Toeplitz is of particular interest, so let us study its phase.

If  $f$  and  $g$  are analytic symbols, then  $T_N^{\text{cov}}(f)T_N^{\text{cov}}(g)$  has the following kernel:

$$(x, z) \mapsto \Psi^N(x, z) \int_M e^{N(2\tilde{\phi}(x,y) - 2\phi(y) + 2\tilde{\phi}(y,z) - 2\tilde{\phi}(x,z))} \times \left( \sum_{k=0}^{cN} N^{d-k} f_k(x, y) \right) \left( \sum_{j=0}^{cN} N^{d-j} g_j(y, z) \right) dy.$$

We let  $\Phi_1$  be the complex extension (with respect to the middle variable) of the phase appearing in the last formula:

$$\Phi_1 : (x, y, \bar{w}, \bar{z}) \mapsto 2\tilde{\phi}(x, \bar{w}) - 2\tilde{\phi}(y, \bar{w}) + 2\tilde{\phi}(y, \bar{z}) - 2\tilde{\phi}(x, \bar{z}).$$

We write  $\Phi_1(x, y, \bar{w}, \bar{z})$  to indicate anti-holomorphic dependence on the two last variables. In particular,  $\Phi_1$  is holomorphic on the open set  $U \times U$  of  $M \times \widetilde{M} \times \overline{M} = M_x \times (M_y \times \overline{M}_{\bar{w}}) \times \overline{M}_{\bar{z}}$ .

The fact that  $\Phi_1$  is a well-behaved phase function is well-known; let us state it in this real-analytic context.

**Proposition 4.3.** *There exists a smooth neighbourhood  $U$  of*

$$\{(x, \bar{z}) \in M \times \overline{M}, \bar{x} = \bar{z}\}$$

*such that function  $\Phi_1$ , on the open set*

$$\{(x, y, \bar{y}, \bar{z}), (x, \bar{w}) \in U, (y, \bar{w}) \in U, (x, \bar{z}) \in U\},$$

is an analytic phase of  $(y, \bar{w})$ , with parameter  $\lambda = (x, \bar{z})$ . The critical point is  $(x, \bar{z})$ .

In particular, after a trivialisation of a tubular neighbourhood of

$$\{(x, y, \bar{w}, \bar{z}) \in M \times \widetilde{M} \times \overline{M}, (x, \bar{z}) \in U, (y, \bar{w}) = (x, \bar{z})\}$$

in

$$\{(x, y, \bar{w}, \bar{z}) \in M \times \widetilde{M} \times \overline{M}, (x, \bar{z}) \in U\}$$

as a vector bundle over the former, the analytic phase  $\Phi_1$  satisfies the assumptions of Definition 3.11.

### 4.3 Composition of covariant Toeplitz operators

In this subsection we study the composition rules for operators with kernels of the form

$$T_N^{\text{cov}}(f)(x, y) = \Psi^N(x, y) \left( \sum_{k=0}^{cN} N^{d-k} f_k(x, y) \right).$$

Here, for a small, smooth neighbourhood  $U$  of the diagonal in  $M \times M$ , one has  $f \in S_m^{r,R}(U)$ , and  $f$  is holomorphic in the first variable and anti-holomorphic in the second variable.

Such operators can be formally composed, that is, there holds

$$T_N^{\text{cov}}(f)T_N^{\text{cov}}(g) = T_N^{\text{cov}}(f\sharp g) + O(N^{-\infty})$$

where  $f\sharp g$  is a classical symbol. This formal calculus satisfies a Wick rule (Proposition 4.4). This allows us, in Proposition 4.7, to prove that, if  $f$  and  $g$  are analytic symbols, then  $f\sharp g$  is also an analytic symbol, so that one can perform an analytic summation (as in Proposition 3.6), and the error in the composition becomes  $O(e^{-cN})$ .

**Proposition 4.4.** (See also [8], Lemmes 2.33 and following) *The composition of two covariant Toeplitz operators can be written as a formal series in  $N^{-1}$ . More precisely, if  $f$  and  $g$  are functions on a neighbourhood of the diagonal in  $M \times M$ , holomorphic in the first variable, anti-holomorphic in the second variable, then*

$$T_N^{\text{cov}}(f)T_N^{\text{cov}}(g) = T_N^{\text{cov}}(f\sharp g) + O(N^{-\infty}),$$

where  $f\sharp g$  is a formal series  $h \sim \sum_{k \geq 0} N^{-k} (f\sharp g)_k$ , holomorphic in the first variable, anti-holomorphic in the second variable. The composition law can be written as

$$(f\sharp g)_k = B_k(f, g),$$

where  $B_k$  is a bidifferential operator of degree at most  $k$  in  $f$  and at most  $k$  in  $g$ .

Since the thesis [8] is in French, we present a self-contained proof of this fact in the appendix.

**Remark 4.5.** Proposition 4.4 is not a trivial consequence of the expression of the phase  $\Phi_1$ . Indeed, the composition rule for  $f\sharp g$  consists in finding, for  $x, \bar{z}$  fixed, a holomorphic change of variables  $(y, \bar{w}) \mapsto (v, \bar{v})$  such that  $\Phi_1 = -v \cdot \bar{v}$ ; then, if  $J$  denotes the Jacobian of this change of variables,

$$(f\sharp g)_k(x, \bar{z}) = \sum_{n=0}^k \frac{\partial_v^n \partial_{\bar{v}}^n}{n!} \left( \sum_{l=0}^{k-n} f_l(x, \bar{w}(x, v, \bar{v}, \bar{z})) g_{k-n-l}(y(x, v, \bar{v}, \bar{z}), \bar{z}) J(x, v, \bar{v}, \bar{z}) \right)_{v=\bar{v}=0}.$$

If the Morse change of variables can be split as  $(y, \bar{w}) \mapsto (v(y), \bar{v}(\bar{w}))$ , then Proposition 4.4 follows immediately from the formula above: holomorphic derivatives (in  $v$ ) will only hit  $g_{k-n-l}$  or  $J$  while anti-holomorphic derivatives (in  $\bar{v}$ ) will only hit  $f_l$  or  $J$ , so that both  $f$  and  $g$  are differentiated at most  $k$  times.

In the model case where  $\Phi_1$  is a quadratic form (such as on Bargmann space), such a splitting is indeed true. However, the property  $\Phi_1 = -v(y) \cdot \bar{v}(\bar{w})$  is, in dimension 1, equivalent to the easily checked identity

$$\partial_y \partial_{\bar{w}} \log(\Phi_1) = 0.$$

This property is false, for instance, if  $\Phi_1$  represents the usual 2-sphere in the stereographic projection. In this case,

$$\begin{aligned} \Phi_1 : (x, y, \bar{w}, \bar{z}) \mapsto & 2(\log(1 + x \cdot \bar{w}) + \log(1 + y \cdot \bar{z}) \\ & - \log(1 + y \cdot \bar{w}) - \log(1 + x \cdot \bar{z})). \end{aligned}$$

Restricting to  $x = z = 0$ , we obtain

$$\partial_y \partial_{\bar{w}} \log(\Phi_1) = \frac{-2}{(1 + y \cdot \bar{w}) \log(1 + y \cdot \bar{w})} + \frac{2y \cdot \bar{w} (\log(1 + y \cdot \bar{w}) + 1)}{((1 + y \cdot \bar{w}) \log(1 + y \cdot \bar{w}))^2},$$

which is obviously non-zero (in particular, it is equal to  $\frac{1-\log(2)}{2\log^2(2)}$  at  $y = \bar{w} = 1$ .)

Proposition 4.4 predicts that, when applying a stationary phase lemma to  $\Phi_1$  in order to study  $T_N^{cov}(f)T_N^{cov}(g)$ , at order  $k$ , only derivatives of  $f$  and  $g$  at order  $k$  will appear. However, in the stationary phase (Lemma 3.12), these derivatives appear in the form of an usual Laplace operator, conjugated by a change of variables. Before proceeding further, let us prove a technical lemma.

**Lemma 4.6.** *Let  $U, V, \Lambda$  be domains in  $\mathbb{C}^d$  containing 0. Let  $\kappa_\lambda$  be a biholomorphism from  $V$  to  $U$ , with real-analytic dependence on  $\lambda \in \Lambda$ , and such*

that  $\kappa_\lambda(0) = 0$  for all  $\lambda \in \Lambda$ . Let  $\kappa(\lambda, v) \mapsto \kappa_\lambda(v)$ , and suppose that there exists  $C_\kappa, r_0, m_0$  such that, for all  $j \in \mathbb{N}_0$ , one has

$$\|\kappa\|_{C^j(V \times \Lambda)} \leq C \frac{r_0^j j!}{(j+1)^{m_0}}.$$

Then the following is true for all  $m \geq m_0, r \geq 8r_0 2^{m-m_0}$ .

Let  $f$  be a real-analytic function on  $U \times \Lambda$ , and suppose that there exists  $C_f$  and  $k \geq 0$  such that

$$\|f\|_{C^j(U \times \Lambda)} \leq C_f \frac{r^j (j+k)!}{(j+k+1)^m}.$$

Let  $n \leq k$  and  $i \leq 2n$ ; let  $\nabla_v^i$  denote the  $i$ -th gradient (as in Definition 2.11) over the first set of variables, acting on  $V \times \Lambda$ ; then

$$g \mapsto (\lambda \mapsto \nabla_v^i g(\kappa_\lambda(v), \lambda)_{v=0})$$

is a differential operator of degree  $i$ , from functions on  $U \times \Lambda$  to vector-valued functions on  $\Lambda$ . Let  $(\nabla_\kappa^i)^{[\leq n]}$  denote the truncation of this differential operator to a differential operator of degree less than  $n$ .

Then, with

$$\gamma = 4Cr,$$

one has, for every  $j \geq 0$ ,

$$\begin{aligned} \|(\nabla_\kappa^i)^{[\leq n]} f\|_{\ell^1(C^j(\Lambda))} &\leq i^{d+1} j^{d+1} \gamma^i C_f \frac{r^{j+i}}{(i+j+l+1)^m} \\ &\times \begin{cases} (i+j+k)! & \text{if } i \leq n \\ \max((n+j+k)!(i-n)!, (j+k)!i!) & \text{otherwise.} \end{cases} \end{aligned}$$

*Proof.* Let us make explicit the operator  $(\nabla_\kappa^i)^{[\leq n]}$ . Given a polyindex  $\mu$  with  $|\mu| = i$ , the Faà di Bruno formula states:

$$\partial_v^\mu (f(\kappa_\lambda(v), \lambda))_{v=0} = \sum_{P \in \Pi(\{1, \dots, i\})} f^{|P|}(0, \lambda) \prod_{E \in P} (\partial^E \kappa_\lambda)(0),$$

where the sum runs among all partitions  $P = \{E_1, \dots, E_{|P|}\}$  of  $\{1, \dots, i\}$ .

When considering the operator  $(\nabla_\kappa^i)^{[\leq n]}$ , we only need to consider partitions  $P$  such that  $|P| \leq n$ . If the sizes  $|E_1| = s_1, \dots, |E_{|P|}| = s_{|P|}$  of the elements of  $P$  are fixed, the number of possible partitions is simply

$$\frac{i!}{(|P|)! s_1! \dots s_{|P|}!}$$

Then, since there are less than  $i^d$  polyindices  $\mu$  with  $|\mu| = i$ , one has, for all  $\rho \in \mathbb{N}_0^d$  with  $|\rho| = j$ , by differentiation of the Faà di Bruno formula and



Proposition 3.2,

$$\|\partial^\rho((\nabla_\kappa^i)^{[\leq n]} f)\|_{\ell^1} \leq i^d \sum_{|P|=1}^{\min(n,i)} \sum_{\substack{e_0+\dots+e_{|P|}=j \\ s_1+\dots+s_{|P|}=|P|}} \frac{j!}{e_0!e_1!\dots e_{|P|}!} \frac{i!}{(|P|)!s_1!\dots s_{|P|}!} \|f\|_{C^{|P|+e_0}} \prod_{i=1}^{|P|} \|\kappa\|_{C^{s_i+e_i}}.$$

Here  $\kappa$  denotes the real-analytic function  $(\lambda, v) \mapsto \kappa_\lambda(v)$ .

In particular, since there are less than  $j^d$  polyindices  $\rho$  such that  $|\rho| = j$ , one has

$$\|\partial^\rho((\nabla_\kappa^i)^{[\leq n]} f)\|_{\ell^1} \leq i^d j^d \times \sum_{|P|=1}^{\min(n,i)} \sum_{\substack{e_0+\dots+e_{|P|}=j \\ s_1+\dots+s_{|P|}=|P|}} \frac{j!}{e_0!e_1!\dots e_{|P|}!} \frac{i!}{(|P|)!s_1!\dots s_{|P|}!} \|f\|_{C^{|P|+e_0}} \prod_{i=1}^{|P|} \|\kappa\|_{C^{s_i+e_i}}. \quad (3)$$

Since, for all  $j \geq 0$ , one has

$$\|\kappa\|_{C^j(V \times \Lambda)} \leq C \frac{r_0^j j!}{(j+1)^{m_0}},$$

by Lemma 2.12, for all  $m \geq m_0, r \geq 8r_0 2^{m-m_0}$ , one has

$$\|\kappa\|_{C^j} \leq C \frac{(r/8)^j j!}{(j+1)^m}.$$

In particular, if  $j \geq 1$ , there holds

$$\|\kappa\|_{C^j} \leq C \frac{(r/4)^j (j-1)!}{j^m} j \left(\frac{j}{j+1}\right)^m 2^{-j} \leq C \frac{(r/4)^j (j-1)!}{j^m},$$

since

$$j \left(\frac{j}{j+1}\right)^m 2^{-j} \leq j 2^{-j} \leq 1.$$

Let us suppose further that

$$\|f\|_{C^j(U \times \Lambda)} \leq C_f \frac{r^j R^l (j+l)!}{(j+l+1)^m}.$$

Then, the contribution of one term in the sum (3) is

$$\begin{aligned} & \frac{j!}{e_0!e_1!\dots e_{|P|}!} \frac{i!}{(|P|)!s_1!\dots s_{|P|}!} \|f\|_{C^{|P|+e_0}} \prod_{i=1}^{|P|} \|\kappa\|_{C^{s_i+e_i}} \\ & \leq C_f C^{|P|} \frac{r^{|P|+e_0} (r/4)^{i+j-e_0} R^l (|P|+e_0+l)! i!}{(|P|+e_0+l+1)^m (|P|)!s_1!\dots s_{|P|}!} \\ & \quad \times \frac{j!(s_1+e_1-1)!\dots (s_{|P|}+e_{|P|}-1)!}{e_0!e_1!\dots e_{|P|}!(s_1+e_1)^m \dots (s_{|P|}+e_{|P|})^m}. \end{aligned}$$

As  $e_0 + \dots + e_{|P|} = j$  and  $s_1 + \dots + s_{|P|} = i$ , and since, as soon as  $x \geq 0$ ,  $y \geq 0$ , there holds

$$(1+x)(1+y) = 1+x+y+xy \geq 1+x+y,$$

one has

$$\begin{aligned} & (|P| + e_0 + l + 1)^m (s_1 + e_1)^m \dots (s_{|P|} + e_{|P|})^m \\ & \geq (|P| + j + i + l - |P| + 1)^m = (j + i + l + 1)^m, \end{aligned}$$

so that one can simplify

$$\begin{aligned} & C_f C^{|P|} \frac{r^{|P|+e_0} (r/4)^{i+j-e_0} R^l (|P| + e_0 + l)! i!}{(|P| + e_0 + l + 1)^m (|P|)! s_1! \dots s_{|P|}!} \\ & \quad \times \frac{j!(s_1 + e_1 - 1)! \dots (s_{|P|} + e_{|P|} - 1)!}{e_0! e_1! \dots e_{|P|}! (s_1 + e_1)^m \dots (s_{|P|} + e_{|P|})^m} \\ & \leq C_f C^{|P|} \frac{r^{|P|+e_0} (r/4)^{i+j-e_0} R^l (|P| + e_0 + l)!}{(j + i + l + 1)^m} \\ & \quad \times \frac{i! j!(s_1 + e_1 - 1)! \dots (s_{|P|} + e_{|P|} - 1)!}{e_0! (|P|)! s_1! \dots s_{|P|}! e_1! \dots e_{|P|}!}. \end{aligned}$$

By Lemma 2.5, one has

$$\frac{(s_1 + e_1 - 1)! \dots (s_{|P|} + e_{|P|} - 1)!}{s_1! \dots s_{|P|}! e_1! \dots e_{|P|}!} \leq \frac{(i - |P| + j - e_0)!}{(i - |P| + 1)! (j - e_0)!}.$$

Hence, the contribution of one term in the sum (3) is smaller than

$$\begin{aligned} & C_f C^{|P|} \frac{i!}{(|P|)! (i - |P| + 1)!} \\ & \quad \times \frac{r^{|P|+e_0} (r/4)^{i+j-e_0} R^l (|P| + e_0 + l)! j! (i - |P| + j - e_0)!}{(j + i + l + 1)^m e_0! (j - e_0)!}. \end{aligned}$$

As  $(i - |P| + j - e_0)! \leq (j - e_0)! (i - |P|)! 2^{i+j-e_0}$  and  $i! \leq 2^i (|P|)! (i - |P|)!$ , we control each term in the sum (3) with

$$\begin{aligned} & C_f 2^{e_0-j} C^{|P|} r^i \frac{r^{j+|P|} R^l (|P| + e_0 + l)! j! (i - |P|)!}{(j + i + l + 1)^m e_0!} \\ & \leq C_f 2^{e_0-j} (Cr)^i \frac{r^{j+i} R^l (|P| + e_0 + l)! j! (i - |P|)!}{(j + i + l + 1)^m e_0!}. \end{aligned}$$

There are  $\binom{i}{|P|} \leq 2^i$  choices for positive  $s_1, \dots, s_{|P|}$  such that their sum is  $i$ ; similarly, there are  $\binom{j-e_0+|P|}{|P|} \leq 2^{j-e_0+|P|}$  choices for non-negative

$e_1, \dots, e_{|P|}$  such that their sum is  $j - e_0$ . Hence

$$\begin{aligned} & \|(\nabla_{\kappa}^i)^{[\leq n]} f\|_{\ell^1(C^j)} \\ & \leq i^d j^d \sum_{|P|=1}^{\min(n,i)} \sum_{e_0=0}^j 2^{j+|P|-e_0} 2^i C_f 2^{e_0-j} (Cr)^i \frac{r^{j+i} R^l (|P| + e_0 + l)! j!(i - |P|)!}{(j + i + l + 1)^m e_0!} \\ & \leq i^d j^d \sum_{|P|=1}^{\min(n,i)} \sum_{e_0=0}^j C_f (4Cr)^i \frac{r^{j+i} R^l (|P| + e_0 + l)! j!(i - |P|)!}{(j + i + l + 1)^m e_0!}. \end{aligned}$$

The terms in the sum above are increasing with respect to  $e_0$ , so that

$$\begin{aligned} & \|\nabla_v^i f(x, \kappa(x, v, \bar{z}))_{v=0}\|_{\ell^1(C^j)} \\ & \leq i^d j^{d+1} \sum_{|P|=1}^{\min(n,i)} C_f (4Cr)^i \frac{r^{j+i} R^l (|P| + j + l)!}{(i + j + l + 1)^m} (i - |P|)!. \end{aligned}$$

Observe that the quantity in the sum above is log-convex with respect to  $|P|$  as it is a product of factorials, so that

$$\begin{aligned} & \|(\nabla_{\kappa}^i)^{[\leq n]} f\|_{\ell^1(C^j)} \\ & \leq i^{d+1} j^{d+1} C_f \frac{r^{j+i} R^l}{(i + j + l + 1)^m} (4Cr)^i \max((n + j + l)!(i - n)!, (j + l)!i!) \end{aligned}$$

if  $i \geq n$ , and

$$\|(\nabla_{\kappa}^i)^{[\leq n]} f\|_{\ell^1(C^j)} \leq i^{d+1} j^{d+1} C_f \frac{r^{j+i} R^l}{(i + j + l + 1)^m} (4Cr)^i (i + j + l)!$$

if  $i \leq n$ . This concludes the proof, with  $\gamma = 4Cr$ .  $\square$

We are in position to prove the first part of Theorem B, which does not use the structure of the Bergman kernel. Let us prove that the composition of two covariant Toeplitz operators with analytic symbols also admits an analytic symbol, up to an exponentially small error.

**Proposition 4.7.** *There exists a small neighbourhood  $U$  of the diagonal in  $M \times M$ , and constants  $C, m_0, r_0$  such that, for every  $m \geq m_0, r \geq r_0, R \geq Cr^3$ , there exists  $c' > 0$  such that, for every  $f \in S_m^{r,R}(U)$  and  $g \in S_m^{2r,2R}(U)$ , holomorphic in the first variable, anti-holomorphic in the second variable, there exists  $f \sharp g \in S_m^{2r,2R}(U)$  with the same properties, such that*

$$\|T_N^{\text{cov}}(f)T_N^{\text{cov}}(g) - T_N^{\text{cov}}(f \sharp g)\|_{L^2 \rightarrow L^2} \leq C e^{-c'N} \|g\|_{S_m^{2r,2R}(U)} \|f\|_{S_m^{r,R}(U)}.$$

Moreover

$$\|f \sharp g\|_{S_m^{2r,2R}(U)} \leq C \|g\|_{S_m^{2r,2R}(U)} \|f\|_{S_m^{r,R}(U)}.$$

*Proof.* The kernel of  $T_N^{cov}(f)T_N^{cov}(g)$  can be written as

$$(x, z) \mapsto \Psi^N(x, z) \int_{y \in M} e^{N\Phi_1(x, y, \bar{y}, z)} \left( \sum_{k=0}^{cN} N^{d-k} f_k(x, \bar{y}) \right) \times \left( \sum_{j=0}^{cN} N^{d-j} g_j(y, \bar{z}) \right) dy.$$

Here, and until the end of the proof, we write  $f_k(x, \bar{y})$  to indicate that  $f_k$  is holomorphic in the first variable and anti-holomorphic in the second variable. We similarly write  $g_j(y, \bar{z})$ .

Since  $\Phi_1$  is an analytic phase (Proposition 4.3), let us apply the stationary phase lemma (Proposition 3.12). There exists a biholomorphism on a neighbourhood of  $x$  in  $\widetilde{M}$ , of the form

$$\kappa_{(x, \bar{z})} : (y, \bar{y}) \mapsto v(x, y, \bar{y}, \bar{z}),$$

with holomorphic dependence on  $(x, \bar{z})$  (that is, holomorphic in  $x$  and anti-holomorphic in  $z$ ), in which the phase  $\Phi_1$  can be written as  $-|v|^2$ . In particular,

$$v(x, x, \bar{z}, \bar{z}) = 0.$$

Let  $J$  denote the Jacobian of this change of variables. Then

$$\begin{aligned} T_N^{cov}(f)T_N^{cov}(g)(x, z) &= \\ \Psi^N(x, z) \sum_{k, j, n=0}^{\dots} N^{d-k-j-n} \frac{\Delta_v^n}{n!} (f_k(x, \bar{y}(x, v, \bar{z}))g_j(y(x, v, \bar{z}), \bar{z})J(x, v, \bar{z}))_{v=0} & \\ &+ \dots \end{aligned}$$

We will make sense of this sum later on; that is, prove that one can sum until  $k, j$  and  $n$  are equal to  $cN$ , up to an exponentially small error. For the moment, let us treat this formula in decreasing powers of  $N$ . Writing

$$T_N^{cov}(f)T_N^{cov}(g)(x, z) = T_N^{cov}(f\sharp g)(x, \bar{z}) = \Psi^N(x, z) \sum_{k=0}^{\dots} N^{d-k} (f\sharp g)_k(x, \bar{z}) + \dots$$

the symbol  $f\sharp g$  must be holomorphic in the first variable, anti-holomorphic in the second variable, and such that

$$(f\sharp g)_k(x, \bar{z}) = \sum_{n=0}^k \frac{\Delta_v^n}{n!} \left( \sum_{l=0}^{k-n} f_l(x, \bar{y}(x, v, \bar{z}))g_{k-n-l}(y(x, v, \bar{z}), \bar{z})J(x, v, \bar{z}) \right)_{v=0}.$$

Here the Laplace operator acts on  $v$ .

The proof proceeds now in two steps. In the first step, we write a control of the formal symbol  $f\sharp g$  using the analytic symbol structure of  $f$  and  $g$  and

Lemma 4.6. This control involves a complicated quotient of factorials as well as a rational expression similar to the one appearing in Lemma 3.7. The second step is a control the quotients of factorials, thus reducing the proof that  $f\sharp g \in S_m^{2r,2R}$  to Lemma 3.7. It is then standard [41, 34] to check that, if  $f, g, f\sharp g$  are analytic symbols then one can perform an analytic summation to prove that

$$T_N^{cov}(f)T_N^{cov}(g) = T_N^{cov}(f\sharp g) + O(e^{-cN}).$$

**First step.**

We wish to control  $\|(f\sharp g)_k\|_{C^j(U)}$ , which amounts to control, for each  $0 \leq n \leq k, 0 \leq l \leq k - n$ , the  $C^j$ -norm of

$$(x, z) \mapsto \Delta_v^n (f_l(x, \bar{y}(x, v, \bar{z}))g_{k-n-l}(y(x, v, \bar{z}), \bar{z})J(x, v, \bar{z}))_{v=0}.$$

This bidifferential operator acting on  $f_l$  and  $g_{k-n-l}$  coincides, up to a multiplicative factor, with the operator  $B_n$  considered in Proposition 4.4. Indeed, if  $f = f_0$  and  $g = g_0$ , then

$$(f\sharp g)_k(x, \bar{z}) = \frac{\Delta_v^k}{k!} (f_0(x, \bar{y}(x, v, \bar{z}))g_0(y(x, v, \bar{z}))J(x, v, \bar{z}))_{v=0} = B_k(f_0, g_0),$$

where  $(B_k)_{k \geq 0}$  is the sequence of bidifferential operators appearing in Proposition 4.4. In particular, when expanding

$$\Delta_v^n (f_l(x, \bar{y}(x, v, \bar{z}))g_{k-n-l}(y(x, v, \bar{z}), \bar{z})J(x, v, \bar{z}))_{v=0},$$

using the Leibniz and Faà di Bruno formulas, no derivative of  $f_l$  and  $g_{k-n-l}$  of order greater than  $n$  will appear. Let us write this expansion.

Until the end of the proof,  $C^j$  or analytic norms of functions are implicitly on the domain  $U$  or  $U \times U$ .

For every  $n \in \mathbb{N}_0$ , by the multinomial formula, there holds

$$\Delta_v^n = \left( \sum_{i=1}^{2d} \frac{\partial^2}{\partial v_j^2} \right)^n = \sum_{\substack{\mu \in \mathbb{N}_0^{2d} \\ |\mu|=n}} \frac{n!}{\mu!} \partial_v^{2\mu}.$$

Applying the generalised Leibniz rule twice, one has then

$$\begin{aligned} & \Delta_v^n (f_l(x, \bar{y}(x, v, \bar{z}))g_{k-n-l}(y(x, v, \bar{z}), \bar{z})J(x, v, \bar{z}))_{v=0} \\ &= \sum_{\substack{|\mu|=n \\ \nu_1 + \nu_2 \leq 2\mu}} \frac{n!(2\mu)!}{\mu! \nu_1! \nu_2! (2\mu - \nu_1 - \nu_2)!} \partial_v^{\nu_1} f_l(x, \bar{y}(x, v, \bar{z}))_{v=0} \\ & \quad \times \partial_v^{\nu_2} g_{k-n-l}(y(x, v, \bar{z}), \bar{z})_{v=0} \partial_v^{2\mu - \nu_1 - \nu_2} J_{v=0}. \end{aligned}$$

By Proposition 4.4, in the formula above one can replace  $\partial_v^{\nu_1} f(x, \bar{y}(x, v, \bar{z}))_{v=0}$  by its truncation into a differential operator of degree less than  $n$ , applied on

$f$ , which we denote  $(\partial_\kappa^{\nu_1})^{[\leq n]} f(x, \bar{z})$  (similarly as in Lemma 4.6). Similarly one can replace  $\partial_v^{\nu_2} g(y(x, v, \bar{z}), \bar{z})_{v=0}$  by  $(\partial_\kappa^{\nu_2})^{[\leq n]} g(x, \bar{z})$ . Then

$$\begin{aligned} & \Delta_v^n (f_l(x, \bar{y}(x, v, \bar{z})) g_{k-n-l}(y(x, v, \bar{z}), \bar{z}) J(x, v, \bar{z}))_{v=0} \\ &= \sum_{\substack{|\mu|=n \\ \nu_1+\nu_2 \leq 2\mu}} \frac{n!(2\mu)!}{\mu! \nu_1! \nu_2! (2\mu - \nu_1 - \nu_2)!} (\partial_\kappa^{\nu_1})^{[\leq n]} f_l(x, \bar{z}) \\ & \quad \times (\partial_\kappa^{\nu_2})^{[\leq n]} g_{k-n-l}(x, \bar{z}) \partial_v^{2\mu - \nu_1 - \nu_2} J_{v=0}, \end{aligned}$$

with, by Lemma 2.4,

$$\begin{aligned} \frac{n! \mu_1!}{\nu_1! \nu_2! (2\mu - \nu_1 - \nu_2)!} &= \frac{n!}{\mu!} \frac{(2\mu)!}{\nu_1! (2\mu - \nu_1)!} \frac{(2\mu - \nu_1)!}{\nu_2! (2\mu - \nu_1 - \nu_2)!} \\ &\leq \frac{n!}{\mu!} \frac{(2n)!}{|\nu_1|! (2n - |\nu_1|)!} \frac{(2n - |\nu_1|)!}{|\nu_2|! (2n - |\nu_1| - |\nu_2|)!} \\ &= \frac{n!}{\mu!} \binom{2n}{|\nu_1|, |\nu_2|} \leq (2d)^n \binom{2n}{|\nu_1|, |\nu_2|}. \end{aligned}$$

Moreover, applying Proposition 3.2 twice,

$$\begin{aligned} & \|(\partial_\kappa^{\nu_1})^{[\leq n]} f_l(x, \bar{z}) (\partial_\kappa^{\nu_2})^{[\leq n]} g_{k-n-l}(x, \bar{z}) \partial_v^{2\mu - \nu_1 - \nu_2} J_{v=0}\|_{C^j} \\ & \leq \sum_{j_1+j_2 \leq j} \binom{j}{j_1, j_2} \|(\partial_\kappa^{\nu_1})^{[\leq n]} f_l(x, \bar{z})\|_{C^{j_1}} \\ & \quad \times \|(\partial_\kappa^{\nu_2})^{[\leq n]} g_{k-n-l}(x, \bar{z})\|_{C^{j_2}} \|\partial_v^{2\mu - \nu_1 - \nu_2} J_{v=0}\|_{C^{j-j_1-j_2}}. \end{aligned}$$

In particular, using the notation  $(\nabla_\kappa^j)^{[\leq n]}$  as introduced in Lemma 4.6, one has

$$\begin{aligned} & \|n! B_n(f_l, g_{k-n-l})\|_{C^j} \\ &= \|\Delta_v^n (f_l(x, \bar{y}(x, v, \bar{z})) g_{k-n-l}(y(x, v, \bar{z}), \bar{z}) J(x, v, \bar{z}))_{v=0}\|_{C^j} \\ &\leq (2d)^n \sum_{\substack{j_1+j_2 \leq j \\ i_1+i_2 \leq 2n}} \binom{j}{j_1, j_2} \binom{2n}{i_1, i_2} \|(\nabla_\kappa^{i_1})^{[\leq n]} f_l(x, \bar{z})\|_{\ell^1(C^{j_1})} \\ & \quad \times \|(\nabla_\kappa^{i_2})^{[\leq n]} g_{k-n-l}(x, \bar{z})\|_{\ell^1(C^{j_2})} \|\nabla_v^{2n-i_1-i_2} J\|_{\ell^1(C^{j-j_1-j_2})}. \end{aligned}$$

By Lemma 4.6, for some  $\gamma_r$  depending linearly on  $r$  (but independent of  $R, m$ ), one has

$$\begin{aligned} & \|(\nabla_\kappa^{i_1})^{[\leq n]} f_l(x, \bar{z})\|_{\ell^1(C^{j_1})} \\ & \leq i_1^{d+1} j_1^{d+1} \|f\|_{S_m^{r,R}} \gamma_r^{i_1} \frac{r^{j_1+i_1} R^l}{(i_1 + j_1 + l + 1)^m} A(i_1, j_1, l, n), \end{aligned}$$

and

$$\begin{aligned} & \|(\nabla_{\kappa}^{i_2})^{[\leq n]} g_{k-n-l}(x, \bar{z})\|_{\ell^1(C^{j_2})} \\ & \leq i_2^{d+1} j_2^{d+1} \|g\|_{S_m^{2r, 2R}} \gamma_r^{i_2} \frac{(2r)^{j_2+i_2} (2R)^{k-n-l}}{(i_2 + j_2 + l + 1)^m} A(i_2, j_2, k-n-l, n), \end{aligned}$$

where

$$A(i, j, l, n) = \begin{cases} (i + j + l)! & \text{if } i \leq n, \\ \max((n + j + l)!(i - n)!, (j + l)!i!) & \text{otherwise,} \end{cases}$$

The real-analytic function  $J$  belongs to some fixed analytic space, so that there exists  $r_0, m_0$  such that.

$$\|J\|_{C^j} \leq C_J \frac{r_0^j j!}{(j+1)^{m_0}},$$

If  $r \geq 2r_0 2^{m-m_0}$ , by Proposition 2.12, one has

$$\|J\|_{C^j} \leq C_J \frac{(r/2)^j j!}{(j+1)^m},$$

hence

$$\begin{aligned} & \|(f \# g)_k\|_{C^j} \leq \\ & C_J \|f\|_{S_m^{r, R}} \|g\|_{S_m^{2r, 2R}} \frac{(2r)^j (2R)^k (j+k)!}{(k+j+1)^m} \sum_{n=0}^k \left(\frac{\gamma_r r^2}{R}\right)^n \sum_{l=0}^{k-n} \sum_{i_1+i_2 \leq 2n} \sum_{j_1+j_2 \leq j} \\ & \frac{(2n)! j! A(i_1, j_1, l, n) A(i_2, j_2, k-l, n) (2n+j-j_1-j_2-i_1-i_2)!}{2^{2n+j-j_1-j_2-i_1-i_2} 2^{j_1+i_1+l} i_1! i_2! j_1! j_2! (2n-i_1-i_2)! (j-j_1-j_2)! n! (k+j)!} \\ & \frac{i_1^d i_2^d j_1^d j_2^d (k+j+1)^m}{(j_1+i_1+l+1)^m (j_2+i_2+k-n-l+1)^m (j+2n-i_1-i_2-j_1-j_2+1)^m}. \end{aligned}$$

**Second step.**

Let us control the quotient of factorials above. There holds

$$\frac{(2n+j-j_1-j_2-i_1-i_2)!}{2^{2n+j-j_1-j_2-i_1-i_2} (j-j_1-j_2)! (2n-i_1-i_2)!} = \frac{\binom{2n+j-j_1-j_2-i_1-i_2}{j-j_1-j_2}}{2^{2n+j-j_1-j_2-i_1-i_2}} \leq 1.$$

Thus, the middle line in the control on  $\|(f \# g)_k\|_{C^j}$  is smaller than

$$\frac{(2n)! j! A(i_1, j_1, l, n) A(i_2, j_2, k-l, n)}{2^{j_1+i_1+l} i_1! i_2! j_1! j_2! n! (k+j)!}.$$

Let us prove that, if  $i_1 \leq 2n, i_2 \leq 2n, 0 \leq l \leq k-n, j_1+j_2 \leq j$ , then

$$\frac{(2n)! j! A(i_1, j_1, l, n) A(i_2, j_2, k-l, n)}{2^{j_1+i_1+l} i_1! i_2! j_1! j_2! n! (k+j)!} \leq 4^n.$$

For the moment, let us focus on the  $i_1 \leq n, i_2 \leq n$  case. As  $i_1 \geq 0$  one has  $\frac{1}{2^{i_1}} \leq 1$  and it remains to control

$$\frac{(2n)!j!(j_1 + i_1 + l)(j_2 + i_2 + k - n - l)!}{2^{j_1+l}i_1!i_2!j_1!j_2!n!(k+j)!}.$$

This expression is increasing with respect to  $i_1$  and  $i_2$ , so that we only need to control the  $i_1 = i_2 = n$  case, which is

$$\frac{(2n)!j!(j_1 + n + l)(j_2 + k - l)!}{2^{j_1+l}(n!)^3j_1!j_2!(k+j)!}$$

Moreover, the expression above is log-convex with respect to  $l$ , so that we only need to control the  $l = 0$  and  $l = k - n$  case.

If  $l = 0$  we are left with

$$\frac{(2n)!j!(j_1 + n)(k + j_2)!}{2^{j_1}(n!)^3j_1!j_2!(k+j)!} = 2^n \binom{2n}{n} \frac{\binom{j_1+n}{n} \binom{k+j_2+j_2}{j_2}}{2^{j_1+n} \binom{k+j_2+j_2}{j}} \leq 4^n \frac{\binom{k+j_2+j_2}{j_2}}{\binom{k+j_2+j_2}{j}}.$$

To conclude,  $j$  is closer from  $\frac{k+j+j_2}{2}$  than  $j_2$  since  $j \geq j_2$ , so that  $\frac{\binom{k+j_2+j_2}{j_2}}{\binom{k+j_2+j_2}{j}} \leq 1$ , hence the claim.

If  $l = k - n$ , one has

$$\frac{(2n)!j!(j_1 + k)(j_2 + n)!}{2^{j_1+k-n}(n!)^3j_1!j_2!(k+j)!} = 2^n \binom{2n}{n} \frac{\binom{j_1+k}{k} \binom{j_2+n}{n}}{2^{j_1+k} \binom{j+k}{k}} \leq 4^n.$$

We now consider the case  $i_1 \geq n$  or  $i_2 \geq n$ . We need to replace  $(i_1 + j_1 + l)!$  with either  $(j_1 + l)i_1!$  or  $(j_1 + l + n)(i_1 - n)!$ . One has

$$\begin{aligned} \frac{(j_1 + l)i_1!}{i_1!} &= (j_1 + l)! \leq \frac{(j_1 + l + n)!}{n!} \\ \frac{(j_1 + l + n)(i_1 - n)!}{i_1!} &\leq \frac{(j_1 + l + n)i_1!}{i_1!n!} = \frac{(j_1 + l + n)!}{n!}. \end{aligned}$$

The same inequalities apply with  $i_1, j_1$  replaced with  $i_2, j_2$ . Hence, in all cases, we are left with

$$\frac{(2n)!j!(j_1 + n + l)(j_2 + k - l)!}{2^{j_1+l}(n!)^3j_1!j_2!(k+j)!},$$

which we just proved to be smaller than  $4^n$ .

This yields

$$\begin{aligned} \|(f \# g)_k\|_{C^j} &\leq \\ C_J \|f\|_{S_m^{r,R}} \|g\|_{S_m^{2r,2R}} &\frac{(2r)^j (2R)^k (j+k)!}{(k+j+1)^m} \sum_{n=0}^k \left( \frac{4\gamma_r r^2}{R} \right)^n \sum_{l=0}^{k-n} \sum_{i_1, i_2=0}^n \sum_{j_1+j_2 \leq j} \\ &\frac{(k+j+1)^m i_1^d i_2^d j_1^d j_2^d}{(j_1 + i_1 + l + 1)^m (j_2 + i_2 + k - n - l + 1)^m (j + 2n - i_1 - i_2 - j_1 - j_2 + 1)^m}. \end{aligned}$$



We are almost in position to apply Lemma 3.7; since

$$(k + j + n + 1)^m \geq (k + j + 1)^m,$$

one has

$$\begin{aligned} \|(f \# g)_k\|_{C^j} &\leq \\ C_J \|f\|_{S_m^{r,R}} \|g\|_{S_m^{2r,2R}} &\frac{(2r)^j (2R)^k (j+k)!}{(k+j+1)^m} \sum_{n=0}^k \left(\frac{4\gamma_r r^2}{R}\right)^n \sum_{l=0}^{k-n} \sum_{i_1, i_2=0}^n \sum_{j_1+j_2 \leq j} \\ &\frac{i_1^{d_1} i_2^{d_2} j_1^{d_1} j_2^{d_2} (k+j+n+1)^m}{(j_1+i_1+l+1)^m (j_2+i_2+k-n-l+1)^m (j+2n-i_1-i_2-j_1-j_2+1)^m}. \end{aligned}$$

Applying Lemma 3.7 yields, for  $m$  large enough depending on  $d$ ,

$$\|(f \# g)_k\|_{C^j} \leq C_J \|f\|_{S_m^{r,R}} \|g\|_{S_m^{2r,2R}} \frac{(2r)^j (2R)^k (j+k)!}{(k+j+1)^m} \sum_{n=0}^k \left(\frac{4\gamma_r r^2}{R}\right)^n.$$

As long as  $R \geq 4\gamma_r r^2$ , which is possible if  $R$  is chosen large enough since  $\gamma_r$  depends only on  $r$ , one can conclude:

$$\|(f \# g)_k\|_{C^j} \leq 2^m C_J \|f\|_{S_m^{r,R}} \|g\|_{S_m^{2r,2R}} \frac{(2r)^j (2R)^k (j+k)!}{(k+j+1)^m}.$$

At this stage, we are almost done with the proof: we obtained that the formal series which corresponds, in the  $C^\infty$  class, to the composition  $T_N^{cov}(f)T_N^{cov}(g)$ , belongs to the same analytic symbol class than  $g$ .

This concludes the proof.  $\square$

#### 4.4 Inversion of covariant Toeplitz operators and the Bergman kernel

In this subsection we prove Theorem A as well as the second part of Theorem B. To do so, we first show in Proposition 4.8, as a reciprocal to Proposition 4.7, that if  $f$  and  $h$  are analytic symbols of covariant Toeplitz operators with  $f_0$  non-vanishing, then there exists an analytic symbol  $g$  such that

$$T_N^{cov}(f)T_N^{cov}(g) = T_N^{cov}(h) + O(e^{-cN}).$$

We then prove in Proposition 4.9 that, under the same hypotheses,  $T_N^{cov}(f)$ , whose image is almost contained in  $H^0(M, L^{\otimes N})$  by Remark 4.2, is invertible on this space up to an exponentially small error. Thus, one can conclude that, on  $H^0(M, L^{\otimes N})$ , there holds

$$T_N^{cov}(g) = T_N^{cov}(h)(T_N^{cov}(f))^{-1} + O(e^{-cN}).$$

This allows us to prove Theorem A, since by setting  $h = f$  one recovers that the Bergman kernel can be written as  $T_N^{cov}(f)(T_N^{cov}(f))^{-1} = T_N(a)$ . Then, the second part of Theorem B follows from Proposition 4.8 by setting  $h = a$ .

Following the lines of Proposition 4.7, let us try to construct inverses for analytic symbols.

**Proposition 4.8.** *Let  $U$  denote a small neighbourhood of the diagonal in  $M \times M$  and let  $f, h \in S_{m_0}^{r_0, R_0}(U)$  be analytic symbols, holomorphic in the first variable and anti-holomorphic in the second variable, for some  $r_0, R_0, m_0$ . Suppose that the principal symbol  $f_0$  of  $f$  is bounded away from zero on  $U$ .*

*Then there exists  $r, R, m$  as well as  $g \in S_m^{r, R}(U)$ , holomorphic in the first variable, anti-holomorphic in the second variable, such that*

$$T_N^{\text{cov}}(f)T_N^{\text{cov}}(g) = T_N^{\text{cov}}(h) + O(e^{-cN}).$$

*Proof.* Recalling the proof of Proposition 4.7, let us recover  $g$  from  $f$  and  $h = f\sharp g$ . By definition of  $h_k$ , one has

$$g_k(x, \bar{z})f_0(x, \bar{z})J(x, x, \bar{z}, \bar{z}) = h_k(x, \bar{z}) - \sum_{n=0}^k \frac{\Delta_v^n}{n!} \left( \sum_{\substack{l=0 \\ l+n>0}}^{k-n} f_l(x, \bar{y}(x, v, \bar{z}))g_{k-n-l}(y(x, v, \bar{z}), \bar{z})J(x, v, \bar{z}) \right)_{v=0}. \quad (4)$$

As  $f_0$  is bounded away from zero, this indeed defines  $g_k$  by induction. Let us try to control  $g$  in an analytic space.

We first let  $m$  large enough, and  $r \geq 2r_0 2^{m-m_0}$  as well as  $R \geq 2R_0 2^{m-m_0}$ . Then, by Lemma 3.4, there exist  $C_f, C_h, C_J$  independent of  $m, r, R$  such that, for every  $k \geq 0, j \geq 0$ ,

$$\begin{aligned} \|f_k\|_{C^j(U)} &\leq C_f \frac{(r/2)^j (R/2)^k (j+k)!}{(j+k+1)^m} \\ \|h_k\|_{C^j(U)} &\leq C_h \frac{r^j r^k (j+k)!}{(j+k+1)^m} \\ \|J\|_{C^j(U \times U)} &\leq C_J \frac{(r/2)^j j!}{(j+1)^m}. \end{aligned}$$

Here  $J$  denotes again the Jacobian in the change of variables corresponding to the Morse lemma for the phase  $\Phi_1$ .

We first note that

$$g_0(x, \bar{z}) = f_0(x, \bar{z})^{-1} h_0(x, \bar{z}) J(x, x, \bar{z}, \bar{z}),$$

so that, by Lemma 2.14, there exists  $C_0$  such that, for every  $r \geq 2r_0 2^{m-m_0}$  and  $R \geq 2R_0 2^{m-m_0}$ , for every  $j \geq 0$ ,

$$\|g_0\|_{C^j(U)} \leq C_0 \frac{r^j j!}{(j+1)^m}.$$

Let us prove by induction on  $l \geq 1$  that, for some fixed  $C_g, m, r, R$ , for every  $j \geq 0$ , one has

$$\|g_l\|_{C^j} \leq C_g \frac{r^j R^l (j+l)!}{(j+l+1)^m}.$$

Over the course of the induction, we will fix the values of  $C_g, m, r, R$ .

Suppose that a control above is true for indices up to  $l = k - 1$ . Then, from the recursive formula (4), if we repeated the proof of Proposition 4.7, we would obtain

$$\|g_k\|_{C^j} \leq C(C_h + C_g C_f C_J) \frac{r^j R^k (j+k)!}{(j+k+1)^m}.$$

This is not enough, as the constant  $C(C_h + C_g C_f C_J)$  appearing here might be greater than  $C_g$ . However, as we will see, the constant can be made arbitrarily small by choosing  $C_g$  large enough, as well as  $m$  large enough, depending on  $f$ , and  $R/r^2$  large enough.

Let  $C_1 = C\|(f_0 J)^{-1}\|_{H(m,r,U)}$  where  $C$  is the constant appearing in Proposition 2.14.

There holds

$$C_h \leq \frac{C_g}{4C_1}$$

if  $C_g$  is large enough with respect to  $C_h, C_f, C_J, C_0$ . It remains to estimate the second term on the right-hand side of (4).

Let us isolate the  $n = 0, l = k$  term in (4). This term is  $-g_0 J f_k$ , and the  $S_m^{r,R}(U)$ -norm of  $g_0 J f$  is again smaller than  $\frac{C_g}{4C_1}$  if  $C_g$  is large enough with respect to  $C_f C_0 C_J$ .

Repeating the proof of Proposition 4.7, the  $n = 0, l < k$  terms in (4) are bounded in  $C^j$ -norm by

$$\begin{aligned} & C C_J C_f C_g \frac{r^j R^k (j+k)!}{(j+k+1)^m} \\ & \times \sum_{l=1}^{k-1} \sum_{j_1+j_2 \leq j} \frac{(j+k+1)^m}{(j_1+l+1)^m (j_2+k-l+1)^m (j-j_1-j_2+1)^m}. \end{aligned}$$

By Lemma 3.7, since no term in the sum

$$\begin{aligned} & \sum_{\substack{1 \leq l \leq k-1 \\ j_1+j_2 \leq j}} \frac{(j+k+1)^m}{(j_1+l+1)^m (j_2+k-l+1)^m (j-j_1-j_2+1)^m} \\ & = \sum_{\substack{i_1+i_2+i_3=j+k \\ i_1 \geq 1 \\ i_2 \geq 1}} \frac{(j+k+1)^m}{(i_1+1)^m (i_2+1)^m (i_3+1)^m} \end{aligned}$$

contribute as 1, by Lemma 3.7 (with  $d = 0$  and  $n = 3$ ), this sum is smaller than  $C(3/4)^m$  for some  $C > 0$ . Hence, if  $m$  is large enough, this contribution is also smaller than  $\frac{C_g}{4C_1}$ . Now  $m$  is fixed.

It remains to control the  $n \geq 1$  terms in (4). From the proof of Proposition 4.7, their sum is smaller than

$$C C_J C_f C_g \sum_{n=1}^k \frac{r^j R^k (j+k)!}{(j+k+1)^m} \left( \frac{4\gamma_r r^2}{R} \right)^n.$$

As long as  $R/r^2$  is large enough with respect to  $\gamma_r C_J C_f$ , (which is possible if  $R$  is large enough since  $\gamma_r = Cr$  for some fixed  $C$ ), this is again smaller than  $\frac{C_g}{4C_1}$ .

In conclusion,

$$\|g_k f_0 J\|_{C^j} \leq \frac{C_g}{C_1} \frac{r^j R^k (j+k)!}{(j+k+1)^m}.$$

In particular, by Lemma 2.14, and since  $\|(f_0 J)^{-1}\|_{H(m,r,U)} = C_1/C$ , one has

$$\|g_k\|_{C^j} = \|g_k f_0 J (f_0 J)^{-1}\|_{C^j} \leq C_g \frac{r^j R^k (j+k)!}{(j+k+1)^m}.$$

This concludes the induction.

Once the formal series  $g$  is controlled in an analytic symbol space, the composition  $T_N(g)T_N(f)$  coincides with  $T_N(h)$  up to an exponentially small error as in the end of the proof of Proposition 4.7, hence the claim.  $\square$

**Proposition 4.9.** *Let  $f$  be a function on  $U$ , holomorphic with respect to the first variable, anti-holomorphic with respect to the second variable. If  $f$  is nonvanishing then  $S_N T_N^{cov}(f)$  has an inverse on  $H^0(M, L^{\otimes N})$ , with operator norm bounded independently of  $N$ .*

*Proof.* One can invert  $S_N T_N^{cov}(f)$  by a formal covariant symbol, that is, up to an  $O(N^{-K})$  error for any fixed  $K$ . In particular, there exists an operator  $A_N$  on  $H^0(M, L^{\otimes N})$  such that  $A_N S_N T_N^{cov}(f) = S_N + O(N^{-1})$ , and such that the operator norm of  $A_N$  is bounded independently on  $N$ .

Since  $A_N S_N T_N^{cov}(f)$  is invertible on  $H^0(M, L^{\otimes N})$ , so is  $S_N T_N^{cov}(f)$ , and the operator norm of this inverse is  $\|A_N\|_{L^2 \rightarrow L^2} (1 + O(N^{-1}))$ , which is bounded independently on  $N$ , hence the claim.  $\square$

Let us now conclude the proofs of Theorems A and B.

Let  $U$  be a small neighbourhood of the diagonal in  $M \times M$  and let  $f$  be any function on  $U$  bounded away from zero, holomorphic in the first variable, anti-holomorphic in the second variable. From Proposition 4.8 there exists an analytic symbol  $a$  with the same properties, such that

$$T_N^{cov}(f) T_N^{cov}(a) = T_N^{cov}(f) + O(e^{-cN}). \quad (5)$$

Let  $A_N = (S_N T_N^{cov}(f))^{-1}$  on  $H^0(M, L^{\otimes N})$ ; we know from Proposition 4.9 that  $A_N$  is well-defined and bounded independently on  $N$ . Then, for any  $u \in H^0(M, L^{\otimes N})$ , one has

$$T_N^{cov}(a)u = u + O(e^{-cN}).$$

Indeed, one can write  $u = A_N v$  and apply the adjoint of (5) to  $v$ .

Moreover, by Remark 4.2, there holds

$$(I - S_N) T_N^{cov}(a) = O(e^{-cN}).$$

To conclude, one has  $T_N^{cov}(a) = S_N + O(e^{-cN})$ . In other terms,

$$S_N(x, \bar{y}) = \Psi^N(x, y) \sum_{k=0}^{cN} N^{d-k} a_k(x, \bar{y}) + O(e^{-cN}).$$

This concludes the proof of Theorem A.

Let us complete the proof of Theorem B. Its first part is Proposition 4.7. For the second part, we apply Proposition 4.8 with  $h = a$ , the symbol of the Bergman kernel.

We remark that, from this proof,  $a$  may depend on  $f$ , but necessarily  $a$  coincides with the formal symbol of the Bergman kernel, so that it is uniquely defined.

**Remark 4.10** (Normalised covariant Toeplitz operators). Let  $T_N^{cov}(a)$  denote the approximate Bergman kernel constructed in the previous proposition. Once the symbol  $a$  is known, one can study, as in the proof of Proposition 4.4, *normalised covariant* Toeplitz operators, of the form

$$\Psi^N(x, y) \left( \sum_{k=0}^{cN} N^{-k} a_k(x, \bar{y}) \right) \left( \sum_{k=0}^{cN} N^{d-k} f_k(x, \bar{y}) \right).$$

Under this convention, the operator associated with the function  $f = 1$  is  $S_N + O(e^{-cN})$ , as in contravariant Toeplitz quantization.

Propositions 4.7 and 4.8 can be adapted to normalised covariant Toeplitz operators, for which the algebra product is

$$(f, g) \mapsto ((f * a) \sharp (g * a)) * a^{*-1}.$$

For instance, since the Cauchy product is continuous on each symbol class, there holds, for  $m$  large enough,  $r > 2^m$  and  $R > Cr^3$ ,

$$\|((f * a) \sharp (g * a)) * a^{*-1}\|_{S_m^{2r, 2R}(U)} \leq C_a \|f\|_{S_m^{r, R}(U)} \|g\|_{S_m^{2r, 2R}(U)}.$$

To conclude this section, we prove that analytic contravariant Toeplitz operators are contained within analytic covariant Toeplitz operators.

**Proposition 4.11.** *Let  $f$  be a real-analytic function on  $M$ . There exists an analytic symbol  $g$  and  $c > 0$  such that*

$$T_N(f) = T_N^{cov}(g) + O(e^{-cN}).$$

*Proof.* Recall from Theorem A that there exists an analytic symbol  $a$  such that

$$S_N = T_N^{cov}(a) + O(e^{-cN}).$$

Letting  $\tilde{f}$  be a holomorphic extension of  $f$ , the kernel of  $T_N(f) = S_N f S_N$  is then

$$(x, z) \mapsto \Psi^N(x, z) \int_{y \in M} e^{-N\Phi_1(x, y, \bar{y}, \bar{z})} \sum_{k, j=0}^{cN} N^{2d-k-j} a_k(x, \bar{y}) a_j(y, \bar{z}) \tilde{f}(y, \bar{y}) dy + O(e^{-cN}).$$

One can then repeat the proof of Proposition 4.7 with  $J$  replaced with  $(x, y, \bar{y}, \bar{z}) \mapsto J(x, y, \bar{y}, z) \tilde{f}(y, \bar{y})$ . This yields an analytic symbol  $g$  such that

$$g_k(x, \bar{z}) = \sum_{n=0}^k \frac{\tilde{\Delta}_v^n}{n!} \left( \sum_{l=0}^{k-n} a_l(x, \bar{y}(x, v, \bar{z})) a_{k-n-l}(y(x, v, \bar{z}), \bar{z}) \times J(x, (y, \bar{y})(x, v, \bar{z}), \bar{z}) \tilde{f}((y, \bar{y})(x, v, \bar{z})) \right)_{v=0},$$

that is,

$$T_N^{cov}(g) = S_N f S_N + O(e^{-cN}).$$

□

## 4.5 Exponential decay of low-energy states

Since covariant analytic Toeplitz operators form an algebra up to exponentially small error terms (Theorem B), and since contravariant Toeplitz operators are a subset of covariant analytic Toeplitz operators (Proposition 4.11), one can study exponential localisation for eigenfunctions of contravariant analytic Toeplitz operators. In this subsection we prove Theorem C.

Let  $f$  be a real-analytic, real-valued function on  $M$ , let  $E \in \mathbb{R}$  and let  $(u_N)_{N \geq 1}$  be a normalized family of eigenstates of  $T_N(h)$  with eigenvalue  $\lambda_N = E + o(1)$ . Let  $V$  be an open set at positive distance from  $\{f = E\}$ . Let  $a \in C^\infty(M, \mathbb{R}^+)$  be such that  $\text{supp}(a) \cap \{f = E\} = \emptyset$  and  $a = 1$  on  $V$ . The function  $a$  is of course not real-analytic; we will nevertheless prove that

$$T_N(a)u_N = O(e^{-cN}).$$

This implies Theorem C, since

$$\int_V |u_N|^2 = \langle u_N, \mathbb{1}_V u_N \rangle \leq \langle u_N, a u_N \rangle = \langle u_N, T_N(a) u_N \rangle = O(e^{-cN}).$$

Let  $W$  be an open set of  $M$  such that

$$\text{supp}(a) \subset\subset W \subset\subset \{f \neq E\}.$$

On  $W$ , the function  $b - E$  is bounded away from zero. Let us consider, on a neighbourhood of  $\text{diag}(W)$  in  $M \times M$ , the analytic covariant symbol  $g$  which is such that  $T_N^{\text{cov}}(g)$  is the analytic inverse (on this neighbourhood) of  $T_N(f - \lambda(N))$ . This symbol is well-defined: one can check that the construction of an inverse symbol in Proposition 4.8 only relies on local properties. The function  $f - \lambda(N)$  might not be a classical analytic symbol, since we made no assumption on the eigenvalue  $\lambda(N)$ . However, for every  $t$  close to  $E$  one can define the microlocal inverse  $g_t$  of  $f - t$  near  $W$ , in an analytic class independent of  $t$ , so that we define the microlocal inverse of  $T_N(f - \lambda(N))$  as the operator with kernel

$$T_N^{\text{cov}}(g) : (x, y) \mapsto \Psi^N(x, y)g_{\lambda(N)}(N)(x, y).$$

We arbitrarily cut off  $g$  outside a neighbourhood of  $\text{diag}(W_1)$ , where  $W \subset\subset W_1 \subset\subset \{f \neq E\}$  so that  $T_N^{\text{cov}}(g)$  is a well-defined operator. Let us prove that, for some  $c > 0$  small, one has

$$T_N(a)T_N^{\text{cov}}(g)T_N(f - \lambda_N) = T_N(a) + O(e^{-cN}).$$

By construction, uniformly on  $x \in W_1$  and  $z \in M$ , one has

$$\int_{y \in M} T_N^{\text{cov}}(g)(x, y)T_N(f - \lambda_N)(y, z) = S_N(x, z) + O(e^{-cN}).$$

In particular, since  $T_N(a)$  is bounded by  $O(e^{-cN})$  on  $W \times (M \setminus W_1)$ , for  $x \in W$  one has

$$\begin{aligned} & \int_{y_1 \in M, y_2 \in M} T_N(a)(x, y_1)T_N^{\text{cov}}(g)(y_1, y_2)T_N(f)(y_2, z) \\ &= \int_{y_1 \in W_1, y_2 \in M} T_N(a)(x, y_1)T_N^{\text{cov}}(g)(y_1, y_2)T_N(f - \lambda_N)(y_2, z) + O(e^{-cN}) \\ &= \int_{y_1 \in W_1} T_N(a)(x, y_1)S_N(y_1, z) + O(e^{-cN}) \\ &= \int_{y_1 \in M} T_N(a)(x, y_1)S_N(y_1, z) + O(e^{-cN}) = T_N(a)(x, z) + O(e^{-cN}). \end{aligned}$$

Moreover, uniformly on  $(x \notin W, y \in M)$  there holds  $T_N(a)(x, y_1) = O(e^{-cN})$  so that, finally,

$$T_N(a)T_N^{\text{cov}}(g)T_N(f - \lambda_N) = T_N(a) + O(e^{-cN}).$$

In particular,

$$0 = T_N(a)T_N^{\text{cov}}(g)T_N(f - \lambda(N))u_N = T_N(a)u_N + O(e^{-cN}),$$

which concludes the proof.

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## A The Wick rule

Here we present a self-contained proof of Proposition 4.4.

It is well-known (see [9], Theorem 2) that there exists an invertible formal series  $a$  of functions defined on a neighbourhood of the diagonal in  $M \times M$ , holomorphic in the first variable and anti-holomorphic in the second variable, which correspond to the Bergman kernel, that is, such that

$$T_N^{cov}(a) = S_N + O(N^{-\infty}).$$

In Theorem A, we prove that  $a$  is in fact an analytic symbol; but for the moment, it is sufficient to know that  $a$  exists as a formal series.

Let us deform covariant Toeplitz operators by this formal symbol  $a$ , into *normalised* covariant Toeplitz operators of the form  $T_N^{cov}(f * a)$ . Here  $*$  denotes the Cauchy product of symbols (Proposition 3.8). Since in this case  $f$  and  $g$  are simply holomorphic functions one has  $f * a = fa$  and  $g * a = ga$ .

We will first prove our claim for this modified quantization: that is, there exists a sequence of bidifferential operators  $(C_k)_{k \geq 0}$  acting on functions on a neighbourhood of the diagonal in  $M \times M$ , such that, given two such functions  $f$  and  $g$ , if we let

$$f \sharp g = \sum_{k=0}^{+\infty} N^{-k} C_k(f, g) + O(N^{-\infty}),$$

then

$$T_N^{cov}((f \sharp g) * a) = T_N^{cov}(fa)T_N^{cov}(ga) + O(N^{-\infty}).$$

Moreover,  $C_k$  is of order at most  $k$  in each of its arguments. Then, we will relate the coefficients  $C_k$  with the coefficients  $B_k$  in the initial claim.

The claim is easier to prove for the coefficients  $C_k$  because normalised covariant Toeplitz quantization follows the Wick rule. Indeed, if the function  $f$ , near a point  $x_0$ , depends only on the first variable (that is, the restriction of  $f$  to the diagonal is, near this point, a holomorphic function on  $M$ ), then the kernel  $T_N^{cov}(af)(x, y)$ , for  $x$  close to  $x_0$ , can be written as  $f(x)T_N^{cov}(a)(x, y) = f(x)S_N(x, y) + O(N^{-\infty})$ . In particular, for  $x$  close to  $x_0$  the Wick rule holds:

$$T_N^{cov}(af)T_N^{cov}(ag)(x, y) = T_N^{cov}(afg)(x, y) + O(N^{-\infty}),$$

since by Remark 4.2 the kernel of  $T_N^{cov}(ag)$  is almost holomorphic in the first variable, up to an  $O(N^{-\infty})$  error. Thus, locally where  $f$  depends only on the first variable, there holds

$$\forall k \geq 1, C_k(f, g) = 0.$$



More generally, we wish to compute

$$N^{2d}\Psi^N(x, z) \int_M \exp(N\Phi_1(x, y, \bar{y}, \bar{z})) (fa)(N)(x, \bar{y})(ga)(N)(y, \bar{z}) dy,$$

where we recall that

$$\Phi_1(x, y, \bar{w}, \bar{z}) = -2\tilde{\phi}(x, \bar{w}) + 2\tilde{\phi}(y, \bar{w}) - 2\tilde{\phi}(y, \bar{z}) + 2\tilde{\phi}(x, \bar{z}).$$

Here, we write  $(fa)(N)(x, \bar{y})$  to indicate that  $fa$  is holomorphic in the first variable and anti-holomorphic in the second variable. Similarly, we write  $\Phi_1(x, y, \bar{w}, \bar{z})$  to indicate that  $\Phi_1$  is a function on  $M_x \times \widetilde{M}_{y, \bar{w}} \times M_z$ , holomorphic in its two first arguments and anti-holomorphic in the third argument; we integrate over  $M$  which is the subset of  $\widetilde{M}$  such that  $\bar{w} = \bar{y}$ .

First of all, let us prove a Schur test: operator with kernels of the form

$$(x, z) \mapsto N^{2d} \int_M \exp(N\Phi_1(x, y, \bar{y}, \bar{z})) b(x, y, \bar{y}, z) dy$$

are bounded from  $L^2(M, L^{\otimes N})$  to itself independently on  $N$ ; in particular, successive integration by parts on  $(y, \bar{y})$ , which will introduce negative powers of  $N$  in the symbol, will lead to a control of the operator.

Since for any  $(x, z) \in U$  one has  $|\Psi^N(x, z)| \leq e^{-cN \text{dist}(x, z)^2}$ , then there exists  $C > 0$  such that, for any analytic symbol  $b$  on  $U \times U$ , there holds

$$\begin{aligned} & N^{2d} \sup_x \int_M \left| \Psi^N(x, z) \int_M \exp(N\Phi_1(x, y, \bar{y}, \bar{z})) b(N)(x, y, \bar{y}, z) dy \right| dz \\ & \leq N^{2d} \sup_{U \times U} |b(N)| \sup_x \int_M \int_M |\Psi^N(x, y)| |\Psi^N(y, z)| dy dz \\ & \leq \sup_{U \times U} |b(N)| N^{2d} \sup_x \int_{M \times M} e^{-Nc \text{dist}(x, y)^2 - Nc \text{dist}(y, z)^2} dy dz \\ & \leq C \sup_{U \times U} |b(N)|. \end{aligned}$$

In particular, by the Schur test, the operator with the kernel above is bounded independently on  $N$ .

As  $\partial_y \Phi_1$  vanishes in a non-degenerate way at  $\bar{w} = \bar{z}$ , one can write

$$f(x, \bar{w}) = f(x, \bar{z}) - \partial_y \Phi_1 \cdot F_1(x, \bar{z}, y, \bar{w}).$$

Thus,

$$\begin{aligned} & N^{2d}\Psi^N(x, z) \int_M e^{N\Phi_1(x, y, \bar{y}, \bar{z})} (fa)(N)(x, \bar{y})(ga)(N)(y, \bar{z}) dy \\ & = N^{2d}\Psi^N(x, z) f(x, \bar{z}) \left( \int_M e^{N\Phi_1(x, y, \bar{y}, \bar{z})} a(N)(x, \bar{y})(ga)(N)(y, \bar{z}) dy \right. \\ & \quad \left. + N^{-1} \int_M e^{N\Phi_1(x, y, \bar{y}, \bar{z})} a(N)(x, \bar{y}) \partial_M [F_1(x, \bar{z}, y, \bar{y})(ga)(N)(y, \bar{z})] dy \right). \end{aligned}$$

The first term in the right-hand side above is equal to

$$f(x, \bar{z}) \int_M T_N^{\text{cov}}(a)(x, \bar{y}) T_N^{\text{cov}}(ga)(y, \bar{z}) dy = f(x, \bar{z}) T_N^{\text{cov}}(ga)(x, \bar{z}) + O(N^{-\infty}),$$

since  $T_N^{\text{cov}}(a) = S_N + O(N^{-\infty})$ .

In the second line, which is of order  $N^{-1}$  by a Schur test, derivatives of  $g$  of order at most 1 appear. This remainder can be written as

$$\begin{aligned} & N^{2d-1} \Psi^N(x, z) \int_M e^{N\Phi_1(x, y, \bar{y}, \bar{z})} a(N)(x, \bar{y}) [\partial_y F_1(x, \bar{z}, y, \bar{y})] (ga)(N)(y, \bar{z}) dy \\ & + N^{2d-1} \Psi^N(x, z) \int_M e^{N\Phi_1(x, y, \bar{y}, \bar{z})} a(N)(x, \bar{y}) F_1(x, \bar{z}, y, \bar{y}) [\partial_y (ga)(N)(y, \bar{z})] dy. \end{aligned}$$

We recover the initial expression, where  $f$  has been replaced with either  $F_1$  or  $\partial_y F_1$ , and  $g$  has potentially been differentiated once. Thus, by induction, the coefficient  $C_k(f, g)$  only differentiates at most  $k$  times on  $g$ . By duality,  $C_k(f, g)$  only differentiates at most  $k$  times on  $f$ .

Let us now relate the coefficients  $C_k$  and  $B_k$ . Let  $a^{*-1}$  denote the inverse of  $a$  for the Cauchy product. One has

$$\begin{aligned} T_N^{\text{cov}}(f) T_N^{\text{cov}}(g) &= T_N^{\text{cov}}((fa^{*-1}) * a) T_N^{\text{cov}}((ga^{*-1}) * a) + O(N^{-\infty}) \\ &= T_N^{\text{cov}}((C_k(f, g))_{k \geq 0} * a) + O(N^{-\infty}), \end{aligned}$$

so that the coefficients  $B_k$  in the initial claim are recovered as

$$B_k(f, g) = \sum_{j+l+m \leq k} a_j C_{k-j-l-m}(fa_l^{*-1}, ga_m^{*-1}),$$

thus  $B_k$  itself differentiates at most  $k$  times on  $f$  and at most  $k$  times on  $g$ .

## References

- [1] Y. Ameur, H. Hedenmalm, and N. Makarov. Fluctuations of eigenvalues of random normal matrices. *Duke mathematical journal*, 159(1):31–81, 2011.
- [2] R. Berman, B. Berndtsson, and J. Sjöstrand. A direct approach to Bergman kernel asymptotics for positive line bundles. *Arkiv för Matematik*, 46(2):197–217, 2008.
- [3] M. Bordemann, E. Meinrenken, and M. Schlichenmaier. Toeplitz quantization of Kähler manifolds and  $\text{gl}(N)$ ,  $N \rightarrow \infty$  limits. *Communications in Mathematical Physics*, 165(2):281–296, 1994.
- [4] D. Borthwick. Introduction to Kähler quantization. *Contemporary Mathematics*, 260:91, 2000.

- [5] L. Boutet de Monvel and V. Guillemin. *The Spectral Theory of Toeplitz Operators*. Number 99 in Annals of Mathematics Studies. Princeton University Press, 1981.
- [6] L. Boutet de Monvel and P. Krée. Pseudo-differential operators and Gevrey classes. *Annales de l'Institut Fourier*, 17(1):295–323, 1967.
- [7] L. Boutet de Monvel and J. Sjöstrand. Sur la singularité des noyaux de Bergman et de Szegő. *Journées équations aux dérivées partielles*, 34-35:123–164, 1975.
- [8] L. Charles. *Aspects Semi-Classiques de La Quantification Géométrique*. PhD thesis, Université Paris 9, 2000.
- [9] L. Charles. Berezin-Toeplitz Operators, a Semi-Classical Approach. *Communications in Mathematical Physics*, 239(1-2):1–28, Aug. 2003.
- [10] L. Charles. Analytic Berezin-Toeplitz operators. *arXiv:1912.06819*, 2019.
- [11] L. Charles and B. Estienne. Entanglement entropy and Berezin-Toeplitz operators. *Communications in Mathematical Physics*, Oct. 2018.
- [12] E. M. Chudnovsky and L. Gunther. Quantum tunneling of magnetization in small ferromagnetic particles. *Physical review letters*, 60(8):661, 1988.
- [13] A. Deleporte. Low-energy spectrum of Toeplitz operators: The case of wells. *Journal of Spectral Theory*, 9:79–125, 2019.
- [14] A. Deleporte. Low-energy spectrum of Toeplitz operators with a mini-well. *Communications in Mathematical Physics*, (to appear), 2019.
- [15] A. Deleporte. WKB eigenmode construction for analytic Toeplitz operators. *arXiv*, 1901.07215, 2019.
- [16] A. Deleporte. Fractional exponential decay in the forbidden region for Toeplitz operators. *arXiv*, 2001.07921, Jan. 2020.
- [17] F. Faure. Prequantum chaos: Resonances of the prequantum cat map. *Journal of Modern Dynamics*, 1(2):255–285, 2007.
- [18] B. Helffer and J. Sjöstrand. Multiple wells in the semi-classical limit I. *Communications in Partial Differential Equations*, 9(4):337–408, 1984.
- [19] B. Helffer and J. Sjöstrand. Résonances en limite semi-classique. *Mémoires de la Société Mathématique de France*, 24:1–228, 1986.
- [20] H. Hezari, Z. Lu, and H. Xu. Off-diagonal asymptotic properties of Bergman kernels associated to analytic Kähler potentials. *International Mathematics Research Notices*, rny081, 2018.

- [21] H. Hezari and H. Xu. Quantitative upper bounds for Bergman kernels associated to smooth Kähler potentials. *arXiv:1807.00204*, June 2018.
- [22] H. Hezari and H. Xu. On a property of Bergman kernels when the Kähler potential is analytic. *arXiv:1912.11478*, 2019.
- [23] M. Hitrik and J. Sjöstrand. Non-selfadjoint perturbations of selfadjoint operators in 2 dimensions I. *Annales Henri Poincaré*, 5(1):1–73, Feb. 2004.
- [24] M. Hitrik and J. Sjöstrand. Rational invariant tori, phase space tunneling, and spectra for non-selfadjoint operators in dimension 2. In *Annales Scientifiques de l'ÉNS*, volume 41, pages 513–573, 2008.
- [25] M. Hitrik and J. Sjöstrand. Two minicourses on analytic microlocal analysis. In *Algebraic and Analytic Microlocal Analysis*, pages 483–540. Springer, 2013.
- [26] S. Klevtsov. Random normal matrices, Bergman kernel and projective embeddings. *Journal of High Energy Physics*, 2014(1):133, 2014.
- [27] J. J. Kohn. Harmonic integrals on strongly pseudo-convex manifolds: I. *Annals of mathematics*, pages 112–148, 1963.
- [28] J. J. Kohn. Harmonic integrals on strongly pseudo-convex manifolds: II. *Annals of mathematics*, pages 450–472, 1964.
- [29] Y. A. Kordyukov. Semiclassical spectral analysis of Toeplitz operators on symplectic manifolds: The case of discrete wells. *arXiv:1809.06799*, Sept. 2018.
- [30] Y. Le Floch. *Théorie Spectrale Inverse Pour Les Opérateurs de Toeplitz 1D*. PhD thesis, Rennes 1, 2014.
- [31] A. Martinez. Estimations sur l'effet tunnel microlocal. *Séminaire Équations aux dérivées partielles (Polytechnique)*, 1991:1–10, 1992.
- [32] A. Martinez. Estimates on complex interactions in phase space. *Mathematische Nachrichten*, 167(1):203–254, 1994.
- [33] A. Martinez. Precise exponential estimates in adiabatic theory. *Journal of Mathematical Physics*, 35(8):3889–3915, 1994.
- [34] A. Martinez. *An Introduction to Semiclassical and Microlocal Analysis. 2002*. Universitext. Springer, 2002.
- [35] A. Martinez and V. Sordoni. Microlocal WKB expansions. *Journal of functional analysis*, 168(2):380–402, 1999.

- [36] A. Melin and J. Sjöstrand. Bohr-Sommerfeld quantization condition for non-selfadjoint operators in dimension 2. *Astérisque*, 284:181–244, 2003.
- [37] S. A. Owerre and M. B. Paranjape. Macroscopic quantum tunneling and quantum-classical phase transitions of the escape rate in large spin systems. *Physics Reports*, 546:1–60, Jan. 2015.
- [38] J. Ross and M. Singer. Asymptotics of Partial Density Functions for Divisors. *The Journal of Geometric Analysis*, pages 1–52, Sept. 2016.
- [39] O. Rouby. Bohr–Sommerfeld Quantization Conditions for Nonselfadjoint Perturbations of Selfadjoint Operators in Dimension One. *International Mathematics Research Notices*, rnw309, 2017.
- [40] O. Rouby, J. Sjöstrand, and S. Vũ Ngọc. Analytic Bergman operators in the semiclassical limit. *hal-01851770*, July 2018.
- [41] J. Sjöstrand. *Singularites Analytiques Microlocales*, volume 95 of *Astérisque*. Soc. Math. de France, 1982.
- [42] J. Sjöstrand. Geometric bounds on the density of resonances for semiclassical problems. *Duke mathematical journal*, 60(1):1–57, 1990.
- [43] J. Sjöstrand. Resonances associated to a closed hyperbolic trajectory in dimension 2. *Asymptotic Analysis*, 36(2):93–113, 2003.
- [44] H. Tsuji. Dynamical construction of Kähler-Einstein metrics. *Nagoya Mathematical Journal*, 199:107–122, Sept. 2010.
- [45] A. Voros. Wentzel-Kramers-Brillouin method in the Bargmann representation. *Physical Review A*, 40(12):6814, 1989.
- [46] H. Whitney and F. Bruhat. Quelques propriétés fondamentales des ensembles analytiques-réels. *Commentarii Mathematici Helvetici*, 33(1):132–160, Dec. 1959.
- [47] S. Zelditch. Szegő kernels and a theorem of Tian. *Int. Math. Research Notices*, 6, 2000.
- [48] S. Zelditch and P. Zhou. Interface asymptotics of partial Bergman kernels on S1-symmetric Kähler manifolds. *Journal of Symplectic Geometry*, 17(3):793–856, 2019.
- [49] M. Zworski. *Semiclassical Analysis*, volume 138. American Mathematical Soc., 2012.