# Analysis of stability and bifurcation in non linear-mechanics with dissipation

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Two sources of non linearities The energy of the spring  $w(\theta) = \frac{1}{2}C_1\theta^2 + \frac{1}{3}C_2\theta^3 + \frac{1}{4}C_3\theta_4 + \dots$ The potential energy due to external loading :  $\lambda L(1 - \cos \theta)$  Equilibrium position : stationnarity of the total potential energy

$$\frac{\partial \mathcal{E}}{\partial \theta} \delta \theta = 0$$

Two positions of equilibrium

$$\theta = 0, \forall \lambda \tag{1}$$

$$\lambda(\theta) = \lambda_c + \lambda_1 \theta + \lambda_2 \theta^2 + \dots$$
 (2)

$$\lambda_c = rac{C_1}{L}, \quad \lambda_1 = \lambda_c rac{C_2}{C_1}, \quad \lambda_2 = \lambda_c (rac{C_3}{C_1} + rac{1}{6})$$

Generic diagram of stability Stability : Dirichlet criterion : positivity of second derivative of total potential energy

# Generic diagram of bifurcation



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The energy depends now of internal parameters  $\alpha$ (plastic strain, cracks length, damage, ...)  $\boldsymbol{u}$  displacement,  $\lambda$  loading parameters Total potential energy :  $\mathcal{E}(\boldsymbol{u}, \alpha, \mathbf{T}^d) = \int_{\Omega} w(\varepsilon, \alpha) \, \mathrm{d}\Omega - \int_{\partial\Omega_{\mathrm{T}}} \mathbf{T}^d \cdot \boldsymbol{u} \, \mathrm{d}s$ , Equilibrium  $\frac{\partial \mathcal{E}}{\partial \boldsymbol{u}} \cdot \delta \boldsymbol{u} = 0$ Dissipation  $D_m = -\frac{\partial w}{\partial \alpha} \dot{\alpha} = A \dot{\alpha} \ge 0$ Evolution of  $\alpha$  : normality rule  $f(A) \le 0$ 

$$\mu \ge 0, \quad f(A) \le 0, \quad \mu f(A) = 0$$

The evolution of the system depends essentially on the evolution of internal parameters.

We assume that the solution at equilibrium  $u(\lambda, \alpha)$  is unique.

#### The constitutive law

• free energy :

$$w(\varepsilon, \alpha) = \frac{1}{2}(\varepsilon - \alpha) : \mathbb{C} : (\varepsilon - \alpha) + w_b(\alpha)$$

equations of state

$$\boldsymbol{\sigma} = \frac{\partial w}{\partial \boldsymbol{\varepsilon}} = \mathbb{C} : (\boldsymbol{\varepsilon} - \alpha),$$
$$\boldsymbol{A} = -\frac{\partial w}{\partial \alpha} = \boldsymbol{\sigma} - \frac{\partial w_b}{\partial \alpha}$$

#### Dissipation

$$D_m = A.\dot{\alpha} \ge 0$$

Yield function

 $f(A) \leq 0$ 

• Normality rule :

$$\dot{\alpha} = \mu \; \frac{\partial f}{\partial A}, \quad \mu f = 0, \; \mu \ge 0$$

• Constitutive law :

$$\boldsymbol{\sigma} = \rho \frac{\partial w}{\partial \boldsymbol{\varepsilon}} = \mathbb{C} : (\boldsymbol{\varepsilon} - \alpha)$$

Compatibility

$$\boldsymbol{\varepsilon} = \frac{1}{2}(\operatorname{grad} \boldsymbol{u} + \operatorname{grad}^T \boldsymbol{u})$$

• Boundary conditions :

over 
$$\partial \Omega_{\mathbf{u}}$$
:  $\boldsymbol{u} = \boldsymbol{u}^d$ , over  $\partial \Omega_{\mathbf{T}}$ :  $\boldsymbol{\sigma}.\boldsymbol{n} = \lambda \mathbf{T}^d$ 

• Equilibrium conditions :

div 
$$\boldsymbol{\sigma} = 0$$

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In fact the energy depends upon fields of mechanical quantities

$$\mathcal{E}(\widetilde{\boldsymbol{u}},\widetilde{\alpha},\mathbf{T}^d) = \int_{\Omega} \rho w(\boldsymbol{\varepsilon}(\boldsymbol{u}),\alpha) \, \mathrm{d}\omega - \int_{\partial \Omega_{\mathrm{T}}} \mathbf{T}^d \boldsymbol{.} \boldsymbol{u} \, \mathrm{d}\Omega$$

 At a given state, the equilibrium is determined by a displacement u which minimize £. The energy at equilibrium is then :

$$\mathcal{E}_{E}(\widetilde{\boldsymbol{u}}(\widetilde{\alpha},\widetilde{\mathbf{T}^{d}},\widetilde{\boldsymbol{\nu}}^{d}),\widetilde{\alpha},\widetilde{\mathbf{T}^{d}})=E(\widetilde{\alpha},\widetilde{\mathbf{T}^{d}},\widetilde{\boldsymbol{\nu}}^{d})$$

- We must solve the problem of evolution of internal state.
- Generalization : it is also true for non-linear external loading given by a potential in u, if we assume stability of reversible path.

 $\Omega$  is splitted in two parts

- elastic domain : f < 0
- plastic domain  $\Omega_p$  : f = 0

Characterisation of  $\mu$ 

$$\widetilde{\mu} \in \mathcal{K}$$

$$\mathcal{K} = \{ \widetilde{eta} / eta({m x}) \geq 0, ext{over} \ \Omega_p, \ eta({m x}) = 0, ext{over} \ \Omega / \Omega_p \}$$

Yield function

$$f(A) \le 0$$
$$\mathcal{N} = \frac{\partial f}{\partial A}$$

Normality rule implies :

$$\dot{\alpha} = \mu \mathcal{N}, \mu f(A) = 0, \ \mu \ge 0$$

#### Consistency condition.

$$\mathcal{K} = \{ \widetilde{\beta} / \beta(\mathbf{x}) \ge 0, \text{over } \Omega_p, \ \beta(\mathbf{x}) = 0, \text{over } \Omega / \Omega_p \}$$
  
 $\Omega_p = \{ \mathbf{x} / f = 0 \}$   
 $\widetilde{\mu} \in \mathcal{K}$ 

Deriving this condition with respect to time

$$\frac{\mathrm{d}}{\mathrm{dt}}(\mu f) = 0 \Rightarrow \mu \dot{f} = 0,$$

for 
$$\boldsymbol{x} \in \Omega_p$$
  $f = 0$ ,  $\dot{f} \leq 0$ .

This condition is rewritten as

for 
$$\boldsymbol{x} \in \Omega_p$$
,  $\forall \beta \ge 0$ ,  $(\mu - \beta)\dot{f} \ge 0$ 

and by integration over  $\Omega$  : towards variational inequality !

$$\forall \widetilde{\beta} \in \mathcal{K} \quad \int_{\Omega} (\lambda - \beta) \dot{f} \; \mathrm{d} \omega \geq 0$$

#### Characterization of the evolution

Constitutive law :

$$\dot{\boldsymbol{\sigma}} = \frac{\partial^2 w}{\partial \boldsymbol{\varepsilon} \partial \boldsymbol{\varepsilon}} : \dot{\boldsymbol{\varepsilon}} + \frac{\partial^2 w}{\partial \boldsymbol{\varepsilon} \partial \alpha} \dot{\alpha}$$

Compatibility

$$\dot{\boldsymbol{\varepsilon}} = \frac{1}{2}(\operatorname{grad} \boldsymbol{\nu} + \operatorname{grad}^T \boldsymbol{\nu})$$

Boundary conditions :

over 
$$\partial \Omega_{\mathbf{u}}$$
:  $\mathbf{v} = \mathbf{v}^d$ , over  $\partial \Omega_{\mathbf{T}}$ :  $\dot{\boldsymbol{\sigma}} \cdot \boldsymbol{n} = \mathbf{T}^d$ 

• Equilibrium conditions :

div 
$$\dot{\boldsymbol{\sigma}} = 0$$

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Evolution of the equilibrium.

 $\dot{\boldsymbol{\sigma}} \text{ is } \mathcal{S}.\mathcal{A}.$   $\operatorname{div} \dot{\boldsymbol{\sigma}} = 0, \quad \dot{\boldsymbol{\sigma}}.\boldsymbol{n} = \dot{\mathbf{T}}^{d} \text{ over } \partial\Omega_{\mathrm{T}}$   $\int_{\Omega} \dot{\boldsymbol{\sigma}} : \boldsymbol{\varepsilon}(\boldsymbol{u}^{*}) \, \mathrm{d}\Omega \quad - \quad \int_{\partial\Omega} \dot{\mathbf{T}}^{d}.\boldsymbol{u}^{*} \, \mathrm{d}S = 0$   $\int_{\Omega} \boldsymbol{\varepsilon}(\boldsymbol{u}^{*}) : \left(\frac{\partial^{2} w}{\partial \boldsymbol{\varepsilon} \partial \boldsymbol{\varepsilon}} : \dot{\boldsymbol{\varepsilon}} + \frac{\partial^{2} w}{\partial \boldsymbol{\varepsilon} \partial \alpha} \dot{\alpha}\right) \, \mathrm{d}\Omega - \int_{\partial\Omega} \dot{\mathbf{T}}^{d}.\boldsymbol{u}^{*} \, \mathrm{d}s = 0$ 

Normality rule and evolution of internal state

$$\int_{\Omega} (\lambda - \beta) \dot{f} \, \mathrm{d}\Omega \ge 0$$

Variation of the yielding function

$$\dot{f} = \frac{\partial f}{\partial A}\dot{A}$$

Constitutive law:

$$\dot{A} = -\frac{\partial^2 w}{\partial \varepsilon \partial \alpha} : \dot{\varepsilon} - \frac{\partial^2 w}{\partial \alpha \partial \alpha} \dot{\alpha}$$

where  $\alpha$  satisfies:

 $\dot{\alpha} = \mu \mathcal{N}$ 

#### The evolution of the system

$$\int_{\Omega} \boldsymbol{\varepsilon}^{*} : \frac{\partial^{2} w}{\partial \boldsymbol{\varepsilon} \partial \boldsymbol{\varepsilon}} : \dot{\boldsymbol{\varepsilon}} + \boldsymbol{\varepsilon}^{*} : \frac{\partial^{2} w}{\partial \boldsymbol{\varepsilon} \partial \alpha} \cdot \mathcal{N} \mu \, \mathrm{d}\Omega - \int_{\partial \Omega_{\mathrm{T}}} \dot{\mathbf{T}}^{d} \cdot \boldsymbol{u}^{*} \, \mathrm{d}s = 0$$
$$\int_{\Omega} (\mu - \beta) (\mathcal{N} \cdot \frac{\partial^{2} w}{\partial \alpha \partial \boldsymbol{\varepsilon}} : \dot{\boldsymbol{\varepsilon}} + \mathcal{N} \cdot \frac{\partial^{2} w}{\partial \alpha \partial \alpha} \cdot \mathcal{N} \mu) \, \mathrm{d}\Omega \leq 0$$

Potential upon the rate

$$\begin{aligned} \mathcal{F}(\widetilde{\mathbf{v}},\widetilde{\mu}) &= \int_{\Omega} \frac{1}{2} \dot{\boldsymbol{\varepsilon}} : \frac{\partial^2 w}{\partial \boldsymbol{\varepsilon} \partial \boldsymbol{\varepsilon}} : \dot{\boldsymbol{\varepsilon}} + \dot{\boldsymbol{\varepsilon}} : \frac{\partial^2 w}{\partial \boldsymbol{\varepsilon} \partial \alpha} \cdot \mathcal{N} \mu \\ &+ \frac{1}{2} \mathcal{N} \cdot \frac{\partial^2 w}{\partial \alpha \partial \alpha} \cdot \mathcal{N} \mu^2 \, \mathrm{d}\Omega - \int_{\partial \Omega_{\mathrm{T}}} \dot{\mathbf{T}}^d \cdot \boldsymbol{v} \, \mathrm{d}S \end{aligned}$$

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The response  $\tilde{v}, \tilde{\mu}$  is solution of

$$\frac{\partial \mathcal{F}}{\partial \boldsymbol{\nu}}(\boldsymbol{\nu}-\boldsymbol{\nu}^*)+\frac{\partial \mathcal{F}}{\partial \boldsymbol{\mu}}(\boldsymbol{\mu}-\boldsymbol{\mu}^*)\leq 0$$

for any admissible fields  $v^*, \mu^*$ 

$$\widetilde{\mu}, \widetilde{\mu}^* \in \mathcal{K}, \quad \widetilde{\nu}, \widetilde{\nu}^* \in \mathcal{C}.\mathcal{A}$$

$$\mathcal{K} = \{\widetilde{eta}/eta(\mathbf{x}) \ge 0, \text{over } \Omega_p, \ eta(\mathbf{x}) = 0, \text{over } \Omega/\Omega_p\}$$
  
 $\mathcal{C}.\mathcal{A}. = \{\widetilde{\mathbf{v}}/\mathbf{v} = \mathbf{v}^d \text{ along}\partial\Omega_u\}$ 

Two possibilities: consider the problem upon v or the problem upon  $\dot{\alpha}$ 

A (10) > A (10) > A (10)

Classical elimination of  $\mu$  by consistency condition :

$$\mathbb{H} = \mathcal{N} \cdot \frac{\partial^2 w}{\partial \alpha \partial \alpha} \cdot \mathcal{N} \ge 0, \ \mu = \frac{1}{\mathbb{H}} < -\mathcal{N} \cdot \frac{\partial^2 w}{\partial \alpha \partial \varepsilon} : \dot{\varepsilon} >_+$$

Hill's formulation, the material is hypo-elastic in sense of Hill :

$$\begin{split} \mathcal{U}(\widetilde{\mathbf{v}}) &= \int_{\Omega} \frac{1}{2} \dot{\boldsymbol{\varepsilon}} : \frac{\partial^2 w}{\partial \boldsymbol{\varepsilon} \partial \boldsymbol{\varepsilon}} : \dot{\boldsymbol{\varepsilon}} - \frac{1}{2\mathbb{H}} < -\mathcal{N} \cdot \frac{\partial^2 w}{\partial a \partial \boldsymbol{\varepsilon}} \dot{\boldsymbol{\varepsilon}} >^2_+ \ \mathrm{d}\Omega \\ &- \int_{\partial \Omega_{\mathrm{T}}} \dot{\mathbf{T}}^d . \mathbf{v} \ \mathrm{d}S \end{split}$$

The solution v satisfies

$$\frac{\partial \mathcal{U}}{\partial \widetilde{\boldsymbol{v}}} \cdot (\boldsymbol{v} - \boldsymbol{v}^*) = 0$$

for all  $v^* \in \mathcal{C}.\mathcal{A}$ 

Elimination of v :

$$\mathcal{F} = \int_{\Omega} \frac{1}{2} \mu \cdot \mathbb{B} \cdot \mu - \bar{Q} . \mu$$

Existence of solution

$$\int_{\Omega} \mu^*.\mathbb{B}.\mu^* \; \mathrm{d}\Omega > 0, \widetilde{\mu}^* \in \mathcal{K}$$

Uniqueness

$$\int_{\Omega} \mu^*.\mathbb{B}.\mu^* \ \mathrm{d}\Omega > 0, \widetilde{\mu}^* \neq \widetilde{0}$$

A (10) > A (10) > A (10)

Two results :

- Elimination of *u*, the variational inequality gives two conditions : one for existence of a solution (Stability condition), second for uniqueness (no-bifurcation).
- Elimination of μ : Hill's formulation : If U(ν) is convex : uniqueness of the evolution ν.

Examination of the results on an example.

#### The Shanley column



Displacement  $u, \theta$ , Energy  $w = \frac{1}{2}(\varepsilon - \alpha)^2 + \frac{1}{2}H\alpha^2$ Strain :  $\varepsilon_1 = u + l\theta$ ,  $\varepsilon_2 = u - l\theta$ Stress :  $\sigma_i = E(\varepsilon_i - \alpha_i)$ ,  $f_i = (\sigma_i - H\alpha_i)^2 - \sigma_y \le 0$ Normality rule :  $\dot{\alpha}_i = \mu_i \frac{\partial f}{\partial \sigma_i}$ ,  $\mu_i \ge 0$ ,  $\mu_i f_i = 0$ 

Evolution law :

$$\dot{\sigma}_i = E_i \dot{\varepsilon}_i \text{ with } \begin{cases} E_i = E, & f_i \le 0\\ E_i = E_T, & f_i = 0, \dot{f}_i = 0 \end{cases}$$
(3)

#### Equilibrium and discussion

Equilibrium  $(\lambda, \theta = 0)$  is solution

$$-\lambda = \sigma_1 + \sigma_2 \tag{4}$$

$$-\lambda L\theta = l(\sigma_2 - \sigma_1) \tag{5}$$

For the evolution we derive the equation with respect to time, and taking account of the local behaviour

$$\dot{u} = -\frac{\dot{\lambda}}{E_1 + E_2} + \frac{E_2 - E_1}{E_1 + E_2} l\dot{\theta}$$
(6)

$$(\dot{\lambda}\theta) = C_o\dot{\lambda} + C_1\dot{\theta}$$
 (7)

Along a path of equilibrium with the same condition of loading (the  $E_i$  are given), the response of the system is obtained as

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$$\lambda(\theta) = \frac{\lambda_o C_o - C_1 \theta}{C_o - \theta} \tag{8}$$

$$C_o = rac{l}{L} rac{E_2 - E_1}{E_1 + E_2}, C_1 = 4 rac{l^2}{L} rac{E_1 E_2}{E_1 + E_2}.$$

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#### Discussion

• Elastic domain:  $E_1 = E_2 = E$ : (critical value : Euler)

$$\lambda = \lambda_E = E \frac{2l^2}{L}, \forall \theta$$

• Plastic regime:  $E_1 = E_2 = E_T$ 

$$\lambda = \lambda_T = E_T \frac{2l^2}{L}, \forall \theta$$

Compatible with the loading if and only if  $\dot{\theta} = 0$  !

• Elastic- Plastic regime:  $E_1 = E, E_2 = E_T (\theta, \lambda(\theta))$  is an hyperbole with asymptote  $\lambda_R = E_R \frac{2l^2}{L}$  with the reduced modulus  $E_R = \frac{2EE_T}{E + E_T}$ 

#### This example shows that

for  $\lambda_T \leq \lambda \leq \lambda_R$  a bifurcated equilibrium path exists. There is no transition from one bifurcated path to a close one

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Elastic stability For  $\lambda < \lambda_E$  the displacement can be eliminate and the value of potential energy is given by

$$W(\alpha_i, \lambda) = \mathcal{E}(u(\lambda, \alpha_i), \theta(\lambda, \alpha_i), \alpha_i, \lambda)$$
(9)

with

$$u = -\frac{\lambda}{2E} + \frac{\alpha_1 + \alpha_2}{2}, \quad \theta(\lambda_E - \lambda) = \frac{El}{L}(\alpha_2 - \alpha_1)$$
(10)

The problem of evolution is now defined by

$$A_{i} = -\frac{\partial W}{\partial \alpha_{i}}, f(A_{i}) \le 0, \mu_{i} \ge 0, \mu_{i}f(A_{i}) = 0$$
(11)

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The normality rule If f = 0 then  $\dot{f} \le 0$  and the normality rule ( $\mu f = 0$ ) implies

$$\mu_i f = 0, \quad \forall \beta_i \ge 0 \quad (\mu_i - \beta_i) f \ge 0.$$
(12)

The problem of evolution satisfies  $(N_i = \frac{\partial f}{\partial A_i})$ 

$$\sum_{i} N_i \dot{A}_i (\mu_i - \beta_i) \ge 0 \tag{13}$$

 $\mu_i, \beta_i \in \mathcal{C}$ 

$$\mathcal{C} = \{(\beta_1, \beta_2) / \beta_i = 0 \text{ if } f(A_i) < 0, \beta_i \ge 0 \text{ if } f(A_i) = 0\}$$

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#### On the problem of evolution

$$\mathcal{F}(\mu_1,\mu_2,\dot{\lambda}) = \sum_{ij} \frac{1}{2} \mu_i N_i \frac{\partial^2 W}{\partial \alpha_j \partial \alpha_i} N_j \mu_j + \sum_i N_i \frac{\partial^2 W}{\partial \lambda \partial \alpha_i} \dot{\lambda}$$
(14)

The solution  $\mu_i$  satisfies the variational inequality

$$\sum_{i} \frac{\partial \mathcal{F}}{\partial \mu_{i}} (\mu_{i} - \beta_{i}) \le 0$$
(15)

on the admissible set  $\mu_i, \beta_i \in C$ 

$$\mathcal{C} = \{ (\beta_1, \beta_2) / \beta_i = 0 \text{ if } f(A_i) < 0, \beta_i \ge 0 \text{ if } f(A_i) = 0 \}$$

The solution depends upon the quadratic form :  $Q_{ij} = N_i \cdot \frac{\partial^2 W}{\partial \alpha_j \partial \alpha_i} N_j$ The solution exists if

$$eta_i Q_{ij} eta_j > 0, orall eta \in \mathcal{C}$$

The solution is unique if

$$\beta_i Q_{ij} \beta_j > 0, \forall \beta_i \neq 0$$

# Results

The stability is obtained for  $\lambda \leq \lambda_R$ 

The uniqueness (no -bifurcation)  $\lambda \leq \lambda_T$ 

There is the main difference with conservative system : the

fundamental and the bifurcated paths are stable in the range

 $\lambda_T \leq \lambda \leq \lambda_R.$ 

Connection can be made with the criterion of Hill :

The solution of the rate boundary value problem is a stationary point of the functional

$$U(\dot{u},\dot{ heta},\dot{\lambda})$$

on the set of admissible displacement.

Uniqueness if the functional is convex on the set of displacement.

On an equilibrium path  $(u(\lambda), \theta(\lambda))$ , *U* is convex for  $\lambda < \lambda_T$  the solution is unique.

At point  $\lambda_T$ , U is not strictly convex, two solutions gives the same U. For  $\lambda > \lambda_T$ , U is no more convex.

Complete discussion : Petryk (1991)

# **Bifurcation diagram**



#### A simple continuum : the Shanley column

Now the springs are uniformly distributed along the segment [-l, l]. The domain  $\Omega_p(t)$  is assumed to be [d(t), l]. The value of *d* is determined by the condition of neutral loading

$$\dot{\varepsilon}(d) = 0 = \dot{u} - d \dot{\theta}, \quad [\dot{\alpha}](d(t)) = 0$$
(16)

The equations of equilibrium are deduced from the potential energy

$$\mathcal{E}(u(x),\alpha(x),T^d) = \int_{-l}^{l} w(\varepsilon,\alpha)dx + T^d(u + L(1-\theta^2/2))$$
(17)

as previously,  $\varepsilon(x) = u - x\theta$ , then the state of equilibrium obeys to

$$0 = T^d + \int_{-l}^{+l} E(x)(\varepsilon - \alpha) dx$$
 (18)

$$0 = -T^{d}L\theta + \int_{-l}^{+l} E(x)x(\varepsilon - \alpha)dx$$
 (19)

Then, we obtain

$$0 = \dot{T}^{d} + \int_{-l}^{d} E\dot{\varepsilon}dx + \int_{d}^{l} E_{T}\dot{\varepsilon}dx$$
(20)  
$$0 = -L\overline{T^{d}\theta} + \int_{-l}^{+d} E_{X}\dot{\varepsilon}dx + \int_{d}^{l} E_{T}x\dot{\varepsilon}dx$$
(21)

A non trivial solution in  $\theta$  is obtained by introducing the time-scale  $\tau$  such that the velocity  $x_1 = \dot{d}$  of propagation of the unloading domain is finite. The domain  $\Omega_p(\tau) = [x_\tau, l]$  is parametrized with

$$d = x_{\tau} = \sum_{i} x_{i} \tau^{i}.$$
 (22)

At point  $x_{\tau}$ , the condition  $[\dot{\alpha}(x_{\tau}, \tau)] = 0$  where  $\alpha(x, t) = \sum_{i} \alpha_{i}(x)\tau_{i}$  gives conditions on the asymptotic expansion.

$$0 = [\alpha_1(x_o)]$$
  

$$0 = [\alpha_2(x_o)] + x_1[\alpha'_1(x_o)]$$
  

$$0 = [\alpha_3(x_o)] + 2x_1[\alpha'_2(x_o)] + (2x_1^2 + x_2)[\alpha'_1(x_o)]$$
(23)

A non trivial solution is then obtained as

$$\alpha^*(\tau) = \alpha(\tau) + m(\tau)x \tag{24}$$

We can take the time-derivative of the equilibrium equations taking account of the position of  $x_{\tau}$  and of discontinuities of the mechanical quantities on this boundary. It is obvious that we have:

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{-l}^{l} f(x,\tau) dx = \int_{-l}^{l} \dot{f}(x,\tau) dx + \left[f(x,\tau)\right]_{x_{\tau}^{-}}^{x_{\tau}^{+}} \dot{x}_{\tau}$$
(25)

We find 
$$T_c = T_T = \frac{2l^3E}{3L} = \frac{H}{E+H}T_E$$
,  $m_1 = 0$  and  $m_2 = -T_2/2Hl^2$ ,

$$x_{1}^{2} = \frac{4l^{2}}{3} \frac{T^{2}}{T_{E} - T_{T}}$$

$$T = T_{T} + T_{2} \frac{\tau^{2}}{2}$$

$$\theta = \frac{ElT_{2}}{3LH(T_{E} - T_{T})} \frac{\tau^{2}}{2} + \dots$$

This is a bifurcated path. The condition of stability of the fundamental path ( $\theta = 0, T$ ) is preserved for loading near  $T = T_c$  but for  $T \ge T_c$  another path exists which is also a stable path. Hutchinson (1978)

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Example contain essentially two main ideas

- The existence of the potential energy
- The evolution law for the internal variable given by the normality rule associated with a convex domain of reversibility.
- Elastoplastic beam has been studied by many authors (Hutchinson,...Nguyen, Potier Ferry, CS,...) with different techniques, critical loading and bifurcated solution have been obtained : along the bifurcated path the equilibrium solution is governed by the shape of the unloading part.
- In the same idea, other mechanisms of dissipation have been discussed : delamination of laminates, a set of cracks, damage zones, moving interfaces.
- The critical load is easy to obtained. The post-critical behaviour must be studied to confirm existence of bifurcated path.

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Consider *n* cracks in an elastic body.

$$\mathcal{E}(\boldsymbol{u}, l_i, \lambda) = \int_{\Omega} \frac{1}{2} \boldsymbol{\varepsilon}(\boldsymbol{u}) : \mathbb{C}_o : \boldsymbol{\varepsilon}(\boldsymbol{u}) \, \mathrm{d}\Omega - \int_{\partial \Omega_{\mathrm{T}}} \lambda T^d . \boldsymbol{u} \, \mathrm{d}S$$
(26)

The driving force associated to the propagation of a cracks is the release rate of energy  $G_i$ , and we have

$$G_i = \int_{\Gamma} \boldsymbol{n}.(w \mathbf{I} - \boldsymbol{\sigma}.\nabla \boldsymbol{u}).\boldsymbol{e}_i \, \mathrm{d}\boldsymbol{s} = -\frac{\partial \mathcal{E}}{\partial l_i}$$
(27)

The quadratic functional for the rate of displacement and propagation of cracks has been given in Eur.J. Mech. A/Solids 9(2),157-173,1990

#### A clamped beam

A beam in flexure, clamped at extremities. Vertical displacement *w* and local strain is  $\epsilon = w^{"}(x)y$ .

Loading w(o) = V is imposed or T(o) = F is prescribed.

For an elastic beam, the energy due to flexure is given by

$$W(l_1, l_2, V) = \frac{3}{2} EIV^2 \frac{(l_1 + l_2)^3}{l_1^3 l_2^3} = \frac{1}{2} K(l_1, l_2) V^2, G_i = -\frac{\partial W}{\partial l_i}$$

The evolution is governed by the quadratic form

$$Q_{ij} = \frac{\partial^2 W}{\partial l_i l_j} \tag{28}$$

which is always positive, when  $l_1 = l_2$ , the propagation is stable with no bifurcation

$$W^*(l_1, l_2, F) = -\frac{1}{2K}F^2$$
 (29)

The quadratic form Q is always negative definite . The symmetric equilibrium is unstable with possible bifurcation.

A material 1 changes to material 2 along a moving interface  $\Gamma$ Consider two linear isotropic elastic materials with moduli  $\mathbb{C}_i$ Along the interface

$$\llbracket \boldsymbol{u} \rrbracket_{\Gamma} = 0, \quad \llbracket \boldsymbol{\sigma} \rrbracket_{\Gamma} \boldsymbol{.} \boldsymbol{n} = 0, \llbracket \boldsymbol{w} \rrbracket_{\Gamma} \neq 0$$
(30)

When the surface is moving with normal velocity  $\phi(s)$ , a change of the global strain energy occurs

$$D_m = \int_{\Gamma} (\llbracket w \rrbracket_{\Gamma} - \llbracket \boldsymbol{\sigma} \rrbracket_{\Gamma} : \llbracket \nabla \boldsymbol{u} \rrbracket_{\Gamma}) \phi(s) \, \mathrm{d}s = \int_{\Gamma} G(s) \phi(s) \, \mathrm{d}s \qquad (31)$$

Given a normality rule (Griffith) to define the evolution  $\phi(s)$ 

$$\phi(s) \ge 0, \quad G(s) \le G_c, \quad (G(s) - G_c) \ \phi(s) = 0$$
 (32)

we obtain a quadratic functional to described the evolution of damage.



To follow the evolution of mechanical quantity f along  $\Gamma$  the convected derivative  $\mathcal{D}_{\phi}(f)$  is introduced

$$\mathcal{D}_{\phi}(f) = \lim_{\tau \to 0^+} \frac{f(X_{\Gamma} + a\mathbf{n}\tau, t + \tau) - f(X_{\Gamma}, t)}{\tau}$$
(33)

Then the normality rules implies

$$G(s) = G_c, \quad \int_{\Gamma} (a(s) - \beta(s)) \mathcal{D}_{\phi}(G)(s, t) \, \mathrm{d}s \le 0. \tag{34}$$

and  $\mathcal{D}_{\phi}(\llbracket \boldsymbol{u} 
bracket_{\Gamma}) = 0, \mathcal{D}_{\phi}(\llbracket \boldsymbol{\sigma} 
bracket_{\Gamma}.\boldsymbol{n}) = 0$ 

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#### Models of fracture and damage

Linear Frac. Mech.	Total damage	Graded Damage
$W(\boldsymbol{u},l)$	$W(\boldsymbol{u},\Gamma)$	$W(\boldsymbol{u},\phi)$
$\mathcal{G} = -\frac{\partial W}{\partial l}$	$\mathcal{G}(s) = -\frac{\partial W}{\partial \Gamma}$	$\bar{G} = -\frac{\partial W}{\partial \phi}$
$\mathcal{G} \leq G_c^{Ol}$	$\mathcal{G} \leq G_c^{O1}$	$ar{G} \leq ar{G}_c^{ar{O}\psi}$
$a \ge 0$	$a(s) \ge 0$	$a(s) \ge 0, \dot{\phi} + a\nabla\phi.\mathbf{n} = 0$
$aD_l(\mathcal{G}-G_c)=0.$	$a(s)\mathcal{D}_{\phi}(\mathcal{G}-G_c)=0.$	$a(s)(\mathcal{D}_{\phi}\bar{G}-\mathcal{D}_{\phi}\bar{G}_{c})=0.$

Solution of evolution v, a satisfies a variational inequality

$$\frac{\partial F}{\partial \boldsymbol{v}}(\boldsymbol{v}-\boldsymbol{v}^*)+\frac{\partial F}{\partial a}(a-a^*)+(\frac{\partial F}{\partial \frac{\mathrm{d}a}{\mathrm{d}s}},\frac{\mathrm{d}(a-a^*)}{\mathrm{d}s})\geq 0$$

Analysis of stability and bifurcation (existence and uniqueness) (CS 1985-2012)

- Modelization
- Rate boundary value problem
- Analysis of the solution

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#### Modelization



$$\mathcal{E}(u,S,p) = \int_{S} \frac{1}{2} K \nabla u^{2} - pu \, \mathrm{d}S$$

Boundary condition

$$u = 0$$
, over  $\partial S$ 

The equilibrium gives

$$K\Delta u + p = 0$$

#### Evolution of the interface

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E} = \int_{S} K\nabla u \cdot \nabla v - pv - \dot{p}u \, \mathrm{d}S + \int_{\partial S} (\frac{1}{2}K\nabla u^{2} - pu)\phi \, \mathrm{d}s$$

On the boundary u = 0

$$\mathcal{D}_{\phi}(u) = 0, \quad v + \phi \nabla u. \boldsymbol{n} = 0$$

Dissipation and energy release rate :

$$\mathcal{G} = -(\frac{1}{2}K\nabla u^2 - pu) + K\nabla u.\boldsymbol{n}\nabla u.\boldsymbol{n}, \quad \mathcal{G} = \frac{1}{2}K(\nabla u.\boldsymbol{n})^2$$

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$$\mathcal{D}_{\phi}\mathcal{G} = K\nabla u.\boldsymbol{n}(\nabla v.\boldsymbol{n} + \phi\nabla\nabla u.\boldsymbol{n}.\boldsymbol{n})$$

As  $\mathcal{D}_{\phi}u = v + \phi \nabla u.\mathbf{n}$  we obtain

$$\mathcal{D}_{\phi}\mathcal{G} = K(v + \nabla u.\boldsymbol{n}\phi)\nabla\nabla u.\boldsymbol{n}.\boldsymbol{n}$$

Introducing the potential

$$\mathcal{F} = \int_{S} \frac{1}{2} K \nabla v \cdot \nabla v - \dot{p} v \, \mathrm{d}S + \int_{\Gamma} \phi K v \, \nabla \nabla u \cdot \mathbf{n} \cdot \mathbf{n}$$
$$+ \int_{\Gamma} \frac{1}{2} \phi^{2} K \nabla u \cdot \mathbf{n} \, \nabla \nabla u \cdot \mathbf{n} \cdot \mathbf{n} \, \mathrm{d}s$$

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# Equilibrium $T\Delta v + \dot{p} = 0$ Continuity $\mathbf{v} + \phi \nabla u.\mathbf{n} = 0$ Criterion

 $\sigma = K \nabla u \cdot \boldsymbol{n} = \sigma_c$ 

Evolution law

$$K\nabla v.\boldsymbol{n} + K\phi\nabla\nabla u.\boldsymbol{n}.\boldsymbol{n} + K\nabla u\mathcal{D}_{\phi}\boldsymbol{n} = 0$$

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#### Properties of solutions.

For circular initial geometry:  $u = -p/4K(r^2 - R^2)$ 

$$pR < p_c R = 2\sqrt{2KG_c}.$$

Analysis by Fourier series

$$\delta \phi = a_o + \sum_i a_i \cos(i\theta) + b_i \sin(i\theta)$$

The displacement solution is given by

$$u = \frac{pR}{2K}(a_o + \sum_i (a_i \cos(i\theta) + b_i \sin(i\theta))(\frac{r}{R})^i$$

Then the quadratic form  $W(\phi) = \mathcal{F}(v(\phi), \phi)$  can be evaluated.

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Under prescribed pressure The condition of stability is

$$W = 2\pi G_c(-2a_o^2 + \sum_i (i-1)(a_i^2 + b_i^2)) \ge 0$$

then the circular shape is unstable Under controlled volume  $\int_{S} u dS = V$  the changes to

$$W = 2\pi G_c (2a_o^2 + \sum_i (i-1)(a_i^2 + b_i^2)) \ge 0$$

the position is stable.

But bifurcation is possible :  $(i = 1) \phi = a_o + a_1 \cos\theta + b_1 \sin\theta \ge 0$  can be solution.

The post-critical behaviour must be studied.

In presence of dissipation the concept of stability-bifurcation has different properties than for conservative system.

Two types of non linearities are present : the local behaviour and the potential of external forces.

For standard generalized materials (Plasticity, Fracture, Damage) the rate boundary value problem is governed by a variational inequality and hypo-elastic potential in sense of Hill is obtained : hence the critical value for loss of uniqueness can be evaluated.

Post critical behaviour must be studied to ensure the existence of bifurcated path.

The analysis shows that fundamental and bifurcated path can be stable paths

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The presented work has been in continuity of many works due to many author

- Hill, Mandel
- Hutchinson, Budiansky, Sewell
- Petrick, Nguyen Quoc Son, Triantaphyllidis, Storakers, Potier-Ferry- Cochelin, Ben Allal,...