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# Exponential stability of PI control for Saint-Venant equations with a friction term

Georges Bastin\* and Jean-Michel Coron†

*Dedicated to Roland Glowinski, a master and a friend, on the occasion of his 80th birthday*

## Abstract

We consider open channels represented by Saint-Venant equations that are monitored and controlled at the downstream boundary and subject to unmeasured flow disturbances at the upstream boundary. We address the issue of feedback stabilization and disturbance rejection under Proportional-Integral (PI) boundary control. For channels with uniform steady states, the analysis has been carried out previously in the literature with spectral methods as well as with Lyapunov functions in Riemann coordinates. In this article, our main contribution is to show how the analysis can be extended to channels with non-uniform steady states with a Lyapunov function in physical coordinates.

## Introduction

The hyperbolic Saint-Venant equations are commonly used for the description of water flow dynamics in open channels and for the design of management and control systems in irrigation networks and navigable rivers. In particular, the exponential stabilization of Saint-Venant equations by boundary feedback control has been a recurring research topic in the literature for more than twenty years.

The earlier results dealt with *static proportional control*. In the simplest case of horizontal channels with negligible friction, the stability analysis was carried out in [6] with an entropy Lyapunov function, in [16, 11] with the method of characteristics, and in [7, Section VI] with a Lyapunov function in Riemann coordinates. The stability analysis was then extended to channels with slope and friction. In the special case of a uniform steady state, the stability analysis was carried out with a spectral method for linearized equations in [17, Section 6]. However the linearized system stability does not directly imply the stability of the steady state for the nonlinear Saint-Venant equations (see e.g. [8]). For this nonlinear case, the stability analysis is done in [4, 13] with a Lyapunov function in Riemann coordinates. More recently, the case of channels with friction and slope and non-uniform steady state was considered in [3] and [15] with dedicated Lyapunov functions expressed in physical coordinates.

The boundary feedback stabilization of Saint-Venant equations by *Proportional-Integral (PI) control* has received much less attention in the literature. It has been analyzed for channels with uniform steady states in [5] with a spectral method and in [14, Section 4], [2, Section 5.5] with Lyapunov functions in Riemann coordinates. *In the present article, our main contribution is to show how the analysis of [3] can be extended to channels with non-uniform steady states under PI control, using a Lyapunov function in physical coordinates.*

Obviously, in principle, stabilization is also possible with more sophisticated control laws. In particular, the recent backstepping method for  $2 \times 2$  hyperbolic systems, see e.g. [10, 1, 12],

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allows to design stabilizing boundary output feedbacks in *observer-controller form* for Saint-Venant equations. However, it is clear that such advanced solutions are far from being used in practice and that PI controllers are the only regulators that are really implemented in the vast majority of field applications. The reason is obviously that PI regulators, besides their great ease of implementation, are the simplest solution to cancel off-set errors and attenuate load disturbances. In a PI regulator, the parameter  $k_i$  is a measure of the disturbance attenuation efficiency, but too large values may produce instability. The analysis of the stability of a closed-loop system under PI control, as we present in this article, is therefore an important and relevant issue.

## Saint-Venant equations

We consider a pool of a prismatic horizontal open channel with a rectangular cross section, as shown in Fig.1. The dynamics of the system are described by the Saint-Venant equations

$$H_t + (HV)_x = 0, \quad (1a)$$

$$V_t + \left( gH + \frac{1}{2}V^2 \right)_x + S_f(H, V) = 0, \quad (1b)$$

with the state variables  $H(t, x)$  = water depth and  $V(t, x)$  = horizontal water velocity at the time instant  $t$  and the location  $x$  along the channel.  $L$  is the length of the pool and  $g$  is the gravity acceleration.  $S_f(H, V)$  is the friction term for which various empirical models are available in the engineering literature. In this article, we adopt the simple model

$$S_f(H, V) \triangleq C \frac{V^2}{H} \quad (2)$$

with  $C$  a constant friction coefficient.

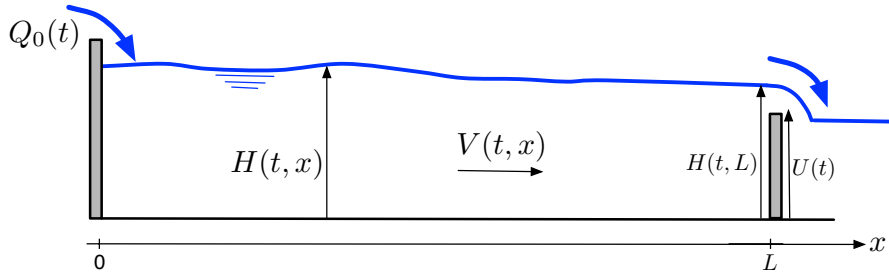


Figure 1: Pool of an open channel with an overflow gate at the downstream side.

The system is subject to the following boundary conditions:

$$H(t, 0)V(t, 0) = Q_0(t), \quad (3a)$$

$$H(t, L)V(t, L) = v_G(H(t, L) - U(t)). \quad (3b)$$

The first boundary condition (3a) imposes the value of the canal inflow rate which is an unknown disturbance denoted  $Q_0(t)$ . The second boundary condition (3b) is a simple linear model of a spillway outflow gate with  $U(t)$  the gate elevation used as control input and  $v_G$  a constant gate parameter.

## Proportional-Integral control

In this article we are concerned with the case where the outflow gate is provided with a Proportional-Integral (PI) control law

$$U(t) \triangleq U_r + k_p(H_{sp} - H(t, L)) + k_i \int_0^t (H_{sp} - H(\tau, L))d\tau \quad (4)$$

where  $H_{sp}$  denotes the set-point for the downstream level  $H(t, L)$  which is assumed to be measured on line. The first term  $U_r$  is an arbitrary constant value for the gate elevation. The second term is the proportional correction action with the tuning parameter  $k_p$ . The last term is the integral action with the tuning parameter  $k_i$ .

With this control law, defining  $Z(t) \triangleq U(t) + k_p H(t, L)$ , the boundary conditions are written in differential form as follows:

$$H(t, 0)V(t, 0) = Q_0(t), \quad (5a)$$

$$H(t, L)V(t, L) = v_G [(1 + k_p)H(t, L) - Z(t)], \quad (5b)$$

$$\frac{dZ}{dt} = k_i(H_{sp} - H(t, L)). \quad (5c)$$

When  $U(t)$  is the feedback command signal (4), the system (1), (5) is a closed loop boundary control system.

In this article, our main purpose is to analyze the exponential stability of this closed loop control system.

**Remark 1.** *It would be interesting to know if, as in the case of a single linear transport equation (see [9, Theorem 2.2]), the control system (1) subject to Proportional-Integral-Derivative (PID) boundary controls is always unstable.*

## Fluvial steady state

In case of a constant positive disturbance  $Q_0 > 0$  and a constant positive set point  $H_{sp} > 0$ , a steady state of the closed loop control system is a time-invariant solution  $H^*(x), V^*(x), Z^*$ ,  $x \in [0, L]$ , given by:

$$H^*(x) \text{ solution of } (gH^{*3} - Q_0^2)H_x^* + CQ_0^2 = 0, \quad H^*(L) = H_{sp}, \quad (6a)$$

$$V^*(x) = \frac{Q_0}{H^*(x)}, \quad (6b)$$

$$Z^* = (1 + k_p)H_{sp} - \frac{Q_0}{v_G}. \quad (6c)$$

The existence of a solution to (6a) requires that  $gH_{sp}^3 \neq Q_0^2$ . If  $gH_{sp}^3 > Q_0^2$ , then (6a) has a solution (note that  $H^*$  is then decreasing) and the steady state flow is subcritical (or fluvial). In such case, from (6a) and (6b), according to the physical evidence, the state  $(H^*, V^*)$  is positive :

$$H^*(x) > 0, \quad V^*(x) > 0, \quad \text{for all } x \in [0, L], \quad (7)$$

and satisfies the following inequality:

$$0 < gH^*(x) - V^{*2}(x) = gH^{*3}(x) - Q_0^2, \quad \forall x \in [0, L]. \quad (8)$$

In the case where  $gH_{sp}^3 < Q_0^2$ , the steady state, if it exists, is said to be supercritical (or torrential). We do not consider that case in the present article.

## Linearization

In order to linearize the model, we define the deviations of the states  $H(t, x)$ ,  $V(t, x)$  and  $Z(t)$  with respect to the steady states  $H^*(x)$ ,  $V^*(x)$  and  $Z^*$ :

$$h(t, x) \triangleq H(t, x) - H^*(x), \quad v(t, x) \triangleq V(t, x) - V^*(x), \quad z = Z(t) - Z^*. \quad (9)$$

Then the linearized Saint-Venant equations around the steady-state are

$$\begin{pmatrix} h_t \\ v_t \end{pmatrix} + \begin{pmatrix} V^* & H^* \\ g & V^* \end{pmatrix} \begin{pmatrix} h_x \\ v_x \end{pmatrix} + \begin{pmatrix} V_x^* & H_x^* \\ -C \frac{V^{*2}}{H^{*2}} & V_x^* + 2C \frac{V^*}{H^*} \end{pmatrix} \begin{pmatrix} h \\ v \end{pmatrix} = 0, \quad (10)$$

and the linearized boundary conditions are

$$v(t, 0) = -b_0 h(t, 0) \quad \text{with} \quad b_0 = \frac{V^*(0)}{H^*(0)}, \quad (11a)$$

$$v(t, L) = b_L h(t, L) - b_z z(t), \quad b_L = \frac{v_G(1 + k_p) - V^*(L)}{H^*(L)}, \quad b_z = \frac{v_G}{H^*(L)}, \quad (11b)$$

$$z_t = -k_i h(t, L). \quad (11c)$$

## Exponential stability of the linearized system

Let us consider the linearized system (10), (11) under an initial condition

$$h(0, x) = h^o(x), \quad v(0, x) = v^o(x), \quad z(0) = z^o, \quad (12)$$

such that

$$(h^o, v^o) \in L^2((0, L); \mathbb{R}^2), \quad z^o \in \mathbb{R}. \quad (13)$$

The Cauchy problem (10)-(11)-(12) is well-posed (see [2, Appendix A]).

Our concern is to analyze the exponential stability of the system (10)-(11) according to the following definition.

**Definition 1.** *The system (10)-(11) is exponentially stable (for the  $L^2$ -norm) if there exist  $\nu > 0$  and  $C_o > 0$  such that, for every initial condition  $(h^o, v^o) \in L^2((0, L); \mathbb{R}^2)$ ,  $z^o \in \mathbb{R}$ , the solution to the Cauchy problem (10), (11), (12) satisfies*

$$\|(h(t, \cdot), v(t, \cdot))\|_{L^2} + |z(t)| \leq C_o e^{-\nu t} \left[ \|(h^o, v^o)\|_{L^2} + |z^o| \right]. \quad (14)$$

We now prove that the linearized control system (10)-(11) is exponentially stable if the steady state is subcritical and the control tuning parameters are positive:  $k_p > 0$  and  $k_i > 0$ . For this stability analysis, the following candidate Lyapunov function is considered:

$$\mathbf{V}(h, v, z) = \int_0^L (gh^2 + H^*v^2)dx + qz^2, \quad (15)$$

where  $q > 0$  is chosen later on. Note that there exists  $C_1 > 0$  such that

$$\frac{1}{C_1} (\|(h, v)\|_{L^2}^2 + |z|^2) \leq \mathbf{V}(h, v, z) \leq C_1 (\|(h, v)\|_{L^2}^2 + |z|^2), \quad \forall (h, v, z) \in L^2(0, L) \times L^2(0, L) \times \mathbb{R}. \quad (16)$$

The time derivative of this function  $\mathbf{V}$  along the  $C^1$  solutions of the Cauchy problem (10), (11), (12) is

$$\frac{d\mathbf{V}}{dt} = 2 \int_0^L (ghh_t + H^*vv_t)dx + 2qzz_t. \quad (17)$$

Using the system equation (10) and the boundary condition (11c), we have

$$\begin{aligned} \frac{d\mathbf{V}}{dt} = & -2 \int_0^L \left( gh(V^*h_x + H^*v_x + V_x^*h + H_x^*v) \right. \\ & \left. + H^*v(gh_x + V^*v_x - C \frac{V^{*2}}{H^{*2}}h + (V_x^* + 2C \frac{V^*}{H^*}v)) \right) dx - 2qk_i z h(t, L). \end{aligned} \quad (18)$$

Then, using integration by parts together with (6a), we have

$$\frac{d\mathbf{V}}{dt} = - \left[ (h \ v) M(x) \begin{pmatrix} h \\ v \end{pmatrix} \right]_0^L - \int_0^L (h \ v) N(x) \begin{pmatrix} h \\ v \end{pmatrix} dx - 2qk_i z h(t, L), \quad (19)$$

with

$$M(x) = \begin{pmatrix} gV^*(x) & gH^*(x) \\ gH^*(x) & H^*(x)V^*(x) \end{pmatrix}, \quad (20)$$

and

$$N(x) = \begin{pmatrix} \frac{gCV^{*3}}{H^*(gH^* - V^{*2})} & -\frac{CV^{*2}}{H^*} \\ -\frac{CV^{*2}}{H^*} & \frac{2CV^{*3}}{(gH^* - V^{*2})} + 4CV^* \end{pmatrix}. \quad (21)$$

We introduce the notations

$$h_0 = h(t, 0), \quad h_L = h(t, L), \quad v_0 = v(t, 0), \quad v_L = v(t, L), \quad (22)$$

$$H_0 = H^*(0), \quad V_0 = V^*(0), \quad H_L = H^*(L), \quad V_L = V^*(L). \quad (23)$$

Then, using the boundary conditions (11a), (11b), we have

$$\begin{aligned} \left[ (h \ v) M(x) \begin{pmatrix} h \\ v \end{pmatrix} \right]_0^L &= gV_L h_L^2 + 2gH_L h_L v_L + Q_0 v_L^2 - gV_0 h_0^2 - 2gH_0 h_0 v_0 - Q_0 v_0^2 \\ &= gV_L h_L^2 + 2gH_L h_L (b_L h_L - b_z z) + Q_0 (b_L h_L - b_z z)^2 - gV_0 h_0^2 + 2gH_0 h_0 (b_0 h_0) - Q_0 (b_0 h_0)^2 \\ &= (gV_L + 2gb_L H_L + Q_0 b_L^2) h_L^2 + (-gV_0 + 2gb_0 H_0 - Q_0 b_0^2) h_0^2 + Q_0 b_z^2 z^2 \\ &\quad + (-2gb_z H_L - 2Q_0 b_L b_z) h_L z. \end{aligned} \quad (24)$$

Consequently

$$\frac{d\mathbf{V}}{dt} = -m_0 h_0^2 - (h_L \ z) \mathcal{M} \begin{pmatrix} h_L \\ z \end{pmatrix} - \int_0^L (h \ v) N(x) \begin{pmatrix} h \\ v \end{pmatrix} dx - 2aqk_i z^2, \quad (25)$$

with

$$m_0 = -gV_0 + 2gb_0 H_0 - Q_0 b_0^2, \quad (26)$$

$$\mathcal{M} = \begin{pmatrix} gV_L + 2gb_L H_L + Q_0 b_L^2 & -gb_z H_L - Q_0 b_L b_z + qk_i \\ -gb_z H_L - Q_0 b_L b_z + qk_i & Q_0 b_z^2 - 2aqk_i \end{pmatrix}, \quad (27)$$

and  $a$  is a real positive constant to be determined.

Under the subcritical flow condition (8), using the definition of  $b_0$  (11a), we have that

$$m_0 = -gV_0 + 2gb_0 H_0 - H_0 V_0 b_0^2 = b_0 (gH_0 - V_0^2) > 0 \quad (28)$$

and that the matrix  $N(x)$  is positive definite for all  $x \in [0, L]$  since

$$\det[N(x)] = \left( \frac{CV^{*2}}{H^*} \right)^2 \left( \frac{2gH^*V^{*2}}{(gH^* - V^{*2})^2} + \frac{4gH^*}{(gH^* - V^{*2})} - 1 \right) > 0. \quad (29)$$

On the other hand,  $\mathcal{M}$  is positive definite if

$$(a) \quad gV_L + 2gb_LH_L + Q_0b_L^2 > 0, \quad (30)$$

$$(b) \quad \det(\mathcal{M}) = (gV_L + 2gb_LH_L + Q_0b_L^2)(Q_0b_z^2 - 2aqk_i) - (gb_zH_L + Q_0b_Lb_z - qk_i)^2 > 0. \quad (31)$$

It follows from (3b) that  $v_G > V_L$ . Hence, since  $k_p > 0$ , we have from (11b)

$$b_L = \frac{(v_G - V_L) + v_Gk_p}{H^*(L)} > 0 \quad (32)$$

and Condition (a) is satisfied.

Regarding condition (b), using the definition of  $b_L$ , we have

$$\det(\mathcal{M}) = -\alpha + 2\beta k_i q - k_i^2 q^2 = \mathcal{P}(q), \quad (33)$$

with

$$\alpha = gb_z^2 H_L (gH_L - V_L^2) \quad (34)$$

and

$$\beta = gb_z H_L + Q_0 b_L b_z - a(gV_L + 2gb_L H_L + Q_0 b_L^2). \quad (35)$$

$\mathcal{P}(q)$  is a degree-2 polynomial in  $q$  with discriminant

$$\Delta = 4k_i^2(\beta^2 - \alpha). \quad (36)$$

We observe that  $\alpha > 0$  under the subcritical flow condition (8). Moreover, it is easy to check that the positive parameter  $a$  can be selected sufficiently small so that  $\beta > 0$  and  $\beta^2 - \alpha > 0$ . Hence, if  $k_i > 0$ ,  $\mathcal{P}(q)$  has two positive real roots and there exists a positive value of  $q$  (depending on  $k_i$ ) such that  $\det(\mathcal{M}) > 0$  and condition (b) is satisfied. Then, it follows directly from the definition (15) of  $\mathbf{V}$  and from (25) that there exists a positive real constant  $\mu$  such that

$$\frac{d\mathbf{V}}{dt} \leq -\mu \mathbf{V} \quad (37)$$

along the  $C^1$ -solutions of the system. However, since the  $C^1$ -solutions are dense in the set of  $L^2$ -solutions, inequality (37) is also satisfied in the sense of distributions for  $L^2$ -solutions (see [2] for details). Consequently,  $\mathbf{V}$  is an exponentially decaying Lyapunov function for the  $L^2$ -norm and the system (10)-(11) is exponentially stable in the sense of Definition 1.

## Exponential stability of the steady state of the Saint-Venant equations

In the previous section, we have shown that the PI controller (4) stabilizes the **linearized** Saint-Venant equations if the steady state is subcritical and the control tuning parameters are positive:  $k_p > 0$  and  $k_i > 0$ . In this section, we briefly explain how it can be shown that the same PI controller is also sufficient to guarantee the local exponential stability for the  $H^2$ -norm of the steady state  $H^*(x)$ ,  $V^*(x)$  of the **nonlinear** system of Saint-Venant equations (1), (2) under the nonlinear boundary conditions (5).

Let us rewrite the Saint-Venant equations in the  $(h, v)$  coordinates (see (9)),

$$\begin{pmatrix} h_t \\ v_t \end{pmatrix} + \begin{pmatrix} V^*(x) + v & H^*(x) + h \\ g & V^*(x) + v \end{pmatrix} \begin{pmatrix} h_x \\ v_x \end{pmatrix} + \begin{pmatrix} V_x^*(x) & H_x^*(x) \\ -C \frac{V^{*2}(x)}{H^*(x)(H^*(x) + h)} & V_x^*(x) + C \frac{2V^*(x) + v}{H^*(x) + h} \end{pmatrix} \begin{pmatrix} h \\ v \end{pmatrix} = 0, \quad (38)$$

with the boundary conditions (using the notations (22) and (23))

$$v_0 = -b_0 h_0 + \frac{V_0}{H_0(H_0 + h_0)} h_0^2, \quad (39a)$$

$$v_L = b_L h_L - b_z z + \frac{(V_L - v_G(1 + k_P))h_L^2 + v_G h_L z}{H_L(H_L + h_L)}, \quad (39b)$$

$$z_t = -k_i h_L. \quad (39c)$$

Then, we transform the system into Riemann coordinates which are defined as follows:

$$\mathbf{R} = \begin{pmatrix} R^+ \\ R^- \end{pmatrix} = \begin{pmatrix} v + 2\eta(h) \\ v - 2\eta(h) \end{pmatrix} \quad \text{with} \quad \eta(h) = \sqrt{g(H^* + h)} - \sqrt{gH^*}. \quad (40)$$

With these coordinates, the system (38) is written in the following characteristic form:

$$\mathbf{R}_t + \Lambda(\mathbf{R}, x)\mathbf{R}_x + \mathbf{B}(\mathbf{R}, x) = \mathbf{0}, \quad (41)$$

with the diagonal matrix

$$\Lambda(\mathbf{R}, x) = \begin{pmatrix} \lambda^+(\mathbf{R}, x) & 0 \\ 0 & \lambda^-(\mathbf{R}, x) \end{pmatrix} \quad \text{with} \quad \lambda^\pm(\mathbf{R}, x) = V^* \pm \sqrt{gH^*} + v \pm \eta(h), \quad (42)$$

and an appropriate definition of  $\mathbf{B}(\mathbf{R}, x)$ .

The goal is to prove the  $H^2$  exponential stability of the zero steady state for the system (42) under the boundary conditions (39) and under an initial condition

$$\mathbf{R}(0, x) = \mathbf{R}_o(x), \quad z(0) = z^o. \quad (43)$$

according to the following definition.

**Definition 2.** *The steady state  $\mathbf{R}(t, x) \equiv 0$  of the system (39) and (41) is exponentially stable for the  $H^2$ -norm if there exist  $\delta > 0$ ,  $\nu > 0$  and  $C_0 > 0$  such that, for every initial condition  $\mathbf{R}_o \in H^2((0, L); \mathbb{R}^n)$  satisfying  $\|\mathbf{R}_o\|_{H^2((0, L); \mathbb{R}^n)} \leq \delta$  and compatibility<sup>1</sup> conditions of order 1, the solution  $\mathbf{R}$  of the Cauchy problem (39), (41), (43) is defined on  $[0, +\infty) \times [0, L]$  and satisfies*

$$\|\mathbf{R}(t, \cdot)\|_{H^2((0, L); \mathbb{R}^n)} + |z(t)| \leq C_0 e^{-\nu t} \left[ \|\mathbf{R}_o\|_{H^2((0, L); \mathbb{R}^n)} + |z^o| \right]. \quad (44)$$

With this definition our main result is the following theorem

**Theorem 1.** *The steady state  $\mathbf{R}(t, x) \equiv 0$  of the system (39) and (41) is exponentially stable for the  $H^2$ -norm.*

<sup>1</sup>For an explanation of the concept of compatibility of initial conditions, see [2, Section 4.5.2]

The proof of Theorem 1 is built in a way similar to the proof given in [2, Chapter 6] for a general class of quasi-linear hyperbolic systems with static boundary conditions. Here we limit ourselves to the key points of the proof and we refer the reader to [2, Section 6.2] for a comprehensive development.

First, we consider an augmented system with state  $(\mathbf{R}, \mathbf{R}_t, \mathbf{R}_{tt})$  where the dynamics of  $\mathbf{R}_t$  and  $\mathbf{R}_{tt}$  are simply obtained by taking partial time derivatives of the system equation (42) and the boundary conditions (39).

Then the candidate Lyapunov function is defined as

$$\mathbf{V}_{\text{NL}} = \mathbf{V}_1(\mathbf{R}, z) + \mathbf{V}_2(\mathbf{R}_t, z_t) + \mathbf{V}_3(\mathbf{R}_{tt}, z_{tt}), \quad (45)$$

with

$$\mathbf{V}_1(\mathbf{R}, z) = \int_0^L \frac{1}{2} H^* \mathbf{R}^T \mathbf{R} dx + qz^2, \quad (46)$$

$$\mathbf{V}_2(\mathbf{R}_t, z_t) = \int_0^L \frac{1}{2} H^* \mathbf{R}_t^T \mathbf{R}_t dx + qz_t^2, \quad (47)$$

$$\mathbf{V}_3(\mathbf{R}_{tt}, z_{tt}) = \int_0^L \frac{1}{2} H^* \mathbf{R}_{tt}^T \mathbf{R}_{tt} dx + qz_{tt}^2. \quad (48)$$

For a vector  $\boldsymbol{\xi} = (\xi_1, \dots, \xi_n)^T \in \mathbb{R}^n$ , we denote  $|\boldsymbol{\xi}|_\infty = \max\{|\xi_j|; j \in \{1, \dots, n\}\}$ . For a map  $f \in C^0([0, L]; \mathbb{R}^n)$ , we denote  $|f|_0 = \max\{|f(x)|_\infty; x \in [0, L]\}$ . We remark that, for small  $|h|_0$ , the function  $\mathbf{V}_1(\mathbf{R}, z)$  can be viewed as a perturbation of the Lyapunov function  $\mathbf{V}(h, v, z)$  of the linearized system (see equation (15)). More precisely, for  $|h|_0$  sufficiently small,

$$\begin{aligned} \mathbf{V}_1(\mathbf{R}, z) &= \int_0^L \frac{1}{2} H^* \mathbf{R}^T \mathbf{R} dx + qz^2 \\ &= \int_0^L (4H^* \eta^2(h) + H^* v^2) dx + qz^2 \\ &= \int_0^L (gh^2 + H^* v^2 + O(h^3)) dx + qz^2 \\ &= \mathbf{V}(h, v, z) + \int_0^L O(|h|^3) dx. \end{aligned} \quad (49)$$

Similar expressions of  $\mathbf{V}_2$  and  $\mathbf{V}_3$  are obtained as follows: for  $|h|_0$  sufficiently small

$$\mathbf{V}_2(\mathbf{R}_t, z_t) = \mathbf{V}(h_t, v_t, z_t) + \int_0^L O(|hh_t^2|) dx, \quad (51)$$

$$\mathbf{V}_3(\mathbf{R}_{tt}, z_{tt}) = \mathbf{V}(h_{tt}, v_{tt}, z_{tt}) + \int_0^L O(|h_t^2 h_{tt}| + |hh_{tt}^2| + |h_t|^4) dx. \quad (52)$$

Let us now introduce a notation to deal with ‘‘higher order terms’’ in the time derivative of the Lyapunov function. We denote by  $\mathcal{O}(X_1; X_2)$ , with  $X_1 \geq 0$  and  $X_2 \geq 0$ , quantities for which there exist  $C_0 > 0$  and  $\varepsilon > 0$  independent of  $\mathbf{R}$ ,  $\mathbf{R}_t$  and  $\mathbf{R}_{tt}$ , such that

$$(X_2 \leq \varepsilon) \implies (|\mathcal{O}(X_1; X_2)| \leq C_0 X_1).$$

It follows that the time derivatives of  $\mathbf{V}_1$ ,  $\mathbf{V}_2$  and  $\mathbf{V}_3$  along the system solutions can be expressed

in the following form

$$\begin{aligned} \frac{d\mathbf{V}_1}{dt} = & - \left[ (h \ v) M(x) \begin{pmatrix} h \\ v \end{pmatrix} \right]_0^L - \int_0^L (h \ v) N(x) \begin{pmatrix} h \\ v \end{pmatrix} dx - 2qz z_t \\ & + \mathcal{O}(|\mathbf{R}(t,0)|^3 + |\mathbf{R}(t,L)|^3; |\mathbf{R}(t,0)| + |\mathbf{R}(t,L)|) \\ & + \mathcal{O}\left(\int_0^L (|\mathbf{R}|^3 + |\mathbf{R}_t||\mathbf{R}|^2)dx; |\mathbf{R}(t,\cdot)|_0\right), \end{aligned} \quad (53)$$

$$\begin{aligned} \frac{d\mathbf{V}_2}{dt} = & - \left[ (h_t \ v_t) M(x) \begin{pmatrix} h_t \\ v_t \end{pmatrix} \right]_0^L - \int_0^L (h_t \ v_t) N(x) \begin{pmatrix} h_t \\ v_t \end{pmatrix} dx - 2qz_t z_{tt} \\ & + |\mathbf{R}_t(t,0)|^2 |\mathbf{R}(t,0)| + |\mathbf{R}_t(t,L)|^2 |\mathbf{R}(t,L)|; |\mathbf{R}(t,0)| + |\mathbf{R}(t,L)| \\ & + \mathcal{O}\left(\int_0^L |\mathbf{R}_t|^2 (|\mathbf{R}_t| + |\mathbf{R}|) dx; |\mathbf{R}(t,\cdot)|_0\right), \end{aligned} \quad (54)$$

$$\begin{aligned} \frac{d\mathbf{V}_3}{dt} = & - \left[ (h_{tt} \ v_{tt}) M(x) \begin{pmatrix} h_{tt} \\ v_{tt} \end{pmatrix} \right]_0^L - \int_0^L (h_t \ v_t) N(x) \begin{pmatrix} h_t \\ v_t \end{pmatrix} dx - 2qz_{tt} z_{ttt} \\ & + \mathcal{O}(|\mathbf{R}_{tt}(t,0)|^2 |\mathbf{R}(t,0)| + |\mathbf{R}_{tt}(t,0)||\mathbf{R}_t(t,0)|^2 + |\mathbf{R}_t(t,0)|^4 \\ & + |\mathbf{R}_{tt}(t,L)|^2 |\mathbf{R}(t,L)| + |\mathbf{R}_{tt}(t,L)||\mathbf{R}_t(t,L)|^2 + |\mathbf{R}_t(t,L)|^4; |\mathbf{R}(t,0)| + |\mathbf{R}(t,L)|) \\ & + \mathcal{O}\left(\int_0^L (|\mathbf{R}_{tt}|^2 (|\mathbf{R}_t| + |\mathbf{R}|) + |\mathbf{R}_{tt}||\mathbf{R}_t|^2) dx; |\mathbf{R}(t,\cdot)|_0 + |\mathbf{R}_t(t,\cdot)|_0\right). \end{aligned} \quad (55)$$

We observe that, in each case, we recover the quadratic formula of the linear case augmented with (at least) cubic terms that are negligible for small  $|\mathbf{R}(t,\cdot)|_0 + |\mathbf{R}_t(t,\cdot)|_0$ . It is therefore not surprising that the local  $H^2$  stability of the nonlinear steady state can be deduced from the global  $L^2$  stability of the linear system. By proceeding similarly to [2, Chapter 6], it can be shown that there exist positive constants  $\alpha$  and  $\delta$  such that, for every  $\mathbf{R}$  such that  $|\mathbf{R}(t,\cdot)|_0 + |\mathbf{R}_t(t,\cdot)|_0 < \delta$ , we have

$$\frac{d\mathbf{V}_{\text{NL}}}{dt} \leq -\alpha \mathbf{V}_{\text{NL}} \quad (56)$$

along the system solutions. Compared to [2, Chapter 6], note that, by (39b),  $v_L$  (and its first and second time derivatives) can be expressed in terms of  $h_L$  and  $z$  (and their first and second time derivatives) as we did in (24) for the linear case. It follows that the system steady-state is locally exponentially stable for the  $H^2$ -norm in the sense of Definition 2. This concludes the proof of Theorem 1.

## Conclusion

In this article, our main contribution was to exhibit a Lyapunov function which allows to study the exponential stability of nonuniform steady-states for Saint-Venant equations with a friction term under boundary feedback PI control.

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