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CLOSED STRING AMPLITUDES FROM SINGLE-VALUED CORRELATION FUNCTIONS

by

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Abstract. — We argue in two different ways that the low-energy expansion of any tree-level closed string amplitudes only involves single-valued multiple zeta values. First, we identify the building blocks of any closed string amplitudes with the value at $z = 1$ of single-valued correlation functions in two dimensional conformal field theory. We use the single-valuedness condition to determine uniquely the correlation function and determine the role of the momentum kernel in the single-valued projection. The second argument is more technical and leads to a mathematical proof of the statement. It is obtained through a direct analysis of the α'-expansion of closed string integrals, whose coefficients are given by multiple integrals of single-valued hyperlogarithms over the complex plane. The main new tool is the extension to the single-valued case of some technical results used for multiple integration of holomorphic hyperlogarithms.

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Contents

1. Introduction and overview of the results

String theory amplitudes display remarkable properties that are still being uncovered. It has been noticed that the low-energy (or small inverse tension α') expansion of closed string theory involves only single-valued multiple zeta values [1] at tree-level (on a sphere) [2–5], at genus one (on a torus) [6–11] and in some limit of the genus two amplitude [12, 13]. It is tempting to conjecture that this will be true at all orders of the genus expansion. This is of particular importance, as from string theory amplitudes one can extract the ultraviolet behaviour of supergravity amplitudes in various dimensions [14–24]. The ultraviolet behaviour of the four-graviton maximal supergravity amplitudes up to three-loop order [25, 26] have been obtained from appropriate limits of string theory amplitudes [14, 18, 19, 22, 24]. Another application is the mapping of discontinuities of $\mathcal{N} = 4$ super-Yang-Mills correlation functions to contributions in the low-energy expansion of the flat-space limit of $AdS_5 \times S^5$ string theory amplitudes [27, 28] at tree-level or one-loop [29].

On the other hand, the low-energy expansion of open string amplitudes, at least up to genus one, involves all kinds of multiple zeta values [30, 31]. Extensive computations [2] have lead to the conjecture that the low-energy expansion of tree-level closed string amplitudes [3–5] is obtained by the application of the single-valued map of [1] applied on open string amplitudes.

Open string amplitudes can be written in terms of ordered integrals on the boundary of the disc [32], which are generalised Selberg integrals. Tree-level closed string amplitudes involve integrals on the punctured sphere, which can be seen as complex analogues of generalised Selberg integrals. It has been shown in [33,34] that the disc integrals are mapped to the sphere integrals by the single-valued map. The single-trace sector of tree-level heterotic amplitudes can be obtained by the application of the single-valued map on the single-trace open string amplitudes [4,5] and the double-trace sector by expressing the double-trace partial tree-level

amplitudes in terms of single-trace partial amplitudes [35]. The applicability of the single-valued map to obtain multiple-trace contributions from heterotic-string amplitudes is still an open question [36]. A generalization of the single-valued map to genus-one string amplitudes was recently proposed in [10], and it was used to relate four-point one-loop open string amplitudes to genus-one heterotic-string amplitudes in [11]. It is however still a conjecture that all the modular graph functions and forms appearing in the low-energy expansion of genus-one closed string amplitudes in [8,10,11] are in the image of the single-valued map defined in [37,38] on iterated integrals of modular forms.

In this work we take a totally different route: we directly analyse the closed string integrals and we show that the coefficients of the low-energy expansion are single-valued multiple zeta values without having to use the single-valued map from disc amplitudes.

In part I we consider the decomposition of any tree-level closed string amplitude as a finite linear combination of partial amplitudes ($N \geq 1$)

$$M_{N+3}(\boldsymbol{s},\boldsymbol{\epsilon}) = \sum_r c_r(\boldsymbol{s},\boldsymbol{\epsilon}) M_{N+3}(\boldsymbol{s},\boldsymbol{n}^r,\bar{\boldsymbol{n}}^r)\,. \tag{1.1}$$

The coefficients $c_r(\boldsymbol{s},\boldsymbol{\epsilon})$ are rational functions of the kinematic invariants $\boldsymbol{s} = (2\alpha' k_i \cdot k_j)_{1\leq i<j\leq N+3}$ (where k_i are external momenta), the polarisation tensors $\boldsymbol{\epsilon} = (\epsilon_i)_{1\leq i\leq N+3}$ and the colour factors for the heterotic string amplitudes (see §3 for conventions and notations). The kinematic coefficients determine the specifics of the closed string theory (bosonic string, superstring or heterotic string) or of the external states (polarisation tensors and colour factors).

The partial amplitudes $M_N(\boldsymbol{s},\boldsymbol{n},\bar{\boldsymbol{n}})$ are generic integrals given by

$$M_{N+3}(\boldsymbol{s},\boldsymbol{n},\bar{\boldsymbol{n}}) = \int_{\mathbb{C}^N} \prod_{i=2}^{N+1} d^2 w_i \prod_{2\leq i<j\leq N+1} |w_i - w_j|^{2\alpha' k_i\cdot k_j} (w_i - w_j)^{n_{ij}} (\bar{w}_i - \bar{w}_j)^{\bar{n}_{ij}}$$

$$\times \prod_{i=2}^{N+1} |w_i|^{2\alpha' k_1\cdot k_i} |1-w_i|^{2\alpha' k_i\cdot k_{N+2}} w_i^{n_{1i}} \bar{w}_i^{\bar{n}_{1i}} (1-w_i)^{n_{i\,(N+2)}} (1-\bar{w}_i)^{\bar{n}_{i\,(N-1)}} \,. \tag{1.2}$$

(Using conformal invariance we have fixed the points $w_1 = 0$, $w_{N+2} = 1$ and $w_{N+3} = \infty$.) We explain that these partial amplitudes are the value at $z = 1$, $M_{N+3}(\boldsymbol{s}, \boldsymbol{n}, \bar{\boldsymbol{n}}) = \mathcal{G}_N(1,1)$, of correlation functions defined in §3.1 as

$$\mathcal{G}_N(z,\bar{z}) := \int_{\mathbb{C}^N} \prod_{i=1}^{N} w_i^{a_i} (w_i-1)^{b_i} (w_i-z)^{c_i} \bar{w}_i^{\bar{a}_i} (\bar{w}_i-1)^{\bar{b}_i} (\bar{w}_i-\bar{z})^{\bar{c}_i} \times \prod_{1\leq i<j\leq N} (w_i-w_j)^{g_{ij}} (\bar{w}_i-\bar{w}_j)^{\bar{g}_{ij}} \prod_{i=1}^{N} d^2 w_i \,. \tag{1.3}$$

The correlation functions do not have to be associated to a physical string theory process, only specific linear combinations of their $z = 1$-values need to give a physical amplitudes. It is not necessary (and in general not true) that the total amplitude arises as the $z = 1$-value of a single-valued correlation function. It is enough that each partial amplitude is associated to a correlation function of the type in (1.3). Some of these integrals may be related by integration by parts, but for our arguments it is better to not use a minimal set of integral functions. We only need in our analysis that (1.3) arises as a correlation function in a conformal field theory.

As a conformal-field-theory correlation function, $\mathcal{G}_N(z,\bar{z})$ has to be a single-valued function of the position z. Using the contour deformation techniques of [39–42], $\mathcal{G}_N(z,\bar{z})$ has the holomorphic factorisation

$$\mathcal{G}_N(z,\bar{z}) = \sum_{r,s=1}^{(N+1)!} G_{r,s} I_r(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c};\boldsymbol{g};z) I_s(\bar{\boldsymbol{a}},\bar{\boldsymbol{b}},\bar{\boldsymbol{c}};\bar{\boldsymbol{g}};\bar{z}) \tag{1.4}$$

where $I_r(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c};\boldsymbol{g};z)$ and $I_s(\bar{\boldsymbol{a}},\bar{\boldsymbol{b}},\bar{\boldsymbol{c}};\bar{\boldsymbol{g}};\bar{z})$ are Aomoto-Gel'fand hypergeometric functions [43, 44] defined in §4.1.

Using the techniques introduced by Dotsenko and Fateev in [39, 40] we determine the matrix $G = (G_{ab})$ such that the function does not have monodromies around $z = 0$ and $z = 1$. The proof is constructive and shows that the coefficients of the matrix G are rational functions of $\sin(\pi\alpha' x)$ where x are integer linear combinations of kinematic invariants. It was shown in [39, 40] that the single-valuedness condition determines the correlation function $\mathcal{G}_N(z, \bar{z})$ up to an overall coefficient. In our construction the overall coefficient is fixed by matching the string building blocks, and there is no freedom in determining the coefficients of the matrix G in the holomorphic factorisation (1.4) for each building blocks. The $z = 1$-value gives a version of the Kawai, Lewellen and Tye (KLT) relations [45] which is a symmetric sum over the $N!$ permutations of the colour-ordered open string amplitudes

$$M_{N+3}(\boldsymbol{s}, \boldsymbol{n}, \bar{\boldsymbol{n}}) = \sum_{\sigma,\rho\in\mathfrak{S}_N} G_{\sigma,\rho}\, A_N(\sigma(2,\dots,N+1), 1, N+2, N+3; \boldsymbol{n}) \times \bar{A}_N(\rho(2,\dots,N+1), 1, N+2, N+3; \bar{\boldsymbol{n}})\,, \quad (1.5)$$

where $A_N(\sigma(1,\dots,N+1), 1, N+2, N+3; \boldsymbol{n})$ are colour-ordered open string amplitudes

$$A_N(\sigma(2,\dots,N{+}1), 1, N{+}2, N{+}3; \boldsymbol{n}) = \int_{\delta(\sigma)} \prod_{2\le i<j\le N+1} |w_i{-}w_j|^{\alpha' k_i\cdot k_j} (w_i{-}w_j)^{n_{ij}} \times \prod_{i=2}^{N+1} |w_i|^{\alpha' k_1\cdot k_i} |1-w_i|^{\alpha' k_i\cdot k_{N+2}} w_i^{n_{1i}} (1-w_i)^{n_{i\,(N+2)}} \prod_{i=2}^{N+1} dw_i\,, \quad (1.6)$$

integrated over the simplex

$$\delta(\sigma) := \{w_{\sigma(2)} \le \cdots \le w_{\sigma(N+1)} \le w_1 \le w_{N+2} \le w_{N+3}\}\,. \quad (1.7)$$

Such a form, which is equivalent to the standard KLT relations, was derived for the four-point ampliutude with $N = 1$ and the five-point amplitudes with $N = 2$ in [41, eq (15)-(16)]. The matching with the holomorphic factorisation of the closed string integral gives a precise relation, given in §6, between some coefficients of the matrix G and the

momentum kernel $S_{\alpha'}(\sigma|\rho)$ (see Appendix C for a definition). The matrix elements $G_{\sigma,\rho}$ in (1.5) are non-local coefficients with denominators, but they are related to the local momentum kernel $S_{\alpha'}(\sigma|\rho)$ using the linear relations between the colour-ordered open string amplitudes [32, 41]. In order to construct a single-valued correlation function in the complex plane one needs to enlarge the momentum kernel to the full matrix G. It would be interesting to relate the present construction with the derivation of the inverse momentum kernel from intersection numbers of twisted cycles [46].

We argue in §5, using results from part II, that the α'-expansions of all Aomoto-Gel'fand hypergeometric functions are multiple zeta values-linear combinations of multiple polylogarithms. Since $\mathcal{G}_N(z,\bar{z})$ has no monodromies, it follows from the Theorem 9.2 proved by Francis Brown in [47] that its small α'-expansion involves only multiple zeta values-linear combinations of single-valued multiple polylogarithms. The general expectation is that actually the coefficients of these combinations should be just single-valued multiple zeta values. Assuming that this is true, we conclude that the coefficient of the kinematics factors in the small α'-expansions of closed string amplitudes are rational combinations of single-valued multiple zeta values. This is a natural expectation since each of the closed string partial amplitudes in eq. (1.1) is the value at $z=1$ of such a single-valued function in the complex plane ($M_{N+3}(\boldsymbol{s},\boldsymbol{n}^r,\bar{\boldsymbol{n}}^r)=\mathcal{G}^r_N(1,1)$), and since the kinematic coefficients $c_r(\boldsymbol{s},\boldsymbol{\epsilon})$ are rational functions of α' times the kinematics invariants (see [36, 48–50] for various amplitudes in bosonic and superstring theory). Notice that due to the presence of the kinematic coefficients $c_r(\boldsymbol{s},\boldsymbol{\epsilon})$, a given order in the α'-expansion of the total closed string amplitude can mix single-valued multiple zeta values of different weight as can easily be seen from the expansion of the heterotic-string amplitudes evaluated in [48, 49].

This physically motivated argument, supported by numerical checks, does not constitute a proof of the single-valuedness of the expansion.

For this reason, we dedicate part II of the paper to develop alternative methods which lead to a full proof (under some technical assumptions on the convergence of the integrals) of the single-valuedness of the multiple zeta values in the small α'-expansions, confirming our expectations.

More precisely, in part II of this paper we recall and extend the theory of integration of single-valued hyperlogarithms. These functions, introduced by Brown in [47], generalise single-valued multiple polylogarithms to the case of iterated integrals over the n-punctured complex plane, and constitute the natural framework to describe the sphere integrals appearing in closed string amplitudes. We generalise at the same time results of Panzer for the integration of (multi-valued) hyperlogarithms [51] and results of Schnetz for the integration of single-valued multiple polylogarithms [52]. This allows to obtain an alternative and elementary mathematical proof that the α'-expansion of massless superstring theory amplitudes only involves single-valued multiple zeta values. Our results are quite general, and apply to a broad class of integrals. Here we content ourselves to show how to implement our methods for one family of multiple complex integrals, namely we demonstrate the following (see Theorem 7.1 for a more precise statement):

Theorem 1.1. — *The coefficients of the low-energy expansion of the integral*

$$\int_{(\mathbb{P}^1_{\mathbb{C}})^k} \prod_{i=1}^{k} |z_i|^{2\alpha_i-2}|z_i-1|^{2\beta_i-2} \prod_{1\leq i<j\leq k} |z_j-z_i|^{2\gamma_{i,j}}\, d^2z_1\cdots d^2z_k \qquad (1.8)$$

are single-valued multiple zeta values.

We will argue that this family of integrals is a convenient prototye of the integrals

$$J_{\rho,\sigma}(a_1,\ldots,a_k,b_1,\ldots,b_k,c_{1,2},\ldots,c_{k-1,k}) =$$

$$\int_{(\mathbb{P}^1_{\mathbb{C}})^k} \frac{\prod_{1\leq i<j\leq k}|z_j-z_i|^{2c_{i,j}}\prod_{i=1}^k|z_i|^{2a_i}|z_i-1|^{2b_i}\,d^2z_i}{z_{\rho(1)}\,\overline{z}_{\sigma(1)}(1-z_{\rho(k)})(1-\overline{z}_{\sigma(k)})\prod_{i=2}^k(z_{\rho(i)}-z_{\rho(i-1)})(\overline{z}_{\sigma(i)}-\overline{z}_{\sigma(i-1)})} \tag{1.9}$$

appearing in closed superstring theory (ρ and σ are permutations), and hence explain that the analogue of Theorem 1.1 for the $J_{\rho,\sigma}$'s can be obtained by a straightforward adaptation of our methods. As anticipated before, (a stronger version of) this result was partly demonstrated in [33], assuming some transcendentality assumptions on multiple zeta values, and was rigorously proven immediately afterwards in [34] with different methods. However, since both proofs rely on abstract constructions in algebraic geometry, we believe that our approach to demonstrate the weaker statement, whose nature is much more elementary, may be of particular interest for the physics community.

Let us now come back to the general picture of string amplitudes, and discuss the possibility to generalise our methods to higher genera. The decomposition in (1.4) is the generic expansion of a correlation function in conformal field theory on conformal blocks which exists for any higher-genus Riemann surface. At genus one, the low-energy expansion of closed string amplitudes has been obtained from the special value of single-valued modular functions and modular forms on the torus [8, 11, 13]. This suggest that one could extend the present arguments to higher-genus Riemann surfaces. It would be interesting to use this construction to connect with the flat space limit of AdS/CFT correlation functions analysed in the works [27–29]. The methods explained in the second part of the paper, on the other side, would generalise (at least) to the genus-one (configuration-space) integrals, as long as one develops a theory of single-valued elliptic (or higher-genus) multiple polylogarithms.

The paper is organised as follows. The first part gives a construction of the single-valued correlation function $\mathcal{G}_N(z,\bar{z})$ and its relation to the

closed string partial amplitudes of any closed string integrals. In Section 2 we review generic properties of conformal-field-theory correlation functions and in Section 3 we detail the relation with the closed string amplitudes. The single-valued conditions on the correlation function are derived in Section 4 and we show in Section 5 how this implies that the low-energy expansion of the closed string amplitudes are given in by single-valued multiple zeta values. Section 6 details the relation between the monodromy matrix and the momentum kernel, and explains how the symmetric form of the KLT relations matches the single-valued expression for the closed string amplitudes.

The second part of the paper begins with Section 7, which contains a precise version of the main result, the computation of the region of convergence of the family of integrals (1.8) and the details about the connection of these integrals to closed superstring amplitudes. The next two sections, which constitute the technical heart of part II, are dedicated to the theory of multi-valued and single-valued hyperlogarithms, respectively. The new results that we obtain on single-valued hyperlogarithms play a crucial role in the last three sections of this part, which are dedicated to the proof of the main result. In particular, in Section 10 and Section 11 we explain in great details how to prove the statement for $k = 1$ and $k = 2$, respectively. This paves the way towards the proof of the general case, given in Section 12, where we omit some computations already discussed in the special cases. Finally, at the end of Section 12 and in Appendix D we explain how to adapt our methods to the superstring integrals (1.9).

PART I

SINGLE-VALUED CORRELATION FUNCTIONS

2. Single-valuedness of CFT correlators

Conformal field theories are local two dimensional quantum field theories. Correlation functions are vacuum expectation values of the product of dynamical composite fields (or vertex operators in string theory) $V_i(\boldsymbol{x})$ with $\boldsymbol{x} = (x, y) \in \mathbb{R}^2$ (see [53, 54] for some review and introduction)

$$\mathcal{G}(\boldsymbol{x}_1, \dots, \boldsymbol{x}_n) = \langle \prod_{i=1}^{n} V_i(\boldsymbol{x}_i) \rangle = \frac{1}{Z} \int \prod \mathcal{D}V \prod_{i=1}^{n} V_i(\boldsymbol{x}_i)\, e^{-S}\,. \tag{2.1}$$

The partition function is Z. The action S is a function of the elementary fields and of the two-dimensional metric. This metric is determined by the geometry of the Riemann surface Σ on which the theory is considered.

One axiom of conformal field theories is the requirement of single-valuedness of the correlation functions as functions of the positions $\boldsymbol{x}_i = (x_i, y_i)$ of the vertex operators in the euclidean plane $\mathbb{R}^2$. That means any physical correlators should not have monodromies when one varies the positions of a given operator around the position other operators.

After complexification one can consider that the composite fields are functions $V_i(z, \tilde{z})$ on $\mathbb{C} \times \mathbb{C}$. The correlation function becomes a multi-valued function of the doubled coordinates $\mathcal{G}(z_1, \dots, z_n, \tilde{z}_1, \dots, \tilde{z}_n)$ in $\mathbb{C}^n \times \mathbb{C}^n$. The euclidean real space is recovered when $\tilde{z}$ is identified with the complex conjugate of z, so that $V_i(z, \bar{z}) = V_i(x, y)$ and $z = x + iy$ and $\bar{z} = x - iy$ with $x, y \in \mathbb{R}$. It is only on the real slice that the correlation functions are single-valued functions free of monodromies.

This is particularly clear on the case of the two-point correlation functions determined by the $SL(2, \mathbb{C}) \times SL(2, \mathbb{C})$ Ward identities

$$\langle V_1(z_1, \tilde{z}_1) V_2(z_2, \tilde{z}_2) \rangle = \frac{\delta_{\Delta_1 = \Delta_2} \delta_{\tilde{\Delta}_1 = \tilde{\Delta}_2} N_{12}}{(z_1 - z_2)^{2\Delta_1} (\tilde{z}_1 - \tilde{z}_2)^{2\tilde{\Delta}_1}}\,, \tag{2.2}$$

between two fields of conformal dimensions[(1)] Δ_1 and Δ_2 and N_{12} is a constant determined by the normalisation of the fields. The correlation function is non-vanishing only for $\Delta_i = \tilde{\Delta}_i$ with $i = 1, 2$ as indicated by the kronecker delta functions. The three-point correlation function is as well determined by the Ward identity

$$\langle \prod_{i=1}^{3} V_i(z_i, \tilde{z}_i) \rangle = \frac{C_{123}}{(z_1 - z_2)^{\Delta_1+\Delta_2-\Delta_3}(z_1 - z_3)^{\Delta_1+\Delta_3-\Delta_2}(z_2 - z_3)^{\Delta_2+\Delta_3-\Delta_1}} \times \frac{1}{(\tilde{z}_1 - \tilde{z}_2)^{\tilde{\Delta}_1+\tilde{\Delta}_2-\tilde{\Delta}_3}(\tilde{z}_1 - \tilde{z}_3)^{\tilde{\Delta}_1+\tilde{\Delta}_3-\tilde{\Delta}_2}(\tilde{z}_2 - \tilde{z}_3)^{\tilde{\Delta}_2+\tilde{\Delta}_3-\tilde{\Delta}_1}}, \quad (2.3)$$

where C_{123} is a constant.

The single-valuedness of the correlation functions (2.2) and (2.3) on the real slice, $\tilde{z} = \bar{z}$, imposes the spins, given by the different of the conformal weights, $\Delta_i - \tilde{\Delta}_i$ to be integral. The single-valued condition does not determine the value of the constants N_{12} nor C_{123}.

The four points correlation function is a non-trivial function of the unique independent cross-ratio in two dimensions

$$\eta = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_3)(z_2 - z_4)} \quad (2.4)$$

and reads

$$\mathcal{G}_4(\eta, \bar{\eta}) = \langle \prod_{i=1}^{4} V_i(z_i, \tilde{z}_i) \rangle = \frac{G_{12|34}(\eta, \tilde{\eta})}{\prod_{1 \le i < j \le 4}(z_i - z_j)^{\Delta_i+\Delta_j-\Delta}(\tilde{z}_i - \tilde{z}_j)^{\tilde{\Delta}_i+\tilde{\Delta}_j-\tilde{\Delta}}} \quad (2.5)$$

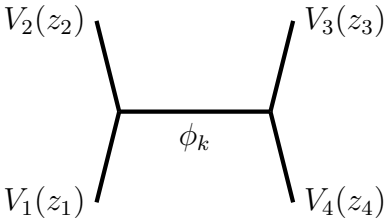
with $\Delta = \frac{1}{3}\sum_{i=1}^{4} \Delta_i$ and $\tilde{\Delta} = \frac{1}{3}\sum_{i=1}^{4} \tilde{\Delta}_i$.

By using the associativity of the operator product expansion one can expand the four-point function on the conformal blocks $\mathcal{F}_{1234}(k; \eta)$ and $\widetilde{\mathcal{F}}_{1234}(k; \tilde{\eta})$ as

$$G_{12|34}(\eta, \tilde{\eta}) = \sum_{k,\tilde{k}} G^{k,\tilde{k}}_{12|34} \mathcal{F}_{12|34}(k; \eta) \widetilde{\mathcal{F}}_{12|34}(\tilde{k}; \tilde{\eta}) \,. \quad (2.6)$$

[(1)] The conformal dimension is defined the exponents Δ and $\tilde{\Delta}$ under a coordinate transformation $V(z, \tilde{z}) \to \left(\frac{\partial f(z)}{\partial z}\right)^{\Delta} \left(\frac{\partial \tilde{f}(\tilde{z})}{\partial \tilde{z}}\right)^{\tilde{\Delta}} V(f(z), \tilde{f}(\tilde{z}))$.

With the holomorphic factorisation of the composite fields $V(z,\tilde{z}) = V(z) \times \tilde{V}(\tilde{z})$, the four-point conformal block arises from the exchange of the field $V_k(z)$ between the pair of fields (V_1, V_2) and (V_3, V_4). For this it necessary that the field V_k is in the operator product expansion of the fields $V_1(z_1)$ and $V_2(z_2)$ and in the operator product expansion of the fields $V_3(z_3)$ and $V_4(z_4)$. This can be graphically represented as

$\mathcal{F}_{1234}(k;\eta) =$ 

(2.7)

with an identical interpretation for the anti-holomorphic conformal block $\tilde{\mathcal{F}}_{1234}(\tilde{k};\tilde{\eta})$.

The four points conformal blocks $\mathcal{F}_{1234}(k;\eta)$ and $\tilde{\mathcal{F}}_{1234}(\tilde{k};\tilde{\eta})$ are not monodromy invariant as their expression changes when one moves η or $\tilde{\eta}$ in the complex plane around the points $\eta = 0$ and $\eta = 1$. The coefficients $G^{k,\tilde{k}}_{12|34}$ need to be such that the correlation function evaluated on the real slice $\tilde{\eta} = \bar{\eta}$, $\mathcal{G}_{12|34}(\eta,\bar{\eta})$ is free of monodromies. For the minimal model described by a Coulomb-gas, Dotsenko and Fateev in [39, 40] showed that the single-valuedness condition of the four-point correlation function determines the correlation functions up to an overall constant. The construction generalises to higher-point correlation functions.

We will apply the same logic to the correlations functions that evaluate to closed string partial amplitudes at a special value, but with the important difference that we will determine the overall constant. The multi-valued conformal blocks will be Aomoto-Gel'fand hypergeometric functions which reduce to the colour-ordered open string amplitudes at a special value.

3. Closed string theory amplitudes

Generic closed string tree-level amplitudes are finite linear combinations (with $N \geq 0$)

$$\mathcal{M}_{N+3}(\boldsymbol{s},\boldsymbol{\epsilon}) = \sum_r c_r(\boldsymbol{s},\boldsymbol{\epsilon})\, M_{N+3}(\boldsymbol{s},\boldsymbol{n}^r,\bar{\boldsymbol{n}}^r) \tag{3.1}$$

of the partial amplitudes [55–58]

$$M_{N+3}(\boldsymbol{s},\boldsymbol{n},\bar{\boldsymbol{n}}) = \int_{\mathbb{C}^N} \prod_{i=2}^{N+2} d^2w_i \prod_{2\leq i<j\leq N+1} |w_i{-}w_j|^{2\alpha' k_i\cdot k_j} (w_i{-}w_j)^{n_{ij}} (\bar{w}_i{-}\bar{w}_j)^{\bar{n}_{ij}}$$
$$\times \prod_{i=2}^{N+1} |w_i|^{2\alpha' k_1\cdot k_i} |1-w_i|^{2\alpha' k_i\cdot k_{N+2}} w_i^{n_{1i}} \bar{w}_i^{\bar{n}_{1i}} (1-w_i)^{n_{i\,(N+2)}} (1-\bar{w}_i)^{\bar{n}_{i\,(N+2)}} \,. \tag{3.2}$$

With have fixed the conformal gauge by setting $w_1 = 0$, $w_{N+2} = 1$ and $w_{N+3} = \infty$. The exponents $\boldsymbol{n} := \{n_{ij}\}_{1\leq i<j\leq N+3}$ and $\bar{\boldsymbol{n}} := \{\bar{n}_{ij}\}_{1\leq i<j\leq N+3}$ are integers. The set of kinematic invariants $\boldsymbol{s} := \{s_{ij} = 2\alpha' k_i{\cdot}k_j\}_{1\leq i\leq j\leq N+3}$ is given by the scalar product of the external momenta k_i, which are vectors of the D-dimensional Minkowsky space-time $\mathbb{R}^{1,D-1}$ with metric $(+-\cdots-)$. The external momenta satisfy the conservation condition

$$k_1 + \cdots + k_{N+3} = 0 \tag{3.3}$$

and the on-shell conditions $\alpha' k_i^2 \in -2 + \mathbb{N}$.

The amplitudes are defined in any space-time dimension $D \leq 26$ for the bosonic string theory and $D \leq 10$ for the superstring theory and heterotic strings.

For instance, the simplest tree-level bosonic closed string theory amplitude is that between N-tachyon fields which is given by [55–58]

$$\mathcal{M}_{N+3}^{\text{tachyon}}(\boldsymbol{s}) = \int \prod_{1\leq i<j\leq N} |z_i - z_j|^{2\alpha' k_i\cdot k_j} \prod_{i=2}^{N+1} d^2z_i \tag{3.4}$$

with $\alpha' k_i^2 = -2$ for $1 \leq i \leq N+3$.

The coefficients $c_r(\boldsymbol{s},\boldsymbol{\epsilon})$ are rational functions in the kinematic invariants $\boldsymbol{s} = (2\alpha' k_i \cdot k_j)_{1\leq i\leq j\leq N}$, the product of the external momenta and the polarisation vectors $(\sqrt{\alpha'} k_i \cdot \epsilon_j)_{1\leq i<j\leq N}$ and the product of the polarisation vectors $(\epsilon_i \cdot \epsilon_j)_{1\leq i<j\leq N}$. For the heterotic string with external gauge fields they would depend as well on the colour factors through a product of traces. The precise form of the coefficients $c_r(\boldsymbol{s},\boldsymbol{\epsilon})$ depends on the closed string theory one considers—i.e. the bosonic string, or the type-II superstring or the heterotic string—and the spin of the external particles and traces over the colour factors for the heterotic string. Expression for the four-point amplitude can be found in e.g [55, 56]. For higher points open superstring amplitudes a pure spinor formalism based derivation of the polarisation tensors has been given in [50]. For open and closed bosonic string and heterotic string in [36].

3.1. Closed strings from single-valued correlation function. — The closed string theory partial amplitudes (3.2) are only functions of kinematic parameters, they do not depend on the position of the vertex operators on the world-sheet. Therefore we consider the generalisation by introducing a dependence on the complex variable z

$$\mathcal{G}_N(z,\bar{z}) := \int_{\mathbb{C}^N} \prod_{i=1}^{N} w_i^{a_i}(w_i-1)^{b_i}(w_i-z)^{c_i}\bar{w}_i^{\bar{a}_i}(\bar{w}_i-1)^{\bar{b}_i}(\bar{w}_i-\bar{z})^{\bar{c}_i} \times \prod_{1\leq i<j\leq N}(w_i-w_j)^{g_{ij}}(\bar{w}_i-\bar{w}_j)^{\bar{g}_{ij}} \prod_{i=1}^{N} d^2w_i\,. \quad (3.5)$$

where the exponents are given by

$$\begin{aligned}
&\boldsymbol{a} = (A_i + n_i, 1\leq i\leq N), && \bar{\boldsymbol{a}} = (A_i + \bar{n}_i, 1\leq i\leq N), && n_i,\bar{n}_i \in \mathbb{Z},\\
&\boldsymbol{b} = (B_i + m_i, 1\leq i\leq N), && \bar{\boldsymbol{b}} = (B_i + \bar{m}_i, 1\leq i\leq N), && m_i,\bar{m}_i \in \mathbb{Z},\\
&\boldsymbol{c} = (C_i + p_i, 1\leq i\leq N), && \bar{\boldsymbol{c}} = (C_i + \bar{p}_i, 1\leq i\leq N), && p_i,\bar{p}_i \in \mathbb{Z},\\
&\boldsymbol{g} = (G_{ij} + q_{ij}, 1\leq i<j\leq N), && \bar{\boldsymbol{g}} = (G_{ij} + \bar{q}_{ij}, 1\leq i<j\leq N), && q_{ij},\bar{q}_{ij} \in \mathbb{Z}.
\end{aligned} \quad (3.6)$$

We assume that $\boldsymbol{A} := (A_1, \ldots, A_N) \in \mathbb{R}^N$, $\boldsymbol{B} := (B_1, \ldots, B_N) \in \mathbb{R}^N$, $\boldsymbol{C} := (C_1, \ldots, C_N) \in \mathbb{R}^N$ and $\boldsymbol{G} := (G_{ij}, 1 \leq i < j \leq N) \in \mathbb{R}^{\frac{N(N-1)}{2}}$ are real numbers. It is important for the single-valuedness of the correlation function and the string theory amplitudes that the difference of the exponents between the holomorphic and anti-holomorphic factors are integers (see Remark 4.5).

The correlation function $\mathcal{G}_N(z, \bar{z})$ when evaluated at $z = \bar{z} = 1$ matches with the partial amplitudes of any closed string tree-level amplitudes in (3.2)

$$M_{N+3}(\boldsymbol{s}, \boldsymbol{n}, \bar{\boldsymbol{n}}) = \mathcal{G}_N(1, 1)\,, \tag{3.7}$$

with the following identifications of the parameters

$$\begin{aligned} \boldsymbol{A} &:= \{\alpha' k_1 \cdot k_i, \quad 2 \leq i \leq N+1\}, \\ \boldsymbol{B} + \boldsymbol{C} &:= \{\alpha' k_{N-1} \cdot k_i, \quad 2 \leq i \leq N+1\}, \\ \boldsymbol{G} &:= \{\alpha' k_i \cdot k_j, \quad 2 \leq i < j \leq N+1\}, \end{aligned} \tag{3.8}$$

and the momentum conversation condition (3.3). For the value at $z = 1$ only the combination (3.8) matters, and the freedom in choosing the values for $b_i, c_i, \bar{b}_i$ and $\bar{c}_i$ disappears. These generalisations are needed in order to get a single-valued function of z that reduces to the closed string partial amplitudes.

As explained in the introduction it is important for the argument that the functions in (3.5) and (3.9) are valid conformal field theory correlation functions. We obtain (3.5) as the correlation function of $N+1$ vertex operators

$$\mathcal{G}_N(z, \bar{z}) = \int_{\mathbb{C}^N} \prod_{i=2}^{N+1} d^2 w_i \langle \prod_{i=1}^{N} V_i(w_i, \bar{w}_i) V_1(0) V_{N+2}(1) V_{N+3}(\infty) U(z, \bar{z}) \rangle \tag{3.9}$$

between N integrated physical vertex operators $V_i(w_i, \bar{w}_i)$ and one auxiliary unintegrated vertex operator $U(z, \bar{z})$. As usual for a correlation function on the sphere we fix three points $w_1 = 0$, $w_{N+2} = 1$ and $w_{N+3} = \infty$, but we keep z as a free complex variable.

We take for auxiliary vertex operator the simple plane-wave tachyonic vector operator

$$U(z, \bar{z}) =: e^{ik_* \cdot X(z,\bar{z})} : \ . \tag{3.10}$$

The generic form in (3.5) can be obtained by considering the operator product expansion of $U(z, \bar{z})$ with the vertex operator, gauged-fixed at the position $z_{N-1} = 1$:

$$V_{N-1} =: h_{\mu_1 \cdots \mu_r \nu_1 \cdots \nu_s} \prod_{i=1}^{r} \partial X^{\mu_i}(1) \prod_{j=1}^{s} \bar{\partial} X^{\nu_j}(1) e^{ik_{N-1} \cdot X(1,1)} : \tag{3.11}$$

for a higher-spin field in bosonic string. We choose the tensors $h_{\mu_1 \cdots \mu_r \nu_1 \cdots \nu_s}$ so that contractions between the vertex operators singles out the contribution in $\mathcal{G}_N(z, \bar{z})$. The momentum k_* enters only for $z \neq 1$ as it is only needed to define $\boldsymbol{B}$ and $\boldsymbol{C}$ in (3.8), as $\boldsymbol{B} = \{2\alpha' k_{N-1} \cdot (k_i - k_*), i = 2, \ldots, N-2\}$ and $\boldsymbol{C} = \{2\alpha' k_{N-1} \cdot k_*, i = 2, \ldots, N-2\}$ but disappear in the final result.

4. Single-valued correlation functions

A correlation function in a conformal field theory (3.5) is a single-valued function of z in the complex plane [54]. In this section we derive the conditions for the single-valuedness on the function $\mathcal{G}_N(z, \bar{z})$. We assume that the integral converges. For considerations on the domain of convergence we refer to §7.1.

4.1. Aomoto-Gel'fand hypergeometric functions. — if X_n is a set of indices $\{1, \ldots, n\}$ of cardinality n, we consider for $r + s = N$ all inclusions $\sigma : X_r \hookrightarrow X_N$ and $\rho : X_s \hookrightarrow X_N$ such that $X_N = \sigma(X_r) \sqcup \rho(X_s)$. We denote by $\mathfrak{S}_r$ the set of permutations of r elements. From now

on, by abuse of notation, we will consider σ and ρ also as elements of $\mathfrak{S}_r$ and $\mathfrak{S}_s$, respectively. We introduce a special case of the Aomoto-Gel'fand hypergeometric functions [43, 44] for $(\sigma,\rho)\in\mathfrak{S}_r\times\mathfrak{S}_s$

$$I_{(\sigma,\rho)}(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c};\boldsymbol{g};z)=\int_{\Delta(\sigma,\rho)}\prod_{m=1}^{r}\prod_{n=1}^{s}(w_{\sigma(m)}-w_{\rho(n)})^{g_{\sigma(m)\sigma(n)}}\times\prod_{1\le m<n\le r}(w_{\sigma(n)}-w_{\sigma(m)})^{g_{\sigma(n)\sigma(m)}}\prod_{1\le m<n\le s}(w_{\rho(n)}-w_{\rho(m)})^{g_{\rho(n)\rho(m)}}\times\prod_{m=1}^{r}w_{\sigma(m)}^{a_{\sigma(m)}}(w_{\sigma(m)}-1)^{b_{\sigma(m)}}(w_{\sigma(m)}-z)^{c_{\sigma(m)}}\times\prod_{n=1}^{s}w_{\rho(n)}^{a_{\rho(n)}}(1-w_{\rho(n)})^{b_{\rho(n)}}(z-w_{\rho(n)})^{c_{\rho(n)}}\prod_{j=1}^{N}dw_j\,,\quad(4.1)$$

integrated over the simplex

$$\Delta(\sigma,\rho):=\{0\le w_{\rho(1)}\le\cdots\le w_{\rho(s)}\le z\le 1\le w_{\sigma(1)}\le\cdots\le w_{\sigma(r)}\}\,,\quad(4.2)$$

and the integrals[(2)] $J_{(\sigma,\rho)}(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c};\boldsymbol{g};z):=I_{(\sigma,\rho)}(\boldsymbol{b},\boldsymbol{a},\boldsymbol{c};\boldsymbol{g};1-z)$

$$J_{(\sigma,\rho)}(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c};\boldsymbol{g};z)=\int_{\tilde\Delta(\sigma,\rho)}\prod_{m=1}^{r}\prod_{n=1}^{s}(w_{\sigma(n)}-w_{\rho(m)})^{g_{\sigma(n)\rho(m)}}\times\prod_{1\le m<n\le r}(w_{\sigma(m)}-w_{\sigma(n)})^{g_{\sigma(m)\sigma(n)}}\prod_{1\le m<n\le s}(w_{\rho(n)}-w_{\rho(m)})^{g_{\rho(n)\rho(m)}}\times\prod_{m=1}^{r}(-w_{\sigma(m)})^{a_{\sigma(m)}}(1-w_{\sigma(m)})^{b_{\sigma(m)}}(z-w_{\sigma(m)})^{c_{\sigma(m)}}\times\prod_{n=1}^{s}(w_{\rho(n)})^{a_{\rho(n)}}(1-w_{\rho(n)})^{b_{\rho(n)}}(w_{\rho(n)}-z)^{c_{\rho(n)}}\prod_{j=1}^{N}dw_j\,,\quad(4.3)$$

integrated over the simplex

$$\tilde\Delta(\sigma,\rho):=\{w_{\sigma(1)}\le\cdots\le w_{\sigma(r)}\le 0\le z\le w_{\rho(1)}\le\cdots\le w_{\rho(s)}\le 1\}\,.\quad(4.4)$$

(2) Since the notation is very similar, we warn the reader that these integrals should not be confused with the closed superstring integrals $J_{\rho,\sigma}$ defined by eq. (1.9) and considered in part II of the paper.

See §5.1 for the integrals relevant for the four-point amplitudes and §5.2 for the integral relevant for the five-point amplitudes.

Remark 4.1. — *There are* $(N+1)!$ *such integral functions: for given* $r \geq 0$ *and* $s = N - r \geq 0$ *because there are* $r!$ *permutations* σ *and* $s!$ *permutations* ρ *therefore there are* $r! \times s! \times \binom{N}{r} = N!$ *ordered integrals. The total number of integrals is* $N! \times (N+1) = (N+1)!$. *A quicker way to obtain this counting is to realise that the set* $\alpha = \{\rho(1), \ldots, \rho(s), N+1, \sigma(1), \ldots, \sigma(r)\}$ *runs over all permutations of* $\{1, \ldots, N+1\}$ *as the permutations* ρ *and* σ *vary. These integral functions can therefore be labelled with permutations* $\alpha \in \mathfrak{S}_{N+1}$ *as* $I_\alpha(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}; \boldsymbol{g}; z) = I_{(\sigma,\rho)}(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}; \boldsymbol{g}; z)$ *and* $J_\alpha(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}; \boldsymbol{g}; z) = J_{(\sigma,\rho)}(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}; \boldsymbol{g}; z)$.

Remark 4.2. — *These integrals functions are not all independent as there exist linear combinations between them. The dimension of the linear system is* $(N+1)!$ *as shown in [32, 41]. However, in the construction of the single valued correlation function it is preferable to use this larger set of integral functions.*

Remark 4.3. — *Aomoto-Gel'fand hypergeometric functions include the auxiliary function* $F(z_0, s_{0k})$ *used in [59]. The multi-valued conformal blocks of §2 entering string theory amplitude and Coulomb-gas correlation functions are special cases of Aomoto-Gel'fand hypergeometric functions.*

4.2. Change of basis. — We derive the linear transformation between the two set of integral functions

$$I_\alpha(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}; \boldsymbol{g}; z) = \sum_{\beta \in \mathfrak{S}_{N+1}} S(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}; \boldsymbol{G})_\alpha{}^\beta J_\beta(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}; \boldsymbol{g}; z), \ , \tag{4.5}$$

where we made use of the notation introduced in Remark 4.1.

We will show that the elements of the matrix $S(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}; \boldsymbol{G})$ are rational functions of sines of linear combinations of elements of the sets $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$

and $\boldsymbol{G}$.[3] The transformation is invertible with

$$S(\boldsymbol{A},\boldsymbol{B},\boldsymbol{C};\boldsymbol{G})^{-1} = S(\boldsymbol{B},\boldsymbol{A},\boldsymbol{C};\boldsymbol{G})\,, \tag{4.6}$$

because $J_\alpha(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c};\boldsymbol{g};z) = I_\alpha(\boldsymbol{b},\boldsymbol{a},\boldsymbol{c};\boldsymbol{g};1-z)$ for all $\alpha \in \mathfrak{S}_{N+1}$.

This relation is obtained as an application of the contour deformation method used in [39–41]. We introduce the notation for the integral functions

$$I(\alpha;\beta;\gamma;\delta) = \int_{\Delta(\alpha;\beta;\gamma;\delta)} f(\alpha;\beta;\gamma;\delta) \tag{4.7}$$

where the set of indices are $\alpha := (i_1,\ldots,i_\alpha)$, $\beta := (j_1,\ldots,j_\beta)$, $\gamma := (k_1,\ldots,k_\gamma)$ and $\delta := (l_1,\ldots,l_\delta)$ and

$$\begin{aligned}
f(\alpha;\beta;\gamma;\delta) := & \prod_{1\le r<s\le\alpha}(w_{i_s}-w_{i_r})^{g_{i_s i_r}} \prod_{1\le r<s\le\beta}(w_{j_s}-w_{j_r})^{g_{j_s j_r}} \\
& \times \prod_{1\le r<s\le\gamma}(w_{k_s}-w_{k_r})^{g_{k_s k_r}} \prod_{1\le r<s\le\delta}(w_{l_s}-w_{l_r})^{g_{l_s l_r}} \\
\times \prod_{r=1}^{\alpha}\prod_{s=1}^{\beta}\prod_{t=1}^{\gamma}\prod_{u=1}^{\delta} & (w_{l_u}-w_{i_r})^{g_{l_u i_r}}(w_{l_u}-w_{j_s})^{g_{l_u j_s}}(w_{l_u}-w_{k_t})^{g_{l_u k_t}}(w_{k_t}-w_{i_r})^{g_{k_t i_r}} \\
& \times (w_{k_t}-w_{j_s})^{g_{k_t j_s}}(w_{j_s}-w_{i_r})^{g_{j_s i_r}} \\
& \times \prod_{r=1}^{\alpha}(-w_{i_r})^{a_{i_r}} \prod_{r=1}^{\beta} w_{j_r}^{a_{j_r}} \prod_{r=1}^{\gamma} w_{k_r}^{a_{k_r}} \prod_{r=1}^{\delta} w_{l_r}^{a_{l_r}} \\
\times \prod_{r=1}^{\alpha}(1-w_{i_r})^{b_{i_r}} & \prod_{r=1}^{\beta}(1-w_{j_r})^{b_{j_r}} \prod_{r=1}^{\gamma}(1-w_{k_r})^{b_{k_r}} \prod_{r=1}^{\delta}(w_{l_r}-1)^{b_{l_r}} \\
\prod_{r=1}^{\alpha}(z-w_{i_r})^{c_{i_r}} & \prod_{r=1}^{\beta}(z-w_{j_r})^{c_{j_r}} \prod_{r=1}^{\gamma}(w_{k_r}-z)^{c_{k_r}} \prod_{r=1}^{\delta}(w_{l_r}-z)^{c_{l_r}}
\end{aligned} \tag{4.8}$$

for the ordered variables

$$\begin{aligned}
\Delta(\alpha;\beta;\gamma;\delta) := \{ & w_{i_1} \le \cdots \le w_{i_\alpha} \le 0 \le w_{j_1} \le \cdots \le w_{j_\beta} \le z \le w_{k_1} \le \\
& \cdots \le w_{k_\gamma} \le 1 \le w_{l_1} \le \cdots \le w_{l_\delta}\}\,.
\end{aligned} \tag{4.9}$$

[3] The relation between this matrix and the string theory momentum kernel which relates the different order open string amplitudes [42] is discussed in §6.

We remark that this ordered integral is equivalent to integral over nested contours of integrations. The notation $C_x^{[a,b]} \succ C_y^{[a,b]}$ means that the contour C_x lies above the contour C_y and start at the point a and end at the point b. Once projected on the real line, $C_x^{[a,b]} \succ C_y^{[a,b]}$ implies that $a \leq x \leq y \leq b$. The contours are all ordered in the upper half-plane or the lower-half plane. The integral functions (4.1) correspond to the case $I(\emptyset;\alpha;\emptyset;\beta)$ and the integral functions (4.3) correspond to the case $I(\alpha;\emptyset;\beta;\emptyset)$ respectively.

Starting with $I(\emptyset;\alpha;\emptyset;\beta)$ with $\beta = (j_1,\ldots,j_\beta)$. The contours of integration for this set are nested as follows $C_{j_1} \succ C_{j_2} \succ \cdots \succ C_{j_\beta}$. If the contours are ordered in the upper half-plane one can rotate C_{j_1} counter-clockwise in the upper half-plane. If the contours are ordered in the lower half-plane one can rotate C_{j_1} clockwise in the lower half-plane. For $\alpha \in \mathbb{C}$ and x a real number we fix the following determination of the logarithm:

$$(-|x|)^\alpha = |x|^\alpha \times \begin{cases} e^{+i\pi\alpha} & \Im\mathrm{m}(x) \geq 0 \\ e^{-i\pi\alpha} & \Im\mathrm{m}(x) < 0 \end{cases} . \tag{4.10}$$

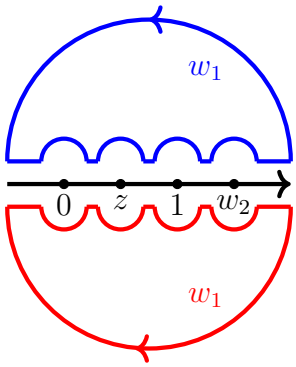


FIGURE 4.1. Contours deformation for the integration over w_1. The rotation of the contour in the upper half-plane (blue contour) leads to the relation (4.11) with the + sign in the phase factors, and the rotation in the lower half-plane (red contour) leads to the relation (4.11) with the − sign in the phase factors.

Setting $\beta = (j_1, \beta')$ we get from a rotation of the contour C_{i_1} in either the upper half-plane or the lower half-plane

$$I(\emptyset;\alpha;\emptyset;\beta) + e^{\pm i\pi B_{j_1}} I(\emptyset;\alpha;j_1;\beta') + e^{\pm i\pi(B_{j_1}+C_{j_1})} I(\emptyset;(j_1,\alpha);\emptyset;\beta') + e^{\pm i\pi(A_{j_1}+B_{j_1}+C_{j_1})} I(j_1;\alpha;\emptyset;\beta') = 0\,. \quad (4.11)$$

Notice that the phase factors only depend on the complex numbers A_i, B_i and C_i and do not depend on the integers n_{ij} which do not contribute to the branch cuts of the integrand. This implies that

$$I(\emptyset;\alpha;\emptyset;\beta) = \frac{\sin(\pi C_{j_1})}{\sin(\pi(B_{j_1}+C_{j_1}))} I(\emptyset;\alpha;i_1;\beta') - \frac{\sin(\pi A_{j_1})}{\sin(\pi(B_{j_1}+C_{j_1}))} I(i_1;\alpha;\emptyset;\beta')\,. \quad (4.12)$$

By iterating the procedure we rotate all the contours in the β set, and we find that

$$I(\emptyset;\alpha;\emptyset;\beta) = \sum_{\sigma,\rho} c_{\sigma,\rho} I(\sigma;\alpha;\rho;\emptyset)\,, \quad (4.13)$$

where the coefficients $c_{\sigma,\rho}$ are real number given by rational functions of sines, and the sum is over the partitions of the set $\beta = \sigma \cup \rho$ into two set of indices $\sigma = (i_r, \ldots, i_i)$ and $\rho = (i_\beta, \ldots, i_{r+1})$.

Similarly setting $\alpha = (i_1, \alpha')$ we have

$$I(\sigma;\alpha';\rho;i_1) + e^{\pm i\pi B_{i_1}} I(\sigma;\alpha';(i_1,\rho);\emptyset) + e^{\pm i\pi(B_{i_1}+C_{i_1})} I(\sigma;\alpha;\rho;\emptyset) + e^{\pm i\pi(A_{i_1}+B_{i_1}+C_{i_1})} I((\sigma,i_1);\alpha';\rho;\emptyset) = 0\,, \quad (4.14)$$

which implies that

$$I(\sigma;\alpha;\rho;\emptyset) = -\frac{\sin(\pi B_1)}{\sin(\pi(B_1+C_1))} I(\sigma;\alpha';(i_1,\rho);\emptyset) - \frac{\sin(\pi(A_1+B_1+C_1))}{\sin(\pi(B_1+C_1))} I((i_1,\sigma);\alpha';\rho;\emptyset)\,. \quad (4.15)$$

By recursively moving all the contour of integration from the set α we get that

$$I(\sigma;\alpha;\rho;\emptyset) = \sum_{\gamma,\eta} \tilde{c}_{\gamma,\eta} I(\gamma;\emptyset;\eta;\emptyset)\,. \tag{4.16}$$

Combining the relations (4.13) and (4.16) gives the relation between the two set of integral functions in (4.5).

Remark 4.4. — *The above discussion implies that the matrix* $S(\boldsymbol{A},\boldsymbol{B},\boldsymbol{C};\boldsymbol{G})$ *has for element rational functions of* $\sin(\pi x)$ *where* x *are linear combination of* A_i, B_i, C_i *and* G_{ij} *with coefficients in* $\{-1,0,1\}$.

We give some examples of such matrices in Appendix A for $N=1$ and Appendix B for $N=2$.

4.3. Monodromy operators. — We consider the monodromy of the Aomoto-Gel'fand hypergeometric functions around $z=0$ and $z=1$. The analytic continuation of these integrals along paths around $z=0$ and $z=1$ lead to monodromies §8.2.

We consider the closed loop γ_0 around the point $z=0$ with based point z_*. The circulation of the Aomoto-Gel'fand hypergeometric functions $I_\alpha(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c};\boldsymbol{g};z)$ around the loop lead to a monodromy matrix g_0. Using the notation of Remark 4.1,

$$I_\alpha(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c};\boldsymbol{g};z) \xrightarrow{\gamma_0} \sum_{\beta\in\mathfrak{S}_{N+1}} (g_0)_\alpha{}^\beta I_\beta(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c};\boldsymbol{g};z). \tag{4.17}$$

Likewise for a closed loop γ_1 around the point $z=1$ with based point z_*: the circulation of the Aomoto-Gel'fand hypergeometric functions $I_\alpha(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c};\boldsymbol{g};z)$ around this loop leads to a monodromy matrix g_1. Using the notation of Remark 4.1,

$$I_\alpha(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c};\boldsymbol{g};z) \xrightarrow{\gamma_1} \sum_{\beta\in\mathfrak{S}_{N+1}} (g_1)_\alpha{}^\beta I_\beta(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c};\boldsymbol{g};z). \tag{4.18}$$

It is a classical theorem that the monodromy group does not depend on the base point of z_* of the loop.

4.3.1. The monodromy around $z = 0$. — The monodromy matrix around $z = 0$ of the integral functions in (4.1) is given by a diagonal matrix g_0 of size $(N+1)!$ with diagonal elements

$$(g_0)^{(\sigma,\rho)}_{(\sigma,\rho)} = \prod_{m=1}^{s} \exp\left(A_{\rho(m)} + C_{\rho(m)}\right) \times \prod_{1\leq m<n\leq s} \exp\left(G_{\rho(m)\rho(n)}\right) , \quad (4.19)$$

where the σ runs over the permutations of the r elements and ρ runs over the permutations of the s elements such that $\{\sigma(1),\dots,\sigma(r)\} \cup \{\rho(1),\dots,\rho(s)\} = \{1,\dots,N\}$ and $r+s=N$ with $r,s\geq 0$.

This is easily seen by performing the change of variable $w_{\rho(m)} = z\hat{w}_{\rho(m)}$ for $1\leq m\leq s$: one gets

$$\begin{aligned} I_{(\sigma,\rho)}(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c};\boldsymbol{g};z) = \prod_{m=1}^{s} z^{1+a_{\rho(m)}+c_{\rho(m)}} \prod_{1\leq m<n\leq s} z^{g_{\rho(m)\rho(n)}} \int_{\delta(\sigma,\rho)} \prod_{m=1}^{r} dw_{\sigma(m)} \prod_{n=1}^{s} d\hat{w}_{\rho(n)} \\ \times \prod_{m=1}^{r} w_{\sigma(m)}^{a_{\sigma(m)}} (1-w_{\sigma(m)})^{b_{\sigma(m)}} \prod_{1\leq m<n\leq r} |w_{\sigma(m)} - w_{\sigma(n)}|^{g_{\sigma(m)\sigma(n)}} \\ \times \prod_{m=1}^{s} \hat{w}_{\rho(m)}^{a_{\rho(m)}} (1-\hat{w}_{\rho(m)})^{b_{\rho(m)}} \prod_{1\leq m<n\leq s} |\hat{w}_{\rho(m)} - \hat{w}_{\rho(n)}|^{g_{\rho(m)\rho(n)}} \\ \times \prod_{m=1}^{r}\prod_{n=1}^{s} (w_{\sigma(m)} - z\hat{w}_{\rho(n)})^{g_{\sigma(m)\rho(n)}} , \quad (4.20) \end{aligned}$$

integrated over the simplex

$$\delta(\rho,\sigma) := \{0\leq w_{\sigma(1)}\leq\cdots\leq w_{\sigma(r)}\leq 1\leq w_{\rho(1)}\leq\cdots\leq w_{\rho(s)}\leq 1\} . \quad (4.21)$$

Only the powers of z in front of the integral give monodromies when z makes a loop around $z=0$. Since $\boldsymbol{a} = \boldsymbol{A}+\mathbb{Z}$, $\boldsymbol{c} = \boldsymbol{C}+\mathbb{Z}$ and $\boldsymbol{g} = \boldsymbol{G}+\mathbb{Z}$ this proves the claim (4.19).

4.3.2. The monodromy around $z = 1$. — The monodromy matrix around $z=1$ is not diagonal for the integral functions (4.1) but it is diagonal for the other set of integrals in (4.3).

The monodromy matrix around $z=1$ of the integral functions in (4.3) is given by a diagonal matrix g_1 of size $(N+1)!$ with diagonal elements

$$(g_1)^{(\sigma,\rho)}_{(\sigma,\rho)} = \prod_{m=1}^{s} \exp\left(B_{\rho(m)} + C_{\rho(m)}\right) \times \prod_{1\leq m<n\leq s} \exp\left(G_{\rho(m)\rho(n)}\right) , \quad (4.22)$$

where σ runs over the permutations of the r elements and ρ runs over the permutations of the s elements such that $\{\sigma(1),\dots,\sigma(r)\}\cup\{\rho(1),\dots,\rho(s)\} = \{1,\dots,N\}$ and $r+s=N$.

Since $J_{(\sigma,\rho)}(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c};\boldsymbol{g};z) = I_{(\sigma,\rho)}(\boldsymbol{b},\boldsymbol{a},\boldsymbol{c};\boldsymbol{g};1-z)$ the derivation of the monodromy operator is similar to the one given previously. Performing the change of variable $w_{\rho(m)} = (1-z)\hat{w}_{\rho(m)}$ for $1\leq m\leq s$ one gets

$$\begin{aligned} J_{(\sigma,\rho)}(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c};\boldsymbol{g};1-z) = {} & \prod_{m=1}^{s}(1-z)^{1+b_{\rho(m)}+c_{\rho(m)}} \prod_{1\leq m<n\leq s}(1-z)^{g_{\rho(m)\rho(n)}} \\ & \int_{\delta(\sigma,\rho)} \prod_{m=1}^{r} dw_{\sigma(m)} \prod_{n=1}^{s} d\hat{w}_{\rho(n)} \prod_{m=1}^{r}\prod_{n=1}^{s}(w_{\sigma(m)} - z\hat{w}_{\rho(n)})^{g_{\sigma(m)\rho(n)}} \\ & \times \prod_{m=1}^{r} w_{\sigma(m)}^{b_{\sigma(m)}}(1-w_{\sigma(m)})^{a_{\sigma(m)}} \prod_{1\leq m<n\leq r}|w_{\sigma(m)} - w_{\sigma(n)}|^{g_{\sigma(m)\sigma(n)}} \\ & \times \prod_{m=1}^{s} \hat{w}_{\rho(m)}^{b_{\rho(m)}}(1-\hat{w}_{\rho(m)})^{a_{\rho(m)}} \prod_{1\leq m<n\leq s}|\hat{w}_{\rho(m)} - \hat{w}_{\rho(n)}|^{g_{\rho(m)\rho(n)}} , \quad (4.23) \end{aligned}$$

integrated over the simplex (4.21). Only the powers of z in front of the integral give monodromies when z makes a loop around $z=1$. Since $\boldsymbol{a} = \boldsymbol{A}+\mathbb{Z}$, $\boldsymbol{c} = \boldsymbol{C}+\mathbb{Z}$ and $\boldsymbol{g} = \boldsymbol{G}+\mathbb{Z}$ this proves the claim (4.22).

Remark 4.5. — *Because the differences between the parameters of the integral functions are integers* $\boldsymbol{a}-\bar{\boldsymbol{a}}\in\mathbb{Z}$, $\boldsymbol{b}-\bar{\boldsymbol{b}}\in\mathbb{Z}$, $\boldsymbol{c}-\bar{\boldsymbol{c}}\in\mathbb{Z}$, $\boldsymbol{g}-\bar{\boldsymbol{g}}\in\mathbb{Z}$, *the holomorphic integral functions* $I_{(\sigma,\rho)}(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c};\boldsymbol{g};z)$ *and anti-holomorphic integral functions* $I_{(\sigma,\rho)}(\bar{\boldsymbol{a}},\bar{\boldsymbol{b}},\bar{\boldsymbol{c}};\bar{\boldsymbol{g}};\bar{z})$ *have the same monodromies around* $z=0$ *and* $z=1$. *This is be a important for constructing single-valued correlation functions from bilinear in these integrals in the following sections.*

4.4. Holomorphic factorisation. — We show that the integral function $\mathcal{G}_N(z,\bar{z})$ in (3.5) can be expressed as the sesquilinear combination

$$\mathcal{G}_N(z,\bar{z}) = \sum_{(\alpha,\beta)\in\mathfrak{S}_{N+1}\times\mathfrak{S}_{N+1}} G_{\alpha,\beta} I_\alpha(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c};\boldsymbol{g};z)\, I_\beta(\bar{\boldsymbol{a}},\bar{\boldsymbol{b}},\bar{\boldsymbol{c}};\bar{\boldsymbol{g}};\bar{z}) \qquad (4.24)$$

of the multi-valued Aomoto-Gel'fand hypergeometric functions in (4.1). In this expression we are using the notation of Remark 4.1 with $\alpha = \{\rho(1),\dots,\rho(s),N+1,\sigma(1),\dots,\sigma(r)\}$ with $r+s=N$ and $r,s\geq 0$.

We have the equivalent expression, using the other set of Aomoto-Gel'fand hypergeometric functions (4.3)

$$\mathcal{G}_N(z,\bar{z}) = \sum_{(\alpha,\beta)\in\mathfrak{S}_{N+1}\times\mathfrak{S}_{N+1}} \hat{G}_{\alpha,\beta} J_\alpha(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c};\boldsymbol{g};z)\, J_\beta(\bar{\boldsymbol{a}},\bar{\boldsymbol{b}},\bar{\boldsymbol{c}};\bar{\boldsymbol{g}};\bar{z})\,. \qquad (4.25)$$

We follow the constructive proof given in [42]. If $w_r = w_r^{(1)} + i w_r^{(2)}$, with $w_r^{(1)}$ and $w_r^{(2)}$ real numbers, we rotate the contour of integration for $w_r^{(2)}$ contour-clockwise by performing the change of variable $w_r^{(2)} = i e^{-2i\epsilon} y_r$ with $\epsilon > 0$ and we preserve the contour of integration on the real axis for $w_r^{(1)} = x_r$. For $\epsilon \ll 1$ we have

$$w_r = x_r - e^{-2i\epsilon} y_r = x_r - y_r + 2i\epsilon y_r + O(\epsilon^2)\,. \qquad (4.26)$$

Let us introduce the notation

$$v_r^{\pm} := x_r \pm y_r; \qquad \delta_r := v_r^+ - v_r^- \qquad (4.27)$$

so that

$$w_r = v_r^- + i\epsilon\delta_r; \qquad \bar{w}_r = v_r^+ - i\epsilon\delta_r\,. \qquad (4.28)$$

With these new variables the integral in (3.5) reads

$$\begin{aligned}\mathcal{G}_N^\epsilon(z,\bar{z}) = \int_{\mathbb{R}^{2N}} \prod_{r=1}^N dv_r^+ dv_r^- \prod_{r=1}^N (v_r^- + i\epsilon\delta_r)^{a_r} (v_r^+ - i\epsilon\delta_r)^{\bar{a}_r} \\ \times \prod_{r=1}^N (v_r^- - 1 + i\epsilon\delta_r)^{b_r} (v_r^+ - 1 - i\epsilon\delta_r)^{\bar{b}_r} (v_r^- - z + i\epsilon\delta_r)^{c_r} (v_r^+ - \bar{z} - i\epsilon\delta_r)^{\bar{c}_r} \\ \times \prod_{1\leq r<s\leq N} (v_r^- - v_s^- + i\epsilon(\delta_r - \delta_s))^{g_{rs}} (v_r^+ - v_s^+ - i\epsilon(\delta_r - \delta_s))^{\bar{g}_{rs}}\,. \qquad (4.29)\end{aligned}$$

The integral we want to evaluate is obtained as $\mathcal{G}_N(z,\bar{z}) = \lim_{\epsilon\to 0} \mathcal{G}_N^{\epsilon}(z,\bar{z})$.

4.4.1. The main claim. — We formulate the main claim about the holomorphic factorisation. This will be illustrated on the $N=1$ and $N=2$ cases in the following section. We denote by $\alpha \in \mathfrak{S}_N$ the permutation of the variables v_i^+ so that $0 \leq v^+_{\alpha(1)} \leq v^+_{\alpha(2)} \leq \cdots \leq v^+_{\alpha(i)} \leq z \leq v^+_{\alpha(i+1)} \leq \cdots \leq v^+_{\alpha(N)} \leq 1$, and we sum over all such possible orderings. We decompose this set into two sets : $\sigma := (\alpha(1),\ldots,\alpha(i))$ and $\rho = (\alpha(i+1),\ldots,\alpha(N))$. The integrals over the v_i^+ variables are given by

$$I^{(+)}_{\alpha}(\bar{z}) := \int_{\Delta^{(+)}_{(\sigma,\rho)}} \prod_{i=1}^{N} |v_i^+|^{\bar{a}_i} |1-v_i^+|^{\bar{b}_i} |\bar{z}-v_i^+|^{\bar{c}_i} \prod_{1\leq i<j\leq N} |v_i^+ - v_j^+|^{\bar{g}_{ij}} \prod_{i=1}^{N} dv_i^+ \tag{4.30}$$

where we have introduced the domain of integration with $r+s=N$

$$\Delta^{(+)}_{(\sigma,\rho)} := \{0 \leq v^+_{\sigma(1)} \leq \cdots \leq v^+_{\sigma(r)} \leq z \leq v^+_{\rho(1)} \leq \cdots \leq v^+_{\rho(s)} \leq 1\}. \tag{4.31}$$

We now proceed with the integrals over the v_i^- variables. The integration is over all possible permutation of the v_i^- variables in $[0,1]$ for each ordering of the v_i^+ variables. As before, we pull the contour of integration to the left for $v^-_{\sigma(i)}$ as $i=1,\ldots,r$ and the contour of integration to the right for $v^-_{\eta(i)}$ as $i=1,\ldots,s$, and we get

$$\mathcal{G}_N(z,\bar{z}) = \sum_{\alpha,\beta\in\mathfrak{S}_{N+1}} \tilde{G}_{\alpha,\beta} I^{(+)}_{\alpha}(\bar{z}) I^{(-)}_{\beta}(z)\,. \tag{4.32}$$

The permutation α is the union of the permutations σ with $0 \leq v^+_{\sigma(1)} \leq \cdots \leq v^+_{\sigma(r)} \leq z$ and the permutation ρ with $z \leq v^+_{\rho(1)} \leq \cdots \leq v^+_{\rho(s)} \leq 1$ and $r+s=N$ and $r,s \geq 0$. The permutation β is the union of the permutation $\tilde{\sigma}$ and $\tilde{\rho}$ with

$$I^{(-)}_{\beta}(z) := \int_{\Delta^{(-)}_{(\tilde{\sigma},\tilde{\rho})}} \prod_{i=1}^{N} |v_i^-|^{\bar{a}_i} |1-v_i^-|^{\bar{b}_i} |\bar{z}-v_i^-|^{\bar{c}_i} \prod_{1\leq i<j\leq N} |v_i^- - v_j^-|^{\bar{g}_{ij}} \prod_{i=1}^{N} dv_i^- \tag{4.33}$$

and

$$\Delta^{(-)}_{(\tilde\sigma,\tilde\rho)} := \{v^-_{\tilde\sigma(r)} \leq \cdots \leq v^-_{\tilde\sigma(1)} \leq 0 \leq 1 \leq v^-_{\tilde\rho(1)} \leq \cdots \leq v^-_{\tilde\rho(s)}\}. \qquad (4.34)$$

We use the method of the previous section to express $\mathcal{G}_N(z,\bar z)$ as a bilinear form in the elements of the set of integrals $I_{(\sigma,\rho)}(z)$ or $J_{(\sigma,\rho)}(z)$ to get the holomorphic factorisations in (4.24) and (4.25). Since this form is obtained using the linear relations between the integral functions discussed in previous section, it is clear that the coefficients of G_N and $\hat G_N$ only involve sine function $\sin(\pi x)$ where x is a linear combination of the coefficients of the vectors $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$, and $\boldsymbol{G}$ with coefficients in $\{-1,0,1\}$.

In all cases we have studied the matrices $G_N := (G_{\alpha,\beta})_{\alpha,\beta\in\mathfrak{S}_N}$ and $\hat G_N := (\hat G_{\alpha,\beta})_{\alpha,\beta\in\mathfrak{S}_N}$ admit the block diagonal form

$$G_N = \begin{pmatrix} G_N^{(1)} & 0 & 0 \\ 0 & G_N^{(2)} & 0 \\ 0 & 0 & G_N^{(3)} \end{pmatrix}, \qquad \hat G_N = \begin{pmatrix} \hat G_N^{(1)} & 0 & 0 \\ 0 & \hat G_N^{(2)} & 0 \\ 0 & 0 & \hat G_N^{(3)} \end{pmatrix}, \qquad (4.35)$$

where $G_N^{(i)}$ and $\hat G_N^{(i)}$ for $i=1,3$ are real square-matrices of size $N!$ and $G_N^{(2)}$ and $\hat G_N^{(2)}$ are diagonal matrices of size $(N-1)N!$

We are not proving this claim, but a proof at a given order N is done by explicitly constructing the matrices. This has to be true because the form of these matrices is the one required for the correlation function $\mathcal{G}_N(z,\bar z)$ to be free of monodromies around $z=0$ and $z=1$, as shown in §4.3. The fact that the correlation function $\mathcal{G}_N(z,\bar z)$ is a single-valued function of z in the complex plane was explained in §3.1 to be a consequence of the fact that physical correlation functions are single valued functions by construction in any physical conformal field theories in Euclidean space.

4.4.2. The $N=1$ case. — To illustrate how the factorisation works we consider the case $N=1$. The integral is given by

$$\mathcal{G}_1^\epsilon(z,\bar{z}) = \int_{\mathbb{R}^2} dv_1^+ dv_1^- (v_1^- + i\epsilon\delta_1)^{a_1} (v_1^+ - i\epsilon\delta_1)^{\bar{a}_1} (v_1^- - 1 + i\epsilon\delta_1)^{b_1} \\ (v_1^+ - 1 - i\epsilon\delta_1)^{\bar{b}_1} (v_1^- - z + i\epsilon\delta_1)^{c_1} (v_1^+ - \bar{z} - i\epsilon\delta_1)^{\bar{c}_1} \,. \quad (4.36)$$

Recall that $\delta_1 > 0$ for $v_1^+ > v_1^-$ and $\delta_1 < 0$ for $v_1^+ < v_1^-$.

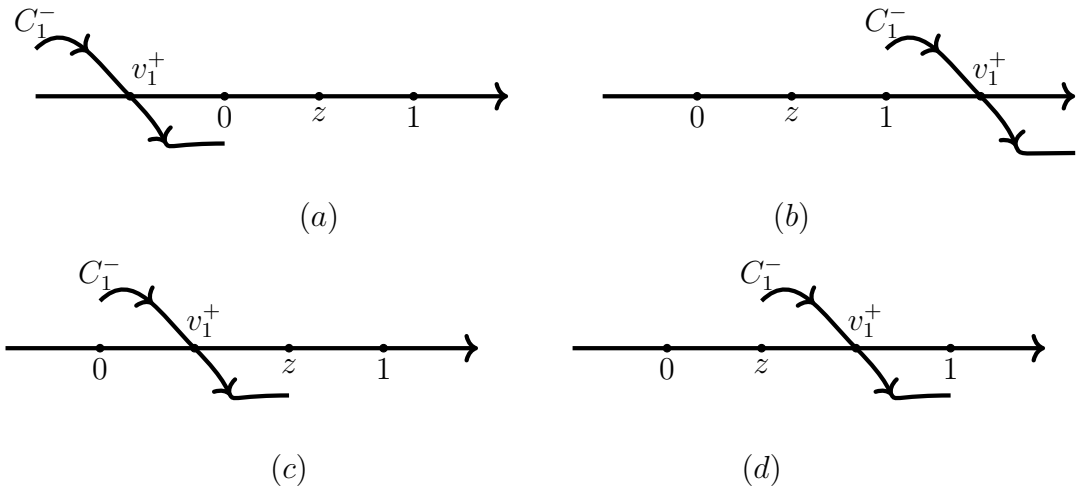


FIGURE 4.2. The contour of integration C_1^- for the variable v_1^- depending on the position of v_1^+ on the real axis.

We consider the four different cases

- If $v_1^+ \in]-\infty, 0]$ then the contour integral for v_1^- is given in figure 4.2(a). By pulling the lower part of the contour of integration for v_1^- in the lower half-plane to the left, and because there are no poles, we obtain that the integral vanishes.
- If $v_1^+ \in [1, +\infty[$ then the contour integral for v_1^- is given in figure 4.2(b). By pulling the lower part of the contour of integration for v_1^- in the lower half-plane to the right, and because the integral has no poles, we obtain that the integral vanishes.
- If $v_1^+ \in [0, z]$ then the contour integral for v_1^- is given in figure 4.2(c). By deforming the lower part of the contour of integration for v_1^- in the lower half-plane to the left, one picks the contribution from

$v_1^- = 0$ and the integral gives

$$\lim_{\epsilon\to 0}\int_{C_1^-} dv_1^-(v_1^- + i\epsilon\delta_1)^{a_1}(v_1^- - 1 + i\epsilon\delta_1)^{b_1}(v_1^- - z + i\epsilon\delta_1)^{c_1}$$
$$= \sin(\pi A_1)\int_{-\infty}^{0} dv_1^-(-v_1^-)^{a_1}(1-v_1^-)^{b_1}(z-v_1^-)^{c_1}, \quad (4.37)$$

where we made use of the determination of the logarithm in (4.10) and the fact that $a_1 - A_1 \in \mathbb{Z}$, so it does not contribute to the monodromy.

- If $v_1^+ \in [z,1]$ then the contour integral for v_1^- is given in figure 4.2(d). By deforming the the lower part of the contour of integration for v_1^- in the lower half-plane to the right, one picks the contribution from $v_1^- = 1$ and the integral gives

$$\lim_{\epsilon\to 0}\int_{C_1^-} dv_1^-(v_1^- + i\epsilon\delta_1)^{a_1}(v_1^- - 1 + i\epsilon\delta_1)^{b_1}(v_1^- - z + i\epsilon\delta_1)^{c_1}$$
$$= \sin(\pi B_1)\int_{-\infty}^{0} dv_1^-(-v_1^-)^{a_1}(1-v_1^-)^{b_1}(z-v_1^-)^{c_1} \quad (4.38)$$

where we made use of the determination of the logarithm in (4.10) and the fact that $b_1 - B_1 \in \mathbb{Z}$, so it does not contribute to the monodromy.

Since $\mathcal{G}_1(z,\bar z) = \lim_{\epsilon\to 0}\mathcal{G}_1^\epsilon(z,\bar z)$ we obtain that

$$\mathcal{G}_1(z,\bar z) = \sin(\pi A_1)\int_0^z (v_1^+)^{\bar a_1}(1-v_1^+)^{\bar b_1}(\bar z - v_1^+)^{\bar c_1}dv_1^+ \int_{-\infty}^0 (-v_1^-)^{a_1}(1-v_1^-)^{b_1}(z-v_1^-)^{c_1}dv_1^-$$
$$+\sin(\pi B_1)\int_z^1 (v_1^+)^{\bar a_1}(1-v_1^+)^{\bar b_1}(v_1^+ - \bar z)^{\bar c_1}dv_1^+ \int_1^{+\infty}(v_1^-)^{a_1}(v_1^- - 1)^{b_1}(v_1^- - z)^{c_1}dv_1^- . \quad (4.39)$$

Using the notation in (4.1) and (4.3) for the Aomoto-Gel'fand hypergeometric functions

$$I_{((1),\emptyset)}(a_1,b_1,c_1;;z) = \int_1^{+\infty} w^{a_1}(w-1)^{b_1}(w-z)^{c_1}dw,$$

$$
\begin{aligned}
I_{(\emptyset,(1))}(a_1,b_1,c_1;;z) &= \int_0^z w^{a_1}(1-w)^{b_1}(z-w)^{c_1}dw,\\
J_{((1),\emptyset)}(a_1,b_1,c_1;;z) &= \int_{-\infty}^0 (-w)^{a_1}(1-w)^{b_1}(z-w)^{c_1}dw,\\
J_{(\emptyset,(1))}(a_1,b_1,c_1;;z) &= \int_z^1 w^{a_1}(1-w)^{b_1}(w-z)^{c_1}dw\,,
\end{aligned}
\tag{4.40}
$$

and the vector notation

$$
\begin{aligned}
\mathcal{I}(a_1,b_1,c_1;z) &= \begin{pmatrix} I_{((1),\emptyset)}(a_1,b_1,c_1;;z)\\ I_{(\emptyset,(1))}(a_1,b_1,c_1;;z)\end{pmatrix},\\
\mathcal{J}(a_1,b_1,c_1;z) &= \begin{pmatrix} J_{((1),\emptyset)}(a_1,b_1,c_1;;z)\\ J_{(\emptyset,(1))}(a_1,b_1,c_1;;z)\end{pmatrix}
\end{aligned}
\tag{4.41}
$$

we have

$$
\begin{aligned}
\mathcal{G}_1(z,\bar z) = \sin(\pi A_1)\mathcal{J}_1(a_1,b_1,c_1;z)\mathcal{I}_2(\bar a_1,\bar b_1,\bar c_1;\bar z)\\
+\sin(\pi B_1)\mathcal{I}_1(a_1,b_1,c_1;z)\bar{\mathcal{J}}_2(\bar a_1,\bar b_1,\bar c_1;\bar z),.
\end{aligned}
\tag{4.42}
$$

With the notation that $\bar{\mathcal{I}}_r := I_r(\bar a,\bar b,\bar c;;\bar z)$ and $\bar{\mathcal{J}}_r := J_r(\bar a,\bar b,\bar c;;\bar z)$. The relation between this expression and holomorphic factorisation of string theory closed string amplitudes in [42, 45] is discussed in §6. The arbitrariness in choosing to close the contour of integration to the left or to the right is taken care by the linear relations derived earlier between the various integrals appeared in §4.2. For the $N=1$-case the relation between the two sets of integral functions is given in Appendix A. Using these relations we find that the correlation function is given by

$$
\mathcal{G}_1(z,\bar z) = \mathcal{I}(\bar a_1,\bar b_1,\bar c_1;\bar z)^T G_1 \mathcal{I}(a_1,b_1,c_1;z),
\tag{4.43}
$$

with

$$
G_1 = \frac{-1}{\sin(\pi(A_1+C_1))}\begin{pmatrix} \sin(\pi(A_1+B_1+C_1))\sin(\pi B_1) & 0\\ 0 & \sin(\pi C_1)\sin(\pi A_1)\end{pmatrix}
\tag{4.44}
$$

and

$$\mathcal{G}_1(z,\bar{z}) = \mathcal{J}(\bar{a}_1,\bar{b}_1,\bar{c}_1;\bar{z})^T \hat{G}_1 \mathcal{J}(a_1,b_1,c_1;z), \tag{4.45}$$

with

$$\hat{G}_1 = \frac{-1}{\sin(\pi(B_1+C_1))} \begin{pmatrix} \sin(\pi(A_1+B_1+C_1))\sin(\pi A_1) & 0 \\ 0 & \sin(\pi C_1)\sin(\pi B_1) \end{pmatrix}. \tag{4.46}$$

4.4.3. The $N=2$ case. — For the case of $N=2$ the integral we need to consider reads

$$\begin{aligned} \mathcal{G}_2^{\epsilon}(z,\bar{z}) = \int_{\mathbb{R}^4} \prod_{r=1}^{2} dv_i^+ dv_i^- (v_r^- + i\epsilon\delta_r)^{a_r} (v_r^+ - i\epsilon\delta_r)^{\bar{a}_r} (v_r^- - 1 + i\epsilon\delta_r)^{b_r} \\ \times (v_r^+ - 1 - i\epsilon\delta_r)^{\bar{b}_r} (v_r^- - z + i\epsilon\delta_r)^{c_r} (v_r^+ - \bar{z} - i\epsilon\delta_r)^{\bar{c}_r} \\ \times (v_1^- - v_2^- + i\epsilon(\delta_1-\delta_2))^{g_{12}} (v_1^+ - v_2^+ - i\epsilon(\delta_1-\delta_2))^{\bar{g}_{12}}. \end{aligned} \tag{4.47}$$

We introduce the vectors of integrals adapted to the monodromy around $z=0$ and $z=1$ analysed in §4.3:

$$\mathcal{I}(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c};g_{12};z) := \begin{pmatrix} I_{((12),\emptyset)}(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c};g_{12};z) \\ I_{((21),\emptyset)}(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c};g_{12};z) \\ I_{((1),(2))}(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c};g_{12};z) \\ I_{((2),(1))}(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c};g_{12};z) \\ I_{(\emptyset,(12))}(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c};g_{12};z) \\ I_{(\emptyset,(21))}(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c};g_{12};z) \end{pmatrix}, \tag{4.48}$$

and

$$\mathcal{J}(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c};g_{12};z) := \begin{pmatrix} J_{((12),\emptyset)}(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c};g_{12};z) \\ J_{((21),\emptyset)}(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c};g_{12};z) \\ J_{((1),(2))}(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c};g_{12};z) \\ J_{((2),(1))}(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c};g_{12};z) \\ J_{(\emptyset,(12))}(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c};g_{12};z) \\ J_{(\emptyset,(21))}(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c};g_{12};z) \end{pmatrix}. \tag{4.49}$$

These two vectors are related by the linear relation $\mathcal{J} = S(\boldsymbol{A},\boldsymbol{B},\boldsymbol{C};G_{12})\mathcal{I}$ derived in §4.2. The matrix $S(\boldsymbol{A},\boldsymbol{B},\boldsymbol{C};G_{12})$ is given in Appendix B. We

recall that the matrix does not depend on the integers in (3.6), and is the same for the holomorphic and anti-holomorphic integrals.

The derivation of this holomorphic factorisation is an application of the contour-deformation method of [42]. The explicit derivation is rather long so we will refrain from doing it explicitly but we present a numerical approach that shows that the holomorphic factorisation has the the form claimed in (4.35).

The two variables v_1^+ and v_2^+ need to be ordered on the real axis. In the following we assume that $v_1^+ < v_2^+$, but one also needs to consider the analogous case $v_2^+ < v_1^+$. With the same reasoning as for the $N = 1$-case, we get that only for $0 \leq v_1^+ \leq v_2^+ \leq z$, $0 \leq v_1^+ \leq z \leq v_2^+ \leq 1$, or $1 \leq v_1^+ \leq v_2^+$ the integrals are non vanishing.

The antiholomorphic integrals are given by $\bar{I}_{\delta^+} = \lim_{\epsilon\to 0} \bar{I}^\epsilon_{\delta^+}$ with

$$\bar{I}^\epsilon_{\delta^+} := \int_{\delta^+} \prod_{r=1}^{2} dv_i^+ (v_r^+ - i\epsilon\delta_r)^{\bar{a}_r} (v_r^+ - 1 - i\epsilon\delta_r)^{\bar{b}_r} (v_r^+ \bar{z} - i\epsilon\delta_r)^{\bar{c}_r} \times (v_1^+ - v_2^+ - i\epsilon(\delta_1 - \delta_2))^{\bar{g}_{12}} \quad (4.50)$$

with the domain of integration δ^+ is runing over the six domains of integrations

$$\Delta^+ = \{0 \leq v_1^+ \leq v_2^+ \leq z, 0 \leq v_1^+ \leq z \leq v_2^+ \leq 1, z \leq v_1^+ \leq v_2^+ \leq 1 \\ , 0 \leq v_2^+ \leq v_1^+ \leq z, 0 \leq v_2^+ \leq z \leq v_1^+ \leq 1, z \leq v_2^+ \leq v_1^+ \leq 1\} . \quad (4.51)$$

The holomorphic integrals are given by the contours of integrations $I_{\delta^-} = \lim_{\epsilon\to 0} I^\epsilon_{\delta^-}$ where

$$I^\epsilon_{\delta^-} = \int_{\delta^-} \prod_{r=1}^{2} dv_r^- (v_r^- + i\epsilon\delta_r)^{a_r} (v_r^- - 1 + i\epsilon\delta_r)^{b_r} (v_r^- - z + i\epsilon\delta_r)^{c_r} \times (v_1^- - v_2^- + i\epsilon(\delta_1 - \delta_2))^{g_{12}} \quad (4.52)$$

and the integration is over each of the six contours for v_1^- and v_2^- in figure 4.3 and 4.4. The holomorphic integral $I_{\delta_i^-}$ over the contour of

integration δ_i^- is multiplied by the antiholomorphic contribution $\bar{I}_{\delta_i^+}$ in the holomorphic factorisation

$$\mathcal{G}_2(z,\bar{z}) = \lim_{\epsilon\to 0}\mathcal{G}_2^\epsilon(z,\bar{z}) = \sum_{i,j=1}^{6} G_{ij}\bar{I}_{\delta_i^+}I_{\delta_j^-}\,. \tag{4.53}$$

By applying the methods of §4.2 one gets that I_{δ^-} is a linear combination of either the integral functions in $\mathcal{I}$ of (4.48) or in $\mathcal{J}$ of (4.49), and each integral $\bar{I}_{\delta^+}$ is a linear combination of the integrals in $\bar{\mathcal{I}}$ or $\bar{\mathcal{J}}$ obtained by replacing the parameters $(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c},\boldsymbol{g},z)$ by $(\bar{\boldsymbol{a}},\bar{\boldsymbol{b}},\bar{\boldsymbol{c}},\bar{\boldsymbol{g}},\bar{z})$ in (4.48)-(4.49).

The expression for the holomorphic factorisation is then given by

$$\begin{aligned}\mathcal{G}_2(z,\bar{z}) &= \mathcal{I}(\bar{\boldsymbol{a}},\bar{\boldsymbol{b}},\bar{\boldsymbol{c}};\bar{g}_{12};\bar{z})^T G_2 \mathcal{I}(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c};g_{12};z)\\ &= \mathcal{J}(\bar{\boldsymbol{a}},\bar{\boldsymbol{b}},\bar{\boldsymbol{c}};\bar{g}_{12};\bar{z})^T \hat{G}_2 \mathcal{J}(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c};g_{12};z)\,,\end{aligned} \tag{4.54}$$

with the relation $\hat{G}_2 = S(\boldsymbol{A},\boldsymbol{B},\boldsymbol{C};G_{12})^T G_2 S(\boldsymbol{A},\boldsymbol{B},\boldsymbol{C};G_{12})$ obtained using the linear relations between the two vectors (4.48)-(4.49) derived in §4.2. The explicit expression of the matrix $S(\boldsymbol{A},\boldsymbol{B},\boldsymbol{C};g_{12})$ is given in Appendix B. The relation between the matrices G_2 and $\hat{G}_2$ and the momentum kernel introduced in [42] is discussed in §6.

By construction is it clear that the elements in the matrices G_2 and $\hat{G}_2$ are rational functions of $\sin(\pi x)$ where x is a linear combination of the components of $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$, G_{12} with coefficients in $\{-1,0,1\}$.

For the correlation function $\mathcal{G}_2(z,\bar{z})$ these matrices should have the block-diagonal form

$$G_2 = \begin{pmatrix} G_2^{(1)} & 0 & 0\\ 0 & G_2^{(2)} & 0\\ 0 & 0 & G_2^{(3)}\end{pmatrix}, \qquad \hat{G}_2 = \begin{pmatrix} \hat{G}_2^{(1)} & 0 & 0\\ 0 & \hat{G}_2^{(2)} & 0\\ 0 & 0 & \hat{G}_2^{(3)}\end{pmatrix}, \tag{4.55}$$

where $G_2^{(i)}$, $\hat{G}_2^{(i)}$ with $i=1,2$ are two-by-two square matrices, and $G_2^{(2)}$ and $\hat{G}_2^{(2)}$ are diagonal two-by-two matrices.

Assuming the block diagonal form for G_2 as in (4.55), we have numerically solved the system by imposing that the matrix $\hat{G}_2 = S(\boldsymbol{A},\boldsymbol{B},\boldsymbol{C};g_{12})^T\times$

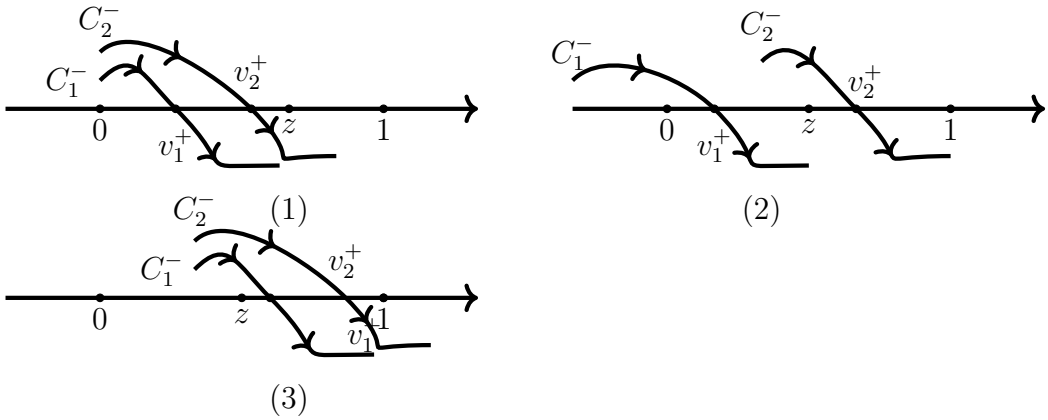


FIGURE 4.3. Contour of integration C_1^- for v_1^- and C_2^- and v_2^- for the ordering $v_1^+ < v_2^+$.

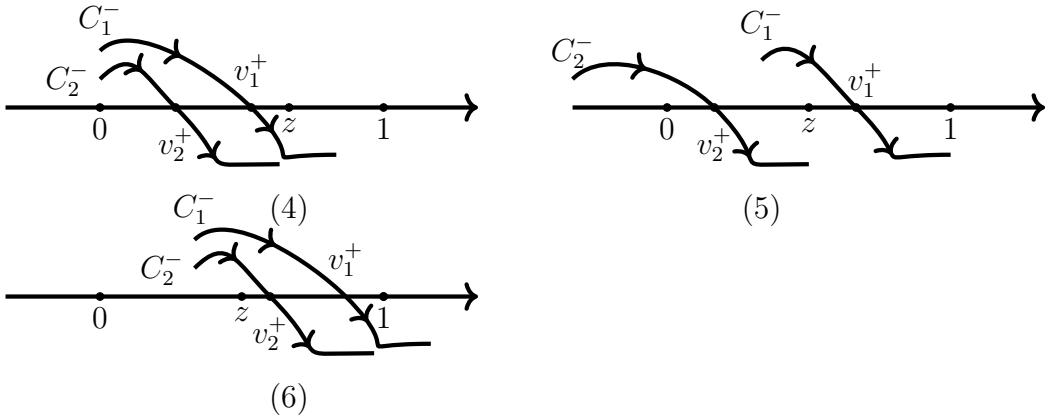


FIGURE 4.4. Contour of integration C_1^- for v_1^- and C_2^- and v_2^- for the ordering $v_2^+ < v_1^+$.

$\times G_2 S(\boldsymbol{A},\boldsymbol{B},\boldsymbol{C};g_{12})$ has the block diagonal form as in (4.55). We have used the expression for the matrix $S(\boldsymbol{A},\boldsymbol{B},\boldsymbol{C};g_{12})$ given in Appendix B. On various numerical values we have confirmed that the holomorphic factorisation in (4.55) has a unique solution up to an overall factor. The overall factor is uniquely fix by considering a special value of the correlation function $\mathcal{G}_2(z,\bar{z})$ and there is no arbitrariness in the holomorphic factorisation.

5. The single-valued α'-expansion of closed string amplitudes

We can now show that the α'-expansion of the partial closed string amplitudes only involves single-valued multiple zeta values[(4)].

With the identification of the parameters in (3.6) the small α'-expansion of the single-valued function $\mathcal{G}_N(z,\bar{z})$ reads

$$\mathcal{G}_N(z,\bar{z}) = \sum_{r\geq 0} (\alpha')^r \sigma_N^{(r)} \mathcal{G}_N^{(r)}(z,\bar{z})\,, \tag{5.1}$$

where $\sigma_N^{(r)}$ are polynomials of degree r (with rational coefficients) in the kinematic invariants $k_i\cdot k_j$ with $1\leq i<j\leq N$. The coefficients $\mathcal{G}_N^{(r)}(z,\bar{z})$ are single-valued functions of z because the expansion in α' cannot bring branch cuts. Similarly to the way the coefficients of the α' expansion of generalised Selberg integrals are proven in [60, 61] to be polynomials in multiple zeta values and $2\pi i$ with rational coefficients, one can use Proposition 8.7 to demonstrate that the coefficients of the α' expansion of Aomoto-Gel'fand hypergeometric functions are linear combinations of multiple polylogarithms with coefficients given by polynomials in multiple zeta values and $2\pi i$ with rational coefficients. Extra powers of π are introduced by the small α' expansion of the matrix $G=(G_{\alpha,\beta})$ in (4.24) and the matrix $\hat{G}=(\hat{G}_{\alpha,\beta})$ (4.25). By Theorem 9.2 and Lemma 9.1 by Francis Brown and Proposition 9.5, we conclude that the coefficients of $\mathcal{G}_N^{(r)}(z,\bar{z})$ are linear combinations of single-valued multiple polylogarithms with coefficients given by polynomials in multiple zeta values and $2\pi i$ with rational coefficients [47]. Based on analytic evaluation of four-point amplitudes and various numerical checks on five-point amplitudes, we expect that the coefficients of the single-valued multiple polylogarithms should actually belong to the ring of single-valued multiple zeta values $\mathcal{Z}^{\rm sv}$ defined in §9.1. This would imply that the coefficients of the α'-expansion of tree-level closed string amplitudes are single-valued

(4) See §9 for a precise definition.

multiple zeta values. This will be confirmed by the proof of this fact[(5)] contained in part II of the paper.

The integrals $J_{(\sigma,\rho)}(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c};\boldsymbol{g};z)$ in (4.3) evaluated at $z=1$ vanishes, $J_{(\sigma,\rho)}(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c};\boldsymbol{g};1)=0$, when ρ is not the empty set. When ρ is not empty one integrates over the variables $w_{\rho(m)}\in[z,1]$ with $1\leq m\leq s$ in (4.4). Setting $z=1$ the range of integration of goes to zero and therefore the integral vanishes. Therefore in (4.25) we have that the partial amplitude of the closed string amplitude $M_{N+3}(\boldsymbol{s},\boldsymbol{n},\bar{\boldsymbol{n}})=\mathcal{G}_N(1,1)$ is given by

$$M_{N+3}(\boldsymbol{s},\boldsymbol{n},\bar{\boldsymbol{n}})=\sum_{(\sigma,\tilde{\sigma})\in\mathfrak{S}_N\times\mathfrak{S}_N}\hat{G}_{(\sigma,\emptyset),(\tilde{\sigma},\emptyset)}J_{(\sigma,\emptyset)}(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c};\boldsymbol{g};1)J_{(\tilde{\sigma},\emptyset)}(\bar{\boldsymbol{a}},\bar{\boldsymbol{b}},\bar{\boldsymbol{c}};\bar{\boldsymbol{g}};1)\,, \tag{5.2}$$

and has only single-valued multiple zeta values in its α'-expansion.

Remark 5.1. — *Although the single-valued function $\mathcal{G}_N(z,\bar{z})$ depends explicitly on the values of b_i and c_i independently, the value at $z=1$ only involves the combination b_i+c_i, as it is obvious from the value $z=1$ of the integral functions in (4.1) and (4.3). There is a family of single-valued correlation functions that leads to the same value at $z=1$. This parallels the construction of the single-valued version of multiple polylogarithms, which is not unique, but for $z=1$ they all lead to the same space of single-valued multiple zeta values over the rational numbers $\mathbb{Q}$ [62].*

5.1. The single-valued expansion of the four-point amplitude. — The colour-ordered open string amplitudes are defined as

$$\begin{aligned}A_4(1,2,3,4;\boldsymbol{n})&=\int_0^1 w^{2\alpha' k_1\cdot k_2+n_1}(1-w)^{2\alpha' k_1\cdot k_3+n_2+n_3}\,dw,\\ A_4(1,3,2,4;\boldsymbol{n})&=\int_1^\infty w^{2\alpha' k_1\cdot k_2+n_1}(1-w)^{2\alpha' k_2\cdot k_3+n_2+n_3}\,dw\end{aligned} \tag{5.3}$$

(5) We do not provide a proof in full generality, but only for some special cases, which include the integrals investigated in the recent superstring theory literature [33, 34].

$$A_4(2,1,3,4;\boldsymbol{n}) = \int_{-\infty}^{1} (-w)^{2\alpha' k_1\cdot k_2+n_1}(1-w)^{2\alpha' k_2\cdot k_3+n_2+n_3}\, dw,$$

where the labels refer to the ordering of the external states on the boundary of the disc. The integrals (4.1) at $z=1$ map to[(6)]

$$I_{(\emptyset,(1))}(a,b,c;1) = A_4(1,2,3,4;\boldsymbol{n}); \quad I_{((1),\emptyset)}(a,b,c;1) = A_4(1,3,2,4;\boldsymbol{n})\,, \tag{5.4}$$

and the integrals (4.3) at $z=1$ map to

$$J_{((1),\emptyset)}(a,b,c;1) = A_4(2,1,3,4;\boldsymbol{n}); \qquad J_{(\emptyset,(1))}a,b,c;1) = 0\,. \tag{5.5}$$

Therefore the evaluation at $z=1$ of the single-valued correlation function in (4.43) and (4.45) gives

$$\mathcal{G}_1(1,1) = -\frac{\sin(\pi A_1)\sin(\pi(A_1+B_1+C_1))}{\sin(\pi(B_1+C_1))}\,|A_4(2,1,3,4;\boldsymbol{n})|^2\,. \tag{5.6}$$

By using [41, eq. (8)] one can bring the KLT expression in [45, eq. (3.11)] for the closed string amplitude into the non-local symmetric form

$$M_4(\boldsymbol{s},\boldsymbol{n},\bar{\boldsymbol{n}}) = \frac{\sin(2\pi\alpha' k_1\cdot k_2)\sin(2\pi\alpha' k_2\cdot k_4)}{\sin(2\pi\alpha' k_2\cdot k_3)}|A_4(2,1,3,4;\boldsymbol{n})|^2\,. \tag{5.7}$$

This form is equivalent to the one given in [41, eq. (15)] by using the relations [41, eq. (8)].

The closed string four-point partial amplitude $M_4(\boldsymbol{s},\boldsymbol{n},\bar{\boldsymbol{n}})$ equals the value at $z=1$ of the single-valued correlation function $\mathcal{G}_1(z,\bar{z})$

$$M_4(\boldsymbol{s},\boldsymbol{n},\bar{\boldsymbol{n}}) = \mathcal{G}_1(1,1)\,, \tag{5.8}$$

using the identification of the parameters

$$A_1 = 2\alpha' k_1\cdot k_2, \quad B_1+C_1 = 2\alpha' k_2\cdot k_3, \quad A_1+B_1+C_1 = -2\alpha' k_1\cdot k_4\,, \tag{5.9}$$

together with the momentum conservation relation $k_1+\cdots+k_4=0$ and the on-shell conditions $k_i^2=0$ for $i=1,\ldots,4$.

[(6)] The relation between the parameters of the Selberg integral and the open string amplitudes is given in (3.6) and (3.8) with $N=4$. The integers n_1, n_2 and n_3 are arbitrary.

Notice that it is not necessary that the total amplitude is given by the special value at $z = 1$ of a single-valued correlation function. It is enough that each partial amplitude arises this way, because the kinematic coefficients $c_r(\boldsymbol{s}, \boldsymbol{\epsilon})$ in the expansion of the amplitude in (3.1) are rational functions of the kinematic invariants (see [50] for some expressions using the pure spinor formalism, and [36] for bosonic open and closed string and heterotic string amplitudes). This is the case, for instance, of the single- and double-trace contributions to the heterotic-string amplitude given in [48].

5.2. The single-valued expansion of the five-point amplitude. — The advantage of the expression in (5.2) is that at $z = 1$ the only non vanishing integrals are $J_1(\boldsymbol{b}, \boldsymbol{a}, \boldsymbol{c}; g; 1)$ and $J_2(\boldsymbol{b}, \boldsymbol{a}, \boldsymbol{c}; g; 1)$. These values correspond to the ordered open string amplitudes

$$J_1(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}; g; 1) = A_5(2, 3, 1, 4, 5; \boldsymbol{n}), \quad J_2(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}; g; 1) = A_5(3, 2, 1, 4, 5; \boldsymbol{n}), \tag{5.10}$$

where the colour-ordered open string amplitudes are

$$\begin{aligned} A_5(2, 3, 1, 4, 5; n) &= \int_{-\infty}^{0} dw_2 \int_{w_2}^{0} dw_3 (w_3 - w_2)^{2\alpha' k_2 \cdot k_3 + n_{23}} \\ &\quad \times \prod_{r=2}^{3} (-w_r)^{-2\alpha' k_1 \cdot k_r + n_{1r}} (1 - w_r)^{2\alpha' k_r \cdot k_4 + n_{r4}} \\ A_5(3, 2, 1, 4, 5; n) &= \int_{-\infty}^{0} dw_3 \int_{w_3}^{0} dw_2 (w_2 - w_3)^{2\alpha' k_2 \cdot k_3 + n_{23}} \\ &\quad \times \prod_{r=2}^{3} (-w_r)^{-2\alpha' k_1 \cdot k_r + n_{1r}} (1 - w_r)^{2\alpha' k_r \cdot k_4 + n_{r4}} \end{aligned} \tag{5.11}$$

with the parameter identification

$$\begin{aligned} a_1 &= 2\alpha'\, k_1 \cdot k_2 + n_{12}, & a_2 &= 2\alpha' k_1 \cdot k_3 + n_{13}, \\ b_1 + c_1 &= 2\alpha'\, k_2 \cdot k_4 + n_{24}, & b_2 + c_2 &= 2\alpha' k_3 \cdot k_4 + n_{34}, \\ g &= 2\alpha'\, k_2 \cdot k_3 + n_{23}\,, \end{aligned} \tag{5.12}$$

with $k_1 + \cdots + k_5 = 0$ and $k_i^2 = 0$ for $1 \leq i \leq 5$, and $n_{23}, n_{1r}, n_{4r} \in \mathbb{Z}$ for $r = 2, 3$.

Therefore the value $z = 1$ of the single-valued correlation function is given by

$$\mathcal{G}_2(1,1) = \begin{pmatrix} A_5(2,3,1,4,5;\boldsymbol{n}) & A_5(3,2,1,4,5;\boldsymbol{n}) \end{pmatrix} \begin{pmatrix} \tilde{G}_{11} & \tilde{G}_{12} \\ \tilde{G}_{12} & \tilde{G}_{22} \end{pmatrix} \begin{pmatrix} \bar{A}_5(2,3,1,4,5;\bar{\boldsymbol{n}}) \\ \bar{A}_5(3,2,1,4,5;\bar{\boldsymbol{n}}) \end{pmatrix}. \tag{5.13}$$

It was shown in [41, eq. (16)] that the five-point closed xstring amplitude takes the same form. Using the kinematic amplitude relations derived in [32, 41] one can change the basis to convert the result in [41, eq. (16)] into

$$\begin{aligned} \sin(2\pi\alpha' k_3 \cdot k_4) \sin(2\pi\alpha' k_1 \cdot k_5) A_5(1,2,3,4,5,\boldsymbol{n}) = \\ - \sin(2\pi\alpha' k_1 \cdot k_3) \sin(2\pi\alpha' k_1 \cdot k_5) A_5(2,3,1,4,5,\boldsymbol{n}) \\ - \sin(2\pi\alpha' k_3 \cdot k_5) \sin(2\pi\alpha' k_1 \cdot (k_2 + k_5)) A_5(3,2,1,4,5,\boldsymbol{n}) \end{aligned} \tag{5.14}$$

and

$$\begin{aligned} \sin(2\pi\alpha' k_2 \cdot k_4) \sin(2\pi\alpha' k_1 \cdot k_5) A_5(1,3,2,4,5,\boldsymbol{n}) = \\ \sin(2\pi\alpha' k_1 \cdot k_2) \sin(2\pi\alpha' k_3 \cdot k_5) A_5(3,2,1,4,5,\boldsymbol{n}) \\ - \sin(2\pi\alpha' k_2 \cdot k_5) \sin(2\pi\alpha' k_1 \cdot (k_2 + k_4)) A_5(2,3,1,4,5,\boldsymbol{n}). \end{aligned} \tag{5.15}$$

Hence we find that the five-point closed string partial amplitudes are given by the single-valued correlation functions evaluated at $z = 1$

$$M_5(\boldsymbol{s}, \boldsymbol{n}, \bar{\boldsymbol{n}}) = \mathcal{G}_2(1,1)\,. \tag{5.16}$$

This can be checked against the explicit evaluation of the five-point heterotic-string amplitude in [49].

6. Relation with the momentum kernel

The fundamental ingredient entering the cancellation of the monodromy is the matrix $S(\boldsymbol{A},\boldsymbol{B},\boldsymbol{C};\boldsymbol{G})$. This matrix realises the linear transformation between the set of integral functions $I_{(\sigma,\rho)}(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c};\boldsymbol{g};z)$ in (4.1) to the set of integral functions $J_{(\sigma,\rho)}(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c};\boldsymbol{g};z)$ in (4.3).

When evaluated at $z=1$ the ordered integrals in (4.1) evaluate to N-point ordered open string amplitudes with the parameter dictionary in (3.6)

$$I_{(\sigma,\rho)}(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c};\boldsymbol{g};1)=A_N(1,\rho(1),\ldots,\rho(s),N-1,\sigma(1),\ldots,\sigma(r),N;\boldsymbol{n}), \tag{6.1}$$

with the permutations $\sigma\in\mathfrak{S}_r$ and $\rho\in\mathfrak{S}_s$ such that $\{\sigma(1),\ldots,\sigma(r)\}\sqcup\{\rho(1),\ldots,\rho(s)\}=\{2,\ldots,N-2\}$ with $r,s\geq 0$. The integrals $J_{(\sigma,\rho)}(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c};\boldsymbol{g};1)$ vanish unless ρ is the empty set because the integration over the range $[z,1]$ vanishes when $z=1$ in (4.3). The non-vanishing integrals are given by the ordered open string amplitudes

$$J_{(\sigma,\emptyset)}(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c};\boldsymbol{g};1)=A_N(\sigma(2),\ldots,\sigma(N-2),1,N-1,N;\boldsymbol{n}), \tag{6.2}$$

where σ is a permutation of the $N-3$ elements $\{2,\ldots,N-2\}$.

The ordered open string amplitudes satisfy the following kinematic identities

$$\sum_{\alpha\in\mathfrak{S}_{N-2}}\mathcal{S}_{\alpha'}(\alpha|\beta)A_N(1,\alpha(2),\ldots,\alpha(N-1),N;\boldsymbol{n})=0 \tag{6.3}$$

for all permutation $\beta\in\mathfrak{S}_{N-2}$, where $\mathcal{S}_{\alpha'}(\alpha|\beta)$ is the momentum kernel [42]. We have recalled its definitions and main properties in Appendix C.

The relation with the momentum kernel and the matrix $\hat{G}$ in the holomorphic factorisation of §4.4 is obtained by comparing with equation (4.32) the generic formula for the holomorphic factorisation of the colour-ordered open string amplitudes using the momentum kernel formalism

$$M_N(\boldsymbol{s},\boldsymbol{n}^r,\bar{\boldsymbol{n}}^r) = -\sum_{\sigma,\eta\in\mathfrak{S}_{N-3}} \mathcal{S}_{\alpha'}(\sigma|\eta) \times A_N(1,\sigma(2,\ldots,N-2),N-1,N)\tilde{A}_N(1,\eta(2,\ldots,N-2),N,N-1)\,, \tag{6.4}$$

which is a combination of $I_{(\sigma,\rho)}(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c};\boldsymbol{g};1)$ and $J_{(\sigma,\rho)}(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c};\boldsymbol{g};1)$ integrals. Since for $z=1$ we have that $I^{(+)}_{(\sigma,\rho)}(1) = J_{(\sigma,\rho)}(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c};\boldsymbol{g};1) \neq 0$ only when $\rho=\emptyset$, we then deduce that

$$\begin{aligned}\mathcal{S}_{\alpha'}(\sigma|\eta) &= G_{(\sigma|\emptyset)(\rho|\lambda)} S(\boldsymbol{A},\boldsymbol{B},\boldsymbol{C};\boldsymbol{G})^{(\rho|\lambda)}{}_{(\eta|\emptyset)}\,,\\ &= S(\boldsymbol{B},\boldsymbol{A},\boldsymbol{C};\boldsymbol{G})_{(\sigma|\emptyset)}{}^{(\rho|\lambda)}\hat{G}_{(\rho|\lambda)(\eta|\emptyset)}\,.\end{aligned} \tag{6.5}$$

The non-localities in the matrices G and $\hat{G}$ are removed with the multiplication by change of basis matrix $S(\boldsymbol{A},\boldsymbol{B},\boldsymbol{C};\boldsymbol{G})$ or its inverse $S(\boldsymbol{B},\boldsymbol{A},\boldsymbol{C};\boldsymbol{G})$.

The linear system in (6.3) on the colour-ordered open string amplitudes captures all the kinematic relations between the open strings. It has rank $(N-3)!$, which implies that any ordered amplitudes can be expressed in a basis of $(N-3)!$ ordered open string amplitudes [32, 41]. Choosing the amplitudes $\{A_N(\eta(2,\ldots,N-2),1,N-1,N;\boldsymbol{n}),\eta\in\mathfrak{S}_{N-3}\}$ as basis elements we have, for all permutations $\eta\in\mathfrak{S}_{N-3}$,

$$\begin{aligned}A_N(1,\rho(1),\ldots,\rho(s),N-1,\sigma(1),\ldots,\sigma(r),N;\boldsymbol{n}) =\\ M_{(\sigma,\rho)}{}^{\eta} A_N(\eta(2),\ldots,\eta(N-2),1,N-1,N;\boldsymbol{n})\,,\end{aligned} \tag{6.6}$$

with $\sigma\in\mathfrak{S}_r$, and $\rho\in\mathfrak{S}_s$ such that $\{\sigma(1),\ldots,\sigma(r)\}\sqcup\{\rho(1),\ldots,\rho(s)\} = \{2,\ldots,N-2\}$ with $r,s\geq 0$. The matrix $M_{(\sigma,\rho)}{}^{\eta}$ is the product of the inverse momentum kernel in [63, eq. (5.1)].

The relation between the two sets of integral functions $I_\alpha(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c};\boldsymbol{g};1)$ and $J_\alpha(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c};\boldsymbol{g};1)$ reads

$$I_{(\sigma,\eta)}(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c};\boldsymbol{g};1) = S(\boldsymbol{A},\boldsymbol{B},\boldsymbol{C};\boldsymbol{G})^{(\eta,\emptyset)}_{(\sigma,\eta)} J_{(\eta,\emptyset)}(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c};\boldsymbol{g};1), \tag{6.7}$$

which means that

$$M_{(\sigma,\rho)}{}^{\eta} = S(\boldsymbol{A},\boldsymbol{B},\boldsymbol{C};\boldsymbol{G})^{(\eta,\emptyset)}_{(\sigma,\rho)}\,. \tag{6.8}$$

- At four points we have

$$A_4(1,2,3,4;\boldsymbol{n}) = \frac{\sin(2\pi\alpha' k_1\cdot k_3)}{\sin(2\pi\alpha' k_1\cdot k_4)} A_4(2,1,3,4;\boldsymbol{n}), \tag{6.9}$$

$$A_4(1,3,2,4;\boldsymbol{n}) = \frac{\sin(2\pi\alpha' k_1\cdot k_2)}{\sin(2\pi\alpha' k_1\cdot k_4)} A_4(2,1,3,4;\boldsymbol{n}), \tag{6.10}$$

  which matches the relation (A.4) with the parameter identification in (5.9).

- At five points we can express all six ordered amplitudes $A_5(1,\sigma(2,3,4),5;\boldsymbol{n})$ on the basis of $A_5(2,3,1,4,5,\boldsymbol{n})$ and $A_5(3,2,1,4,5;\boldsymbol{n})$:

$$\begin{pmatrix} A_5(1,4,2,3,5;\boldsymbol{n})\\ A_5(1,4,3,2,5;\boldsymbol{n})\\ A_5(1,3,4,2,5;\boldsymbol{n})\\ A_5(1,2,4,3,5;\boldsymbol{n})\\ A_5(1,2,3,4,5;\boldsymbol{n})\\ A_5(1,3,2,4,5;\boldsymbol{n}) \end{pmatrix} = \frac{\begin{pmatrix} M_5^1 & M_5^2 \end{pmatrix}}{s(k_1\cdot k_5)s(k_2\cdot k_4)s(k_3\cdot k_4)} \begin{pmatrix} A_5(2,3,1,4,5;\boldsymbol{n})\\ A_5(3,2,1,4,5;\boldsymbol{n}) \end{pmatrix}, \tag{6.11}$$

  with the notation $s(x) := \sin(2\pi\alpha' x)$ and $k_{ij} := k_i + k_j$ and with the column vectors

$$M_5^1 = \begin{pmatrix} s(k_1\cdot k_3)s(k_2\cdot k_4)s(k_1\cdot k_2 - k_3\cdot k_4)\\ -s(k_1\cdot k_3)s(k_2\cdot k_5)s(k_3\cdot k_4)\\ s(k_3\cdot k_5)s(k_2\cdot k_{13})s(k_4\cdot k_{23}) + s(k_2\cdot k_3)s(k_2\cdot k_4)s(k_4\cdot k_5)\\ s(k_1\cdot k_3)s(k_2\cdot k_5)s(k_2\cdot k_4 + k_3\cdot k_4)\\ -s(k_1\cdot k_3)s(k_2\cdot k_4)s(k_2\cdot k_5)\\ -s(k_2\cdot k_5)s(k_3\cdot k_4)s(k_1\cdot k_{35}) \end{pmatrix} \tag{6.12}$$

  and

$$M_5^2 = \begin{pmatrix} -s(k_1\cdot k_2)s(k_2\cdot k_4)s(k_3\cdot k_5) \\ s(k_1\cdot k_2)s(k_3\cdot k_4)s(k_1\cdot k_3 - k_2\cdot k_4) \\ s(k_1\cdot k_2)s(k_3\cdot k_5)s(k_4\cdot k_{23}) \\ -s(k_1\cdot k_5)s(k_3\cdot k_{12})s(k_2\cdot k_{14}) - s(k_1\cdot k_2)s(k_2\cdot k_3)s(k_3\cdot k_5) \\ -s(k_2\cdot k_4)s(k_3\cdot k_5)s(k_1\cdot k_{25}) \\ -s(k_1\cdot k_2)s(k_3\cdot k_4)s(k_3\cdot k_5) \end{pmatrix}. \tag{6.13}$$

This matches the expression given in Appendix B with the parameter identification in (5.12).

PART II

MULTIPLE COMPLEX INTEGRALS AND SINGLE-VALUED MULTIPLE ZETA VALUES

7. A family of complex integrals

Let us consider the multiple complex integrals given for $k \geq 1$ by

$$R_k(\alpha_1,\ldots,\alpha_k,\beta_1,\ldots,\beta_k,\gamma_{1,2},\ldots\gamma_{1,k},\gamma_{2,3},\ldots\gamma_{2,k},\ldots,\gamma_{k-1,k}) =$$
$$\int_{(\mathbb{P}^1_{\mathbb{C}})^k} \prod_{i=1}^{k} |z_i|^{2\alpha_i-2}|z_i-1|^{2\beta_i-2} \prod_{1\leq i<j\leq k} |z_j-z_i|^{2\gamma_{i,j}}\, d^2z_1\cdots d^2z_k, \tag{7.1}$$

where $d^2z := dxdy/\pi$ for $z = x+iy$. This a special case of the multiple integral $\mathcal{G}_k(z,\overline{z})$ considered in part I (see eq. (3.5)), obtained by setting $z=1$, $a_i = \overline{a}_i = \alpha_i - 1$, $b_i + c_i = \overline{b}_i + \overline{c}_i = \beta_i - 1$ and $g_{i,j} = \overline{g}_{i,j} = \gamma_{i,j}$. As we will see in Proposition 7.2, the integrals (7.1) are not convergent at the origin $(0,\ldots,0) \in \mathbb{C}^{k(k+3)/2}$, but this point is situated right at the boundary of the region of convergence $\mathcal{C}_k$ in which R_k defines a holomorphic function. The aim of this part of the paper is to prove the following:

Theorem 7.1. — *For each k there exist a neighborhood U_k of the origin of $\mathbb{C}^{k(k+3)/2}$, a rational function P_k (the polar part) and a function H_k holomorphic on U_k (the holomorphic part) such that:*

- $R_k = P_k + H_k$ *on* $U_k \cap \mathcal{C}_k$[(7)].
- *The poles of P_k are situated along hyperplanes which contain the origin*[(8)].
- *The coefficients of the rational function P_k and of the power-series expansion around the origin of H_k belong to the ring of single-valued multiple zeta values.*

It is in principle possible, following the steps of the proof, to algorithmically compute all the coefficients. Our main motivation to study this expansion comes from the calculation of closed string amplitudes at tree level. For instance, the integral (7.1) is a prototype of the integrals $J_{\rho,\sigma}$ (see eq. (7.9)) appearing in superstring theory [4, 33], as we will argue at the end of this section. In Section 12 and Appendix D we will explain how our proof of Theorem 7.1 can be easily adapted to demonstrate that the coefficients of the low-energy expansion of all $J_{\rho,\sigma}$'s are single-valued multiple zeta values. In order to obtain the same statement for the more general closed string integrals (1.2) considered in part I, one must overcome the problem that the low-energy expansion of such integrals is usually defined by analytic continuation. Indeed, they are typically expanded around a point which lies outside of the region of convergence (see the next subsection). Studying such analytic continuations goes beyond the scope of this paper, so we content ourselves to work with subfamilies of integrals where this is not necessary, like the $J_{\rho,\sigma}$'s. We believe, however, that the standard approach in string theory (using integration-by-part

(7) This implies that R_k can be uniquely extended to a meromorphic function on $U_k \cup \mathcal{C}_k$.

(8) This uniquely determines P_k (up to the addition of a polynomial). Since our focus is rather on the coefficients of the holomorphic part H_k, we will not provide an explicit formula for the polar part. Such a formula can be worked out from our (constructive) proof, or can alternatively deduced from the results obtained in [34].

identities) to analytically continue tree-level string integrals would allow to prove with the techniques developed in the next sections that the coefficients of the low-energy expansion of all integrals (1.2) are single-valued multiple zeta values, as claimed in part I.

7.1. The region of convergence. — We will write down the region of convergence of the integral for the more natural set of variables $a_i := \alpha_i + 1$, $b_i := \beta_i + 1$, $c_{i,j} := \gamma_{i,j}$.

Proposition 7.2. — *Let $P = (a_1, \dots, a_k, b_1, \dots, b_k, c_{1,2}, \dots, c_{k-1,k})$ belong to $\mathbb{C}^{k(k+3)/2}$. For each subset of indices $I = \{i_1, \dots, i_h\} \subset \{1, \dots k\}$, with $i_1 < i_2 < \cdots < i_h$, we define*

$$U_{I,0} := \{P = (\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}) \in \mathbb{C}^{k(k+3)/2} \,:\, \mathrm{Re}\Big(\sum_{s=1}^{h} a_{i_s} + \sum_{1\leq s<r\leq h} c_{i_s,i_r}\Big) > -h\}, \tag{7.2}$$

$$U_{I,1} := \{P = (\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}) \in \mathbb{C}^{k(k+3)/2} \,:\, \mathrm{Re}\Big(\sum_{s=1}^{h} b_{i_s} + \sum_{1\leq s<r\leq h} c_{i_s,i_r}\Big) > -h\} \tag{7.3}$$

Morever, if $|I| = h \geq 2$ we define

$$D_I := \{P = (\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}) \in \mathbb{C}^{k(k+3)/2} \,:\, \mathrm{Re}\Big(\sum_{1\leq s<r\leq h} c_{i_s,i_r}\Big) > 1-h\}. \tag{7.4}$$

Finally, let

$$U_\infty := \{P = (\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}) \in \mathbb{C}^{k(k+3)/2} \,:\, \mathrm{Re}\Big(\sum_{i=1}^{k}(a_i+b_i) + \sum_{1\leq i<j\leq k} c_{i,j}\Big) < -h\}. \tag{7.5}$$

The region of absolute convergence $\mathcal{C}_k \subset \mathbb{C}^{k(k+3)/2}$ of the integral

$$\int_{(\mathbb{P}^1_{\mathbb{C}})^k} \prod_{1\leq i<j\leq k} |z_j - z_i|^{2c_{i,j}} \prod_{i=1}^{k} |z_i|^{2a_i} |z_i - 1|^{2b_i}\, d^2 z_i \tag{7.6}$$

is given by the intersection of all the domains $U_{I,0}, U_{I,1}, D_I, U_\infty$. In this region, the integral (7.6) defines a holomorphic function.

A proof of this result can be obtained by dividing the domain of integration into all possible regions $0 \leq |z_{i_k}| \leq \cdots \leq |z_{i_1}|$ and operating the change of variables $u_j = z_{i_j}/z_{i_{j-1}}$ (setting $z_{i_0} = 0$). This is enough to isolate the domains $U_{I,0}$, D_I and U_∞, where the integral is convergent near $(0,\dots,0)$, near the diagonals and at infinity, resepctively. The region $U_{I,1}$ where the integral is convergent near $(1,\dots,1)$ is then obtained by substituting $a_i \leftrightarrow b_i$. We leave the details to the reader.

One can immediately see that the point $P = (\mathbf{a},\mathbf{b},\mathbf{c})$ with $\mathbf{a} = (-1,\dots,-1)$, $\mathbf{b} = (-1,\dots,-1)$ and $\mathbf{c} = (0,\dots,0)$ belongs to the boundary of $\mathcal{C}_k$, and that for $\mathrm{Re}(\varepsilon) > 0$ small enough all the points $P = (\mathbf{a},\mathbf{b},\mathbf{c})$ with $\mathbf{a} = (-1+\varepsilon,\dots,-1+\varepsilon)$, $\mathbf{b} = (-1+\varepsilon,\dots,-1+\varepsilon)$ and $\mathbf{c} = (0,\dots,0)$ are contained in $\mathcal{C}_k$ (which in particular in always non-empty). This means that our integral (7.1) is absolutely convergent for all $\gamma_{i,j} = 0$ and for any $\alpha_1,\dots,\alpha_k,\beta_1,\dots,\beta_k$ with positive small enough real part. For example, for $k = 1$ and $k = 2$ the regions of convergence of (7.6) are

$$\mathcal{C}_1 = \{(a_1,b_1) : \mathrm{Re}(a_1), \mathrm{Re}(b_1) > -1,\ \mathrm{Re}(a_1+b_1) < -1\}, \tag{7.7}$$

$$\begin{aligned}\mathcal{C}_2 =& \{(a_1,a_2,b_1,b_2,c_{1,2}) : \mathrm{Re}(a_1), \mathrm{Re}(a_2), \mathrm{Re}(b_1), \mathrm{Re}(b_2), \mathrm{Re}(c_{1,2}) > -1\\ & \mathrm{Re}(a_1+b_1+c_{1,2}), \mathrm{Re}(a_2+b_2+c_{1,2}) > -2,\\ & \mathrm{Re}(a_1+a_2+b_1+b_2+c_{1,2}) < -2\}\end{aligned} \tag{7.8}$$

7.2. Multiple complex integrals in superstring amplitudes. — Let us now comment on the integrals which appear in the computation of closed superstring amplitudes (see for instance [2]) and on their relationship with our integrals R_k. Let ρ,σ be two permutations in the symmetric group $\mathfrak{S}_k$, and let us consider the integrals $J_{\rho,\sigma}(\mathbf{a},\mathbf{b},\mathbf{c})$ given for $(\mathbf{a},\mathbf{b},\mathbf{c}) \in \mathbb{C}^{k(k+3)/2}$ as above by

$$\int_{(\mathbb{P}^1_{\mathbb{C}})^k} \frac{\prod_{1\leq i<j\leq k} |z_j - z_i|^{2c_{i,j}} \prod_{i=1}^k |z_i|^{2a_i} |z_i - 1|^{2b_i}\, d^2 z_i}{z_{\rho(1)}\,\overline{z}_{\sigma(1)}(1-z_{\rho(k)})(1-\overline{z}_{\sigma(k)})\prod_{i=2}^k (z_{\rho(i)} - z_{\rho(i-1)})(\overline{z}_{\sigma(i)} - \overline{z}_{\sigma(i-1)})}. \tag{7.9}$$

Since the proof of Proposition 7.2 can be straightforwardly modified to find the region of convergence of integrals with arbitrary powers of $(z_i - z_j)$ and $(\overline{z}_i - \overline{z}_j)$ (possibly with $z_i, z_j = 0, 1$), one can check that the origin is always either inside or at the boundary of the region of convergence of $J_{\rho,\sigma}(\mathbf{a}, \mathbf{b}, \mathbf{c})$, which therefore do not need to be analytically continued. After subtracting a suitable rational function which encodes its divergence, each $J_{\rho,\sigma}$ can be expanded as a power series around the origin; this is called the *low energy expansion*[(9)]. It was conjectured in [2,3] that the coefficients of the low-energy expansion of $J_{\rho,\sigma}(\mathbf{a}, \mathbf{b}, \mathbf{c})$ should be single-valued multiple zeta values[(10)]. We claim that this can be proven using the machinery developed in Section 9, along the same lines of the proof of Theorem 7.1 and using the above-mentioned fact that the origin lies (at worse) on the boundary of the region of convergence. Details will be given at the end of Section 12, after the proof of Theorem 7.1, as well as in Appendix D.

We now want to clarify the precise relationship between our integrals R_k and the integrals $J_{\rho,\sigma}$ from superstring amplitudes. One can show by induction the identity

$$\sum_{\rho\in\mathfrak{S}_k} \frac{1}{z_{\rho(1)}(1-z_{\rho(k)})\prod_{i=2}^{k}(z_{\rho(i)}-z_{\rho(i-1)})} = \prod_{i=1}^{k}\frac{1}{z_i(1-z_i)}. \tag{7.10}$$

This implies that

$$\begin{aligned} &R_k(\alpha_1,\dots,\alpha_k,\beta_1,\dots,\beta_k,\gamma_{1,2},\dots\gamma_{1,k},\gamma_{2,3},\dots\gamma_{2,k},\dots,\gamma_{k-1,k}) = \\ &\sum_{(\rho,\sigma)\in\mathfrak{S}_k^2} J_{\rho,\sigma}(\alpha_1,\dots,\alpha_k,\beta_1,\dots,\beta_k,\gamma_{1,2},\dots\gamma_{1,k},\gamma_{2,3},\dots\gamma_{2,k},\dots,\gamma_{k-1,k}), \end{aligned} \tag{7.11}$$

which tells us that we are indeed considering an integral which comes from genus-zero closed superstring theory.

(9) A rigorous proof of this fact, together with an explicit description of the rational functions involved, can be found in [34].
(10) This is now a theorem, thanks to [33,34].

8. Hyperlogarithms

Hyperlogarithms are holomorphic multi-valued functions given by homotopy invariant iterated integrals on the punctured complex plane. In this section we will define them and recall some of their properties from the literature. Our main references will be [47] and [51].

8.1. Definition and first properties. — Let M be a smooth manifold, let $\omega_1, \ldots, \omega_r$ denote smooth complex-valued 1-forms on M and let $\gamma : [0,1] \to M$ be a parametrization of a piecewise smooth path. We can write $\gamma^*\omega_i = f_i(t)dt$ for some piecewise smooth function $f_i : [0,1] \to \mathbb{C}$, where $1 \leq i \leq r$. The *iterated integral* of $\omega_1, \ldots, \omega_r$ along γ is

$$\int_\gamma \omega_1 \cdots \omega_r := \int_{1 \geq t_1 \geq \cdots \geq t_r \geq 0} f_1(t_1) \cdots f_r(t_r)\, dt_1 \cdots dt_r. \tag{8.1}$$

We will call r the *length* of the iterated integral.

Let $\sigma_0, \sigma_1, \sigma_2 \ldots, \sigma_n$ be $n+1$ distinct complex numbers. We will think of them also as formal non-commutative letters of an *alphabet* $X = \{\hat{\sigma}_0, \ldots \hat{\sigma}_n\}$, denote X^* the free non-commutative monoid given by all possible words in this alphabet and the empty word e, and denote by $\mathbb{C}\langle X\rangle$ the free $\mathbb{C}$-algebra generated by X^* and equipped with the (commutative) shuffle product. A string of r consecutive repetition of the same letter $\hat{\sigma}_i$ will be denoted by $\hat{\sigma}_i^r$, and the length of a word $w = \hat{\sigma}_{i_1} \cdots \hat{\sigma}_{i_r}$ is $|w| = r$. From now on we will write σ both meaning the complex number and the associated formal letter, except for those formulae where we need to distinguish between the two meanings. Moreover, we set $\sigma_0 = 0$ and $\sigma_1 = 1$. Let us fix a simply connected domain U obtained from $\mathbb{C}$ by cutting out closed half lines $l(\sigma_i)$ starting at the points σ_i and not intersecting among themselves, and let us choose a branch of the logarithms on U. It is known that any iterated integral involving the differential forms $dx/(x-\sigma_i)$ is homotopy invariant in $D := \mathbb{P}^1_{\mathbb{C}} \setminus (X \cup \{\infty\})$ [64]. Using the fact that U is simply connected, we define *hyperlogarithms* in

the following way: if $i_r \neq 0$ and $w = \sigma_{i_1} \cdots \sigma_{i_r}$ we set

$$L_w(z) = \int_{[0,z]} \frac{dx}{x - \sigma_{i_1}} \cdots \frac{dx}{x - \sigma_{i_r}}, \tag{8.2}$$

where $[0, z]$ is any path contained in U starting at 0 and ending at z. Homotopy invariance allows us to extend these functions to holomorphic multi-valued functions on D. In order to define hyperlogarithms for any word $w \in X^*$, we require $w \to L_w(z)$ to be a homomorphism (i.e. it respects the shuffle product), and we set $L_{0^r}(z) = \log^r(z)/r!$, $L_e(z) = 1$. Note that, for all $i \neq 0$,

$$L_{\sigma_i^r}(z) = \frac{1}{r!} \log^r \left(1 - \frac{z}{\sigma_i}\right). \tag{8.3}$$

Moreover, for any $w \in X^*$ and for any $\sigma_i \in X$ one locally has for some non-negative integer K_i

$$L_w(z) = \sum_{k=0}^{K_i} \sum_{j \geq 0} c_{k,j}^{(i)}(w)(z - \sigma_i)^j \log^k(z - \sigma_i), \tag{8.4}$$

as well as, for some non-negative integer K_∞,

$$L_w(z) = \sum_{k=0}^{K_\infty} \sum_{j \geq 0} c_{k,j}^{(\infty)}(w) z^{-j} \log^k(z - \sigma_i), \tag{8.5}$$

where the coefficients $c_{j,k}^{(\bullet)}(w)$ are complex numbers. For $\alpha \in X \cup \{\infty\}$ we define the *regularized value* of $L_w(z)$ at α to be $L_w(\alpha) := c_{0,0}^{(\alpha)}(w)$. From now on, whenever we talk of special values of hyperlogarithms, we implicitly mean regularized values. It will be useful later to introduce the following notation: for an alphabet $X = \{\hat{\sigma}_j\}_{0 \leq j \leq n}$, a ring $R \leq \mathbb{C}$ and $0 \leq j \leq n$ we define the free R-module $\mathcal{S}_{X,R,j} = R\langle L_w(\sigma_j) : w \in X^* \rangle$. Each of these modules is actually a subring of $\mathbb{C}$, because hyperlogarithms satisfy shuffle relations. Then we define $\mathcal{S}_{X,R}$ to be the smallest subring of $\mathbb{C}$ containing all $\mathcal{S}_{X,R,j}$. If $X = \{0, 1\}$, the special values $\zeta_w := L_w(1)$ are called *multiple zeta values*. Multiple zeta values are real numbers, they form a $\mathbb{Q}$-algebra $\mathcal{Z} := \mathcal{S}_{\{0,1\},\mathbb{Q},1} \equiv \mathcal{S}_{\{0,1\},\mathbb{Q}}$ and are usually defined

using the nested sum notation

$$\zeta(k_1,\dots,k_r)=\sum_{0<v_1<\cdots<v_r}\frac{1}{v_1^{k_1}\cdots v_r^{k_r}}, \tag{8.6}$$

where $k_i\in\mathbb{N}$, $k_r\geq 2$. It is a simple exercise to verify that $\zeta(k_1,\dots,k_r)=\zeta_{0^{k_r-1}1\cdots 0^{k_1-1}1}$.

One can characterize hyperlogarithms on any simply-connected dense open subset of the punctured complex plane D in the following way:

Theorem 8.1 **(Brown, [47]).** — *Let $D=\mathbb{P}^1_{\mathbb{C}}\backslash(X\cup\{\infty\})$, and let U be a simply-connected dense open subset of D. The hyperlogarithms $\{L_w(z):w\in X^*\}$ are the unique family of holomorphic functions satisfying for $z\in U$*

$$\frac{\partial}{\partial z}L_{\sigma_i w}(z)=\frac{L_w(z)}{z-\sigma_i} \tag{8.7}$$

such that $L_e(z)=1$, $L_{0^r}(z)=\log^r(z)/r!$ for all $n\in\mathbb{N}$ and $L_w(z)\to 0$ as $z\to 0$ for every other word w. We say that w is the label and z is the argument of $L_w(z)$.

8.2. Generating series and monodromy. — It is often convenient to consider the generating series of hyperlogarithms

$$L_X(z)=\sum_{w\in X^*}L_w(z)\,w. \tag{8.8}$$

Eq. (8.7) is then equivalent to

$$\frac{\partial}{\partial z}L_X(z)=\sum_{i=0}^{n}\frac{\hat{\sigma}_i}{z-\sigma_i}L_X(z), \tag{8.9}$$

and L_X is the unique solution of (8.9) on $U=\mathbb{C}\setminus\bigcup_i l(\sigma_i)$ taking values in $\mathbb{C}\langle\langle X\rangle\rangle$ (non-commutative formal power series in X^*) such that

$$L_X(z)=f_0(z)\exp(\hat{\sigma}_0\log(z)), \tag{8.10}$$

where $f_0(z)$ is a holomorphic function on $\mathbb{C}\setminus\bigcup_{i\neq 0}l(\sigma_i)$ which satisfies $f_0(0)=1$ [65]. For the alphabet $X=\{0,1\}$, the generating series of

regularized special values at $z = 1$

$$Z(\hat{\sigma}_0, \hat{\sigma}_1) := L_{\{0,1\}}(1) \tag{8.11}$$

is known as the *Drinfel'd associator*, and equation (8.9) is known as the *Knizhnik-Zamolodchikov equation*. More generally, we will later be interested in all (regularized) special values $L_X(\sigma_i)$. The following lemma is a consequence of the path-concatenation property of iterated integrals[(11)].

Lemma 8.2. — *For any* $1 \leq i \leq n$ *let* $Y(X,i) := \{\hat{\sigma}_i - \hat{\sigma}_0, \ldots, \hat{\sigma}_i - \hat{\sigma}_n\}$. *We have*

$$L_X(\sigma_i - z) = L_{Y(X,i)}(z) L_X(\sigma_i) \tag{8.12}$$

Let now $C_m^\infty(D)$ denote the algebra of multi-valued real analytic functions on the punctured plane D, and let us fix $z_0 \in D$. The fundamental group $\pi_1(D, z_0)$ is the free group on generators $\gamma_0, \ldots, \gamma_1$, where each γ_i is a loop based at z_0 and winding once around σ_i in the positive direction. For each $0 \leq i \leq n$ we write $M_{\sigma_i} : C_m^\infty(D) \to C_m^\infty(D)$ for the monodromy operator given by analytic continuation of functions around γ_i. The maps M_{σ_i} are homomorphisms of algebras which commute with $\partial/\partial z$ and $\partial/\partial \overline{z}$.

Proposition 8.3 **(Brown, [47])**. — *For each* $1 \leq i \leq n$ *we have*

$$M_{\sigma_i} L_X(z) = L_X(z)(L_X(\sigma_i))^{-1} e^{2\pi i \hat{\sigma}_i} L_X(\sigma_i). \tag{8.13}$$

This implies that non-constant hyperlogarithms are not single-valued on D.

8.3. Integration. — We conclude this section by stating two fundamental results about hyperlogarithms which imply that we can always find primitives of hyperlogarithms, and that we can do it algorithmically. First of all, we need to introduce some notation. For any subring R of $\mathbb{C}$

(11) If $\gamma_1(1) = \gamma_2(0)$ then $\int_{\gamma_1\gamma_2} \omega_1 \cdots \omega_n = \sum_{i=0}^n \int_{\gamma_2} \omega_1 \cdots \omega_i \int_{\gamma_1} \omega_{i+1} \cdots \omega_n$.

and any alphabet X let us define $\mathcal{H}_{X,R}$ to be the ring generated by R-linear combinations of hyperlogarithms w.r.t. the alphabet X, graded[(12)] by the length $|w| := n$ of words $w = \sigma_{i_1} \cdots \sigma_{i_n}$. Moreover, let

$$\mathcal{O}_{X,R} = R\left[z, \sigma_i, \frac{1}{z-\sigma_i}, \frac{1}{\sigma_j - \sigma_i} : \sigma_i, \sigma_j \in X, \sigma_i \neq \sigma_j\right] \tag{8.14}$$

and let $\mathcal{A}_{X,R}$ be the free $\mathcal{O}_{X,R}$-module generated by $\mathcal{H}_{X,R}$. Note that $\mathcal{A}_{X,R}$ is closed under differentiation (when R is a field, it is a differential algebra).

***Theorem 8.4* (Brown, [47]).** — *Hyperlogarithms $L_w(z)$ form an $\mathcal{O}_{X,\mathbb{C}}$-basis for $\mathcal{A}_{X,\mathbb{C}}$. The only algebraic relations with coefficients in $\mathcal{O}_{X,\mathbb{C}}$ between hyperlogarithms are given by the shuffle product. Every function $f(z) \in \mathcal{A}_{X,\mathbb{C}}$ has a primitive in $\mathcal{A}_{X,\mathbb{C}}$ which is unique up to a constant.*

In order to calculate integrals of hyperlogarithms algorithmically, it is useful to consider $L_w(z)$ as multivariable functions depending also on the punctures $\sigma_i \in \mathbb{C}$ appearing in the label. First of all, one can compute the total derivative explicitly from the definition, obtaining (see [51]):

Lemma 8.5. — *If we denote $\sigma_{i_0} := z$, $\sigma_{i_{r+1}} := 0$, $d\log(0) := 0$, we have for any $L_{\sigma_{i_1}\cdots\sigma_{i_r}}(z)$*

$$dL_w(z) = \sum_{k=1}^{n} L_{\sigma_{i_1}\cdots\sigma_{i_{k-1}}\sigma_{i_{k+1}}\cdots\sigma_{i_r}}(z) d\log\left(\frac{\sigma_{i_k} - \sigma_{i_{k-1}}}{\sigma_{i_{k+1}} - \sigma_{i_k}}\right). \tag{8.15}$$

Since some of the terms in the formula may tend to ∞ when $z \to \sigma_i$, it is important to mention that we implicitly consider regularized values. For each $2 \leq i \leq n$, let us define the alphabet $X_i = X \setminus \{\sigma_i\}$. Comparing the two sides of eq. (8.15) leads to the following:

Corollary 8.6. — *For each $w \in X^*$*

$$\frac{\partial}{\partial \sigma_i} L_w(z) = \sum_{\substack{\tau \in X_i \cup \{z\} \\ v \in X^*}} \frac{\lambda_{\tau,v}}{\sigma_n - \tau} L_v(z), \tag{8.16}$$

(12) The fact that the length is a grading follows from Theorem 8.4.

where $\lambda_{\tau,v} \in \mathbb{Z}$, $|v| < |w|$ *and the sum is finite.*

The main consequence of this fact is that, if we consider hyperlogarithms as functions of a puncture σ_i, they can be rewritten as hyperlogarithms in σ_i[(13)]:

Proposition 8.7. — *For any* $w \in X^*$ *and any* $1 \leq i, j \leq n$

$$L_w(\sigma_j) = \sum_u c_u L_u(\sigma_i), \tag{8.17}$$

where $c_u \in \mathcal{S}_{X_i,\mathbb{Q}[2\pi i]}$ *and the sum is over a finite number of words* $u \in X_i^*$.

Proof. This can be proven by induction on $|w|$. If $|w| = 1$ then one needs to rewrite $\log(1 - \sigma_j/\sigma_i)$ in terms of $L_0(\sigma_i)$ and $L_{\sigma_j}(\sigma_i)$. This introduces an integer multiple of πi which depends on the chosen branch of the logarithm. In the general case, by the inductive hypothesis and Corollary 8.6 we get

$$\frac{\partial}{\partial \sigma_i} L_w(\sigma_j) = \sum_{\substack{\tau \in X_i \cup \{z\} \\ u \in X_i^*}} \frac{c_u}{\sigma_i - \tau} L_u(\sigma_i), \tag{8.18}$$

where $c_u \in \mathcal{S}_{X_i,\mathbb{Q}[2\pi i]}$. Hence we have

$$L_w(\sigma_j) = \sum_{\substack{\tau \in X_i \cup \{z\} \\ u \in X_i^*}} c_u L_{\tau u}(\sigma_i) + c, \tag{8.19}$$

and $c = \lim_{\sigma_i \to 0} L_w(\sigma_j) \in \mathcal{S}_{X_i,\mathbb{Q}[2\pi i]}$[(14)].

□

We conclude this section by mentioning that, if we consider the complex plane's punctures as variables, hyperlogarithms can be written in

(13) This is the key ingredient of the Maple program `HyperInt` developed by Panzer [66] which allows to compute multiple integrals of hyperlogarithms.

(14) We always consider regularized limits. In the case where $|w| = 1$, this regularized limit introduces the extra πi.

terms of *multiple polylogarithms*

$$\mathrm{Li}_{k_1,\ldots,k_r}(z_1,\ldots,z_r) := \sum_{0<v_1<\cdots<v_r} \frac{z_1^{v_1}\cdots z_r^{v_r}}{v_1^{k_1}\cdots v_r^{k_r}}, \tag{8.20}$$

where $k_1,\ldots,k_r \in \mathbb{N}$, and vice versa. Multiple polylogarithms play a fundamental role in the theory of iterated integrals on the moduli space $\mathfrak{M}_{0,n}$ of punctured Riemann spheres[(15)]. In particular, all statements presented in this subsection can also be seen as corollaries of the machinery developed by Brown in [61] to study iterated integrals on $\mathfrak{M}_{0,n}$.

9. Single-valued hyperlogarithms

Let us consider the map $\sim\colon X^* \to X^*$ which sends $w = \sigma_{i_1}\cdots\sigma_{i_r}$ to $\tilde{w} = \sigma_{i_r}\cdots\sigma_{i_1}$. This map can be extended by linearity to $\mathbb{C}\langle\langle X\rangle\rangle$. We define the *single-valued hyperlogarithm* $\mathcal{L}_w(z)$ to be the coefficient of the word $w \in X^*$ in the generating series

$$\mathcal{L}_X(z) := L_X(z)\widetilde{\overline{L_{X'}}}(z), \tag{9.1}$$

where X' is the alphabet given by the letters $\hat{\sigma}'_i$ determined by the system of equations

$$\widetilde{\overline{L_{X'}}}(\sigma_i)\hat{\sigma}'_i\widetilde{\overline{L_{X'}}}(\sigma_i)^{-1} = L_X(\sigma_i)^{-1}\hat{\sigma}_i L_X(\sigma_i). \tag{9.2}$$

This construction was first proposed by Brown in [47], where it was shown that eq. (9.2) admits a unique solution in terms of elements of $\mathbb{C}\langle\langle X\rangle\rangle$. In particular, one has $\hat{\sigma}'_0 = \hat{\sigma}_0$ and, for $i \neq 0$, $\hat{\sigma}'_i = \hat{\sigma}_i$ modulo words $w \in X^*$ with $|w| \geq 4$. Knowing that such a solution exists, we get by Proposition 8.3 that for all $0 \leq i \leq n$

$$\begin{aligned} M_{\sigma_i}\mathcal{L}_X(z) &= L_X(z)L_X(\sigma_i)^{-1}e^{2\pi i\hat{\sigma}_i}L_X(\sigma_i)\widetilde{\overline{L_{X'}}}(\sigma_i)e^{-2\pi i\hat{\sigma}'_i}\widetilde{\overline{L_{X'}}}(\sigma_i)^{-1}\widetilde{\overline{L_{X'}}}(z) \\ &= L_X(z)\widetilde{\overline{L_{X'}}}(z) = \mathcal{L}_X(z), \end{aligned} \tag{9.3}$$

(15) They generate all periods of the pro-unipotent fundamental groupoid of $\mathfrak{M}_{0,n}$ [61].

which implies that $\mathcal{L}_X(z)$ is indeed single-valued. Simple examples of single-valued hyperlogarithms are given by $\mathcal{L}_{0^n}(z) = \log^n|z|^2/n!$ and (for $i \neq 0$) by

$$\mathcal{L}_{\sigma_i^n}(z) = \frac{1}{n!}\log^n\left|1-\frac{z}{\sigma_i}\right|^2. \tag{9.4}$$

These examples reflect the fact that $\log(z)$ is not single-valued on the whole complex plane, but one can obtain a single valued function by adding its complex conjugate $\overline{\log(z)}$, which has the opposite monodromies.

For any subring $R \leq \mathbb{C}$, we denote by $\mathcal{H}^{\mathrm{sv}}_{X,R}$ the ring generated over R by single-valued hyperlogarithms. Moreover, we denote by $\mathcal{A}^{\mathrm{sv}}_{X,R}$ the free $\mathcal{O}_{X,R}\otimes_R\overline{\mathcal{O}_{X,R}}$-module generated by $\mathcal{H}^{\mathrm{sv}}_{X,R}$, where $\overline{\mathcal{O}_{X,R}}$ is the ring of functions generated over R by the complex conjugates of the generators of $\mathcal{O}_{X,R}$. Notice that $\mathcal{H}^{\mathrm{sv}}_{X,\mathbb{C}} \subset \mathcal{H}_{X,\mathbb{C}}\otimes\overline{\mathcal{H}_{X,\mathbb{C}}}$ and $\mathcal{A}^{\mathrm{sv}}_{X,\mathbb{C}} \subset \mathcal{A}_{X,\mathbb{C}}\otimes\overline{\mathcal{A}_{X,\mathbb{C}}}$.

Lemma 9.1 **(Brown, [47]).** — *The only algebraic relations in the space $\mathcal{A}_{X,\mathbb{C}}\otimes\overline{\mathcal{A}_{X,\mathbb{C}}}$ are given by the shuffle product.*

Theorem 9.2 **(Brown, [47]).** — *The series $\mathcal{L}_X(z)$ is single-valued, and is the unique solution to the differential equations*

$$\frac{\partial}{\partial z}\mathcal{L}_X(z) = \sum_{i=0}^{n}\frac{\hat{\sigma}_i}{z-\sigma_i}\mathcal{L}_X(z) \tag{9.5}$$

and

$$\frac{\partial}{\partial \overline{z}}\mathcal{L}_X(z) = \mathcal{L}_X(z)\sum_{i=0}^{n}\frac{\hat{\sigma}_i'}{\overline{z}-\overline{\sigma}_i} \tag{9.6}$$

such that $\mathcal{L}_X(z) \sim \exp(\hat{\sigma}_0\log|z|^2)$ as $z\to 0$. The functions $\mathcal{L}_w(z)$ are linearly independent over $\mathcal{O}_{X,\mathbb{C}}\otimes\overline{\mathcal{O}_{X,\mathbb{C}}}$ and satisfy the shuffle relations. Every element in $\mathcal{A}^{\mathrm{sv}}_{X,\mathbb{C}}$ has a primitive with respect to $\partial/\partial z$ and $\partial/\partial\overline{z}$, and every single-valued function $f(z)\in\mathcal{A}_{X,\mathbb{C}}\otimes\overline{\mathcal{A}_{X,\mathbb{C}}}$ can be written as a unique $\mathcal{O}_{X,\mathbb{C}}\otimes\overline{\mathcal{O}_{X,\mathbb{C}}}$-linear combination of functions $\mathcal{L}_w(z)$.

To recapitulate, we now know that all $\mathcal{L}_w(z)$ can be written in a unique way as $\mathbb{C}$-linear combinations of products $L_{w_1}(z)\overline{L_{w_2}(z)}$ which are single-valued in z. Looking more carefully at the equations defining single-valued hyperlogarithms and the alphabet X', one can see that the coefficients of these linear combinations must in fact belong to $\mathcal{S}_{X,\mathbb{Q}} \otimes_{\mathbb{Q}} \overline{\mathcal{S}_{X,\mathbb{Q}}} \subset \mathbb{C}$. It will be crucial for us to demonstrate that these coefficients belong to an even smaller ring (see Proposition 9.5).

9.1. Special values. — As a consequence of the asymptotic expansions (8.4) and (8.5) of hyperlogarithms one can deduce the following:

Lemma 9.3. — *Let $f(z) \in \mathcal{A}^{\rm sv}_{X,\mathbb{C}}$, then for each $\sigma_i \in X$ we have locally*

$$f(z) = \sum_{k=0}^{K_i} \sum_{m=M_i}^{\infty} \sum_{n=N_i}^{\infty} c^{(i)}_{k,m,n} (\log|z-\sigma_i|^2)^k (z-\sigma_i)^m (\overline{z}-\overline{\sigma}_i)^n, \tag{9.7}$$

as well as

$$f(z) = \sum_{k=0}^{K_\infty} \sum_{m=-\infty}^{M_\infty} \sum_{n=-\infty}^{N_\infty} c^{(\infty)}_{k,m,n} (\log|z|^2)^k z^m \overline{z}^n, \tag{9.8}$$

where $c^{(\bullet)}_{k,m,n} \in \mathbb{C}$, $M_\bullet, N_\bullet \in \mathbb{Z}$. Moreover, if $f(z) = \mathcal{L}_w(z)$ then $M_\bullet = 0$ and $N_\bullet = 0$.

Using this lemma one can define, just as we did in the holomorphic case, regularized special values of single-valued hyperlogarithms at points $\sigma_i \in X$. The special values

$$\zeta^{\rm sv}_w := \mathcal{L}_w(1) \tag{9.9}$$

for the alphabet $X = \{0,1\}$ are called *single-valued multiple zeta values*. These numbers form a $\mathbb{Q}$-sub-algebra $\mathcal{Z}^{\rm sv}$ of the algebra $\mathcal{Z}$ of multiple zeta values, which we can think of (assuming standard conjectures on multiple zeta values) as the image of an endomorphism "sv" of $\mathcal{Z}$. If we use the notation $\zeta(k_1,\ldots,k_r)$ associated with the nested sum representation of multiple zeta values, one has for instance $\zeta^{\rm sv}(2k) = 0$ and $\zeta^{\rm sv}(2k+1) = 2\zeta(2k+1)$ for all $k \geq 1$. The structure of this algebra was first studied

in [1]. More generally, we define $\mathcal{S}^{\mathrm{sv}}_{X,R,j} = R\langle \mathcal{L}_w(\sigma_j) : w \in X^* \rangle$ and $\mathcal{S}^{\mathrm{sv}}_{X,R}$ to be the ring generated over R by all $\mathcal{S}^{\mathrm{sv}}_{X,R,j}$. In particular, $\mathcal{Z}^{\mathrm{sv}} = \mathcal{S}^{\mathrm{sv}}_{\{0,1\},\mathbb{Q},1} \equiv \mathcal{S}^{\mathrm{sv}}_{\{0,1\},\mathbb{Q}}$. We want to assign a weight to certain elements of $\mathcal{S}^{\mathrm{sv}}_{X,R}$ in the following way: if $c \in R$ then $W(c) = 0$, if $c = \mathcal{L}_w(\sigma_j)$ then $W(c) = |w|$, if $c = a \cdot b$ then $W(c) = W(a) + W(b)$. Finally, if $c \in \mathcal{S}^{\mathrm{sv}}_{X,R}$ is an R-linear combination of monomials given by products of special values of single-valued hyperlogarithms, such that each monomial has the same weight α, we say that c has homogeneous weight $W(c) = \alpha$. It is important to remark that, a priori, the weight is well-defined on elements of $\mathcal{S}^{\mathrm{sv}}_{X,R}$ only if we think of them as abstract symbols, without taking into account the relations among them. Indeed, while the length is a grading for the algebra of single-valued hyperlogarithms (because of Lemma 9.1), the weight just defined on their special values a priori is not, and it may in principle happen that there are linear relations among elements of different weight[(16)]. In other words, the weight can only be proven to be a filtration on $\mathcal{S}^{\mathrm{sv}}_{X,R}$, but for our purposes it is enough to know that it is well defined on single-valued hyperlogarithms when considered as abstract symbols.

Once again, we will need to consider the generating series of regularized special values $\mathcal{L}_X(\sigma_i)$ of hyperlogarithms.

Lemma 9.4. — *For any $1 \leq i \leq n$ let $Y(X,i) = \{\hat{\sigma}_i - \hat{\sigma}_0, \dots, \hat{\sigma}_i - \hat{\sigma}_n\}$. We have*

$$\mathcal{L}_X(\sigma_i - z) = \mathcal{L}_{Y(X,i)}(z)\mathcal{L}_X(\sigma_i) \tag{9.10}$$

Proof. By Lemma 8.2 we know that

$$\mathcal{L}_X(\sigma_i - z) = L_{Y(X,i)}(z)L_X(\sigma_i)\widetilde{\overline{L_{X'}}}(\sigma_i)\widetilde{\overline{L_{(Y(X,i))'}}}(z) \tag{9.11}$$

[(16)] A famous conjecture of Zagier predicts that this will never happen in the case of multiple zeta values.

The right-hand sides of (9.10) and (9.11) both satisfy the differential equation

$$\frac{\partial}{\partial z}F(z) = \sum_{j=0}^{n} \frac{\hat{\tau}_j}{z-\tau_j} F(z), \tag{9.12}$$

where $\tau_j := \sigma_i - \sigma_j$. Since their asymptotic behaviour at $z = 0$ is the same, they must coincide.

□

The following proposition is one of the main technical points of this section. It is an easy generalization of the analogous result for the alphabet $\{0, 1\}$ demonstrated by Schnetz in [52], but because of its relevance for our argument we prefer to repeat the proof here.

Proposition 9.5. — *For all* $w \in X^*$

$$\mathcal{L}_w(z) = \sum_{w_1,w_2\in X^*} c_{w_1,w_2} L_{w_1}(z)\overline{L_{w_2}(z)}, \tag{9.13}$$

where all c_{w_1,w_2} *belong to* $\mathcal{S}^{\mathrm{sv}}_{X,\mathbb{Q}}$ *and have homogeneous weight* $W(c_{w_1,w_2}) = |w| - |w_1| - |w_2|$.

Proof. First of all, by Theorem 9.2 we have for all $0 \leq j \leq n$

$$\begin{aligned}\lim_{z\to\sigma_j}(\overline{z}-\overline{\sigma}_j)\frac{\partial}{\partial\overline{z}}\mathcal{L}_X(z) &= \lim_{z\to\sigma_j}(\overline{z}-\overline{\sigma}_j)\mathcal{L}_X(z)\left(\frac{\hat{\sigma}'_0}{\overline{z}}+\cdots+\frac{\hat{\sigma}'_n}{\overline{z}-\overline{\sigma}_n}\right)\\ &= \mathcal{L}_X(\sigma_j)\hat{\sigma}'_j.\end{aligned} \tag{9.14}$$

In particular, since for $j = 0$ we know that $\hat{\sigma}'_0 = \hat{\sigma}_0$ and that $\mathcal{L}_X(0) = 1$, we get

$$\lim_{z\to 0}\overline{z}\frac{\partial}{\partial\overline{z}}\mathcal{L}_X(z) = \hat{\sigma}_0, \tag{9.15}$$

which implies that

$$\lim_{z\to 0}\overline{z}\frac{\partial}{\partial\overline{z}}\mathcal{L}_w(z) = \delta_{w,\hat{\sigma}_0} \tag{9.16}$$

Since by Lemma 9.4 we know that $\mathcal{L}_X(\sigma_j - z) = \mathcal{L}_{Y(X,j)}(z)\mathcal{L}_X(\sigma_j)$ we conclude (using also eq. (9.16)) that for any $w \in X^*$

$$\lim_{z\to\sigma_j}(\overline{z}-\overline{\sigma}_j)\frac{\partial}{\partial\overline{z}}\mathcal{L}_w(z) = \lim_{z\to 0}\overline{z}\frac{\partial}{\partial\overline{z}}\mathcal{L}_w(\sigma_j - z) \tag{9.17}$$

belongs to $\mathcal{S}^{\mathrm{sv}}_{X,\mathbb{Q},j}$. Let us now denote the coefficient of any word w in the series $\hat{\sigma}'_j \in \mathbb{C}$ by $(\hat{\sigma}'_j|w)$. By eq. (9.14) we can also write

$$\lim_{z\to\sigma_j}(\overline{z}-\overline{\sigma}_j)\frac{\partial}{\partial\overline{z}}\mathcal{L}_w(z) = (\hat{\sigma}'_j|w) + \sum_{\substack{uv=w\\ |v|<|w|}} \mathcal{L}_u(\sigma_j)(\hat{\sigma}'_j|v). \tag{9.18}$$

Comparing eqs. (9.17) and (9.18) and using induction on the length of the words we conclude that the coefficients of $\hat{\sigma}'_j$ belong to $\mathcal{S}^{\mathrm{sv}}_{X,\mathbb{Q},j}$ (recall that $\hat{\sigma}'_j = \hat{\sigma}_j +$ higher-length terms). This immediately implies the first statement of the proposition. To demonstrate the second statement about the homogeneity of the weight, we observe that if we assign weight -1 to each letter $\hat{\sigma}_j$ of the alphabet X, we obviously have that $L_X(z)$ has weight zero. Therefore, by the definition of $\mathcal{L}_X(z)$, it is enough to show that each $\hat{\sigma}'_j$ has homogeneous weight -1, i.e. that $(\hat{\sigma}'_j|w)$ has weight $|w|-1$ for any word w. Once again, this follows by comparing eqs. (9.17) and (9.18) and using induction.
□

Corollary 9.6. — *Let $F(z)$ be a single-valued function which is a $\mathcal{S}^{\mathrm{sv}}_{X,\mathbb{Q}}$-linear combination of products $L_{w_1}(z)\overline{L_{w_2}(z)}$. Then $F(z) \in \mathcal{H}^{\mathrm{sv}}_{X,\mathcal{S}^{\mathrm{sv}}_{X,\mathbb{Q}}}$.*

Proof. Let

$$F(z) = \sum_{w_1,w_2\in X^*} c_{w_1,w_2} L_{w_1}(z)\overline{L_{w_2}(z)} \tag{9.19}$$

with $c_{w_1,w_2} \in \mathcal{S}^{\mathrm{sv}}_{X,\mathbb{Q}}$. Since $F(z)$ is single-valued we know by Theorem 9.2 that

$$F(z) = \sum_{u\in X^*} k_u \mathcal{L}_u(z), \tag{9.20}$$

with $k_u \in \mathbb{C}$. By Proposition 9.5 we have that

$$F(z) = \sum_{u\in X^*} k_u \sum_{u_1,u_2\in X^*} l^{(u)}_{u_1,u_2} L_{u_1}(z)\overline{L_{u_2}(z)}, \tag{9.21}$$

with $l^{(u)}_{u_1,u_2} \in \mathcal{S}^{\mathrm{sv}}_{X,\mathbb{Q}}$. Comparing eq. (9.19) with eq. (9.21) and using Lemma 9.1 we conclude that all k_u must belong to the field of fractions of $\mathcal{S}^{\mathrm{sv}}_{X,\mathbb{Q}}$. However, since $\hat{\sigma}'_i = \hat{\sigma}_i$ modulo words $w \in X^*$ with $|w| \geq 4$, for

each u one always has the coefficient $l^{(u)}_{u,e} = 1$, which implies that in fact $k_u \in \mathcal{S}^{\rm sv}_{X,\mathbb{Q}}$.
□

9.2. Integration. — For $f(z) \in \mathcal{A}^{\rm sv}_{X,\mathbb{C}}$ we define the *holomorphic and anti-holomorphic residues* of f at a point $\sigma_i \in X$ as $\mathrm{Res}_{z=\sigma_i} f(z) := c^{(i)}_{0,-1,0}$, $\overline{\mathrm{Res}}_{z=\sigma_i} f(z) := c^{(i)}_{0,0,-1}$, respectively, where we are referring to the coefficients $c^{(i)}_{k,m,n}$ in the expansions given by Lemma 9.3. Similarly, we can define the residues at ∞ by $\mathrm{Res}_{z=\infty} f(z) := c^{(\infty)}_{0,-1,0}$ and $\overline{\mathrm{Res}}_{z=\infty} f(z) := c^{(\infty)}_{0,0,-1}$. The following theorem is an adaptation of a theorem of Schnetz to our context.

Theorem 9.7. — *(Schnetz, [52]) Suppose that $f(z) \in \mathcal{A}^{\rm sv}_{X,\mathbb{C}}$ and that $\int_{\mathbb{P}^1_{\mathbb{C}}} f(z) d^2z < \infty$, where $d^2z := dxdy/\pi$ for $z = x + iy$. Then*

$$\int_{\mathbb{P}^1_{\mathbb{C}}} f(z) d^2z = \mathrm{Res}_{z=\infty} G(z) - \sum_{i=0}^{n} \mathrm{Res}_{z=\sigma_i} G(z) \tag{9.22}$$

$$= \overline{\mathrm{Res}}_{z=\infty} F(z) - \sum_{i=0}^{n} \overline{\mathrm{Res}}_{z=\sigma_i} F(z) \tag{9.23}$$

for any $F, G \in \mathcal{A}^{\rm sv}_{X,\mathbb{C}}$ such that $\partial_z F(z) = \partial_{\overline{z}} G(z) = f(z)$.

Proof. We will show only the first equality, as the second follows by repeating exactly the same steps in the anti-holomorphic case. One has

$$f(z) d^2z = -\frac{f(z)}{2\pi i} dz \wedge d\overline{z} = d\left(\frac{G(z)}{2\pi i}\right) dz,$$

therefore $f(z)d^2z$ is exact on $\mathbb{P}^1_{\mathbb{C}} \setminus (X \cup \{\infty\})$. Let $B_a(r)$ denote the ball centered in a of radius r, $S^{\pm}_a(r) = \partial^{\pm} B_a(r)$ and $\varepsilon > 0$ such that $B_\eta(\varepsilon) \cap B_\theta(\varepsilon) = \emptyset$ for all finite $\eta, \theta \in X$. Then $f(z)d^2z$ is exact on the oriented manifold $V_\varepsilon := \mathbb{P}^1_{\mathbb{C}} \setminus (\bigcup_{\eta \in X} B_\eta(\varepsilon) \cup B_0(\varepsilon^{-1}))$ with boundaries $S^+_0(\varepsilon^{-1})$, $S^-_\eta(\varepsilon)$, and so by Stokes's theorem we have

$$\int_{V_\varepsilon} f(z) d^2z = -\frac{1}{2\pi i} \left(\int_{S^+_0(\varepsilon^{-1})} + \sum_{\eta \in X} \int_{S^-_\eta(\varepsilon)} \right) G(z) dz.$$

Parametrizing $S_\eta(\varepsilon) = \{z = \eta + \varepsilon e^{i\theta} \,:\, 0 \leq \theta < 2\pi\}$ and integrating term by term the expansion of the integrand given in Lemma 9.3 we get our claim.
□

This theorem gives us the first powerful instrument to deal with integrals of single-valued hyperlogarithms over the complex plane. However, in order to compute multiple integrals like (7.1) we need to treat the letters in the label as variables and to have at disposal an analogue of Theorem 8.7 for the single-valued case. To obtain such a result, we must first get an analogue of Corollary 8.6.

Lemma 9.8. — *For any $w \in X^*$, $z \in \mathbb{C}$ and $2 \leq i \leq n$, we have*

$$\frac{\partial}{\partial \sigma_i}\mathcal{L}_w(z) = \sum_{\substack{\tau \in X_i \cup \{z\} \\ v \in X^*}} \frac{\lambda_{\tau,v}}{\sigma_i - \tau}\mathcal{L}_v(z), \tag{9.24}$$

where the sum is finite and $\lambda_{\tau,v} \in \mathcal{S}^{\mathrm{sv}}_{X,\mathbb{Q}}$ have homogenous weight $W(\lambda_{\tau,v}) = |w| - |v| - 1$.

Proof. We will use induction on the length $|w|$. If $|w| = 1$ the statement clearly holds. For $|w| \geq 2$ let

$$\mathcal{L}_w(z) = \sum_{w_1,w_2} c_{w_1,w_2} L_{w_1}(z)\overline{L_{w_2}(z)}. \tag{9.25}$$

By Proposition 9.5 $c_{w_1,w_2} \in \mathcal{S}^{\mathrm{sv}}_{X,\mathbb{Q}}$ and $W(c_{w_1,w_2}) = |w| - |w_1| - |w_2|$. Since some c_{w_1,w_2} may depend on σ_i and $\frac{\partial}{\partial \sigma_i}\overline{L_w(z)} = 0$ for all w, we have

$$\begin{aligned}\frac{\partial}{\partial \sigma_i}\mathcal{L}_w(z) = &\sum_{w_1,w_2} \left(\frac{\partial}{\partial \sigma_i} c_{w_1,w_2}\right) L_{w_1}(z)\overline{L_{w_2}(z)} \\ &+ \sum_{w_1,w_2} c_{w_1,w_2}\left(\frac{\partial}{\partial \sigma_i} L_{w_1}(z)\right)\overline{L_{w_2}(z)}.\end{aligned} \tag{9.26}$$

It is trivial to see from the definition of single-valued hyperlogarithms that $|w_1| + |w_2| > 0$. Therefore we can use the inductive hypothesis on the single-valued hyperlogarithms evaluated at points of the alphabet X

appearing in c_{w_1,w_2}, obtaining that

$$\frac{\partial}{\partial\sigma_i}c_{w_1,w_2}=\sum_{\tau\in X_i}\frac{\mu_{\tau,w_1,w_2}}{\sigma_i-\tau}, \tag{9.27}$$

where each $\mu_{\tau,w_1,w_2}\in\mathcal{S}^{\rm sv}_{X,\mathbb{Q}}$ has homogeneous weight given by $W(\mu_{\tau,w_1,w_2})=W(c_{w_1,w_2})-1$. On the other side, by Corollary 8.6 we have

$$\frac{\partial}{\partial\sigma_i}L_{w_1}(z)=\sum_{\substack{\tau\in X_i\cup\{z\}\\ v\in X^*}}\frac{\lambda_{\tau,v}}{\sigma_i-\tau}L_v(z) \tag{9.28}$$

with $\lambda_{\tau,v}\in\mathbb{Z}$ and $|v|=|w|-1$. Therefore we are left with

$$\frac{\partial}{\partial\sigma_i}\mathcal{L}_w(z)=\sum_{\tau\in X_i\cup\{z\}}\frac{f_\tau(z)}{\sigma_i-\tau}, \tag{9.29}$$

where each $f_\tau(z)$ is a single-valued $\mathcal{S}^{\rm sv}_{X,\mathbb{Q}}$-linear combination of products $L_{v_1}(z)\overline{L_{v_2}(z)}$ of homogeneous weight $|w|-1$. By Corollary 9.6 we conclude that each $f_\tau(z)$ is a $\mathcal{S}^{\rm sv}_{X,\mathbb{Q}}$-linear combination of single-valued hyperlogarithms $\sum_v\lambda_{\tau,v}\mathcal{L}_v(z)$ and $W(\lambda_{\tau,v})+|v|=|w|+1$.
□

If we want to consider a point σ_i as a variable, it is also crucial to notice the following important property[(17)].

Lemma 9.9. — *For each $w\in X^*$, each $2\leq i\leq n$ and each $z\in\mathbb{C}$ the single-valued hyperlogarithm $\mathcal{L}_w(z)$ is single-valued as a function of σ_i.*

Proof. We need to show that $\sigma_i\to\mathcal{L}_w(z)$ is a well-defined function for all $w\in X^*$ and all $\sigma_i,z\in\mathbb{C}$. Recall that, by Theorem 9.2, $\mathcal{L}_w(z)$ is a uniquely determined single-valued function of $z\in\mathbb{C}\setminus X$ for any $\sigma_i\in\mathbb{C}$, which we have extended to a function on $\mathbb{C}$. This precisely means that $\mathcal{L}_w(z)$ takes a unique value for any z and any w, so $\sigma_i\to\mathcal{L}_w(z)$ is well-defined.
□

[(17)] A (different) proof of this lemma was already given in [33].

We are now ready to state the main result of this section, which is the crucial ingredient to perform multiple integrals of single-variable hyperlogarithms. We recall that we have denoted $X_i = X \setminus \{\sigma_i\}$.

Theorem 9.10. — *For any $w \in X^*$ and any $2 \leq i, j \leq n$*

$$\mathcal{L}_w(\sigma_j) = \sum_u c_u \mathcal{L}_u(\sigma_i), \tag{9.30}$$

where $c_u \in \mathcal{S}^{\mathrm{sv}}_{X_i,\mathbb{Q}}$ and the sum is finite.

Proof. If $|w| = 1$ we have $\mathcal{L}_{\sigma_k}(\sigma_j) = \log|1 - \sigma_j/\sigma_k|^2$. Therefore if $k \neq i$ then $\mathcal{L}_{\sigma_k}(\sigma_j) \in \mathcal{S}^{\mathrm{sv}}_{X_i,\mathbb{Q}}$ and if $k = i$ then $\mathcal{L}_{\sigma_i}(\sigma_j) = \mathcal{L}_{\sigma_j}(\sigma_i) - \mathcal{L}_0(\sigma_i)$. Let us proceed by induction on $|w|$. By Lemma 9.8 we have

$$\frac{\partial}{\partial\sigma_i}\mathcal{L}_w(\sigma_j) = \sum_{\substack{\tau\in X_i \\ v\in X^*}} \frac{\lambda_{\tau,v}}{\sigma_i - \tau}\mathcal{L}_v(\sigma_j), \tag{9.31}$$

with $\lambda_{\tau,v} \in \mathcal{S}^{\mathrm{sv}}_{X,\mathbb{Q}}$ of homogenous weight such that $W(\lambda_{\tau,v}) + |v| = |w| - 1$. We can therefore apply the inductive hypothesis both on the coefficients $\lambda_{\tau,v}$ and on the hyperlogarithms $\mathcal{L}_v(\sigma_j)$, obtaining (after performing enough shuffle products)

$$\frac{\partial}{\partial\sigma_i}\mathcal{L}_w(\sigma_j) = \sum_{\substack{\tau\in X_i \\ u\in X_i^*}} \frac{\mu_{\tau,u}}{\sigma_i - \tau}\mathcal{L}_u(\sigma_i) \tag{9.32}$$

with $\mu_{\tau,u} \in \mathcal{S}^{\mathrm{sv}}_{X_i,\mathbb{Q}}$, which implies by Theorem 9.2 that

$$\mathcal{L}_w(\sigma_j) = \sum_{\substack{\tau\in X_i \\ u\in X_i^*}} \mu_{\tau,u}\mathcal{L}_{\tau u}(\sigma_i) + f(\overline{\sigma_i}) \tag{9.33}$$

for some function f not depending on σ_i. Because of Proposition 8.7, the function $f(\overline{\sigma_i})$ must belong to $\overline{\mathcal{H}_{X,\mathbb{C}}}$. By Lemma 9.9 we know that f is a single-valued function of σ_i, but then by Proposition 8.3 it has to be constant in σ_i. Since $|\tau u| \geq 1$ we have $\mathcal{L}_{\tau u}(0) = 0$. Therefore we conclude that

$$f = \lim_{\sigma_i\to 0} \mathcal{L}_w(\sigma_j), \tag{9.34}$$

which belongs to $\mathcal{S}^{\text{sv}}_{X_i,\mathbb{Q}}$.
□

Once again, we conclude the section by recalling that, when we consider the letters of the alphabet as variables, hyperlogarithms can be seen as iterated integrals on $\mathfrak{M}_{0,n}$. It is possible to construct single-valued real-analytic versions of these iterated integrals [67]. They form an algebra which is isomorphic to the algebra of their holomorphic counterparts. We believe that also all our results about single-valued hyperlogarithms as functions of the alphabet's letters should be consequences of the structure of the abstract algebra of iterated integrals on $\mathfrak{M}_{0,n}$ studied in [61]. A very nice dictionary between the two possible approaches was presented in [68].

10. The case $k = 1$

In this case the integral R_k coincides precisely with the Virasoro bosonic string amplitude [69], i.e. the *complex beta function* defined for $\alpha, \beta \in \mathbb{C}$, $\text{Re}(\alpha) > 0$, $\text{Re}(\beta) > 0$ and $\text{Re}(\alpha + \beta) < 1$ by the absolutely convergent integral

$$\int_{\mathbb{P}^1_{\mathbb{C}}} |z|^{2\alpha-2}|1-z|^{2\beta-2}d^2z. \tag{10.1}$$

Until the very recent appearence of [33, 34], this was the only case where the low energy expansion (i.e. the asymptotic expansion for $\alpha, \beta \to 0$) of a genus-zero closed string amplitude was proven to only involve single-valued multiple zeta values, and more precisely only odd Riemann zeta values. The classical way to do this computation is known at least since the 1960s and consists in writing the integral as a quotient of Γ-functions[(18)]:

$$R_1(\alpha,\beta) = \frac{(\alpha+\beta)}{\alpha\beta}\frac{\Gamma(1+\alpha)\Gamma(1+\beta)\Gamma(1-\alpha-\beta)}{\Gamma(1-\alpha)\Gamma(1-\beta)\Gamma(1+\alpha+\beta)}$$

[(18)] See for instance [70] for the details of this computation.

$$= \frac{(\alpha+\beta)}{\alpha\beta}\exp\Big(-2\sum_{n\geq 1}\frac{\zeta(2n+1)}{(2n+1)}(\alpha^{2n+1}+\beta^{2n+1}-(\alpha+\beta)^{2n+1})\Big). \tag{10.2}$$

In order to warm up for the next unproven cases, we will now give an alternative proof of the appearence of single-valued multiple zeta values in the Laurent expansion of R_1 at the origin.

In the region of convergence, we can rewrite (10.1) as

$$\lim_{\varepsilon\to 0}\int_{U_\varepsilon}|z|^{2\alpha-2}|1-z|^{2\beta-2}d^2z, \tag{10.3}$$

where $U_\varepsilon := \mathbb{P}^1_{\mathbb{C}}\setminus(B_0(\varepsilon)\cup B_1(\varepsilon)\cup B_0(\varepsilon^{-1}))$ and we denote by $B_x(r)$ the ball centered at x of radius r. If we expand the integrand as a power series in α and β, we can interchange summation and integration and get an infinite sum of absolutely convergent integrals

$$I_4^{\mathrm{cl}}(\alpha,\beta) = \lim_{\varepsilon\to 0}\sum_{p,q\geq 0}\alpha^p\beta^q\int_{U_\varepsilon}\frac{\mathcal{L}_{0^p}(z)\mathcal{L}_{1^q}(z)}{|z|^2|1-z|^2}d^2z, \tag{10.4}$$

where we recall that $\mathcal{L}_{0^p}(z) = (\log|z|^2)^p/p!$ and $\mathcal{L}_{1^q}(z) = (\log|1-z|^2)^q/q!$.

For $p\geq 1$ and $q\geq 1$ we can write

$$\int_{U_\varepsilon}\frac{\mathcal{L}_{0^p}(z)\mathcal{L}_{1^q}(z)}{|z|^2|1-z|^2}d^2z = \Big(\int_{\mathbb{P}^1_{\mathbb{C}}}-\int_{\mathbb{P}^1_{\mathbb{C}}\setminus U_\varepsilon}\Big)\frac{\mathcal{L}_{0^p}(z)\mathcal{L}_{1^q}(z)}{|z|^2|1-z|^2}d^2z, \tag{10.5}$$

because both integrals are absolutely convergent. The second integral vanishes as ε tends to zero, so we only need to care of the first one. Since by Theorem 9.2 single-valued hyperlogarithms satisfy the shuffle product, we have

$$\frac{\mathcal{L}_{0^p}(z)\mathcal{L}_{1^q}(z)}{|z|^2|1-z|^2} = \sum_{w=0^p\sqcup\!\sqcup 1^q}\frac{\mathcal{L}_w(z)}{|z|^2|1-z|^2} = \sum_{w=0^p\sqcup\!\sqcup 1^q}\frac{1}{\overline{z}(1-\overline{z})}\Big(\frac{\mathcal{L}_w(z)}{z}-\frac{\mathcal{L}_w(z)}{z-1}\Big), \tag{10.6}$$

hence again by Theorem 9.2 we have that

$$F(z) = \sum_{w=0^p\sqcup\!\sqcup 1^q}\frac{\mathcal{L}_{0w}(z)-\mathcal{L}_{1w}(z)}{\overline{z}(1-\overline{z})} \tag{10.7}$$

satisfies

$$\partial_z F(z) = \frac{\mathcal{L}_{0^p}(z)\mathcal{L}_{1^q}(z)}{|z|^2|1-z|^2}. \tag{10.8}$$

Therefore by Theorem 9.7 we conclude that

$$\begin{aligned}\int_{\mathbb{P}^1_{\mathbb{C}}} \frac{\mathcal{L}_{0^p}(z)\mathcal{L}_{1^q}(z)}{|z|^2|1-z|^2} d^2z &= \overline{\mathrm{Res}}_{z=\infty} F(z) - \overline{\mathrm{Res}}_{z=0} F(z) - \overline{\mathrm{Res}}_{z=1} F(z) \\ &= \sum_{w=0^p \sqcup\!\sqcup 1^q} \mathcal{L}_{0w}(1) - \mathcal{L}_{1w}(1),\end{aligned}$$

which belongs to the algebra $\mathcal{Z}^{\mathrm{sv}}$ for any $p, q \geq 0$.

We are left with the cases where $q = 0$ or $p = 0$. We will only show the details of the first case, as the second goes along the same lines. We have

$$\begin{aligned}\int_{U_\varepsilon} \frac{\mathcal{L}_{0^p}(z)}{|z|^2|1-z|^2} d^2z &= \int_{U_\varepsilon} \frac{1}{\overline{z}(1-\overline{z})}\left(\frac{\mathcal{L}_{0^p}(z)}{z} - \frac{\mathcal{L}_{0^p}(z)}{z-1}\right) d^2z \\ &= \int_{U_\varepsilon} \frac{\partial_z(\mathcal{L}_{0^{p+1}}(z) - \mathcal{L}_{10^p}(z))}{\overline{z}(1-\overline{z})} \frac{i dz d\overline{z}}{2\pi},\end{aligned} \tag{10.9}$$

and by the Stokes Theorem we can write (10.9) as

$$\left(\int_{\partial^+ B_0(\varepsilon^{-1})} + \int_{\partial^- B_0(\varepsilon)} + \int_{\partial^- B_1(\varepsilon)}\right) \frac{\mathcal{L}_{0^{p+1}}(z) - \mathcal{L}_{10^p}(z)}{\overline{z}(1-\overline{z})} \frac{i d\overline{z}}{2\pi}. \tag{10.10}$$

It is easy to see by a residue-like computation (see the proof of Theorem 2.29 in [52]) that

$$\lim_{\varepsilon \to 0} \int_{\partial^+ B_0(\varepsilon^{-1})} \frac{\mathcal{L}_{0^{p+1}}(z) - \mathcal{L}_{10^p}(z)}{\overline{z}(1-\overline{z})} \frac{i d\overline{z}}{2\pi} = 0 \tag{10.11}$$

and that

$$\lim_{\varepsilon \to 0} \int_{\partial^- B_1(\varepsilon)} \frac{\mathcal{L}_{0^{p+1}}(z) - \mathcal{L}_{10^p}(z)}{\overline{z}(1-\overline{z})} \frac{i d\overline{z}}{2\pi} \in \mathcal{Z}^{\mathrm{sv}}. \tag{10.12}$$

The last integral diverges as ε tends to zero:

$$\begin{aligned}&\int_{\partial^- B_0(\varepsilon)} \frac{\mathcal{L}_{0^{p+1}}(z) - \mathcal{L}_{10^p}(z)}{\overline{z}(1-\overline{z})} \frac{i d\overline{z}}{2\pi} = \int_{\partial^- B_0(\varepsilon)} \frac{\mathcal{L}_{0^{p+1}}(z)}{\overline{z}(1-\overline{z})} \frac{i d\overline{z}}{2\pi} \\ &= \mathcal{L}_{0^{p+1}}(\varepsilon) \int_{\partial^- B_0(\varepsilon)} \left(\frac{1}{\overline{z}} - \frac{1}{\overline{z}-1}\right) \frac{i d\overline{z}}{2\pi}\end{aligned}$$

$$= -\mathcal{L}_{0^{p+1}}(\varepsilon)\left(\int_0^{2\pi}\frac{d\theta}{2\pi} - \int_0^{2\pi}\frac{\varepsilon e^{-i\theta}d\theta}{2\pi(\varepsilon e^{-i\theta}-1)}\right)$$
$$= -\mathcal{L}_{0^{p+1}}(\varepsilon) = -\frac{(\log\varepsilon^2)^{p+1}}{(p+1)!}, \tag{10.13}$$

therefore in this case we cannot exchange the summation with the limit. If we consider the limit of the summation over all $p \geq 0$ we get

$$\lim_{\varepsilon\to 0} -\sum_{p\geq 0}\frac{(\log\varepsilon^2)^{p+1}}{(p+1)!}\alpha^p = \lim_{\varepsilon\to 0}\frac{1}{\alpha}(1-\varepsilon^{2\alpha}) = \frac{1}{\alpha}, \tag{10.14}$$

so we recover the simple pole in α of (10.1). One can treat the case $p = 0$ in the same way, recovering the behaviour $1/\beta$ as β goes to zero. Therefore, if we choose $P_1(\alpha,\beta) = \alpha^{-1} + \beta^{-1}$ and $H_1 = R_1 - P_1$, we have a proof of Theorem 7.1 in the case $k = 1$.

11. The case $k = 2$

The $k = 2$-case presents difficulties which do not arise in the $k = 1$-case; it constitutes the simplest non-trivial example on which we can see at work, step by step, the method of the proof given in the next section for the general case.

We want to study the behaviour of the function $R_2(\alpha_1,\alpha_2,\beta_1,\beta_2,\gamma)$, given by the integral

$$\int_{(\mathbb{P}^1_{\mathbb{C}})^2} |z|^{2\alpha_1-2}|u|^{2\alpha_2-2}|1-z|^{2\beta_1-2}|1-u|^{2\beta_2-2}|z-u|^{2\gamma}d^2zd^2u, \tag{11.1}$$

near the origin, which is situated, as pointed out in Section 7.1, at the boundary of the region of convergence given by Proposition 7.2. First of all, wherever (11.1) is absolutely convergent we can rewrite it as

$$\lim_{\varepsilon\to 0}\left(\int_{U_{\varepsilon,1}} + \int_{U_{\varepsilon,2}}\right)|z|^{2\alpha_1-2}|u|^{2\alpha_2-2}|1-z|^{2\beta_1-2}|1-u|^{2\beta_2-2}|z-u|^{2\gamma}d^2zd^2u, \tag{11.2}$$

where

$$U_{\varepsilon,1} = \{z,u \in \mathbb{P}^1_{\mathbb{C}} : |z|,|1-z|,|u|,|1-u|,|z|-|u| > \varepsilon,\ |z|,|u| < \varepsilon^{-1}\}, \tag{11.3}$$

$$U_{\varepsilon,2} = \{z,u \in \mathbb{P}^1_{\mathbb{C}} : |z|,|1-z|,|u|,|1-u|,|u|-|z| > \varepsilon,\ |z|,|u| < \varepsilon^{-1}\}. \tag{11.4}$$

Inside the region of convergence, sufficiently close to the origin, we can expand the integrand as a power series and we can interchange summation and integration to get an infinite sum of absolutely convergent integrals:

$$R_2(\alpha_1,\alpha_2,\beta_1,\beta_2,\gamma) = \lim_{\varepsilon\to 0} \sum_{p_1,p_2,q_1,q_2,r\geq 0} \alpha_1^{p_1}\alpha_2^{p_2}\beta_1^{q_1}\beta_2^{q_2}\frac{\gamma^r}{r!}\times$$
$$\times \left(\int_{U_{\varepsilon,1}} + \int_{U_{\varepsilon,2}}\right) \frac{\mathcal{L}_{0^{p_1}}(z)\mathcal{L}_{0^{p_2}}(u)\mathcal{L}_{1^{q_1}}(z)\mathcal{L}_{1^{q_2}}(u)\Big(\log|z-u|^2\Big)^r}{|z|^2|1-z|^2|u|^2|1-u|^2} d^2z d^2u. \tag{11.5}$$

Writing $\log|z-u|^2 = \mathcal{L}_z(u) - \mathcal{L}_0(z)$ on $U_{\varepsilon,1}$ and $\log|z-u|^2 = \mathcal{L}_u(z) - \mathcal{L}_0(u)$ on $U_{\varepsilon,2}$ we can rewrite $R_2(\alpha_1,\alpha_2,\beta_1,\beta_2,\gamma)$ as

$$\lim_{\varepsilon\to 0} \sum_{p_1,p_2,q_1,q_2,r_1,r_2\geq 0} \alpha_1^{p_1}\alpha_2^{p_2}\beta_1^{q_1}\beta_2^{q_2}\gamma^{r_1+r_2}(-1)^{r_2}\binom{r_1+r_2}{r_2}\times \tag{11.6}$$
$$\times \left(\int_{U_{\varepsilon,1}} \frac{\mathcal{L}_{0^{p_1+r_2}}(z)\mathcal{L}_{0^{p_2}}(u)\mathcal{L}_{1^{q_1}}(z)\mathcal{L}_{1^{q_2}}(u)\mathcal{L}_{z^{r_1}}(u)}{|z|^2|1-z|^2|u|^2|1-u|^2} d^2z d^2u \right.$$
$$\left. + \int_{U_{\varepsilon,2}} \frac{\mathcal{L}_{0^{p_1}}(z)\mathcal{L}_{0^{p_2+r_2}}(u)\mathcal{L}_{1^{q_1}}(z)\mathcal{L}_{1^{q_2}}(u)\mathcal{L}_{u^{r_1}}(z)}{|z|^2|1-z|^2|u|^2|1-u|^2} d^2z d^2u \right).$$

We need now to distinguish between different cases.

The case $p_1,p_2,q_1,q_2 \geq 1$. In this case, we have that

$$\left(\int_{(\mathbb{P}^1_{\mathbb{C}})^2} - \int_{U_{\varepsilon,1}} - \int_{U_{\varepsilon,2}}\right) \frac{\mathcal{L}_{0^{p_1}}(z)\mathcal{L}_{0^{p_2}}(u)\mathcal{L}_{1^{q_1}}(z)\mathcal{L}_{1^{q_2}}(u)\Big(\log|z-u|^2\Big)^r}{|z|^2|1-z|^2|u|^2|1-u|^2} d^2z d^2u, \tag{11.7}$$

tends to zero as $\varepsilon \to 0$. Therefore, we just need to explain how to compute the absolutely convergent integrals

$$\int_{(\mathbb{P}^1_{\mathbb{C}})^2} \frac{\mathcal{L}_{0^{p_1+r_2}}(z)\mathcal{L}_{0^{p_2}}(u)\mathcal{L}_{1^{q_1}}(z)\mathcal{L}_{1^{q_2}}(u)\mathcal{L}_{z^{r_1}}(u)}{|z|^2|1-z|^2|u|^2|1-u|^2} d^2z d^2u \tag{11.8}$$

$$= \int_{\mathbb{P}^1_{\mathbb{C}}} \frac{J_{p_2,q_2,r_1}(z)\mathcal{L}_{0^{p_1+r_2}}(z)\mathcal{L}_{1^{q_1}}(z)}{|z|^2|1-z|^2} d^2z, \tag{11.9}$$

where

$$J_{p_2,q_2,r_1}(z) = \int_{\mathbb{P}^1_{\mathbb{C}}} \frac{\mathcal{L}_{0^{p_2}}(u)\mathcal{L}_{1^{q_2}}(u)\mathcal{L}_{z^{r_1}}(u)}{|u|^2|1-u|^2} d^2u. \tag{11.10}$$

We can assume that $r_1 \geq 1$, otherwise the integral factors into the product of two integrals of the same kind as those treated in the $k=1$-case. By Theorem 9.7 we can write $J_{p_2,q_2,r_1}(z)$ as

$$\overline{\mathrm{Res}}_{u=\infty} F(z,u) - \overline{\mathrm{Res}}_{u=0} F(z,u) - \overline{\mathrm{Res}}_{u=1} F(z,u) - \overline{\mathrm{Res}}_{u=z} F(z,u), \tag{11.11}$$

where

$$F(z,u) = \sum_{w=0^{p_2} \sqcup\!\sqcup 1^{q_2} \sqcup\!\sqcup z^{r_1}} \Big(\mathcal{L}_{1w}(u) - \mathcal{L}_{0w}(u)\Big)\left(\frac{1}{\overline{u}-1} - \frac{1}{\overline{u}}\right) \tag{11.12}$$

is such that $\partial_u F(z,u)$ is equal to the integrand of (11.10). It is easy to see that the only non-vanishing residues in (11.11) are obtained at $u=1$, thus

$$J_{p_2,q_2,r_1}(z) = \sum_{w=0^{p_2} \sqcup\!\sqcup 1^{q_2} \sqcup\!\sqcup z^{r_1}} \Big(\mathcal{L}_{0w}(1) - \mathcal{L}_{1w}(1)\Big). \tag{11.13}$$

By Theorem 9.10 this is a $\mathcal{Z}^{\mathrm{sv}}$-linear combination of single-valued multiple polylogarithms in the variable z. Moreover, since $p_2, q_2 \geq 1$, both $\lim_{z\to 0} J_{p_2,q_2,r_1}(z)$ and $\lim_{z\to 1} J_{p_2,q_2,r_1}(z)$ are finite[(19)], which means that the integral

$$\int_{\mathbb{P}^1_{\mathbb{C}}} \frac{J_{p_2,q_2,r_1}(z)\mathcal{L}_{0^{p_1+r_2}}(z)\mathcal{L}_{1^{q_1}}(z)}{|z|^2|1-z|^2} d^2z, \tag{11.14}$$

is absolutely convergent and by the same argument used in the $k=1$-case we get single-valued multiple zeta values.

[(19)] It would have been sufficient to have at most logarithmic singularities.

The case $q_1 = q_2 = 0$. First of all, this case is completely similar to the case $p_1 = p_2 = 0$, which we omit. We want now to consider the integrals over $U_{\varepsilon,1}$ and $U_{\varepsilon,2}$ separately. Since the situation is symmetric, we will just focus on $\int_{U_{\varepsilon,1}}$, i.e. the integral where $|u| < |z|$. This integral (as well as $\int_{U_{\varepsilon,2}}$) is not convergent because of the singularity of the integrand at the origin. Let us consider the change of variables $t = z$, $st = u$, which means that $s = u/z$ and that $d^2z\, d^2u = |t|^2\, d^2t\, d^2s$. Moreover, let us denote by $\tilde{U}_{\varepsilon,1}$ the image of $U_{\varepsilon,1}$ under the change of coordinates. By deforming the shape of U_ε, we can suppose that $\tilde{U}_{\varepsilon,1}$ is obtained by removing a neighborhood of the origin which is a sphere of radius ε. What we need to compute is

$$\sum_{p_1,p_2,r} \frac{\alpha_1^{p_1}\alpha_2^{p_2}\gamma^r}{p_1!\,p_2!\,r!} \int_{\tilde{U}_{\varepsilon,1}} \frac{(\log|t|^2)^{p_1}(\log|s|^2+\log|t|^2)^{p_2}(\log|1-s|^2+\log|t|^2)^r}{|t|^2|s|^2|1-t|^2|1-ts|^2}$$

$$= \sum_{p_1,p_2,r} \frac{\alpha_1^{p_1}\alpha_2^{p_2}\gamma^r}{p_1!} \sum_{\substack{i+j=p_2\\ k+l=r}} \frac{1}{i!\,j!\,k!\,l!} \int_{\tilde{U}_{\varepsilon,1}} \frac{(\log|t|^2)^{p_1+i+k}(\log|s|^2)^j(\log|1-s|^2)^l}{|t|^2|s|^2|1-t|^2|1-ts|^2}$$

$$= \sum_{n,j,l} (\alpha_1+\alpha_2+\gamma)^n \alpha_2^j \gamma^l \int_{\tilde{U}_{\varepsilon,1}} \frac{\mathcal{L}_{0^n}(t)\mathcal{L}_{0^j}(s)\mathcal{L}_{1^l}(s)}{|t|^2|s|^2|1-t|^2|1-ts|^2}. \tag{11.15}$$

In the last integral we can separate the variables and use the same method seen in the $k = 1$-case to obtain the polar part in the limit $\varepsilon \to 0$, producing the quadratic-denominator term $\alpha_2^{-1}(\alpha_1+\alpha_2+\gamma)^{-1}$ as well as other contributions to the polar part with linear denominator (note that the coefficients of the linear-denominator contributions will belong to $\mathcal{Z}^{\rm sv}$, but they are not rational numbers anymore). It is at this point an easy exercise to see that the remaining contributions around $u = 1$ or $z = 1$ give rise to power series with coefficients in $\mathcal{Z}^{\rm sv}$.

The case $q_1 = p_2 = 0$. This case is simpler than the one considered above, because the singularities along the diagonal $z = u$ do not matter anymore, as the only problem occurs at $(z, u) = (0, 1)$. One can therefore simply consider the integral over the union of $U_{\varepsilon,1}$ and $U_{\varepsilon,2}$ and split it locally into a product of integrals like those considered in the case

$k=1$, obtaining the quadratic-denominator contribution $\alpha_1^{-1}\beta_2^{-1}$ to the polar part P_k as well as linear-denominator contributions and power series contributions with coefficients in $\mathcal{Z}^{\mathrm{sv}}$. We omit the case $q_2 = p_1 = 0$ because it is completely similar.

The other possible cases with just one p_i or q_i vanishing are analogous but simpler; they give linear-denominator contributions to the polar part as well as power series contributions, both with coefficients in $\mathcal{Z}^{\mathrm{sv}}$. We conclude that we can write $R_2 = P_2 + H_2$, with H_k a power series with coefficients in $\mathcal{Z}^{\mathrm{sv}}$ and

$$P_2(\alpha_1,\alpha_2,\beta_1,\beta_2,\gamma) = \frac{1}{\alpha_1\beta_2} + \frac{1}{\alpha_2\beta_1} + \frac{1}{\alpha_1(\alpha_1+\alpha_2+\gamma)} + \frac{1}{\alpha_2(\alpha_1+\alpha_2+\gamma)}$$
$$+ \frac{1}{\beta_1(\beta_1+\beta_2+\gamma)} + \frac{1}{\beta_2(\beta_1+\beta_2+\gamma)} + P_2^{\mathrm{lin}}(\alpha_1,\alpha_2,\beta_1,\beta_2,\gamma), \qquad (11.16)$$

where P_2^{lin} is a rational function with at most linear denominators and coefficients in $\mathcal{Z}^{\mathrm{sv}}$. This proves Theorem 7.1 in the $k=2$-case.

12. The general case

The aim of this section is to prove Theorem 7.1. In the previous sections we have already demonstrated the statement for $k \leq 1, 2$. In particular, all the delicate points in the proof of the theorem were already encountered and treated in details in the case $k=2$, and one just needs to repeat the same steps in the general case. For this reason, we will only give a sketch of the proof, commenting very briefly on how to separate the polar part P_k from the holomorphic part H_k and contenting ourselves to show with a recursive method that the coefficients of the most general terms in the expansion are single-valued multiple zeta values.

Proof of Theorem 7.1. Since the origin is situated at the boundary of the region of convergence, we can place ourselves where the integral converges absolutely and write

$$R_k(\alpha_1,\ldots,\alpha_k,\beta_1,\ldots,\beta_k,\gamma_{1,2},\ldots\gamma_{1,k},\gamma_{2,3},\ldots\gamma_{2,k},\ldots,\gamma_{k-1,k}) =$$

$$\lim_{\varepsilon\to 0}\left(\sum_{\tau\in\mathfrak{S}_k}\int_{U_{\varepsilon,\tau}}\right)\prod_{i=1}^{k}|z_i|^{2\alpha_i-2}|z_i-1|^{2\beta_i-2}\prod_{1\leq i<j\leq k}|z_j-z_i|^{2\gamma_{i,j}}\,d^2z_1\cdots d^2z_k, \tag{12.1}$$

where $\mathfrak{S}_k$ is the group of permutation of k letters and

$$U_{\varepsilon,\tau}=\{z_1,\ldots,z_k\in\mathbb{P}^1_{\mathbb{C}}:|z_i|,|1-z_i|,|z_{\tau(i)}|-|z_{\tau(i+1)}|>\varepsilon,\,|z_i|<\varepsilon^{-1}\}. \tag{12.2}$$

Expanding the integrand as a power series and interchanging summation and integration we get

$$\begin{aligned}R_k(\alpha_1,\ldots,\gamma_{k-1,k})=\lim_{\varepsilon\to 0}\sum_{p_i,q_i,r_{i,j}\geq 0}\prod_{i=1}^{k}\frac{\alpha_i^{p_i}\beta_i^{q_i}}{p_i!q_i!}\prod_{1\leq i<j\leq k}\frac{\gamma_{i,j}^{r_{i,j}}}{r_{i,j}!}\left(\sum_{\tau\in\mathfrak{S}_k}\int_{U_{\varepsilon,\tau}}\right)\times\\ \times\prod_{i=1}^{k}\frac{(\log|z_i|^2)^{p_i}(\log|1-z_i|^2)^{q_i}}{|z_i|^2|1-z_i|^2}\prod_{1\leq i<j\leq k}(\log|z_i-z_j|^2)^{r_{i,j}}d^2z_1\cdots d^2z_k.\end{aligned} \tag{12.3}$$

Then one needs to distinguish the treatment of the cases where the integrals appearing in (12.3) do not have a finite limit as $\epsilon\to 0$, and therefore we cannot interchange the limit with the summation, from the other cases. In the first situation we get contributions to the polar part, and we really need to use the decomposition of $(\mathbb{P}^1_{\mathbb{C}})^k$ into the cones $U_{\varepsilon,\tau}$, while in the second case we can simply compute the integrals appearing in (12.3) over $(\mathbb{P}^1_{\mathbb{C}})^k$:

$$\int_{(\mathbb{P}^1_{\mathbb{C}})^k}\prod_{i=1}^{k}\frac{(\log|z_i|^2)^{p_i}(\log|1-z_i|^2)^{q_i}}{|z_i|^2|1-z_i|^2}\prod_{1\leq i<j\leq k}(\log|z_i-z_j|^2)^{r_{i,j}}d^2z_1\cdots d^2z_k. \tag{12.4}$$

The integrals (12.4) are absolutely convergent precisely when $p_i,q_i\geq 1$ for all i. Let us also assume that the $r_{i,j}$'s are greater than 1, because the cases where some of them vanish are simpler and go along the same lines. Let us actually consider for $m_i,n_i,l_{i,j}\geq 1$ the class of integrals

$$\int_{(\mathbb{P}^1_{\mathbb{C}})^k}\prod_{i=1}^{k}\frac{\mathcal{L}_{0^{m_i}}(z_i)\mathcal{L}_{1^{n_i}}(z_i)}{|z_i|^2|1-z_i|^2}\prod_{1\leq i<j\leq k}\mathcal{L}_{z_i^{l_{i,j}}}(z_j)d^2z_1\cdots d^2z_k, \tag{12.5}$$

Since $\log|z_i - z_j|^2 = \mathcal{L}_{z_i}(z_j) - \mathcal{L}_0(z_i)$, the statement that these integrals belong to $\mathcal{Z}^{\mathrm{sv}}$ is equivalent to the statement of the theorem. The main ingredient of our proof is the following:

Lemma 12.1. — *Let X be the alphabet $\{0,1,\sigma_2,\dots,\sigma_N\}$, and let $X_N = \{0,1,\sigma_2,\dots,\sigma_{N-1}\}$. Let $f(z) = \sum_{u\in X^*} c_u \mathcal{L}_u(z)$ be a finite linear combination of single-valued hyperlogarithms with coefficients $c_u \in \mathcal{S}^{\mathrm{sv}}_{X,\mathbb{Q}}$. Then there exists a finite linear combination $g(\sigma_N) = \sum_{v\in X_N^*} k_v \mathcal{L}_v(\sigma_N)$ with $k_v \in \mathcal{S}^{\mathrm{sv}}_{X_N,\mathbb{Q}}$ such that*

$$\int_{\mathbb{P}^1_{\mathbb{C}}} \frac{\mathcal{L}_{0^m}(z)\mathcal{L}_{1^n}(z)}{|z|^2|1-z|^2} \prod_{2\le i\le N} \mathcal{L}_{\sigma_i^{l_i}}(z) f(z)\, d^2z = g(\sigma_N) \tag{12.6}$$

for all $m,n,l_i \ge 1$.

Proof of the lemma. First of all, we remark that since $m,n \ge 1$ this integral is absolutely convergent. Using the shuffle product the integrand is a finite linear combination

$$\sum_w \frac{c_w \mathcal{L}_w(z)}{|z|^2|1-z|^2} \tag{12.7}$$

with words $w \in X^*$. Because of our assumption on $f(z)$, we know that the coefficients c_w belong to $\mathcal{S}^{\mathrm{sv}}_{X,\mathbb{Q}}$. We have already seen in the previous sections that

$$\int_{\mathbb{P}^1_{\mathbb{C}}} \sum_w \frac{c_w \mathcal{L}_w(z)}{|z|^2|1-z|^2}\, d^2z = \sum_w c_w(\mathcal{L}_{0w}(1) - \mathcal{L}_{1w}(1)). \tag{12.8}$$

Theorem 9.10 concludes the proof.
□

We can now choose any order of integration of the k variables, and since we can choose $f(z) = 1$ for the first integration, we conclude recursively using the lemma that each of the integrals (12.5) belongs to $\mathcal{S}^{\mathrm{sv}}_{\{0,1\},\mathbb{Q}}$, which is nothing but $\mathcal{Z}^{\mathrm{sv}}$.

The treatment of the integrals (12.4) which are not convergent (at least one p_i or q_i is zero) is completely similar to that explained in the case

$k = 2$: one needs to make the same change of variables and use the Stokes Theorem. After this, as in the $k = 2$-case, one is left with a contribution to the holomorphic part (coming from those ε-balls where the integral is convergent) and a contribution to the polar part (coming from those ε-balls where the integral is divergent), and all coefficients belong to $\mathcal{Z}^{\rm sv}$ by the same kind of argument exploited above.
□

Finally, let us come back to the integrals $J_{\rho,\sigma}$ appearing in tree-level closed superstring amplitudes. Because we have remarked that the origin lies inside or at the boundary of the region of convergence, each $J_{\rho,\sigma}$ can be treated as in eq. (12.1). This allows to isolate the singularities and extract the polar part. In order to demonstrate the main statement, i.e. that the coefficients of the holomorphic part are single-valued multiple zeta values, one simply needs to use Theorem 9.10 to prove a version of Lemma 12.1 with more general denominators. This is done in Appendix D. A closed formula for the polar part can be found in [34]. In conclusion, using the methods developed in this part of the paper we are able to demonstrate that the small α'-expansion of tree-level closed superstring amplitudes only involves single-valued multiple zeta values.

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Appendix A

The $N = 1$ matrix $S(A, B, C; G)$

We introduce the vectors

$$\mathcal{I}(a_1, b_1, c_1; z) = \begin{pmatrix} I_{((1),\emptyset)}(a_1, b_1, c_1; ; z) \\ I_{(\emptyset,(1))}(a_1, b_1, c_1; ; z) \end{pmatrix},$$
$$\mathcal{J}(a_1, b_1, c_1; z) = \begin{pmatrix} J_{((1),\emptyset)}(a_1, b_1, c_1; ; z) \\ J_{(\emptyset,(1))}(a_1, b_1, c_1; ; z) \end{pmatrix} \tag{A.1}$$

have for components the Aomoto-Gel'fand hypergeometric functions

$$\begin{aligned} I_{((1),\emptyset)}(a_1, b_1, c_1; ; z) &= \int_1^{+\infty} w^{a_1}(w-1)^{b_1}(w-z)^{c_1} dw, \\ I_{(\emptyset,(1))}(a_1, b_1, c_1; ; z) &= \int_0^{z} w^{a_1}(1-w)^{b_1}(z-w)^{c_1} dw, \\ J_{((1),\emptyset)}(a_1, b_1, c_1; ; z) &= \int_{-\infty}^{0} (-w)^{a_1}(1-w)^{b_1}(z-w)^{c_1} dw, \\ J_{(\emptyset,(1))}(a_1, b_1, c_1; ; z) &= \int_z^{1} w^{a_1}(1-w)^{b_1}(w-z)^{c_1} dw\,, \end{aligned} \tag{A.2}$$

The relations derived in §4.2 read in this case (with $\underline{a} := (a_1, b_1, c_1)$)

$$\begin{aligned} e^{-i\pi(B_1+C_1)}\mathcal{I}_1(\underline{a}; z) + e^{i\pi A_1}\mathcal{J}_1(\underline{a}; z) + e^{-i\pi C_1}\mathcal{J}_2(\underline{a}; z) + \mathcal{I}_2(\underline{a}; z) &= 0, \\ e^{i\pi(B_1+C_1)}\mathcal{I}_1(\underline{a}; z) + e^{-i\pi A_1}\mathcal{J}_1(\underline{a}; z) + e^{i\pi C_1}\mathcal{J}_2(\underline{a}; z) + \mathcal{I}_2(\underline{a}; z) &= 0\,, \end{aligned} \tag{A.3}$$

where $a_1 - A_1 = n_1 \in \mathbb{Z}$, $b_1 - B_1 = m_1 \in \mathbb{Z}$, $c_1 - C_1 = p_1 \in \mathbb{Z}$. The above linear relations can be rewritten in the matrix form as

$$\mathcal{I}(a_1, b_1, c_1; z) = S(A_1, B_1, C_1)\mathcal{J}(a_1, b_1, c_1; z)\,, \tag{A.4}$$

with the monodromy matrix

$$S(A_1, B_1, C_1) := \frac{-1}{\sin(\pi(B_1 + C_1))} \begin{pmatrix} -\sin(\pi A_1) & \sin(\pi C_1) \\ \sin(\pi(A_1 + B_1 + C_1)) & \sin(\pi B_1) \end{pmatrix} \tag{A.5}$$

which does not depends on the integers n_1, m_1 and p_1.

Appendix B

The $N = 2$ matrix $S(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}; G)$

In this appendix we give details on the determination of the matrix $S(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}; g)$ of §4.2 such that it holds the identity

$$\mathcal{I} = S(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}; G_{12})\mathcal{J} \Longleftrightarrow \mathcal{J} = S(\boldsymbol{B}, \boldsymbol{A}, \boldsymbol{C}; G_{12})\mathcal{I} \tag{B.1}$$

between the vectors of integral functions (4.1) and (4.3)

$$\mathcal{I} := \begin{pmatrix} I_{((12),\emptyset)}(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}; g_{12}; z) \\ I_{((21),\emptyset)}(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}; g_{12}; z) \\ I_{((1),(2))}(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}; g_{12}; z) \\ I_{((2),(1))}(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}; g_{12}; z) \\ I_{(\emptyset,(12))}(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}; g_{12}; z) \\ I_{(\emptyset,(21))}(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}; g_{12}; z) \end{pmatrix}; \qquad \mathcal{J} := \begin{pmatrix} J_{((12),\emptyset)}(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}; g_{12}; z) \\ J_{((21),\emptyset)}(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}; g_{12}; z) \\ J_{((1),(2))}(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}; g_{12}; z) \\ J_{((2),(1))}(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}; g_{12}; z) \\ J_{(\emptyset,(12))}(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}; g_{12}; z) \\ J_{(\emptyset,(21))}(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}; g_{12}; z) \end{pmatrix}. \tag{B.2}$$

The method is based on the contour deformation method of [39–41].

We deform the contour of integration of w_1 in $\mathcal{I}_1$ by rotating in the upper half-plane (the blue contour in figure 4.1) and obtain

$$\mathcal{I}_1 + e^{i\pi G}\mathcal{I}_2 + e^{i\pi(B_1+G)}\hat{\mathcal{I}}_1 + e^{i\pi(B_1+C_1+G)}\mathcal{I}_4 + e^{i\pi(A_1+B_1+C_1+G)}\hat{\mathcal{I}}_2 = 0, \tag{B.3}$$

where

$$\hat{\mathcal{I}}_1 = \int_z^1 dw_1 \int_1^\infty dw_2 |w_1 - w_2|^g \prod_{i=1}^2 |w_i|^{a_i} |1 - w_i|^{b_i} |z - w_i|^{c_i},$$
$$\hat{\mathcal{I}}_2 = \int_{-\infty}^0 dw_1 \int_1^\infty dw_2 |w_1 - w_2|^g \prod_{i=1}^2 |w_i|^{a_i} |1 - w_i|^{b_i} |z - w_i|^{c_i}. \quad \text{(B.4)}$$

By convergence of the integral the contour at infinity does not contribute. The deformation of the contour of integration for w_1 in $\mathcal{I}_1$ in the lower half-plane (the red contour in figure 4.1) gives the equation

$$\mathcal{I}_1 + e^{-i\pi G}\mathcal{I}_2 + e^{-i\pi(B_1+G)}\hat{\mathcal{I}}_1 + e^{-i\pi(B_1+C_1+G)}\mathcal{I}_4 + e^{-i\pi(A_1+B_1+C_1+G)}\hat{\mathcal{I}}_2 = 0. \quad \text{(B.5)}$$

Deforming the contour of integration for w_2 in the upper half-plane we obtain

$$\hat{\mathcal{I}}_1 + e^{i\pi B_2}\mathcal{J}_6 + e^{i\pi(B_2+G)}\mathcal{J}_5 + e^{i\pi(B_2+C_2+G)}\hat{\mathcal{I}}_3 + e^{i\pi(A_2+B_2+C_2+G)}\mathcal{J}_4 = 0. \quad \text{(B.6)}$$

The deformation of the contour in the lower half-plane gives

$$\hat{\mathcal{I}}_1 + e^{-i\pi B_2}\mathcal{J}_6 + e^{-i\pi(B_2+G)}\mathcal{J}_5 + e^{-i\pi(B_2+C_2+G)}\hat{\mathcal{I}}_3 + e^{-i\pi(A_2+B_2+C_2+G)}\mathcal{J}_4 = 0, \quad \text{(B.7)}$$

with

$$\hat{\mathcal{I}}_3 := \int_z^1 dw_1 \int_0^z dw_2 |w_1 - w_2|^g \prod_{i=1}^2 |w_i|^{a_i} |1 - w_i|^{b_i} |z - w_i|^{c_i}. \quad \text{(B.8)}$$

The deformation of the contour of integration over w_2 in the upper half-plane in $\hat{\mathcal{I}}_2$ gives

$$\hat{\mathcal{I}}_2 + e^{i\pi B_2}\mathcal{J}_3 + e^{i\pi(B_2+C_2)}\hat{\mathcal{I}}_4 + e^{i\pi(A_2+B_2+C_2)}\mathcal{J}_2 + e^{i\pi(A_2+B_2+C_2+G)}\mathcal{J}_1 = 0, \quad \text{(B.9)}$$

and the deformation of the contour in the lower half-plane gives

$$\hat{\mathcal{I}}_2 + e^{-i\pi B_2}\mathcal{J}_3 + e^{-i\pi(B_2+C_2)}\hat{\mathcal{I}}_4 + e^{-i\pi(A_2+B_2+C_2)}\mathcal{J}_2 + e^{-i\pi(A_2+B_2+C_2+G)}\mathcal{J}_1 = 0, \quad \text{(B.10)}$$

with

$$\hat{\mathcal{I}}_4 = \int_{-\infty}^{1} dw_1 \int_0^z dw_2 |w_1 - w_2|^g \prod_{i=1}^{2} |w_i|^{a_i} |1 - w_i|^{b_i} |z - w_i|^{c_i}. \quad \text{(B.11)}$$

From these equations one can easily solve the $\hat{\mathcal{I}}_1$, $\hat{\mathcal{I}}_2$, $\hat{\mathcal{I}}_3$ and $\hat{\mathcal{I}}_4$ in terms of the $\mathcal{J}_i$ integrals with the result (using $s(x) := \sin(\pi x)$)

$$\begin{aligned}
s(B_2+C_2+G)\hat{\mathcal{I}}_1 &= s(A_2)\mathcal{J}_4 - s(C_2)\mathcal{J}_5 - s(C_2+G)\mathcal{J}_6 \quad \text{(B.12)}\\
s(B_2+C_2)\hat{\mathcal{I}}_2 &= s(A_2+G)\mathcal{J}_1 + s(A_2)\mathcal{J}_2 - s(C_2)\mathcal{J}_3\\
-s(B_2+C_2+G)\hat{\mathcal{I}}_3 &= s(A_2+B_2+C_2+G)\mathcal{J}_4 + s(B_2+G)\mathcal{J}_5 + s(B_2)\mathcal{J}_6\\
-s(B_2+C_2)\hat{\mathcal{I}}_4 &= -s(A_2+B_2+C_2+G)\mathcal{J}_1 + s(A_2+B_2+C_2)\mathcal{J}_2 + s(B_2)\mathcal{J}_3\,.
\end{aligned}$$

Plugging these expressions back into (B.3) and (B.5) gives a system of equations relating the integrals $\mathcal{I}_1$, $\mathcal{I}_2$, $\mathcal{I}_3$ and $\mathcal{I}_4$ and the integrals $\tilde{\mathcal{I}}_i$ with $i = 1, \ldots, 6$. We get a similar system by considering the contour deformation of the other $\mathcal{I}_i$ integrals in (B.2). With similar equations, obtained after deforming the w_1 contour of integration in $\mathcal{I}_3$ and the contour of w_2 in $\mathcal{I}_2$ and $\mathcal{I}_4$, we get a linear system relating the integrals $\mathcal{I}_i$ and the integrals $\mathcal{J}_i$ with $i = 1, \ldots, 6$. The expression for the matrix $S(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}; \boldsymbol{G})$ is given below and in the attached file `S5-matrix.txt`.

The first line of the matrix is given by using the notation $s(x) := \sin(\pi x)$

$$S(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}; G)_{1i} = \begin{pmatrix} \frac{s(A_2)s(A_1-B_2-C_2))}{s(B_2+C_2))s(B_1+B_2+C_1+C_2+G))} \\ \frac{s(A_1)s(A_2+B_2+C_2+G))}{s(B_2+C_2))s(B_1+B_2+C_1+C_2+G))} \\ -\frac{s(A_1)s(C_2)}{s(B_2+C_2))s(B_1+C_1+G))} \\ 0 \\ -\frac{s(B_1)s(C_2)}{s(B_1+C_1+G))s(B_1+B_2+C_1+C_2+G))} \\ \frac{s(C_1)s(B_1+C_1+C_2+G))}{s(B_1+C_1+G))s(B_1+B_2+C_1+C_2+G))} \end{pmatrix}. \quad \text{(B.13)}$$

The second line of the matrix is given by

$$S(\boldsymbol{A},\boldsymbol{B},\boldsymbol{C};G)_{2i} = \begin{pmatrix} \frac{s(A_2)s(A_1+B_1+C_1+G))}{s(B_1+C_1))s(B_1+B_2+C_1+C_2+G))} \\ \frac{s(A_1)s(A_2-B_1-C_1))}{s(B_1+C_1))s(B_1+B_2+C_1+C_2+G))} \\ 0 \\ -\frac{s(A_2)s(C_1)}{s(B_1+C_1))s(B_2+C_2+G))} \\ \frac{s(C_2)s(B_2+C_1+C_2+G))}{s(B_2+C_2+G))s(B_1+B_2+C_1+C_2+G))} \\ -\frac{s(B_2)s(C_1)}{s(B_2+C_2+G))s(B_1+B_2+C_1+C_2+G))} \end{pmatrix}. \tag{B.14}$$

The third line of the matrix is given by

$$S(\boldsymbol{A},\boldsymbol{B},\boldsymbol{C};G)_{3i} =$$

$$\begin{pmatrix} \frac{s(G)s((B_1+C_1)s((A_1+A_2+G)-s((A_1+G)s((A_2+B_2+C_2+G)s((B_1+B_2+C_1+C_2)}{s((B_1+C_1)s((B_2+C_2)s((B_1+B_2+C_1+C_2+G)} \\ -\frac{s(A_1)s(B_1+B_2+C_1+C_2))s(A_2+B_2+C_2+G))}{s(B_1+C_1))s(B_2+C_2))s(B_1+B_2+C_1+C_2+G))} \\ -\frac{s(A_1)s(B_2)}{s(B_2+C_2))s(B_1+C_1+G))} \\ \frac{s(C_1)s(A_2+B_2+C_2+G))}{s(B_1+C_1))s(B_2+C_2+G))} \\ \frac{s(G)s((B_1+B_2+C_1+G)s((B_2+C_1+C_2+G)+s(B_2)s(C_1)s((B_1+B_2+C_1+C_2+G)}{s((B_1+C_1+G)s((B_2+C_2+G)s((B_1+B_2+C_1+C_2+G)} \\ \frac{s(B_2)s(C_1)s(B_1+B_2+C_1+C_2+2G))}{s(B_1+C_1+G))s(B_2+C_2+G))s(B_1+B_2+C_1+C_2+G))} \end{pmatrix}. \tag{B.15}$$

The fourth line of the matrix is given by

$$S(\boldsymbol{A},\boldsymbol{B},\boldsymbol{C};G)_{4i} =$$

$$\begin{pmatrix} -\frac{s(A_2)s(B_1+B_2+C_1+C_2))s(A_1+B_1+C_1+G))}{s(B_1+C_1))s(B_2+C_2))s(B_1+B_2+C_1+C_2+G))} \\ \frac{s(A_1)s(g)s(A_2+B_2+C_2+G))-s(A_1+B_1+C_1))s(A_2+G))s(B_1+B_2+C_1+C_2+G))}{s(B_1+C_1))s(B_2+C_2))s(B_1+B_2+C_1+C_2+G))} \\ \frac{s(C_2)s(A_1+B_1+C_1+G))}{s(B_2+C_2))s(B_1+C_1+G))} \\ -\frac{s(A_2)s(B_1)}{s(B_1+C_1))s(B_2+C_2+G))} \\ \frac{s(B_1)s(C_2)s(B_1+B_2+C_1+C_2+2G))}{s(B_1+C_1+G))s(B_2+C_2+G))s(B_1+B_2+C_1+C_2+G))} \\ \frac{s(C_1)s(g)s(B_2+C_2+G))s(B_1+C_1+C_2+G))+s(B_1)s(B_1+C_1+G))s(C_2+G))s(B_1+B_2+C_1+C_2+G))}{s(B_1+C_1))s(B_1+C_1+G))s(B_2+C_2+G))s(B_1+B_2+C_1+C_2+G))} \end{pmatrix}. \tag{B.16}$$

The fifth line of the matrix is given by

$$S(\boldsymbol{A},\boldsymbol{B},\boldsymbol{C};G)_{5i}=\begin{pmatrix}\frac{s(A_2)s(A_1+B_1+C_1+G))}{s(B_2+C_2))s(B_1+B_2+C_1+C_2+G))}\\ \frac{s(A_2+B_2+C_2+G))s(A_1+B_1+B_2+C_1+C_2+G))}{s(B_2+C_2))s(B_1+B_2+C_1+C_2+G))}\\ \frac{s(B_2)s(A_1+B_1+C_1+G))}{s(B_2+C_2))s(B_1+C_1+G))}\\ 0\\ \frac{s(B_1)s(B_1+B_2+C_1+G))}{s(B_1+C_1+G))s(B_1+B_2+C_1+C_2+G))}\\ -\frac{s(B_2)s(C_1)}{s(B_1+C_1+G))s(B_1+B_2+C_1+C_2+G))}\end{pmatrix}. \tag{B.17}$$

The sixth line of the matrix is given by

$$S(\boldsymbol{A},\boldsymbol{B},\boldsymbol{C};G)_{6i}=\begin{pmatrix}\frac{s(A_1+B_1+C_1+G))s(A_2+B_1+B_2+C_1+C_2+G))}{s(B_1+C_1))s(B_1+B_2+C_1+C_2+G))}\\ \frac{s(A_1)s(A_2+B_2+C_2+G))}{s(B_1+C_1))s(B_1+B_2+C_1+C_2+G))}\\ 0\\ \frac{s(B_1)s(A_2+B_2+C_2+G))}{s(B_1+C_1))s(B_2+C_2+G))}\\ -\frac{s(B_1)s(C_2)}{s(B_2+C_2+G))s(B_1+B_2+C_1+C_2+G))}\\ \frac{s(B_2)s(B_1+B_2+C_2+G))}{s(B_2+C_2+G))s(B_1+B_2+C_1+C_2+G))}\end{pmatrix}. \tag{B.18}$$

Appendix C

The momentum kernel in string theory

The momentum kernel $\mathcal{S}_{\alpha'}(\alpha(i_1,\ldots,i_r)|\beta(i_1,\ldots,i_r))|_p$ depends on two permutations α and β, and p a reference momentum. The momentum kernel encodes all kinematic relations between multi-particle ordered open string amplitudes [42, 71–73]. We review briefly the basic properties of the momentum kernel listed in [42, §3] and refer to this work for a proof.

1. **Reflection symmetry:**

$$\mathcal{S}_{\alpha'}[\sigma(1,\ldots,r)|\beta(1,\ldots,r)]_p=\mathcal{S}_{\alpha'}[\gamma(r,\ldots,1)|\beta(r,\ldots,1)]_p \tag{C.1}$$

for any massless external momentum p and for σ and β arbitrary permutations of the k labels $\{1,\ldots,r\}$.

2. **Factorisation property:** For any permutations of the external legs α, β, γ and δ, the momentum kernel factorises as

$$\begin{aligned}&\mathcal{S}_{\alpha'}[\gamma(r+1,\dots,p),\sigma(2,\dots,r)|\beta(2,\dots,r),\delta(r+1,\dots,p)]_{k_1}\\&=\mathcal{S}_{\alpha'}[\sigma(2,\dots,r)|\beta(2,\dots,r)]_{k_1}\times\mathcal{S}_{\alpha'}[\gamma(r+1,\dots,p)|\delta(r+1,\dots,p)]_P\,,\end{aligned}\tag{C.2}$$

for all on-shell massless, $P^2=0$ momentum $P=k_1+k_2+\cdots+k_p$.

This allows to determine recursively the momentum kernel to all order, starting from the four-point momentum kernel

$$\begin{aligned}&\mathcal{S}_{\alpha'}[23|23]_{k_1}=\mathcal{S}_{\alpha'}[32|32]_{k_1}=\sin(\pi\alpha k_1\cdot k_2)\sin(\pi\alpha' k_1\cdot k_3)\\&\mathcal{S}_{\alpha'}[23|32]_{k_1}=-\sin(\pi\alpha k_1\cdot k_3)^2,\mathcal{S}_{\alpha'}[32|23]_{k_1}=-\sin(\pi\alpha' k_1\cdot k_2)^2\,,\end{aligned}\tag{C.3}$$

where the external momenta are massless $k_i^2=0$, for $1\le i\le 4$ and satisfy the momentum conservation condition $k_1+\cdots+k_4=0$.

3. **Annihilation of amplitudes:**

$$\sum_\sigma \mathcal{S}_{\alpha'}[\sigma(2,\dots,N-1)|\beta(2,\dots,N-1)]_{k_1}A_N(N,\sigma(2,\dots,N-1),1)=0\,,\tag{C.4}$$

where β is any permutation of the legs $\{2,\dots,N-1\}$ and A_n are colour-ordered tree-level string amplitudes. The annihilation property provides all possible kinematic relations between ordered open string amplitudes.

4. **The shifting-formula:** For any $2\le j\le n-2$:

$$\begin{aligned}&\sum_{\gamma,\beta}\mathcal{S}_{\alpha'}[\gamma(i_2,\dots,i_j)|i_2,\dots,i_j]_{k_1}\mathcal{S}_{\alpha'}[i_{j+1},\dots,i_{n-2}|\beta(i_{j+1},\dots,i_{n-2})]_{k_{N-1}}\\&\qquad\times A_n(\gamma(i_2,\dots,i_j),1,N-1,\beta(i_{j+1},\dots,i_{n-2}),N)\\&=\sum_{\gamma',\beta'}\mathcal{S}_{\alpha'}[\gamma'(i_2,\dots,i_{j-1})|i_2,\dots,i_{j-1}]_{k_1}\mathcal{S}_{\alpha'}[i_j,\dots,i_{n-2}|\beta'(i_j,\dots,i_{n-2})]_{k_{N-1}}\\&\qquad\times A_n(\gamma'(i_2,\dots,i_{j-1}),1,N-1,\beta'(i_j,\dots,i_{n-2}),N)\,.\end{aligned}$$

Appendix D

The closed superstring integrals $J_{\rho,\sigma}$

In this appendix we prove that the absolutely convergent integrals contributing to the non-polar part of the α'-expansion of the tree-level closed superstring amplitudes $J_{\rho,\sigma}$ are single-valued multiple zeta values. More precisely, we demonstrate that for $m_i, n_i, l_{i,j} \geq 1$ and for arbitrary $\rho, \sigma \in \mathfrak{S}_k$ the (absolutely convergent) integrals

$$\int_{(\mathbb{P}^1_{\mathbb{C}})^k} \frac{\prod_{1\leq i<j\leq k} \mathcal{L}_{z_i}^{l_{i,j}}(z_j) \prod_{i=1}^k \mathcal{L}_0^{m_i}(z_i)\mathcal{L}_1^{n_i}(z_i)\, d^2z_i}{z_{\rho(1)}\, \overline{z}_{\sigma(1)}(1-z_{\rho(k)})(1-\overline{z}_{\sigma(k)}) \prod_{i=2}^k (z_{\rho(i)}-z_{\rho(i-1)})(\overline{z}_{\sigma(i)}-\overline{z}_{\sigma(i-1)})} \tag{D.1}$$

belong to $\mathcal{Z}^{\mathrm{sv}}$. We will need the following:

Lemma D.1. — *Let $X = \{0, 1, \sigma_2, \ldots, \sigma_N\}$, $X_N = \{0, 1, \sigma_2, \ldots, \sigma_{N-1}\}$, $m, n \leq N$ and $\{\sigma_{i_r}\}_{r=1}^m, \{\sigma_{j_s}\}_{s=1}^n \subset X$ (possibly with non-empty intersection). Let $f(z) = \sum_u c_u \mathcal{L}_u(z)$ be a finite linear combination of single-valued hyperlogarithms with coefficients $c_u \in \mathcal{S}^{\mathrm{sv}}_{X,\mathbb{Q}}$ such that the integral*

$$I := \int_{\mathbb{P}^1_{\mathbb{C}}} \frac{f(z)\, d^2z}{\prod_{r=1}^m (z-\sigma_{i_r}) \prod_{s=1}^n (\overline{z}-\overline{\sigma}_{j_s})} \tag{D.2}$$

is absolutely convergent. Then there exists a finite linear combination $g(\sigma_N) = \sum_{v\in X_n^*} k_v \mathcal{L}_v(\sigma_N)$ with $k_v \in \mathcal{S}^{\mathrm{sv}}_{X_N,\mathbb{Q}}$ such that

$$I = \sum_{r=1}^m \sum_{s=1}^n h_r \overline{h}_s g(\sigma_N), \tag{D.3}$$

where

$$h_r := \prod_{\substack{k=1\\k\neq r}}^m \frac{1}{\sigma_{i_r}-\sigma_{i_k}}, \quad \overline{h}_s := \prod_{\substack{k=1\\k\neq s}}^n \frac{1}{\overline{\sigma}_{j_s}-\overline{\sigma}_{j_k}}. \tag{D.4}$$

Proof. First of all, we recall the partial-fraction identities

$$\prod_{r=1}^m \frac{1}{z-\sigma_{i_r}} = \sum_{r=1}^m \frac{h_r}{z-\sigma_{i_r}}, \tag{D.5}$$

$$\prod_{s=1}^{n} \frac{1}{\overline{z} - \overline{\sigma}_{j_s}} = \sum_{s=1}^{n} \frac{\overline{h}_s}{\overline{z} - \overline{\sigma}_{j_s}}, \tag{D.6}$$

with $h_r, \overline{h}_s$ as in eq.(D.4). A single-valued primitive with respect to $\partial/\partial z$ of the integrand of (D.2) is therefore given by

$$\sum_{r=1}^{m} \sum_{s=1}^{n} h_r \frac{\overline{h}_s}{\overline{z} - \overline{\sigma}_{j_s}} \sum_{u} c_u \mathcal{L}_{\sigma_{i_r} u}(z), \tag{D.7}$$

and so by Theorem 9.7 we have

$$I = -\sum_{r=1}^{m} \sum_{s=1}^{n} h_r \overline{h}_s \sum_{u} c_u \mathcal{L}_{\sigma_{i_r} u}(\sigma_{j_s}). \tag{D.8}$$

Theorem 9.10 concludes the proof.
□

We can now integrate (D.1) one variable at a time, and we claim that we can use Lemma D.1 at each step. This follows from the following two remarks:

(i) The special cyclic structure of the denominator of (D.1) implies that after each integration, even though h_r and $\overline{h}_s$ introduce new factors, we always get a square-free denominator, as required by the assumptions of the lemma.

(ii) The fact that $m_i, n_i, l_{i,j} \geq 1$, together with the previous remark, implies that at each step we get a numerator $f(z)$ of the integrand such that the integral converges absolutely, as required by the assumptions of the lemma.

These two remarks also provide a double-check that the integrals (D.1) are indeed absolutely convergent: suppose for instance that at some integration step we could get a factor $(z_i - z_j)^2(\overline{z}_i - \overline{z}_j)$, then the next integral in z_i would be divergent. To conclude, after k integrations and k applications of the lemma we land on a number belonging to $\mathcal{Z}^{\mathrm{sv}} = \mathcal{S}^{\mathrm{sv}}_{\{0,1\},\mathbb{Q}}$, as claimed.

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P. Vanhove, Institut de Physique Théorique, CEA, IPhT, F-91191 Gif-sur-Yvette, France, CNRS, URA 2306, F-91191 Gif-sur-Yvette, France • National Research University Higher School of Economics, Russian Federation

F. Zerbini, Institut de Physique Théorique, CEA, IPhT, F-91191 Gif-sur-Yvette, France, CNRS, URA 2306, F-91191 Gif-sur-Yvette, France