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On the Omega Lemma

Nadji HERMAS[‡], Nabila BEDIDA, Slimane AMRAOUI

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Abstract

Dans cet article, on donne des nouvelles versions, dans le cadre des espaces d'Hölder, au lemme qui est nommé dans l'analyse globale 'the ω -lemma', et on croit que ces versions auront des applications dans plusieurs disciplines telles que 'la théorie des variétés des applications' et 'la théorie des groupes de difféomorphismes'.

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1 Introduction

The aim of this paper is to give some new formulations in the context of the Hölder spaces of the Omega lemma, which is a known lemma in the global analysis. The terminology 'Omega lemma' was invented in 1963 by Abraham [1].

This lemma has various important applications in the global analysis, especially in the theory of manifolds of mappings. Sobolev [8] is the first who gave some formulations and applications of the omega lemma in the end of the thirties of the twentieth century. In [1], [2], and [5], we find classical versions of this result formulated for classical Banach spaces of functions and sections of smooth vector bundles. In [3], the authors used a suitable version of the omega lemma in the study of the structures of some Hilbert diffeomorphism groups based on Sobolev spaces. These groups have played a crucial role in the use of the techniques of the Riemann geometry in the study of the Euler equations for a perfect fluid. This is probably one of the successful applications of Riemann geometry in partial differential equations. For further information about the omega lemma in the context of Sobolev mappings, the reader can consult [4] and [7].

The main results of this paper are Theorem 2.5 and Proposition 2.6. The proofs of these two results, which are based on the famous theorem of

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Taylor, were presented with greater accuracy and greater clarity, and this is to facilitate the readers' understanding of this paper. For the same purpose, we gave an appropriate text for the Taylor's theorem.

Theorem 2.8 is an important adjustment for Theorem 2.5, while Theorem 2.10, which is the result most similar to the classical version for the omega lemma given in [1] and [2], is a suitable formulation in the case of smooth vector bundles, for Theorem 2.5. We also gave here the proofs of these two theorems.

Always in order to facilitate the reading of the paper, we gave a short and precise text containing the necessary definitions of Hölder spaces and their natural topologies. The three simple results 'Propositions 2.3 and 2.4 and Lemma 2.7' about these spaces have been given here because we need them and also because they are not, as we see, very much presented in the literature of these spaces. Moreover, we believe that the results 2.4 and 2.7 have not yet been given in this literature.

We expect that the results obtained in this paper have appropriate uses in several disciplines as Theory of manifolds of mappings, Theory of groups of diffeomorphisms, and Finsler Geometry.

2 Some formulations of the Omega Lemma

2.1 Taylor's theorem

In this subsection, we give a suitable text of the famous theorem of Taylor.

Let E_1, \dots, E_n and F be normed spaces. We provide the space

$$\mathcal{L}(E_1, \dots, E_n; F),$$

of all bounded n -linear mappings from $E_1 \times \dots \times E_n$ to F with the norm

$$\|\ell\|_{\mathcal{L}(E_1, \dots, E_n; F)} = \sup_{\|u_1\|_{E_1}, \dots, \|u_n\|_{E_n} \leq 1} \|\ell(u_1, \dots, u_n)\|_F, \ell \in \mathcal{L}(E_1, \dots, E_n; F),$$

which defines the topology of uniform convergence on bounded sets in $E_1 \times \dots \times E_n$. If $E_1 = \dots = E_n = E$, we simply write

$$\mathcal{L}^n(E; F) = \mathcal{L}(E_1, \dots, E_n; F).$$

Let E be a normed space. For $k \geq 1$, we denote by $\text{Poly}^k(E; F)$ the space of homogeneous polynomials of degree k from E to F . Recall that $\text{Poly}^k(E; F)$ is defined as follows:

$$\begin{aligned} & \text{Poly}^k(E; F) \\ &= \left\{ P : E \rightarrow F : \exists B \in \mathcal{L}^k(E; F), P(u) = B(u, \dots, u), u \in E \right\}. \end{aligned}$$

We equip $\text{Poly}^k(E; F)$ with the norm

$$\sup \{ \|P(u)\|_F : \|v\|_E \leq 1 \}, P \in \text{Poly}^k(E; F).$$

Let $\mathcal{L}_{\text{sym}}^k(E; F)$ denote the subspaces of symmetric elements of $\mathcal{L}^k(E; F)$. Then for each $P \in \text{Poly}^k(E; F)$, there is an unique element $\tilde{P} \in \mathcal{L}_{\text{sym}}^k(E; F)$ such that

$$P(u) = \tilde{P}(u, \dots, u) = \tilde{P}(u)^k, u \in E.$$

Furthermore, the mapping $P \in \text{Poly}^k(E; F) \mapsto \tilde{P} \in \mathcal{L}_{\text{sym}}^k(E; F)$ is a linear homeomorphism.

Let Ω be a non-empty open subset of E . We put

$$\tilde{\Omega} = \{(x, u) \in \Omega \times E : \|u\|_E < \text{dist}(x, \partial\Omega)\},$$

where

$$\text{dist}(x, \partial\Omega) = \inf \{ \|y - x\|_E, y \in \partial\Omega \},$$

is the distance between x and $\partial\Omega$. This set is open in $\Omega \times E$ by the continuity of the function $x \in E \mapsto \text{dist}(x, \partial\Omega) \in \mathbb{R}_+$, and has the property: for each $(x, u, t) \in \tilde{\Omega} \times [0, 1]$, $x + tu \in \Omega$.

If $\partial\Omega = \phi$, then $\Omega = E$ and $\tilde{\Omega} = E \times E$.

Let $f : \Omega \rightarrow F$ be a given function, which is k times differentiable at $x \in \Omega$, where $k \geq 1$. For $j \in \{1, \dots, k\}$, we note by $f^{(j)}(x)$ the derivative of order j of f at x . Knowing that $f^{(j)}(x) \in \mathcal{L}_{\text{sym}}^j(E; F)$, we put

$$f^{(j)}(x)(u)^j = f^{(j)}(x)(u, \dots, u), u \in E.$$

For $j = 0$, we have $f^{(0)} = f$ and then $f^{(0)}(x)(u)^0 = f(x)$ for all $u \in E$.

Theorem 2.1 (Taylor's theorem) *Let E be normed space, let F be a Banach space, and let $f : \Omega \subset E \rightarrow F$ be a given function defined on an open set $\Omega \subset E$. The following two statements are true.*

1. *If f is C^k in Ω with $k \geq 1$, then*

$$f(x + u) = \sum_{0 \leq j \leq k-1} \frac{f^{(j)}(x)(u)^j}{j!} + \int_0^1 \frac{(1-t)^{k-1}}{(k-1)!} f^{(k)}(x + tu)(u)^k dt,$$

for all $(x, u) \in \tilde{\Omega}$.

2. *If there are functions $P_j \in C^0(\Omega; \text{Poly}^j(E; F))$ ($1 \leq j \leq k$) and $R_k \in C^0(\tilde{\Omega}; \text{Poly}^k(E; F))$ such that*

$$f(x + u) = f(x) + \sum_{1 \leq j \leq k} P_j(x)(u) + R_k(x, u)(u), R_k(x, 0) = 0,$$

for all $(x, u) \in \tilde{\Omega}$, then f is C^k , and we necessarily have:

$$P_j(x)(u) = \frac{f^{(j)}(x)(u)^j}{j!}, 1 \leq j \leq k,$$

$$R_k(x, u)(v) = \int_0^1 \frac{(1-t)^{k-1}}{(k-1)!} f^{(k)}(x+tu)(v)^k dt - \frac{f^{(k)}(x)(v)^k}{k!},$$

for $(x, u) \in \tilde{\Omega}$ and $v \in E$.

The proof of the statement 1 is easy. Concerning the statement 2, which is known by the name 'the converse to Taylor's theorem', we can consult [5] (Theorem 2.4.15) and [6] (Theorem 3, p. 7).

2.2 Hölder spaces

In this subsection, we discuss the Hölder spaces and their linear topologies.

For any subset A of a topological space, we denote by $\bar{A} = \text{cl}(A)$ the closure of A and by $\mathring{A} = \text{int}(A)$ the interior of A . For $s \in \mathbb{R}$, $[s]$ denotes the integer part of s , that is the unique integer checking the double inequality $[s] \leq s < [s] + 1$.

Let E and F be normed spaces and let Ω be an open set of E . For a positive integer $s \in \mathbb{N}$, we denote by $\mathcal{C}^s(\Omega; F)$ the space of the functions $u : \Omega \rightarrow F$ of class C^s such that all the derivatives $u^{(j)} : \Omega \rightarrow \mathcal{L}^j(E; F)$, $0 \leq j \leq s$, of u are bounded.

For $s \in]0, 1[$, we define the Hölder space $\mathcal{C}^s(\Omega; F)$ to consist of functions $u \in \mathcal{C}^0(\Omega; F)$ such that

$$\sup_{x, y \in \Omega, x \neq y} \frac{\|u(x) - u(y)\|_F}{\|x - y\|_E^s} < \infty.$$

If $s \in \mathbb{R}_+ \setminus \mathbb{N} = [0, +\infty[\setminus \mathbb{N}$, we define $\mathcal{C}^s(\Omega; F)$ to consist of functions $u \in \mathcal{C}^{[s]}(\Omega; F)$ such that the derivative $u^{([s])}$ belongs to $\mathcal{C}^{s-[s]}(\Omega; \mathcal{L}^{[s]}(E; F))$.

The standard norm of $\mathcal{C}^s(\Omega; F)$, $s \in \mathbb{R}_+$, is given by

$$\|u\|_{\mathcal{C}^s(\Omega; F)} = \sum_{0 \leq j \leq s} \sup_{x \in \Omega} \|u^{(j)}(x)\|_{\mathcal{L}^j(E; F)}, u \in \mathcal{C}^s(\Omega; F),$$

when $s \in \mathbb{N}$, and by

$$\|u\|_{\mathcal{C}^s(\Omega; F)} = \sum_{0 \leq j \leq s} \sup_{x \in \Omega} \|u^{(j)}(x)\|_{\mathcal{L}^j(E; F)} + \sup_{x, y \in \Omega, x \neq y} \frac{\|u^{([s])}(x) - u^{([s])}(y)\|_{\mathcal{L}^{[s]}(E; F)}}{\|x - y\|_E^{s-[s]}},$$

when $s \in \mathbb{R}_+ \setminus \mathbb{N}$.

If $E = \mathbb{R}^n$, the norm $\|\cdot\|_{\mathcal{C}^s(\Omega;F)}$ can also be given as follows:

$$\|u\|_{\mathcal{C}^s(\Omega;F)} = \begin{cases} \sum_{|\alpha| \leq s} \sup_{x \in \Omega} \|\partial^\alpha u(x)\|_F, & \text{for } s \in \mathbb{N}, \\ \sum_{\alpha \in \mathbb{N}^n, |\alpha| \leq [s]} \sup_{x \in \Omega} \|\partial^\alpha u(x)\|_F \\ + \sup_{\substack{x, y \in \Omega, x \neq y \\ \alpha \in \mathbb{N}^n, |\alpha| = [s]}} \frac{\|\partial^\alpha u(x) - \partial^\alpha u(y)\|_F}{\|x - y\|_E^{s - [s]}}, & \text{for } s \in \mathbb{R}_+ \setminus \mathbb{N}, \end{cases}$$

with $u \in \mathcal{C}^s(\Omega;F)$, where $\partial^\alpha u$ denote the partial derivative of order $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ of u and $|\alpha| = \alpha_1 + \dots + \alpha_n$.

When F is a Banach space, $\mathcal{C}^s(\Omega;F)$, $s \in \mathbb{R}_+$, provided with its standard norm is again a Banach space.

Also, for $s \in \mathbb{N}$, we denote by $\mathcal{C}^s(\overline{\Omega};F)$ the space of the functions $u : \Omega \rightarrow F$ of class \mathcal{C}^s such that all the derivatives $u^{(j)}$, $0 \leq j \leq s$, have bounded, continuous extensions defined in $\overline{\Omega}$. If $s \in]0, 1[$, we take $\mathcal{C}^s(\Omega;F)$ to consist of functions $u \in \mathcal{C}^0(\overline{\Omega};F)$ such that

$$\sup_{x, y \in \overline{\Omega}, x \neq y} \frac{\|u(x) - u(y)\|_F}{\|x - y\|_E^s} < \infty,$$

and if $s \in [1, +\infty[\setminus \mathbb{N}$, we take $\mathcal{C}^s(\overline{\Omega};F)$ to consist of functions $u \in \mathcal{C}^{[s]}(\overline{\Omega};F)$ such that $u^{([s])} \in \mathcal{C}^{s - [s]}(\overline{\Omega};\mathcal{L}^{[s]}(E;F))$.

The standard norm of $\mathcal{C}^s(\overline{\Omega};F)$, $s \in \mathbb{R}_+$, is given by the same expression of the norm of $\mathcal{C}^s(\Omega;F)$ where Ω is replaced by $\overline{\Omega}$.

For given $s \in \mathbb{R}_+$, the relationship between the two spaces $\mathcal{C}^s(\Omega;F)$ and $\mathcal{C}^s(\overline{\Omega};F)$ is easy to detect. First, we note that the linear mapping of restriction

$$R_s : u \in \mathcal{C}^s(\overline{\Omega};F) \mapsto u|_\Omega \in \mathcal{C}^s(\Omega;F),$$

is an isometry. In addition, if F is a Banach space, then, for each $s \in \mathbb{R}_+ \setminus \mathbb{N}$, the mapping R_s is an isometric linear homeomorphism. This statement comes from the fact that, for every function $u \in \mathcal{C}^s(\Omega;F)$, the derivatives $u^{(j)}$, $0 \leq j \leq [s]$, are uniformly continuous in Ω , so that they have bounded, continuous extensions defined in $\overline{\Omega}$.

Let $s \in \mathbb{N}$. If $\overline{\Omega}$ is compact (so, by the Riesz lemma, E has finite dimension), then the space $\mathcal{C}^s(\overline{\Omega};F)$ coincides with the space $\mathcal{C}^s(\overline{\Omega};F)$ of functions $u \in \mathcal{C}^s(\Omega;F)$ such that all the derivatives $u^{(j)}$, $0 \leq j \leq s$, have continuous extensions defined in $\overline{\Omega}$. Hence, if F is a Banach space, then $\mathcal{C}^s(\overline{\Omega};F)$ is exactly the space of functions $u \in \mathcal{C}^s(\Omega;F)$ such that all the derivatives $u^{(j)}$, $0 \leq j \leq s$, are uniformly continuous in Ω .

Let U be a subset of E satisfying $\text{int}(U) \subset U \subset \overline{\text{int}(U)}$. Since every continuous function in $\text{int}(U)$ admits at most one continuous extension in U , we define the spaces $\mathcal{C}^s(U;F)$, $s \in \mathbb{R}_+$, just like the spaces $\mathcal{C}^s(\Omega;F)$, $s \in \mathbb{R}_+$.

We recall that if U is a subset of E having the property $\text{int}(U) \subset U \subset \overline{\text{int}(U)}$, then each open subset V of U checks the same property $\text{int}(V) \subset V \subset \overline{\text{int}(V)}$.

We now define the local Hölder spaces.

Let U be a subset of E satisfying $\text{int}(U) \subset U \subset \overline{\text{int}(U)}$. For $s \in \mathbb{N}$, we put $\mathcal{C}_{loc}^s(U; F) = C^s(U; F)$, where $C^s(U; F)$ is the space of functions $u \in C^s(\text{int}(U); F)$ such that all the derivatives $u^{(j)}$, $0 \leq j \leq s$, have continuous extensions defined in U .

If $s \in \mathbb{R}_+ \setminus \mathbb{N}$, we take $\mathcal{C}_{loc}^s(U; F)$ to consist of functions $u \in C^{[s]}(U; F)$ such that, for each $x_0 \in U$, there exists an open neighbourhood $V(x_0)$ in U of x_0 with $u|_{V(x_0)} \in C^s(V(x_0); F)$.

In the following, we assume that E is finite dimensional and U is a locally compact subset of E having the property $\text{int}(U) \subset U \subset \overline{\text{int}(U)}$.

It is not hard to verify that, for $s \in \mathbb{R}_+ \setminus \mathbb{N}$, $\mathcal{C}_{loc}^s(U; F)$ is the space of functions $u \in C^{[s]}(U; F)$ such that

$$\sup_{x, y \in K, x \neq y} \frac{\|u^{([s])}(x) - u^{([s])}(y)\|_{\mathcal{L}^{[s]}(E; F)}}{\|x - y\|_E^{s - [s]}} < \infty,$$

for every compact set K of U . Indeed, if $u \in C^{[s]}(U; F)$ checks the previous condition, then $u \in \mathcal{C}_{loc}^s(U; F)$ since U is locally compact. Now, let $u \in \mathcal{C}_{loc}^s(U; F)$ and let K be compact set of U . There is an open covering $\mathcal{U} = (U_j)_{1 \leq j \leq N}$ in U of K such that, for all $j \in \{1, \dots, N\}$, $K_j = \overline{U_j}$ is a compact subset of U and

$$\sup_{x, y \in K_j, x \neq y} \frac{\|u^{([s])}(x) - u^{([s])}(y)\|_{\mathcal{L}^{[s]}(E; F)}}{\|x - y\|_E^{s - [s]}} < \infty$$

By Lebesgue covering theorem, there exists $\lambda > 0$ such that, for all $(x, y) \in K \times K$ with $\|x - y\|_E \leq \lambda$, there is a set in \mathcal{U} containing both x and y . Thus, for $(x, y) \in K \times K$ with $\|x - y\|_E > \lambda$, we have

$$\frac{\|u^{([s])}(x) - u^{([s])}(y)\|_F}{\|x - y\|_E^s} \leq \frac{2}{\lambda^s} \|u^{([s])}\|_{\mathcal{C}^0(K; \mathcal{L}^{[s]}(E; F))},$$

and for $(x, y) \in K \times K$ with $0 < \|x - y\|_{\mathbb{R}^d} \leq \lambda$, there is $j_0 \in \{1, \dots, N\}$, such that $(x, y) \in K_{j_0}$, and so,

$$\begin{aligned} & \frac{\|u^{([s])}(x) - u^{([s])}(y)\|_F}{\|x - y\|_E^{s - [s]}} \\ & \leq \|u^{([s])}\|_{\mathcal{C}^s(K_{j_0}; \mathcal{L}^{[s]}(E; F))}^* = \sup_{x, y \in K_{j_0}, x \neq y} \frac{\|u^{([s])}(x) - u^{([s])}(y)\|_{\mathcal{L}^{[s]}(E; F)}}{\|x - y\|_E^{s - [s]}}. \end{aligned}$$

Hence, we get

$$\begin{aligned} & \left\| u^{([s])} \right\|_{\mathcal{C}^s(K; \mathcal{L}^{[s]}(E; F))}^* = \sup_{x, y \in K, x \neq y} \frac{\|u^{([s])}(x) - u^{([s])}(y)\|_{\mathcal{L}^{[s]}(E; F)}}{\|x - y\|_E^{s - [s]}} \\ & \leq \max \left\{ \frac{2}{\lambda^s} \left\| u^{([s])} \right\|_{\mathcal{C}^0(K; \mathcal{L}^{[s]}(E; F))}, \max_{1 \leq j \leq N} \left\| u^{([s])} \right\|_{\mathcal{C}^s(K_j; \mathcal{L}^{[s]}(E; F))}^* \right\} < \infty. \end{aligned}$$

The standard topology of $\mathcal{C}_{loc}^s(U; F)$ is the locally convex topology generated by the seminorms

$$\|u\|_{\mathcal{C}^s(K; F)} = \begin{cases} \sum_{0 \leq j \leq s} \sup_{x \in K} \|u^{(j)}(x)\|_{\mathcal{L}^j(E; F)}, & \text{if } s \in \mathbb{N}, \\ \sum_{0 \leq j \leq [s]} \sup_{x \in K} \|u^{(j)}(x)\|_{\mathcal{L}^j(E; F)} \\ + \sup_{x, y \in K, x \neq y} \frac{\|u^{([s])}(x) - u^{([s])}(y)\|_{\mathcal{L}^{[s]}(E; F)}}{\|x - y\|_E^{s - [s]}}, & \text{if } s \in \mathbb{R}_+ \setminus \mathbb{N}, \end{cases}$$

where $u \in \mathcal{C}_{loc}^s(U; F)$ and K is a compact set in U .

The completeness of F implies that of $\mathcal{C}_{loc}^s(U; F)$. So, if F is a Banach space, then $\mathcal{C}_{loc}^s(U; F)$ is a Fréchet space.

If U is compact, then $U = \overline{\text{int}(U)}$, and thus $\mathcal{C}_{loc}^s(U; F) = \mathcal{C}^s(\overline{\text{int}(U)}; F)$ with $s \in \mathbb{R}_+$.

The topology of the space

$$C^\infty(U; F) = \bigcap_{s \in \mathbb{R}_+} \mathcal{C}_{loc}^s(U; F) = \bigcap_{j \in \mathbb{N}} C^j(U; F)$$

is generated by the seminorms

$$\sup_{x \in K} \left\| u^{(j)}(x) \right\|_{\mathcal{L}^j(E; F)},$$

where K is a compact set in U and $j \in \mathbb{N}$.

We now discuss the Hölder spaces on the manifolds.

Let M be a finite dimensional manifold (possibly with corners) modeled on the space \mathbb{R}^n with $n \geq 1$ and let (U, φ) be a chart of M . Then $\varphi(U)$ is an open subset of a quadrant

$$Q = \{x \in \mathbb{R}^n : \ell_1(x) \geq 0, \dots, \ell_k(x) \geq 0\},$$

of \mathbb{R}^n . Here $\{\ell_1, \dots, \ell_k\}$ is a linearly independent subset of the dual $(\mathbb{R}^n)^*$ and $k \in \{0, 1, \dots, n\}$. This quadrant is a convex closed subset of \mathbb{R}^n satisfying $Q = \overline{\text{int}(Q)}$ where

$$\text{int}(Q) = \{x \in \mathbb{R}^n : \ell_1(x) > 0, \dots, \ell_k(x) > 0\}.$$

So $\varphi(U)$ has the property $\text{int}(\varphi(U)) \subset \varphi(U) \subset \overline{\text{int}(\varphi(U))}$. In addition, $\varphi(U)$ provided with the restriction of the Euclidean topology of \mathbb{R}^n is locally compact and locally convex.

This fact allows us to define the Hölder spaces over the manifold M . Indeed, let N be a smooth manifold modeled on normed spaces and let $s \in \mathbb{R}_+ \cup \{\infty\}$. We define the functional set $\mathcal{C}_{loc}^s(M; N)$ to consist of functions $u : M \rightarrow N$ such that, for each chart (U, φ) of M and each chart (V, ψ) of N with $u(U) \subset V$, $\psi \circ u \circ \varphi^{-1} \in \mathcal{C}_{loc}^s(\varphi(U); E_V)$, where E_V is the normed space associated with the chart (V, ψ) (that is $\psi(V) \subset E_V$).

If M is compact, we put $\mathcal{C}^s(M; N) = \mathcal{C}_{loc}^s(M; N)$.

There is an important situation here when $N = E$ is a normed space. In this case, $\mathcal{C}_{loc}^s(M; E)$ is a vector space. We equip it with the initial topology with respect to the mappings

$$\varphi_* : u \in \mathcal{C}_{loc}^s(M; E) \mapsto u \circ \varphi^{-1} \in \mathcal{C}_{loc}^s(\varphi(U); E),$$

where (U, φ) is a chart of M . Thus, we obtain a locally convex space, which is complete when E is a Banach space.

If M is compact, we consider an atlas $((U_k, \varphi_k))_{1 \leq k \leq n}$ of M and then we consider a partition of unity $(\lambda_k)_{1 \leq k \leq n}$ of M dominated by $(U_k)_{1 \leq k \leq n}$. For $s \in \mathbb{R}_+$, the space $\mathcal{C}^s(M; E) = \mathcal{C}_{loc}^s(M; E)$ provided with the norm

$$u \in \mathcal{C}^s(M; E) \mapsto \sum_{1 \leq k \leq n} \|(\lambda_k u) \circ \varphi_k^{-1}\|_{\mathcal{C}^s(\text{supp} \lambda_k \circ \varphi_k^{-1}; E)} \in \mathbb{R}_+,$$

is a normed space, which is complete when E is complete. In addition, the topology associated with this norm is exactly the standard topology of $\mathcal{C}^s(M; E)$.

Now let E be a smooth vector bundle of standard fiber F (which is a normed space) over M . For $s \in \mathbb{R}_+ \cup \{\infty\}$, we put

$$\mathcal{C}_{loc}^s(M \rightarrow E) = \{X \in \mathcal{C}_{loc}^s(M; E) : \forall x \in M, X(x) \in E_x\},$$

where $E_x = \pi_E^{-1}(\{x\})$ is the fiber of E at x and $\pi_E : E \rightarrow M$ is the bundle projection. This set has a real vector structure and is called the space of sections of class \mathcal{C}_{loc}^s of E . It is not hard to see that a continuous section $X \in C^0(M \rightarrow E)$ belongs to $\mathcal{C}_{loc}^s(M \rightarrow E)$ if, and only if, for each bundle chart (U, ψ) of E , $\text{pr}_2 \circ \psi \circ (X|_U) \in \mathcal{C}_{loc}^s(U; F)$, where pr_2 is the projection $(x, v) \in U \times F \mapsto v \in F$. The standard topology of $\mathcal{C}_{loc}^s(M \rightarrow E)$, which is a locally convex topology, is the initial topology with respect to the mappings

$$X \in \mathcal{C}_{loc}^s(M \rightarrow E) \mapsto \text{pr}_2 \circ \psi \circ (X|_U) \in \mathcal{C}_{loc}^s(U; F),$$

where (U, ψ) is a bundle chart of E .

This topology is also the initial topology with respect to the mappings

$$X \in \mathcal{C}_{loc}^s(M \rightarrow E) \mapsto \text{pr}_2 \circ \psi \circ (X|_U) \circ \varphi^{-1} \in \mathcal{C}_{loc}^s(\varphi(U); F),$$

where (U, φ) is a chart of M and (U, ψ) is a bundle chart of E .

Assume that M is compact and let $s \in \mathbb{R}_+$. Let $((U_k, \varphi_k))_{1 \leq k \leq n}$ be an atlas of M , let $((U_k, \psi_k))_{1 \leq k \leq n}$ be a bundle atlas of E , and let $(\lambda_k)_{1 \leq k \leq n}$ be a partition of unity of M dominated by $(U_k)_{1 \leq k \leq n}$. The standard topology of the space $\mathcal{C}^s(M \rightarrow E) = \mathcal{C}_{loc}^s(M \rightarrow E)$ is the topology associated with this norm

$$\begin{aligned} X &\in \mathcal{C}^s(M \rightarrow E) \\ &\mapsto \sum_{1 \leq k \leq n} \left\| \left\{ \lambda_k \text{pr}_2 \circ \psi_k \circ \left(X|_{U_k} \right) \right\} \circ \varphi_k^{-1} \right\|_{\mathcal{C}^s(\text{supp} \lambda_k \circ \varphi_k^{-1}; F)} \in \mathbb{R}_+. \end{aligned}$$

The proof of the following simple proposition is easy.

Proposition 2.2 1. Let E, E_1, \dots, E_k , and F be normed spaces. Let $U \subset E$ be a subset having the property $\text{int}(U) \subset U \subset \overline{\text{int}(U)}$ and let $s \in \mathbb{R}_+$. If $L \in \mathcal{C}_{loc}^s(U; \mathcal{L}(E_1, \dots, E_k; F))$, $u_1 \in \mathcal{C}_{loc}^s(U; E_1)$, \dots , $u_k \in \mathcal{C}_{loc}^s(U; E_k)$, then $L(\cdot)(u_1(\cdot), \dots, u_k(\cdot)) \in \mathcal{C}_{loc}^s(U; F)$. In addition, if E is finite dimensional and U is a locally compact, then the mapping $(u_1, \dots, u_k, L) \mapsto L(\cdot)(u_1(\cdot), \dots, u_k(\cdot))$ is continuous from $\mathcal{C}_{loc}^s(U; E_1) \times \dots \times \mathcal{C}_{loc}^s(U; E_k) \times \mathcal{C}_{loc}^s(U; \mathcal{L}(E_1, \dots, E_k; F))$ to $\mathcal{C}_{loc}^s(U; F)$. In particular, if $E_1 = \dots = E_k = F = \mathbb{C}$, then the mapping $(u_1, \dots, u_k) \mapsto u_1 \cdots u_k$ is continuous from $\mathcal{C}_{loc}^s(U; \mathbb{C})^k$ to $\mathcal{C}_{loc}^s(U; \mathbb{C})$.

2. Let E, F and G be three normed spaces, let $U \subset E$ and $V \subset F$ two subsets with $\text{int}(U) \subset U \subset \overline{\text{int}(U)}$ and $\text{int}(V) \subset V \subset \overline{\text{int}(V)}$. We assume that U is locally convex. If $u \in \mathcal{C}_{loc}^s(U; F)$ and $v \in \mathcal{C}_{loc}^s(V; G)$ with $u(U) \subset V$ and $s \in]1, +\infty[- \mathbb{N}$, then $v \circ u \in \mathcal{C}_{loc}^s(U; G)$.

We note that the composition of two functions of class \mathcal{C}_{loc}^s with $s \in]0, 1[$ is not generally a function of class \mathcal{C}_{loc}^s . For example, the function $|\cdot|^{1/2} : x \in \mathbb{R} \mapsto |x|^{1/2} \in \mathbb{R}$ belongs to the space $\mathcal{C}^{1/2}(\mathbb{R}; \mathbb{R}) \subset \mathcal{C}_{loc}^{1/2}(\mathbb{R}; \mathbb{R})$, however $|\cdot|^{1/2} \circ |\cdot|^{1/2} = |\cdot|^{1/4} \notin \mathcal{C}^{1/2}(-\varepsilon, \varepsilon; \mathbb{R})$ for every $\varepsilon > 0$. Thus, the statement 2 is wrong for $s \in]0, 1[$.

The approximation of functions of class \mathcal{C}_{loc}^s or of class \mathcal{C}^s by smooth functions is an important problem. In this context, we present the following.

Proposition 2.3 Let F be a Banach space, and let $\varphi \in \mathcal{C}^s(\mathbb{R}^n; F)$ with $s \in \mathbb{R}_+$. We consider a function $\theta \in \mathcal{C}_c^\infty(\mathbb{R}^n; \mathbb{R})$ having the properties $\theta \geq 0$ and $\int_{\mathbb{R}^n} \theta(y) dy = 1$, and then for $j \geq 1$, we put $\theta_j(x) = j^n \theta(jx)$ and

$$\varphi_j(x) = \theta_j * \varphi(x) = \int_{\mathbb{R}^n} \theta_j(x-y) \varphi(y) dy = \int_{\mathbb{R}^n} \theta_j(y) \varphi(x-y) dy, x \in \mathbb{R}^n.$$

Then $(\varphi_j)_{j \geq 1} \subset \mathcal{C}^\infty(\mathbb{R}^n; F) = \bigcap_{s \in \mathbb{R}_+} \mathcal{C}^s(\mathbb{R}^n; F)$ and:

1. For $s \in \mathbb{N}$, if the derivatives $\partial^\alpha \varphi : \mathbb{R}^n \rightarrow F$, $|\alpha| = [s]$, are uniformly continuous, then $\varphi_j \rightarrow \varphi$ in $\mathcal{C}^s(\mathbb{R}^n; F)$.
2. For $s \in \mathbb{R}_+ \setminus \mathbb{N}$, if the functions

$$(x, y) \in \mathbb{R}^n \times \mathbb{R}^n - \Delta_{\mathbb{R}^n} \mapsto \frac{\partial^\alpha \varphi(x) - \partial^\alpha \varphi(y)}{\|x - y\|_{\mathbb{R}^n}^{s - [s]}} \in F, |\alpha| = [s],$$

where $\Delta_{\mathbb{R}^n} = \{(x, x) : x \in \mathbb{R}^n\}$, are uniformly continuous, then we have $\varphi_j \rightarrow \varphi$ in $\mathcal{C}^s(\mathbb{R}^n; F)$.

Proof. Let $j \geq 1$. It is obvious that $\varphi_j \in C^\infty(\mathbb{R}^n; F)$ with

$$\partial^\alpha \varphi_j(x) = \int_{\mathbb{R}^n} \partial^\alpha \theta_j(y) \varphi(x - y) dy, x \in \mathbb{R}^n, \alpha \in \mathbb{N}^n.$$

So,

$$\|\partial^\alpha \varphi_j\|_{\mathcal{C}^0(\mathbb{R}^n; F)} \leq \|\varphi\|_{\mathcal{C}^0(\mathbb{R}^n; F)} \int_{\mathbb{R}^n} \|\partial^\alpha \theta_j(y)\|_E dy < \infty, \alpha \in \mathbb{N}^n,$$

which proves that $\varphi_j \in \mathcal{C}^\infty(\mathbb{R}^n; F)$. On the other hand, since we have

$$\partial^\alpha \varphi_j(x) = \int_{\mathbb{R}^n} \theta_j(y) \partial^\alpha \varphi(x - y) dy, x \in \mathbb{R}^n, \alpha \in \mathbb{N}^n, |\alpha| \leq [s],$$

it is enough to treat the case $s \in [0, 1[$.

Indeed, for $j \geq 1$ and $x \in \mathbb{R}^n$, we write

$$\begin{aligned} \varphi_j(x) - \varphi(x) &= \int_{\mathbb{R}^n} \theta_j(y) \{\varphi(x - y) - \varphi(x)\} dy \\ &= \int_{\mathbb{R}^n} \theta(y) \left\{ \varphi\left(x - \frac{1}{j}y\right) - \varphi(x) \right\} dy, \end{aligned}$$

and therefore,

$$\|\varphi_j - \varphi\|_{\mathcal{C}^0(\mathbb{R}^n; F)} \leq \sup_{x \in \mathbb{R}^n, y \in \text{supp}\theta} \left\| \varphi\left(x - \frac{1}{j}y\right) - \varphi(x) \right\|_F.$$

Hence, if φ is uniformly continuous in \mathbb{R}^n , it follows, since $\text{supp}\theta$ is bounded, that

$$\lim_{j \rightarrow \infty} \sup_{x \in \mathbb{R}^n, y \in \text{supp}\theta} \left\| \varphi\left(x - \frac{1}{j}y\right) - \varphi(x) \right\|_F = 0,$$

and then $\lim_{j \rightarrow \infty} \|\varphi_j - \varphi\|_{\mathcal{C}^0(\mathbb{R}^n; F)} = 0$. This proves the statement 1.

Now let $s \in]0, 1[$. By the above, $\varphi_j \rightarrow \varphi$ in $\mathcal{C}^0(\mathbb{R}^n; F)$ since φ is uniformly continuous in \mathbb{R}^n .

For $j \geq 1$ and $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$ with $x \neq y$, we have

$$\begin{aligned} \frac{\psi_j(x) - \psi_j(y)}{\|x - y\|_{\mathbb{R}^n}^s} &= \int_{\mathbb{R}^n} \theta(z) \left\{ \frac{\varphi\left(x - \frac{1}{j}z\right) - \varphi\left(y - \frac{1}{j}z\right)}{\|x - y\|_{\mathbb{R}^n}^s} \right. \\ &\quad \left. - \frac{\varphi(x) - \varphi(y)}{\|x - y\|_{\mathbb{R}^n}^s} \right\} dz \\ &= \int_{\mathbb{R}^n} \theta(z) \left\{ m\left(x - \frac{1}{j}z, y - \frac{1}{j}z\right) - m(x, y) \right\} dz, \end{aligned}$$

where $\psi_j = \varphi_j - \varphi$ and

$$m(x, y) = \frac{\varphi(x) - \varphi(y)}{\|x - y\|_{\mathbb{R}^n}^s}, \quad x, y \in \mathbb{R}^n, x \neq y,$$

So,

$$\begin{aligned} \|\psi_j\|_{\mathcal{C}^s(\mathbb{R}^n; F)}^* &= \sup_{x, y \in \mathbb{R}^n, x \neq y} \frac{\|\psi_j(x) - \psi_j(y)\|_F}{\|x - y\|_{\mathbb{R}^n}^s} \\ &\leq \sup_{\substack{x, y \in \mathbb{R}^n, x \neq y, \\ z \in \text{supp}\theta}} \left\| m\left(x - \frac{1}{j}z, y - \frac{1}{j}z\right) - m(x, y) \right\|_F. \end{aligned}$$

Hence, if m is uniformly continuous from $\mathbb{R}^n \times \mathbb{R}^n - \Delta_{\mathbb{R}^n}$ to F , it follows that

$$\lim_{j \rightarrow \infty} \sup_{\substack{x, y \in \mathbb{R}^n, x \neq y, \\ z \in \text{supp}\theta}} \left\| m\left(x - \frac{1}{j}z, y - \frac{1}{j}z\right) - m(x, y) \right\|_F = 0,$$

and then $\lim_{j \rightarrow \infty} \|\psi_j\|_{\mathcal{C}^s(\mathbb{R}^n; F)}^* = \lim_{j \rightarrow \infty} \|\varphi_j - \varphi\|_{\mathcal{C}^s(\mathbb{R}^n; F)}^* = 0$, meaning that $\varphi_j \rightarrow \varphi$ in $\mathcal{C}^s(\mathbb{R}^n; F)$. ■

We adjust the previous proposition for the local Hölder spaces as follows.

Proposition 2.4 *Let F be a Banach space, let $\varphi \in \mathcal{C}_{loc}^s(\mathbb{R}^n; F)$ with $s \in \mathbb{R}_+$, and let $(\varphi_j)_{j \geq 1}$ be the sequence defined in Proposition 2.3. The following statements are true.*

1. *If $s \in \mathbb{N}$, then $\varphi_j \rightarrow \varphi$ in $\mathcal{C}_{loc}^s(\mathbb{R}^n; F) = C^s(\mathbb{R}^n; F)$.*
2. *If $s \in \mathbb{R}_+ \setminus \mathbb{N}$, then $\varphi_j \rightarrow \varphi$ in $\mathcal{C}_{loc}^s(\mathbb{R}^n; F)$ if, and only if, for all $x_0 \in \mathbb{R}^n$ and all $\alpha \in \mathbb{N}^n$ with $|\alpha| = [s]$,*

$$\lim_{(x, y) \rightarrow (x_0, x_0)} \frac{\partial^\alpha \varphi(x) - \partial^\alpha \varphi(y)}{\|x - y\|_{\mathbb{R}^n}^{s - [s]}} = 0.$$

Proof. The statement 1 follows immediately from the local uniform continuity of the derivatives $\partial^\alpha \varphi$, $|\alpha| \leq s$, and the fact that

$$\|\partial^\alpha \varphi_j - \partial^\alpha \varphi\|_{\mathcal{C}^0(K;F)} \leq \sup_{x \in K, y \in \text{supp}\theta} \left\| \partial^\alpha \varphi \left(x - \frac{1}{j}y \right) - \partial^\alpha \varphi(x) \right\|_F, j \geq 1,$$

for all compact subset K of \mathbb{R}^n and all $\alpha \in \mathbb{N}^n$ with $|\alpha| \leq s$.

As in the proof of Proposition 2.3, for the statement 2, it suffices to treat the case $s \in]0, 1[$. For all $j \geq 1$ and all $(x, y) \in \mathbb{R}^2$ with $x \neq y$, we have

$$\frac{\varphi_j(x) - \varphi_j(y)}{\|x - y\|_{\mathbb{R}^n}^s} = \frac{\int_0^1 \varphi_j'(y + t(x - y))(x - y) dt}{\|x - y\|_{\mathbb{R}^n}^s},$$

and then

$$\frac{\|\varphi_j(x) - \varphi_j(y)\|_F}{\|x - y\|_{\mathbb{R}^n}^s} \leq \|x - y\|_{\mathbb{R}^n}^{1-s} \int_0^1 \|\varphi_j'(y + t(x - y))\|_{\mathcal{L}(\mathbb{R}^n;F)} dt,$$

which shows that

$$\lim_{(x,y) \rightarrow (x_0, x_0)} \frac{\varphi_j(x) - \varphi_j(y)}{\|x - y\|_{\mathbb{R}^n}^s} = 0,$$

in F for all $x_0 \in \mathbb{R}^n$. If $\varphi_j \rightarrow \varphi$ in $\mathcal{C}_{loc}^s(\mathbb{R}^n; F)$, then

$$\lim_{j \rightarrow \infty} \sup_{x, y \in B(x_0, R), x \neq y} \left\| \frac{\varphi_j(x) - \varphi_j(y)}{\|x - y\|_{\mathbb{R}^n}^s} - \frac{\varphi(x) - \varphi(y)}{\|x - y\|_{\mathbb{R}^n}^s} \right\|_F = 0,$$

for every closed ball $B(x_0, R) = \{z \in \mathbb{R}^n : \|z - x_0\|_{\mathbb{R}^n} \leq R\}$ of \mathbb{R}^n . From this, it is not difficult to deduce that

$$\lim_{(x,y) \rightarrow (x_0, x_0)} \frac{\varphi(x) - \varphi(y)}{\|x - y\|_{\mathbb{R}^n}^s} = 0, \forall x_0 \in \mathbb{R}^n.$$

Now, assume that

$$\lim_{(x,y) \rightarrow (x_0, x_0)} \frac{\varphi(x) - \varphi(y)}{\|x - y\|_{\mathbb{R}^n}^s} = 0, \forall x_0 \in \mathbb{R}^n,$$

and let K be a compact subset of \mathbb{R}^n . Since the function

$$(x, y) \in \mathbb{R}^n \times \mathbb{R}^n - \Delta_{\mathbb{R}^n} \mapsto m(x, y) = \frac{\varphi(x) - \varphi(y)}{\|x - y\|_{\mathbb{R}^n}^s} \in F,$$

has a continuous extension defined in $\mathbb{R}^n \times \mathbb{R}^n$, it follows that it is uniformly continuous in $K \times K - \Delta_K$, and then

$$\lim_{j \rightarrow \infty} \sup_{\substack{x, y \in K, x \neq y \\ z \in \text{supp}\theta}} \left\| m \left(x - \frac{1}{j}z, y - \frac{1}{j}z \right) - m(x, y) \right\|_F = 0.$$

Thus, using the inequality

$$\begin{aligned} \|\psi_j\|_{\mathcal{C}^s(K;F)}^* &= \sup_{x,y \in K, x \neq y} \frac{\|\psi_j(x) - \psi_j(y)\|_F}{\|x - y\|_{\mathbb{R}^n}^s} \\ &\leq \sup_{\substack{x,y \in K, x \neq y \\ z \in \text{supp}\theta}} \left\| m\left(x - \frac{1}{j}z, y - \frac{1}{j}z\right) - m(x, y) \right\|_F, j \geq 1, \end{aligned}$$

we get $\lim_{j \rightarrow \infty} \|\psi_j\|_{\mathcal{C}^s(K;F)}^* = 0$, which means that $\varphi_j \rightarrow \varphi$ in $\mathcal{C}_{loc}^s(\mathbb{R}^n; F)$, since K is arbitrary. ■

In the end of this subsection, we recall that the sequence $(u_j)_{j \in \mathbb{N}} \subset \mathcal{C}_{loc}^s(\mathbb{R}^n; F)$ converges, in $\mathcal{C}_{loc}^s(\mathbb{R}^n; F)$, to $u \in \mathcal{C}_{loc}^s(\mathbb{R}^n; F)$ if, and only if, for all function $\eta \in C_c^\infty(\mathbb{R}^n; F)$, the sequence $(\eta u_j)_{j \in \mathbb{N}}$ converges, in $\mathcal{C}^s(\mathbb{R}^n; F)$, to ηu .

2.3 Some results

In this subsection, we present a several convenient variants of Omega lemma.

Let S be a topological space and let E be a normed space. For given compact subset $K \subset S$, we provide the space

$$C^0(S; E)|_K = \{u|_K : u \in C^0(S; E)\},$$

which is a vector subspace of $C^0(K; E)$, with the norm of the uniform convergence on K defined by

$$\|u\|_{C^0(K; E)} = \sup_{x \in K} \|u(x)\|_E, u \in C^0(S; E)|_K.$$

If Ω is a subset of E , we put

$$\begin{aligned} C^0(S, K; \Omega) &= \{u \in C^0(S; E)|_K : u(K) \subset \Omega\} \\ &= \{v|_K : v \in C^0(S; E), v(K) \subset \Omega\}. \end{aligned}$$

This set is a subset of the set $C^0(K; \Omega)$ and contains the set

$$C^0(S; \Omega)|_K = \{u|_K : u \in C^0(S; \Omega)\}.$$

If S is compact, then $C^0(S, S; \Omega) = C^0(S; \Omega)$.

We equip the space $C^0(S; E)$ with its compact-open topology (or CO-topology) generated by the family of semi-norms

$$u \in C^0(S; E) \mapsto \|u|_K\|_{C^0(K; E)} \in \mathbb{R}_+,$$

where K is a compact subset of S .

Theorem 2.5 *Let E be a normed space and let F be a Banach space. Let Ω be an open set of E , let S be a topological space, and let M be a compact manifold (possibly with corners). Then the following statements holds.*

1. Let K be a compact subset of S . The set $C^0(S, K; \Omega)$ is open in $C^0(S; E)|_K$, and if $\varphi \in C^k(\Omega; F)$ with $k \in \mathbb{N} \cup \{\infty\}$, then the mapping $\Phi : u \mapsto \varphi \circ u$ is C^k from $C^0(S, K; \Omega)$ to $C^0(K; F)$. In particular, if S is compact, then $C^0(S; \Omega)$ is open in $C^0(S; E)$ and Φ is C^k from $C^0(S; \Omega)$ to $C^0(S; F)$.
2. If $\varphi \in C^0(\Omega; F)$, then $\Phi : u \mapsto \varphi \circ u$ is continuous from $C^0(S; \Omega)$ equipped with the trace of the compact-open topology of $C^0(S; E)$ to $C^0(S; F)$.
3. If $\varphi \in C^{j+k}(\Omega; F)$ with $j \in \mathbb{N}$ and $k \in \mathbb{N} \cup \{\infty\}$, then Φ is C^k from $C^j(M; \Omega)$ to $C^j(M; F)$.
4. Assume that E is finite dimensional and let $\varphi \in C^{[s]+k}(\Omega; F)$, where $s \in \mathbb{R}_+ \setminus \mathbb{N}$ and $k \in \mathbb{N}^* \cup \{\infty\}$. Then Φ is C^{k-1} from $C^s(M; \Omega)$ to $C^s(M; F)$.

Proof. We first treat the case $k = 0$ in the statement 1. Let $u_0 \in C^0(S, K; \Omega)$ and let $\varepsilon > 0$. Since $u_0(K)$ is compact, there exists $\delta = \delta(u_0(K), \varepsilon) > 0$ such that

$$\forall (y, z) \in u_0(K) \times \Omega : \|y - z\|_E \leq \delta \Rightarrow \|\varphi(z) - \varphi(y)\|_F \leq \varepsilon.$$

So, if $u \in C^0(S, K; \Omega)$ with $\|u - u_0\|_{C^0(K; E)} \leq \delta$, then

$$\|\varphi(u(x)) - \varphi(u_0(x))\|_F \leq \varepsilon, \forall x \in K,$$

and thus

$$\|\Phi(u) - \Phi(u_0)\|_{C^0(K; F)} = \sup_{x \in K} \|\varphi(u(x)) - \varphi(u_0(x))\|_F \leq \varepsilon,$$

which proves that $\Phi : u \in C^0(S, K; \Omega) \mapsto \varphi \circ u \in C^0(K; F)$ is continuous at u_0 .

On the other hand, due to the compactness of $u_0(K)$, there is $\rho > 0$ such that $\{z \in E : \text{dist}(z, u_0(K)) < \rho\} \subset \Omega$, where

$$\text{dist}(z, u_0(K)) = \inf_{y \in u_0(K)} \|z - y\|_E.$$

This implies that the open ball $\left\{u \in C^0(S; E)|_K : \|u - u_0\|_{C^0(K; E)} < \rho\right\}$ is included in $C^0(S, K; \Omega)$. Hence $C^0(S, K; \Omega)$ is open in $C^0(S; E)|_K$ since u_0 is arbitrary in $C^0(S, K; \Omega)$.

Let $(u, h) \in C^0(\widetilde{S, K; \Omega})$ and let $k \geq 1$. According to Taylor's formula applied to the function φ , we write

$$\begin{aligned} \Phi(u+h)(x) &= \sum_{j=0}^{k-1} \frac{\varphi^{(j)}(u(x))(h(x))^j}{j!} \\ &+ \int_0^1 \frac{(1-t)^{k-1}}{(k-1)!} \varphi^{(k)}(u(x)+th(x))(h(x))^k dt \\ &= \Phi(u)(x) + \sum_{j=1}^k \Phi_j(u)(h)(x) + R_k(u, h)(h)(x), \quad x \in K, \end{aligned}$$

where Φ_j and R_k are defined as follows:

$$\begin{aligned} \Phi_j(u)(h)(x) &= \frac{\varphi^{(j)}(u(x))(h(x))^j}{j!}, \quad 1 \leq j \leq k, \\ R_k(u, h)(\tilde{h})(x) &= \int_0^1 \frac{(1-t)^{k-1}}{(k-1)!} \varphi^{(k)}(u(x)+th(x))(\tilde{h}(x))^k dt \\ &\quad - \frac{\varphi^{(k)}(u(x))(\tilde{h}(x))^k}{j!}. \end{aligned}$$

with $x \in S$ and $\tilde{h} \in C^0(S; E)|_K$. We verify just like the case $k = 0$ that the mappings $\Phi_j, 1 \leq j \leq k$, are continuous from $C^0(S, K; \Omega)$ to $\text{Poly}^j(C^0(S; E)|_K; C^0(K; F))$ and the mapping R_k is continuous from the set $C^0(\widetilde{S, K; \Omega})$ to $\text{Poly}^k(C^0(S; E)|_K; C^0(K; F))$. Hence by Theorem 2.1, Φ is C^k from $C^0(S, K; \Omega)$ to $C^0(K; F)$. This proves the statement 1.

By the above, for each compact subset K of S , Φ is continuous from $C^0(S; \Omega)|_K$ to $C^0(K; F)$. So, it is continuous from $C^0(S; \Omega)$ to $C^0(S; F)$. This gives the statement 2.

Now, let $s \in \mathbb{R}_+$. Since the inclusion mapping $\mathcal{C}^s(M; E) \hookrightarrow C^0(M; E)$ is continuous and $C^0(M; \Omega)$ is open in $C^0(M; E)$ by the statement 1, it follows that the set $\mathcal{C}^s(M; \Omega) = \mathcal{C}^s(M; E) \cap C^0(M; \Omega)$ is open in $\mathcal{C}^s(M; E)$.

As in the proof of the statement 1, it is sufficient to treat the case $k = 0$ in the statement 3 and the case $k = 1$ in the statement 4. Recall that, according to the definitions of topologies of spaces $\mathcal{C}^s(M; E)$ and $\mathcal{C}^s(M; F)$, to prove that Φ is continuous from $\mathcal{C}^s(M; \Omega)$ to $\mathcal{C}^s(M; F)$, it is enough to verify that, for all local chart (U, f) of M , the mapping $f_* \circ \Phi \circ f^* : u \mapsto \varphi \circ u$ is continuous from $\mathcal{C}_{loc}^s(f(U); \Omega) \subset \mathcal{C}_{loc}^s(f(U); E)$ to $\mathcal{C}_{loc}^s(f(U); F)$, where

$$\begin{aligned} f^* & : u \in \mathcal{C}_{loc}^s(f(U); \Omega) \mapsto f^*u = u \circ f \in \mathcal{C}_{loc}^s(U; \Omega), \\ f_* & : v \in \mathcal{C}_{loc}^s(U; F) \mapsto v \circ f^{-1} \in \mathcal{C}_{loc}^s(f(U); F). \end{aligned}$$

So, let (U, f) be a local chart of M . By Proposition 2.2, the statement 2, and the chain rule, all mappings $u \mapsto \partial^\alpha \{\varphi \circ u\}$, $|\alpha| \leq [s]$, are continuous from

$\mathcal{C}_{loc}^s(f(U); \Omega)$ to $C^0(f(U); F)$. This implies the continuity of the mapping $f_* \circ \Phi \circ f^*$ from $\mathcal{C}_{loc}^s(f(U); \Omega)$ to $C^{[s]}(f(U); F)$. Hence, the statement 3 is true for $k = 0$. On the other hand, to prove the statement 4 for $k = 1$, it suffices to check that, for $\varphi \in C^1(\Omega; F)$ and $s \in]0, 1[$, $f_* \circ \Phi \circ f^*$ is continuous from $\mathcal{C}_{loc}^s(f(U); \Omega)$ to $\mathcal{C}_{loc}^s(f(U); F)$.

We assume that $E = \mathbb{R}^n$, and let $\varphi \in C^1(\Omega; F)$, $u_0 \in \mathcal{C}_{loc}^s(f(U); \Omega)$ with $s \in]0, 1[$. Let K be a compact subset of $f(U)$ and let V be an open set in Ω such that $u_0(K) \subset V$ and $\bar{V} = \text{cl}(V)$ is a compact set in Ω . We choose a function $\eta \in C_c^\infty(\Omega; \mathbb{R})$ with the property that $\eta \equiv 1$ on \bar{V} . As in the proof of the statement 1, we check that the set

$$\mathcal{C}_{loc}^s(f(U), K; V) = \{u|_K : u \in \mathcal{C}_{loc}^s(f(U); \mathbb{R}^n), u(K) \subset V\},$$

is open in $\mathcal{C}_{loc}^s(f(U); \mathbb{R}^n)|_K$ equipped, of course, with the norm

$$\|u\|_{\mathcal{C}^s(K; \mathbb{R}^n)} = \sup_{x \in K} \|u(x)\|_{\mathbb{R}^n} + \sup_{x, y \in K, x \neq y} \frac{\|u(x) - u(y)\|_{\mathbb{R}^n}}{\|x - y\|_{\mathbb{R}^d}^s},$$

where $u \in C^1(f(U); \mathbb{R}^n)|_K$ and $d = \dim M$. So, the set

$$W = \mathcal{C}_{loc}^s(f(U), K; V) \cap \mathcal{C}_{loc}^s(f(U); \Omega)|_K,$$

is an open neighborhood of u_0 in $\mathcal{C}_{loc}^s(f(U); \Omega)|_K$, and for any $u \in W$, we have $\varphi \circ u = (\eta\varphi) \circ u$ in K with $\eta\varphi \in C_c^1(\Omega; F)$. Therefore, we can assume that $\varphi \in C_c^1(\Omega; F) \subset C_c^1(\mathbb{R}^n; F)$ (that is to say φ is in $C^1(\Omega; F)$ and has compact support). Let θ and θ_j be as in Proposition 2.3. For $j \geq 1$, we put $\varphi_j = \theta_j * \varphi$. If j is sufficiently large, we have $\varphi_j \in C_c^\infty(\Omega; F)$. In addition, by Proposition 2.3, the sequence (φ_j) converges in $C^1(\mathbb{R}^n; F)$ to φ since the derivative $\varphi' : \mathbb{R}^n \rightarrow \mathcal{L}(\mathbb{R}^n; F)$ is uniformly continuous, which means that $C_c^\infty(\Omega; F)$ is dense in $C_c^1(\Omega; F)$ with respect to the norm of $C^1(\Omega; F)$. Hence, we can treat first the case $\varphi \in C_c^\infty(\Omega; F)$.

Let $u \in \mathcal{C}_{loc}^s(f(U); \Omega)$. By the mean value theorem, we have

$$\varphi \circ u - \varphi \circ u_0 = \int_0^1 \varphi'(u_0 + \eta(u - u_0))(u - u_0) d\eta.$$

Putting $v = u - u_0$ and $w = \varphi \circ u - \varphi \circ u_0$, we claim that we have the estimate

$$\begin{aligned} & \|w\|_{\mathcal{C}^s(K; F)} \\ & \leq \|\varphi'\|_{C^1(\mathbb{R}^n; \mathcal{L}(\mathbb{R}^n; F))} \left\{ 1 + \|u_0\|_{\mathcal{C}^s(K; \mathbb{R}^n)} + \|v\|_{\mathcal{C}^s(K; \mathbb{R}^n)} \right\} \|v\|_{\mathcal{C}^s(K; \mathbb{R}^n)}, \end{aligned} \tag{1}$$

where

$$\|\varphi'\|_{C^1(\mathbb{R}^n; \mathcal{L}(\mathbb{R}^n; F))} = \sup_{x \in \mathbb{R}^n} \|\varphi'(x)\|_{\mathcal{L}(\mathbb{R}^n; F)} + \sup_{x \in \mathbb{R}^n} \|\varphi''(x)\|_{\mathcal{L}^2(\mathbb{R}^n; F)}.$$

Indeed, we have

$$\begin{aligned}\|w\|_{\mathcal{C}^0(K;F)} &= \left\| \int_0^1 \varphi'(u_0 + \eta v)(v) d\eta \right\|_{\mathcal{C}^0(K;F)} \\ &\leq \|\varphi'\|_{\mathcal{C}^0(\mathbb{R}^n; \mathcal{L}(\mathbb{R}^n; F))} \|v\|_{\mathcal{C}^0(K; \mathbb{R}^n)}.\end{aligned}$$

On the other hand, for $(x, y) \in K \times K$ with $x \neq y$, we write

$$\begin{aligned}&\frac{w(x) - w(y)}{\|x - y\|_{\mathbb{R}^d}^s} \\ &= \frac{\int_0^1 \varphi'(u_0(x) + \eta v(x))(v(x)) d\eta - \int_0^1 \varphi'(u_0(y) + \eta v(y))(v(y)) d\eta}{\|x - y\|_{\mathbb{R}^d}^s} \\ &= \int_0^1 \frac{\varphi'(u_0(x) + \eta v(x)) - \varphi'(u_0(y) + \eta v(y))}{\|x - y\|_{\mathbb{R}^d}^s} (v(x)) d\eta \\ &\quad + \int_0^1 \varphi'(u_0(y) + \eta v(y)) \left(\frac{v(x) - v(y)}{\|x - y\|_{\mathbb{R}^d}^s} \right) d\eta,\end{aligned}$$

with $d = \dim M$. So,

$$\begin{aligned}\|w\|_{\mathcal{C}^s(K;F)}^* &= \sup_{x, y \in K, x \neq y} \frac{\|w(x) - w(y)\|_F}{\|x - y\|_{\mathbb{R}^d}^s} \\ &\leq \|v\|_{\mathcal{C}^0(K; \mathbb{R}^n)} \int_0^1 \|\varphi'(u_0 + \eta v)\|_{\mathcal{C}^s(K; \mathcal{L}(\mathbb{R}^n; F))}^* d\eta \\ &\quad + \|\varphi'\|_{\mathcal{C}^0(\mathbb{R}^n; \mathcal{L}(\mathbb{R}^n; F))} \|v\|_{\mathcal{C}^s(K; \mathbb{R}^n)}^* \\ &\leq \|v\|_{\mathcal{C}^0(K; \mathbb{R}^n)} \|\varphi''\|_{\mathcal{C}^0(\mathbb{R}^n; \mathcal{L}^2(\mathbb{R}^n; F))} \int_0^1 \|u_0 + \eta v\|_{\mathcal{C}^s(K; \mathbb{R}^n)}^* d\eta \\ &\quad + \|\varphi'\|_{\mathcal{C}^0(\mathbb{R}^n; \mathcal{L}(\mathbb{R}^n; F))} \|v\|_{\mathcal{C}^s(K; \mathbb{R}^n)}^*,\end{aligned}$$

and then

$$\begin{aligned}\|w\|_{\mathcal{C}^s(K;F)}^* &\leq \|v\|_{\mathcal{C}^0(K; \mathbb{R}^n)} \|\varphi''\|_{\mathcal{C}^0(\mathbb{R}^n; \mathcal{L}^2(\mathbb{R}^n; F))} \left(\|u_0\|_{\mathcal{C}^s(K; \mathbb{R}^n)}^* + \|v\|_{\mathcal{C}^s(K; \mathbb{R}^n)}^* \right) \\ &\quad + \|\varphi'\|_{\mathcal{C}^0(\mathbb{R}^n; \mathcal{L}(\mathbb{R}^n; F))} \|v\|_{\mathcal{C}^s(K; \mathbb{R}^n)}^*\end{aligned}$$

Hence,

$$\begin{aligned}\|w\|_{\mathcal{C}^s(K;F)} &= \|w\|_{\mathcal{C}^0(K;F)} + \|w\|_{\mathcal{C}^s(K;F)}^* \\ &\leq \|\varphi'\|_{\mathcal{C}^0(\mathbb{R}^n; \mathcal{L}(\mathbb{R}^n; F))} \|v\|_{\mathcal{C}^0(K; \mathbb{R}^n)} \\ &\quad + \|v\|_{\mathcal{C}^0(K; \mathbb{R}^n)} \|\varphi''\|_{\mathcal{C}^0(\mathbb{R}^n; \mathcal{L}^2(\mathbb{R}^n; F))} \left(\|u_0\|_{\mathcal{C}^s(K; \mathbb{R}^n)}^* + \|v\|_{\mathcal{C}^s(K; \mathbb{R}^n)}^* \right) \\ &\quad + \|\varphi'\|_{\mathcal{C}^0(\mathbb{R}^n; \mathcal{L}(\mathbb{R}^n; F))} \|v\|_{\mathcal{C}^s(K; \mathbb{R}^n)}^* \\ &\leq \|\varphi'\|_{\mathcal{C}^1(\mathbb{R}^n; \mathcal{L}(\mathbb{R}^n; F))} \left(1 + \|u_0\|_{\mathcal{C}^s(K; \mathbb{R}^n)}^* + \|v\|_{\mathcal{C}^s(K; \mathbb{R}^n)}^* \right) \|v\|_{\mathcal{C}^s(K; \mathbb{R}^n)} \\ &\leq \|\varphi'\|_{\mathcal{C}^1(\mathbb{R}^n; \mathcal{L}(\mathbb{R}^n; F))} \left(1 + \|u_0\|_{\mathcal{C}^s(K; \mathbb{R}^n)} + \|v\|_{\mathcal{C}^s(K; \mathbb{R}^n)} \right) \|v\|_{\mathcal{C}^s(K; \mathbb{R}^n)},\end{aligned}$$

which is the desired estimate (1).

This estimate shows that the mapping

$$u \in \mathcal{C}_{loc}^s(f(U); \Omega)|_K \rightarrow \varphi \circ u \in \mathcal{C}_{loc}^s(f(U); F)|_K$$

is continuous at u_0 .

Let $\varphi \in C_c^1(\Omega; F)$ and let $(\varphi_j) \subset C_c^\infty(\Omega; F)$ with $\varphi_j \rightarrow \varphi$ in $C^1(\mathbb{R}^n; F)$. For all $u \in \mathcal{C}_{loc}^s(f(U); \Omega)$ with $\|u - u_0\|_{\mathcal{C}^s(K; \mathbb{R}^n)} \leq 1$ and all $\psi \in C_c^1(\Omega; F)$, we have

$$\begin{aligned} \|\psi \circ u\|_{\mathcal{C}^s(K; F)} &\leq \left\{1 + \|u\|_{\mathcal{C}^s(K; \mathbb{R}^n)}\right\} \|\psi\|_{C^1(\mathbb{R}^n; F)} \\ &\leq \left\{2 + \|u_0\|_{\mathcal{C}^s(K; \mathbb{R}^n)}\right\} \|\psi\|_{C^1(\mathbb{R}^n; F)} \end{aligned} \quad (2)$$

So, we get

$$\begin{aligned} &\|\varphi \circ u - \varphi \circ u_0\|_{\mathcal{C}^s(K; F)} \\ &\leq \|\varphi_j \circ u - \varphi_j \circ u_0\|_{\mathcal{C}^s(K; F)} + \|(\varphi - \varphi_j) \circ u\|_{\mathcal{C}^s(K; F)} \\ &\quad + \|(\varphi_j - \varphi) \circ u_0\|_{\mathcal{C}^s(K; F)} \\ &\leq \|\varphi_j \circ u - \varphi_j \circ u_0\|_{\mathcal{C}^s(K; F)} \\ &\quad + 2 \left\{2 + \|u_0\|_{\mathcal{C}^s(K; \mathbb{R}^n)}\right\} \|\varphi_j - \varphi\|_{C^1(\mathbb{R}^n; F)}, \end{aligned}$$

where $j \geq 1$ and $u \in \mathcal{C}_{loc}^s(f(U); \Omega)$ with $\|u - u_0\|_{\mathcal{C}^s(K; \mathbb{R}^n)} \leq 1$. Let $\varepsilon > 0$. If we choose $j \geq 1$ and $\delta \in]0, 1]$ such that

$$2 \left\{2 + \|u_0\|_{\mathcal{C}^s(K; \mathbb{R}^n)}\right\} \|\varphi_j - \varphi\|_{C^1(\mathbb{R}^n; F)} \leq \frac{\varepsilon}{2},$$

and

$$\|u - u_0\|_{\mathcal{C}^s(K; \mathbb{R}^n)} \leq \delta \Rightarrow \|\varphi_j \circ u - \varphi_j \circ u_0\|_{\mathcal{C}^s(K; F)} \leq \frac{\varepsilon}{2},$$

we obtain

$$\|u - u_0\|_{\mathcal{C}^s(K; \mathbb{R}^n)} \leq \delta \Rightarrow \|\varphi \circ u - \varphi \circ u_0\|_{\mathcal{C}^s(K; F)} \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

which shows that the mapping

$$u \in \mathcal{C}_{loc}^s(f(U); \Omega)|_K \rightarrow \varphi \circ u \in \mathcal{C}_{loc}^s(f(U); F)|_K$$

is continuous at u_0 . Since u_0 and K are arbitrary, it follows that $f_* \circ \Phi \circ f^*$ is continuous from $\mathcal{C}_{loc}^s(f(U); \Omega)$ to $\mathcal{C}_{loc}^s(f(U); F)$.

The proof of the theorem is finished. ■

In the previous proof, we used the fact that if for all local chart (U, f) of M , $f_* \circ \Phi \circ f^* : u \mapsto \varphi \circ u$ is continuous from $\mathcal{C}_{loc}^s(f(U); \Omega) \subset \mathcal{C}_{loc}^s(f(U); E)$ to $\mathcal{C}_{loc}^s(f(U); F)$, then Φ is continuous from $\mathcal{C}^s(M; \Omega)$ to $\mathcal{C}^s(M; F)$. We mention here that the inverse implication is also true if Ω is a convex open containing 0.

The statement 4 in Theorem 2.5 may be modified in the manner described in the following proposition.

Proposition 2.6 *Let F be a Banach space, let Ω be an open set of \mathbb{R}^n , and let M be a compact manifold (possibly with corners). For $s \in]1, \infty[\cap \mathbb{N}$ and $k \in \mathbb{N}$, let $\varphi \in \mathcal{C}_{loc}^{s+k}(\Omega; F)$ be a function having the property*

$$\lim_{(x,y) \rightarrow (x_0,x_0)} \frac{\partial^\alpha \varphi(x) - \partial^\alpha \varphi(y)}{\|x - y\|_{\mathbb{R}^n}^{s-[\alpha]}} = 0, |\alpha| = [s] + k, x_0 \in \Omega.$$

Then $\Phi : u \mapsto \varphi \circ u$ is C^k from $\mathcal{C}^s(M; \Omega)$ to $\mathcal{C}^s(M; F)$.

Proof. As in the proof of Theorem 2.5, we only treat the case $k = 0$. We can easily notice that to complete the proof, it suffices to simply check that, for $\varphi \in \mathcal{C}_{loc}^s(\Omega; F)$ with $s \in]0, 1[$, $f_* \circ \Phi \circ f^*$ is continuous from $C^1(f(U); \Omega)$ to $\mathcal{C}_{loc}^s(f(U); F)$ for every local chart (U, f) of M .

Let (U, f) be a local chart of M . We can assume, always as in the proof of Theorem 2.5, that $\varphi \in \mathcal{C}_c^s(\Omega; F) \subset \mathcal{C}_c^s(\mathbb{R}^n; F)$. If $(\varphi_j)_{j \geq 1}$ is the sequence defined in Proposition 2.3, then we can find a compact subset $K' \subset \Omega$ such that, for j sufficiently large, $\varphi_j = 0$ and $\varphi = 0$ in $\Omega - K'$. By Proposition 2.4, the sequence (φ_j) converges in $\mathcal{C}_{loc}^s(\mathbb{R}^n; F)$ to φ . So, $\varphi_j = \eta \varphi_j \rightarrow \eta \varphi = \varphi$ in $\mathcal{C}^s(\mathbb{R}^n; F)$, where $\eta \in C_c^\infty(\Omega; \mathbb{R})$ is a function with the property that $\eta \equiv 1$ on K' . This allows us to treat, in the first step, the smooth case $\varphi \in C_c^\infty(\Omega; F)$.

Let $u_0 \in C^1(f(U); \Omega)$ and let K be a compact subset of $f(U)$. The estimate (1) shows that the mapping

$$u \in \mathcal{C}_{loc}^s(f(U); \Omega)|_K \rightarrow \varphi \circ u \in \mathcal{C}_{loc}^s(f(U); F)|_K$$

is continuous at u_0 . So, since the natural injection

$$C^1(f(U); \Omega)|_K \hookrightarrow \mathcal{C}_{loc}^s(f(U); \Omega)|_K,$$

is continuous, it follows that the mapping

$$u \in C^1(f(U); \Omega)|_K \rightarrow \varphi \circ u \in \mathcal{C}_{loc}^s(f(U); F)|_K,$$

is continuous at u_0 .

Now let $\varphi \in \mathcal{C}_c^s(\Omega; F)$ and $(\varphi_j) \subset C_c^\infty(\Omega; F)$ with $\varphi_j \rightarrow \varphi$ in $\mathcal{C}^s(\mathbb{R}^n; F)$. We assume, in the first step, that K is convex, and let 's check that if $u \in C^1(f(U); \Omega)$ with $\|u - u_0\|_{C^1(K; \mathbb{R}^n)} \leq 1$ and $\psi \in \mathcal{C}_c^s(\Omega; F)$, then

$$\begin{aligned} \|\psi \circ u\|_{\mathcal{C}^s(K; F)} &\leq \left\{ 1 + \|u\|_{C^1(K; \mathbb{R}^n)}^s \right\} \|\psi\|_{\mathcal{C}^s(\mathbb{R}^n; F)} \\ &\leq \left\{ 1 + \left(1 + \|u_0\|_{C^1(K; \mathbb{R}^n)} \right)^s \right\} \|\psi\|_{\mathcal{C}^s(\mathbb{R}^n; F)}. \end{aligned} \quad (3)$$

Indeed, for any $(x, y) \in K \times K$ with $x \neq y$ and $u(x) \neq u(y)$, we write

$$\begin{aligned} & \frac{\psi \circ u(x) - \psi \circ u(y)}{\|x - y\|_{\mathbb{R}^d}^s} \\ = & \frac{\psi \circ u(x) - \psi \circ u(y)}{\|u(x) - u(y)\|_{\mathbb{R}^n}^s} \times \frac{\|u(x) - u(y)\|_{\mathbb{R}^n}^s}{\|x - y\|_{\mathbb{R}^d}^s} \\ = & \frac{\psi \circ u(x) - \psi \circ u(y)}{\|u(x) - u(y)\|_{\mathbb{R}^n}^s} \times \frac{\left\| \int_0^1 u'(y + t(x - y))(x - y) dt \right\|_{\mathbb{R}^n}^s}{\|x - y\|_{\mathbb{R}^d}^s}, \end{aligned}$$

and then,

$$\begin{aligned} & \frac{\|\psi \circ u(x) - \psi \circ u(y)\|_F}{\|x - y\|_{\mathbb{R}^d}^s} \\ \leq & \frac{\|\psi \circ u(x) - \psi \circ u(y)\|_F}{\|u(x) - u(y)\|_{\mathbb{R}^n}^s} \times \left(\int_0^1 \|u'(y + t(x - y))\|_{\mathcal{L}(\mathbb{R}^d; \mathbb{R}^n)} dt \right)^s \\ \leq & \frac{\|\psi \circ u(x) - \psi \circ u(y)\|_F}{\|u(x) - u(y)\|_{\mathbb{R}^n}^s} \times \|u'\|_{\mathcal{C}^0(K; \mathcal{L}(\mathbb{R}^d; \mathbb{R}^n))}^s. \end{aligned}$$

So,

$$\|\psi \circ u\|_{\mathcal{C}^s(K; F)}^* \leq \|\psi\|_{\mathcal{C}^s(\mathbb{R}^n; F)}^* \|u'\|_{\mathcal{C}^0(K; \mathcal{L}(\mathbb{R}^d; \mathbb{R}^n))}^s \leq \|\psi\|_{\mathcal{C}^s(\mathbb{R}^n; F)} \|u\|_{\mathcal{C}^1(K; \mathbb{R}^n)}^s,$$

and therefore,

$$\begin{aligned} \|\psi \circ u\|_{\mathcal{C}^s(K; F)} & \leq \|\psi\|_{\mathcal{C}^0(\mathbb{R}^n; F)} + \|\psi\|_{\mathcal{C}^s(\mathbb{R}^n; F)} \|u\|_{\mathcal{C}^1(K; \mathbb{R}^n)}^s \\ & \leq \left\{ 1 + \|u\|_{\mathcal{C}^1(K; \mathbb{R}^n)}^s \right\} \|\psi\|_{\mathcal{C}^s(\mathbb{R}^n; F)} \\ & \leq \left\{ 1 + \left(1 + \|u_0\|_{\mathcal{C}^1(K; \mathbb{R}^n)} \right)^s \right\} \|\psi\|_{\mathcal{C}^s(\mathbb{R}^n; F)}, \end{aligned}$$

which is the estimate (3). Now, for $j \geq 1$, we write

$$\begin{aligned} & \|\varphi \circ u - \varphi \circ u_0\|_{\mathcal{C}^s(K; F)} \\ \leq & \|\varphi_j \circ u - \varphi_j \circ u_0\|_{\mathcal{C}^s(K; F)} + \|(\varphi - \varphi_j) \circ u\|_{\mathcal{C}^s(K; F)} \\ & + \|(\varphi_j - \varphi) \circ u_0\|_{\mathcal{C}^s(K; F)} \\ \leq & \|\varphi_j \circ u - \varphi_j \circ u_0\|_{\mathcal{C}^s(K; F)} \\ & + 2 \left\{ 1 + \left(1 + \|u_0\|_{\mathcal{C}^1(K; \mathbb{R}^n)} \right)^s \right\} \|\varphi_j - \varphi\|_{\mathcal{C}^s(\mathbb{R}^n; F)}. \end{aligned}$$

Let $\varepsilon > 0$. If we choose $j \geq 1$ and $\delta \in]0, 1]$ such that

$$2 \left\{ 1 + \left(1 + \|u_0\|_{\mathcal{C}^1(K; \mathbb{R}^n)} \right)^s \right\} \|\varphi_j - \varphi\|_{\mathcal{C}^s(\mathbb{R}^n; F)} \leq \frac{\varepsilon}{2},$$

and

$$\|u - u_0\|_{\mathcal{C}^1(K; \mathbb{R}^n)} \leq \delta \Rightarrow \|\varphi_j \circ u - \varphi_j \circ u_0\|_{\mathcal{C}^s(K; F)} \leq \frac{\varepsilon}{2},$$

then for any $u \in C^1(f(U); \Omega)$ with $\|u - u_0\|_{C^1(K; \mathbb{R}^n)} \leq \delta$, we have

$$\|\varphi \circ u - \varphi \circ u_0\|_{C^s(K; F)} \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

This proves that the mapping

$$u \in C^1(f(U); \Omega)|_K \mapsto \varphi \circ u \in \mathcal{C}_{loc}^s(f(U); F)|_K$$

is continuous at u_0 .

If K is an arbitrary compact subset of $f(U)$, there is an open covering $\mathcal{U} = (U_j)_{1 \leq j \leq N}$ in $f(U)$ of K such that, for all $j \in \{1, \dots, N\}$, $K_j = \overline{U_j}$ is a convex and compact subset of $f(U)$. By Lebesgue covering theorem, there exists $\lambda > 0$ such that, for all $(x, y) \in K \times K$ with $\|x - y\|_{\mathbb{R}^d} \leq \lambda$, there is a set in \mathcal{U} containing both x and y .

Let $u \in C^1(f(U); \Omega)$ and let $(x, y) \in K \times K$. For $\|x - y\|_{\mathbb{R}^d} > \lambda$, we have

$$\frac{\|w(x) - w(y)\|_F}{\|x - y\|_{\mathbb{R}^d}^s} \leq \frac{2}{\lambda^s} \|w\|_{C^0(K; F)},$$

with $w = \varphi \circ u - \varphi \circ u_0$, and for $0 < \|x - y\|_{\mathbb{R}^d} \leq \lambda$, there is $j_0 \in \{1, \dots, N\}$, such that $(x, y) \in K_{j_0}$, and so,

$$\frac{\|w(x) - w(y)\|_F}{\|x - y\|_{\mathbb{R}^d}^s} \leq \|w\|_{C^s(K_{j_0}; F)}^*.$$

Thus, we get the inequality

$$\|w\|_{C^s(K; F)}^* \leq \max \left\{ \frac{2}{\lambda^s} \|w\|_{C^0(K; F)}, \max_{1 \leq j \leq N} \|w\|_{C^s(K_j; F)}^* \right\},$$

which prove, with the above, that the mapping

$$u \in C^1(f(U); \Omega)|_K \mapsto \varphi \circ u \in \mathcal{C}_{loc}^s(f(U); F)|_K$$

is continuous at u_0 .

Now, since u_0 and K are arbitrary, it follows that $f_* \circ \Phi \circ f^*$ is continuous from $C^1(f(U); \Omega)$ to $\mathcal{C}_{loc}^s(f(U); F)$. ■

We want to make a change to the text of Theorem 2.5. But before that, we need to introduce some concepts.

Let E and F be normed spaces, let Ω be an open set of E , and let M be a finite dimensional manifold (possibly with corners). For $s \in \mathbb{N}$ and $k \in \mathbb{N} \cup \{\infty\}$, we denote by $C^{s, s+k}(M \times \Omega; F)$ the space of functions $\varphi \in C^s(M \times \Omega; F)$ such that, for all $x \in M$, $\varphi(x, \cdot) \in C^{s+k}(\Omega; F)$ with $\partial_y^\ell \varphi \in C^s(M \times \Omega; \mathcal{L}^\ell(E; F))$, $0 \leq \ell < k + 1$. Here, $\partial_y^\ell \varphi$ denote the partial derivative of order ℓ of φ with respect to the second variable y in Ω . For $k = \infty$, we simply write $C^{s, s+\infty}(M \times \Omega; F) = C^{s, \infty}(M \times \Omega; F)$.

Let $s \in \mathbb{R}_+ | \mathbb{N}$ and $k \in \mathbb{N}$. If $\varphi \in C^{[s],[s]+k}(M \times \Omega; F)$ is a function with the property: for every local chart (U, f) of M , every $(x_0, y_0) \in f(U) \times \Omega$, and every $j \in \{0, \dots, k\}$, we have

$$\lim_{\substack{(x,y) \rightarrow (x_0,y_0), \\ (x',y') \rightarrow (x_0,y_0)}} \frac{\varphi_j^{([s])}(x,y) - \varphi_j^{([s])}(x',y')}{\|(x,y) - (x',y')\|_{\mathbb{R}^d \times E}^{s-[s]}} = 0, \quad (4)$$

where $\varphi_j(\cdot, \cdot) = \partial_y^j \varphi(f(\cdot), \cdot)$ and $d = \dim M$, then obviously, the partial derivative $\partial_y^\ell \varphi$ belongs to $\mathcal{C}_{loc}^s(M \times \Omega; \mathcal{L}^\ell(E; F))$ for $0 \leq \ell \leq k$.

Let U be a subset of a normed space G satisfying $\text{int}(U) \subset U \subset \overline{\text{int}(U)}$. It is not difficult to see that we have

$$\begin{aligned} \text{int}(U \times \Omega) &= \text{int}(U) \times \Omega \subset U \times \Omega \\ &\subset \overline{\text{int}(U)} \times \overline{\Omega} = \overline{\text{int}(U) \times \Omega} = \overline{\text{int}(U \times \Omega)}. \end{aligned}$$

Thus, we can define the spaces $C^{s,s+k}(U \times \Omega; F)$ ($s \in \mathbb{N}$, $k \in \mathbb{N} \cup \{\infty\}$) in the same way as before.

We need here the following lemma.

Lemma 2.7 *Let F be a Banach space, let M be a finite dimensional manifold (possibly with corners), and let $\varphi \in C^0(M \times \mathbb{R}^n; F)$. For $j \geq 1$, we put*

$$\varphi_j(x, y) = \theta_j * (\varphi(x, \cdot))(y) = \int_{\mathbb{R}^n} \theta_j(y - z) \varphi(x, z) dz, \quad x \in M, y \in \mathbb{R}^n,$$

where θ_j is as in Proposition 2.3. Then $(\varphi_j)_{j \geq 1} \subset C^{0,\infty}(M \times \mathbb{R}^n; F)$ and:

1. If $\varphi \in C^{s,s+k}(M \times \mathbb{R}^n; F)$ with $s \in \mathbb{N}$ and $k \in \mathbb{N} \cup \{\infty\}$, then $(\varphi_j)_{j \geq 1} \subset C^{s,\infty}(M \times \mathbb{R}^n; F)$, and $\partial_y^\beta \varphi_j \rightarrow \partial_y^\beta \varphi$ in $C^s(M \times \mathbb{R}^n; F)$ for all $\beta \in \mathbb{N}^n$ with $|\beta| < k + 1$.
2. If $\varphi \in C^{[s],[s]+k}(M \times \mathbb{R}^n; F)$ is a function with the property (4), where $s \in \mathbb{R}_+ | \mathbb{N}$ and $k \in \mathbb{N}$, then $(\partial_y^\beta \varphi_j)_{j \geq 1} \subset \mathcal{C}_{loc}^s(M \times \mathbb{R}^n; F)$ and $\partial_y^\beta \varphi_j \rightarrow \partial_y^\beta \varphi$ in $\mathcal{C}_{loc}^s(M \times \mathbb{R}^n; F)$ for all $\beta \in \mathbb{N}^n$ with $|\beta| \leq k$.

Proof. Without loss of generality, we assume that M is an open subset of a quadrant of \mathbb{R}^d . Let $\varphi \in C^{s,s+k}(M \times \mathbb{R}^n; F)$ with $s \in \mathbb{N}$ and $k \in \mathbb{N} \cup \{\infty\}$. For given $j \geq 1$, it is obvious that $\varphi_j \in C^{s,\infty}(M \times \mathbb{R}^n; F)$ with

$$\partial_x^\alpha \partial_y^{\beta+\gamma} \varphi_j(x, y) = \int_{\mathbb{R}^n} \partial_y^\gamma \theta_j(y - z) \partial_y^\beta \partial_x^\alpha \varphi(x, z) dz, \quad x \in M, y \in \mathbb{R}^n,$$

where $\alpha \in \mathbb{N}^d$, $\beta, \gamma \in \mathbb{N}^n$, $|\alpha| \leq s$, $|\beta| < s + k + 1$. So, it is enough to treat the case $s = k = 0$ in the statement 1.

Let K be a compact subset of $M \times \mathbb{R}^n$. For $j \geq 1$ and $(x, y) \in K$, we write

$$\begin{aligned}\varphi_j(x, y) - \varphi(x, y) &= \int_{\mathbb{R}^n} \theta_j(z) \{\varphi(x, y - z) - \varphi(x, y)\} dz \\ &= \int_{\mathbb{R}^n} \theta(z) \left\{ \varphi\left(x, y - \frac{1}{j}z\right) - \varphi(x, y) \right\} dz,\end{aligned}$$

where θ is chosen as in Proposition 2.3. So,

$$\|\varphi_j - \varphi\|_{C^0(K;F)} \leq \sup_{\substack{(x,y) \in K \\ z \in \text{supp}\theta}} \left\| \varphi\left(x, y - \frac{1}{j}z\right) - \varphi(x, y) \right\|_F.$$

Since $\text{supp}\theta$ is bounded, it follows by the local uniform continuity of φ in $M \times \mathbb{R}^n$ that

$$\lim_{j \rightarrow \infty} \sup_{\substack{(x,y) \in K \\ z \in \text{supp}\theta}} \left\| \varphi\left(x, y - \frac{1}{j}z\right) - \varphi(x, y) \right\|_F = 0,$$

and then $\lim_{j \rightarrow \infty} \|\varphi_j - \varphi\|_{C^0(K;F)} = 0$. This proves that we have $\varphi_j \rightarrow \varphi$ in $C^0(M \times \mathbb{R}^n; F)$ because K is arbitrary.

For the statement 2, it suffices to treat the case $s \in]0, 1[$, $k = 0$. So, let $\varphi \in C^0(M \times \mathbb{R}^n; F)$ with the property (4). Let K be a compact subset of $M \times \mathbb{R}^n$ and let $j \geq 1$. For $(x, y) \in K$ and $(x', y') \in K$ with $(x, y) \neq (x', y')$, we have

$$\begin{aligned}\frac{\varphi_j(x, y) - \varphi_j(x', y')}{\|(x, y) - (x', y')\|_{\mathbb{R}^d \times \mathbb{R}^n}^s} &= \int_{\mathbb{R}^n} \theta_j(z) \left\{ \frac{\varphi(x, y - z) - \varphi(x', y' - z)}{\|(x, y) - (x', y')\|_{\mathbb{R}^d \times \mathbb{R}^n}^s} \right\} dz \\ &= \int_{\mathbb{R}^n} \theta_j(z) \left\{ \frac{\varphi(x, y - z) - \varphi(x', y' - z)}{\|(x, y - z) - (x', y' - z)\|_{\mathbb{R}^d \times \mathbb{R}^n}^s} \right\} dz.\end{aligned}$$

and therefore,

$$\begin{aligned}\|\varphi_j\|_{C^s(K;F)}^* &= \sup_{\substack{(x,y), (x',y') \in K, \\ (x,y) \neq (x',y')}} \frac{\|\varphi_j(x, y) - \varphi_j(x', y')\|_F}{\|(x, y) - (x', y')\|_{\mathbb{R}^d \times \mathbb{R}^n}^s} \\ &\leq \|\varphi\|_{C^s(\tilde{K};F)}^* < \infty,\end{aligned}$$

where $\tilde{K} = \{(x, y - z) : (x, y) \in K, z \in \text{supp}\theta_j\}$, which is a compact subset of $M \times \mathbb{R}^n$. Hence, $\varphi_j \in C_{loc}^s(M \times \mathbb{R}^n; F)$ since K is arbitrary.

On the other hand, for $(x, y) \in K$ and $(x', y') \in K$ with $(x, y) \neq (x', y')$, we write

$$\begin{aligned}\frac{\psi_j(x, y) - \psi_j(x', y')}{\|(x, y) - (x', y')\|_{\mathbb{R}^d \times \mathbb{R}^n}^s} &= \int_{\mathbb{R}^n} \theta(z) \left\{ \frac{\varphi\left(x, y - \frac{1}{j}z\right) - \varphi\left(x', y' - \frac{1}{j}z\right)}{\|(x, y) - (x', y')\|_{\mathbb{R}^d \times \mathbb{R}^n}^s} \right. \\ &\quad \left. - \frac{\varphi(x, y) - \varphi(x', y')}{\|(x, y) - (x', y')\|_{\mathbb{R}^d \times \mathbb{R}^n}^s} \right\} dz,\end{aligned}$$

where $\psi_j = \varphi_j - \varphi$. Thus, if we put

$$m(x, y, x', y') = \frac{\varphi(x, y) - \varphi(x', y')}{\|(x, y) - (x', y')\|_{\mathbb{R}^d \times \mathbb{R}^n}^s},$$

for $(x, y), (x', y') \in M \times \mathbb{R}^n$ with $(x, y) \neq (x', y')$, it follows that

$$\|\psi_j\|_{\mathcal{C}^s(K; F)}^* \leq \sup_{\substack{(x, y), (x', y') \in K, \\ (x, y) \neq (x', y'), \\ z \in \text{supp}\theta}} \left\| m\left(x, y - \frac{1}{j}z, x', y' - \frac{1}{j}z\right) - m(x, y, x', y') \right\|_F.$$

By (4), m is locally uniformly continuous from $M \times \mathbb{R}^n \times M \times \mathbb{R}^n - \Delta_{M \times \mathbb{R}^n}$ to F , where

$$\Delta_{M \times \mathbb{R}^n} = \{(x, y, x, y) : (x, y) \in M \times \mathbb{R}^n\},$$

which gives

$$\lim_{j \rightarrow \infty} \sup_{\substack{(x, y), (x', y') \in K, \\ (x, y) \neq (x', y'), \\ z \in \text{supp}\theta}} \left\| m\left(x, y - \frac{1}{j}z, x', y' - \frac{1}{j}z\right) - m(x, y, x', y') \right\|_F = 0,$$

and then $\lim_{j \rightarrow \infty} \|\psi_j\|_{\mathcal{C}^s(K; F)}^* = \lim_{j \rightarrow \infty} \|\varphi_j - \varphi\|_{\mathcal{C}^s(K; F)}^* = 0$. So, $\varphi_j \rightarrow \varphi$ in $\mathcal{C}_{loc}^s(M \times \mathbb{R}^n; F)$ since K is arbitrary. ■

Theorem 2.8 *Let E be a normed space and let F be a Banach space. Let Ω be an open set of E , let S be a topological space, and let M be a compact manifold (possibly with corners). Then we have the following:*

1. *If K is a compact subset of S and if $\varphi \in C^{0, k}(S \times \Omega; F)$ with $k \in \mathbb{N} \cup \{\infty\}$ (that is, $\varphi \in C^0(S \times \Omega; F)$, which is C^k in the second variable), then the mapping $\Phi : u \mapsto \varphi(\cdot, u(\cdot))$ is C^k from $C^0(S, K; \Omega)$ to $C^0(K; F)$. In particular, if S is compact, Φ is C^k from $C^0(S; \Omega)$ to $C^0(S; F)$.*
2. *Φ is continuous from $C^0(S; \Omega)$ to $C^0(S; F)$.*
3. *If $\varphi \in C^{j, j+k}(M \times \Omega; F)$ with $j \in \mathbb{N}$ and $k \in \mathbb{N} \cup \{\infty\}$, then Φ is C^k from $C^j(M; \Omega)$ to $C^j(M; F)$.*
4. *Assume that E is finite dimensional and let $\varphi \in C^{[s], [s]+k}(M \times \Omega; F)$ be a function having the property (4), where $s \in \mathbb{R}_+ \setminus \mathbb{N}$ and $k \in \mathbb{N}$. Then:*
 - (a) *For $s \in]1, \infty[\setminus \mathbb{N}$, Φ is C^k from $\mathcal{C}^s(M; \Omega)$ to $\mathcal{C}^s(M; F)$.*
 - (b) *For $s \in]0, 1[$ and $k \geq 1$, Φ is C^{k-1} from $\mathcal{C}^s(M; \Omega)$ to $\mathcal{C}^s(M; F)$.*

Proof. As in the proof of Theorem 2.5, it suffices to treat the case $k = 0$ in both statements 1 and 3.

Let $u_0 \in C^0(S, K; \Omega)$ and $\varepsilon > 0$. Since K and $u_0(K)$ are compact, there exists $\delta = \delta(u_0(K), \varepsilon) > 0$ such that

$$\forall (y, z) \in u_0(K) \times \Omega : \|y - z\|_E \leq \delta \Rightarrow \forall x \in K : \|\varphi(x, z) - \varphi(x, y)\|_F \leq \varepsilon.$$

So, if $u \in C^0(S, K; \Omega)$ with $\|u - u_0\|_{C^0(K; E)} \leq \delta$, then

$$\|\varphi(x, u(x)) - \varphi(x, u_0(x))\|_F \leq \varepsilon, \forall x \in K,$$

and thus

$$\|\Phi(u) - \Phi(u_0)\|_{C^0(K; F)} = \sup_{x \in K} \|\varphi(x, u(x)) - \varphi(x, u_0(x))\|_F \leq \varepsilon,$$

which proves that $\Phi : u \in C^0(S, K; \Omega) \mapsto \varphi(\cdot, u(\cdot)) \in C^0(K; F)$ is continuous at u_0 . Hence the validity of the statement 1 for $k = 0$.

According to the above, for each compact subset K of S , Φ is continuous from $C^0(S; \Omega)|_K$ to $C^0(K; F)$. So, it is continuous from $C^0(S; \Omega)$ to $C^0(S; F)$, which is the statement 2.

From Proposition 2.2, the statement 2, and the chain rule, it follows that, for any local chart (U, f) of M , the mapping

$$f_* \circ \Phi \circ f^* : u \mapsto \varphi(f^{-1}(\cdot), u(\cdot)),$$

is continuous from $C^j(f(U); \Omega) \subset C^j(f(U); E)$ to $C^j(f(U); F)$. Therefore, Φ is continuous from $C^j(M; E)$ to $C^j(M; F)$, which means that the statement 3 is true for $k = 0$.

As in the proof of Proposition 2.6, to prove the statement (a), it suffices to simply check that, for $s \in]0, 1[$ and $\varphi \in \mathcal{C}_{loc}^s(M \times \Omega; F)$ having the property (4), $f_* \circ \Phi \circ f^*$ is continuous from $C^1(f(U); \Omega)$ to $\mathcal{C}_{loc}^s(f(U); F)$ for every local chart (U, f) of M .

Without loss of generality, we assume that $E = \mathbb{R}^n$ and M is an open subset of a quadrant of \mathbb{R}^d , so $U = M$ and $f = Id_M$. Also, we can assume, always as in the proof of Proposition 2.6, that the function φ is equal to 0 in a subset of the form $M \times (\Omega - K')$, where K' is a compact subset of Ω . Hence, $\varphi \in \mathcal{C}_{loc}^s(M \times \mathbb{R}^n; F)$, and for all $x \in M$, $\varphi(x, \cdot) \in \mathcal{C}_c^s(\Omega; F) \subset \mathcal{C}_c^s(\mathbb{R}^n; F)$. If $(\varphi_j)_{j \geq 1}$ is the sequence defined in Lemma 2.7, then we can find a compact subset $K'' \subset \Omega$ such that, for j sufficiently large, $\varphi_j = 0$ in $M \times (\Omega - K'')$. From this fact and Lemma 2.7, it follows that, for every compact subset $K \subset M$, $\lim_{j \rightarrow \infty} \|\psi_j\|_{\mathcal{C}^s(K \times \mathbb{R}^n; F)} = 0$, where $\psi_j = \varphi_j - \varphi$ and

$$\begin{aligned} & \|\psi_j\|_{\mathcal{C}^s(K \times \mathbb{R}^n; F)} = \|\psi_j\|_{C^0(K \times \mathbb{R}^n; F)} + \|\psi_j\|_{\mathcal{C}^s(K \times \mathbb{R}^n; F)}^* \\ &= \sup_{(x, y) \in K \times \mathbb{R}^n} \|\psi_j(x, y)\|_F + \sup_{\substack{(x, y), (x', y') \in K \times \mathbb{R}^n, \\ (x, y) \neq (x', y')}} \frac{\|\psi_j(x, y) - \psi_j(x', y')\|_F}{\|(x, y) - (x', y')\|_{\mathbb{R}^d \times \mathbb{R}^n}^s}. \end{aligned}$$

As we did in the proof of Proposition 2.6, we first treat the case when $\varphi \in C^{0,\infty}(M \times \mathbb{R}^n; F)$ with the property (4). Let's check that if $u \in \mathcal{C}_{loc}^s(M; \Omega)$ then $v = \varphi(\cdot, u(\cdot)) \in \mathcal{C}_{loc}^s(M; F)$. Indeed, for any compact subset K of M and for $(x, y) \in K \times K$ with $x \neq y$, we write

$$\begin{aligned} \frac{v(x) - v(y)}{\|x - y\|_{\mathbb{R}^d}^s} &= \frac{\varphi(x, u(x)) - \varphi(x, u(y))}{\|x - y\|_{\mathbb{R}^d}^s} + \frac{\varphi(x, u(y)) - \varphi(y, u(y))}{\|x - y\|_{\mathbb{R}^d}^s} \\ &= \int_0^1 \partial_y \varphi(x, u(y) + t(u(x) - u(y))) \left(\frac{u(x) - u(y)}{\|x - y\|_{\mathbb{R}^d}^s} \right) dt \\ &\quad + \frac{\varphi(x, u(y)) - \varphi(y, u(y))}{\|(x, u(y)) - (y, u(y))\|_{\mathbb{R}^d \times \mathbb{R}^n}^s}. \end{aligned}$$

So,

$$\|v\|_{\mathcal{C}^s(K; F)}^* \leq \|\partial_y \varphi\|_{\mathcal{C}^0(K \times \mathbb{R}^n; \mathcal{L}(\mathbb{R}^n; F))} \|u\|_{\mathcal{C}^s(K; F)}^* + \|\varphi\|_{\mathcal{C}^s(K \times \mathbb{R}^n; F)}^* < \infty,$$

which proves that $v \in \mathcal{C}_{loc}^s(M; F)$.

As in the proof of estimate (1), we use the mean value theorem to get the following

$$\begin{aligned} \|w\|_{\mathcal{C}^s(K; F)} &\leq \left\{ \|\varphi\|_{\mathcal{C}^s(K \times \mathbb{R}^n; F)} + \|\partial_y \varphi\|_{\mathcal{C}^0(K \times \mathbb{R}^n; \mathcal{L}(\mathbb{R}^n; F))} \right. \\ &\quad \left. + \|\partial_y^2 \varphi\|_{\mathcal{C}^0(K \times \mathbb{R}^n; \mathcal{L}^2(\mathbb{R}^n; F))} \left(\|u_0\|_{\mathcal{C}^s(K; \mathbb{R}^n)} + \|v\|_{\mathcal{C}^s(K; \mathbb{R}^n)} \right) \right\} \|v\|_{\mathcal{C}^s(K; \mathbb{R}^n)}, \end{aligned}$$

where $u_0, u \in \mathcal{C}_{loc}^s(M; \Omega)$, $v = u - u_0$, $w = \varphi \circ u - \varphi \circ u_0$, and K is a compact subset of M . From this estimate and the fact that the natural injection

$$C^1(M; \Omega)|_K \hookrightarrow \mathcal{C}_{loc}^s(M; \Omega)|_K,$$

is continuous, it follows that the mapping

$$u \in C^1(M; \Omega)|_K \rightarrow \varphi \circ u \in \mathcal{C}_{loc}^s(M; F)|_K,$$

is continuous.

For the treatment of the case $\varphi \in \mathcal{C}_{loc}^s(M \times \mathbb{R}^n; F)$, we first get the following estimate

$$\begin{aligned} \|\psi(\cdot, u(\cdot))\|_{\mathcal{C}^s(K; F)} &\leq \left\{ 2 + \|u\|_{\mathcal{C}^1(K; \mathbb{R}^n)}^s \right\} \|\psi\|_{\mathcal{C}^s(K \times \mathbb{R}^n; F)} \\ &\leq \left\{ 2 + \left(1 + \|u_0\|_{\mathcal{C}^1(K; \mathbb{R}^n)} \right)^s \right\} \|\psi\|_{\mathcal{C}^s(K \times \mathbb{R}^n; F)}. \end{aligned}$$

which holds for every convex and compact subset K of M , every $u, u_0 \in C^1(M; \Omega)$ with $\|u - u_0\|_{\mathcal{C}^1(K; \mathbb{R}^n)} \leq 1$, and every $\psi \in \mathcal{C}_{loc}^s(M \times \mathbb{R}^n; F)$ having the property of being equal to 0 in a subset of the form $M \times (\Omega - K''')$, where K''' is a compact subset of Ω . Then we use this estimate, which is

similar to that (3), and the sequence $(\varphi_j)_{j \geq 1}$ defined in Lemma 2.7 to prove that the mapping

$$u \in C^1(M; \Omega)|_K \rightarrow \varphi \circ u \in \mathcal{C}_{loc}^s(M; F)|_K$$

is continuous.

The statement (b) is proven exactly like the statement 4 in Theorem 2.5, where we use, instead of the estimate (2), the following estimate.

$$\begin{aligned} & \|\psi(\cdot, u(\cdot))\|_{\mathcal{C}^s(K; F)} \\ & \leq \|\psi\|_{\mathcal{C}^s(K \times \mathbb{R}^n; F)} + \|\partial_2 \psi\|_{\mathcal{C}^0(K \times \mathbb{R}^n; \mathcal{L}(\mathbb{R}^n; F))} \|u\|_{\mathcal{C}^s(K; \mathbb{R}^n)} \\ & \leq \|\psi\|_{\mathcal{C}^s(K \times \mathbb{R}^n; F)} + \|\partial_2 \psi\|_{\mathcal{C}^0(K \times \mathbb{R}^n; \mathcal{L}(\mathbb{R}^n; F))} \left(1 + \|u_0\|_{\mathcal{C}^s(K; \mathbb{R}^n)}\right), \end{aligned}$$

where $u, u_0 \in C^1(M; \Omega)$ with $\|u - u_0\|_{\mathcal{C}^s(K; \mathbb{R}^n)} \leq 1$, K is a compact subset of M , and $\psi \in \mathcal{C}_{loc}^s(M \times \mathbb{R}^n; F) \cap C^{0,1}(M \times \mathbb{R}^n; F)$ is a function having the property of being equal to 0 in a subset of the form $M \times (\Omega - K''')$, with K''' is a compact subset of Ω . ■

We note that the statement 4 in Theorem 2.5 and the statements 3 and (b) in Theorem 2.8 remain valid if M is the closure of a bounded open set of a finite dimensional space. If, moreover, M is locally convex, then Proposition 2.6 and the statement (a) in Theorem 2.8 also remain valid. In fact, the proofs in this case are more concise and more clearer.

In this context, we cite the following corollary.

Corollary 2.9 *Let E be normed space, let F be a Banach space, let Ω be an open set of E , and let M be the closure of a bounded open set of \mathbb{R}^d . Then the following statements are true.*

1. *Let $s \in \mathbb{N}$ and $k \in \mathbb{N} \cup \{\infty\}$. Let N be the cardinality of the set $\{\alpha \in \mathbb{N}^d : 1 \leq |\alpha| \leq j\}$ with $j \geq 1$. If $\varphi \in C^{s, s+k}(M \times \Omega \times E^N; F)$ (here the second variable is in $\Omega \times E^N$), then the mapping*

$$\Phi : u \mapsto \Phi(u) = \varphi\left(x, u, (\partial^\alpha u)_{1 \leq |\alpha| \leq j}\right),$$

is C^k from $C^{s+j}(M; \Omega)$ to $C^s(M; F)$.

2. *Assume that E is finite dimensional and let $s \in \mathbb{R}_+ \setminus \mathbb{N}$ and $k \in \mathbb{N}$. If $\varphi \in C^{[s], [s]+k}(M \times \Omega \times E^N; F)$ is a function, which have the property (4), then:*

- (a) *For $k \in \mathbb{N}^*$, Φ is C^{k-1} from $C^{s+j}(M; \Omega)$ to $C^s(M; F)$.*
- (b) *If M is locally convex, then for $s \in]1, \infty[\setminus \mathbb{N}$, Φ is C^k from $C^{s+j}(M; \Omega)$ to $C^s(M; F)$.*

Proof. Let $s \in \mathbb{N}$. By the statement 3 in Theorem 2.8, the mapping

$$\Phi_1 : v \in C^s(M; \Omega \times E^N) \mapsto \varphi(\cdot, v(\cdot)) \in C^s(M; F),$$

is C^k . So, since the linear continuous mapping

$$\Phi_2 : u \in C^{s+j}(M; E) \mapsto \left(u(\cdot), (\partial^\alpha u(\cdot))_{1 \leq |\alpha| \leq j} \right) \in C^s(M; E \times E^N),$$

is C^∞ , it follows that $\Phi = \Phi_1 \circ \left(\Phi_2|_{C^{s+j}(M; \Omega)} \right)$ is C^k .

In the same way, we prove the statement 2. ■

Below, we present a theorem containing two formulations for statements 3 and 4 from Theorem 2.5 in the case of smooth vector bundles.

Let M be a paracompact manifold (possibly with corners) and let E be a smooth vector bundle of standard fiber \mathfrak{E} (which is a normed space) over M . Let Ω be an open subset of E such that for all $x \in M$, $\Omega \cap E_x$ is a convex neighborhood of the zero 0_x of E_x . Under this assumption, we can find a bundle atlas $\{(U_\gamma, \psi_\gamma) : \gamma \in \Gamma\}$ of E and a family of open neighborhoods $(V_\gamma)_{\gamma \in \Gamma}$ of 0 in \mathfrak{E} such that for every $\gamma \in \Gamma$, $U_\gamma \times V_\gamma \subset \psi_\gamma(\Omega \cap \pi_E^{-1}(U_\gamma))$.

Now we consider a partition of unity $(\lambda_\gamma)_{\gamma \in \Gamma}$ of M dominated by $(U_\gamma)_{\gamma \in \Gamma}$, and for $\gamma \in \Gamma$, we choose a function $Y_\gamma \in C^\infty(U_\gamma; V_\gamma)$ and then we put $X_\gamma(x) = \psi^{-1}(x, Y_\gamma(x))$, $x \in U_\gamma$. The function $X = \sum_{\gamma \in \Gamma} \lambda_\gamma X_\gamma$ belongs to $C^\infty(M \rightarrow E)$ and has the property: $X(x) \in \Omega$ for all $x \in M$. Therefore,

$$C^\infty(M \rightarrow \Omega) = \{X \in C^\infty(M \rightarrow E) : X(x) \in \Omega, \forall x \in M\} \neq \emptyset,$$

and hence $C_{loc}^s(M \rightarrow \Omega) \neq \emptyset$ for all $s \in \mathbb{R}_+$. Furthermore, if M is compact, we check that $C^s(M \rightarrow \Omega) = C_{loc}^s(M \rightarrow \Omega)$, $s \in \mathbb{R}_+$, is an open subset of $C^s(M \rightarrow E)$.

Theorem 2.10 *Let M be a compact manifold (possibly with corners), let E and F be two Banach vector bundles over M , and let Ω be an open set of E . We suppose that E is finite dimensional and that for each $x \in M$, $\Omega \cap E_x$ is a convex neighborhood of 0_x in E_x . Then the following assertions holds.*

1. *Let $j \in \mathbb{N}$, $k \in \mathbb{N} \cup \{\infty\}$, and let $\varphi \in C^{j+k}(\Omega; F)$ be a function such that $\pi_E = \pi_F \circ \varphi$, where π_E and π_F are the bundle projections of E and F . Then $\Phi : X \mapsto \varphi \circ X$ is C^k from $C^j(M \rightarrow \Omega)$ to $C^j(M \rightarrow F)$.*
2. *Let $s \in \mathbb{R}_+ \setminus \mathbb{N}$ and $k \in \mathbb{N}^* \cup \{\infty\}$. If $\varphi \in C^{[s]+k}(\Omega; F)$ with $\pi_E = \pi_F \circ \varphi$, then Φ is C^{k-1} from $C^s(M \rightarrow \Omega)$ to $C^s(M \rightarrow F)$.*

Proof. As in the proof of Theorem 2.5, it is sufficient to treat the case $j \in \mathbb{N}$, $k = 0$ in the statement 1 and the case $0 < s < 1$, $k = 1$ in the statement 2.

Let $X_0 \in C^s(M \rightarrow \Omega)$, $\varphi \in C^{[s]+k}(\Omega; F)$ with $s \in \mathbb{R}_+$, and let V be an open set in Ω such that $X_0(M) \subset V$ and $\bar{V} = \text{cl}(V)$ is a compact set in Ω .

Since E is paracompact, we can choose a function $\eta \in C_c^\infty(\Omega; \mathbb{R})$ with the property that $\eta \equiv 1$ on \overline{V} . The set $\mathcal{C}^s(M \rightarrow V)$ is an open neighborhood of X_0 in $\mathcal{C}^s(M \rightarrow \Omega)$, and for any $X \in \mathcal{C}^s(M \rightarrow V)$, we have $\varphi \circ X = (\eta\varphi) \circ X$ with $\eta\varphi \in C_c^{[s]+k}(\Omega; F) \subset C_c^{[s]+k}(E; F)$ and $\pi_E = \pi_F \circ (\eta\varphi)$. So, in the study of the continuity of Φ , we can assume in advance that φ is a function with compact support, which allows us to suppose without loss of generality that $\Omega = E$.

On the other hand, according to the definitions of topologies of spaces $\mathcal{C}^s(M \rightarrow E)$ and $\mathcal{C}^s(M \rightarrow F)$, to prove the continuity of Φ from $\mathcal{C}^s(M \rightarrow E)$ to $\mathcal{C}^s(M \rightarrow F)$, it is enough to verify that, for all bundle charts (U, ψ_E) and (U, ψ_F) of E and F , the mapping $X \mapsto \text{pr}_2 \circ \psi_F \circ \varphi \circ \psi_E^{-1}(\cdot, X(\cdot))$ is continuous from $\mathcal{C}_{loc}^s(U; \mathfrak{E})$ to $\mathcal{C}_{loc}^s(U; \mathfrak{F})$, where \mathfrak{E} and \mathfrak{F} are the standard fibers of E and F .

Let (U, ψ_E) and (U, ψ_F) be bundle charts of E and F . For $\varphi \in C^j(\Omega; F)$ with $j \in \mathbb{N}$, we have $\tilde{\varphi} = \text{pr}_2 \circ \psi_F \circ \varphi \circ \psi_E^{-1} \in C^j(U \times \mathfrak{E}; \mathfrak{F})$. Hence, we verify, exactly like the statement 3 in Theorem 2.8, that the mapping $X \mapsto \tilde{\varphi}(\cdot, X(\cdot))$ is continuous from $C^j(U; \mathfrak{E})$ to $C^j(U; \mathfrak{F})$.

Let $\varphi \in C^1(\Omega; F)$. Since $\tilde{\varphi} \in C^1(U \times \mathfrak{E}; \mathfrak{F}) \subset C^{0,1}(U \times \mathfrak{E}; \mathfrak{F})$ and since $\tilde{\varphi}$ checks the condition (4) for $s \in]0, 1[$ and $k = 0$, it follows that the mapping $X \mapsto \tilde{\varphi}(\cdot, X(\cdot))$ is continuous from $\mathcal{C}_{loc}^s(U; \mathfrak{E})$ to $\mathcal{C}_{loc}^s(U; \mathfrak{F})$. This fact is proved exactly as the assertion (b) in Theorem 2.8. ■

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