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# Symmetric forms for hyperbolic-parabolic systems of multi-gradient fluids

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## Abstract

ä We consider multi-gradient fluids endowed with a volumetric internal energy which is a function of mass density, volumetric entropy and their successive gradients. We obtained the thermodynamic forms of equation of motions and equation of energy, and the motions are compatible with the two laws of thermodynamics.

The equations of multi-gradient fluids belong to the class of dispersive systems. In the conservative case, we can replace the set of equations by a quasi-linear system written in a divergence form. Near an equilibrium position, we obtain a symmetric-Hermitian system of equations in the form of Godunov's systems. The equilibrium positions are proved to be stable when the total volume energy of the fluids is a convex function with respect to convenient conjugated variables - called main field - of mass density, volumetric entropy, their successive gradients, and velocity.

*Keywords:* Multi-gradient fluids - Equation of energy - Hermitian-symmetric form - Dispersive equations  
*2000 MSC:* 76A02, 76E30, 76M30

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## 1. Introduction

In continuum mechanics, Cauchy has only described the behavior of a mechanical system when inhomogeneities have a characteristic length scale much smaller than the macro-scale in which the phenomena are observed. Usually, the mechanical description of other conservative systems needs a higher order stress tensor and they are many physical phenomena described by this generalized continuum theory. For instance Piolas continua need a  $(n+1)$ -uple of hyper stress-tensors where the order is increasing from second to  $n+1$ ; the contact interactions do not reduce to forces per unit area on boundaries, but include  $k$ -forces concentrated on areas, on lines or even in wedges [1, 2]. The  $(n+1)$ -th order models are suitable for describing non-local effects as in bio-mechanical phenomena [3, 4], damage phenomena [5], and internal friction in solids [6]. The range of validity of Nolls theorem is not verified but the second principle of thermodynamics is clearly proved [7, 8]. Many efforts have been made to study these media, theoretically and numerically, where the research of symmetric forms for the equations of processes must be a main subject to verify the well-posed mathematical problems.

For fluids, across liquid-vapor interfaces, pressure  $p$  and volumetric internal energy  $\varepsilon_0$  are non-convex functions of volumetric entropy  $\eta$  and mass density. Consequently, the simplest continuous model allowing to study non-homogeneous fluids inside interface layers considers another volumetric internal energy  $\varepsilon$  as the sum of two terms: the first one defined as  $\varepsilon_0$  and the second one associated with the non-uniformity of the fluid which is approximated by an expansion limited at the first gradient of mass density. This form of energy which describes interfaces as diffuse layers was first introduced by van der Waals [9] and is now widely used in the literature [10]. The model has many applications for inhomogeneous fluids [11, 12] and is extended for different materials in continuum mechanics, which modelizes the behavior of strongly inhomogeneous media [13, 4, 14, 15, 16, 17]. The model yields a constant temperature at equilibrium. Consequently, the volume entropy varies with the mass density in the same way as in the bulks. This first assumption of van der Waals using long-ranged but weak-attractive forces is not exact for realistic intermolecular potentials and the thermodynamics is not completely considered [18].

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For variational principles, it is not possible to take directly the temperature gradient into account : the volume internal energy must be a functional of canonical variables, *i.e.* mass density and volumetric entropy. The simplest model was called *thermocapillary fluid model* when the internal energy depends on mass density, volumetric entropy and their first gradients [19, 20]. Such a behavior has also been considered in models when at equilibrium the temperature is not constant in inhomogeneous parts of complex media [21, 22, 23].

To improve the model accuracy, the general case considers fluids when the volume internal energy depends on mass density, volumetric entropy and their gradients up to a convenient  $n$ -order ( $n \in \mathbb{N}$ ) where continuum models of *gradient theories* are useful in case of strongly inhomogeneous fluids [24, 25]. The models have a justification in the framework of mean-field molecular theories when the van der Waals forces exert stresses on fluid molecules producing surface tension effects [18, 20, 26, 27].

In [44], we obtained the equation of motions for perfect multi-gradient fluids. For dissipative motions, the conservation of mass and balance of entropy implied the equation of energy. The Clausius-Duhem inequality was deduced from viscous-stress dissipation and Fourier's equation.

Moreover, the symmetrization of the equations of mechanical systems is a main subject of study for the structure of solutions of complex media, and being still debated.

First, we present some historical remarks which are detailed in Ref. [28] :

In 1961, Godunov wrote a paper on *an interesting class of quasi-linear system* which proves that with convenient change of variables, the system of Euler fluids becomes symmetric. He also proved that all systems coming from variational principles can be written in symmetric form [29]. In 1971, Friedrichs and Lax proved that all systems compatible with the entropy principle are symmetrizable [30]: after a pre-multiplication with a convenient matrix, systems become symmetric. In 1974, Boillat introduced a new field of variables for which the original system can be written in a symmetric form [31]. He was the first who symmetrized original hyperbolic systems that were compatible with the entropy principle. He called the systems, *Godunov's systems* [32]. The technique of Lagrange multipliers to study the entropy principle was given first by I-Shi Liu [33], and was similar to the work by Ruggeri and Strumia which were interested in extending the previous technique to the relativistic case by using a covariant formulation [34]. In 1982, Boillat extended the symmetrization to the case with constraints [35]; the problem was also considered by Dafermos [36]. In 1983, Ruggeri realized it was possible to construct a symmetrization for parabolic systems and he wrote down the expression of *the main field of variables* for Navier-Stokes-Fourier fluids [37], and in 1989, he proved that symmetrization was compatible with the Galilean invariance [38, 39].

Second, we consider the framework of models which are represented by quasi-linear first-order systems of  $n$  balance laws (we adopt the sum convection on repeated indexes) :

$$\frac{\partial \mathbf{G}^0(\mathbf{v})}{\partial t} + \frac{\partial \mathbf{G}^j(\mathbf{v})}{\partial x^j} = \mathbf{g}(\mathbf{v}), \quad (1)$$

with an additional scalar balance equation corresponding to the energy equation in pure mechanics or the entropy equation in thermodynamics :

$$\frac{\partial h^0(\mathbf{v})}{\partial t} + \frac{\partial h^j(\mathbf{v})}{\partial x^j} = \Sigma(\mathbf{v}),$$

where  $\mathbf{G}^0, \mathbf{G}^j$ ,  $j \in \{1, \dots, n\}$ ,  $\mathbf{g}, \mathbf{v}$  are column vectors of  $\mathbb{R}^n$ , and  $h^0, h^j$ ,  $j \in \{1, \dots, n\}$ ,  $\Sigma$  are scalar functions; scalar  $t$ , and  $\mathbf{x} = (x^1, \dots, x^n)$  are time and  $\mathbb{R}^n$ -space coordinates, respectively. Function  $h^0$  is assumed to be convex with respect to field  $\mathbf{G}^0(\mathbf{v}) \equiv \mathbf{v}$  (see Refs. [29, 30, 40]). Dual-vector field  $\mathbf{v}'$ , associated with Legendre transform  $h'^0$  and potentials  $h'^j$  is such that (see Ref. [31]) :

$$\mathbf{v}' = \left( \frac{\partial h^0}{\partial \mathbf{v}} \right)^*, \quad h'^0 = \mathbf{v}'^* \mathbf{v} - h^0, \quad h'^j = \mathbf{v}'^* \mathbf{G}^j(\mathbf{v}) - h^j,$$

where  $*$  indicates the transposition. By a convexity argument, it is possible to take  $\mathbf{v}'$  as a vector field and we obtain :

$$\mathbf{v} = \left( \frac{\partial h'^0}{\partial \mathbf{v}'} \right)^*, \quad \mathbf{G}^j(\mathbf{v}) = \left( \frac{\partial h'^j}{\partial \mathbf{v}'} \right)^*. \quad (2)$$

Inserting new variables given by Eqs. (2) into System (1), we get :

$$\frac{\partial}{\partial t} \left( \frac{\partial h^{i0}}{\partial v^i} \right) + \frac{\partial}{\partial x^j} \left( \frac{\partial h^{ij}}{\partial v^i} \right) = \mathbf{g}(v'),$$

which is symmetric and equivalent to

$$\mathbf{A}^0 \frac{\partial v'}{\partial t} + \mathbf{A}^j \frac{\partial v'}{\partial x^j} = \mathbf{g}(v'), \quad (3)$$

where matrix  $\mathbf{A}^0 \equiv (\mathbf{A}^0)^\star$  is positive-definite symmetric and matrices  $\mathbf{A}^j \equiv (\mathbf{A}^j)^\star$  are symmetric,

$$\mathbf{A}^0 \equiv (\mathbf{A}^0)^\star = \frac{\partial^2 h^{i0}}{\partial v'^2}, \quad \mathbf{A}^j \equiv (\mathbf{A}^j)^\star = \frac{\partial^2 h^{ij}}{\partial v'^2}, \quad (j = 1, \dots, n). \quad (4)$$

The symmetric form of governing equations implies hyperbolicity. For conservation laws with vanishing production terms, the hyperbolicity is equivalent to the stability of constant solutions with respect to perturbations in form  $e^{i(\mathbf{k}^\star x - \omega t)}$ , where  $i^2 = -1$ ,  $\mathbf{k}^\star = [k_1, \dots, k_n] \in \mathbb{R}^{n^\star}$  and  $\omega$  is a real scalar. Indeed, the symmetric form of governing equations for an unknown vector  $\mathbf{v}$ , ( $\mathbf{v}^\star = [v_1, \dots, v_n]$ ) implies the *dispersion relation* :

$$\det(\mathbf{A}_{(k)} - \omega \mathbf{A}^0) = 0 \quad \text{with} \quad \mathbf{A}_{(k)} = \mathbf{A}^j k_j,$$

which determines real values of  $\omega$  for any *real wave vector*  $\mathbf{k}$ , where operator  $\det$  denotes the determinant. In this case, phase velocities are real and coincide with the characteristic velocities of hyperbolic system [41, 42]. Moreover, right-eigenvectors of  $\mathbf{A}_{(k)}$  with respect to  $\mathbf{A}^0$  are linearly independent and any symmetric system is also automatically hyperbolic. Symmetric form given by Eq. (3) with relations (4) are commonly called *Godunov's systems* [29].

In the case of systems with parabolic structure (*hyperbolic-parabolic systems*), a generalization of symmetric system is written :

$$\mathbf{A}^0 \frac{\partial v'}{\partial t} + \mathbf{A}^j \frac{\partial v'}{\partial x^j} - \frac{\partial}{\partial x^j} \left( \mathbf{B}^{jl} \frac{\partial v'}{\partial x^l} \right) = 0, \quad (5)$$

where matrices  $\mathbf{B}^{jl} \equiv (\mathbf{B}^{jl})^\star$  are symmetric, and  $\mathbf{B}_{(k)} = \mathbf{B}^{jl} k_j k_l$  are non-negative definite.

The compatibility of hyperbolic-parabolic systems given by Eq. (5) with entropy principle and the corresponding determination of main field is given in [37] for Navier-Stokes-Fourier fluids and in general case in [43]. The same authors considered linearized version of System (5) proving that the constant solutions are stable.

These reminders being given, the aim of present paper is to extend the results of symmetrization for *the most general case of multi-gradient fluids*. Using a convenient change of variables – *the main field* – associated with a Legendre's transformation of the total fluid energy, equations of processes can be written in this special divergence form as in Eq. (5). Near an equilibrium position, we obtain a new Hermitian-symmetric form of the system of perturbations. The obtained set belongs to the class of dispersive systems.

The paper is organized as follows : In Section 2, we recall the main results obtained in [44] (equations of conservative motions, balance of energy and compatibility with the two laws of thermodynamics). We additively obtain the existence of a stress tensor which can write the equation of motions in a form similar to those of continuous media. In Section 3, *the main field of variables* – for which the conservative equations of motions are written in divergence form – is obtained. In Section 4, the Hermitian-symmetric form for the equations of perturbations near an equilibrium position is deduced. The perturbations are stable in domains where the total volume energy is a convex function of the main field of variables, which proof confirms that the mathematical problem is well posed. A conclusion ends the paper.

## 2. Multi-gradient fluids and equation of motions

In this section we recall in a new presentation, *adapted for symmetric calculations*, the main results obtained in [44], but subsection 2.3 introduces new calculations allowing to obtain the stress tensor of conservative multi-gradient fluids. In this Section, for the sake of simplicity, we identify vectors and covectors and we always indicate indexes in subscript position without taking account of the tensors' covariance or contravariance.

### 2.1. Definition of multi-gradient fluids

We consider perfect fluids with a volume internal energy  $\varepsilon$  function of volumetric entropy  $\eta$ , mass density  $\rho$ , and their gradients until order  $n \in \mathbb{N}$ ,

$$\varepsilon = \varepsilon(\eta, \rho, \nabla\eta, \nabla\rho, \dots, \nabla^n\eta, \nabla^n\rho),$$

where operators  $\nabla^p$ ,  $p \in \{1, \dots, n\}$ , denote the successive gradient in Euclidian space  $\mathcal{D}_t$ , of Euler variables  $\mathbf{s} \equiv [x_1, x_2, x_3]^*$ , occupied by the fluid at time  $t$ ,

$$\nabla^p \eta \equiv \left\{ \eta_{,x_{j_1} \dots x_{j_p}} \right\} \quad \text{and} \quad \nabla^p \rho \equiv \left\{ \rho_{,x_{j_1} \dots x_{j_p}} \right\}. \quad (6)$$

The subscript comma indicates partial derivatives with respect to variables  $x_{j_1} \dots x_{j_p}$  belonging to the set of Euler variables  $(x_1, x_2, x_3)$ . We deduce,

$$d\varepsilon = \frac{\partial\varepsilon}{\partial\eta} d\eta + \frac{\partial\varepsilon}{\partial\rho} d\rho + \left( \frac{\partial\varepsilon}{\partial\nabla\eta} \vdots d\nabla\eta \right) + \left( \frac{\partial\varepsilon}{\partial\nabla\rho} \vdots d\nabla\rho \right) + \dots + \left( \frac{\partial\varepsilon}{\partial\nabla^n\eta} \vdots d\nabla^n\eta \right) + \left( \frac{\partial\varepsilon}{\partial\nabla^n\rho} \vdots d\nabla^n\rho \right).$$

Notation  $\vdots$  means the complete product of tensors (or scalar product) and

$$\tilde{T} = \frac{\partial\varepsilon(\rho, \eta)}{\partial\eta} \quad \text{and} \quad \tilde{\mu} = \frac{\partial\varepsilon(\rho, \eta)}{\partial\rho},$$

are called the *extended temperature* and *extended chemical potential*, respectively.

### 2.2. Equation of conservative motions

The volume mass satisfies the mass conservation :

$$\frac{\partial\rho}{\partial t} + \text{div}(\rho \mathbf{u}) = 0.$$

where  $\mathbf{u}$  is the fluid velocity and  $\text{div}$  denotes the divergence operator. The motion is supposed to be conservative and consequently, the volumetric entropy verifies :

$$\frac{\partial\eta}{\partial t} + \text{div}(\eta \mathbf{u}) = 0, \quad (7)$$

The specific entropy  $s = \eta/\rho$  is constant along each trajectory. The *extended divergence at order  $p$*  is defined as :

$$\text{div}_p(b_{j_1 \dots j_p}) = (b_{j_1 \dots j_p})_{,x_{j_1} \dots x_{j_p}}, \quad p \in \mathbb{N} \quad \text{with} \quad x_{j_1}, \dots, x_{j_p} \in (x_1, x_2, x_3).$$

Classically, term  $(b_{j_1 \dots j_p})_{,x_{j_1} \dots x_{j_p}}$  corresponds to the summation on the repeated indexes  $j_1 \dots j_p$  of the consecutive derivatives of  $b_{j_1 \dots j_p}$  with respect to  $x_{j_1}, \dots, x_{j_p}$ . Term  $\text{div}_p$  decreases from order  $p$  the tensor order, while term  $\nabla^p$  increases from order  $p$  the tensor order. We denote :

$$\begin{cases} \theta = \tilde{T} - \text{div}\Psi_1 + \text{div}_2\Psi_2 + \dots + (-1)^n \text{div}_n\Psi_n, & \text{with} \quad \Psi_p = \frac{\partial\varepsilon}{\partial\nabla^p\eta}, \\ \Xi = \tilde{\mu} - \text{div}\Phi_1 + \text{div}_2\Phi_2 + \dots + (-1)^n \text{div}_n\Phi_n, & \text{with} \quad \Phi_p = \frac{\partial\varepsilon}{\partial\nabla^p\rho}, \end{cases} \quad (8)$$

where  $\theta$  and  $\Xi$  are called the *generalized temperature* and *generalized chemical potential*. We obtain the equation of conservative motions in Ref. [44], where we can find the proofs of these results :

$$\mathbf{a} + \text{grad}(\Xi + \Omega) + s \text{grad}\theta = 0 \quad \text{or} \quad \mathbf{a} + \text{grad}(H + \Omega) - \theta \text{grad}s = 0, \quad (9)$$

where  $\mathbf{a}$  denotes the acceleration,  $\text{grad}$  the gradient operator,  $\Omega$  the external force potential,  $H = \Xi - s\theta$  is called the *generalized free enthalpy*. Relations (9) are the generalization of relation (29.8) in Ref. [45] and constitutes the *thermodynamic form* of the equation of isentropic motions for perfect fluids.

### 2.3. Complement: the stress tensor of conservative fluids

The new results of the subsection are not useful for the other parts of the paper, but completely extend results obtained in [19]. We have the relation :

$$d\varepsilon = \tilde{T} d\eta + \tilde{\mu} d\rho + \left( \Psi_1 \dot{:} d\nabla\eta \right) + \left( \Phi_1 \dot{:} d\nabla\rho \right) + \dots + \left( \Psi_n \dot{:} d\nabla^n\eta \right) + \left( \Phi_n \dot{:} d\nabla^n\rho \right).$$

The Legendre transformation of  $\varepsilon$  with respect to  $\eta, \rho, \nabla\eta, \nabla\rho, \dots, \nabla^n\eta, \nabla^n\rho$  is denoted by  $\Pi$ . Fonction  $\Pi$  depends on  $\tilde{T}, \tilde{\mu}, \Psi_1, \Phi_1, \dots, \Psi_n, \Phi_n$ .

$$\Pi = \eta \tilde{T} + \rho \tilde{\mu} + \left( \nabla\eta \dot{:} \Psi_1 \right) + \left( \nabla\rho \dot{:} \Phi_1 \right) + \dots + \left( \nabla^n\eta \dot{:} \Psi_n \right) + \left( \nabla^n\rho \dot{:} \Phi_n \right) - \varepsilon, \quad (10)$$

and

$$d\Pi = \eta d\tilde{T} + \rho d\tilde{\mu} + \left( \nabla\eta \dot{:} d\Psi_1 \right) + \left( \nabla\rho \dot{:} d\Phi_1 \right) + \dots + \left( \nabla^n\eta \dot{:} d\Psi_n \right) + \left( \nabla^n\rho \dot{:} d\Phi_n \right),$$

where

$$\frac{\partial\Pi}{\partial\tilde{T}} = \eta, \quad \frac{\partial\Pi}{\partial\tilde{\mu}} = \rho, \quad \text{and} \quad \frac{\partial\Pi}{\partial\Psi_k} = \nabla^k\eta, \quad \frac{\partial\Pi}{\partial\Phi_k} = \nabla^k\rho, \quad k \in \{1, \dots, n\} \quad (11)$$

Consequently,

$$\frac{\partial\Pi}{\partial\mathbf{x}} = \eta \frac{\partial\tilde{T}}{\partial\mathbf{x}} + \rho \frac{\partial\tilde{\mu}}{\partial\mathbf{x}} + \left( \nabla\eta \dot{:} \frac{\partial\Psi_1}{\partial\mathbf{x}} \right) + \left( \nabla\rho \dot{:} \frac{\partial\Phi_1}{\partial\mathbf{x}} \right) + \dots + \left( \nabla^n\eta \dot{:} \frac{\partial\Psi_n}{\partial\mathbf{x}} \right) + \left( \nabla^n\rho \dot{:} \frac{\partial\Phi_n}{\partial\mathbf{x}} \right).$$

Because  $\frac{\partial \operatorname{div}_n \Phi_n}{\partial \mathbf{x}} = \operatorname{div}_n \frac{\partial \Phi_n}{\partial \mathbf{x}}$ , by taking account of identities,

$$\begin{aligned} \left( \nabla\rho \dot{:} \frac{\partial\Phi_1}{\partial\mathbf{x}} \right) &\equiv \operatorname{div} \left( \rho \frac{\partial\Phi_1}{\partial\mathbf{x}} \right) - \rho \frac{\partial \operatorname{div}\Phi_1}{\partial\mathbf{x}} \\ \left( \nabla^2\rho \dot{:} \frac{\partial\Phi_2}{\partial\mathbf{x}} \right) &\equiv \operatorname{div} \left[ \left( \nabla\rho \dot{:} \frac{\partial\Phi_2}{\partial\mathbf{x}} \right) - \rho \frac{\partial \operatorname{div}\Phi_2}{\partial\mathbf{x}} \right] + \rho \frac{\partial \operatorname{div}_2\Phi_2}{\partial\mathbf{x}} \\ &\vdots \\ \left( \nabla^n\rho \dot{:} \frac{\partial\Phi_n}{\partial\mathbf{x}} \right) &\equiv \operatorname{div} \left[ \left( \nabla^{n-1}\rho \dot{:} \frac{\partial\Phi_n}{\partial\mathbf{x}} \right) - \left( \nabla^{n-2}\rho \dot{:} \operatorname{div}\frac{\partial\Phi_n}{\partial\mathbf{x}} \right) + \dots \right. \\ &\quad \left. + (-1)^{p-1} \left( \nabla^{n-p}\rho \dot{:} \operatorname{div}_{p-1}\frac{\partial\Phi_n}{\partial\mathbf{x}} \right) + \dots + (-1)^{n-1} \rho \operatorname{div}_{n-1}\frac{\partial\Phi_n}{\partial\mathbf{x}} \right] + (-1)^n \rho \frac{\partial \operatorname{div}_n\Phi_n}{\partial\mathbf{x}}, \end{aligned} \quad (12)$$

and an analog expression for  $\eta$ , where  $\rho$  and  $\Phi_p$  are replaced by  $\eta$  and  $\Psi_p$ ,  $p \in \{1, \dots, n\}$ . We deduce,

$$\rho \frac{\partial \Xi}{\partial \mathbf{x}} + \eta \frac{\partial \theta}{\partial \mathbf{x}} = \operatorname{div} (\Pi \mathbf{I} - \sigma).$$

The identical transformation is denoted by  $\mathbf{I}$  and stress tensor  $\sigma$  is :

$$\begin{aligned} \sigma &= \rho \frac{\partial\Phi_1}{\partial\mathbf{x}} + \left( \nabla\rho \dot{:} \frac{\partial\Phi_2}{\partial\mathbf{x}} \right) - \rho \frac{\partial \operatorname{div}_2\Phi_2}{\partial\mathbf{x}} + \dots + \left( \nabla^{n-1}\rho \dot{:} \frac{\partial\Phi_n}{\partial\mathbf{x}} \right) - \left( \nabla^{n-2}\rho \dot{:} \operatorname{div}\frac{\partial\Phi_n}{\partial\mathbf{x}} \right) \\ &\quad + \dots + (-1)^{p-1} \left( \nabla^{n-p}\rho \dot{:} \operatorname{div}_{p-1}\frac{\partial\Phi_n}{\partial\mathbf{x}} \right) + \dots + (-1)^{n-1} \rho \operatorname{div}_{n-1}\frac{\partial\Phi_n}{\partial\mathbf{x}} \\ &\quad + \eta \frac{\partial\Psi_1}{\partial\mathbf{x}} + \left( \nabla\eta \dot{:} \frac{\partial\Psi_2}{\partial\mathbf{x}} \right) - \eta \frac{\partial \operatorname{div}_2\Psi_2}{\partial\mathbf{x}} + \dots + \left( \nabla^{n-1}\eta \dot{:} \frac{\partial\Psi_n}{\partial\mathbf{x}} \right) - \left( \nabla^{n-2}\eta \dot{:} \operatorname{div}\frac{\partial\Psi_n}{\partial\mathbf{x}} \right) \\ &\quad + \dots + (-1)^{p-1} \left( \nabla^{n-p}\eta \dot{:} \operatorname{div}_{p-1}\frac{\partial\Psi_n}{\partial\mathbf{x}} \right) + \dots + (-1)^{n-1} \eta \operatorname{div}_{n-1}\frac{\partial\Psi_n}{\partial\mathbf{x}}. \end{aligned}$$

Due to the mass conservation, we get  $\rho \mathbf{a} = \partial(\rho \mathbf{u})/\partial t + \text{div}(\rho \mathbf{u} \otimes \mathbf{u})$  and the equation of motions (9) can be written in the other form :

$$\frac{\partial \rho \mathbf{u}}{\partial t} + \text{div}(\rho \mathbf{u} \otimes \mathbf{u} + \Pi \mathbf{I} - \boldsymbol{\sigma}) + \rho \text{grad} \Omega = 0 \quad (13)$$

The two previous equations are deduced from Hamilton's principle [44], which can be used only for conservative media because Eq. (7) is verified. In this case, Eq. (9) is strictly equivalent to Eq. (13).

Let us note that, for classical fluids, the two equations are two forms of the equation of motions which are written in Eq. (29.8) of [45]:

$$\rho \mathbf{a} + \text{grad} p + \rho \text{grad} \Omega = 0 \quad \iff \quad \mathbf{a} + \text{grad}(\mu + \Omega) + s \text{grad} T = 0,$$

where  $p$  is here the thermodynamical pressure of simple fluids,  $\mu$  the corresponding chemical potential and  $T$  the Kelvin temperature.

The stress tensor  $\boldsymbol{\sigma}$  is only an artifact different from the Cauchy stress tensor, which can be interesting to compare with solid mechanics; the most important conservative equations are expressed by Eq. (9). It is the reason why the entropy law is expressed without dissipative terms.

#### 2.4. Equation of energy for dissipative motions (see the detailed proofs in Ref. [44])

For viscous fluids, the equation of motions can be written as :

$$\frac{\partial \rho \mathbf{u}}{\partial t} + \text{div}(\rho \mathbf{u} \otimes \mathbf{u}) + \rho \text{grad} \Xi + \eta \text{grad} \theta - \text{div} \boldsymbol{\sigma}_v + \rho \text{grad} \Omega = 0,$$

here  $\boldsymbol{\sigma}_v$  denotes the viscous-stress tensor of the fluid. We denote

$$\left\{ \begin{array}{l} \mathbf{M} = \frac{\partial \rho \mathbf{u}}{\partial t} + \text{div}(\rho \mathbf{u} \otimes \mathbf{u}) + \rho \text{grad} \Xi + \eta \text{grad} \theta - \text{div} \boldsymbol{\sigma}_v + \rho \text{grad} \Omega \\ \mathbf{B} = \frac{\partial \rho}{\partial t} + \text{div}(\rho \mathbf{u}) \\ \mathbf{N} = \frac{\partial \rho s}{\partial t} + \text{div}(\rho s \mathbf{u}) + \frac{1}{\theta} (\text{div} \mathbf{q} - r - \text{Tr}(\boldsymbol{\sigma}_v \mathbf{D})) \\ \mathbf{F} = \frac{\partial}{\partial t} \left( \frac{1}{2} \rho \mathbf{u} \cdot \mathbf{u} + \rho \Xi + \eta \theta - \Pi + \rho \Omega \right) \\ + \text{div} \left\{ \left[ \left( \frac{1}{2} \rho \mathbf{u} \cdot \mathbf{u} + \rho \Xi + \eta \theta + \rho \Omega \right) \mathbf{I} - \boldsymbol{\sigma}_v \right] \mathbf{u} + \boldsymbol{\chi} \right\} + \text{div} \mathbf{q} - r - \rho \frac{\partial \Omega}{\partial t}, \end{array} \right.$$

where  $\text{Tr}$  denotes the trace operator and  $\cdot$  the scalar product ( $\mathbf{u} \cdot \mathbf{u} = \mathbf{u}^* \mathbf{u}$ ). Terms  $\mathbf{q}$  and  $r$  represent the heat-flux vector and the heat supply;  $\mathbf{D} = \frac{1}{2} (\partial \mathbf{u} / \partial \mathbf{x} + (\partial \mathbf{u} / \partial \mathbf{x})^*)$  is the velocity gradient. Due to the relaxation time in the dissipative processes, we only consider the case when dissipative viscous stress tensor  $\boldsymbol{\sigma}_v$  takes account of the first derivative of the velocity field: the higher terms are assumed negligible and the viscosity does not take any gradient terms into account.

The equation of motion is written  $\mathbf{M} = 0$  in the dissipative case with addition of viscous stress tensor  $\boldsymbol{\sigma}_v$ . The equation of motion written in the conservative case can be now written for viscous fluids, the conservative motions are written without viscosity. Terms  $\mathbf{q}$  and  $r$  being introduced together with  $\boldsymbol{\sigma}_v$  are adapted into  $\mathbf{N}$  and  $\mathbf{F}$  for the dissipative case. Due to subsection 2.3, the term  $\mathbf{M}$  can be written in two equivalent expressions :

$$\mathbf{M} = \frac{\partial \rho \mathbf{u}}{\partial t} + \text{div}(\rho \mathbf{u} \otimes \mathbf{u}) + \rho \text{grad} \Xi + \eta \text{grad} \theta - \text{div} \boldsymbol{\sigma}_v + \rho \text{grad} \Omega$$

or equivalently,

$$\mathbf{M} = \frac{\partial \rho \mathbf{u}}{\partial t} + \text{div}(\rho \mathbf{u} \otimes \mathbf{u} + \Pi \mathbf{I} - \boldsymbol{\sigma} - \boldsymbol{\sigma}_v) + \rho \text{grad} \Omega$$

The first expression is more adapted to the following. Viscous stress tensor  $\boldsymbol{\sigma}_v$  is classically introduced in the same part than conservative stress tensor  $\boldsymbol{\sigma}$ . In this case,

$$\mathbf{N} = \frac{\partial \rho s}{\partial t} + \text{div}(\rho s \mathbf{u}) + \frac{1}{\theta} (\text{div} \mathbf{q} - r - \text{Tr}(\boldsymbol{\sigma}_v \mathbf{D}))$$

and due to relation,

$$\left( \frac{\partial \rho s}{\partial t} + \operatorname{div}(\rho s \mathbf{u}) \right) \theta + (\operatorname{div} \mathbf{q} - \mathbf{r} - \operatorname{Tr}(\boldsymbol{\sigma}_v \mathbf{D})) = 0,$$

term  $\frac{\partial \rho s}{\partial t} + \operatorname{div}(\rho s \mathbf{u})$  corresponds to the variation of the entropy. Then, we only take account of  $\mathbf{D}$  which is the velocity deformation tensor and (see for proof, Ref. [44]),

$$\begin{aligned} \chi = & \rho \frac{\partial \Phi_1}{\partial t} + \dots + \left( \frac{\partial \Phi_n}{\partial t} : \nabla^{n-1} \rho \right) - \left( \operatorname{div} \frac{\partial \Phi_n}{\partial t} : \nabla^{n-2} \rho \right) + \dots + (-1)^{p-1} \left( \operatorname{div}_{p-1} \frac{\partial \Phi_n}{\partial t} : \nabla^{n-p} \rho \right) + \dots + (-1)^{n-1} \rho \operatorname{div}_{n-1} \frac{\partial \Phi_n}{\partial t} \\ & + \eta \frac{\partial \Psi_1}{\partial t} + \dots + \left( \frac{\partial \Psi_n}{\partial t} : \nabla^{n-1} \eta \right) - \left( \operatorname{div} \frac{\partial \Psi_n}{\partial t} : \nabla^{n-2} \eta \right) + \dots + (-1)^{p-1} \left( \operatorname{div}_{p-1} \frac{\partial \Psi_n}{\partial t} : \nabla^{n-p} \eta \right) + \dots + (-1)^{n-1} \eta \operatorname{div}_{n-1} \frac{\partial \Psi_n}{\partial t}. \end{aligned}$$

Term  $\chi$  is the general extension of the interstitial-working vector obtained in [46]. We obtain the following results (see the proofs in Ref. [44]),

**Theorem 1.** *Relation*

$$F - \mathbf{M} \cdot \mathbf{u} - \left( \frac{1}{2} \mathbf{u} \cdot \mathbf{u} + \Xi + \Omega \right) B - \theta N \equiv 0$$

is an algebraic identity.

$\mathbf{M} = 0$  is the equation of motion,  $B = 0$  is the mass conservation and  $N = 0$  the entropy relation, then  $F = 0$  is the equation of energy for dissipative fluids.

**Corollary 2.** *The equation of energy is*

$$\frac{\partial}{\partial t} \left( \frac{1}{2} \rho \mathbf{u} \cdot \mathbf{u} + \rho \Xi + \eta \theta - \Pi + \rho \Omega \right) + \operatorname{div} \left\{ \left[ \left( \frac{1}{2} \rho \mathbf{u} \cdot \mathbf{u} + \rho \Xi + \eta \theta + \rho \Omega \right) \mathbf{I} - \boldsymbol{\sigma}_v \right] \mathbf{u} + \chi \right\} + \operatorname{div} \mathbf{q} - \mathbf{r} - \rho \frac{\partial \Omega}{\partial t} = 0.$$

For dissipative fluid motions,  $\operatorname{tr}(\boldsymbol{\sigma}_v \mathbf{D}) \geq 0$ . From  $N = 0$  and  $B = 0$ , we deduce the Planck inequality [47] :

$$\rho \theta \frac{ds}{dt} + \operatorname{div} \mathbf{q} - \mathbf{r} \geq 0.$$

We consider the Fourier equation in the form of general inequality :

$$\mathbf{q} \cdot \operatorname{grad} \theta \leq 0,$$

and we obtain,

$$\rho \frac{ds}{dt} + \operatorname{div} \frac{\mathbf{q}}{\theta} - \frac{\mathbf{r}}{\theta} \geq 0,$$

which is the extended form of the Clausius-Duhem inequality. Then, multi-gradient fluids are compatible with the two law of thermodynamics.

### 3. Main field variables

In this section, we use the properties of symmetry and consequently we cannot any more identify covariant and contravariant vectors and tensors. Then, superscript  $\star$  denotes the transposition in  $\mathcal{D}_t$ . When clarity is necessary, we use the notation  $\mathbf{b}^\star \mathbf{c}$  for the scalar product of vectors  $\mathbf{b}$  and  $\mathbf{c}$ , tensor product  $\mathbf{b} \mathbf{c}^\star$  corresponds to  $\mathbf{b} \otimes \mathbf{c}$ . The divergence of a linear transformation  $\mathbf{S}$  denotes the covector  $\operatorname{div}(\mathbf{S})$  such that, for any constant vector  $\mathbf{d}$ ,  $\operatorname{div}(\mathbf{S}) \mathbf{d} = \operatorname{div}(\mathbf{S} \mathbf{d})$ . Now, previous terms  $\nabla^p \eta$  and  $\nabla^p \rho$ , defined in Eqs. (6), are covariant tensors of order  $p$ , while  $\Psi_p$  and  $\Phi_p$ , defined in Eqs. (8), are contravariant tensors of order  $p$ .



### 3.1. Study of conservative motion equation

Without missing the generality, and for the sake of simplicity, we do not consider external-force term. The total energy of the fluid is :

$$E = \frac{\mathbf{j}^* \mathbf{j}}{2\rho} + \varepsilon \quad \text{where} \quad \mathbf{j} = \rho \mathbf{u},$$

and

$$dE = \tilde{T} d\eta + R d\rho + \mathbf{u}^* d\mathbf{j} + \left( d\nabla\eta : \Psi_1 \right) + \left( d\nabla\rho : \Phi_1 \right) + \dots + \left( d\nabla^n \eta : \Psi_n \right) + \left( d\nabla^n \rho : \Phi_n \right),$$

where

$$R = \tilde{\mu} - \frac{\mathbf{u}^* \mathbf{u}}{2}.$$

The Legendre transformation of  $E$  with respect to variables  $\eta, \rho, \mathbf{j}, \nabla\eta, \nabla\rho, \dots, \nabla^n \eta, \nabla^n \rho$  is denoted  $\mathcal{P}$ ;  $\mathcal{P}$  is a function of  $\tilde{T}, R, \mathbf{u}, \Psi_1, \Phi_1, \dots, \Psi_n, \Phi_n$ .

$$\mathcal{P} = \eta \tilde{T} + \rho R + \mathbf{j}^* \mathbf{u} + \left( \nabla\eta : \Psi_1 \right) + \left( \nabla\rho : \Phi_1 \right) + \dots + \left( \nabla^n \eta : \Psi_n \right) + \left( \nabla^n \rho : \Phi_n \right) - E,$$

where

$$\frac{\partial \mathcal{P}}{\partial \tilde{T}} = \eta, \quad \frac{\partial \mathcal{P}}{\partial R} = \rho, \quad \frac{\partial \mathcal{P}}{\partial \mathbf{u}} = \mathbf{j}^*, \quad \text{and} \quad \frac{\partial \mathcal{P}}{\partial \Psi_k} = \nabla^k \eta, \quad \frac{\partial \mathcal{P}}{\partial \Phi_k} = \nabla^k \rho, \quad k \in \{1, \dots, n\}. \quad (14)$$

We notice that

$$\mathcal{P} = \eta \tilde{T} + \rho \tilde{\mu} + \left( \nabla\eta : \Psi_1 \right) + \left( \nabla\rho : \Phi_1 \right) + \dots + \left( \nabla^n \eta : \Psi_n \right) + \left( \nabla^n \rho : \Phi_n \right) - \varepsilon. \quad (15)$$

Consequently, the value of  $\mathcal{P}$  is the same than the value of  $\Pi$  given in Eq. (10), but function  $\mathcal{P}$  is associated with a different field of variables. Motion equation (9) can be written,

$$\frac{\partial \rho \mathbf{u}}{\partial t} + \text{div}(\rho \mathbf{u} \otimes \mathbf{u}) + \rho \text{grad} \Xi + \eta \text{grad} \theta = 0. \quad (16)$$

Due to (14),

$$d(\mathcal{P} \mathbf{u}) = d\mathcal{P} \mathbf{u} + \mathcal{P} d\mathbf{u} \implies \frac{\partial(\mathcal{P} \mathbf{u})}{\partial \mathbf{u}} = \mathbf{u} \otimes \mathbf{j} + \mathcal{P} \mathbf{I} \implies \text{div} \left[ \frac{\partial(\mathcal{P} \mathbf{u})}{\partial \mathbf{u}} \right] = \text{div}(\rho \mathbf{u} \otimes \mathbf{u}) + \frac{\partial \mathcal{P}}{\partial \mathbf{x}}.$$

From (15), we get

$$\frac{\partial \mathcal{P}}{\partial \mathbf{x}} = \eta \frac{\partial \tilde{T}}{\partial \mathbf{x}} + \rho \frac{\partial \tilde{\mu}}{\partial \mathbf{x}} + \left( \nabla\eta : \frac{\partial \Psi_1}{\partial \mathbf{x}} \right) + \left( \nabla\rho : \frac{\partial \Phi_1}{\partial \mathbf{x}} \right) + \dots + \left( \nabla^n \eta : \frac{\partial \Psi_n}{\partial \mathbf{x}} \right) + \left( \nabla^n \rho : \frac{\partial \Phi_n}{\partial \mathbf{x}} \right).$$

Taking account of Eq. (12), we get :

$$\begin{aligned} \frac{\partial \mathcal{P}}{\partial \mathbf{x}} &= \eta \frac{\partial}{\partial \mathbf{x}} \left( \tilde{T} - \text{div} \Psi_1 + \text{div}_2 \Psi_2 + \dots + (-1)^n \text{div}_n \Psi_n \right) + \rho \frac{\partial}{\partial \mathbf{x}} \left( \tilde{\mu} - \text{div} \Phi_1 + \text{div}_2 \Phi_2 + \dots + (-1)^n \text{div}_n \Phi_n \right) \\ &+ \text{div} \left( \eta \frac{\partial \Psi_1}{\partial \mathbf{x}} \right) + \text{div} \left[ \left( \nabla\eta : \frac{\partial \Psi_2}{\partial \mathbf{x}} \right) - \eta \frac{\partial \text{div} \Psi_2}{\partial \mathbf{x}} \right] \\ &\vdots \\ &+ \text{div} \left[ \left( \nabla^{n-1} \eta : \frac{\partial \Psi_n}{\partial \mathbf{x}} \right) - \left( \nabla^{n-2} \eta : \text{div} \frac{\partial \Psi_n}{\partial \mathbf{x}} \right) + \dots + (-1)^{p-1} \left( \nabla^{n-p} \eta : \text{div}_{p-1} \frac{\partial \Psi_n}{\partial \mathbf{x}} \right) + \dots + (-1)^{n-1} \eta \text{div}_{n-1} \frac{\partial \Psi_n}{\partial \mathbf{x}} \right] \\ &+ \text{div} \left( \rho \frac{\partial \Phi_1}{\partial \mathbf{x}} \right) + \text{div} \left[ \left( \nabla\rho : \frac{\partial \Phi_2}{\partial \mathbf{x}} \right) - \rho \frac{\partial \text{div} \Phi_2}{\partial \mathbf{x}} \right] \\ &\vdots \\ &+ \text{div} \left[ \left( \nabla^{n-1} \rho : \frac{\partial \Phi_n}{\partial \mathbf{x}} \right) - \left( \nabla^{n-2} \rho : \text{div} \frac{\partial \Phi_n}{\partial \mathbf{x}} \right) + \dots + (-1)^{p-1} \left( \nabla^{n-p} \rho : \text{div}_{p-1} \frac{\partial \Phi_n}{\partial \mathbf{x}} \right) + \dots + (-1)^{n-1} \rho \text{div}_{n-1} \frac{\partial \Phi_n}{\partial \mathbf{x}} \right] \end{aligned} \quad (17)$$

and consequently,

$$\begin{aligned} \frac{\partial \mathcal{P}}{\partial \mathbf{x}} &= \eta \frac{\partial \theta}{\partial \mathbf{x}} + \rho \frac{\partial \Xi}{\partial \mathbf{x}} + \operatorname{div} (C_n + D_n) \\ &+ \operatorname{div} \left[ \eta \frac{\partial}{\partial \mathbf{x}} (\Psi_1 - \operatorname{div} \Psi_2 + \dots + (-1)^{n-1} \operatorname{div}_{n-1} \Psi_n) \right] + \operatorname{div} \left[ \rho \frac{\partial}{\partial \mathbf{x}} (\Phi_1 - \operatorname{div} \Phi_2 + \dots + (-1)^{n-1} \operatorname{div}_{n-1} \Phi_n) \right], \end{aligned}$$

with

$$C_n = \left( \nabla \eta : \frac{\partial \Psi_2}{\partial \mathbf{x}} \right) + \left( \nabla^2 \eta : \frac{\partial \Psi_3}{\partial \mathbf{x}} \right) - \left( \nabla \eta : \operatorname{div} \frac{\partial \Psi_3}{\partial \mathbf{x}} \right) + \dots + \left( \nabla^{n-1} \eta : \frac{\partial \Psi_n}{\partial \mathbf{x}} \right) - \left( \nabla^{n-2} \eta : \operatorname{div} \frac{\partial \Psi_n}{\partial \mathbf{x}} \right) + \dots \quad (18)$$

$$+ (-1)^{p-1} \left( \nabla^{n-p} \eta : \operatorname{div}_{p-1} \frac{\partial \Psi_n}{\partial \mathbf{x}} \right) + \dots + (-1)^{n-2} \left( \nabla \eta : \operatorname{div}_{n-2} \frac{\partial \Psi_n}{\partial \mathbf{x}} \right),$$

$$D_n = \left( \nabla \rho : \frac{\partial \Psi_2}{\partial \mathbf{x}} \right) + \left( \nabla^2 \rho : \frac{\partial \Psi_3}{\partial \mathbf{x}} \right) - \left( \nabla \rho : \operatorname{div} \frac{\partial \Psi_3}{\partial \mathbf{x}} \right) + \dots + \left( \nabla^{n-1} \rho : \frac{\partial \Psi_n}{\partial \mathbf{x}} \right) - \left( \nabla^{n-2} \rho : \operatorname{div} \frac{\partial \Psi_n}{\partial \mathbf{x}} \right) + \dots \quad (19)$$

$$+ (-1)^{p-1} \left( \nabla^{n-p} \rho : \operatorname{div}_{p-1} \frac{\partial \Psi_n}{\partial \mathbf{x}} \right) + \dots + (-1)^{n-2} \left( \nabla \rho : \operatorname{div}_{n-2} \frac{\partial \Psi_n}{\partial \mathbf{x}} \right).$$

From Eq. (16) and Eqs. (17)-(18)-(19), we finally obtain,

$$\begin{aligned} \frac{\partial \rho \mathbf{u}}{\partial t} + \operatorname{div} \left[ \frac{\partial (\mathcal{P} \mathbf{u})}{\partial \mathbf{u}} \right] - \operatorname{div} (C_n + D_n) \\ - \operatorname{div} \left[ \eta \frac{\partial}{\partial \mathbf{x}} (\Psi_1 - \operatorname{div} \Psi_2 + \dots + (-1)^{n-1} \operatorname{div}_{n-1} \Psi_n) \right] - \operatorname{div} \left[ \rho \frac{\partial}{\partial \mathbf{x}} (\Phi_1 - \operatorname{div} \Phi_2 + \dots + (-1)^{n-1} \operatorname{div}_{n-1} \Phi_n) \right] = 0. \end{aligned}$$

### 3.2. Balances of mass and entropy

For the mass density, we get by successive derivations

$$\begin{cases} \frac{\partial \rho}{\partial t} + \operatorname{div} (\rho \mathbf{u}) = 0, \\ \frac{\partial (\nabla \rho)}{\partial t} + \operatorname{div} [\nabla (\rho \mathbf{u})] = 0, \\ \vdots \\ \frac{\partial (\nabla^n \rho)}{\partial t} + \operatorname{div} [\nabla^n (\rho \mathbf{u})] = 0, \end{cases}$$

where we recall that,

$$\nabla \rho = \frac{\partial \rho}{\partial \mathbf{x}}, \quad \nabla (\rho \mathbf{u}) = \frac{\partial (\rho \mathbf{u})}{\partial \mathbf{x}}, \quad \dots, \quad \nabla^n \rho = \frac{\partial^n \rho}{\partial \mathbf{x}^n}, \quad \nabla^n (\rho \mathbf{u}) = \frac{\partial^n (\rho \mathbf{u})}{\partial \mathbf{x}^n}.$$

If we assume  $\beta_1|_{t=0} = \nabla \rho|_{t=0}$ , one can consider  $\beta_1 = \nabla \rho$  as an independent variable. That is the same for  $\beta_p = \nabla^p \rho$  with  $\beta_p|_{t=0} = \nabla^p \rho|_{t=0}$ . Then, all the previous equations are compatible with the mass conservation. But,

$$\begin{aligned} \nabla (\rho \mathbf{u}) &= \nabla \rho \otimes \mathbf{u} + \rho \nabla \mathbf{u} \\ &\vdots \\ \nabla^n (\rho \mathbf{u}) &= \nabla^n \rho \otimes \mathbf{u} + C_n^1 \nabla^{n-1} \rho \otimes \nabla \mathbf{u} + \dots + C_n^p \nabla^{n-p} \rho \otimes \nabla^p \mathbf{u} + \dots + \rho \nabla^n \mathbf{u}, \end{aligned}$$

Then,

$$\begin{cases} \frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \mathbf{u}) = 0, \\ \frac{\partial \nabla \rho}{\partial t} + \operatorname{div}[\nabla \rho \otimes \mathbf{u} + \rho \nabla \mathbf{u}] = 0, \\ \vdots \\ \frac{\partial \nabla^n \rho}{\partial t} + \operatorname{div}[\nabla^n \rho \otimes \mathbf{u} + C_n^1 \nabla^{n-1} \rho \otimes \nabla \mathbf{u} + \dots + C_n^p \nabla^{n-p} \rho \otimes \nabla^p \mathbf{u} + \dots + \rho \nabla^n \mathbf{u}] = 0. \end{cases}$$

It is the same for the volumetric entropy if we consider  $\eta, \nabla \eta, \dots, \nabla^n \eta$  as independent variables. If we note that

$$\frac{\partial \mathcal{P}}{\partial \Phi_p} = \nabla^p \rho, \quad \frac{\partial \mathcal{P}}{\partial \Psi_p} = \nabla^p \eta,$$

we get,

$$\begin{aligned} C_n &= \left( \frac{\partial \mathcal{P}}{\partial \Psi_1} \cdot \frac{\partial \Psi_2}{\partial \mathbf{x}} \right) + \left( \frac{\partial \mathcal{P}}{\partial \Psi_2} \cdot \frac{\partial \Psi_3}{\partial \mathbf{x}} \right) - \left( \frac{\partial \mathcal{P}}{\partial \Psi_1} \cdot \operatorname{div} \frac{\partial \Psi_3}{\partial \mathbf{x}} \right) + \dots + \left( \frac{\partial \mathcal{P}}{\partial \Psi_{n-1}} \cdot \frac{\partial \Psi_n}{\partial \mathbf{x}} \right) - \left( \frac{\partial \mathcal{P}}{\partial \Psi_{n-2}} \cdot \operatorname{div} \frac{\partial \Psi_n}{\partial \mathbf{x}} \right) + \dots \\ &+ (-1)^{p-1} \left( \frac{\partial \mathcal{P}}{\partial \Psi_{n-p}} \cdot \operatorname{div}_{p-1} \frac{\partial \Psi_n}{\partial \mathbf{x}} \right) + \dots + (-1)^{n-2} \left( \frac{\partial \mathcal{P}}{\partial \Psi_1} \cdot \operatorname{div}_{n-2} \frac{\partial \Psi_n}{\partial \mathbf{x}} \right), \\ D_n &= \left( \frac{\partial \mathcal{P}}{\partial \Phi_1} \cdot \frac{\partial \Phi_2}{\partial \mathbf{x}} \right) + \left( \frac{\partial \mathcal{P}}{\partial \Phi_2} \cdot \frac{\partial \Phi_3}{\partial \mathbf{x}} \right) - \left( \frac{\partial \mathcal{P}}{\partial \Phi_1} \cdot \operatorname{div} \frac{\partial \Phi_3}{\partial \mathbf{x}} \right) + \dots + \left( \frac{\partial \mathcal{P}}{\partial \Phi_{n-1}} \cdot \frac{\partial \Phi_n}{\partial \mathbf{x}} \right) - \left( \frac{\partial \mathcal{P}}{\partial \Phi_{n-2}} \cdot \operatorname{div} \frac{\partial \Phi_n}{\partial \mathbf{x}} \right) + \dots \\ &+ (-1)^{p-1} \left( \frac{\partial \mathcal{P}}{\partial \Phi_{n-p}} \cdot \operatorname{div}_{p-1} \frac{\partial \Phi_n}{\partial \mathbf{x}} \right) + \dots + (-1)^{n-2} \left( \frac{\partial \mathcal{P}}{\partial \Phi_1} \cdot \operatorname{div}_{n-2} \frac{\partial \Phi_n}{\partial \mathbf{x}} \right) \end{aligned}$$

and we obtain,

**Theorem 3.** *The system of equations of processes for multi-gradient fluids can be written in the divergence form :*

$$\left\{ \begin{array}{l} \frac{\partial}{\partial t} \left( \frac{\partial \mathcal{P}}{\partial R} \right) + \operatorname{div} \left[ \frac{\partial(\mathcal{P} \mathbf{u})}{\partial R} \right] = 0 \\ \frac{\partial}{\partial t} \left( \frac{\partial \mathcal{P}}{\partial \Phi_1} \right) + \operatorname{div} \left[ \frac{\partial(\mathcal{P} \mathbf{u})}{\partial \Phi_1} + \frac{\partial \mathcal{P}}{\partial R} \frac{\partial \mathbf{u}}{\partial \mathbf{x}} \right] = 0 \\ \vdots \\ \frac{\partial}{\partial t} \left( \frac{\partial \mathcal{P}}{\partial \Phi_n} \right) + \operatorname{div} \left[ \frac{\partial(\mathcal{P} \mathbf{u})}{\partial \Phi_n} + C_n^1 \left( \frac{\partial \mathcal{P}}{\partial \Phi_{n-1}} \otimes \frac{\partial \mathbf{u}}{\partial \mathbf{x}} \right) + \dots + C_n^p \left( \frac{\partial \mathcal{P}}{\partial \Phi_{n-p}} \otimes \frac{\partial^p \mathbf{u}}{\partial \mathbf{x}^p} \right) + \dots + \frac{\partial \mathcal{P}}{\partial R} \frac{\partial^n \mathbf{u}}{\partial \mathbf{x}^n} \right] = 0 \\ \frac{\partial}{\partial t} \left( \frac{\partial \mathcal{P}}{\partial \tilde{T}} \right) + \operatorname{div} \left[ \frac{\partial(\mathcal{P} \mathbf{u})}{\partial \tilde{T}} \right] = 0 \\ \frac{\partial}{\partial t} \left( \frac{\partial \mathcal{P}}{\partial \Psi_1} \right) + \operatorname{div} \left[ \frac{\partial(\mathcal{P} \mathbf{u})}{\partial \Psi_1} + \frac{\partial \mathcal{P}}{\partial \tilde{T}} \frac{\partial \mathbf{u}}{\partial \mathbf{x}} \right] = 0 \\ \vdots \\ \frac{\partial}{\partial t} \left( \frac{\partial \mathcal{P}}{\partial \Psi_n} \right) + \operatorname{div} \left[ \frac{\partial(\mathcal{P} \mathbf{u})}{\partial \Psi_n} + C_n^1 \left( \frac{\partial \mathcal{P}}{\partial \Psi_{n-1}} \otimes \frac{\partial \mathbf{u}}{\partial \mathbf{x}} \right) + \dots + C_n^p \left( \frac{\partial \mathcal{P}}{\partial \Psi_{n-p}} \otimes \frac{\partial^p \mathbf{u}}{\partial \mathbf{x}^p} \right) + \dots + \frac{\partial \mathcal{P}}{\partial \tilde{T}} \frac{\partial^n \mathbf{u}}{\partial \mathbf{x}^n} \right] = 0 \\ \frac{\partial}{\partial t} \left( \frac{\partial \mathcal{P}}{\partial \mathbf{u}} \right) + \operatorname{div} \left[ \frac{\partial(\mathcal{P} \mathbf{u})}{\partial \mathbf{u}} - \frac{\partial \mathcal{P}}{\partial \tilde{T}} \frac{\partial}{\partial \mathbf{x}} \left( \Psi_1 - \operatorname{div}_2 \Psi_2 + \dots + (-1)^{n-1} \operatorname{div}_{n-1} \Psi_{n-1} \right) \right. \\ \left. - \frac{\partial \mathcal{P}}{\partial \tilde{\mu}} \frac{\partial}{\partial \mathbf{x}} \left( \Phi_1 - \operatorname{div}_2 \Phi_2 + \dots + (-1)^{n-1} \operatorname{div}_{n-1} \Phi_{n-1} \right) - C_n - D_n \right] = 0, \end{array} \right. \quad (20)$$

### 3.3. Symmetric form and stability of constant states

System (20) admits constant solutions ( $\rho_e, \eta_e, \mathbf{u}_e, \nabla \rho_e = 0, \dots, \nabla^n \rho_e = 0, \nabla \eta_e = 0, \dots, \nabla^n \eta_e = 0$ ). Since the governing equations are invariant under Galilean transformation, we can assume that  $\mathbf{u}_e = 0$ . Near equilibrium, we

look for the solutions of the linearized system which are proportional in the direction  $\mathbf{k}$  to  $e^{i(x-\lambda t)}$ , where  $x$  is the scalar coordinate in this spread direction,  $\lambda$  is a constant and  $i^2 = -1$ . We denote  $u$  as a scalar corresponding to the velocity in the direction  $\mathbf{k}$  of  $\mathbf{u}$  ( $\mathbf{u} = u \mathbf{k}$ ). We denote

$$\mathbf{U} = U_0 e^{i(x-\lambda t)},$$

the general form of the perturbations with

$$\mathbf{U}^* = [R, \Phi_1, \dots, \Phi_n, \tilde{T}, \Psi_1, \dots, \Psi_n, u] \quad \text{and} \quad \mathbf{U}_0^* = [R_0, \Phi_{10}, \dots, \Phi_{n0}, \tilde{T}_0, \Psi_{10}, \dots, \Psi_{n0}, u_0].$$

We obtain

$$\frac{\partial}{\partial t} \left( \frac{\partial \mathcal{P}}{\partial \mathbf{U}} \right)_e = \frac{\partial}{\partial \mathbf{U}} \left( \frac{\partial \mathcal{P}}{\partial \mathbf{U}} \right)_e \frac{\partial \mathbf{U}}{\partial t} = -i\lambda \frac{\partial}{\partial \mathbf{U}} \left( \frac{\partial \mathcal{P}}{\partial \mathbf{U}} \right)_e U_0 e^{i(x-\lambda t)},$$

where subscript  $e$  means the values at equilibrium and we denote

$$\mathbf{G} \equiv \frac{\partial}{\partial \mathbf{U}} \left( \frac{\partial \mathcal{P} \mathbf{u}}{\partial \mathbf{U}} \right)_e^*.$$

From

$$\text{div} \left( \frac{\partial \mathcal{P} \mathbf{u}}{\partial \mathbf{U}} \right) = \frac{\partial}{\partial x} \left( \frac{\partial \mathcal{P} \mathbf{u}}{\partial \mathbf{U}} \right)^* = \frac{\partial}{\partial \mathbf{U}} \left( \frac{\partial \mathcal{P} \mathbf{u}}{\partial \mathbf{U}} \right)^* \frac{\partial \mathbf{U}}{\partial x},$$

we get,

$$\text{div} \left( \frac{\partial \mathcal{P} \mathbf{u}}{\partial \mathbf{U}} \right)_e = i \mathbf{G} U_0 e^{i(x-\lambda t)}.$$

At equilibrium,  $\nabla \rho_e = 0, \dots, \nabla^n \rho_e = 0, \nabla \eta_e = 0, \dots, \nabla^n \eta_e = 0$ , which implies

$$\begin{aligned} \left( \frac{\partial \mathcal{P} \mathbf{u}}{\partial \Phi_l} \right)_e = 0, \dots, \left( \frac{\partial \mathcal{P}}{\partial \Phi_n} \right)_e = 0, \quad \left( \frac{\partial \mathcal{P}}{\partial \Psi_l} \right)_e = 0, \dots, \left( \frac{\partial \mathcal{P} \mathbf{u}}{\partial \Psi_n} \right)_e = 0, \\ \left( \frac{\partial \mathcal{P}}{\partial R} \right)_e \frac{\partial^p \mathbf{u}}{\partial x^p} = \rho_e \frac{\partial^p \mathbf{u}}{\partial x^p} = i^p \rho_e \mathbf{u}_0 e^{i(x-\lambda t)} \quad \text{and} \quad \left( \frac{\partial \mathcal{P}}{\partial \tilde{T}} \right)_e \frac{\partial^p \mathbf{u}}{\partial x^p} = \eta_e \frac{\partial^p \mathbf{u}}{\partial x^p} = i^p \eta_e \mathbf{u}_0 e^{i(x-\lambda t)}. \end{aligned}$$

Due to

$$\begin{aligned} \left( \frac{\partial \mathcal{P}}{\partial R} \right)_e &= \rho_e, \quad \frac{\partial}{\partial x} [(-1)^p \text{div}_{p-1} \Phi_p] = (-1)^p i^p \Phi_{p0} e^{i(x-\lambda t)}, \\ \left( \frac{\partial \mathcal{P}}{\partial \tilde{T}} \right)_e &= \eta_e, \quad \frac{\partial}{\partial x} [(-1)^p \text{div}_{p-1} \Psi_p] = (-1)^p i^p \Psi_{p0} e^{i(x-\lambda t)}, \end{aligned}$$

the last equation in system (20) becomes

$$\frac{\partial}{\partial t} \left( \frac{\partial \mathcal{P}}{\partial \mathbf{u}} \right) + \text{div} \left[ \frac{\partial (\mathcal{P} \mathbf{u})}{\partial \mathbf{u}} \right] - \rho_e (i^2 \Phi_1 - i^3 \Phi_2 + \dots + (-1)^{n+1} i^{n+1} \Phi_n) - \eta_e (i^2 \Psi_1 - i^3 \Psi_2 + \dots + (-1)^{n+1} i^{n+1} \Psi_n) = 0.$$

We denote

$$\mathbf{A} = \frac{\partial}{\partial \mathbf{U}} \left( \frac{\partial \mathcal{P}}{\partial \mathbf{U}} \right)_e^*.$$

which is a symmetric matrix. From the relations

$$\text{div} (i^p \rho_e \mathbf{u}) = i^{p+1} \rho_e \mathbf{u}_0 e^{i(x-\lambda t)} = i^{p+1} \rho_e \mathbf{u} \quad \text{and} \quad \text{div} (i^p \eta_e \mathbf{u}) = i^{p+1} \eta_e \mathbf{u}_0 e^{i(x-\lambda t)} = i^{p+1} \eta_e \mathbf{u},$$

System (20) writes

$$\left\{ \begin{array}{l} \frac{\partial}{\partial t} \left( \frac{\partial \mathcal{P}}{\partial \tilde{\mu}} \right) + \operatorname{div} \left[ \frac{\partial (\mathcal{P} \mathbf{u})}{\partial \tilde{\mu}} \right] = 0 \\ \frac{\partial}{\partial t} \left( \frac{\partial \mathcal{P}}{\partial \Phi_l} \right) + \operatorname{div} \left[ \frac{\partial (\mathcal{P} \mathbf{u})}{\partial \Phi_l} \right] + i^2 \rho_e \mathbf{u} = 0 \\ \vdots \\ \frac{\partial}{\partial t} \left( \frac{\partial \mathcal{P}}{\partial \Phi_n} \right) + \operatorname{div} \left[ \frac{\partial (\mathcal{P} \mathbf{u})}{\partial \Phi_n} \right] + i^{n+1} \rho_e \mathbf{u} = 0 \\ \frac{\partial}{\partial t} \left( \frac{\partial \mathcal{P}}{\partial \tilde{T}} \right) + \operatorname{div} \left[ \frac{\partial (\mathcal{P} \mathbf{u})}{\partial \tilde{T}} \right] = 0 \\ \frac{\partial}{\partial t} \left( \frac{\partial \mathcal{P}}{\partial \Psi_l} \right) + \operatorname{div} \left[ \frac{\partial (\mathcal{P} \mathbf{u})}{\partial \Psi_l} \right] + i^2 \eta_e \mathbf{u} = 0 \\ \vdots \\ \frac{\partial}{\partial t} \left( \frac{\partial \mathcal{P}}{\partial \Psi_n} \right) + \operatorname{div} \left[ \frac{\partial (\mathcal{P} \mathbf{u})}{\partial \Psi_n} \right] + i^{n+1} \eta_e \mathbf{u} = 0 \\ \frac{\partial}{\partial t} \left( \frac{\partial \mathcal{P}}{\partial \mathbf{u}} \right) + \operatorname{div} \left[ \frac{\partial (\mathcal{P} \mathbf{u})}{\partial \mathbf{u}} \right] - \rho_e (i^2 \Phi_l - i^3 \Phi_2 + \dots + (-1)^{n+1} i^{n+1} \Phi_n) - \eta_e (i^2 \Psi_l - i^3 \Psi_2 + \dots + (-1)^{n+1} i^{n+1} \Psi_n) = 0, \end{array} \right.$$

which can be written in the form

$$-i \lambda \mathbf{A} \mathbf{U} + i \mathbf{G} \mathbf{U} + i^2 \mathbf{C} \mathbf{U} = \mathbf{0}, \quad (21)$$

where  $\mathbf{C}$  is a matrix with  $2(n+1)+1$  lines and  $2(n+1)+1$  columns which can be written as,

$$\mathbf{C} = \begin{pmatrix} 0, & 0, & 0, & 0, & 0, & 0, & 0, & 0, & 0 \\ 0, & 0, & 0, & 0, & 0, & 0, & 0, & 0, & i^2 \rho_e \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0, & 0, & 0, & 0, & 0, & 0, & 0, & 0, & i^{n+1} \rho_e \\ 0, & 0, & 0, & 0, & 0, & 0, & 0, & 0, & 0 \\ 0, & 0, & 0, & 0, & 0, & 0, & 0, & 0, & i^2 \eta_e \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0, & 0, & 0, & 0, & 0, & 0, & 0, & 0, & i^{n+1} \eta_e \\ 0, & -(-i)^2 \rho_e, & \dots, & -(-i)^{n+1} \rho_e, & 0, & -(-i)^2 \eta_e, & \dots, & -(-i)^{n+1} \eta_e, & 0 \end{pmatrix},$$

and Eq. (21) becomes :

$$i (\mathbf{G} + i \mathbf{C} - \lambda \mathbf{A}) \mathbf{U}_0 e^{i(x-\lambda t)} = 0.$$

Due to  $\overline{i \mathbf{C}}^* = i \mathbf{C}$ , matrix  $i \mathbf{C}$  is Hermitian operator; consequently,  $\mathbf{K} = \mathbf{G} + i \mathbf{C}$  is also an Hermitian operator, but  $\mathbf{A}$  is symmetric. The  $\lambda$ -roots of

$$(\mathbf{K} - \lambda \mathbf{A}) \mathbf{U}_0 = 0,$$

are the solutions of characteristic equation,

$$\det(\mathbf{K} - \lambda \mathbf{A}) = 0,$$

where  $\mathbf{U}_0$  is the eigenvector associated with eigenvalue  $\lambda$ . Near an equilibrium state, and when Legendre transformation  $\mathcal{P}$  of energy  $E$  is locally convex,  $\mathbf{A}$  is a positive definitive matrix and the eigenvalues  $\lambda$  are real. Consequently,

**Corollary 4.** *When  $E$  is locally convex, perturbations  $\mathbf{U}_0 e^{i(x-\lambda t)}$  are stable and the  $U$ -form is dispersive.*

#### 4. Conclusion

We have extended the cases of capillary fluids [48, 49] to the most general case of multi-gradient fluids in density and volumetric entropy. These fluids can be represented by an hyperbolic-parabolic system of equations. The divergence form of governing equations implies a system of Hermitian-symmetric equations constituting the most general

dispersive model of conservative fluids. The perturbations are stable in the domains where the total volumetric internal energy is a convex function of the main field of new variables. The multi-gradient fluids have common properties with simple systems of classical conservative fluids [44, 45]. Multi-gradient fluids correspond to fluid media typified by first integrals represented by Kelvin's theorems [50].

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## References

- [1] F. dell'Isola, P. Seppecher, A. Madeo, Beyond Euler-Cauchy continua: the structure of contact actions in Nth gradient generalized continua: a generalization of the Cauchy tetrahedron argument. In CISM Lecture Notes C-1006, ch. 2., Springer Berlin, 2011.
- [2] F. Dell'Isola, P. Seppecher, A. Della Corte, The postulations à la D'Alembert and à la Cauchy for higher gradient continuum theories are equivalent: a review of existing results, Proc. R. Soc. A, **471**, 2183 (2015).
- [3] U. Andreaus, I. Giorgio, A. Madeo, Modeling of the interaction between bone tissue and resorbable biomaterial as linear elastic materials with voids, J. Appl. Math. Phys., **66**, 209–237 (2015).
- [4] H. Gouin, The watering of tall trees - Embolization and recovery, J. Th. Biology, **369**, 42–50 (2015) & <http://arXiv:1404.4343>
- [5] Y. Yang, W.Y. Ching, A. Misra, Higher-order continuum theory applied to fracture simulation of nano-scale intergranular glassy film. J. Nanomech. Micromech., **1**, 60-71 (2011).
- [6] D. Scerrato, I. Giorgio, A. Madeo, A. Limam, F. Darve, A simple non-linear model for internal friction in modified concrete, Int. J. Eng. Sci. **80**, 136–152 (2014).
- [7] R.A. Toupin, Theories of elasticity with couple-stress. Arch. Ration. Mech. Anal., **17**, 85–112 (1964).
- [8] R.D. Mindlin, Second gradient of strain and surface tension in linear elasticity, Int. J. Solid Struct. **1**, 417–438 (1965).
- [9] J. D. van der Waals, The thermodynamic theory of capillarity under the hypothesis of continuous variation of density. Translation by J.S. Rowlinson, J. Stat. Phys., **20**, 200–244 (1979).
- [10] J. W. Cahn, J. E. Hilliard, Free energy of a nonuniform system. III. Nucleation in a two-component incompressible fluid, J. Chem. Phys., **31**, 688–699 (1959).
- [11] H. Gouin, Energy of interaction between solid surfaces and liquids, J. Phys. Chem. B, **102**, 1212–1218 (1998) & <http://arXiv:0801.4481>
- [12] H. Gouin, Liquid nanofilms. A mechanical model for the disjoining pressure, Int. J. Eng. Sci., **47**, 691–699 (2009) & <http://arXiv:0904.1809>
- [13] M. Găărăjeu, H. Gouin, G. Saccomandi, Scaling Navier-Stokes equation in nanotubes, Physics of fluids, **25**, 082003 (2013) & <http://arXiv:1311.2484>
- [14] H. Gouin, S. Gavrilyuk, Wetting problem for multi-component fluid mixtures, Physica A, **268**, 291–308 (1999) & <http://arXiv:0803.0275>
- [15] H. Gouin, T. Ruggeri, Mixtures of fluids involving entropy gradients and acceleration waves in interfacial layers, Eur. J. Mech. B/Fluids, **24**, 596-613 (2005) & <http://arXiv:0801.2096>
- [16] V.A. Eremeyev, F.D. Fischer, On the phase transitions in deformable solids, Zeitschrift für Angewandte Mathematik und Mechanik (ZAMM), **90**, 535–536 (2010).
- [17] H. Gouin, A. Muracchini, T. Ruggeri, Fluid mixtures in nanotubes, Physical Review E, **97**, 062152 (2018).
- [18] R. Evans, The nature of liquid-vapor interface and other topics in the statistical mechanics of non-uniform classical fluids, Adv. Phys., **28**, 143–200 (1979).
- [19] P. Casal, H. Gouin, Equation of motion of thermocapillary fluids, C. R. Acad. Sci. Paris, II, **306**, 99–104 (1988).
- [20] J.S. Rowlinson, B. Widom, Molecular theory of capillarity, Clarendon Press, Oxford and Google books, 2012.
- [21] M.H. Maitournam, Entropy and temperature gradients thermomechanics: Dissipation, heat conduction inequality and heat equation, C. R. Mécanique, **340**, 434–443 (2012).
- [22] A. Bertram, S. Forest, The thermodynamics of gradient elastoplasticity, Cont. Mech. Therm., **26**, 269–286 (2014).
- [23] H. Gouin, P. Seppecher, Temperature profile in a liquid-vapour interface near the critical point, Proc. R. Soc. A, **473**, 20170229 (2017) & <http://arXiv:1703.07302>
- [24] P. Germain, The method of the virtual power in continuum mechanics - Part 2: microstructure, SIAM J. Appl. Math., **25**, 556–575 (1973).
- [25] F. Dell'Isola, P. Seppecher, A. Della Corte, The postulations à la D'Alembert and à la Cauchy for higher gradient continuum theories are equivalent: a review of existing results, Proc. R. Soc. A, **471**, 2183 (2015).
- [26] B. Widom, What do we know that van der Waals did not know? Physica A, **263**, 500–515 (1999).
- [27] H. Gouin, G. Saccomandi, Travelling waves of density for a fourth-gradient model of fluids, Continuum Mech. Thermodyn. **28**, 1511–1523 (2016) & <http://arXiv:1703.00689>
- [28] T. Ruggeri, M. Sugiyama, Rational Extended Thermodynamics beyond the Monatomic Gas, Springer, Heidelberg, 2015.
- [29] S. Godunov, An interesting class of quasilinear systems, Sov. Math. Dokl., **2**, 947–949 (1961).
- [30] K.O. Friedrichs, P.D. Lax, Systems of conservation equations with a convex extension, Proc. Nat. Acad. Sci. USA, **68**, 1686–1688 (1971).
- [31] G. Boillat, Sur l'existence et la recherche d'équations de conservation supplémentaires pour les systèmes hyperboliques, C. R. Acad. Sci. Paris A, **278**, 909–912 (1974).
- [32] G. Boillat, Non Linear Hyperbolic Fields and Waves, in Lectures Notes in Mathematics 1640 Ed.: T. Ruggeri, pp. 1-47, 1986.
- [33] I.-S. Liu, Method of Lagrange multipliers for exploitation of the entropy principle, Arch. Ration. Mech. Anal., **46**, 131-148 (1972).
- [34] T. Ruggeri, A. Strumia, Main field and convex covariant density for quasi-linear hyperbolic systems, Relativistic fluid dynamics, Ann. Inst. H. Poincaré, Sect A **34**, 65–84 (1981).
- [35] G. Boillat, Involutions des systèmes conservatifs, C. R. Acad. Sci. Paris, **307**, 891–894 (1982).
- [36] C.M. Dafermos, Quasilinear hyperbolic systems with involutions, Arch. Ration. Mech. Anal., **94**, 373-389 (1986).

- [37] T. Ruggeri, Symmetric hyperbolic system of conservative equations for a viscous conducting fluid, *Acta Mechanica* **47**, 167–183 (1983).
- [38] T. Ruggeri, Galilean invariance and entropy principle for systems of balance laws. The structure of the extended thermodynamics, *Continuum Mech. Thermodyn.*, 1, 3–20 (1989).
- [39] T. Ruggeri, Convexity and symmetrization in relativistic theories. Privileged time-like congruence and entropy, *Continuum Mech. Thermodyn.*, 2, 163–177 (1990).
- [40] G. Boillat, T. Ruggeri, Hyperbolic principal subsystems: entropy convexity and subcharacteristic conditions, *Arch. Rat. Mech. Anal.*, **137**, 307–320 (1997).
- [41] A. Muracchini, T. Ruggeri, L. Seccia, Dispersion relation in the high frequency limit and non linear wave stability for hyperbolic dissipative systems, *Wave Motion*, **15**, 143–158 (1992).
- [42] Z. Banach, W. Larecki, T. Ruggeri, Dispersion relation in the limit of high frequency for a hyperbolic system with multiple eigenvalues, *Wave Motion*, **51**, 955–966 (2014).
- [43] S. Kawashima, Y. Shizuta, On the normal form of the symmetric hyperbolic-parabolic systems associated with the conservation laws, *Tohoku Math. Journal*, **40**, 449–464 (1988).
- [44] H. Gouin, Multi-gradient fluids, *Ricerche di Matematica*, Online first, DOI <https://doi.org/10.1007/s11587-018-0397-5>, (2018) & <http://arXiv:1803.06115>
- [45] J. Serrin, *Mathematical Principles of Classical Fluid Mechanics*, *Encyclopedia of Physics VIII/1*, S. Flügge, ed., Springer, Berlin, 1960.
- [46] P. Casal, H. Gouin, Connexion between the energy equation and the motion equation in Korteweg’s theory of capillarity, *C.R. Acad. Sci. Paris, II*, **300**, 231–234 (1985).
- [47] C. Truesdell, *Introduction à la mécanique rationnelle des milieux continus*, Masson, Paris, 1974.
- [48] S. Gavriljuk, H. Gouin, Symmetric form of governing equations for capillary fluids, *Monographs and Surveys in Pure and Applied Mathematics*, vol. **106**, Ed. G. Iooss, O. Guès, A. Nouri, pp. 306–312, Chapman & Hall/CRC, London, 2000 & <http://arXiv:0802.1670>
- [49] S. Benzoni-Gavage, R. Danchin, S. Descombes, D. Jamet, On Korteweg models for fluids exhibiting phase changes. *Proceedings of 10th International Conference on Hyperbolic Problems*, Osaka, Yokohama Publishers, 2004.
- [50] H. Gouin, Noether theorem in fluid mechanics, *Mech. Res. Comm.* **3**, 151–155 (1976).