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Abstract

This paper deals with the sampled-data control problem based on state estimation for uncertain linear sampled-data systems. It is possible to show that the sampled-data control problem based on state estimation may be related with the conditions for the exponential stability of impulsive systems. Thus, a vector Lyapunov function-based approach, derived by means of a $2D$ time domain equivalence, is used for obtaining stability conditions of an impulsive system, and then, a solution to the observer-based control design problem is derived and expressed in terms of LMIs. Some examples illustrate the feasibility of the proposed approach.

Keywords: Observer-based Control, Sampled-Data, Impulsive systems.

1. Introduction

In the last decades, an enormous interest has appeared in the design of controllers and observers for continuous and/or discrete dynamical systems with communication constraints. This interest has its motivations in systems with sampled-data control, quantization and more generally, in networked control systems. However, all the communications constraints, *i.e.* delays, sampling intervals, quantization, packet dropouts, and so on (for details, see [16]); imply additional difficulties in the analysis and design compared to the classical control systems. Regarding the observer design problem, one of the main issues is the scheduling: only a subset of sensors is allowed to send their data to the observer at the transmission instants. The sporadic and partial availability of system measurements requires the development of appropriate observer designs. Moreover, for controller design, it would be unreasonable to assume that all states are measurable. Therefore an observer-based control approach is needed.

In this paper the observer-based control problem will be in the focus for sampled-data systems. Several methods have been developed to study sampled-data systems, *e.g.* the Input/Output stability approach [14], the discrete-time approach [13], but two approaches stand out: the input delay approach, where the system is modeled as a continuous system with a delay in the control input (see, *e.g.* [10, 11]), and the impulsive system approach, where the sampled-data system is treated as an impulsive system (see, *e.g.* [3, 4, 5, 29]).

The input delay approach has been applied in [12] to design a sampled-data output-feedback H_∞ control for linear systems while the impulsive system approach was applied in [17] to sampled-data stabilization of linear uncertain systems in the case of constant sampling based on piecewise linear in time Lyapunov function. The case of variable sampling based on a discontinuous Lyapunov function method was introduced by [23]. Also based on discontinuous Lyapunov functions, in [3] stability and stabilization conditions for periodic and aperiodic sampled-data systems are introduced.

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In the context of observer design, one approach is based on continuous and discrete design. In [7], such an approach is used to design a continuous-discrete version of the high-gain observer for nonlinear systems. In [18] a continuous-discrete observer is proposed for linear and triangular Lipschitz systems based on a sampled-data nonlinear observer that is designed using a continuous-time approach together with an inter-sample output predictor. Applying a small gain approach, in [1] an observer design is proposed for certain classes of nonlinear systems with sampled and delayed measurements. A Luenberger-like observer is proposed by [8] for a class of continuous-time dynamical systems with non-uniformly sampled measurements. In [22], continuous-time systems with sampled uncertain output are considered and the state estimation problem is solved by means of continuous-discrete interval observers that are asymptotically stable in the absence of disturbances. In [21], based on the notion of cooperative systems, a design for continuous-discrete observers is proposed for continuous nonlinear time-varying systems with discrete-time measurements. Using the hybrid system approach, in [6] an observer-protocol pair is designed to estimate the states of a linear system under communication constraints induced by the network. In the same vein, in [9] an observer with jumps triggered by incoming measurements is proposed to deal with the state estimation problem for linear time-invariant systems for which measurements of the output are available sporadically. Adopting a switched observer structure, in [2] decentralized observer-based output-feedback controllers are proposed for linear systems connected via a shared communication network.

In this paper a vector Lyapunov function-based approach [19], derived by means of a $2D$ time domain equivalence (see, *e.g.* [28] and [31]), for stability of impulsive systems is used for designing a robust output-feedback control for linear sampled-data systems. Such an approach, proposed in [26] and [27], provides a stability analysis based on linear matrix inequalities (LMIs) for linear impulsive dynamical systems. Then, it is possible to show that the sampled-data control problem based on state estimation may turn into one of finding conditions for the exponential stability of impulsive systems. Thus, the proposed vector Lyapunov function approach is applied for obtaining stability conditions of the impulsive system, and then, a solution to the robust output-feedback control design problem is obtained and expressed in terms of LMIs. To the best of our knowledge, the output-feedback control design for uncertain sampled-data system is open in the literature and there exist very few works dealing with such a problem. Moreover, it is worth highlighting that a direct application of the methods given in the literature, *e.g.* those ones from [3], [4], or in [15], do not provide a constructive method to solve the robust output-feedback control design problem for uncertain linear sampled-data systems.

The outline of this work is as follows. A motivating problem is given in Section 2. Some stability results for impulsive systems are given in Section 3. The main result is described in Section 4. Some simulation results are depicted in Section 5 while some concluding remarks are discussed in Section 6. The corresponding proofs for the main results are postponed to the Appendix.

2. Motivation

Let us consider the following uncertain sampled-data system

$$\dot{x}(t) = Ax(t) + Bu(t) + f(x(t)),\ x(0) = x_0, \tag{1}$$
$$y(t) = Cx(t_i),\ \forall t \in [t_i, t_{i+1}), \tag{2}$$
$$u(t) = K\hat{x}(t_i),\ \forall t \in [t_i, t_{i+1}), \tag{3}$$

where $x,\ x_0 \in \mathbb{R}^n$ are the state vector and the initial condition, respectively, $u \in \mathbb{R}^m$ is the sampled control vector, and $y \in \mathbb{R}^p$ is the sampled output vector at each time t_i for all $i \in \mathbb{N}$, and $\hat{x} \in \mathbb{R}^n$ represents an estimation of the system state x. The function $f : \mathbb{R}^n \to \mathbb{R}^n$ represents all the parameter uncertainties of the system satisfying $|f(x)|^2 \leq f_0|x|^2$, *i.e.* the function f is Lipschitz. The constant matrices A, B, and C have corresponding dimensions while K is a design control matrix.

The sampling instants t_i are monotonously increasing, such that $\lim_{i\to\infty} t_i = +\infty$, and $T_i := t_{i+1} - t_i \in [T_{\min}, T_{\max}]$, where $T_{\min} > 0$ and $T_{\max} > 0$ are the minimum and maximum sampling intervals, respectively; and $t_0 = 0$. The control u is designed by means of the following sampled-data state observer

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + L(y(t) - C\hat{x}(t_i)),\ \forall t \in [t_i, t_{i+1}),\ \hat{x}(0) = \hat{x}_0, \tag{4}$$

where $\hat{x},\ \hat{x}_0 \in \mathbb{R}^n$ are the estimated state vector and its initial condition, and $L \in \mathbb{R}^{n\times p}$ is a design observer matrix. Define the state estimation error $e(t) := x(t) - \hat{x}(t)$. Then, the closed-loop and state estimation error dynamics are given as follows

$$\dot{x}(t) = Ax(t) + BKx(t_i) - BKe(t_i) + f(x(t)),\ \forall t \in [t_i, t_{i+1}),$$
$$\dot{e}(t) = Ae(t) - LCe(t_i) + f(x(t)),\ \forall t \in [t_i, t_{i+1}).$$

Let us define the extended state vector $\xi(t) := (x^T(t)\ e^T(t)\ x^T(t_i)\ e^T(t_i))^T \in \mathbb{R}^{4n}$ and the timer variable $\tau \in \mathbb{R}_{\geq 0}$. Then, the above dynamics may be written as follows

$$\frac{d}{dt}\begin{pmatrix} \xi \\ \tau \end{pmatrix} = \begin{pmatrix} A_\xi \xi + D_\xi f \\ 1 \end{pmatrix},\ \forall \tau \in [0, T_i),\ i = 0, 1, 2, \ldots,\ (\xi(0), \tau(0)) = (\xi_0, \tau_0), \tag{5}$$
$$\begin{pmatrix} \xi^+ \\ \tau^+ \end{pmatrix} = \begin{pmatrix} I_\xi \xi \\ 0 \end{pmatrix},\ \forall \tau = T_i,\ i = 0, 1, 2, \ldots \tag{6}$$

where $(\xi, \tau),\ (\xi_0, \tau_0) \in \mathbb{R}^{4n} \times [0, T_i]$ represents the current state vector and its initial condition, $(\xi^+, \tau^+) \in \mathbb{R}^{4n} \times T_i$ represents the reset state vector, $T_i \in [T_{\min}, T_{\max}]$ is the sampling interval given for $i = 0, 1, 2, \ldots$, and $f \in \mathbb{R}^n$ denotes the uncertainty. The corresponding matrices have the following structure

$$A_\xi = \begin{pmatrix} A & \mathbf{0} & BK & -BK \\ \mathbf{0} & A & \mathbf{0} & -LC \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix},\ D_\xi = \begin{pmatrix} I_n \\ I_n \\ \mathbf{0} \\ \mathbf{0} \end{pmatrix},\ I_\xi = \begin{pmatrix} I_n & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & I_n & \mathbf{0} & \mathbf{0} \\ I_n & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & I_n & \mathbf{0} & \mathbf{0} \end{pmatrix}.$$

Note that in absence of uncertainties, *i.e.* $f = 0$, the set $\{(\xi, \tau) |\ \xi = 0,\ \tau \in [0, T_i]\}$ is an equilibrium set of (5). The dynamics (5)-(6) describes *periodic/aperiodic* time-triggered jumps, when $\tau = T_i$, governed by the map given by (6), while between the jumps the system behaves according to (5). Note also that, due to the linearity of the system and the facts that f is Lipschitz and $T_i \in [T_{\min}, T_{\max}]$, the existence of a unique forward solution is ensured.

Then, the sampled-data control problem based on state estimation, *i.e.* find the control gain matrix K and the observer gain matrix L, may turn into one of finding conditions for the stability of the impulsive system described by (5)-(6), under arbitrary variations of the sampling intervals.

In the following sections such conditions for the stability of the impulsive system are derived by means of a $2D$ time domain equivalence and a vector Lyapunov function approach. Afterwards, these conditions will be applied to solve the sampled-data control problem based on state estimation. All the proofs are given in the Appendix.

3. Stability Analysis for Impulsive Systems

The stability analysis relies on the embedding of system (5)-(6) into a $2D$ time domain. Indeed, the entire state trajectory (ξ, τ) can be viewed as a sequence of the *diagonal dynamics*[1] of the following $2D$ system:

$$\frac{d}{dt}\begin{pmatrix} \xi_k^t \\ \tau_k^t \end{pmatrix} = \begin{pmatrix} A_\xi \xi_k^t + D_\xi f \\ 1 \end{pmatrix},\ \forall \tau_k^t \in [0, T_i),\ \forall i = k = 0, 1, 2, \ldots,\ (\xi(0,0), \tau(0,0)) = (\xi_0^0, \tau_0^0), \tag{7}$$
$$\begin{pmatrix} \xi_{k+1}^{t_{i+1}} \\ \tau_{k+1}^{t_{i+1}} \end{pmatrix} = \begin{pmatrix} I_\xi \xi_k^{t_{i+1}} \\ 0 \end{pmatrix},\ \forall \tau_k^t = T_i,\ \forall i = k = 0, 1, 2, \ldots \tag{8}$$

where $(\xi_k^t, \tau_k^t) := (\xi(t,k), \tau(t,k)),\ (\xi_0^0, \tau_0^0) \in \mathbb{R}^{4n} \times [0, T_i]$ is the current state vector and its initial condition, $(\xi_{k+1}^{t_{i+1}}, \tau_{k+1}^{t_{i+1}}) := (\xi(t_{i+1}, k+1), \tau(t_{i+1}, k+1)) \in \mathbb{R}^{4n} \times T_i$ represents the reset state vector, while $(\xi_k^{t_{i+1}}, \tau_k^{t_{i+1}}) := (\xi^T(t_{i+1}, k), \tau(t_{i+1}, k))^T \in \mathbb{R}^{4n} \times T_i$ denotes the value of (ξ, τ) just before the jump $k+1$. Taking into account that f is Lipschitz and $T_i \in [T_{\min}, T_{\max}]$, the solutions of (7)-(8) for the *diagonal dynamics, i.e.* for all $i = k$, correspond to the solutions of the system (5)-(6). Note that the discrete time k depicts the number of impulses in the system.

In the present section some definitions and results for the stability of impulsive systems, in the framework of $2D$ systems, are introduced (see [26] and [27]).

[1]The diagonal dynamics make reference only to those dynamics given by (7)-(8) corresponding to $i = k$, for all $i, k \in \mathbb{N}$ and for all $t \in \mathbb{R}_{\geq 0}$.

Let $|q|$ denote the Euclidean norm of a vector q. The following stability definition is introduced:

Definition 1. [26]. *A 2D system described by (7)-(8), is said to be* exponentially diagonal ξ_k^t-stable (EDξ_k^t-S) *if there exist positive constants κ_1, κ_2, κ_3, and c such that $0 < \kappa_1 < 1$ and*

$$|\xi_{k+1}^{t_{i+1}}|^2 \leq c\kappa_1^{k+1}|\xi_0^0|^2, \ \forall \tau_k^t = T_i, \tag{9}$$

$$|\xi_k^t|^2 \leq \kappa_2 |\xi_k^{t_i}|^2, \ \forall \tau_k^t \in [0, T_i], \tag{10}$$

$$|\tau_k^t| \leq \kappa_3, \tag{11}$$

for all $i = k \in \mathbb{N}$.

Note that condition (11) holds by definition, *i.e.* $|\tau_k^t| \leq \kappa_3$, with $\kappa_3 = T_{\max}$. Denote $z_k^t := ((\xi_k^t)^T, \tau_k^t)^T$. In order to give the stability conditions a vector Lyapunov approach is used, *i.e.*

$$V(z_k^t, z_{k+1}^{t_{i+1}}) = \begin{pmatrix} V_1(z_k^t) \\ V_2(z_{k+1}^{t_{i+1}}) \end{pmatrix},$$

where $V_1(\cdot) > 0$, $V_2(\cdot) > 0$, for all z_k^t and $z_{k+1}^{t_{i+1}}$, and $V_1(0) = 0$, $V_2(0) = 0$. Now, let us introduce the following definition.

Definition 2. *The divergence operator of a function V along the trajectories of system (7)-(8) is defined for all $t \in [t_i, t_{i+1})$ as follows*

$$\mathrm{div}V(z_k^t, z_{k+1}^{t_{i+1}}) = \frac{dV_1(z_k^t)}{dt} + V_2(z_{k+1}^{t_{i+1}}) - V_2(z_k^{t_{i+1}}). \tag{12}$$

Note that V_1 is differentiable with respect to continuous time t while the difference in V_2 is calculated in discrete time k. Thus, the following theorem is introduced.

Theorem 1. [26]. *Assume that there exist positive constants ε, c_1, c_2, c_3, c_4 and c_5 such that the vector Lyapunov function $V(z_k^t, z_{k+1}^{t_{i+1}})$ and its divergence along the trajectories of the system (7)-(8) satisfy, for all $\tau_k^t \in [0, T_i]$, $i = k \in \mathbb{N}$, the following inequalities:*

$$c_1|\xi_k^t|^2 \leq V_1(z_k^t) \leq c_2|\xi_k^t|^2, \tag{13}$$

$$c_3|\xi_k^p|^2 \leq V_2(z_k^p) \leq c_4|\xi_k^p|^2, \ \forall p = t_i, t_{i+1} \tag{14}$$

$$\mathrm{div}V \leq -c_5(|\xi_k^t|^2 + |\xi_k^{t_{i+1}}|^2), \tag{15}$$

$$c_2\,(c_4 - c_5) \leq c_1 c_5 \ \vee \ T_i \leq \frac{c_2}{c_5}\alpha, \tag{16}$$

$$\frac{c_2}{c_5}\gamma \leq T_i, \tag{17}$$

where $\gamma = -\ln\left[\frac{c_3(1-\varepsilon)}{c_5+c_3(1-\varepsilon)}\right]$ and $\alpha = -\ln\left[\frac{c_2(c_4-c_5)-c_1c_5}{c_2(c_4-c_5)}\right]$ *for all* $c_2\,(c_4 - c_5) > c_1c_5$. *Then, the 2D system (7)-(8) is* EDξ_k^t-S *for any sequence $\{T_i\}_{i\in\mathbb{N}}$ such that $T_i \in [\frac{c_2}{c_5}\gamma, \frac{c_2}{c_5}\alpha]$.*

The statement given by Theorem 1 relies on a vector Lyapunov function approach in contrast to the results given in [15] (similarly in [24]), where *asymptotic stability* is obtained by means of a single Lyapunov function that needs to have a negative semi-definite derivative. Alternatively, our divergence operator, and not each term, needs to satisfy inequality (15).

Remark 1. *The constructive application of Theorem 1 is illustrated by Algorithm 1 which provides some notions of* minimum *and* maximum or ranged dwell-time *depending on the structure of the system dynamics. In particular, the first and third cases for exponential diagonal stability (pseudo-code lines: 5 and 13, Algorithm 1) give conditions for* minimum dwell-time *while the second case (pseudo-code lines: 7, Algorithm 1) provides conditions for* maximum or ranged dwell-time*.*

Algorithm 1 Exponential Stability

Define the Lyapunov functions V_1 and V_2
Calculate the constants c_1, c_2, c_3, c_4 and c_5
if $c_4 > c_5$ then
    if $c_2\,(c_4 - c_5) \leq c_1 c_5$ and $T_i > \frac{c_2}{c_5}\gamma$ then
        "System is EDξ_k^t-S"
    else if $\frac{c_2}{c_5}\alpha > T_i > \frac{c_2}{c_5}\gamma$ then
        "System is EDξ_k^t-S"
    else
        "No Conclusion"
    end
else
    if $T_i > \frac{c_2}{c_5}\gamma$ then
        "System is EDξ_k^t-S"
    else
        "No Conclusion"
    end
end
end Algorithm

3.1. Exponential Diagonal ξ_k^t-Stability: Quadratic Lyapunov Functions

Consider that V_1 and V_2 take the following quadratic structure

$$V(z_k^t, z_{k+1}^{t_{i+1}}) = \begin{pmatrix} (\xi_k^t)^T P_1(\tau_k^t)\xi_k^t \\ (\xi_{k+1}^{t_{i+1}})^T P_2(\tau_{k+1}^{t_{i+1}})\xi_{k+1}^{t_{i+1}} \end{pmatrix}, \tag{18}$$

where $P_1 \in \mathbb{R}^{4n\times 4n}$ is continuously differentiable with respect to t, symmetric, bounded, and positive definite matrix for all $\tau_k^t \in [0, T_i]$, $i = k \in \mathbb{N}$, while $P_2 \in \mathbb{R}^{4n\times 4n}$ is a symmetric and positive definite matrix, *i.e.*

$$0 < c_1 I \leq P_1(\tau_k^t) \leq c_2 I,\ \forall \tau_k^t \in [0, T_i], \tag{19}$$

$$0 < c_3 I \leq P_2(\tau_k^p) \leq c_4 I,\ p = t_i, t_{i+1}. \tag{20}$$

Thus, based on the previous choice for V_1 and V_2, if Theorem 1 is applied to the ideal and uncertain impulsive system (5)-(6), then the following results are obtained.

Corollary 1. *Consider the vector Lyapunov function $V(z_k^t, z_{k+1}^{t_{i+1}})$ in (18). Assume that there exist matrices $P_1(\tau_k^t) = P_1^T(\tau_k^t) > 0$, continuously differentiable on t and bounded for all $\tau_k^t \in [0, T_i]$, $i = k \in \mathbb{N}$, $P_2(0) = P_2^T(0) > 0$ and $P_2(T_i) = P_2^T(T_i) > 0$ satisfying (19)-(20), $\Lambda = \Lambda^T > 0$ and a constant $c_5 > 0$, such that the following matrix inequality*

$$\begin{pmatrix} P_1(\tau_k^t)A_\xi + A_\xi^T P_1(\tau_k^t) + \frac{dP_1(\tau_k^t)}{dt} + 2f_0\Lambda + c_5 I_{4n} & \mathbf{0} & P_1(\tau_k^t) \\ \star & I_\xi^T P_2(0) I_\xi - P_2(T_i) + c_5 I_{4n} & \mathbf{0} \\ \star & \star & -\Lambda \end{pmatrix} \leq 0, \tag{21}$$

holds for all $\tau_k^t \in [0, T_i]$, $i = k \in \mathbb{N}$, $|f|^2 \leq f_0|x|^2$, and constraints (16)-(17) are satisfied with c_1, c_2, c_3, c_4 and c_5. Then the system (7)-(8) is EDξ_k^t-S *for any sequence $\{T_i\}_{i\in\mathbb{N}}$ such that $T_i \in [\frac{c_2}{c_5}\gamma, \frac{c_2}{c_5}\alpha]$.*

Now, the following result is established for the ideal impulsive system, *i.e.* $f = 0$.

Corollary 2. *Consider the vector Lyapunov function $V(z_k^t, z_{k+1}^{t_{i+1}})$ in (18). Assume that there exist matrices $P_1(\tau_k^t) = P_1^T(\tau_k^t) > 0$, continuously differentiable on t and bounded for all $\tau_k^t \in [0, T_i]$, $i = k \in \mathbb{N}$, $P_2(0) = P_2^T(0) > 0$ and $P_2(T_i) = P_2^T(T_i) > 0$ satisfying (19)-(20), and a constant $c_5 > 0$, such that the following matrix inequality*

$$\begin{pmatrix} P_1(\tau_k^t)A_\xi + A_\xi^T P_1(\tau_k^t) + \frac{dP_1(\tau_k^t)}{dt} + c_5 I_{4n} & \mathbf{0} \\ \mathbf{0} & I_\xi^T P_2(0) I_\xi - P_2(T_i) + c_5 I_{4n} \end{pmatrix} \leq 0, \tag{22}$$

holds for all $\tau_k^t \in [0, T_i]$*, for all* $i = k \in \mathbb{N}$*, and constraints* (16)-(17) *are satisfied with* c_1*,* c_2*,* c_3*,* c_4 *and* c_5*. Then the system* (7)-(8)*, with* $f = 0$*, is* EDξ_k^t-S *for any sequence* $\{T_i\}_{i\in\mathbb{N}}$ *such that* $T_i \in [\frac{c_2}{c_5}\gamma, \frac{c_2}{c_5}\alpha]$.

For the particular case of linear time-invariant systems, our method can be seen as a generalization of the result in [3]. In fact, taking $P_1(\tau_k^t)$ and $P_2(\tau_k^t)$ in the same form for the statements given by Corollary 2, one leads to the conditions given by Theorem 2.2 (ranged dwell-time) and 2.3 (minimum dwell-time) in [3].

It is also worth mentioning that the results given by Theorem 1 and Corollary 2 are consistent with the ones in Proposition 3.24 (Persistent Flowing) and Proposition 3.27 (Persistent Jumping) in [15], respectively.

Note that Corollaries 1 and 2 are able to deal with linear impulsive systems where matrix A_ξ is not Hurwitz, and/or I_ξ is anti-Schur, respectively. In this sense, Corollaries 1 and 2 provide general results to deal with the stability of linear impulsive systems.

In the following, a couple of examples, taken from [5], are presented in the framework of stability, not in the one of output-feedback control design, to illustrate the potential of the proposed method.

Example 1. Let us consider a system as in (7)-(8) with the following matrices

$$A_\xi = \begin{pmatrix} 1 & 3 \\ -1 & 2 \end{pmatrix}, \quad I_\xi = \begin{pmatrix} 0.5 & 0 \\ 0 & 0.5 \end{pmatrix}.$$

Note that the continuous dynamics is unstable while the discrete one is stable, *i.e.* A_ξ is not Hurwitz and I_ξ is Schur. Corollary 2 with

$$P_1(\tau_k^t) = \frac{\tau_k^t P_{11} + (T_i - \tau_k^t)P_{12}}{T_i}, \; P_2(\tau_k^t) = P_{21} + \tau_k^t P_{22},$$

and a bisection algorithm on $\tau_k^t \in [0, T_i]$ can be checked by solving LMIs given by (22). The following results are obtained:

$$P_{11} = \begin{pmatrix} 1.3914 & 0.0139 \\ 0.0139 & 1.3597 \end{pmatrix}, \; P_{12} = 10^4 \begin{pmatrix} 1.2638 & -0.6319 \\ -0.6319 & 3.7917 \end{pmatrix}, \; P_{21} = \begin{pmatrix} 1.0007 & 0 \\ 0 & 1.0007 \end{pmatrix}, \; P_{22} = \begin{pmatrix} 0.9336 & 0 \\ 0 & 0.9336 \end{pmatrix},$$

with $c_1 = 1.3545$, $c_2 = 3.9410 \times 10^4$, $c_3 = 1.0007$, $c_4 = 1.3118$ and $c_5 = 0.5000$. Therefore, according to Corollary 2, the impulsive system is EDξ_k^t-S for all $0.3333 > T_i > 0$. It is easy to check that $c_4 > c_5$ and $c_2(c_4 - c_5) > c_1 c_5$ hold and then, according to Algorithm 1, the second case for EDξ_k^t-S is obtained. When, as in [5], the analysis is restricted to single quadratic Lyapunov functions linear in τ_k^t, *i.e.* $P_1(\tau_k^t) = P_2(\tau_k^t)$, it is possible to show stability for $0.2400 > T_i > 0$.

Example 2. Let us consider a system as in (7)-(8) with the following matrices

$$A_\xi = \begin{pmatrix} -1 & 0 \\ 1 & -2 \end{pmatrix}, \quad I_\xi = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}.$$

For this example the continuous dynamics is stable while the discrete one is unstable, *i.e.* A_ξ is Hurwitz and I_ξ is anti-Schur. Corollary 2 with

$$P_1(\tau_k^t) = \frac{\tau_k^t P_{11} + (T_i - \tau_k^t)P_{12}}{T_i}, \; P_2(\tau_k^t) = P_{21} + \tau_k^t P_{22},$$

and a bisection algorithm on $\tau_k^t \in [0, T_i]$ can be checked by solving LMIs given by (22). The following results are obtained:

$$P_{11} = \begin{pmatrix} 3.3096 & 0.4527 \\ 0.4527 & 1.4411 \end{pmatrix}, \; P_{12} = \begin{pmatrix} 0.6648 & 0.0963 \\ 0.0963 & 0.3941 \end{pmatrix}, \; P_{21} = \begin{pmatrix} 0.1123 & 0 \\ 0 & 0.1123 \end{pmatrix}, \; P_{22} = \begin{pmatrix} 0.0050 & 0.0006 \\ 0.0006 & 0.0056 \end{pmatrix},$$

with $c_1 = 1.3372$, $c_2 = 4.1071$, $c_3 = 0.1123$, $c_4 = 6.05$ and $c_5 = 4.58$. Therefore, according to Corollary 2, the impulsive system is EDξ_k^t-S for all $T_i > 3.3254$. It is easy to check that $c_4 > c_5$ and $c_2(c_4 - c_5) \leq c_1 c_5$ hold and then, according to Algorithm 1, the first case for EDξ_k^t-S is obtained. When, as in [5], the analysis is restricted to single quadratic Lyapunov functions linear in τ_k^t, *i.e.* $P_1(\tau_k^t) = P_2(\tau_k^t)$, it is possible to show stability for $T_i > 5.1000$.

The previous examples show numerically that when the analysis is restricted to the same class of Lyapunov functions, *i.e.* linear with respect to τ_t^k, the vector Lyapunov function approach is less conservative than the scalar one. Nevertheless, theoretically speaking, it is very difficult to ensure that the proposed method provides, in general, less conservative results.

The rest of the paper is devoted to the application of the conditions for *exponential diagonal ξ_k^t-stability* of the impulsive systems (7)-(8), by means of the statements given by Corollaries 1 and 2, in order to solve the sampled-data control problem based on state estimation for system (1)-(3) in a constructive way.

4. Robust Output-Control Design

In this section a particular choice for P_1 and P_2 is proposed. Then, by means of the statements given by Corollaries 1 and 2, the control gain matrix K and the observer gain matrix L will be found to provide a stabilization of the state dynamics x as well as an estimation $\hat{x}$, *i.e.* stabilization of the extended state ξ in (5)-(6), for the ideal and uncertain case, respectively.

Thus, the following proposition gives a solution to the sampled-data control problem based on state estimation for the uncertain linear sampled-data system, *i.e.* $f \neq 0$.

Proposition 1. *Consider that P_1 and P_2 have the following structure for all $\tau_k^t \in [0, T_i]$, $i = k \in \mathbb{N}$*

$$P_1(\tau_k^t) = \frac{\tau_k^t P_{11} + (T_i - \tau_k^t) P_{12}}{T_i}, \quad P_2(\tau_k^t) = P_{21} + \tau_k^t P_{22},$$

where $P_{11} = \operatorname{diag}(P_{11}^{-(1)}, P_{11}^{-(2)}, P_{11}^{-(3)}, P_{11}^{-(3)})$, $P_{12} = \operatorname{diag}(P_{12}^{-(1)}, P_{12}^{-(2)}, \delta I_n, \delta I_n)$, with a fixed $\delta > 0$, $P_{1l}^{(q)} = (P_{1l}^{(q)})^T > 0$, and $P_{2l} = P_{2l}^T > 0$, for $l = 1, 2$, and $q = \overline{1,3}$. If there exist matrices $P_{11}^{(q)}$, $P_{2l} = P_{2l}^T > 0$, Y_K and L, for $l = 1, 2$, and $q = \overline{1,3}$, such that the following matrix inequalities

$$\begin{pmatrix}
\phi_{11}(\Theta)+I_n & \mathbf{0} & BY_K & -BY_K & \mathbf{0} & \mathbf{0} & P_{11}^{(1)} & P_{11}^{(1)} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\star & -R_1+I_n & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & P_{11}^{(2)} & P_{11}^{(2)} & \mathbf{0} & \mathbf{0} \\
\star & \star & \phi_3(\Theta) & \mathbf{0} & I_n & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & P_{11}^{(3)} & \mathbf{0} \\
\star & \star & \star & -R_2 & \mathbf{0} & I_n & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & P_{11}^{(3)} \\
\star & \star & \star & \star & -\Lambda_3 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\star & \star & \star & \star & \star & -\Lambda_4 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\star & \star & \star & \star & \star & \star & -Q_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\star & \star & \star & \star & \star & \star & \star & -\frac{I_n}{2f_0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\star & \star & \star & \star & \star & \star & \star & \star & -Q_2 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\star & \star & \star & \star & \star & \star & \star & \star & \star & -\frac{I_n}{2f_0} & \mathbf{0} & \mathbf{0} \\
\star & \star & \star & \star & \star & \star & \star & \star & \star & \star & -Q_3 & \mathbf{0} \\
\star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & -Q_4
\end{pmatrix} \leq 0, \tag{23}$$

$$\left(\begin{array}{ccccccccc|c}
\phi_{21}(\Theta)+I_n & \mathbf{0} & BY_K & -BY_K & \mathbf{0} & \mathbf{0} & P_{12}^{(1)} & P_{12}^{(1)} & P_{12}^{(1)} & \\
\star & -R_3+I_n & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \\
\star & \star & \phi_3(\Theta) & \mathbf{0} & \delta P_{11}^{(3)} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \\
\star & \star & \star & -R_4 & \mathbf{0} & \delta P_{11}^{(3)} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \\
\star & \star & \star & \star & -\Lambda_3 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathcal{P} \\
\star & \star & \star & \star & \star & -\Lambda_4 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \\
\star & \star & \star & \star & \star & \star & -Q_1 & \mathbf{0} & \mathbf{0} & \\
\star & \star & \star & \star & \star & \star & \star & -\frac{I_n}{2f_0} & \mathbf{0} & \\
\star & \star & \star & \star & \star & \star & \star & \star & -\Theta P_{11}^{(1)} & \\
\hline
\multicolumn{9}{c|}{\star} & \mathcal{Q}
\end{array}\right) \leq 0, \tag{24}$$

$$\mathcal{P} = \left(\begin{array}{cccc|c}
\mathbf{0}^T & P_{12}^{(2)} & \mathbf{0}^T & \mathbf{0}^T & \\
\mathbf{0}^T & P_{12}^{(2)} & \mathbf{0}^T & \mathbf{0}^T & \\
\mathbf{0}^T & P_{12}^{(2)} & \mathbf{0}^T & \mathbf{0}^T & \mathbf{0}^T \\
\mathbf{0}^T & \mathbf{0}^T & P_{11}^{(3)} & \mathbf{0}^T & \\
\mathbf{0}^T & \mathbf{0}^T & \mathbf{0}^T & P_{11}^{(3)} &
\end{array}\right)^T, \quad \mathcal{Q} = \begin{pmatrix}
-Q_2 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\star & -\frac{I_n}{2f_0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\star & \star & -\Theta P_{11}^{(2)} & \mathbf{0} & \mathbf{0} \\
\star & \star & \star & -Q_3 & \mathbf{0} \\
\star & \star & \star & \star & -Q_4
\end{pmatrix},$$

$$\begin{pmatrix} R_1 + \phi_{12}(\Theta) - LC - C^T L^T & I_n \\ I_n & -\bar{\Lambda}_1 \end{pmatrix} \leq 0, \quad \begin{pmatrix} R_2 + \phi_3(\Theta) & P_{11}^{(3)} \\ P_{11}^{(3)} & -\bar{\Lambda}_1 \end{pmatrix} \leq 0, \tag{25}$$

$$\begin{pmatrix} R_3 + \phi_{22}(\Theta) - LC - C^T L^T & I_n \\ I_n & -\bar{\Lambda}_2 \end{pmatrix} \leq 0, \quad \begin{pmatrix} R_4 + \phi_3(\Theta) & P_{11}^{(3)} \\ P_{11}^{(3)} & -\bar{\Lambda}_2 \end{pmatrix} \leq 0, \quad \phi(\Theta) \leq 0, \tag{26}$$

with[2]

$$\phi_{11}(\Theta) = AP_{11}^{(1)} + P_{11}^{(1)}A^T + \frac{P_{11}^{(1)}}{\Theta} - 2P_{11}^{(1)} + \Theta P_{12}^{(1)},$$
$$\phi_{12}(\Theta) = AP_{11}^{(2)} + P_{11}^{(2)}A^T + \frac{P_{11}^{(2)}}{\Theta} - 2P_{12}^{(2)} - \Theta P_{12}^{(2)},$$
$$\phi_{21}(\Theta) = AP_{12}^{(1)} + P_{12}^{(1)}A^T - \frac{P_{12}^{(1)}}{\Theta},$$
$$\phi_{22}(\Theta) = AP_{12}^{(2)} + P_{12}^{(2)}A^T - \frac{P_{12}^{(2)}}{\Theta},$$
$$\phi_3(\Theta) = \frac{P_{11}^{(3)}}{\Theta} - 2P_{11}^{(3)} + \frac{\Theta I_n}{\delta},$$
$$\phi(\Theta) = I_\xi^T P_{21} I_\xi - P_{21} - \Theta P_{22} + Q_5,$$

hold for the finite open set $\Theta \in \left\{0, \frac{c_2}{c_5}\alpha\right\}$, $Q = Q^T = \operatorname{diag}(Q_1^{-1}, Q_2^{-1}, Q_3^{-1}, Q_4^{-1}, Q_5) > 0$, $\Lambda = \Lambda^T = \operatorname{diag}(I_n, I_n, \Lambda_3, \Lambda_4)$, *some* $\bar{\Lambda}_l = \bar{\Lambda}_l{}^T > 0$, $R_j > 0$, *for* $l = 1, 2$, *and* $j = \overline{1,4}$, *respectively, and* $|f(x)|^2 \leq f_0|x|^2$*; and the constraints* (16)-(17) *also hold with* $c_1 = \lambda_{\min}(P_{11})$, $c_2 = \lambda_{\max}(P_{12})$, $c_3 = \lambda_{\min}(P_{21})$, $c_4 = \lambda_{\max}(P_{21} + T_{\max}P_{22})$ *and* $c_5 = \lambda_{\min}(Q)$*; then the system* (7)-(8) *is* EDξ_k^t-S *for any sequence* $\{T_i\}_{i\in\mathbb{N}}$ *such that* $T_i \in \left(0, \frac{c_2}{c_5}\alpha\right]$ *with* $K = Y_K P_{11}^{-(3)}$ *and* L *solution of* (23)-(26).

Now, the following proposition gives a solution to the sampled-data control problem based on state estimation for the ideal linear sampled-data system, *i.e.* $f = 0$.

Proposition 2. *Consider that* P_1 *and* P_2 *have the following structure for all* $\tau_k^t \in [0, T_i]$, $i = k \in \mathbb{N}$

$$P_1(\tau_k^t) = \frac{\tau_k^t P_{11} + (T_i - \tau_k^t)P_{12}}{T_i}, \; P_2(\tau_k^t) = P_{21} + \tau_k^t P_{22},$$

where $P_{11} = \operatorname{diag}(P_{11}^{-(1)}, P_{11}^{-(2)}, P_{11}^{-(3)}, P_{11}^{-(3)})$, $P_{12} = \operatorname{diag}(P_{12}^{-(1)}, P_{12}^{-(2)}, \delta I_n, \delta I_n)$, *with a fixed* $\delta > 0$, $P_{1l}^{(q)} = (P_{1l}^{(q)})^T > 0$, *and* $P_{2l} = P_{2l}^T > 0$, *for* $l = 1, 2$, *and* $q = \overline{1,3}$. *If there exist matrices* $P_{11}^{(q)}$, $P_{2l} = P_{2l}^T > 0$, Y_K *and* L, *for* $l = 1, 2$, *and* $q = \overline{1,3}$, *such that the following matrix inequalities*

$$\begin{pmatrix} \phi_{11}(\Theta) & \mathbf{0} & BY_K & -BY_K & P_{11}^{(1)} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \star & -R_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & P_{11}^{(2)} & \mathbf{0} & \mathbf{0} \\ \star & \star & \phi_3(\Theta) & \mathbf{0} & \mathbf{0} & \mathbf{0} & P_{11}^{(3)} & \mathbf{0} \\ \star & \star & \star & -R_2 & \mathbf{0} & \mathbf{0} & \mathbf{0} & P_{11}^{(3)} \\ \star & \star & \star & \star & -Q_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \star & \star & \star & \star & \star & -Q_2 & \mathbf{0} & \mathbf{0} \\ \star & \star & \star & \star & \star & \star & -Q_3 & \mathbf{0} \\ \star & \star & \star & \star & \star & \star & \star & -Q_4 \end{pmatrix} \leq 0, \tag{27}$$

$$\begin{pmatrix} \phi_{21}(\Theta) & \mathbf{0} & BY_K & -BY_K & P_{12}^{(1)} & P_{12}^{(1)} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \star & -R_3 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & P_{12}^{(2)} & P_{12}^{(2)} & \mathbf{0} & \mathbf{0} \\ \star & \star & \phi_3(\Theta) & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & P_{11}^{(3)} & \mathbf{0} \\ \star & \star & \star & -R_4 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & P_{11}^{(3)} \\ \star & \star & \star & \star & -Q_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \star & \star & \star & \star & \star & -\Theta P_{11}^{(1)} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \star & \star & \star & \star & \star & \star & -Q_2 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \star & \star & \star & \star & \star & \star & \star & -\Theta P_{11}^{(2)} & \mathbf{0} & \mathbf{0} \\ \star & \star & \star & \star & \star & \star & \star & \star & -Q_3 & \mathbf{0} \\ \star & \star & \star & \star & \star & \star & \star & \star & \star & -Q_4 \end{pmatrix} \leq 0, \tag{28}$$

$$\begin{pmatrix} R_1 + \phi_{12}(\Theta) - LC - C^T L^T & I_n \\ I_n & -\bar{\Lambda}_1 \end{pmatrix} \leq 0, \; \begin{pmatrix} R_2 + \phi_3(\Theta) & P_{11}^{(3)} \\ P_{11}^{(3)} & -\bar{\Lambda}_1 \end{pmatrix} \leq 0, \tag{29}$$

---

[2]The variables are $P_{11}^{(1)}$, $P_{11}^{(2)}$, $P_{11}^{(3)}$, $P_{12}^{(1)}$, $P_{12}^{(2)}$ P_{21}, P_{22}, Y_K and L. The matrices Q_i,Λ_k, $\bar{\Lambda}_l$ and R_j, for $i = \overline{1,5}$, $k = 3, 4$, $l = 1, 2$, and $i = \overline{1,4}$, respectively, can be declared as variables or fixed values.

$$\begin{pmatrix} R_3 + \phi_{22}(\Theta) - LC - C^T L^T & I_n \\ I_n & -\bar{\Lambda}_2 \end{pmatrix} \leq 0, \quad \begin{pmatrix} R_4 + \phi_3(\Theta) & P_{11}^{(3)} \\ P_{11}^{(3)} & -\bar{\Lambda}_2 \end{pmatrix} \leq 0, \quad \phi(\Theta) \leq 0, \tag{30}$$

with[3]

$$\begin{aligned}
\phi_{11}(\Theta) &= AP_{11}^{(1)} + P_{11}^{(1)}A^T + \frac{P_{11}^{(1)}}{\Theta} - 2P_{11}^{(1)} + \Theta P_{12}^{(1)}, \\
\phi_{12}(\Theta) &= AP_{11}^{(2)} + P_{11}^{(2)}A^T + \frac{P_{11}^{(2)}}{\Theta} - 2P_{12}^{(2)} - \Theta P_{12}^{(2)}, \\
\phi_{21}(\Theta) &= AP_{12}^{(1)} + P_{12}^{(1)}A^T - \frac{P_{12}^{(1)}}{\Theta}, \\
\phi_{22}(\Theta) &= AP_{12}^{(2)} + P_{12}^{(2)}A^T - \frac{P_{12}^{(2)}}{\Theta}, \\
\phi_3(\Theta) &= \frac{P_{11}^{(3)}}{\Theta} - 2P_{11}^{(3)} + \frac{\Theta I_n}{\delta}, \\
\phi(\Theta) &= I_\xi^T P_{21} I_\xi - P_{21} - \Theta P_{22} + Q_5,
\end{aligned}$$

hold for the finite open set $\Theta \in \left\{0, \frac{c_2}{c_5}\alpha\right\}$, $Q = Q^T = \mathrm{diag}(Q_1^{-1}, Q_2^{-1}, Q_3^{-1}, Q_4^{-1}, Q_5) > 0$, *some* $\bar{\Lambda}_l = \bar{\Lambda}_l{}^T > 0$, $R_j > 0$, *for* $l = 1, 2$, *and* $j = \overline{1, 4}$, *respectively, and constraints* (16)-(17) *also hold with* $c_1 = \lambda_{\min}(P_{11})$, $c_2 = \lambda_{\max}(P_{12})$, $c_3 = \lambda_{\min}(P_{21})$, $c_4 = \lambda_{\max}(P_{21} + T_{\max}P_{22})$ *and* $c_5 = \lambda_{\min}(Q)$*; then the system* (7)-(8) *is* EDξ_k^t-S *for any sequence* $\{T_i\}_{i\in\mathbb{N}}$ *such that* $T_i \in \left(0, \frac{c_2}{c_5}\alpha\right]$ *with* $K = Y_K P_{11}^{-(3)}$ *and* L *solution of* (27)-(30).

Remark 2. *Propositions 1 and 2 provide a particular way to solve the proposed problem, i.e. find the control gain matrix* K *and the gain matrix* L *such that the system* (5)-(6) *is exponentially stable for the ideal case and also for the uncertain case.*

Numerical Aspects: In order to solve the matrix inequalities provided by Propositions 1 and 2, one may use a bisection-like approach using SeDuMi solver among YALMIP in Matlab (see *e.g.* [20] and [30]) on the variable $\Theta \in \left\{0, \frac{c_2}{c_5}\alpha\right\}$. Providing some initialization values, the bisection method is used to establish the maximum value of Θ that satisfies the corresponding matrix inequalities, and in turn, compute the constants c_1, c_2, c_3, c_4 and c_5 that hold constraints (16)-(17). Note that for fixed Θ, δ and f_0, the matrix inequalities given by Propositions 1 and 2 become LMIs.

Note that a different selection for $P_1(\tau_k^t)$ and $P_2(\tau_k^t)$, even for Lyapunov functions with non-quadratic structure, may decrease the conservatism. More complex tools like *sum-of-squares* [3], *looped-functional approach* [5], or *convex characterizations* [4], may be applied to improve the application of this method.

5. Simulation Results

5.1. Ideal Case

Let us consider system (1)-(2) with $f = 0$ and

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 \end{pmatrix}.$$

This example represents a double-integrator that has a wide range of applications. Proposition 2 is applied together with a bisection-like approach using SeDuMi solver among YALMIP in Matlab to find a solution for the LMIs, and the corresponding control and observer gains. The simulations have been done in Matlab with the Euler discretization method, sample time equal to 0.001, and initial conditions $x(0) = (-1, 1)^T$ and $\hat{x}(0) = (0, 0)^T$.

Based on Proposition 2, it is possible to show that the impulsive system (5)-(6), with $f = 0$, is *ED*ξ_k^t*-S* for all $2.55 \geq T_i > 0$, *i.e.* for any sequence $\{T_i\}_{i\in\mathbb{N}}$ such that $T_i \in (0, 2.55] = (T_{\min}, T_{\max}]$, there is a set of feasible control

[3]The variables are $P_{11}^{(1)}$, $P_{11}^{(2)}$, $P_{11}^{(3)}$, $P_{12}^{(1)}$, $P_{12}^{(2)}$ P_{21}, P_{22}, Y_K and L. The matrices Q_i, $\bar{\Lambda}_l$ and R_j, for $i = \overline{1, 5}$, $k = 3, 4$, $l = 1, 2$, and $i = \overline{1, 4}$, respectively, can be declared as variables or fixed values.

and observer gains. The following feasible results are obtained by fixing different constant values of $T_i \in (0, 2.55]$, for all $i = 0, 1, 2, \ldots$:

$$T_i = 0.5$$

$$P_{11}^{(1)} = \begin{pmatrix} 0.0029 & 0.0065 \\ 0.0065 & 0.0570 \end{pmatrix},\ P_{11}^{(2)} = \begin{pmatrix} 0.0078 & 0.0072 \\ 0.0072 & 0.3891 \end{pmatrix},\ P_{11}^{(3)} = \begin{pmatrix} 0.1854 & 0 \\ 0 & 0.0139 \end{pmatrix},$$

$$P_{12}^{(1)} = \begin{pmatrix} 0.0954 & 0.3702 \\ 0.3702 & 1.6643 \end{pmatrix},\ P_{12}^{(2)} = \begin{pmatrix} 0.0095 & 0.0082 \\ 0.0082 & 0.0708 \end{pmatrix},\ \delta = 1,$$

$$P_{21} = \begin{pmatrix} 349.9877 & -0.0226 & -0.0228 & -0.0274 & -125.3336 & 0.0003 & 0.0044 & 0.0033 \\ \star & 349.9973 & -0.0236 & -0.0289 & 0.0040 & -125.3278 & 0.0023 & 0.0027 \\ \star & \star & 349.9921 & -0.0243 & 0.0037 & 0.0003 & -125.3216 & 0.0009 \\ \star & \star & \star & 349.9936 & 0.0021 & 0.0023 & -0.0001 & -125.3348 \\ \star & \star & \star & \star & 234.3587 & 0.0027 & -0.0011 & 0.0006 \\ \star & \star & \star & \star & \star & 234.3513 & 0.0006 & 0.0012 \\ \star & \star & \star & \star & \star & \star & 234.3518 & 0.0019 \\ \star & \star & \star & \star & \star & \star & \star & 234.3589 \end{pmatrix},$$

$$P_{22} = \begin{pmatrix} 581.5497 & -0.0112 & -0.0188 & -0.0164 & 118.6709 & 0.0067 & 0.0032 & 0.0036 \\ \star & 581.5659 & -0.0187 & -0.0145 & 0.0089 & 118.6656 & 0.0036 & 0.0010 \\ \star & \star & 581.5730 & -0.0201 & 0.0076 & 0.0032 & 118.6649 & 0.0064 \\ \star & \star & \star & 581.5502 & 0.0061 & 0.0029 & 0.0040 & 118.6759 \\ \star & \star & \star & \star & 378.7268 & 0.0040 & -0.0033 & 0.0037 \\ \star & \star & \star & \star & \star & 378.7241 & 0.0003 & 0.0023 \\ \star & \star & \star & \star & \star & \star & 378.7310 & 0.0028 \\ \star & \star & \star & \star & \star & \star & \star & 0.0028 \end{pmatrix},$$

$$Y_K = \begin{pmatrix} -0.0062 & -0.0062 \end{pmatrix},\ L = \begin{pmatrix} 254.4294 \\ 33.8006 \end{pmatrix},\ K = \begin{pmatrix} -0.0352 & -0.4698 \end{pmatrix},$$

$$T_i = 1.0$$

$$P_{11}^{(1)} = \begin{pmatrix} 0.0104 & 0.0036 \\ 0.0036 & 0.0309 \end{pmatrix},\ P_{11}^{(2)} = \begin{pmatrix} 0.0294 & 0.0035 \\ 0.0035 & 0.1299 \end{pmatrix},\ P_{11}^{(3)} = \begin{pmatrix} 0.0991 & 0 \\ 0 & 0.0074 \end{pmatrix},$$

$$P_{12}^{(1)} = \begin{pmatrix} 0.0672 & 0.0803 \\ 0.0803 & 0.2420 \end{pmatrix},\ P_{12}^{(2)} = \begin{pmatrix} 0.0382 & 0.0038 \\ 0.0038 & 0.0522 \end{pmatrix},\ \delta = 1,$$

$$P_{21} = \begin{pmatrix} 70.0528 & -0.0047 & -0.0046 & -0.0044 & -17.5110 & 0.0006 & 0.0008 & 0.0011 \\ \star & 70.0528 & -0.0059 & -0.0051 & 0.0011 & -17.5122 & 0.0013 & 0.0007 \\ \star & \star & 70.0521 & -0.0042 & 0.0006 & 0.0015 & -17.5132 & 0.0006 \\ \star & \star & \star & 70.0531 & 0.0009 & 0.0009 & 0.0013 & -17.5134 \\ \star & \star & \star & \star & 50.8534 & -0.0090 & -0.0089 & -0.0088 \\ \star & \star & \star & \star & \star & 50.8526 & -0.0092 & -0.0087 \\ \star & \star & \star & \star & \star & \star & 50.8557 & -0.0088 \\ \star & \star & \star & \star & \star & \star & \star & 50.8532 \end{pmatrix},$$

$$P_{22} = \begin{pmatrix} 112.1137 & -0.0019 & -0.0027 & -0.0020 & 11.6581 & 0.0130 & 0.0124 & 0.0123 \\ \star & 112.1119 & -0.0023 & -0.0022 & 0.0115 & 11.6589 & 0.0122 & 0.0126 \\ \star & \star & 112.1162 & -0.0024 & 0.0125 & 0.0112 & 11.6591 & 0.0126 \\ \star & \star & \star & 112.1113 & 0.0117 & 0.0126 & 0.0111 & 11.6583 \\ \star & \star & \star & \star & 69.9258 & 0.0207 & 0.0203 & 0.0200 \\ \star & \star & \star & \star & \star & 69.9272 & 0.0205 & 0.0202 \\ \star & \star & \star & \star & \star & \star & 69.9231 & 0.0201 \\ \star & \star & \star & \star & \star & \star & \star & 69.9274 \end{pmatrix},$$

$$Y_K = \begin{pmatrix} -0.0079 & -0.0079 \end{pmatrix},\ L = \begin{pmatrix} 50.1074 \\ 21.1055 \end{pmatrix},\ K = \begin{pmatrix} -0.0796 & -1.0652 \end{pmatrix},$$

$$T_i = 2.0$$

$$P_{11}^{(1)} = \begin{pmatrix} 0.0158 & 0.0014 \\ 0.0014 & 0.0281 \end{pmatrix}, \; P_{11}^{(2)} = \begin{pmatrix} 0.0308 & -0.0011 \\ -0.0011 & 0.0698 \end{pmatrix}, \; P_{11}^{(3)} = \begin{pmatrix} 0.1030 & 0 \\ 0 & 0.0077 \end{pmatrix},$$

$$P_{12}^{(1)} = \begin{pmatrix} 0.0777 & 0.0994 \\ 0.0994 & 0.6250 \end{pmatrix}, \; P_{12}^{(2)} = \begin{pmatrix} 0.0448 & 0.0015 \\ 0.0015 & 0.0570 \end{pmatrix}, \; \delta = 1,$$

$$P_{21} = \begin{pmatrix} 57.8022 & 0.0073 & 0.0074 & 0.0067 & -3.9894 & 0.0022 & 0.0023 & 0.0028 \\ \star & 57.8024 & 0.0070 & 0.0065 & 0.0019 & -3.9884 & 0.0019 & 0.0024 \\ \star & \star & 57.8024 & 0.0065 & 0.0020 & 0.0023 & -3.9879 & 0.0025 \\ \star & \star & \star & 57.8027 & 0.0021 & 0.0022 & 0.0027 & -3.9883 \\ \star & \star & \star & \star & 45.6053 & -0.0002 & -0.0002 & -0.0005 \\ \star & \star & \star & \star & \star & 45.6071 & 0.0002 & 0.0001 \\ \star & \star & \star & \star & \star & \star & 45.6062 & 0.0000 \\ \star & \star & \star & \star & \star & \star & \star & 45.6059 \end{pmatrix},$$

$$P_{22} = \begin{pmatrix} 72.0850 & 0.0095 & 0.0094 & 0.0094 & 1.4037 & 0.0084 & 0.0086 & 0.0084 \\ \star & 72.0859 & 0.0087 & 0.0096 & 0.0088 & 1.4030 & 0.0090 & 0.0084 \\ \star & \star & 72.0879 & 0.0093 & 0.0087 & 0.0084 & 1.4027 & 0.0085 \\ \star & \star & \star & 72.0864 & 0.0085 & 0.0085 & 0.0082 & 1.4031 \\ \star & \star & \star & \star & 42.9554 & 0.0051 & 0.0049 & 0.0052 \\ \star & \star & \star & \star & \star & 42.9547 & 0.0049 & 0.0053 \\ \star & \star & \star & \star & \star & \star & 42.9544 & 0.0054 \\ \star & \star & \star & \star & \star & \star & \star & 42.9544 \end{pmatrix},$$

$$Y_K = \begin{pmatrix} -0.0065 & -0.0065 \end{pmatrix}, \; L = \begin{pmatrix} 46.7717 \\ 17.7768 \end{pmatrix}, \; K = \begin{pmatrix} -0.0627 & -0.8393 \end{pmatrix}.$$

The trajectories of the system, the state estimation error and the control signal for different values of T_i are depicted in Figures 1-4. From Fig. 1 it is clear that the trajectories and the estimation worsen whenever the sampling interval increases. However, the proposed approach is capable of stabilizing ξ_k^t for a reasonable sampling interval equal to 2 seconds. For the aperiodic case, from Fig. 2 it is clear that the proposed approach is able to deal also with the aperiodic case and the behavior is very similar with any of the values in the *set of feasible control and observer gains* for the same sequence of T_i. The state estimation error for the ideal case is depicted in Fig. 3 where the estimation error worsens whenever the sampling interval increases. However, the proposed approach is capable of stabilizing ξ_k^t for a reasonable sampling interval equal to 2 seconds. Finally, the control signals are shown in Fig. 4.

5.2. Uncertain Case

Let us consider system (1)-(2) with

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 \end{pmatrix}, \quad f(x) = 0.1 \begin{pmatrix} 0 \\ x_1 + \sin(x_2) \end{pmatrix}.$$

This example represents an uncertain double-integrator where $f(x)$ is a Lipschitz function with $f_0 = 0.1$. Proposition 1 is applied together with a bisection-like approach using SeDuMi solver among YALMIP in Matlab to find a solution for the LMIs, and the corresponding control and observer gains. The simulations have been done in Matlab with the Euler discretization method, sample time equal to 0.001, and initial conditions $x(0) = (-1, 1)^T$ and $\hat{x}(0) = (0, 0)^T$.

Based on Proposition 1, it is possible to show that the impulsive system (5)-(6) is $ED\xi_k^t$-S for all $0.30 > T_i > 0$ *i.e.* for any sequence $\{T_i\}_{i\in\mathbb{N}}$ such that $T_i \in (0, 0.30] = (T_{\min}, T_{\max}]$, there is a set of feasible control and observer gains.

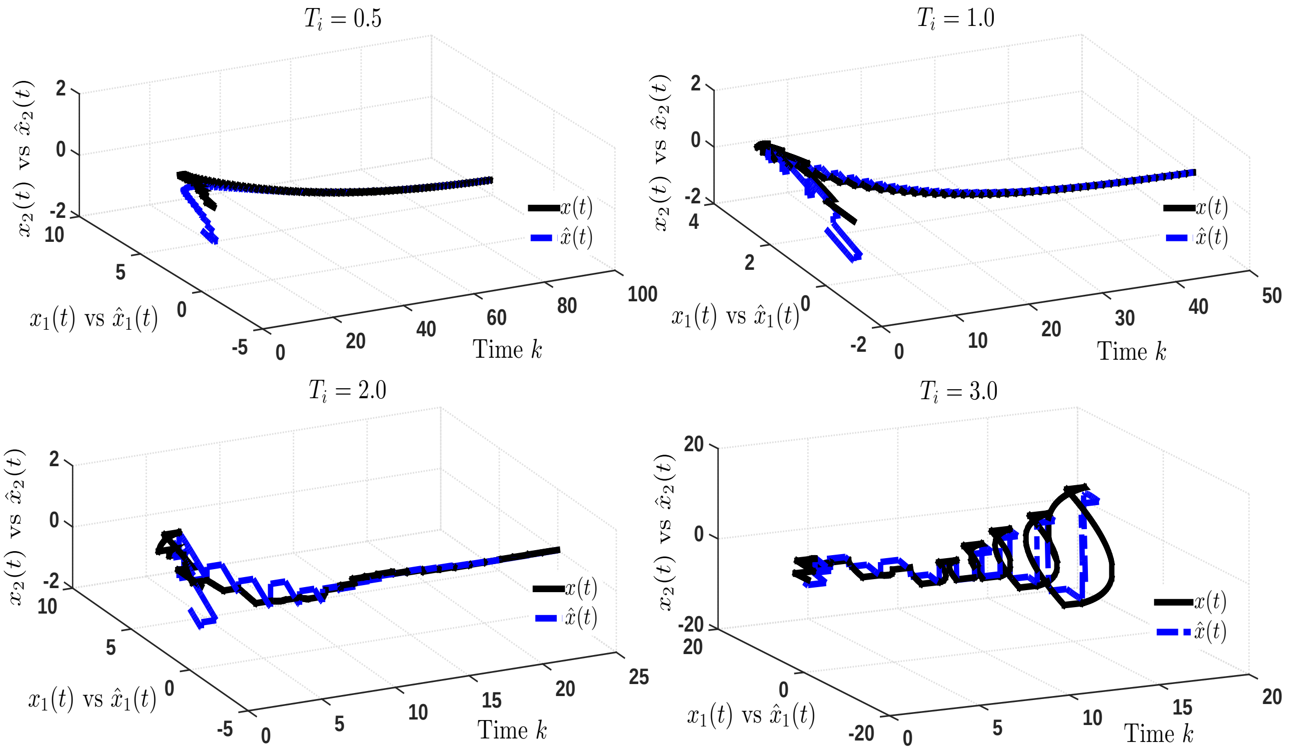

Figure 1: **Real and estimated trajectories of the ideal sampled-data system for different values of T_i**

The following feasible results are obtained by fixing different constant values of $T_i \in (0, 0.30]$, for all $i = 0, 1, 2, \ldots$:

$$T_i = 0.1$$

$$P_{11}^{(1)} = \begin{pmatrix} 0.0464 & 0.0055 \\ 0.0055 & 0.4720 \end{pmatrix}, \; P_{11}^{(2)} = \begin{pmatrix} 0.0158 & 0.0296 \\ 0.0296 & 4.5343 \end{pmatrix}, \; P_{11}^{(3)} = \begin{pmatrix} 0.6860 & -0.0234 \\ -0.0234 & 0.3431 \end{pmatrix},$$

$$P_{12}^{(1)} = \begin{pmatrix} 0.5672 & 0.0264 \\ 0.0264 & 0.4714 \end{pmatrix}, \; P_{12}^{(2)} = \begin{pmatrix} 2.7065 & -0.2887 \\ -0.2887 & 2.9879 \end{pmatrix}, \; \delta = 1,$$

$$P_{21} = \begin{pmatrix} 83.0960 & 0.0266 & 0.0266 & 0.0266 & -37.5004 & -0.0060 & -0.0060 & -0.0060 \\ \star & 83.0960 & 0.0266 & 0.0266 & -0.0060 & -37.5004 & -0.0060 & -0.0060 \\ \star & \star & 83.0960 & 0.0266 & -0.0060 & -0.0060 & -37.5004 & -0.0060 \\ \star & \star & \star & 83.0960 & -0.0060 & -0.0060 & -0.0060 & -37.5004 \\ \star & \star & \star & \star & 43.8472 & -0.0198 & -0.0198 & -0.0198 \\ \star & \star & \star & \star & \star & 43.8472 & -0.0198 & -0.0198 \\ \star & \star & \star & \star & \star & \star & 43.8472 & -0.0198 \\ \star & \star & \star & \star & \star & \star & \star & 43.8472 \end{pmatrix},$$

$$P_{22} = \begin{pmatrix} 127.4672 & 0.9881 & 0.9881 & 0.9881 & 62.8240 & 0.8689 & 0.8689 & 0.8698 \\ \star & 127.4672 & 0.9881 & 0.9881 & 0.8688 & 62.8240 & 0.8689 & 0.8689 \\ \star & \star & 127.4672 & 0.9881 & 0.8689 & 0.8689 & 62.8240 & 0.8689 \\ \star & \star & \star & 127.4672 & 0.8689 & 0.8689 & 0.8689 & 62.8240 \\ \star & \star & \star & \star & 106.2215 & 0.7620 & 0.7619 & 0.7619 \\ \star & \star & \star & \star & \star & 106.2215 & 0.7619 & 0.7619 \\ \star & \star & \star & \star & \star & \star & 106.2215 & 0.7618 \\ \star & \star & \star & \star & \star & \star & \star & 106.2215 \end{pmatrix},$$

$$Y_K = \begin{pmatrix} -0.9182 & -0.9521 \end{pmatrix}, \; L = \begin{pmatrix} 2.6175 \\ 3.7852 \end{pmatrix}, \; K = \begin{pmatrix} -1.4364 & -2.8729 \end{pmatrix},$$

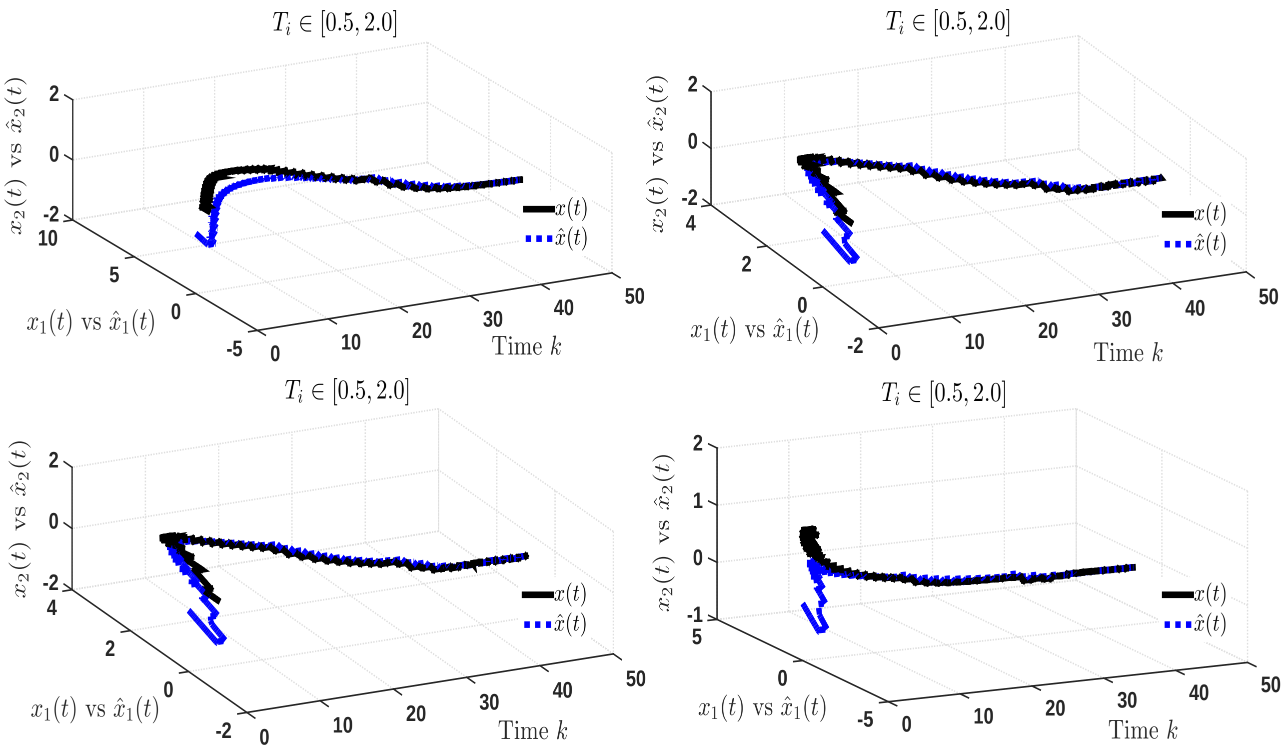

Figure 2: **Real and estimated trajectories of the ideal sampled-data system for the aperiodic case $T_i \in [0.5, 2.0]$ and different values of K and L.** For a fixed sequence in T_i, four different matrix gains, in the set of feasible solutions, for K and L were used, i.e. $K = (-0.0352, -0.4698)$, $L = (254.4294, 33.8006)^T$ - left top; $K = (-0.0796, -1.0652)$, $L = (50.1074, 21.1055)^T$ - right top; $K = (-0.0771, -1.0328)$, $L = (41.5574, 17.6644)^T$ - left bottom; $K = (-0.0627, -0.8393)$, $L = (46.7717, 17.7768)^T$ - right bottom.

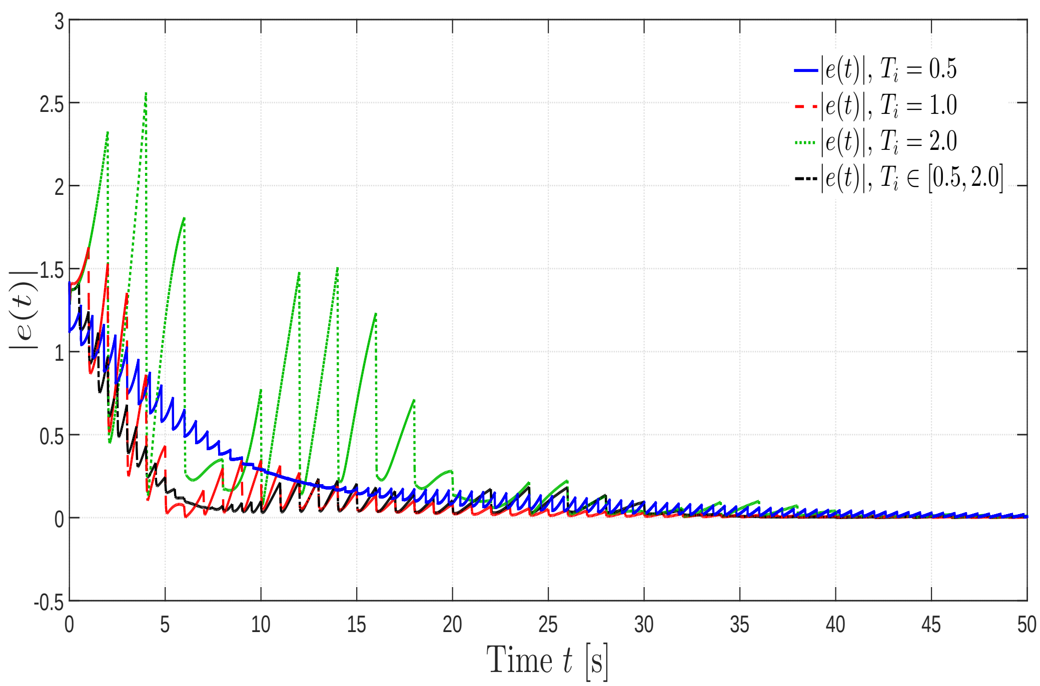

Figure 3: **State estimation error of the ideal sampled-data system for different values of T_i including the aperiodic case.** For the aperiodic case the gains were $K = (-0.0627, -0.8393)$, $L = (46.7717, 17.7768)^T$ corresponding to $T_i = 2.0$.

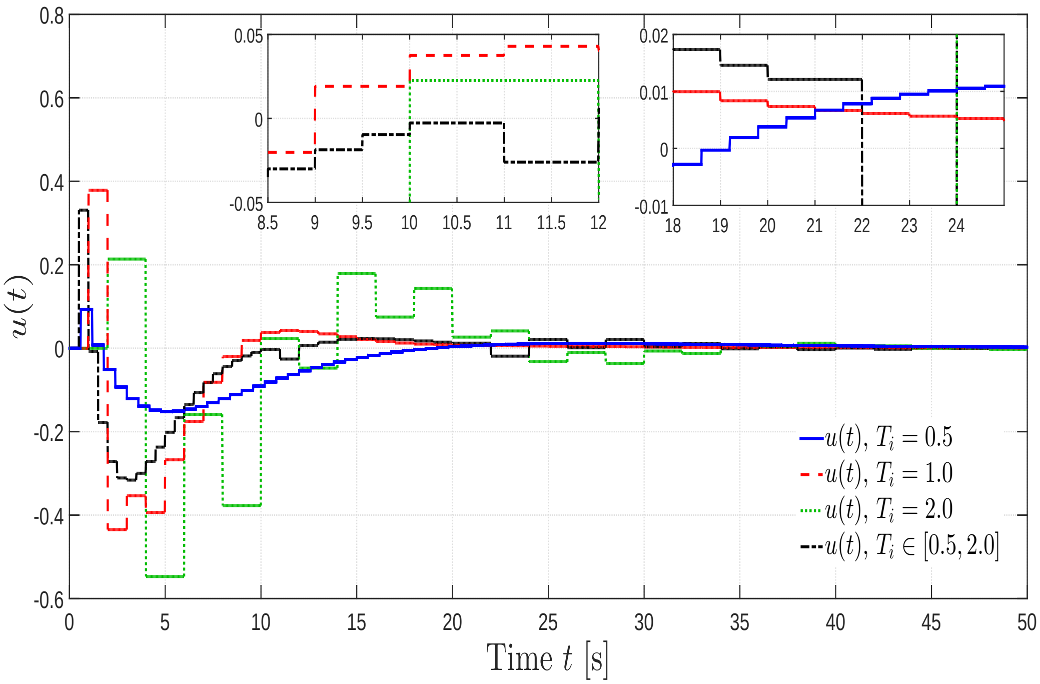

Figure 4: **Control signal of the ideal sampled-data system for different values of T_i including the aperiodic case.** For the aperiodic case the gains were $K = (-0.0627, -0.8393)$, $L = (46.7717, 17.7768)^T$ corresponding to $T_i = 2.0$.

$$T_i = 0.2$$

$$P_{11}^{(1)} = \begin{pmatrix} 0.8291 & -0.1799 \\ -0.1799 & 0.9342 \end{pmatrix},\ P_{11}^{(2)} = \begin{pmatrix} 0.0416 & 0.0238 \\ 0.0238 & 5.7885 \end{pmatrix},\ P_{11}^{(3)} = \begin{pmatrix} 0.3826 & -0.0076 \\ -0.0076 & 0.1913 \end{pmatrix},$$

$$P_{12}^{(1)} = \begin{pmatrix} 2.7765 & -0.1098 \\ -0.1098 & 2.1288 \end{pmatrix},\ P_{12}^{(2)} = \begin{pmatrix} 4.3625 & -0.1738 \\ -0.1738 & 5.7290 \end{pmatrix},\ \delta = 1,$$

$$P_{21} = \begin{pmatrix} 77.3837 & 0.0084 & 0.0084 & 0.0084 & -34.3973 & 0.0002 & 0.0002 & 0.0002 \\ \star & 77.3837 & 0.0084 & 0.0084 & 0.0002 & -34.3973 & 0.0002 & 0.0002 \\ \star & \star & 77.3837 & 0.0084 & 0.0002 & 0.0002 & -34.3973 & 0.0002 \\ \star & \star & \star & 77.3837 & 0.0002 & 0.0002 & 0.0002 & -34.3973 \\ \star & \star & \star & \star & 44.4824 & -0.0124 & -0.0124 & -0.0124 \\ \star & \star & \star & \star & \star & 44.4824 & -0.0124 & -0.0124 \\ \star & \star & \star & \star & \star & \star & 44.4824 & -0.0124 \\ \star & \star & \star & \star & \star & \star & \star & 44.4824 \end{pmatrix},$$

$$P_{22} = \begin{pmatrix} 123.7642 & 0.8964 & 0.8964 & 0.8964 & 51.1589 & 0.7385 & 0.7385 & 0.7385 \\ \star & 123.7642 & 0.8964 & 0.8964 & 0.7385 & 51.1589 & 0.7385 & 0.7385 \\ \star & \star & 123.7642 & 0.8964 & 0.7385 & 0.7385 & 51.1589 & 0.7385 \\ \star & \star & \star & 123.7642 & 0.7385 & 0.7385 & 0.7385 & 51.1589 \\ \star & \star & \star & \star & 93.4763 & 0.6062 & 0.6062 & 0.6062 \\ \star & \star & \star & \star & \star & 93.4763 & 0.6062 & 0.6062 \\ \star & \star & \star & \star & \star & \star & 93.4763 & 0.6062 \\ \star & \star & \star & \star & \star & \star & \star & 93.4763 \end{pmatrix},$$

$$Y_K = \begin{pmatrix} -0.1032 & -0.1054 \end{pmatrix},\ L = \begin{pmatrix} 2.5540 \\ 3.1337 \end{pmatrix},\ K = \begin{pmatrix} -0.2810 & -0.5620 \end{pmatrix},$$

$$T_i = 0.3$$

$$P_{11}^{(1)} = \begin{pmatrix} 0.7794 & 0.1081 \\ 0.1081 & 1.3372 \end{pmatrix},\ P_{11}^{(2)} = \begin{pmatrix} 0.0358 & 0.0284 \\ 0.0284 & 2.4736 \end{pmatrix},\ P_{11}^{(3)} = \begin{pmatrix} 0.3749 & -0.0046 \\ -0.0046 & 0.3749 \end{pmatrix},$$

$$P_{12}^{(1)} = \begin{pmatrix} 2.8366 & -0.2920 \\ -0.2920 & 2.5168 \end{pmatrix},\ P_{12}^{(2)} = \begin{pmatrix} 2.7932 & -0.0059 \\ -0.0059 & 3.6565 \end{pmatrix},\ \delta = 1,$$

$$P_{21} = \begin{pmatrix} 73.0348 & 0.0024 & 0.0024 & 0.0024 & -31.3059 & -0.0001 & -0.0001 & -0.0001 \\ \star & 73.0348 & 0.0024 & 0.0024 & -0.0001 & -31.3059 & -0.0001 & -0.0001 \\ \star & \star & 73.0348 & 0.0024 & -0.0001 & -0.0001 & -31.3059 & -0.0001 \\ \star & \star & \star & 73.0348 & -0.0001 & -0.0001 & -0.0001 & -31.3059 \\ \star & \star & \star & \star & 44.3482 & 0.0003 & 0.0003 & 0.0003 \\ \star & \star & \star & \star & \star & 44.3482 & 0.0003 & 0.0003 \\ \star & \star & \star & \star & \star & \star & 44.3482 & 0.0003 \\ \star & \star & \star & \star & \star & \star & \star & 44.3482 \end{pmatrix},$$

$$P_{22} = \begin{pmatrix} 118.7790 & 0.0300 & 0.0300 & 0.0300 & 40.8381 & 0.0198 & 0.0198 & 0.0198 \\ \star & 118.7790 & 0.0300 & 0.0300 & 0.0198 & 40.8381 & 0.0198 & 0.0198 \\ \star & \star & 118.7790 & 0.0300 & 0.0198 & 0.0198 & 40.8381 & 0.0198 \\ \star & \star & \star & 118.7790 & 0.0198 & 0.0198 & 0.0198 & 40.8381 \\ \star & \star & \star & \star & 83.9838 & 0.0132 & 0.0132 & 0.0132 \\ \star & \star & \star & \star & \star & 83.9838 & 0.0132 & 0.0132 \\ \star & \star & \star & \star & \star & \star & 83.9838 & 0.0132 \\ \star & \star & \star & \star & \star & \star & \star & 83.9838 \end{pmatrix},$$

$$Y_K = \begin{pmatrix} -0.1136 & -0.2315 \end{pmatrix},\ L = \begin{pmatrix} 2.5714 \\ 1.9137 \end{pmatrix},\ K = \begin{pmatrix} -0.3107 & -0.6213 \end{pmatrix}.$$

The trajectories of the system, the state estimation error and the control signal for different values of T_i are depicted in Figures 5-8. From Fig. 5 it is clear that for the uncertain case, the trajectories deteriorate, more than the ideal case, whenever the sampling interval increases. The proposed approach is capable of stabilizing ξ_k^t for a sampling interval less than or equal to 0.30 seconds. For the aperiodic case, from Fig. 6 it is clear that the proposed approach is able to deal also with the aperiodic case and the behavior is very similar with any of the values in the *set of feasible control and observer gains* for the same sequence of T_i but the sampling interval has decreased with respect to the ideal case. The state estimation error for the ideal case is depicted in Fig. 7 where the estimation error deteriorates, more than the ideal case, whenever the sampling interval increases. The proposed approach is capable of stabilizing ξ_k^t for a sampling interval less than or equal to 0.30 seconds. Finally, the control signals are shown in Fig. 8.

6. Conclusions

In this paper a vector Lyapunov function-based approach, derived by means of a $2D$ time domain equivalence, for stability of impulsive systems is used for designing a robust output-feedback control for linear sampled-data systems. This approach provides a stability analysis based on LMIs for linear impulsive dynamical systems. Then, it is possible to show that the sampled-data control problem based on state estimation may turn into one of finding conditions for the exponential stability of impulsive systems. Thus, the proposed vector Lyapunov function approach is applied for obtaining stability conditions of the impulsive system, and then, a solution to the robust output-feedback control design problem is derived and expressed in terms of LMIs. Some numerical examples illustrate the feasibility of the proposed approach. The analysis of uncertain sampled-data nonlinear systems is in the scope of the future research.

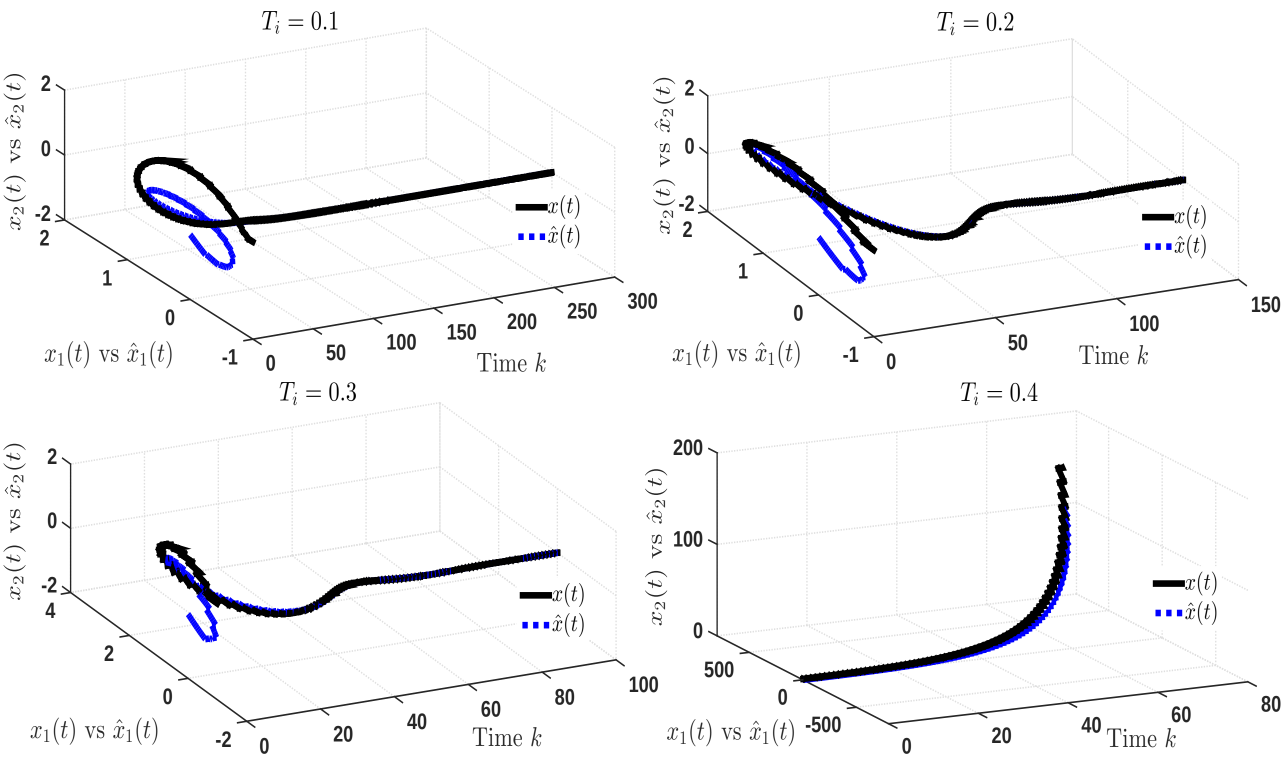

Figure 5: **Real and estimated trajectories of the uncertain sampled-data system for different values of T_i**

Appendix

Proof of Theorem 1: From the divergence definition and inequalities (13), (14) and (15), it follows that

$$\begin{aligned}\frac{dV_1(z_k^t)}{dt} &\leq -c_5(|z_k^t|^2 + |z_k^{t_{i+1}}|^2) - V_2(z_{k+1}^{t_{i+1}}) + V_2(z_k^{t_{i+1}}),\\ &\leq -\beta V_1(z_k^t) + \lambda V_2(z_k^{t_{i+1}}) - V_2(z_{k+1}^{t_{i+1}}),\end{aligned} \tag{31}$$

where $\lambda = 1 - \frac{c_5}{c_4}$ and $\beta = \frac{c_5}{c_2}$. By means of the comparison principle, with respect to the time t, from (31), for all $t \in [t_i, t_{i+1})$, it is obtained that

$$\begin{aligned}V_1\left(z_k^t\right) &\leq e^{-\beta(t-t_i)}V_1(z_k^{t_i}) + \int_{t_i}^{t} e^{-\beta(t-\tau)}\left[\lambda V_2(z_k^{t_{i+1}}) - V_2(z_{k+1}^{t_{i+1}})\right]d\tau,\\ &= e^{-\beta(t-t_i)}V_1(z_k^{t_i}) + \rho_i(t)(\lambda V_2(z_k^{t_{i+1}}) - V_2(z_{k+1}^{t_{i+1}})),\end{aligned} \tag{32}$$

where $\rho_i(t) = \frac{1-e^{-\beta(t-t_i)}}{\beta} > 0$, for all $t \in [t_i, t_{i+1})$. In order to fulfill the statements given by Definition 1 it is necessary to prove *convergence* and *boundedness*. Thus, let us prove each one separately.

1. *Convergence.* Evaluating (32) for $t = t_{i+1}$, it gives

$$V_1(z_k^{t_{i+1}}) \leq e^{-\beta T_i}V_1(z_k^{t_i}) + \rho_i(t_{i+1})\lambda V_2(z_k^{t_{i+1}}) - \rho_i(t_{i+1})V_2(z_{k+1}^{t_{i+1}}), \tag{33}$$

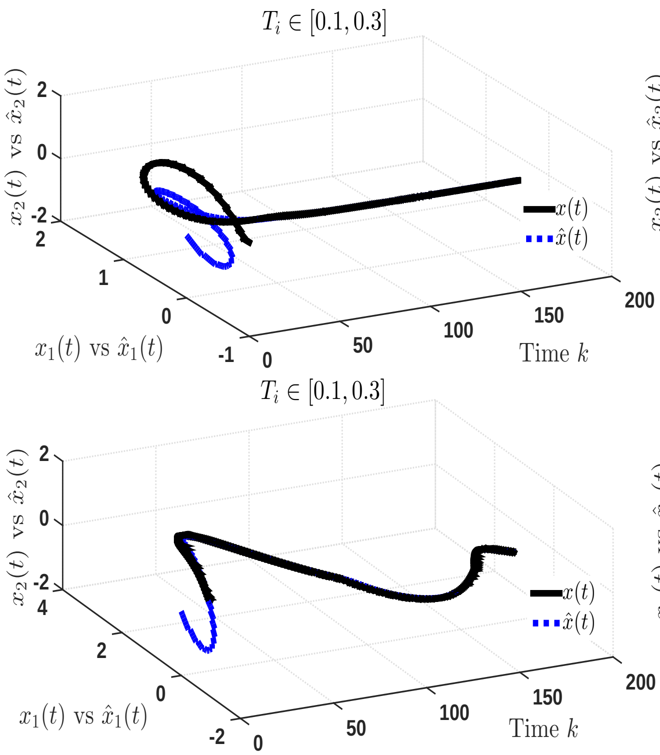

Figure 6: **Real and estimated trajectories of the ideal sampled-data system for the aperiodic case $T_i \in [0.1, 0.3]$ and different values of K and L.** For a fixed sequence in T_i, four different matrix gains, in the set of feasible solutions, for K and L were used, i.e. $K = (-1.4364, -2.8729)$, $L = (2.6175, 3.7852)^T$ - left top; $K = (-0.2810, -0.5620)$, $L = (2.5540, 3.1337)^T$ - right top; $K = (-0.1013, -0.2025)$, $L = (2.5453, 4.7680)^T$ - left bottom; $K = (-0.3107, -0.6213)$, $L = (2.5714, 1.9137)^T$ - right bottom.

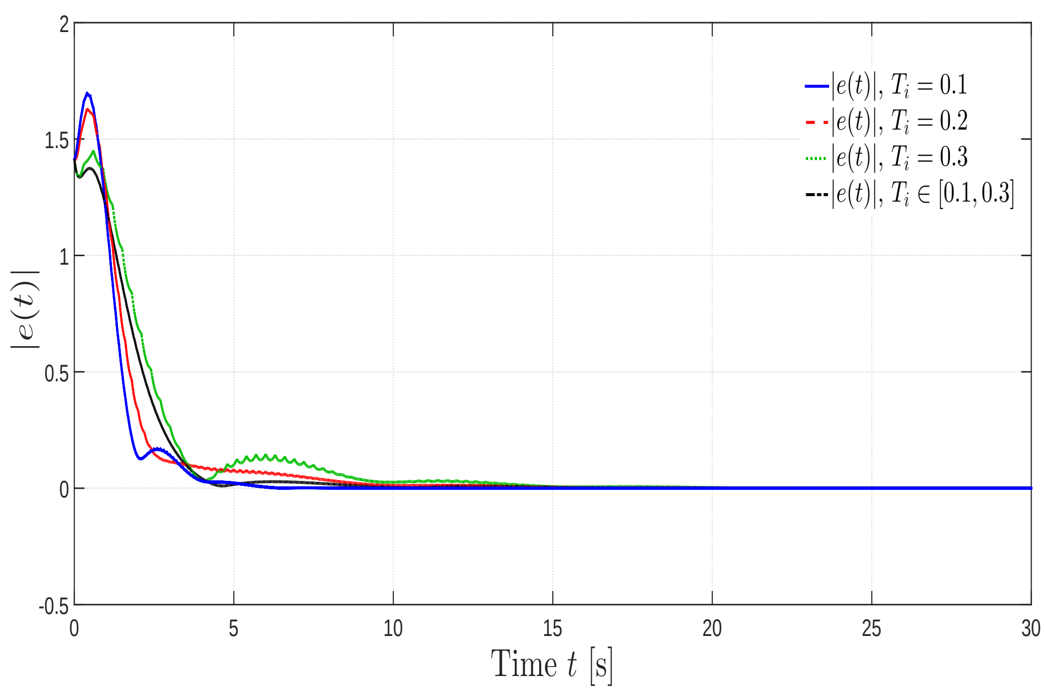

Figure 7: **State estimation error of the uncertain sampled-data system for different values of T_i including the aperiodic case.** For the aperiodic case the gains were $K = (-0.3107, -0.6213)$, $L = (2.5714, 1.9137)^T$ corresponding to $T_i = 0.3$.

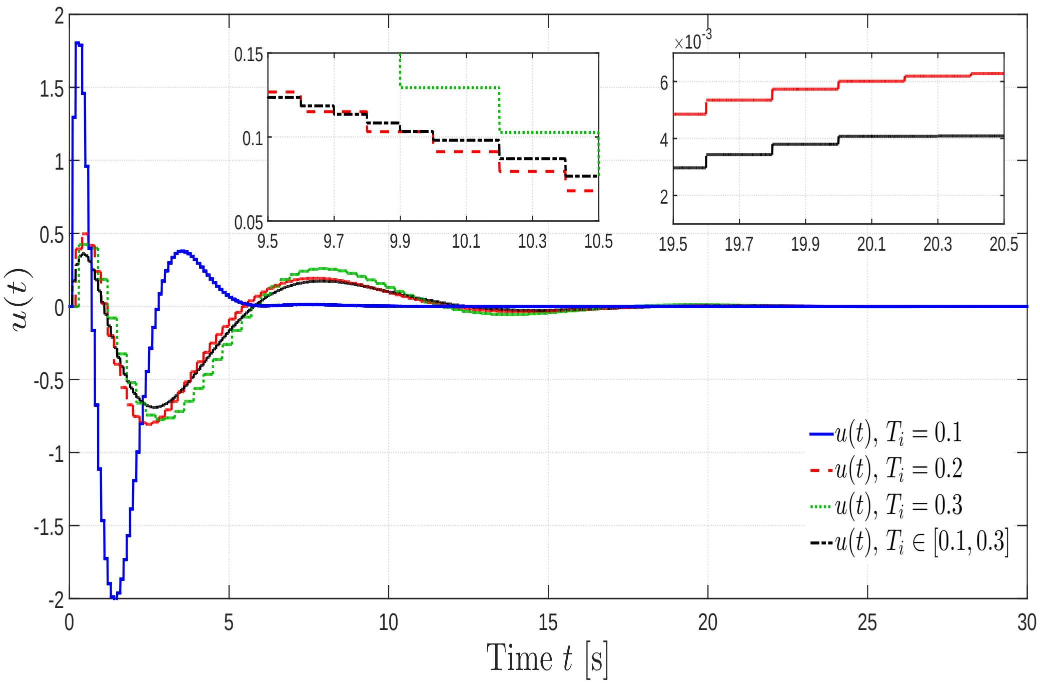

Figure 8: **Control signal of the uncertain sampled-data system for different values of T_i including the aperiodic case.** For the aperiodic case the gains were $K = (-0.3107, -0.6213)$, $L = (2.5714, 1.9137)^T$ corresponding to $T_i = 0.3$.

with $\rho_i(t_{i+1}) = \frac{1-e^{-\beta T_i}}{\beta} > 0$, for all $T_i \geq 0$. From the inequalities (13) and (14), it follows that $\forall i = k \in \mathbb{N}$

$$\frac{V_2(z_k^p)}{c_4} \leq |\xi_k^p|^2 \leq \frac{V_1(z_k^p)}{c_1},\ \forall p = t_i, t_{i+1} \tag{34}$$

$$\frac{V_1(z_k^p)}{c_2} \leq |\xi_k^p|^2 \leq \frac{V_2(z_k^p)}{c_3},\ \forall p = t_i, t_{i+1}. \tag{35}$$

From (34), it is given that $\frac{c_1}{c_4} V_2(z_k^{t_{i+1}}) \leq V_1(z_k^{t_{i+1}})$, and therefore from (33), it is obtained that

$$\rho_i(t_{i+1}) V_2(z_{k+1}^{t_{i+1}}) \leq e^{-\beta T_i} V_1(z_k^{t_i}) + \left(\rho_i(t_{i+1})\lambda - \frac{c_1}{c_4}\right) V_2(z_k^{t_{i+1}}). \tag{36}$$

Let us consider that $c_4 > c_5$, *i.e.* $\lambda \in (0,1)$. Thus, from (35) and (36) it follows that

$$V_2(z_{k+1}^{t_{i+1}}) \leq \frac{c_2 e^{-\beta T_i}}{c_3 \rho_i(t_{i+1})} V_2(z_k^{t_i}) + \left(\frac{\rho_i(t_{i+1})\lambda - \frac{c_1}{c_4}}{\rho_i(t_{i+1})}\right) V_2(z_k^{t_{i+1}}). \tag{37}$$

Note that if the constraint $\rho_i(t_{i+1})\lambda \leq \frac{c_1}{c_4}$ holds, then the term depending on $V_2(z_k^{t_{i+1}})$ can be disregarded. In this sense, in order to satisfy such a constraint, recalling that $\beta = \frac{c_5}{c_2}$, $\lambda = 1 - \frac{c_5}{c_4}$ and $\rho_i(t_{i+1}) = \frac{1-e^{-\beta T_i}}{\beta}$, the following condition is founded

$$c_2\left(1 - e^{-\beta T_i}\right)(c_4 - c_5) \leq c_1 c_5,$$
$$\left(1 - e^{-\beta T_i}\right) \leq \frac{c_1 c_5}{c_2 (c_4 - c_5)}.$$

Then, it is clear that if $c_2\left(c_4-c_5\right)\leq c_1c_5$ holds then $\rho_i(t_{i+1})\lambda\leq\frac{c_1}{c_4}$ is trivially satisfied. Otherwise

$$e^{-\beta T_i}\geq\frac{c_2\left(c_4-c_5\right)-c_1c_5}{c_2\left(c_4-c_5\right)}\Leftrightarrow T_i\leq\frac{c_2}{c_5}\alpha,$$

where $\alpha=-\ln\left[\frac{c_2(c_4-c_5)-c_1c_5}{c_2(c_4-c_5)}\right]>0$, for all $c_2\left(c_4-c_5\right)>c_1c_5$. Note that these two possibilities, *i.e.* $c_2\left(c_4-c_5\right)\leq c_1c_5$ or $T_i\leq\frac{c_2}{c_5}\alpha$, are represented by (16) in Theorem 1. Therefore, if one of them is satisfied, from (37) it is obtained that

$$V_2(z_{k+1}^{t_{i+1}})\leq\frac{c_2e^{-\beta T_i}}{c_3\rho_i(t_{i+1})}V_2(z_k^{t_i}).$$

Then, by induction, it follows that

$$V_2(z_{k+1}^{t_{i+1}})\leq\left(\frac{c_2e^{-\beta T_i}}{c_3\rho_i(t_{i+1})}\right)^{k+1}V_2(z_0^0). \tag{38}$$

Hence, (38) decreases if the following condition holds

$$\begin{aligned}\frac{c_2e^{-\beta T_i}}{c_3\rho_i(t_{i+1})}&\leq 1-\varepsilon,\\ c_5e^{-\beta T_i}&\leq c_3\left(1-\varepsilon\right)\left(1-e^{-\beta T_i}\right),\\ e^{-\beta T_i}&\leq\frac{c_3\left(1-\varepsilon\right)}{c_5+c_3\left(1-\varepsilon\right)}\Leftrightarrow T_i\geq\frac{c_2}{c_5}\gamma,\end{aligned}$$

which is the same that (17), with $\gamma=-\ln\left[\frac{c_3(1-\varepsilon)}{c_5+c_3(1-\varepsilon)}\right]$. Then, from (34), (35), and (38) , it follows that $\forall i=k\in\mathbb{N}$

$$|\xi_{k+1}^{t_{i+1}}|^2\leq c\kappa_1^{k+1}|\xi_0^0|^2,$$

with $c=\frac{c_4}{c_3}>0$ and $0<\kappa_1=\frac{c_5}{c_3(1-e^{-\gamma})}<1-\varepsilon$, for some small positive ε. Thus, the trajectories of system (7)-(8) are convergent under the constraints $c_4>c_5$, $c_2\left(c_4-c_5\right)\leq c_1c_5$ or $T_i\leq\frac{c_2}{c_5}\alpha$, and $T_i\geq\frac{c_2}{c_5}\gamma$, *i.e.* eq. (9) from Definition 1 is obtained. Now, let us take into account that $c_5\geq c_4$, *i.e.* $\lambda\leq 0$. Therefore, from (36), it follows that the term depended on $V_2(z_k^{t_{i+1}})$ can be disregarded, then one gets (38) and just under condition $T_i\geq\frac{c_2}{c_5}\gamma$ convergence is obtained. Thus, it is concluded that the trajectories of system (7)-(8) are convergent under constraints (16)-(17) if $c_4>c_5$, or only under (17) if $c_5\geq c_4$ holds. In order to complete the proof, let us prove boundedness between the impulses, *i.e.* $|\xi_k^t|^2\leq\kappa_2|\xi_k^{t_i}|^2$ for all $t\in[t_i,t_{i+1})$.

2. *Boundedness.* From (32), it is given that

$$V_1(z_k^t)\leq e^{-\beta(t-t_i)}V_1(z_k^{t_i})+\rho_i(t_{i+1})\lambda V_2(z_k^{t_{i+1}}). \tag{39}$$

Let us consider the case $c_5\geq c_4$, *i.e.* $\lambda\leq 0$. Therefore, from (39), it follows that $V_1(z_k^t)\leq e^{-\beta(t-t_i)}V_1(z_k^{t_i})$, $\forall i=k\in\mathbb{N}$, and boundedness is given, *i.e.*

$$|\xi_k^t|^2\leq\kappa_2|\xi_k^{t_i}|^2,\ \forall t\in\left[t_i,t_{i+1}\right),$$

with $\kappa_2=\frac{c_2}{c_1}$. Finally, for the case $c_4>c_5$, *i.e.* $\lambda\in(0,1)$, from (39) and evaluating $t=t_{i+1}$, one gets

$$\begin{aligned}\frac{c_1}{c_4}V_2(z_k^{t_{i+1}})&\leq e^{-\beta T_i}V_1(z_k^{t_i})+\rho_i(t_{i+1})\lambda V_2(z_k^{t_{i+1}}),\\ V_2(z_k^{t_{i+1}})&\leq\frac{e^{-\beta T_i}}{\left(\frac{c_1}{c_4}-\rho_i(t_{i+1})\lambda\right)}V_1(z_k^{t_i}).\end{aligned} \tag{40}$$

Note that $\rho_i(t_{i+1})\lambda<\frac{c_1}{c_4}$ has to hold in order to satisfy inequality (40). However, as it was previously described, if $c_2\left(c_4-c_5\right)\leq c_1c_5$ holds, $\rho_i(t_{i+1})\lambda<\frac{c_1}{c_4}$ is trivially satisfied, otherwise T_i should be less than or equal to $\frac{c_2}{c_5}\alpha$, *i.e.* $T_i\leq\frac{c_2}{c_5}\alpha$ with $\alpha=-\ln\left[\frac{c_2(c_4-c_5)-c_1c_5}{c_2(c_4-c_5)}\right]>0$, for all $c_2\left(c_4-c_5\right)>c_1c_5$. Thus, applying (40) in (39), it is given that

$$
\begin{aligned}
V_1(z_k^t) &\le \left[e^{-\beta(t-t_i)} + \frac{\rho_i(t_{i+1})\lambda e^{-\beta T_i}}{\left(\frac{c_1}{c_4} - \rho_i(t_{i+1})\lambda\right)} \right] V_1(z_k^{t_i}), \\
&\le \left[\frac{c_5\left(c_1 - c_4\rho_i(t_{i+1})\lambda\right) + c_2 c_4}{c_5\left(c_1 - c_4\rho_i(t_{i+1})\lambda\right)} \right] V_1(z_k^{t_i}).
\end{aligned} \tag{41}
$$

Therefore, from (41), boundedness is obtained, *i.e.* $|\xi_k^t|^2 \le \kappa_2 |\xi_k^{t_i}|^2$ for all $t \in [t_i, t_{i+1})$ and

$$
\kappa_2 = \frac{c_2\left(c_1 c_5 + c_2 c_4\right)}{c_1^2 c_5 - c_1 c_2\left(c_4 - c_5\right)\left(1 - e^{-\gamma}\right)},
$$

which clearly is also valid for the case $c_5 \ge c_4$, *i.e.* $\lambda \le 0$.

Thus, during each interval between impulses, the trajectories of the system are bounded by a constant value as in (10), and due to the convergence property given by (9), according to Definition 1, the $2D$ system described by (7)-(8) is $ED\xi_k^t$-S. □

Proof of Corollary 1: Let us calculate the divergence operator for the quadratic vector Lyapunov function $V(z_k^t, z_{k+1}^{t_{i+1}})$ given by (18), *i.e.*

$$
\begin{aligned}
\operatorname{div} V(z_k^t, z_{k+1}^{t_{i+1}}) &= \frac{dV_1(z_k^t)}{dt} + V_2(z_{k+1}^{t_{i+1}}) - V_2(z_k^{t_{i+1}}) \\
&= (\xi_k^t)^T \left(P_1(\tau_k^t) A_\xi + A_\xi^T P_1(\tau_k^t) + \frac{dP_1(\tau_k^t)}{dt} \right) \xi_k^t + (\xi_k^t)^T P_1(\tau_k^t) D_\xi f + f^T D_\xi^T P_1(\tau_k^t) \xi_k^t \\
&\qquad + (\xi_k^{t_{i+1}})^T \left(I_\xi^T P_2(0) I_\xi - P_2(T_i) \right) \xi_k^t.
\end{aligned}
$$

From the Λ-inequality (see, for instance, [25]), it follows that

$$
XY^T + YX^T \le X\Lambda^{-1}X^T + Y\Lambda Y^T,
$$

Therefore, the LMI (23) is obtained when all the elements of $\widehat{\Xi}_3(\Theta)$ are merged, and it is concluded that if the set of LMIs (23) and (25) is feasible then (42) holds, *i.e.* $\Upsilon_1(\Theta) \le 0$. that holds for every $X \in \mathbb{R}^{n\times k}$, $Y \in \mathbb{R}^{n\times k}$, and $0 < \Lambda = \Lambda^T \in \mathbb{R}^{k\times k}$. Applied with $X = (\xi_k^t)^T P_1(\tau_k^t)$ and $Y = f^T D_\xi^T$, it follows that

$$
(\xi_k^t)^T P_1(\tau_k^t) D_\xi f + f^T D_\xi^T P_1(\tau_k^t) \xi_k^t \le (\xi_k^t)^T P_1(\tau_k^t) \Lambda^{-1} P_1(\tau_k^t) \xi_k^t + f^T D_\xi^T \Lambda D_\xi f,
$$

for any $0 < \Lambda = \Lambda^T \in \mathbb{R}^{4n\times 4n}$. Therefore, the divergence can be upper bounded as

$$
\begin{aligned}
\operatorname{div} V(z_k^t, z_{k+1}^{t_{i+1}}) &\le (\xi_k^t)^T \left(P_1(\tau_k^t) A_\xi + A_\xi^T P_1(\tau_k^t) + \frac{dP_1(\tau_k^t)}{dt} + P_1(\tau_k^t)\Lambda^{-1}P_1(\tau_k^t) \right) \xi_k^t \\
&\qquad + f^T D_\xi^T \Lambda D_\xi f + (\xi_k^{t_{i+1}})^T \left(I_\xi^T P_2(0) I_\xi - P_2(T_i) \right) \xi_k^t,
\end{aligned}
$$

and since $f^T f \le f_0 x^T x \le f_0 (\xi_k^t)^T \xi_k^t$ and $D_\xi^T D_\xi = 2I_n \to |D_\xi^T D_\xi| \le 2$, it is obtained that

$$
\begin{aligned}
\operatorname{div} V(z_k^t, z_{k+1}^{t_{i+1}}) &\le (\xi_k^t)^T \left(P_1(\tau_k^t) A_\xi + A_\xi^T P_1(\tau_k^t) + \frac{dP_1(\tau_k^t)}{dt} + P_1(\tau_k^t)\Lambda^{-1}P_1(\tau_k^t) + 2f_0\Lambda \right) \xi_k^t \\
&\qquad + (\xi_k^{t_{i+1}})^T \left(I_\xi^T P_2(0) I_\xi - P_2(T_i) \right) \xi_k^t.
\end{aligned}
$$

By Theorem 1, it follows that the divergence must satisfy (15). Thus, it is given that

$$\begin{pmatrix} \xi_k^t \\ \xi_k^{t_{i+1}} \end{pmatrix}^T \begin{pmatrix} \begin{array}{c} P_1(\tau_k^t)A_\xi + A_\xi^T P_1(\tau_k^t) + \frac{dP_1(\tau_k^t)}{dt} \\ +P_1(\tau_k^t)\Lambda^{-1}P_1(\tau_k^t) + 2f_0\Lambda \end{array} & \mathbf{0} \\ \mathbf{0} & I_\xi^T P_2(0) I_\xi - P_2(T_i) \end{pmatrix} \begin{pmatrix} \xi_k^t \\ \xi_k^{t_{i+1}} \end{pmatrix} \leq -c_5(|\xi_k^t|^2 + |\xi_k^{t_{i+1}}|^2),$$

$$\begin{pmatrix} \xi_k^t \\ \xi_k^{t_{i+1}} \end{pmatrix}^T \begin{pmatrix} \begin{array}{c} P_1(\tau_k^t)A_\xi + A_\xi^T P_1(\tau_k^t) + \frac{dP_1(\tau_k^t)}{dt} \\ +P_1(\tau_k^t)\Lambda^{-1}P_1(\tau_k^t) + 2f_0\Lambda + c_5 I_{4n} \end{array} & \mathbf{0} \\ \mathbf{0} & I_\xi^T P_2(0) I_\xi - P_2(T_i) + c_5 I_{4n} \end{pmatrix} \begin{pmatrix} \xi_k^t \\ \xi_k^{t_{i+1}} \end{pmatrix} \leq 0.$$

By Schur's complement to the previous inequality, one gets the matrix inequality (21). Then, if (21) is feasible for all $\tau_k^t \in [0, T_i]$, $i = k \in \mathbb{N}$, some $P_1(\tau_k^t) = P_1^T(\tau_k^t) > 0$, continuously differentiable on t and bounded, $P_2(0) = P_2^T(0) > 0$ and $P_2(T_i) = P_2^T(T_i) > 0$ satisfying (19)-(20), $\Lambda = \Lambda > 0$ and a constant $c_5 \in \mathbb{R}_{\geq 0}$, the divergence will satisfy (15).

Thus, based on Theorem 1, if the constraints (16)-(17) are satisfied for the given c_1, c_2, c_3, c_4 and c_5; then the system (7)-(8) will be $ED\xi_k^t$-S for any sequence $\{T_i\}_{i\in\mathbb{N}}$ such that $T_i \in [\frac{c_2}{c_5}\gamma, \frac{c_2}{c_5}\alpha]$. □

Proof of Corollary 2: It is straightforward from the proof of Corollary 1. □

Proof of Proposition 1: Due to the linear structure on τ_k^t given for P_1 and P_2, the matrix inequality (21) is affine in τ_k^t and its negative definiteness is given by the negativeness over the finite set $\tau_k^t \in \{0, T_i\}$.

Therefore, taking into account the structure given by A_ξ and D_ξ, and the fact that $P_{11} = \text{diag}(P_{11}^{-(1)}, P_{11}^{-(2)}, P_{11}^{-(3)}, P_{11}^{-(3)})$, $P_{12} = \text{diag}(P_{12}^{-(1)}, P_{12}^{-(2)}, \delta I_n, \delta I_n)$, with $\delta > 0$, after some algebraic manipulations on matrix inequality (21) given by Corollary 1, it is possible to obtain the following inequalities

$$\Upsilon_1(\Theta) = \left(\begin{array}{c|cc} \Omega_1(\Theta) & \mathbf{0} & P_{11} \\ \hline \star & \phi(\Theta) & \mathbf{0} \\ \star & \star & -\Lambda \end{array}\right) \leq 0, \tag{42}$$

$$\Upsilon_2(\Theta) = \left(\begin{array}{c|cc} \Omega_2(\Theta) & \mathbf{0} & P_{12} \\ \hline \star & \phi(\Theta) & \mathbf{0} \\ \star & \star & -\Lambda \end{array}\right) \leq 0, \tag{43}$$

where $\phi(\Theta) = I_\xi^T P_{21} I_\xi - P_{21} - \Theta P_{22} + Q_5$, $\Lambda = \text{diag}(I_n, I_n, \Lambda_3, \Lambda_4)$, and

$$\Omega_1(\Theta) = \begin{pmatrix} \begin{array}{c} P_{11}^{-(1)}A + A^T P_{11}^{-(1)} + 2f_0 I_n \\ + \frac{(P_{11}^{-(1)} - P_{12}^{-(1)})}{\Theta} + Q_1^{-1} \end{array} & \mathbf{0} & P_{11}^{-(1)}BK & -P_{11}^{-(1)}BK \\ \star & \begin{array}{c} P_{11}^{-(2)}A + A^T P_{11}^{-(2)} + 2f_0 I_n \\ + \frac{(P_{11}^{-(2)} - P_{12}^{-(2)})}{\Theta} + Q_2^{-1} \end{array} & \mathbf{0} & -P_{11}^{-(2)}LC \\ \star & \star & \frac{(P_{11}^{-(3)} - \delta I_n)}{\Theta} + Q_3^{-1} & \mathbf{0} \\ \star & \star & \star & \frac{(P_{11}^{-(3)} - \delta I_n)}{\Theta} + Q_4^{-1} \end{pmatrix},$$

$$\Omega_2(\Theta) = \begin{pmatrix} \begin{array}{c} P_{12}^{-(1)}A + A^T P_{12}^{-(1)} + 2f_0 I_n \\ + \frac{(P_{11}^{-(1)} - P_{12}^{-(1)})}{\Theta} + Q_1^{-1} \end{array} & \mathbf{0} & P_{12}^{-(1)}BK & -P_{12}^{-(1)}BK \\ \star & \begin{array}{c} P_{12}^{-(2)}A + A^T P_{12}^{-(2)} + 2f_0 I_n \\ + \frac{(P_{11}^{-(2)} - P_{12}^{-(2)})}{\Theta} + Q_2^{-1} \end{array} & \mathbf{0} & -P_{12}^{-(2)}LC \\ \star & \star & \frac{(P_{11}^{-(3)} - \delta I_n)}{\Theta} + Q_3^{-1} & \mathbf{0} \\ \star & \star & \star & \frac{(P_{11}^{-(3)} - \delta I_n)}{\Theta} + Q_4^{-1} \end{pmatrix},$$

that should be satisfied for the finite set $\Theta \in \{T_{\min}, T_{\max}\}$. Let us begin with the inequality (42). Applying the quadratic non-singular transformation

$$\mathcal{T}_1 = \text{diag}(P_{11}^{(1)}, P_{11}^{(2)}, I_n, I_n, I_{4n}, I_{4n}),$$

to (42), one gets

$$W_1(\Theta) = \mathcal{T}_1 \Upsilon_1(\Theta) \mathcal{T}_1^T = \left(\begin{array}{c|cc} \bar{\Omega}_1(\Theta) & \mathbf{0} & \bar{P}_{11} \\ \hline \star & \phi(\Theta) & \mathbf{0} \\ \star & \star & -\Lambda \end{array}\right), \tag{44}$$

where

$$\bar{\Omega}_1(\Theta) = \begin{pmatrix} \begin{array}{c}\bar{\phi}_{11}(\Theta) \\ -P_{11}^{(1)}(\frac{P_{12}^{-(1)}}{\Theta} - 2f_0 I_n - Q_1^{-1})P_{11}^{(1)}\end{array} & \mathbf{0} & BK & -BK \\ \star & \begin{array}{c}\bar{\phi}_{12}(\Theta) \\ -P_{11}^{(2)}(\frac{P_{12}^{-(2)}}{\Theta} - 2f_0 I_n - Q_2^{-1})P_{11}^{(2)}\end{array} & \mathbf{0} & -LC \\ \star & \star & \frac{P_{11}^{-(3)} - \delta I_n}{\Theta} + Q_3^{-1} & \mathbf{0} \\ \star & \star & \star & \frac{P_{11}^{-(3)} - \delta I_n}{\Theta} + Q_4^{-1} \end{pmatrix},$$

$$\bar{P}_{11} = \mathrm{diag}(I_n, I_n, P_{11}^{-(3)}, P_{11}^{-(3)}),$$

where $\bar{\phi}_{11}(\Theta) = AP_{11}^{(1)} + P_{11}^{(1)}A^T + P_{11}^{(1)}/\Theta$ and $\bar{\phi}_{12}(\Theta) = AP_{11}^{(2)} + P_{11}^{(2)}A^T + P_{11}^{(2)}/\Theta$. From Λ-inequality, applied with $Y = I$, it follows that

$$X + X^T \leq X\Lambda X^T + \Lambda^{-1},$$

which for $\Lambda^{-1} = \Theta P_{12}^{(1)}$ and $\Lambda^{-1} = \Theta P_{12}^{(2)}$ implies that $-P_{11}^{(1)}P_{12}^{-(1)}P_{11}^{(1)}/\Theta \leq -2P_{11}^{(1)} + \Theta P_{12}^{(1)}$ and $-P_{11}^{(2)}P_{12}^{-(2)}P_{11}^{(2)}/\Theta \leq -2P_{11}^{(2)} + \Theta P_{12}^{(2)}$, respectively. Therefore, the matrix $W_1(\Theta)$ in (44) can be upper estimated as $W_1(\Theta) \leq \overline{W}_1(\Theta)$, where $\overline{W}_1(\Theta)$ is defined as

$$\overline{W}_1(\Theta) = \left(\begin{array}{c|cc} \tilde{\Omega}_1(\Theta) & \mathbf{0} & \bar{P}_{11} \\ \hline \star & \phi(\Theta) & \mathbf{0} \\ \star & \star & -\Lambda \end{array}\right), \tag{45}$$

and

$$\tilde{\Omega}_1(\Theta) = \begin{pmatrix} \phi_{11}(\Theta) + P_{11}^{(1)}(2f_0 I_n + Q_1^{-1})P_{11}^{(1)} & \mathbf{0} & BK & -BK \\ \star & \phi_{12}(\Theta) + P_{11}^{(2)}(2f_0 I_n + Q_2^{-1})P_{11}^{(2)} & \mathbf{0} & -LC \\ \star & \star & \frac{P_{11}^{-(3)} - \delta I_n}{\Theta} + Q_3^{-1} & \mathbf{0} \\ \star & \star & \star & \frac{P_{11}^{-(3)} - \delta I_n}{\Theta} + Q_4^{-1} \end{pmatrix},$$

with $\phi_{11}(\Theta) = \bar{\phi}_{11}(\Theta) - 2P_{11}^{(1)} + \Theta P_{12}^{(1)}$ and $\phi_{12}(\Theta) = \bar{\phi}_{12}(\Theta) - 2P_{11}^{(2)} + \Theta P_{12}^{(2)}$. Then, it is clear that

$$\overline{W}_1(\Theta) = \widetilde{W}_1(\Theta) + \bar{P}^{(1)}(2f_0 I_n + Q_1^{-1})(\bar{P}^{(1)})^T + \bar{P}^{(2)}(2f_0 I_n + Q_2^{-1})(\bar{P}^{(2)})^T, \tag{46}$$

where $(\bar{P}^{(1)})^T = (P_{11}^{(1)}, \mathbf{0}, ..., \mathbf{0})$, $(\bar{P}^{(2)})^T = (\mathbf{0}, P_{11}^{(2)}, \mathbf{0}, ..., \mathbf{0})$ and $\widetilde{W}_1(\Theta)$ is given by as

$$\widetilde{W}_1(\Theta) = \left(\begin{array}{c|cc} \hat{\Omega}_1(\Theta) & \mathbf{0} & \bar{P}_{11} \\ \hline \star & \phi(\Theta) & \mathbf{0} \\ \star & \star & -\Lambda \end{array}\right), \tag{47}$$

where

$$\hat{\Omega}_1(\Theta) = \begin{pmatrix} \phi_{11}(\Theta) & \mathbf{0} & BK & -BK \\ \star & \phi_{12}(\Theta) & \mathbf{0} & -LC \\ \star & \star & \frac{P_{11}^{-(3)} - \delta I_n}{\Theta} + Q_3^{-1} & \mathbf{0} \\ \star & \star & \star & \frac{P_{11}^{-(1)} - \delta I_n}{\Theta} + Q_4^{-1} \end{pmatrix}.$$

For the aperiodic case the gains were $K = (-0.3107, -0.6213)$, $L = (2.5714, 1.9137)^T$ corresponding to $T_i = 0.3$.

By Schur's complement to (46), it is obtained that $\overline{W}_1(\Theta) \leq 0$ is equivalent to $W_2(\Theta) \leq 0$, where $W_2(\Theta)$ is defined by

$$W_2(\Theta) = \left(\begin{array}{c|cc|cc} \widetilde{W}_1(\Theta) & \bar{P}^{(1)} & \bar{P}^{(1)} & \bar{P}^{(2)} & \bar{P}^{(2)} \\ \hline \star & -Q_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \star & \star & -\frac{I_n}{2f_0} & \mathbf{0} & \mathbf{0} \\ \hline \star & \star & \star & -Q_2 & \mathbf{0} \\ \star & \star & \star & \star & -\frac{I_n}{2f_0} \end{array}\right). \tag{48}$$

In an analogous manner, applying the equivalent transformation

$$\mathcal{T}_2 = \mathrm{diag}(I_n, I_n, P_{11}^{(3)}, P_{11}^{(3)}, I_{4n}, I_{4n}, I_{4n}),$$

to the matrix $W_2(\Theta)$, it is obtained

$$\Xi_1(\Theta) = \mathcal{T}_2 W_2(\Theta) \mathcal{T}_2^T = \left(\begin{array}{c|cc|cc} \Gamma_1(\Theta) & \bar{P}^{(1)} & \bar{P}^{(1)} & \bar{P}^{(2)} & \bar{P}^{(2)} \\ \hline \star & -Q_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \star & \star & -\frac{I_n}{2f_0} & \mathbf{0} & \mathbf{0} \\ \hline \star & \star & \star & -Q_2 & \mathbf{0} \\ \star & \star & \star & \star & -\frac{I_n}{2f_0} \end{array}\right), \tag{49}$$

where

$$\Gamma_1(\Theta) = \left(\begin{array}{c|cc} \check{\Omega}_1(\Theta) & \mathbf{0} & \tilde{P}_{11} \\ \hline \star & \phi(\Theta) & \mathbf{0} \\ \star & \star & -\Lambda \end{array}\right), \ \check{\Omega}_1(\Theta) = \begin{pmatrix} \phi_{11}(\Theta) & \mathbf{0} & BKP_{11}^{(3)} & -BKP_{11}^{(3)} \\ \star & \phi_{12}(\Theta) & \mathbf{0} & -LCP_{11}^{(3)} \\ \star & \star & P_{11}^{(3)}\left(\frac{P_{11}^{-(3)} - \delta I_n}{\Theta} + Q_3^{-1}\right)P_{11}^{(3)} & \mathbf{0} \\ \star & \star & \star & P_{11}^{(3)}\left(\frac{P_{11}^{-(3)} - \delta I_n}{\Theta} + Q_4^{-1}\right)P_{11}^{(3)} \end{pmatrix}.$$

with $\tilde{P}_{11} = \mathrm{diag}(I_n, I_n, I_n, I_n)$. By Λ-inequality, it follows that $(P_{11}^{(3)} - \delta P_{11}^{(3)} I_n P_{11}^{(3)})/\Theta \leq P_{11}^{(3)}/\Theta - 2P_{11}^{(3)} + \Theta I_n/\delta = \phi_3(\Theta)$. Therefore, the matrix $\Xi_1(\Theta)$ is upper estimated as $\Xi_1(\Theta) \leq \overline{\Xi}_1(\Theta)$, where $\overline{\Xi}_1(\Theta)$ is given by

$$\overline{\Xi}_1(\Theta) = \left(\begin{array}{c|cc|cc} \overline{\Gamma}_1(\Theta) & \bar{P}^{(1)} & \bar{P}^{(1)} & \bar{P}^{(2)} & \bar{P}^{(2)} \\ \hline \star & -Q_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \star & \star & -\frac{I_n}{2f_0} & \mathbf{0} & \mathbf{0} \\ \hline \star & \star & \star & -Q_2 & \mathbf{0} \\ \star & \star & \star & \star & -\frac{I_n}{2f_0} \end{array}\right), \tag{50}$$

where

$$\overline{\Gamma}_1(\Theta) = \left(\begin{array}{c|cc} \tilde{\Gamma}_1(\Theta) & \mathbf{0} & \tilde{P}_{11} \\ \hline \star & \phi(\Theta) & \mathbf{0} \\ \star & \star & -\Lambda \end{array}\right), \ \tilde{\Gamma}_1(\Theta) = \begin{pmatrix} \phi_{11}(\Theta) & \mathbf{0} & BKP_{11}^{(3)} & -BKP_{11}^{(3)} \\ \star & \phi_{12}(\Theta) & \mathbf{0} & -LCP_{11}^{(3)} \\ \star & \star & \phi_{13}(\Theta) + P_{11}^{(3)} Q_3^{-1} P_{11}^{(3)} & \mathbf{0} \\ \star & \star & \star & \phi_3(\Theta) + P_{11}^{(3)} Q_4^{-1} P_{11}^{(3)} \end{pmatrix}.$$

Then, it is given that

$$\overline{\Xi}_1(\Theta) = \widetilde{\overline{\Xi}}_1(\Theta) + \bar{P}^{(3)} Q_3^{-1} (\bar{P}^{(3)})^T + \bar{P}^{(4)} Q_4^{-1} (\bar{P}^{(4)})^T, \tag{51}$$

where $(\bar{P}^{(3)})^T = (\mathbf{0}, \mathbf{0}, P_{11}^{(3)}, \mathbf{0}, ..., \mathbf{0})$, $(\bar{P}^{(4)})^T = (\mathbf{0}, \mathbf{0}, \mathbf{0}, P_{11}^{(3)}, \mathbf{0}, ..., \mathbf{0})$ and $\widetilde{\overline{\Xi}}_1(\Theta)$ defined by

$$\widetilde{\overline{\Xi}}_1(\Theta) = \left(\begin{array}{c|cc|cc} \Pi_1(\Theta) & \bar{P}^{(1)} & \bar{P}^{(1)} & \bar{P}^{(2)} & \bar{P}^{(2)} \\ \hline \star & -Q_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \star & \star & -\frac{I_n}{2f_0} & \mathbf{0} & \mathbf{0} \\ \hline \star & \star & \star & -Q_2 & \mathbf{0} \\ \star & \star & \star & \star & -\frac{I_n}{2f_0} \end{array}\right), \tag{52}$$

where

$$\Pi_1(\Theta) = \left(\begin{array}{c|cc} \tilde{\Pi}_1(\Theta) & \mathbf{0} & \tilde{P}_{11} \\ \hline \star & \phi(\Theta) & \mathbf{0} \\ \star & \star & -\Lambda \end{array}\right), \ \tilde{\Pi}_1(\Theta) = \begin{pmatrix} \phi_{11}(\Theta) & \mathbf{0} & BY_K & -BY_K \\ \star & \phi_{12}(\Theta) & \mathbf{0} & -LCP_{11}^{(3)} \\ \star & \star & \phi_3(\Theta) & \mathbf{0} \\ \star & \star & \star & \phi_3(\Theta) \end{pmatrix},$$

with $Y_K = KP_{11}^{(3)}$. Then, applying Schur's complement to (51), it is obtained that $\overline{\Xi}_1(\Theta) \leq 0$ is equivalent to $\Xi_2(\Theta) \leq 0$, where $\Xi_2(\Theta)$ is defined as

$$\Xi_2(\Theta) = \left(\begin{array}{c|c|c} \tilde{\overline{\Xi}}_1(\Theta) & \bar{P}^{(3)} & \bar{P}^{(4)} \\ \hline \star & -Q_3 & \mathbf{0} \\ \hline \star & \star & -Q_4 \end{array}\right). \tag{53}$$

Then, the bilinear term $-LCP_{11}^{(3)}$ is simplified as follows. By Λ-inequality with

$$X^T = \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & P_{11}^{(3)} & \mathbf{0} & \cdots & \mathbf{0} \end{pmatrix}^T, \ Y = \begin{pmatrix} \mathbf{0} & -LC & \mathbf{0} & \cdots & \mathbf{0} \end{pmatrix},$$

the matrix $\Xi_2(\Theta)$ can be upper estimated as $\Xi_2(\Theta) \leq \overline{\Xi}_2(\Theta)$, where $\overline{\Xi}_2(\Theta)$ is defined by

$$\overline{\Xi}_2(\Theta) = \left(\begin{array}{c|c|c} \Psi_1(\Theta) & \bar{P}^{(3)} & \bar{P}^{(4)} \\ \hline \star & -Q_3 & \mathbf{0} \\ \hline \star & \star & -Q_4 \end{array}\right), \tag{54}$$

where

$$\Psi_1(\Theta) = \left(\begin{array}{c|cc|cc} \overline{\Psi}_1(\Theta) & \bar{P}^{(1)} & \bar{P}^{(1)} & \bar{P}^{(2)} & \bar{P}^{(2)} \\ \hline \star & -Q_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \star & \star & -\frac{I_n}{2f_0} & \mathbf{0} & \mathbf{0} \\ \hline \star & \star & \star & -Q_2 & \mathbf{0} \\ \star & \star & \star & \star & -\frac{I_n}{2f_0} \end{array}\right), \ \overline{\Psi}_1(\Theta) = \left(\begin{array}{c|cc} \Phi_1(\Theta) & \mathbf{0} & \tilde{P}_{11} \\ \hline \star & \phi(\Theta) & \mathbf{0} \\ \star & \star & -\Lambda \end{array}\right),$$

and

$$\Phi_1(\Theta) = \begin{pmatrix} \phi_{11}(\Theta) & \mathbf{0} & BY_K & -BY_K \\ \star & \bar{\phi}_{12}(\Theta) & \mathbf{0} & \mathbf{0} \\ \star & \star & \phi_3(\Theta) & \mathbf{0} \\ \star & \star & \star & \bar{\phi}_3(\Theta) \end{pmatrix},$$

with

$$\begin{aligned} \bar{\phi}_{12}(\Theta) &= \phi_{12}(\Theta) + LC\bar{\Lambda}_1 C^T L^T, \\ \bar{\phi}_3(\Theta) &= \phi_3(\Theta) + P_{11}^{(3)}\bar{\Lambda}_1^{-1}P_{11}^{(3)}, \end{aligned}$$

for any $\bar{\Lambda}_1 = \bar{\Lambda}_1^T > 0$. Let $R_1,\ R_2 > 0$ be new matrix variables. Then, applying Λ-inequality to $\bar{\phi}_{12}$ it follows that

$$\begin{aligned} \phi_{12}(\Theta) + LC\bar{\Lambda}_1 C^T L^T &\leq \phi_{12}(\Theta) - LC - C^T L^T + \bar{\Lambda}_1^{-1} \leq -R_1, \\ \phi_3(\Theta) + P_{11}^{(3)}\bar{\Lambda}_1^{-1}P_{11}^{(3)} &\leq -R_2, \end{aligned}$$

and applying Schur's complement one gets the LMIs (25), *i.e.*

$$\begin{pmatrix} R_1 + \phi_{12}(\Theta) - LC - C^T L^T & I_n \\ I_n & -\bar{\Lambda}_1 \end{pmatrix} \leq 0, \ \begin{pmatrix} R_2 + \phi_3(\Theta) & P_{11}^{(3)} \\ P_{11}^{(3)} & -\bar{\Lambda}_1 \end{pmatrix} \leq 0.$$

Thus the term Φ_1 is upper estimated as $\Phi_1(\Theta) \leq \bar{\Phi}_1(\Theta)$, where $\bar{\Phi}_1(\Theta)$ is defined by

$$\bar{\Phi}_1(\Theta) = \begin{pmatrix} \phi_{11}(\Theta) & \mathbf{0} & BY_K & -BY_K \\ \star & -R_1 & \mathbf{0} & \mathbf{0} \\ \star & \star & \phi_3(\Theta) & \mathbf{0} \\ \star & \star & \star & -R_2 \end{pmatrix}.$$

Therefore, the matrix $\overline{\Xi}_2(\Theta)$ can be upper estimated as $\overline{\Xi}_2(\Theta) \leq \Xi_3(\Theta)$, where $\Xi_3(\Theta)$, is defined by

$$\Xi_3(\Theta) = \left(\begin{array}{c|c|c} \widehat{\Psi}_1(\Theta) & \bar{P}^{(3)} & \bar{P}^{(4)} \\ \hline \star & -Q_3 & \mathbf{0} \\ \hline \star & \star & -Q_4 \end{array}\right), \tag{55}$$

where

$$\widehat{\Psi}_1(\Theta) = \left(\begin{array}{c|cc|cc} \grave{\Psi}_1(\Theta) & \bar{P}^{(1)} & \bar{P}^{(1)} & \bar{P}^{(2)} & \bar{P}^{(2)} \\ \hline \star & -Q_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \star & \star & -\frac{I_n}{2f_0} & \mathbf{0} & \mathbf{0} \\ \hline \star & \star & \star & -Q_2 & \mathbf{0} \\ \star & \star & \star & \star & -\frac{I_n}{2f_0} \end{array}\right), \quad \grave{\Psi}_1(\Theta) = \left(\begin{array}{c|cc} \bar{\Phi}_1(\Theta) & \mathbf{0} & \tilde{P}_{11} \\ \hline \star & \phi(\Theta) & \mathbf{0} \\ \star & \star & -\Lambda \end{array}\right).$$

Therefore, the LMI (23) is obtained when all the elements of $\Xi_3(\Theta)$ are merged, and it is concluded that if the set of LMIs (23) and (25) is feasible then (42) holds, *i.e.* $\Upsilon_1(\Theta) \leq 0$.

To conclude the proof, it is clear that a similar method may be used to obtain the LMIs (24) and (26) by means of inequality (43), and prove that if the set of LMIs (24) and (26) is feasible then (42) also holds, *i.e.* $\Upsilon_2(\Theta) \leq 0$. This procedure is omitted for the sake of brevity. Thus, the theorem is proven. $\square$

Proof of Proposition 2: Due to the linear structure on τ_k^t given for P_1 and P_2, the matrix inequality (21) is affine in τ_k^t and its negative definiteness is given by the negativeness over the finite set $\tau_k^t \in \{0, T_i\}$.

Thus, given the structure of A_ξ and D_ξ, and the fact that $P_{11} = \mathrm{diag}(P_{11}^{-(1)}, P_{11}^{-(2)}, P_{11}^{-(3)}, P_{11}^{-(3)})$, $P_{12} = \mathrm{diag}(P_{12}^{-(1)}, P_{12}^{-(2)}, \delta I_n, \delta I_n)$, with $\delta > 0$, after some algebraic manipulations on matrix inequality (22) given by Corollary 2, it is possible to obtain the following inequality

$$\Upsilon_1(\Theta) = \left(\begin{array}{c|c} \Omega_1(\Theta) & \mathbf{0} \\ \mathbf{0} & \phi(\Theta) \end{array}\right) \leq 0, \tag{56}$$

$$\Upsilon_2(\Theta) = \left(\begin{array}{c|c} \Omega_2(\Theta) & \mathbf{0} \\ \mathbf{0} & \phi(\Theta) \end{array}\right) \leq 0, \tag{57}$$

where $\phi(\Theta) = I_\xi^T P_{21} I_\xi - P_{21} - \Theta P_{22} + Q_5$ and

$$\Omega_1(\Theta) = \begin{pmatrix} \begin{array}{c} P_{11}^{-(1)} A + A^T P_{11}^{-(1)} \\ + \frac{(P_{11}^{-(1)} - P_{12}^{-(1)})}{\Theta} + Q_1^{-1} \end{array} & \mathbf{0} & P_{11}^{-(1)} BK & -P_{11}^{-(1)} BK \\ \star & \begin{array}{c} P_{11}^{-(2)} A + A^T P_{11}^{-(2)} \\ + \frac{(P_{11}^{-(2)} - P_{12}^{-(2)})}{\Theta} + Q_2^{-1} \end{array} & \mathbf{0} & -P_{11}^{-(2)} LC \\ \star & \star & \frac{(P_{11}^{-(3)} - \delta I_n)}{\Theta} + Q_3^{-1} & \mathbf{0} \\ \star & \star & \star & \frac{(P_{11}^{-(3)} - \delta I_n)}{\Theta} + Q_4^{-1} \end{pmatrix},$$

$$\Omega_2(\Theta) = \begin{pmatrix} \begin{array}{c} P_{12}^{-(1)} A + A^T P_{12}^{-(1)} \\ + \frac{(P_{11}^{-(1)} - P_{12}^{-(1)})}{\Theta} + Q_1^{-1} \end{array} & \mathbf{0} & P_{12}^{-(1)} BK & -P_{12}^{-(1)} BK \\ \star & \begin{array}{c} P_{12}^{-(2)} A + A^T P_{12}^{-(2)} \\ + \frac{(P_{11}^{-(2)} - P_{12}^{-(2)})}{\Theta} + Q_2^{-1} \end{array} & \mathbf{0} & -P_{12}^{-(2)} LC \\ \star & \star & \frac{(P_{11}^{-(3)} - \delta I_n)}{\Theta} + Q_3^{-1} & \mathbf{0} \\ \star & \star & \star & \frac{(P_{11}^{-(3)} - \delta I_n)}{\Theta} + Q_4^{-1} \end{pmatrix},$$

that should hold for the finite set $\Theta \in \{T_{\min}, T_{\max}\}$. Let us begin with the inequality (56). Applying the quadratic non-singular transformation

$$\mathcal{T}_1 = \mathrm{diag}(P_{11}^{(1)}, P_{11}^{(2)}, I_n, I_n, I_{4n}),$$

to (56), one gets

$$W_1(\Theta) = \mathcal{T}_1 \Upsilon_1(\Theta) \mathcal{T}_1^T = \left(\begin{array}{c|c} \bar{\Omega}_1(\Theta) & \mathbf{0} \\ \mathbf{0} & \phi(\Theta) \end{array}\right), \tag{58}$$

where

$$\bar{\Omega}_1(\Theta) = \begin{pmatrix} \bar{\phi}_{11}(\Theta) - P_{11}^{(1)}(\frac{P_{12}^{-(1)}}{\Theta} - Q_1^{-1})P_{11}^{(1)} & \mathbf{0} & BK & -BK \\ \star & \bar{\phi}_{12}(\Theta) - P_{11}^{(2)}(\frac{P_{12}^{-(2)}}{\Theta} - Q_2^{-1})P_{11}^{(2)} & \mathbf{0} & -LC \\ \star & \star & \frac{P_{11}^{-(3)} - \delta I_n}{\Theta} + Q_3^{-1} & \mathbf{0} \\ \star & \star & \star & \frac{P_{11}^{-(3)} - \delta I_n}{\Theta} + Q_4^{-1} \end{pmatrix},$$

where $\bar{\phi}_{11}(\Theta) = AP_{11}^{(1)} + P_{11}^{(1)}A^T + P_{11}^{(1)}/\Theta$ and $\bar{\phi}_{12}(\Theta) = AP_{11}^{(2)} + P_{11}^{(2)}A^T + P_{11}^{(2)}/\Theta$. From Λ-inequality, it follows that $-P_{11}^{(1)}P_{12}^{-(1)}P_{11}^{(1)}/\Theta \leq -2P_{11}^{(1)} + \Theta P_{12}^{(1)}$ and $-P_{11}^{(2)}P_{12}^{-(2)}P_{11}^{(2)}/\Theta \leq -2P_{11}^{(2)} + \Theta P_{12}^{(2)}$, respectively. Therefore, the matrix $W_1(\Theta)$ in (58) can be upper estimated as $W_1(\Theta) \leq \overline{W}_1(\Theta)$, where $\overline{W}_1(\Theta)$ is defined as

$$\overline{W}_1(\Theta) = \left(\begin{array}{c|c} \tilde{\Omega}_1(\Theta) & \mathbf{0} \\ \hline \mathbf{0} & \phi_1(\Theta) \end{array}\right), \tag{59}$$

and

$$\tilde{\Omega}_1(\Theta) = \begin{pmatrix} \phi_{11}(\Theta) + P_{11}^{(1)}Q_1^{-1}P_{11}^{(1)} & \mathbf{0} & BK & -BK \\ \star & \phi_{12}(\Theta) + P_{11}^{(2)}Q_2^{-1}P_{11}^{(2)} & \mathbf{0} & -LC \\ \star & \star & \frac{P_{11}^{-(3)} - \delta I_n}{\Theta} + Q_3^{-1} & \mathbf{0} \\ \star & \star & \star & \frac{P_{11}^{-(3)} - \delta I_n}{\Theta} + Q_4^{-1} \end{pmatrix},$$

with $\phi_{11}(\Theta) = \bar{\phi}_{11}(\Theta) - 2P_{11}^{(1)} + \Theta P_{12}^{(1)}$ and $\phi_{12}(\Theta) = \bar{\phi}_{12}(\Theta) - 2P_{11}^{(2)} + \Theta P_{12}^{(2)}$. Then, it is clear that

$$\overline{W}_1(\Theta) = \widetilde{W}_1(\Theta) + \bar{P}^{(1)}Q_1^{-1}(\bar{P}^{(1)})^T + \bar{P}^{(2)}Q_2^{-1}(\bar{P}^{(2)})^T, \tag{60}$$

where $(\bar{P}^{(1)})^T = (P_{11}^{(1)}, \mathbf{0}, ..., \mathbf{0})$, $(\bar{P}^{(2)})^T = (\mathbf{0}, P_{11}^{(2)}, \mathbf{0}, ..., \mathbf{0})$ and $\widetilde{W}_1(\Theta)$ is given by as

$$\widetilde{W}_1(\Theta) = \left(\begin{array}{c|c} \hat{\Omega}_1(\Theta) & \mathbf{0} \\ \hline \mathbf{0} & \phi_1(\Theta) \end{array}\right), \tag{61}$$

where

$$\hat{\Omega}_1(\Theta) = \begin{pmatrix} \phi_{11}(\Theta) & \mathbf{0} & BK & -BK \\ \star & \phi_{12}(\Theta) & \mathbf{0} & -LC \\ \star & \star & \frac{P_{11}^{-(3)} - \delta I_n}{\Theta} + Q_3^{-1} & \mathbf{0} \\ \star & \star & \star & \frac{P_{11}^{-(1)} - \delta I_n}{\Theta} + Q_4^{-1} \end{pmatrix}.$$

Then, by Schur's complement to (60) twice, it is obtained that $\overline{W}_1(\Theta) \leq 0$ is equivalent to $W_2(\Theta) \leq 0$, where $W_2(\Theta)$ is defined as

$$W_2(\Theta) = \left(\begin{array}{c|c|c} \widetilde{W}_1(\Theta) & \bar{P}^{(1)} & \bar{P}^{(2)} \\ \hline \star & -Q_1 & \mathbf{0} \\ \hline \star & \star & -Q_2 \end{array}\right). \tag{62}$$

In a similar way, applying the equivalent transformation

$$\mathcal{T}_2 = \mathrm{diag}(I_n, I_n, P_{11}^{(3)}, P_{11}^{(3)}, I_n, I_{4n}, I_n),$$

to the matrix $W_2(\Theta)$, it is obtained

$$\Xi_1(\Theta) = \mathcal{T}_2 W_2(\Theta)\mathcal{T}_2^T = \left(\begin{array}{c|c|c} \Gamma_1(\Theta) & \bar{P}^{(1)} & \bar{P}^{(2)} \\ \hline \star & -Q_1 & \mathbf{0} \\ \hline \star & \star & -Q_2 \end{array}\right), \tag{63}$$

where

$$\Gamma_1(\Theta) = \left(\begin{array}{c|c} \check{\Omega}_1(\Theta) & \mathbf{0} \\ \hline \mathbf{0} & \phi(\Theta) \end{array}\right),\ \check{\Omega}_1(\Theta) = \begin{pmatrix} \phi_{11}(\Theta) & \mathbf{0} & BKP_{11}^{(3)} & -BKP_{11}^{(3)} \\ \star & \phi_{12}(\Theta) & \mathbf{0} & -LCP_{11}^{(3)} \\ \star & \star & P_{11}^{(3)}\left(\frac{P_{11}^{-(3)}-\delta I_n}{\Theta} + Q_3^{-1}\right)P_{11}^{(3)} & \mathbf{0} \\ \star & \star & \star & P_{11}^{(3)}\left(\frac{P_{11}^{-(3)}-\delta I_n}{\Theta} + Q_4^{-1}\right)P_{11}^{(3)} \end{pmatrix}.$$

By Λ-inequality, it follows that $(P_{11}^{(3)} - \delta P_{11}^{(3)} I_n P_{11}^{(3)})/\Theta \leq P_{11}^{(3)}/\Theta - 2P_{11}^{(3)} + \Theta I_n/\delta = \phi_3(\Theta)$. Therefore, the matrix $\Xi_1(\Theta)$ is upper estimated as $\Xi_1(\Theta) \leq \overline{\Xi}_1(\Theta)$, where $\overline{\Xi}_1(\Theta)$ is given by

$$\overline{\Xi}_1(\Theta) = \left(\begin{array}{c|c|c} \overline{\Gamma}_1(\Theta) & \bar{P}^{(1)} & \bar{P}^{(2)} \\ \hline \star & -Q_1 & \mathbf{0} \\ \hline \star & \star & -Q_2 \end{array}\right), \tag{64}$$

where

$$\overline{\Gamma}_1(\Theta) = \left(\begin{array}{c|c} \tilde{\Gamma}_1(\Theta) & \mathbf{0} \\ \hline \mathbf{0} & \phi(\Theta) \end{array}\right),\ \tilde{\Gamma}_1(\Theta) = \begin{pmatrix} \phi_{11}(\Theta) & \mathbf{0} & BKP_{11}^{(3)} & -BKP_{11}^{(3)} \\ \star & \phi_{12}(\Theta) & \mathbf{0} & -LCP_{11}^{(3)} \\ \star & \star & \phi_{13}(\Theta) + P_{11}^{(3)}Q_3^{-1}P_{11}^{(3)} & \mathbf{0} \\ \star & \star & \star & \phi_3(\Theta) + P_{11}^{(3)}Q_4^{-1}P_{11}^{(3)} \end{pmatrix}.$$

Then, it is given that

$$\overline{\Xi}_1(\Theta) = \widetilde{\Xi}_1(\Theta) + \bar{P}^{(3)}Q_3^{-1}(\bar{P}^{(3)})^T + \bar{P}^{(4)}Q_4^{-1}(\bar{P}^{(4)})^T, \tag{65}$$

where $(\bar{P}^{(3)})^T = (\mathbf{0}, \mathbf{0}, P_{11}^{(3)}, \mathbf{0}, ..., \mathbf{0})$, $(\bar{P}^{(4)})^T = (\mathbf{0}, \mathbf{0}, \mathbf{0}, P_{11}^{(3)}, \mathbf{0}, ..., \mathbf{0})$ and $\widetilde{\Xi}_1(\Theta)$ defined by

$$\widetilde{\Xi}_1(\Theta) = \left(\begin{array}{c|c|c} \Pi_1(\Theta) & \bar{P}^{(1)} & \bar{P}^{(2)} \\ \hline \star & -Q_1 & \mathbf{0} \\ \hline \star & \star & -Q_2 \end{array}\right), \tag{66}$$

where

$$\Pi_1(\Theta) = \left(\begin{array}{c|c} \tilde{\Pi}_1(\Theta) & \mathbf{0} \\ \hline \mathbf{0} & \phi(\Theta) \end{array}\right),\ \tilde{\Pi}_1(\Theta) = \begin{pmatrix} \phi_{11}(\Theta) & \mathbf{0} & BY_K & -BY_K \\ \star & \phi_{12}(\Theta) & \mathbf{0} & -LCP_{11}^{(3)} \\ \star & \star & \phi_3(\Theta) & \mathbf{0} \\ \star & \star & \star & \phi_3(\Theta) \end{pmatrix},$$

with $Y_K = KP_{11}^{(3)}$. Thus, applying Schur's complement to (66), it is obtained that $\overline{\Xi}_1(\Theta) \leq 0$ is equivalent to $\Xi_2(\Theta) \leq 0$, where $\Xi_2(\Theta)$ is defined as

$$\Xi_2(\Theta) = \left(\begin{array}{c|c|c} \widetilde{\Xi}_1(\Theta) & \bar{P}^{(3)} & \bar{P}^{(4)} \\ \hline \star & -Q_3 & \mathbf{0} \\ \hline \star & \star & -Q_4 \end{array}\right). \tag{67}$$

Then, the bilinear term $-LCP_{11}^{(3)}$ is simplified as follows. By Λ-inequality with

$$X^T = \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & P_{11}^{(3)} & \mathbf{0} & \cdots & \mathbf{0} \end{pmatrix}^T,\ Y = \begin{pmatrix} \mathbf{0} & -LC & \mathbf{0} & \cdots & \mathbf{0} \end{pmatrix},$$

the matrix $\Xi_2(\Theta)$ can be upper estimated as $\Xi_2(\Theta) \leq \overline{\Xi}_2(\Theta)$, where $\overline{\Xi}_2(\Theta)$ is defined by

$$\overline{\Xi}_2(\Theta) = \left(\begin{array}{c|c|c} \Psi_1(\Theta) & \bar{P}^{(3)} & \bar{P}^{(4)} \\ \hline \star & -Q_3 & \mathbf{0} \\ \hline \star & \star & -Q_4 \end{array}\right), \tag{68}$$

where

$$\Psi_1(\Theta) = \left(\begin{array}{c|c|c} \overline{\Psi}_1(\Theta) & \bar{P}^{(1)} & \bar{P}^{(2)} \\ \hline \star & -Q_1 & \mathbf{0} \\ \hline \star & \star & -Q_2 \end{array}\right),\ \overline{\Psi}_1(\Theta) = \left(\begin{array}{c|c} \Phi_1(\Theta) & \mathbf{0} \\ \hline \star & \phi(\Theta) \end{array}\right),$$

and

$$\Phi_1(\Theta) = \begin{pmatrix} \phi_{11}(\Theta) & \mathbf{0} & BY_K & -BY_K \\ \star & \bar{\phi}_{12}(\Theta) & \mathbf{0} & \mathbf{0} \\ \star & \star & \phi_3(\Theta) & \mathbf{0} \\ \star & \star & \star & \bar{\phi}_3(\Theta) \end{pmatrix},$$

with

$$\bar{\phi}_{12}(\Theta) = \phi_{12}(\Theta) + LC\bar{\Lambda}_1 C^T L^T,$$
$$\bar{\phi}_3(\Theta) = \phi_3(\Theta) + P_{11}^{(3)}\bar{\Lambda}_1^{-1}P_{11}^{(3)},$$

for any $\bar{\Lambda}_1 = \bar{\Lambda}_1^T > 0$. Let $R_1,\ R_2 > 0$ be new matrix variables. Then, applying Λ-inequality to $\bar{\phi}_{12}$ it follows that

$$\phi_{12}(\Theta) + LC\bar{\Lambda}_1 C^T L^T \leq \phi_{12}(\Theta) - LC - C^T L^T + \bar{\Lambda}_1^{-1} \leq -R_1,$$
$$\phi_3(\Theta) + P_{11}^{(3)}\bar{\Lambda}_1^{-1}P_{11}^{(3)} \leq -R_2,$$

and applying Schur's complement one gets the LMIs (29), *i.e.*

$$\begin{pmatrix} R_1 + \phi_{12}(\Theta) - LC - C^T L^T & I_n \\ I_n & -\bar{\Lambda}_1 \end{pmatrix} \leq 0, \quad \begin{pmatrix} R_2 + \phi_3(\Theta) & P_{11}^{(3)} \\ P_{11}^{(3)} & -\bar{\Lambda}_1 \end{pmatrix} \leq 0.$$

Thus the term Φ_1 is upper estimated as $\Phi_1(\Theta) \leq \bar{\Phi}_1(\Theta)$, where $\bar{\Phi}_1(\Theta)$ is defined by

$$\bar{\Phi}_1(\Theta) = \begin{pmatrix} \phi_{11}(\Theta) & \mathbf{0} & BY_K & -BY_K \\ \star & -R_1 & \mathbf{0} & \mathbf{0} \\ \star & \star & \phi_3(\Theta) & \mathbf{0} \\ \star & \star & \star & -R_2 \end{pmatrix}.$$

Therefore, the matrix $\overline{\Xi}_2(\Theta)$ can be upper estimated as $\overline{\Xi}_2(\Theta) \leq \Xi_3(\Theta)$, where $\Xi_3(\Theta)$, is defined by

$$\Xi_3(\Theta) = \left(\begin{array}{c|c|c} \widehat{\Psi}_1(\Theta) & \bar{P}^{(3)} & \bar{P}^{(4)} \\ \hline \star & -Q_3 & \mathbf{0} \\ \hline \star & \star & -Q_4 \end{array}\right), \tag{69}$$

where

$$\widehat{\Psi}_1(\Theta) = \left(\begin{array}{c|c|c} \grave{\Psi}_1(\Theta) & \bar{P}^{(1)} & \bar{P}^{(2)} \\ \hline \star & -Q_1 & \mathbf{0} \\ \hline \star & \star & -Q_2 \end{array}\right), \quad \grave{\Psi}_1(\Theta) = \left(\begin{array}{c|c} \bar{\Phi}_1(\Theta) & \mathbf{0} \\ \hline \star & \phi(\Theta) \end{array}\right).$$

Therefore, the LMI (27) is obtained when all the elements of $\widehat{\Xi}_3(\Theta)$ are merged, and one conclude that if the set of LMIs (27) and (29) is feasible then $\Upsilon_1(\Theta) \leq 0$.

To conclude the proof, it is clear that a similar method may be used to obtain the LMIs (28) and (30) by means of inequality $\Upsilon_2(\Theta) \leq 0$. However, this procedure is omitted for the sake of brevity. Hence, the theorem is proven. □

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