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# ABEL AND GALOIS CANNOT SHARE A CAKE IN A SIMPLE AND EQUITABLE WAY

GUILLAUME CHÈZE

ABSTRACT. In this note we study a fair division problem. We show that there exist explicit couples of measures for which no algorithm outputs an equitable fair division with connected parts.

In order to state this result we have defined a new model of computation: the *algebraic Robertson-Webb model*. In this model the mediator can ask the same queries as in the usual model but he or she can only perform algebraic operations. All existing algorithms described in the classical Robertson-Webb model can be described in this new model.

The main tool of our approach is Galois theory.

## INTRODUCTION

In 1837, Pierre Wantzel has shown that there exists no general construction using only compass and straightedge which divides an angle into three equal angles. The proof relies on algebra and field theory. The angle trisection problem can be seen as a fair division problem: we have a portion of pizza and we want to divide it in an equitable way between three friends (by using only compass and straightedge constructions...). Wantzel's theorem says that this problem has no solution.

In this article, we are going to study a similar fair division problem and we are going to use similar tools.

In the following, we consider an heterogeneous good, for example: a cake, land, time or computer memory, represented by the interval  $X = [0, 1]$  and  $n$  players with different points of view. We associate to each player a non-atomic probability measure  $\mu_i$  on the interval  $X = [0, 1]$ . These measures represent the utility functions of the player. This means that if  $[a, b] \subset X$  is a part of the cake then  $\mu_i([a, b])$  is the value associated by the  $i$ -th player to this part of the cake. As  $\mu_i$  are probability measures, we have  $\mu_i(X) = 1$  for all  $i$ .

A division of  $X$  is a partition  $X = \sqcup_i X_i$  where  $X_i$  is the part given to the  $i$ -th player. A division is *simple* when each  $X_i$  is an interval.

Several notions of fair division exist.

We say that a division is *proportional* when  $\mu_i(X_i) \geq 1/n$ .

We say that a division is *envy-free* when for  $i \neq j$ , we have  $\mu_i(X_i) \geq \mu_i(X_j)$ .

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We say that a division is *equitable* when for all  $i \neq j$ , we have  $\mu_i(X_i) = \mu_j(X_j)$ .

The problem of fair division (theoretical existence of fair division and construction of algorithms) has been studied in several papers [Ste48, DS61, EP84, EP11, BT95, RW97, Pik00, Tho06, Pro13, BJK13, AM16], and books about this topic, see e.g. [RW98, BT96, Pro16, Bar05]. These results appear in the mathematics, economics, political science, artificial intelligence and computer science literature. Recently, the cake cutting problem has been studied intensively by computer scientists for solving resource allocation problems in multi agents systems, see e.g. [CDE+06, CLPP13, KPS13, BM15].

In this note we are going to study simple equitable fair divisions. This topic has been less studied than proportional and envy-free divisions. However, there exist some results showing the existence of such fair divisions [CDP13, SHS18, Chè17].

A practical problem is the computation of fair divisions. In order to describe algorithms we thus need a model of computation. There exist two main classes of cake cutting algorithms: discrete and continuous protocols (also called moving knife methods). Here, we study only discrete algorithms. These kinds of algorithms can be described thanks to the classical model introduced by Robertson and Webb and formalized by Woeginger and Sgall in [WS07]. In this model we suppose that a mediator interacts with the agents. The mediator asks two type of queries: either cutting a piece with a given value, or evaluating a given piece. More precisely, the two type of queries allowed are:

- (1)  $eval_i(x, y)$ : Ask agent  $i$  to evaluate the interval  $[x, y]$ . This means compute  $\mu_i([x, y])$ .
- (2)  $cut_i(x, a)$ : Asks agent  $i$  to cut a piece of cake  $[x, y]$  such that  $\mu_i([x, y]) = a$ . This means: for given  $x$  and  $a$ , solve  $\mu_i([x, y]) = a$ .

In the Robertson-Webb model the mediator can adapt the queries from the previous answers given by the players. In this model, the complexity counts the finite number of queries necessary to get a fair division. For a rigorous description of this model we can consult: [WS07, BN17].

In the fair division literature some impossibility results have been already given. Stromquist in [Str08] has proved that there exists no algorithm giving a simple and envy-free fair division for  $n \geq 3$  players. When  $n = 2$ , the classical ‘‘Cut and Choose’’ algorithm gives a simple and envy-free fair division.

Cechlárová et al. have shown, in [CP12], that there exists no algorithm computing a simple and equitable fair division for  $n \geq 3$  players.

The strategy used in these articles is the following: we suppose that an algorithm computing the desired division exists and then by an iteration process we construct from this algorithm a set of measures giving a contradiction. Thus we obtain a result of this kind: for all algorithms in the Robertson-Webb model there exists a set of measures for which the desired fair division cannot be given.

It must be noticed that this approach gives for each algorithm a set of measures leading to a contradiction. Thus the set of measures is related to the algorithm. Moreover, the measures are not explicitly given. Therefore, we can imagine that these sets of measure correspond to very complicated situations not appearing in

practice and that for “reasonable” sets of measures the contradiction does not occur.

Furthermore, this strategy does not give an explicit set of measures such that no algorithm in the Robertson-Webb model can return the desired fair division with this input.

In this note we study simple equitable divisions when we have  $n = 2$  players. It must be noticed that in general we cannot deduce a result for two players from a result about  $n \geq 3$  players, see the envy-free situation. Furthermore, the situation  $n = 2$  is not taken into account by the theorem given in [CP12]. Thus, the case  $n = 2$  is not a trivial case.

Now, in order to state our main result, we introduce our model of computation.

### 1. THE ALGEBRAIC ROBERTSON-WEBB MODEL

In the Robertson-Webb model of computation the computational power of the mediator is not specified. It is not mentioned what kind of computations the mediator can perform with the results of the queries. Furthermore, the number of elementary operations done by the mediator (equality and inequality tests and arithmetic operations  $+$ ,  $-$ ,  $\times$ ,  $\div$ ) is not taken into account in the complexity. This point has been discussed in [Chè18].

Here, we suppose as in the classical model that the mediator can use the  $cut_i$  and  $eval_i$  queries. However, we also suppose that the mediator can *only* perform equality and inequality tests and the usual algebraic operations:  $+$ ,  $-$ ,  $\times$ ,  $\div$  on the results of queries. We also suppose that the mediator can use freely the rational numbers.

These assumptions are not restrictive. Indeed, no known algorithm uses the computation of a logarithm or of an exponential by the mediator or more generally the computation of a transcendental function.

Furthermore, when the mediator needs a constant during the algorithm this constant is always a rational number. Indeed, in practice the mediator never asks a query of the form  $cut_i(0, \frac{e}{4}\mu_i(X))$ , where  $e = 2,718\dots$  is Napier’s constant. Queries have the form  $cut_i(0, \frac{\mu_i(X)}{n})$  or  $cut_i(0, \frac{p}{q}\mu_i(X))$ , where  $p, q, n$  are integers.

Therefore, if we suppose that the answer to the first three queries are denoted by  $\alpha_1, \alpha_2, \alpha_3$ , then in this new model, the fourth query is of the form  $cut_i(\beta_1, \beta_2)$  or  $eval_i(\beta_1, \beta_1)$  where  $\beta_1, \beta_2 \in \mathbb{Q}(\alpha_1, \alpha_2, \alpha_3)$ . This means that  $\beta_1$  and  $\beta_2$  are rational expressions in terms of  $\alpha_1, \alpha_2, \alpha_3$ .

The algebraic assumption is not restrictive and to author’s knowledge all algorithms written in the classical Robertson-Webb model can be written in this algebraic Robertson-Webb model. However, these precisions are important for our study. Indeed, if the algorithm uses  $k$  queries with answers  $\alpha_1, \dots, \alpha_k$  for computing a fair division, then the cutpoints used in the output of the algorithm must belong to  $\mathbb{Q}(\alpha_1, \alpha_2, \dots, \alpha_k)$ . This gives an algebraic condition for the final cutpoints. Using this algebraic condition, we can show:

**Theorem 1.** *In the algebraic Robertson-Webb model of computation no algorithm returns a simple and equitable division when the measures  $(\mu_1, \mu_2)$  are given by*

$$\mu_1([0, x]) = x, \quad \mu_2([0, x]) = x^5.$$

The strategy used to prove this theorem is the following: We are going to show that if there exists an equitable and simple division  $X = [0, t] \sqcup [t, 1]$  then the final cutpoint  $t$  must satisfy a polynomial equation. Then with elementary field theory we can show that  $t$  cannot be computed with the algebraic Robertson-Webb model. Now, if we use Abel's impossibility theorem and Galois' theory showing that some polynomials are not solved by radicals, then we obtain other examples as stated in the next theorem:

**Theorem 2.** *In the algebraic Robertson-Webb model of computations there exist measures  $(\mu_1, \mu_2)$  such that no algorithm returns a simple and equitable division for these measures.*

*Furthermore, we can take  $(\mu_1, \mu_2)$  in the following way:*

$$\mu_1([0, x]) = x, \quad \mu_2([0, x]) = x^d$$

where

- $d \geq 5$  is even,
- or  $d \geq 5$  is odd with  $d \not\equiv 2 \pmod{3}$ ,
- or  $d \geq 5$  is prime and  $d \equiv 2 \pmod{3}$ .

Thus, when we have two players, that we can call Abel and Galois, we can give *easy and explicit* couples of measures for which no algorithm in the algebraic Robertson-Webb model gives a simple and equitable fair division.

**Notations.** In order to explain the strategy used to prove this theorem we introduce some notations:

For a couple of measure  $(\mu_1, \mu_2)$  we denote by  $f_i$ ,  $i = 1, 2$  the function

$$f_i(x) = \mu_i([0, x]).$$

Let  $\alpha_j$  be the result of the  $j$ -th query, then we set

$$\mathbb{K}_j = \mathbb{Q}(\alpha_1, \dots, \alpha_j).$$

We thus have  $\mathbb{K}_j = \mathbb{K}_{j-1}(\alpha_j)$  and  $\mathbb{K}_0 = \mathbb{Q}$ .

We recall that when a field  $\mathbb{F}$  is a subfield of a field  $\mathbb{K}$  then we say that we have a field extension and this is denoted by  $\mathbb{K}/\mathbb{F}$ . Furthermore, the dimension of  $\mathbb{K}$  seen as a  $\mathbb{F}$ -vector space is called the degree of the extension and is denoted by  $[\mathbb{K} : \mathbb{F}]$ . When the degree is finite we say that the extension is finite. Moreover, when we have the inclusion  $\mathbb{F} \subset \mathbb{K} \subset \mathbb{L}$ , this gives two extensions  $\mathbb{L}/\mathbb{K}$  and  $\mathbb{K}/\mathbb{F}$ . If the degree of these two extensions are finite then the extension  $\mathbb{L}/\mathbb{F}$  is also finite and we have the following equality:  $[\mathbb{L} : \mathbb{F}] = [\mathbb{L} : \mathbb{K}][\mathbb{K} : \mathbb{F}]$ , see e.g. [Tig01, Lemma 15.3]. Furthermore, we recall that if  $\alpha$  is a root of an irreducible polynomial in  $\mathbb{K}[X]$  with degree  $d$  then  $[\mathbb{K}(\alpha) : \mathbb{K}] = d$ , see [Tig01, Proposition 12.15].

## 2. PROOFS

The idea used to prove our theorems is the following:

If there exists a simple and equitable fair division  $X = [0, t] \sqcup [t, 1]$ , then we have

$$\mu_1([0, t]) = \mu_2([t, 1]) \text{ or } \mu_2([0, t]) = \mu_1([t, 1]).$$

This gives

$$f_1(t) = 1 - f_2(t) \text{ or } f_2(t) = 1 - f_1(t).$$

These two equations are equivalent to the following one:

$$(E) \quad f_1(t) + f_2(t) - 1 = 0.$$

This equation gives a first condition on  $t$ .

The following lemma gives another condition.

**Lemma 3.** *If  $f_1(x) = x$ , and  $f_2(x) = x^p$  with  $p \geq 3$  a prime number then the degree of the field extension  $\mathbb{K}_j/\mathbb{K}_{j-1}$  is equal to  $p$  or 1.*

*Proof.* By definition we have  $\mathbb{K}_j = \mathbb{K}_{j-1}(\alpha_j)$ .

Two situations appears:

First,  $\alpha_j = eval_i(x, y)$ , where  $x, y \in \mathbb{K}_{j-1}$ .

As,  $eval_1(x, y) = y - x$  and  $eval_2(x, y) = y^p - x^p$ , we deduce that in this case  $\alpha_j \in \mathbb{K}_{j-1}$ . Thus  $\mathbb{K}_j = \mathbb{K}_{j-1}$  and the degree of the extension is equal to one.

Second,  $\alpha_j = cut_i(x, a)$ , where  $x, a \in \mathbb{K}_{j-1}$ .

If  $i = 1$  then  $\alpha_j$  is the solution of  $\alpha_j - x = a$  then  $\alpha_j \in \mathbb{K}_{j-1}$  and the degree of the field extension is equal to one.

If  $i = 2$  then  $\alpha_j$  is the solution of  $\alpha_j^p - x^p = a$ . If this equation has a solution in  $\mathbb{K}_{j-1}$  then the degree of the extension is equal to one, else by a classical lemma, see e.g. [Tig01, Lemma 13.9], the polynomial  $X^p - x^p - a \in \mathbb{K}_{j-1}[X]$  is irreducible. In this last case, the degree of the extension is equal to  $p$ , see [Tig01, Proposition 12.5].  $\square$

Now, we can proof Theorem 1.

*Proof of Theorem 1.* If an algorithm in the algebraic Robertson-Webb model computes an equitable and simple fair division in  $k$  steps then the final cutpoint  $t$  belongs to  $\mathbb{K}_k = \mathbb{Q}(\alpha_1, \dots, \alpha_k)$ . We have thus the inclusion  $\mathbb{Q} \subset \mathbb{Q}(t) \subset \mathbb{K}_k$ .

As  $t$  satisfies the equation (E) we have here

$$t^5 + t - 1 = 0.$$

We can factorize this expression and we obtain:

$$t^5 + t - 1 = (t^2 - t + 1)(t^3 + t^2 - 1) = 0.$$

As the polynomial  $X^2 - X + 1$  has no real roots we deduce that we have

$$t^3 + t^2 - 1 = 0.$$

Furthermore, the polynomial  $X^3 + X^2 - 1$  is irreducible in  $\mathbb{Q}[X]$ . This can be shown by considering the reduction of this polynomial in  $\mathbb{Z}/2\mathbb{Z}[X]$ . Thus

$$[\mathbb{Q}(t) : \mathbb{Q}] = 3.$$

However, by Lemma 3 we have

$$[\mathbb{K}_k : \mathbb{Q}] = 5^l,$$

with  $l \leq k$ . Therefore the equality

$$[\mathbb{K}_k : \mathbb{Q}] = [\mathbb{K}_k : \mathbb{Q}(t)][\mathbb{Q}(t) : \mathbb{Q}]$$

is impossible and this concludes the proof.  $\square$

In order to prove Theorem 2, we need some tools.

**Lemma 4.** *If  $f_i(x) = x^{e_i}$ , for  $i = 1, 2$  then for all  $j \geq 1$ ,  $\mathbb{K}_j = \mathbb{K}_{j-1}$  or  $\mathbb{K}_j$  is a radical extension of  $\mathbb{K}_{j-1}$ .*

Roughly speaking, this lemma says that the field  $\mathbb{K}_j$  is of the following form:  $\mathbb{K}_{j-1}(\sqrt[n]{\alpha})$ , where  $n$  is an integer and  $\alpha \in \mathbb{K}_{j-1}$ .

*Proof.* If the  $j$ -th query is of the form  $eval_i([x, y])$  with  $x, y \in \mathbb{K}_{j-1}$  then  $eval_i([x, y])$  is equal to  $y^{e_i} - x^{e_i}$ . Thus the result to this query  $\alpha_j = y^{e_i} - x^{e_i} \in \mathbb{K}_{j-1}$ . In this situation we have then  $\mathbb{K}_j := \mathbb{K}_{j-1}$ .

If the  $j$ -th query is of the form  $cut_i(x, a)$  with  $x, a \in \mathbb{K}_{j-1}$  then the result  $\alpha_j$  to this query is the unique solution in  $[0, 1]$  of the following equation:

$$\alpha_j^{e_i} - x^{e_i} = a.$$

This implies  $\alpha_j = \sqrt[e_i]{a + x^{e_i}}$  and  $\mathbb{K}_j := \mathbb{K}_{j-1}(\sqrt[e_i]{a + x^{e_i}})$ .

The extension  $\mathbb{K}_j/\mathbb{K}_{j-1}$  is thus a radical extension.  $\square$

As  $\mathbb{Q} \subset \mathbb{K}_1 \subset \dots \subset \mathbb{K}_k$  we have by definition of a radical extension, see [Tig01, Chapter 13], the following corollary:

**Corollary 5.** *For all  $j \geq 1$ , the extension  $\mathbb{K}_j/\mathbb{Q}$  is radical.*

Now, we recall a result about the irreducibility and the Galois group of certain trinomials.

**Proposition 6** (Selmer [Sel56]). *The polynomials  $X^d - X - 1$  are irreducible in  $\mathbb{Q}[X]$  for all  $d$ .*

*The polynomials  $X^d + X + 1$  are irreducible in  $\mathbb{Q}[X]$  for  $d \not\equiv 2 \pmod{3}$ , but have a factor  $X^2 + X + 1$  when  $d \equiv 2 \pmod{3}$ . In the latter case,  $X^d + X + 1$  has another factor which is irreducible.*

**Proposition 7** (Osada [Osa87]). *Let  $f(X) = X^d + aX + b \in \mathbb{Z}[X]$ , where  $a = a_0c^d$  and  $b = b_0c^d$  for some integer  $c$ . Then the Galois group over  $\mathbb{Q}$  of this polynomial is isomorphic to  $\mathcal{S}_d$  if the following conditions are satisfied:*

- (1)  $f(X)$  is irreducible over  $\mathbb{Q}$ ,
- (2)  $\gcd(a_0c(d-1), db_0) = 1$ .

*Proof of Theorem 2.* The strategy of the proof for the first two items is the following: We suppose that there exists an algorithm in the algebraic Robertson-Webb model computing an equitable and simple fair division  $X = [0, t] \cup [t, 1]$ . Then,  $t$  must satisfy the equation (E). Here this equation is:

$$t^d + t - 1 = 0.$$

As  $t$  must belong to  $\mathbb{K}_k$  and, by Corollary 5,  $\mathbb{K}_k$  is a radical extension of  $\mathbb{Q}$ , we deduce that  $t$  has a radical expression over  $\mathbb{Q}$ . Thus if the polynomial  $X^d + X - 1$  is irreducible then it can be solved by radicals over  $\mathbb{Q}$ , see [Tig01, Proposition 14.33]. However, the Galois group of  $X^d + X - 1$  is isomorphic to  $\mathcal{S}_d$ .

Indeed, Proposition 6 and the change of variables  $Y = -X$  shows that  $X^d + X - 1$  is irreducible in  $\mathbb{Q}[X]$ , when  $d$  satisfies the hypothesis of the first two items. Then,

Proposition 7 with  $a_0 = b_0 = c = 1$  entails that the Galois group of  $X^d + X - 1$  is isomorphic to  $\mathcal{S}_d$ .

Therefore Galois' theory implies that this polynomial cannot be solved by radicals over  $\mathbb{Q}$ , see [Tig01, Chapter 14]. This gives the desired contradiction.

Now, we suppose that  $d \geq 5$  is prime and  $d \equiv 2 \pmod{3}$ . In this case, the proof is a generalization of the proof of Theorem 1.

Proposition 6 and the change of variables  $Y = -X$  shows that  $X^d + X - 1$  is reducible. Moreover, this polynomial has an irreducible factor with degree 2 and another one with degree  $d - 2$ . This gives

$$[\mathbb{Q}(t) : \mathbb{Q}] = 2 \text{ or } [\mathbb{Q}(t) : \mathbb{Q}] = d - 2.$$

Furthermore, thanks to Lemma 3 we have

$$[\mathbb{K}_k : \mathbb{Q}] = d^l,$$

where  $l \in \mathbb{N}$ .

The equality

$$[\mathbb{K}_k : \mathbb{Q}] = [\mathbb{K}_k : \mathbb{Q}(t)][\mathbb{Q}(t) : \mathbb{Q}]$$

is then impossible since  $d$  is prime. This concludes the proof.  $\square$

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