Decay of the Local Energy for the Charged Klein-Gordon Equation in the Exterior De Sitter-Reissner-Nordström Spacetime

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Abstract

We show decay of the local energy of solutions of the charged Klein-Gordon equation in the exterior De Sitter-Reissner-Nordström spacetime by means of a resonance expansion of the local propagator.

1 Introduction

There has been enormous progress in our understanding of scattering properties of solutions of hyperbolic equations on black hole type backgrounds over the last years. The aim of such studies is multifold. First of all these equations and thus their scattering properties are very important in their own right. Secondly, the understanding of dispersive properties of the solutions of these equations is a first step in the understanding of stability properties of the underlying spacetime. Eventually understanding the classical equation is also a first step in understanding the quantization of the field. The most important spacetime in this context is the (De Sitter) Kerr spacetime, which is conjectured to be the unique solution of the Einstein equations describing a rotating black hole (for uniqueness results see [1] and references therein, and also [17] for the charged case). In the case of positive cosmological constant, nonlinear stability of the De Sitter Kerr spacetime is now known thanks to the seminal result of Hintz and Vasy [18]. The case of zero cosmological constant is still open, but see [19] for recent progress in this direction. Scattering theories for classical equations are also at the origin of many results in quantum field theory, see e.g. the mathematically rigorous description of the Hawking effect in [15].

When studying linear waves on a black hole type spacetime, one encounters several difficulties. The first is linked to trapping and it is already present in the case of the (De Sitter) Schwarzschild spacetime, which describes spherically symmetric black holes. The second is superradiance, which means that there is no positive conserved quantity for spin 1 equations on the (De Sitter) Kerr metric. Whereas this difficulty is not present for the wave equation on the (De Sitter) Schwarzschild metric, it also appears when one considers a charged Klein Gordon field on the (De Sitter) Reissner-Nordström metric which describes a spherically symmetric charged black hole. In this context the phenomenon is linked to the charge of the black hole and the test particle and thus different from the Kerr case, where it is linked to the geometry of the spacetime. Superradiance already appears in flat spacetime when one considers a charged Klein-Gordon field which evolves in a strong electric field. In this context the natural setting seems to be the one of Krein spaces, see e.g. [9]. This setting however is not available in the context of black holes, see [10].

In the present paper we show a resonance expansion for the solutions of the charged Klein-Gordon equation on the De Sitter Reissner-Norström metric. As a corollary we obtain exponential decay of the local energy for these solutions. We restrict our study to the case where the product of charges is small. Such a resonance expansion for the solutions of the wave equation has been obtained first by Bony-Häfner for the wave equation on the De Sitter Schwarzschild metric [5]. This result has been generalized to much more complicated situations which include perturbations of the De Sitter Kerr

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metric by Vasy [30]. This last paper has developed new methods including a Fredholm theory for non elliptic problems. These methods could probably also be applied to the present case. In the present paper however we use the more elementary methods of Bony-Häfner [5] and Georgescu-Gérard-Häfner [10].

The results of this paper hold if the product of the charges of the field and the black hole is sufficiently small. This allows at many places of the paper to use perturbation arguments with respect to the non charged case. As far as we are aware the absence of growing modes for the present system is not known for general charge products. In contrast to that absence of growing modes is known for the wave equation on the Kerr metric for general angular momentum of the black hole, see [31]. The question of the existence or not of such modes is a very subtle question and growing modes appear for example for the Klein-Gordon equation on the Kerr metric, see [25].

Let us also mention that it is crucial for our results that the cosmological constant is strictly positive in order to define resonances as the poles of the meromorphic extension of the weighted resolvent. The low frequency behaviour is more complicated in the zero cosmological constant case and only polynomial decay of the local energy is expected in this case.

The paper is organized as follows. In Section 2 we give an introduction to the De Sitter Reissner-Nordström metric and the charged Klein-Gordon equation on it. In Section 3, a meromorphic extension result is shown for the cut-off resolvent and resonances are introduced. The resonance expansion is presented in Section 4. Suitable resolvent-type estimates are obtained in Section 5. In section 6 we prove the main theorem by a suitable contour deformation and using the resolvent-type estimates of Section 5. The appendix contains a semiclassical limiting absorption principle for a class of generalized resolvents which might have some independent interest.

Notations. The set $\{z \in \mathbb{C} \mid \Im z \gtrless 0\}$ will be denoted by $\mathbb{C}^{\pm}$. For any complex number $\lambda \in \mathbb{C}$, we will write $\langle\lambda\rangle := \sqrt{1+|\lambda|^2}$, $D(\lambda, R)$ will be the disc centered at $\lambda \in \mathbb{C}$ of radius $R > 0$ and $D(\lambda, R)^{\complement}$ its complementary set. For all $\omega = |\omega|\mathrm{e}^{\mathrm{i}\theta} \in \mathbb{C}\backslash]-\infty, 0]$, $\theta \in \mathbb{R}$, we will use the branch of the square root defined by $\sqrt{\omega} := \sqrt{|\omega|}\mathrm{e}^{\mathrm{i}\theta/2}$.

The notation $\mathcal{C}_{\mathrm{c}}^k$ will be used to denote the space of compactly supported $\mathcal{C}^k$ functions. Also, the Schwartz space on $\mathbb{R}$ will be noted $\mathscr{S}$. If V, W are complex vector spaces, then $\mathcal{L}(V, W)$ will denote the space of bounded linear operators $V \to W$. All the scalar products $\langle\cdot,\cdot\rangle$ will be linear in the second component. For any function f, the range (respectively the support) of f will be noted $\operatorname{Ran} f$ (respectively $\operatorname{Supp} f$). If A is an operator, we will note $\mathscr{D}(L)$ its domain, $\sigma(A)$ its spectrum and $\rho(A)$ its resolvent set.

Now we define the symbol classes on $\mathbb{R}^{2d}$

$$S^{m,n} := \left\{a \in \mathcal{C}^{\infty}(\mathbb{R}^{2d}, \mathbb{C}) \mid \forall(\alpha,\beta) \in \mathbb{N}^{2d}, \exists\, C_{\alpha,\beta} > 0, |\partial_\xi^\alpha \partial_x^\beta a(x,\xi)| \leq C_{\alpha,\beta}\langle\xi\rangle^{m-|\alpha|}\langle x\rangle^{n-|\beta|}\right\}$$

for any $(m,n) \in \mathbb{Z}^{2d}$. We then define the semiclassical pseudodifferential operators classes

$$\Psi^{m,n} := \{a^{\mathrm{w}}(x, h\mathrm{D}) \mid a \in S^{m,n}\}, \qquad \Psi^{-\infty,n} := \bigcap_{m\in\mathbb{Z}} \Psi^{m,n}$$

with $a^{\mathrm{w}}(x, h\mathrm{D})$ the Weyl quantization of the symbol a. For any $c > 0$, the notation $P \in c\Psi^{m,n}$ means that $P \in \Psi^{m,n}$ and the norm of P is bounded by a positive multiple of c.

2 Functional framework

2.1 The charged Klein-Gordon equation on the De Sitter-Reissner-Nordström metric

Let

$$F(r) := 1 - \frac{2M}{r} + \frac{Q^2}{r^2} - \frac{\Lambda r^2}{3}.$$

with $M > 0$ the mass of the black hole, $Q \in \mathbb{R} \setminus \{0\}$ its electric charge and $\Lambda > 0$ the cosmological constant. We assume that the parameters

$$\Delta := 1 - 4\Lambda Q^2, \qquad m_k := \sqrt{\frac{1 + (-1)^k\sqrt{\Delta}}{2\Lambda}}, \qquad M_k := m_k - \frac{2}{3}\Lambda m_k^3$$

satisfy for any $k \in \{1, 2\}$ the relations

$$4\Lambda Q^2 < 1, \qquad M_1 < M < M_2 \tag{1}$$

so that F has four distinct zeros $-\infty < r_n < 0 < r_c < r_- < r_+ < +\infty$ and is positive for all $r \in]r_-, r_+[$ (see [23], Proposition 1 with Λ replaced by $\Lambda/3$ in our setting). We also assume that $9\Lambda M^2 < 1$ so that we can use the work of Bony-Häfner [5] (the condition (1) only ensures that $9\Lambda M^2 < 2$). The exterior De Sitter-Reissner-Nordström spacetime is the lorentzian manifold $(\mathcal{M}, g)$ with

$$\mathcal{M} = \mathbb{R}_t \times]r_-, r_+[_r \times \mathbb{S}^2_\omega, \qquad g = F\left(r\right)\mathrm{d}t^2 - F\left(r\right)^{-1}\mathrm{d}r^2 - r^2\mathrm{d}\omega^2$$

where $\mathrm{d}\omega^2$ is the standard metric on the three-dimensional unit sphere $\mathbb{S}^2$.

Let $A^\mu := (Q/r, 0, 0, 0)$. Then the charged wave operator on $(\mathcal{M}, g)$ is

$$\Box_g = \left(\nabla_\mu - \mathrm{i}qA_\mu\right)\left(\nabla^\mu - \mathrm{i}qA^\mu\right) = \frac{1}{F}\left(\left(\partial_t - \mathrm{i}\frac{qQ}{r}\right)^2 - \frac{F}{r^2}\partial_r r^2 F\partial_r - \frac{F}{r^2}\Delta_{\mathbb{S}^2}\right)$$

and the corresponding charged Klein-Gordon equation reads

$$\Box_g u + m^2 u = 0 \qquad m > 0.$$

We set $s := qQ \in \mathbb{R}$ the charge product (which appears in the perturbation term of the standard wave operator), $X :=]r_-, r_+[_r \times \mathbb{S}^2_\omega$ and $V\left(r\right) := r^{-1}$ so that the above equation reads

$$\left(\partial_t - \mathrm{i}sV\right)^2 u + \hat{P}u = 0 \tag{2}$$

with

$$\hat{P} = -\frac{F}{r^2}\partial_r\left(r^2F\partial_r\right) - \frac{F}{r^2}\Delta_{\mathbb{S}^2} + m^2F = -F^2\partial_r^2 - F\left(\frac{2F}{r} + \frac{\partial F}{\partial r}\right)\partial_r - \frac{F}{r^2}\Delta_{\mathbb{S}^2} + m^2F \tag{3}$$

defined on $\mathscr{D}(\hat{P}) := \left\{u \in L^2\left(X, F^{-1}r^2\mathrm{d}r\mathrm{d}\omega\right) \mid \hat{P}u \in L^2\left(X, F^{-1}r^2\mathrm{d}r\mathrm{d}\omega\right)\right\}$ (this is the spatial operator in [5] with the additional mass term m^2F). It turns out that the positive mass makes the study of the equation easier. Besides the fact that massless charged particles do not exist in physics, it is not excluded that the resonance 0 for the case $s = 0$ (see [5]) can lift up to $\mathbb{C}^+$.

2.2 The Regge-Wheeler coordinate

We introduce the Regge-Wheeler coordinate $x \equiv x\left(r\right)$ defined by the differential relation

$$\frac{\mathrm{d}x}{\mathrm{d}r} := \frac{1}{F\left(r\right)}. \tag{4}$$

Using the four roots r_α of F, $\alpha \in I := \{n, c, -, +\}$, we can write

$$\frac{1}{F\left(r\right)} = -\frac{3r^2}{\Lambda}\sum_{\alpha\in I}\frac{A_\alpha}{r - r_\alpha}$$

where $A_\alpha = \prod_{\beta\in I\setminus\{\alpha\}}(r_\alpha - r_\beta)^{-1}$ for all $\alpha \in I$, and $\pm A_\pm > 0$. Integrating (4) then yields

$$x\left(r\right) = -\frac{3}{\Lambda}\sum_{\alpha\in I}A_\alpha r_\alpha^2\ln\left|\frac{r - r_\alpha}{\mathfrak{r} - r_\alpha}\right| \tag{5}$$

with $\mathfrak{r} := \frac{1}{2}\left(3M+\sqrt{9M^2-8Q^2}\right)$ (we will explain this choice below); note that $|Q| < \frac{3}{\sqrt{8}}M$ if (1) holds (see the discussion below (17) in [23]). Therefore, we have

$$|r-r_\alpha| = |\mathfrak{r}-r_\alpha| \prod_{\beta\in I\setminus\{\alpha\}} \left|\frac{r-r_\beta}{\mathfrak{r}-r_\beta}\right|^{-A_\beta r_\beta^2/\left(A_\alpha r_\alpha^2\right)} \exp\left(-\frac{\Lambda}{3A_\alpha r_\alpha^2}x\right) \qquad \forall\alpha\in I$$

which entails the asymptotic behaviours

$$F\left(r\left(x\right)\right) + \left|r\left(x\right)-r_\pm\right| \lesssim \exp\left(-\frac{\Lambda}{3A_\pm r_\pm^2}x\right) \qquad x\to\pm\infty. \tag{6}$$

Note here that

$$\frac{\Lambda}{3A_\pm r_\pm^2} = F'(r_\pm) = 2\kappa_\pm \tag{7}$$

where $\kappa_- > 0$ is the surface gravity at the event horizon and $\kappa_+ < 0$ is the surface gravity at the cosmological horizon. Recall that $\kappa_\pm$ is defined by the relation

$$X^\mu\nabla_\mu X^\nu = -2\kappa_\pm X^\nu \qquad X=\partial_t$$

where the above equation is to be considered at the corresponding horizon.

In the appendix A, we follow Proposition IV.2 in [4] to show the extension result :

Proposition 2.1. *There exists a constant $\mathscr{A}>0$ such that the function $x\mapsto r(x)$ extends analytically to $\{\lambda\in\mathbb{C} \mid |\Re\lambda| > \mathscr{A}\}$.*

On $L^2\left(X,\mathrm{d}x\mathrm{d}\omega\right)$, define the operator $P := r\hat{P}r^{-1}$, given in the coordinates (x,ω) by the expression

$$P = -r^{-1}\partial_x r^2\partial_x r^{-1} - \frac{F\left(r\right)}{r^2}\Delta_{\mathbb{S}^2} + m^2F\left(r\right) = -\partial_x^2 - W_0\Delta_{\mathbb{S}^2} + W_1$$

where

$$W_0\left(x\right) := \frac{F\left(r\left(x\right)\right)}{r\left(x\right)^2}, \quad W_1\left(x\right) := \frac{F\left(r\left(x\right)\right)}{r\left(x\right)}\frac{\partial F}{\partial r}\left(r\left(x\right)\right) + m^2F\left(r\left(x\right)\right).$$

It will happen in the sequel that we write $F\left(x\right)$ for $F\left(r\left(x\right)\right)$ and also $V\left(x\right)$ for $V\left(r\left(x\right)\right)$. Observe that the potentials W_0 and W_1 satisfy the same estimate as in (6).

As

$$\frac{\mathrm{d}W_0}{\mathrm{d}x} = F(r)\frac{\mathrm{d}W_0}{\mathrm{d}r} = \frac{2F(r)}{r^5}\left(3Mr-2Q^2-r^2\right),$$

we see that the maximum of W_0 occurs when $x=0$, which corresponds to $\mathfrak{r} = \frac{1}{2}\left(3M+\sqrt{9M^2-8Q^2}\right)$.

We stress here that the radius $\mathfrak{r}$ has a geometrical meaning. Indeed, consider $\gamma=(t,r,\omega)$ a null geodesic. We can assume that γ satisfies the equatorial initial conditions so that the motion lies in the plane defined by $\theta=\pi/2$ (see [22], Chapter 13, Proposition 11). Because ∂_t and ∂_ϕ are Killing vector fields for the metric g, Noether's theorem implies that

$$g\left(\dot\gamma,\partial_t\right) = F(r)\,\dot t = E\in\mathbb{R}, \qquad g\left(\dot\gamma,\partial_\phi\right) = r^2\,\dot\phi = L\in\mathbb{R}$$

where the constants E and L are respectively the energy and the angular momentum of the null geodesic. Then the mass equation $g\left(\dot\gamma,\dot\gamma\right)=0$ yields the relation

$$E^2 = \dot r^2 + F(r)\frac{L^2}{r^2}$$

which is equivalent to

$$\left(\frac{1}{r^2}\frac{\mathrm{d}r}{\mathrm{d}\phi}\right)^2 + \frac{F(r)}{r^2} = \frac{1}{b^2}$$

where $b = |L/E|$ is the impact parameter of the null geodesic. A trivial solution is given by $r = \mathfrak{r}$, since the quantity

$$\frac{\mathrm{d}}{\mathrm{d}r}\frac{r}{\sqrt{F(r)}} = \frac{1}{\sqrt{F(r)}}\left(1 - \frac{r}{2}\frac{F'(r)}{F(r)}\right)$$

cancels if and only if $r = \mathfrak{r}$. Observe that this solution corresponds to the energy $E = W_0(0) = \frac{F(\mathfrak{r})}{\mathfrak{r}^2}$. The sphere in $\mathbb{R}^3$ centered at 0 with radius $\mathfrak{r}$ is called the photon sphere : it is an unstable equilibrium of the potential W_0, describing closed null geodesic. In [23], it is shown this is the only trapping orbit for null geodesics in the exterior De Sitter-Reissner-Nordström spacetime.

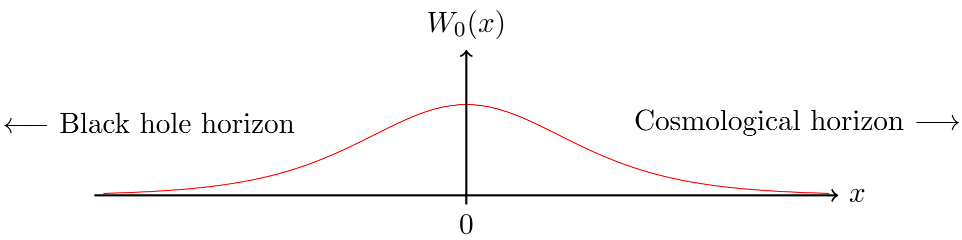

Figure 1: The potential W_0 in the Regge-Wheeler coordinates.

2.3 The charge Klein-Gordon operator

Taking advantage of the spherical symmetry, we write

$$L^2\left(\mathbb{R}\times\mathbb{S}^2, \mathrm{d}x\mathrm{d}\omega\right) \simeq \bigoplus_{\ell\in\mathbb{N}}\left(L^2\left(\mathbb{R}, \mathrm{d}x\right)\otimes Y_\ell\right) =: \bigoplus_{\ell\in\mathbb{N}}\mathcal{V}_\ell$$

where for all $\ell \in \mathbb{N}$, Y_ℓ is the $(2\ell+1)$-dimensional eigenspace of the operator $\left(-\Delta_{\mathbb{S}^2}, H^2(\mathbb{S}^2, \mathrm{d}\omega)\right)$ associated to the eigenvalue $\ell\,(\ell+1)$. On each $\mathcal{V}_\ell$, we define P_ℓ as the restriction of P onto $\mathcal{V}_\ell$ which will be identified with an operator acting on $L^2(\mathbb{R}, \mathrm{d}x)$, i.e.

$$P_\ell = -\partial_x^2 + \ell\,(\ell+1)\,W_0 + W_1 \tag{8}$$

and we set $\mathscr{D}\,(P_\ell) := H^2(\mathbb{R}, \mathrm{d}x)$ so that P_ℓ is self-adjoint. In the sequel, we will use the following (self-adjoint) realization of the total operator P:

$$P := \bigoplus_{\ell\in\mathbb{N}} P_\ell, \qquad \mathscr{D}\,(P) := \left\{u = (u_\ell)_{\ell\in\mathbb{N}} \in \bigoplus_{\ell\in\mathbb{N}}\mathcal{V}_\ell \mid \forall\ell\in\mathbb{N},\ u_\ell\in\mathscr{D}(P_\ell)\right\}.$$

Now the charged Klein-Gordon equation reads

$$\left(\partial_t - \mathrm{i}sV\right)^2 u + Pu = 0. \tag{9}$$

The point is to see that if u is a solution of (9), then $v := (u, -\mathrm{i}\partial_t u - sVu)$ solves the first order equation

$$-\mathrm{i}\partial_t v = \hat{K}(s)v \tag{10}$$

where

$$\hat{K}\,(s) := \begin{pmatrix} sV & \mathrm{Id} \\ P & sV \end{pmatrix} \tag{11}$$

is the charge Klein-Gordon operator. Conversely, if $v = (v_0, v_1)$ solves (10), then v_0 solves (9). We also define $\hat{K}_\ell \equiv \hat{K}_\ell(s)$[1] with P_ℓ in place of P for any $\ell \in \mathbb{N}$. Following [10], we realize $\hat{K}_\ell$ with the domain

$$\mathscr{D}(\hat{K}_\ell) := \left\{ u \in P_\ell^{-1/2} L^2(\mathbb{R}, \mathrm{d}x) \oplus L^2(\mathbb{R}, \mathrm{d}x) \mid \hat{K}_\ell u \in P_\ell^{-1/2} L^2(\mathbb{R}, \mathrm{d}x) \oplus L^2(\mathbb{R}, \mathrm{d}x) \right\}$$

and realize the operator $\hat{K}$ as the direct sum on $\mathbb{N} \ni \ell$ of the $\hat{K}_\ell$.

Let $\dot{\mathcal{E}}_\ell$ be the completion of $P_\ell^{-1/2} L^2(\mathbb{R}, \mathrm{d}x) \oplus L^2(\mathbb{R}, \mathrm{d}x)$ for the norm[2]

$$\|u\|^2_{\dot{\mathcal{E}}_\ell} := \langle u_0, P_\ell u_0 \rangle_{L^2(\mathbb{R}, \mathrm{d}x)} + \|u_1 - sVu_0\|^2_{L^2(\mathbb{R}, \mathrm{d}x)} \qquad\qquad u = (u_0, u_1) \in \dot{\mathcal{E}}_\ell$$

and define $\left(\dot{\mathcal{E}}, \|.\|_{\dot{\mathcal{E}}}\right)$ as the direct sum of the spaces $\dot{\mathcal{E}}_\ell$. Lemma 3.19 of [10] shows that $\hat{K}_\ell$ generates a continuous one-parameter group $(\mathrm{e}^{-\mathrm{i}t\hat{K}_\ell})_{t\in\mathbb{R}}$ on $(\dot{\mathcal{E}}_\ell, \|.\|_{\dot{\mathcal{E}}_\ell})$. We similarly construct the spaces $\left(\mathcal{E}_\ell, \|.\|_{\mathcal{E}_\ell}\right)$ and $\left(\mathcal{E}, \|.\|_{\mathcal{E}}\right)$ with $\langle P_\ell \rangle$ instead of P_ℓ. Let us mention here that for any $n \in \mathbb{R}$ the quantity

$$\langle v \,|\, v \rangle_n := \langle v_1 - nv_0, v_1 - nv_0 \rangle_{L^2(\mathbb{R}, \mathrm{d}x)} + \langle (P - (sV - n)^2\,) u_0, u_0 \rangle_{L^2(\mathbb{R}, \mathrm{d}x)}$$

is formally conserved if $v = (u, -\mathrm{i}\partial_t u)$ with u solution of (9) and is continuous with respect to the norm $\|.\|_{\mathcal{E}}$. However, it is in general not positive nor continuous with respect to the norm $\|.\|_{\dot{\mathcal{E}}}$ (see paragraph 3.4.3 in [10] for more details) : this is superradiance. When $\Lambda = 0$ (that is, when the cosmological horizon is at infinity), the natural energy $\langle . \,|\, . \rangle_{sV_-}$ is positive for s small enough and it can be used to define a Hilbert framework.

An important observation is the fact that the norms $\|.\|_{\dot{\mathcal{E}}_\ell}$ and $\|.\|_{\mathcal{E}_\ell}$ are locally equivalent, meaning that for any $v \in \dot{\mathcal{E}}$ and any cut-off $\chi \in \mathcal{C}_c^\infty(\mathbb{R}, \mathbb{R})$, we have

$$\|\chi v\|_{\dot{\mathcal{E}}} \lesssim \|\chi v\|_{\mathcal{E}} \lesssim \|\chi v\|_{\dot{\mathcal{E}}}. \tag{12}$$

The first inequality is obvious, and the second one is established with the Hardy type estimate $\|\chi v\|_{L^2} \lesssim \|P^{1/2} v\|_{L^2}$ (see Lemma 9.5 in [10]; the validity of this result in our setting is discuted in the subsection 3.4 below).

2.4 The quadratic pencil

Let u be a solution of (9). If we look for u of the form $u = \mathrm{e}^{\mathrm{i}zt} v$ with $z \in \mathbb{C}$ for some v, then v satisfies the equation $(P - (z - sV)^2)v = 0$. We define the harmonic quadratic pencil

$$p_\ell\,(z, s) := P_\ell - (z - sV)^2\,, \qquad\qquad \mathscr{D}(p_\ell(z, s)) := \langle P_\ell \rangle^{-1} L^2(\mathbb{R}, \mathrm{d}x) = H^2(\mathbb{R}, \mathrm{d}x)$$

and realize the total quadratic pencil as

$$p\,(z, s) := \bigoplus_{\ell \in \mathbb{N}} p_\ell\,(z, s)\,,$$

$$\mathscr{D}(p(z, s)) := \left\{ u = (u_\ell)_{\ell\in\mathbb{N}} \in \bigoplus_{\ell\in\mathbb{N}} \mathcal{V}_\ell \mid \forall \ell \in \mathbb{N},\ u_\ell \in \mathscr{D}(p_\ell(z, s)),\ \sum_{\ell\in\mathbb{N}} \|p_\ell(z, s) u_\ell\|_{\mathcal{V}_\ell} < +\infty \right\}.$$

Proposition 3.15 in [10] sets the useful relations

$$\rho(\hat{K}_\ell) \cap \mathbb{C} \setminus \mathbb{R} = \left\{ z \in \mathbb{C} \setminus \mathbb{R} \mid p_\ell\,(z, s) : H^2(\mathbb{R}, \mathrm{d}x) \to L^2(\mathbb{R}, \mathrm{d}x) \text{ is bijective} \right\} \tag{13}$$

and

$$\hat{R}\,(z, s) := (\hat{K}(s) - z)^{-1} = \begin{pmatrix} p\,(z, s)^{-1}\,(z - sV) & p\,(z, s)^{-1} \\ \mathrm{Id} + (z - sV)\,p\,(z, s)^{-1}\,(z - sV) & (z - sV)\,p\,(z, s)^{-1} \end{pmatrix} \tag{14}$$

for all $z \in \rho(\hat{K}_\ell) \cap \mathbb{C} \setminus \mathbb{R}$. In comparison, the relation (1.7) in [5] involves the resolvent of P_ℓ, which corresponds to the case $s = 0$ for us. Proposition 3.12 in [10] shows that (14) is valid for $z \in \rho(\hat{K}_\ell) \cap \mathbb{R}$; by using the local equivalence (12) of the norms $\|.\|_{\dot{\mathcal{E}}_\ell}$ and $\|.\|_{\mathcal{E}_\ell}$, we can use (14) for $z \in \rho(\hat{K}_\ell) \cap \mathbb{R}$ if we consider the cut-off resolvent $\chi \hat{R}\,(z, s)\,\chi$ with $\chi \in \mathcal{C}_c^\infty(\mathbb{R}, \mathbb{R})$. In the sequel, we will simply call $p_\ell(z, s)$ the quadratic pencil when $\ell \in \mathbb{N}$ will be fixed.

[1] We will often drop the dependance in s.

[2] Note that the norm $\|.\|^2_{\dot{\mathcal{E}}_\ell}$ is conserved if $[P_\ell, sV] = 0$; it is the case if $s = 0$.

3 Meromorphic extension and resonances

In this Section, we are concerned with some meromorphic extension results as well as analyticity near 0 of weighted resolvents associated to $\hat{K}(s)$. Using [10], we first define asymptotic hamiltonians which are somehow comparable to the well-known case of $s=0$ studied in [5]. From there, we "glue" together the asymptotic operators in order to construct a meromorphic extension for the charge Klein-Gordon operator. We then deduce the existence of a meromorphic extension for the cut-off inverse of the quadratic pencil.

3.1 Notations

We introduce some notations following Section 2.1 in [10]. First observe that if u solves (2), then $v := \mathrm{e}^{-\mathrm{i}sr_+^{-1}t}u$ satisfies

$$(\partial_t^2 - 2\mathrm{i}s(V - r_+^{-1})\partial_t - s^2(V - r_+^{-1})^2 + P)v = 0.$$

We can therefore work with the potential $\tilde{V} := V - V_+ = \mathcal{O}_{r\to r_+}(r_+ - r)$ in this Section. In order not to overload notations, we will still denote $\tilde{V}$ by V.

Let us define $\mathcal{H} := L^2(X, \mathrm{d}r\mathrm{d}\omega)$ and

$$\mathscr{P} := rF^{-1/2}\hat{P}r^{-1}F^{1/2} = -r^{-1}F^{1/2}\partial_r r^2 F\partial_r r^{-1}F^{1/2} - \frac{F}{r^2}\Delta_{\mathbb{S}^2} + m^2 F \tag{15}$$

with $\hat{P}$ given by (3). Since $u \mapsto r^{-1}F^{1/2}u$ is an unitary isomorphism from $\mathcal{H}$ to $L^2\left(X, F^{-1}r^2\mathrm{d}r\mathrm{d}\omega\right)$, the results obtained below on $\mathscr{P}$ will also apply to $\hat{P}$ (and thus to P). Observe that the space $\dot{\mathcal{E}}$ has been defined in our setting with the operator P which is $r\hat{P}r^{-1}$ expressed with the Regge-Wheeler coordinate, and $\hat{P}$ is equivalent to $\mathscr{P}$ as explained above; in the sequel, we will denote by $\dot{\mathscr{E}}$ the completion of $\mathscr{P}^{-1/2}\mathcal{H} \oplus \mathcal{H}$ for the norm $\|(u_0,u_1)\|^2_{\dot{\mathscr{E}}} := \langle u_0, \mathscr{P}u_0\rangle_{\mathcal{H}} + \|u_1 - sVu_0\|^2_{\mathcal{H}}$. Let $i_\pm, j_\pm \in \mathcal{C}^\infty(]r_-, r_+[, \mathbb{R})$ such that

$$\begin{gathered} i_\pm = j_\pm = 0 \text{ close to } r_\mp, \qquad i_\pm = j_\pm = 1 \text{ close to } r_\pm, \\ i_-^2 + i_+^2 = 1, \qquad i_\pm j_\pm = j_\pm, \qquad i_- j_+ = i_+ j_- = 0. \end{gathered}$$

We then define the operators

$$k_\pm := s(V \mp j_\mp^2 r_-^{-1}), \qquad \mathscr{P}_\pm := \mathscr{P} - k_\pm^2, \qquad \tilde{\mathscr{P}}_- := \mathscr{P} - (sr_-^{-1} - k_-)^2.$$

We now define the isomorphism on $\dot{\mathscr{E}}$ (see comments above Lemma 3.13 in [10])

$$\Phi(sV) := \begin{pmatrix} \mathrm{Id} & 0 \\ sV & \mathrm{Id} \end{pmatrix}$$

and we introduce the energy Klein-Gordon operator

$$\hat{H}(s) = \Phi(sV)\hat{K}(s)\Phi^{-1}(sV) = \begin{pmatrix} 0 & \mathrm{Id} \\ \mathscr{P} - s^2V^2 & 2sV \end{pmatrix}$$

with domain

$$\mathscr{D}(\hat{H}(s)) = \left\{ u \in \mathscr{P}^{-1/2}\mathcal{H} \oplus \mathcal{H} \mid \hat{H}(s)u \in \mathscr{P}^{-1/2}\mathcal{H} \oplus \mathcal{H} \right\}$$

and the asymptotic hamiltonians

$$\hat{H}_\pm(s) = \begin{pmatrix} 0 & \mathrm{Id} \\ \mathscr{P}_\pm & 2k_\pm \end{pmatrix}$$

with domains

$$\mathscr{D}(\hat{H}_+(s)) = \left(\mathscr{P}_+^{-1/2}\mathcal{H} \cap \mathscr{P}_+^{-1}\mathcal{H}\right) \oplus \langle\mathscr{P}_+\rangle^{-1/2}\mathcal{H},$$
$$\mathscr{D}(\hat{H}_-(s)) = \Phi(sr_-^{-1})\left(\tilde{\mathscr{P}}_-^{-1/2}\mathcal{H} \cap \tilde{\mathscr{P}}_-^{-1}\mathcal{H}\right) \oplus \langle\tilde{\mathscr{P}}_-\rangle^{-1/2}\mathcal{H}.$$

These operators are self-adjoint on the following spaces (see the beginning of the paragraph 5.2 in [10]) :

$$\dot{\mathscr{E}}_+ := \mathscr{P}_+^{-1/2}\mathcal{H} \oplus \mathcal{H},$$
$$\dot{\mathscr{E}}_- := \Phi(sr_-^{-1})\left(\tilde{\mathscr{P}}_-^{-1/2}\mathcal{H} \oplus \mathcal{H}\right).$$

In the sequel, we will also use the spaces $\mathscr{E}_\pm$ defined as above but with the operators $\langle\mathscr{P}_\pm\rangle$ instead of $\mathscr{P}_\pm$. Finally, we define the weight $w(r) := \sqrt{(r-r_-)(r_+-r)}$.

3.2 Abstract setting

Meromorphic extensions in our setting follow from the works of Mazzeo-Melrose [21] and Guillarmou [13], as stated in Proposition 5.3 of [10]. The abstract setting in which this result can be used is recalled in this paragraph.

We first recall for the reader convenience the Abstract assumptions (A1)-(A3), the Meromorphic Extensions assumptions (ME1)-(ME2) as well as the "Two Ends" assumptions (TE1)-(TE3) of [10] :

$$\mathscr{P} > 0, \tag{A1}$$

$$\begin{cases} sV \in \mathcal{B}(\mathscr{P}^{-1/2}L^2) > 0, \\ \text{if } z \neq \mathbb{R} \text{ then } (z-sV)^{-1} \in \mathcal{B}(\mathscr{P}^{-1/2}L^2) \text{ and there exists } n > 0 \\ \text{such that } \|(z-sV)^{-1}\|_{\mathcal{B}(\mathscr{P}^{-1/2}L^2)} \lesssim |\Im z|^{-n}, \\ \text{there exists } c > 0 \text{ such that } \|(z-sV)^{-1}\|_{\mathcal{B}(\mathscr{P}^{-1/2}L^2)} \lesssim \big||z| - \|sV\|_{L^\infty}\big| \\ \text{if } |z| \geq c\|sV\|_{L^\infty} \end{cases}, \tag{A2}$$

$$\begin{cases} \text{(a) } wVw \in L^\infty, \\ \text{(b) } [V,w] = 0 \\ \text{(c) } (\mathscr{P}-s^2V^2)^{-1/2}[(\mathscr{P}-s^2V^2), w^{-\epsilon}]w^{\epsilon/2} \in \mathcal{B}(L^2) \text{ for all } 0 < \epsilon \leq 1, \\ \text{(d) if } \epsilon > 0 \text{ then } \|w^{-\epsilon}u\|_{L^2} \lesssim \|(\mathscr{P}-s^2V^2)^{1/2}u\|_{L^2} \text{ for all } u \in (\mathscr{P}-s^2V^2)^{-1/2}L^2, \\ \text{(e) } w^{-1}\langle(\mathscr{P}-s^2V^2)\rangle^{-1} \in \mathcal{B}(L^2) \text{ is compact} \end{cases}, \tag{ME1}$$

$$\begin{cases} \text{For all } \epsilon \text{ there exists } \delta_\epsilon > 0 \text{ such that } w^{-\epsilon}(\mathscr{P}-z^2-s^2V^2)^{-1}w^{-\epsilon} \text{ extends from } \mathbb{C}^+ \\ \text{to } \{z \in \mathbb{C} \mid \Im z > -\delta_\epsilon\} \text{ as a finite meromorphic function with values} \\ \text{in compact operators acting on } L^2 \end{cases}, \tag{ME2}$$

$$\begin{cases} [x, sV] = 0, \\ x \mapsto w(x) \in \mathcal{C}^\infty(\mathbb{R},\mathbb{R}), \\ \chi_1(x)\mathscr{P}\chi_2(x) = 0 \text{ for all } \chi_1,\chi_2 \in \mathcal{C}^\infty(\mathbb{R},\mathbb{R}) \text{ bounded with all their derivatives} \\ \text{and such that } \operatorname{Supp}\chi_1 \cap \operatorname{Supp}\chi_2 = \emptyset \end{cases}, \tag{TE1}$$

$$\text{There exists } \ell_- \in \mathbb{R} \text{ such that } (\mathscr{P}_+, k_+) \text{ and } (\tilde{\mathscr{P}}_-, (k_- - \ell_-)) \text{ satisfy (A2)}, \tag{TE2}$$

$$\begin{cases} \text{(a) } wi_+sVi_+w, wi_-(sV-\ell_-)i_-w \in L^\infty, \\ \text{(b) } [\mathscr{P}-s^2V^2, i_\pm] = \tilde{i}[\mathscr{P}-s^2V^2, i_\pm]\tilde{i} \text{ for some } \tilde{i} \in \mathcal{C}^\infty_c(]-2,2[\,,\mathbb{R}) \text{ such that } \tilde{i}_{|[-1,1]} \equiv 1 \\ \text{(c) } (\mathscr{P}_+, k_+, w) \text{ and } (\mathscr{P}_-, (k_- - \ell_-), w) \text{ fulfill (ME1) and (ME2)}, \\ \text{(d) } \mathscr{P}_\pm^{1/2} i_\pm \mathscr{P}_\pm^{-1/2}, \mathscr{P}^{1/2} i_\pm \mathscr{P}^{-1/2} \in \mathcal{B}(L^2), \\ \text{(e) } w[(\mathscr{P}-s^2V^2), i_\pm]w\mathscr{P}_\pm^{-1/2}, w[(\mathscr{P}-s^2V^2), i_\pm]w\mathscr{P}^{-1/2}, [(\mathscr{P}-s^2V^2), i_\pm]\mathscr{P}_\pm^{-1/2}, \\ \quad [(\mathscr{P}-s^2V^2), i_\pm]\mathscr{P}^{-1/2}, \mathscr{P}^{-1/2}[w^{-1}, \mathscr{P}]w \text{ are bounded operators on } L^2, \\ \text{(e) if } \epsilon > 0 \text{ then } \|w^{-\epsilon}u\|_{L^2} \lesssim \|\mathscr{P}^{1/2}u\|_{L^2} \text{ for all } u \in \mathscr{P}^{-1/2}L^2 \end{cases} . \quad \text{(TE3)}$$

Section 9 in [10] shows that all the above hypotheses actually follow from some geometric assumptions (the assumptions (G1)-(G7) of the paragraph 2.1.1 in [10]). We show here that the charged Klein-Gordon equation in the exterior De Sitter-Reissner-Nordstöm spacetime can be dealt within this geometric setting :

(G1) The operator P in [10] is $-\Delta_{\mathbb{S}^2}$ for us, and satisfies of course $[\Delta_{\mathbb{S}^2}, \partial_\phi] = 0$.

(G2) The operator $h_{0,s}$ in [10] is $\mathscr{P}$ for us, that is $\alpha_1(r) = \alpha_3(r) = r^{-1}F(r)^{1/2}$, $\alpha_2(r) = rF(r)^{1/2}$ and $\alpha_4(r) = mF(r)^{1/2}$. These last coefficients are clearly smooth in r. Furthermore, since we can write $F(r) = g(r)w(r)^2$ with $g(r) = \frac{\Lambda}{3r^2}(r-r_n)(r-r_c) \gtrsim 1$ for all $r \in]r_-, r_+[$, it comes for all $j \in \{1,2,3,4\}$ as $r \to r_\pm$

$$\alpha_j(r) - w(r)\left(i_-(r)\,\alpha_j^- + i_+(r)\,\alpha_j^+\right) = w(r)\left(g(r)^{1/2} - \alpha_j^\pm\right) = \mathcal{O}_{r\to r_\pm}(w(r)^2),$$
$$\alpha_1^\pm = \alpha_3^\pm = \frac{1}{r_\pm^2}\sqrt{\frac{\Lambda(r_\pm - r_n)(r_\pm - r_c)}{3}},$$
$$\alpha_2^\pm = \sqrt{\frac{\Lambda(r_\pm - r_n)(r_\pm - r_c)}{3}},$$
$$\alpha_4^\pm = \frac{m}{r_\pm}\sqrt{\frac{\Lambda(r_\pm - r_n)(r_\pm - r_c)}{3}}.$$

Also, we clearly have $\alpha_j(r) \gtrsim w(r)$. Direct computations show that

$$\partial_r^m \partial_\omega^n \left(\alpha_j - w\left(i_-\,\alpha_j^- + i_+\,\alpha_j^+\right)\right)(r) = \mathcal{O}_{r\to r_\pm}\left(w(r)^{2-2m}\right)$$

for all $m, n \in \mathbb{N}$.

(G3) The operator k_s in [10] is $sV(r)$ for us, so $k_s = k_{s,v}$ and $k_{s,r} = 0$. We have $V(r) = \mathcal{O}_{r\to r_+}(r_+ - r) = \mathcal{O}_{r\to r_+}\left(w(r)^2\right)$ (recall the discussion at the beginning of Subsection 3.1), $V(r) - r_-^{-1} = \mathcal{O}_{r\to r_-}(r - r_-) = \mathcal{O}_{r\to r_-}\left(w(r)^2\right)$ and $\partial_r^m \partial_\omega^n V(r)$ is bounded for any $m, n \in \mathbb{N}$.

(G4) The perturbation k in [10] is simply $k = k_s = sV$ for us, so that this assumption is trivially verified.

(G5) The operator h_0 in [10] is simply $h_0 = h_{0,s} = \mathscr{P}$ for us, and we have

$$\begin{aligned} \mathscr{P} &= -\alpha_1(r)\partial_r w(r)^2 r^2 g(r)\partial_r \alpha_1(r) - \alpha_1(r)^2\Delta_{\mathbb{S}^2} + \alpha_1(r)^2 m^2 r^2 \\ &= \alpha_1(r)\left(-\partial_r w(r)^2 r^2 g(r)\partial_r - \Delta_{\mathbb{S}^2} + m^2r^2\right)\alpha_1(r) \\ &\gtrsim \alpha_1(r)\left(-\partial_r w(r)^2\partial_r - \Delta_{\mathbb{S}^2} + 1\right)\alpha_1(r). \end{aligned}$$

(G6) This assumption is trivial in our setting.

(G7) We check that $(\mathscr{P}_+, k_+)$ and $(\tilde{\mathscr{P}}_-, k_- - sr_-^{-1})$ satisfy (G5). Since $\alpha_1(r), k_+(r) = \mathcal{O}_{r\to r_\pm}(|r_\pm - r|)$, we can write for $|s| < mr_-$

$$\begin{aligned} \mathscr{P}_+ &= -\alpha_1(r)\partial_r w(r)^2 r^2 g(r)\partial_r\alpha_1(r) - \alpha_1(r)^2\Delta_{\mathbb{S}^2} + \alpha_1(r)^2m^2r^2 - k_+(r)^2 \\ &= \alpha_1(r)\left(-\partial_r w(r)^2 r^2 g(r)\partial_r - \Delta_{\mathbb{S}^2} + m^2r^2 - \frac{k_+(r)^2}{\alpha_1(r)^2}\right)\alpha_1(r) \\ &\gtrsim \alpha_1(r)\left(-\partial_r w(r)^2\partial_r - \Delta_{\mathbb{S}^2} + 1\right)\alpha_1(r). \end{aligned}$$

As $k_-(r) - sr_-^{-1} = \mathcal{O}_{r\to r_\pm}(|r_\pm - r|)$ too, we get the same conclusion with $\tilde{\mathscr{P}}_-$.

To end this Subsection, we recall from Section 9 of [10] that

$$(\mathrm{G3}) \implies (\mathrm{A1})\text{-}(\mathrm{A3}), \qquad (\mathrm{G3}) \implies (\mathrm{ME1}), \qquad (\mathrm{G3})\text{-}(\mathrm{G5}) \implies (\mathrm{TE1})\text{-}(\mathrm{TE3})$$

and (ME2) is satisfied by assumptions (G1), (G2) and (G7) on the form of the operator $\mathscr{P}$ using Mazzeo-Melrose standard result (see paragraph 9.2.2 in [10] and also [21] for the original work of Mazzeo-Melrose).

3.3 Study of the asymptotic hamiltonians

The aim of this paragraph is to show the existence of a meromorphic continuation of the weighted resolvent $w^\delta(\hat{H}_\pm(s) - z)^{-1}w^\delta$ form $\mathbb{C}^+$ into a strip in $\mathbb{C}^-$ which is analytic in z in a tight box near 0. We start with the meromorphic extension.

Lemma 3.1. *For all $\delta > 0$, there exists $\varepsilon_\delta > 0$ such that, for all $s \in \mathbb{R}$, $w^\delta(\hat{H}_\pm(s) - z)^{-1}w^\delta$ has a meromorphic extension from $\mathbb{C}^+$ to $\{\omega \in \mathbb{C} \mid \Im\omega > -\varepsilon_{\delta/2}\}$ with values in compact operators acting on $\dot{\mathscr{E}}_\pm$.*

Proof. Since hypotheses (G) are satisfied, we can apply Lemma 9.3 in [10] which shows that we can apply Mazzeo-Melrose result : the meromorphic extension of $w^\delta(\mathscr{P}_\pm - z^2)^{-1}w^\delta$ exists from $\mathbb{C}^+$ to a strip $\mathcal{O}_\delta$. This strip is explicited in the work of Guillarmou (Theorem 1.1 of [13]) :

$$\mathcal{O}_\delta = \left\{ z \in \mathbb{C} \mid z^2 = \lambda(3-\lambda),\, \Re\lambda > \frac{3}{2} - \delta \right\}.$$

The absence of essential singularity is due to the fact that the metric g is even (see Theorem 1.4 and also Definition 1.2 in [13]). We have to check that the set $\mathcal{O}_\delta$ contains a strip in $\overline{\mathbb{C}^-}$. To see this, write $\lambda = \alpha + \mathrm{i}\beta$ and $z = a + \mathrm{i}b$ with $\alpha, \beta, a, b \in \mathbb{R}$, $b \leq 0$ and $z^2 = \lambda(3-\lambda)$. Solving for

$$\begin{cases} a^2 - b^2 &= \alpha(3-\alpha) + \beta^2 \\ 2ab &= (3-2\alpha)\beta \end{cases} \tag{16}$$

we find

$$\begin{cases} \beta &= \pm\sqrt{\frac{1}{2}(a^2 - b^2 - 9/4) + \frac{1}{2}\sqrt{(a^2-b^2-9/4)^2 + 4a^2b^2}} \\ \alpha &= \frac{3}{2} - \frac{ab}{\beta} \end{cases}$$

and these expressions make sense since $\beta = 0$ can happen only if $ab = 0$, and

$$\beta = \pm\frac{|a||b|}{\sqrt{b^2 + 9/4}} + \mathcal{O}_{a\to 0}(a), \qquad\qquad \beta = \mathcal{O}_{b\to 0}(b).$$

If $b = 0$ then $\alpha = 3/2$ and β solves $a^2 = 9/4 + \beta^2$, and conversely $\alpha = 3/2$ implies $b = 0$. Hence $\alpha = 3/2$ allows all $z \in \mathbb{R}$. We may now assume $b < 0$ (hence $\alpha \neq 0$). The condition $\Re\lambda = \alpha > 3/2 - \delta$ reads $\frac{ab}{\beta} < \delta$, and this condition is trivially satisfied if $\alpha \geq 3/2$ since (16) implies that $\frac{ab}{\beta} \leq 0 < \delta$. Otherwise, if $\alpha < 3/2$ then (16) implies that $\frac{ab}{\beta} > 0$ and

$$b > -\left|\frac{\beta}{a}\right|\delta.$$

We compute

$$\left(\frac{\beta}{a}\right)' = \frac{a\beta' - \beta}{a^2}$$

where $'$ denotes here the derivative with respect to a, and

$$\beta' = \frac{a}{2\beta}\left(1 + \frac{(a^2 - b^2 - 9/4) + 2b^2}{\sqrt{(a^2 - b^2 - 9/4)^2 + 4a^2 b^2}}\right)$$

so that

$$\begin{aligned}
a\beta' - \beta = 0 \iff & \; a^2\left(1 + \frac{(a^2 - b^2 - 9/4) + 2b^2}{\sqrt{(a^2 - b^2 - 9/4)^2 + 4a^2 b^2}}\right) = 2\beta^2 \\
\iff & \; a^2(a^2 - b^2 - 9/4) + 2a^2 b^2 \\
& = -(b^2 + 9/4)\sqrt{(a^2 - b^2 - 9/4)^2 + 4a^2 b^2} + (a^2 - b^2 - 9/4)^2 + 4a^2 b^2 \\
\iff & \; (b^2 + 9/4)^2((a^2 - b^2 - 9/4)^2 + 4a^2 b^2) = (b^4 + 81/16 + a^2 b^2 - 9a^2/4 + 9b^2/2)^2.
\end{aligned}$$

After some tedious simplifications, we obtain the very simple condition

$$a\beta' - \beta = 0 \iff 9a^4 b^2 = 0.$$

Thus $a = 0$ is the only possible extremum of β when $b < 0$. One can check that $\beta \to 1$ as $a \to \pm\infty$, whence

$$\{z \in \mathbb{C} \mid 0 \geq \Im z > -\delta\} \subset \mathcal{O}_\delta.$$

From there, we deduce the existence of the meromorphic extension of $w^\delta(\hat{H}_\pm(s) - z)^{-1} w^\delta$ for $z \in \{\omega \in \mathbb{C} \mid \Im\omega > -\varepsilon_{\delta/2}\}$ thanks to Lemma 4.3 and Proposition 4.4 of [10]. $\square$

Before proving the analyticity near 0 of the weighted resolvent, we need to prove the following result :

Lemma 3.2. *For all $\delta > 0$, $w^\delta(\mathscr{P} - z^2)^{-1} w^\delta$ has no pole in $\mathbb{R}$.*

Proof. We can work with the operator P expressed in the Regge-Wheeler coordinate since $P \mapsto \mathscr{P}$ is an unitary transform (as explained at the beginning of Subsection 3.1). Applying the arguments of Subsection 2.1 in [5], we know that there is no eigenvalue on $\mathbb{R}$ and only 0 could be a pole.

We introduce then the Jost solutions following [3]. Fix $\ell \in \mathbb{N}$ and $s \in \mathbb{R}$. Let $\tilde{W}_\ell := \ell(\ell+1)W_0 + W_1$. Recall that $\kappa_- > 0$ is the surface gravity at the event horizon (cf. (7)). For any $\alpha \in \,]0, 2\kappa_-[$,

$$\int_{-\infty}^{+\infty} \left((r_-^{-1} - V(x))\, \mathbb{1}_{]-\infty,0]}(x) + V(x)\, \mathbb{1}_{[0,+\infty[}(x) + |\tilde{W}_\ell(x)|\right) \mathrm{e}^{\alpha|x|} \mathrm{d}x < +\infty.$$

The convergence of the above integral comes from the exponential decay of $r - r_\pm$ and $\tilde{W}_\ell$ at infinity. For all $z \in \mathbb{C}$ such that $\Im z > -\kappa_-$, Proposition 2.1 in [3] shows that there exist two unique $\mathcal{C}^2$ functions $x \mapsto e^\pm(x, z, s, \ell)$, that we will simply write $e_\pm(x)$ or $e_\pm(x, z)$, satisfying the Schrödinger equation

$$(\partial_x^2 + (z - sV)^2 - \tilde{W}_\ell(x))e_\pm(x) = 0 \qquad\qquad \forall x \in \mathbb{R}.$$

with $\partial_x e_\pm \in L^\infty_{\ell oc}(\mathbb{R}_x, \mathbb{C})$, and such that if $\Im z > -\kappa_-$, then $\partial_x^j e_\pm$ is analytic in z for all $0 \leq j \leq 1$ and

$$\lim_{x \to \pm\infty} \left(|e_\pm(x) - \mathrm{e}^{\pm \mathrm{i}zx}| + |\partial_x e_\pm(x) \mp \mathrm{i}z\mathrm{e}^{\pm \mathrm{i}zx}|\right) = 0. \tag{17}$$

By checking the formula on $\mathcal{C}^2(\mathbb{R}, \mathbb{C})$ first and then extending it on $H^2(\mathbb{R}, \mathrm{d}x)$ by density, one easily shows that the kernel K of $(P_\ell - z^2)^{-1}$ for $\Im z > -\kappa_-$ is given by

$$K(x, y, z) = \frac{1}{W(z)}\big(e_+(x, z)e_-(y, z)\mathbb{1}_{x \geq y}(x, y) + e_+(y, z)e_-(x, z)\mathbb{1}_{y \geq x}(x, y)\big)$$

where $W(z) = e_+(x)(e_-)'(x) - (e_+)'(x)e_-(x)$ is the wronskian between e_+ and e_- (independent of $x \in \mathbb{R}$, see the very beginning of the proof of Proposition 2.1 in [3]). In particular, a pole z of order

$n > 0$ for $(P-z^2)^{-1}$ with $\Im z > -\kappa_-$ is a zero of order n of the wronskian W, and $e_+(\cdot, z)$ and $e_-(\cdot, z)$ are then collinear and in $L^2(\mathbb{R}, \mathrm{d}x)$.

We now reproduce the computation (2.14) in [5]. Assume that $z = 0$ is a pole. Then for all $\ell \in \mathbb{N}$, $e_+(\cdot, 0, s, \ell) \in L^2(\mathbb{R}, \mathrm{d}x)$. Using that e_+ solves $(P_\ell - s^2V^2)e_+ = 0$, we can write

$$\begin{aligned}
0 &= \int_{-R_0}^{R_0} \left((P_\ell - s^2V^2)e_+\right) \overline{e_+}\mathrm{d}x \\
&= \left[r\overline{e_+}\partial_x\left(r^{-1}e_+\right)\right]_{-R_0}^{R_0} + \int_{-R_0}^{R_0} \left|r\partial_x\left(r^{-1}e_+\right)\right|^2 \mathrm{d}x + \ell\left(\ell+1\right)\int_{-R_0}^{R_0} F\left(r\right)\left|r^{-1}e_+\right|^2 \mathrm{d}x \\
&+ m^2\int_{-R_0}^{R_0} F\left(r\right)\left|e_+\right|^2 \mathrm{d}x.
\end{aligned}$$

Letting $R_0 \to +\infty$ and using the decay of the derivative of e_+ in (17) show that $e_+ = 0$, a contradiction. $\square$

We are now ready to prove the analyticity.

Proposition 3.3. *Let $0 < \delta < \kappa_-$ and $R > 0$. There exists $\varepsilon_0 \equiv \varepsilon_0(\delta) > 0$ such that the extension of $w^\delta(\hat{H}_\pm(s) - z)^{-1}w^\delta$ is holomorphic in (s, z) for $s \in \mathbb{R}$ sufficiently small and $z \in]-R, R[+\mathrm{i}]-\varepsilon_0, \varepsilon_0[$.*

The restriction $\delta < \kappa_-$ comes from the fact that the extension of $(\hat{H}_\pm - z)^{-1}$ depends itself on δ (see formula (18) in the proof).

Proof. Observe that $(\hat{H}_-(s)-z)^{-1} = (\Phi(-sr_-^{-1})\hat{H}_-(s)\Phi(sr_-^{-1})-z)^{-1}$ on $\dot{\mathscr{E}}_-$, so it is sufficient to prove the announced results for $w^\delta(\hat{H}_+(s) - z)^{-1}w^\delta$ and $w^\delta(\tilde{\hat{H}}_-(s) - z)^{-1}w^\delta$ with

$$\tilde{\hat{H}}_-(s) := \Phi(-sr_-^{-1})\hat{H}_-(s)\Phi(sr_-^{-1}) = \begin{pmatrix} sr_-^{-1} & 1 \\ \tilde{\mathscr{P}}_- & 2k_- - sr_-^{-1} \end{pmatrix}.$$

Proceeding as in the proof of Proposition 4.4 in [10], we can work on the operators $\mathscr{P} - (z - \tilde{k}_\pm)^2$ with

$$\tilde{k}_+ := k_+, \qquad\qquad \tilde{k}_- := k_- - sr_-^{-1}$$

so that $\tilde{k}_\pm$ are now exponentially decaying potentials at infinity (and are polynomial in s).

We reproduce the perturbation argument of Lemma 4.3 in [10]. Choose $\varepsilon_0 \in]0, \varepsilon_\delta[$, ε_δ as in Lemma 3.1, and pick $z \in]-R, R[+\mathrm{i}]-\varepsilon_0, \varepsilon_0[$ so that $w^\delta(\mathscr{P} - z^2)^{-1}w^\delta$ is holomorphic (it is possible since there is no pole in $\mathbb{R}$ by Lemma 3.2) and $w^\delta(\mathscr{P} - (z - \tilde{k}_\pm)^2)^{-1}w^\delta$ is meromorphic[3] in z. Write then in $\mathcal{H}$

$$w^\delta(\mathscr{P} - z^2)^{-1}w^\delta = w^\delta(\mathscr{P} - (z - \tilde{k}_\pm)^2)^{-1}w^\delta\left(\mathrm{Id} + \mathcal{K}_\pm(s, z)\right) \tag{18}$$

with

$$\mathcal{K}_\pm(s, z) := w^{-\delta}\tilde{k}_\pm(2z - \tilde{k}_\pm)w^{-\delta}w^\delta(\mathscr{P} - z^2)^{-1}w^\delta.$$

$\mathcal{K}_\pm(s, z)$ is clearly analytic in $s \in D(0, 1)$ and in $z \in]-R, R[+\mathrm{i}]-\varepsilon_0, \varepsilon_0[$. Since $\tilde{k}_\pm = \mathcal{O}_{r\to r_-\pm\infty}(w^{2\kappa_-})$ by (6) (we use here that $\kappa_- < |\kappa_+|$), $\delta < \kappa_-$ and $w^\delta(\mathscr{P} - z^2)^{-1}w^\delta$ is compact by Lemma 3.1, we see that $\mathcal{K}_\pm(s, z)$ is compact. By two-dimensional analytic Fredholm theory, there exists a subvariety $S \subset D(0, 1) \times \left(]-R, R[+\mathrm{i}]-\varepsilon_0, \varepsilon_0[\right)$ such that $(\mathrm{Id} + \mathcal{K}_\pm(s, z))^{-1}$ exists and is holomorphic in $(s, z) \in \Big(D(0, 1) \times \left(]-R, R[+\mathrm{i}]-\varepsilon_0, \varepsilon_0[\right)\Big) \setminus S$. We then get the representation formula for the extension :

$$w^\delta(\mathscr{P} - (z - \tilde{k}_\pm)^2)^{-1}w^\delta = w^\delta(\mathscr{P} - z^2)^{-1}w^\delta\left(\mathrm{Id} + \mathcal{K}_\pm(s, z)\right)^{-1}. \tag{19}$$

[3]Lemma 3.1 of course applies if we replace k_- by $\tilde{k}_-$.

We claim that for $\sigma > 0$ sufficiently small, we have

$$\Big(D(0,\sigma)\times\big(\,]-R,R[\,+\mathrm{i}\,]-\varepsilon_0,\varepsilon_0[\,\big)\Big)\cap S=\emptyset. \tag{20}$$

Otherwise, for every $n\in\mathbb{N}\setminus\{0\}$, there is a couple $(s_n,z_n)\in D(0,1/n)\times\big(\,]-R,R[\,+\mathrm{i}\,]-\varepsilon_0,\varepsilon_0[\,\big)$ such that $\mathrm{Id}+\mathcal{K}(s_n,z_n)$ is not invertible. By compactness, we can assume that $(s_n,z_n)\to(0,z_0)$ as $n\to+\infty$ for some $z_0\in[-R,R]+\mathrm{i}[-\varepsilon_0,\varepsilon_0]$. But $\mathrm{Id}+\mathcal{K}(0,z_0)=\mathrm{Id}$ is invertible for all $z\in\mathbb{C}$, so $\mathrm{Id}+\mathcal{K}_\pm(s,z)$ must be invertible too for all (s,z) in a small neighbourhood of $(0,z_0)$, a contradiction.

Assuming now that s is sufficiently small so that (20) is true, we deduce by (19) that the poles of $w^\delta(\mathscr{P}-(z-\tilde{k}_\pm)^2)^{-1}w^\delta$ are exactly the poles of $w^\delta(\mathscr{P}-z^2)^{-1}w^\delta$. Since for $z\in\,]-R,R[\,+\mathrm{i}\,]-\varepsilon_0,\varepsilon_0[$, $w^\delta(\mathscr{P}-z^2)^{-1}w^\delta$ has no pole, the same conclusion applies for $w^\delta(\mathscr{P}-(z-\tilde{k}_\pm)^2)^{-1}w^\delta$. □

3.4 Construction of the meromorphic extension of the weighted resolvent

The aim of this paragraph is to show the existence of a meromorphic extension for $w^\delta(\hat{K}(s)-z)^{-1}w^\delta$ in a strip near 0 of width uniform in s. Since the operators $\hat{K}(s)$ and $\hat{H}(s)$ are equivalent modulo the isomorphism $\Phi(sV)$ (by (3.19) in [10]), we will work with the latter one.

We first need some preliminary results. The starting point is the following result :

Proposition 3.4 (Proposition 5.5 in [10])**.** *There is a finite set $Z\in\mathbb{C}\setminus\mathbb{R}$ with $\bar{Z}=Z$ such that the spectrum of $\hat{H}(s)$ is included in $\mathbb{R}\cup Z$ and the resolvent has a meromorphic extension to $\mathbb{C}\setminus\mathbb{R}$. Moreover, the set Z consists in eigenvalues of finite multiplicity of $\hat{H}(s)$.*

An important fact is that Proposition 3.6 in [10] shows that $Z\equiv Z(s)$ is contained in the disc $D(0,C|s|)$ for some constant $C>0$.

We will henceforth use the Regge-Wheeler coordinate x introduced in Subsection 2.2 and will abusively note $\mathscr{P}$ and $\mathcal{H}$ respectively the operator $\mathscr{P}$ expressed in the coordinates (x,ω)

$$\mathscr{P}=-F^{-1/2}(r(x))\,\partial_x^2\,F^{-1/2}(r(x))-W_0(x)\Delta_{\mathbb{S}^2}+W_1(x)$$

and the space $L^2(\mathbb{R}\times\mathbb{S}^2_\omega,F(r(x))\mathrm{d}x\mathrm{d}\omega)$.

Lemma 3.5. *For all $\delta>0$, w^δ sends $\dot{\mathscr{E}}$ into $\dot{\mathscr{E}}_\pm$ and $\dot{\mathscr{E}}$ into itself.*

Proof. Let $u=(u_0,u_1)\in\dot{\mathscr{E}}$. We only show that $w^\delta\dot{\mathscr{E}}\subset\dot{\mathscr{E}}_-$, the proof of the other statements being slightly easier. We thus look for $v=(v_0,v_1)\in\tilde{\mathscr{P}}_-^{-1/2}\mathcal{H}\oplus\mathcal{H}$ such that $(w^\delta u_0,w^\delta u_1)=(v_0,sr_+^{-1}v_0+v_1)$. Since w^δ is bounded on $\mathbb{R}$, $w^\delta u_1\in\mathcal{H}$. Next, using the facts that $(w^\delta)'u_0$, $V_+w^\delta u_0$ and $W_j^{1/2}w^\delta u_0$ are in $\mathcal{H}$ thanks to (ME1) (d) $(0\le j\le1)$, we compute

$$\|\tilde{\mathscr{P}}_-^{1/2}w^\delta u_0\|^2_{\mathcal{H}}=\big\langle\mathscr{P}w^\delta u_0,w^\delta u_0\big\rangle_{\mathcal{H}}-\underbrace{\|s^2(r_-^{-1}-k_-)^2w^\delta u_0\|^2_{\mathcal{H}}}_{\lesssim\|\mathscr{P}^{1/2}u_0\|^2_{\mathcal{H}}}$$

and working with the operators $\mathscr{P}_\ell$ defined as P_ℓ $(\ell\in\mathbb{N})$, we get

$$\big\langle\mathscr{P}_\ell w^\delta u_0,w^\delta u_0\big\rangle_{\mathcal{H}}=\|\partial_x w^\delta u_0\|^2_{\mathcal{H}}+\underbrace{\big\langle(-\ell(\ell+1)W_0+W_1)w^\delta u_0,w^\delta u_0\big\rangle_{\mathcal{H}}}_{\lesssim\|\mathscr{P}_\ell^{1/2}u_0\|^2_{\mathcal{H}}},$$

$$\|\partial_x w^\delta u_0\|^2_{\mathcal{H}}\lesssim\|(w^\delta)'u_0\|^2_{\mathcal{H}}+\|w^\delta u_0'\|^2_{\mathcal{H}}\lesssim\|\mathscr{P}_\ell^{1/2}u_0\|^2_{\mathcal{H}}.$$

This proves that $w^\delta u_0\in\tilde{\mathscr{P}}_-^{-1/2}\mathcal{H}$. Hence $v_0:=w^\delta u_0\in\tilde{\mathscr{P}}_-^{1/2}\mathcal{H}$, and the problem boils down to show that $v_1:=w^\delta u_1-sr_+^{-1}v_0=w^\delta u_1-sr_+^{-1}w^\delta u_0$ is in $\mathcal{H}$; this is a consequence of (ME1) (d) which implies that $sr_+^{-1}w^\delta u_0\in\mathcal{H}$. □

For $z\in\mathbb{C}^+$ and $s\in\mathbb{R}$, we introduce for the operator

$$Q(s,z):=i_-^2(\hat{H}_-(s)-z)^{-1}+i_+^2(\hat{H}_+(s)-z)^{-1}=\sum_\pm i_\pm^2(\hat{H}_\pm(s)-z)^{-1}.$$

By Lemma 3.5, $Q(s,z)w^\delta$ is well-defined on $\dot{\mathscr{E}}$. A direct computation shows that on $\dot{\mathscr{E}}$

$$(\hat{H}(s)-z)Q(s,z)w^\delta = \left(\mathrm{Id} + \sum_\pm [\hat{H}(s), i_\pm^2](\hat{H}_\pm(s)-z)^{-1}\right) w^\delta$$

whence for $z \notin Z$ (see Proposition 3.4)

$$w^\delta Q(s,z)w^\delta = w^\delta(\hat{H}(s)-z)^{-1}w^\delta\big(\mathrm{Id} + \hat{\mathcal{K}}_\pm(s,z)\big). \tag{21}$$

with

$$\hat{\mathcal{K}}_\pm(s,z) := w^{-\delta}\sum_\pm [\hat{H}(s), i_\pm^2](\hat{H}_\pm(s)-z)^{-1}w^\delta$$

Lemma 3.6. *The operators on the left and right-hand sides of (21) send $\dot{\mathscr{E}}$ into itself.*

Proof. For the left-hand side of (21), we successively use Lemma 3.5, the facts that $(\hat{H}_\pm(s)-z)^{-1}$ sends $\dot{\mathscr{E}}_\pm$ into $\mathscr{D}(\hat{H}_\pm(s)) \subset \dot{\mathscr{E}}_\pm$ and $i_\pm$ sends $\dot{\mathscr{E}}_\pm$ into $\dot{\mathscr{E}}$ (by Lemma 5.4 of [10]), and again Lemma 3.5.

We now deal with the right-hand side of (21). By Lemma 3.5, we only have to show that $w^{-\delta}[\hat{H}(s), i_\pm](\hat{H}_\pm(s)-z)^{-1}$ sends $\dot{\mathscr{E}}_\pm$ into $\dot{\mathscr{E}}$. Let $u \in \dot{\mathscr{E}}_\pm$ and write $v = (v_0, v_1) : (\hat{H}_\pm(s)-z)^{-1}u \in \dot{\mathscr{E}}_\pm$. We have

$$w^{-\delta}[H(s), i_\pm](H_\pm(s)-z)^{-1}u = w^{-\delta}\begin{pmatrix} 0 & 0 \\ [\mathscr{P}, i_\pm] & 0 \end{pmatrix}(v_0, v_1) = \begin{pmatrix} 0 \\ w^{-\delta}\,[\mathscr{P}, i_\pm]\, v_0 \end{pmatrix} = \begin{pmatrix} 0 \\ w^{-\delta}\,[\mathscr{P}, i_\pm]\, w^{-\delta} w^\delta v_0 \end{pmatrix}.$$

Since $w^\delta \dot{\mathscr{E}}_\pm \subset \dot{\mathscr{E}}_\pm$, we can use (TE3) (e) to conclude that the second component is in $\mathcal{H}$, whence $w^{-\delta}[\hat{H}(s), i_\pm](\hat{H}_\pm(s)-z)^{-1}\dot{\mathscr{E}}_\pm \subset \dot{\mathscr{E}}$ (when $\pm = -$, we use that $\mathscr{P}^{1/2}\tilde{\mathscr{P}}_-^{-1/2}$ is bounded on $\mathcal{H}$). ☐

Lemma 3.7. *Let $0 < \delta < \kappa_-$ and $R > R_0$. $\mathrm{Id} + \hat{\mathcal{K}}_\pm(s,z)$ is a holomorphic family of Fredholm operators acting on $\dot{\mathscr{E}}$ for $(s,z) \in D(0,\sigma) \times \big(]-R,R[+ \mathrm{i}]-\varepsilon_0, \varepsilon_0[\big)$, with $\sigma > 0$ sufficiently small and $\varepsilon_0 > 0$ as in Proposition 3.3.*

Proof. Write

$$\hat{\mathcal{K}}_\pm(s,z) = \sum_\pm w^{-\delta}[\hat{H}(s), i_\pm^2]w^{-\delta}w^\delta(\hat{H}_\pm(s)-z)^{-1}w^\delta.$$

By Lemma 3.1, $w^\delta(\hat{H}_\pm(s)-z)^{-1}w^\delta$ is compact on $\dot{\mathscr{E}}_\pm$ and Proposition 3.3 shows that the extension is holomorphic in (s,z). Furthermore,

$$w^{-\delta}[\hat{H}(s), i_\pm^2]w^{-\delta} = \begin{pmatrix} 0 & 0 \\ w^{-\delta}\,[\mathscr{P}, i_\pm^2]\, w^{-\delta} & 0 \end{pmatrix}$$

is bounded on $\dot{\mathscr{E}}_\pm$ (as a consequence of (TE3) (e), see the end of the proof of Lemma 3.6). Hence $\hat{\mathcal{K}}_\pm(s,z)$ is compact and thus $\mathrm{Id} + \hat{\mathcal{K}}_\pm(s,z)$ is Fredholm. ☐

We are now ready to construct the meromorphic extension of the weighted resolvent. We keep in mind that Proposition 3.4 and the remark below show that for $|s| < s_0$, $s_0 > 0$ fixed, and $R > R_0 := 2Cs_0$, we have $Z \subset D(0, R/2)$.

Theorem 3.8. *Let $0 < \delta < \kappa_-$.*

1. *For all $s \in \mathbb{R}$, $w^\delta(\hat{H}(s)-z)^{-1}w^\delta$ has a meromorphic extension from $\mathbb{C}^+ \setminus Z$ to $\{\omega \in \mathbb{C} \mid \Im\omega > -\varepsilon_{\delta/2}\}$ with values in compact operators acting on $\dot{\mathscr{E}}$, with $\varepsilon_{\delta/2} > 0$ as in Lemma 3.1.*

2. *For all $R > R_0$ and $s \in]-s_0, s_0[$ sufficiently small, the extension of $w^\delta(\hat{H}(s)-z)^{-1}w^\delta$ is analytic in $z \in]-R,R[+ \mathrm{i}]-\varepsilon_0, \varepsilon_0[$ with $\varepsilon_0 > 0$ as in Proposition 3.3.*

Proof. Our strategy is to compare $\hat{H}(s)$ with the assymptotic hamiltonians $\hat{H}_\pm(s)$ in order to invert $\mathrm{Id} + \hat{\mathcal{K}}_\pm(s,z)$ in formula (21) by analytic Fredholm theory (the argument is similar to the one used for Proposition 3.3).

We first show Part 1. Fix $s \in \mathbb{R}$ and let $z \in \,]-R,R[\,+\mathrm{i}\,]-\varepsilon_{\delta/2},\varepsilon_{\delta/2}[$ with $\varepsilon_{\delta/2} > 0$ as in Propositon 3.3. Since $\hat{H}_\pm(0) = \hat{H}(0)$, we observe that $\hat{\mathcal{K}}_\pm(0,z) = 0$ and $Q(0,z) = (\hat{H}(0)-z)^{-1}$. Hence the operator $\mathrm{Id} + \hat{\mathcal{K}}_\pm(0,z) = \mathrm{Id}$ is invertible for all $z \in \mathbb{C}$. Finally, Lemma 3.1 shows that $w^\delta(\hat{H}_\pm(s) - z)^{-1}w^\delta$ is meromorphic in z. We can therefore use the meromorphic Fredholm theory to invert $\mathrm{Id} + \hat{\mathcal{K}}_\pm(s,z)$ on $\dot{\mathscr{E}}$. Using (21) to write the representation formula

$$w^\delta(\hat{H}(s) - z)^{-1}w^\delta = w^\delta Q(s,z)w^\delta\big(\mathrm{Id} + \hat{\mathcal{K}}_\pm(s,z)\big)^{-1} \tag{22}$$

which valid for $z \in \,]-R,R[\,+\mathrm{i}\,]-\varepsilon_{\delta/2},\varepsilon_{\delta/2}[$. This shows that $w^\delta(\hat{H}(s) - z)^{-1}$ has a meromorphic extension in this strip and Part 1 is settled.

Let us show Part 2. of the theorem. We pick this time $(s,z) \in D(0,\sigma) \times \big(]-R,R[\,+\mathrm{i}\,]-\varepsilon_0,\varepsilon_0[\big)$. Lemma 3.7 shows that, if σ is small enough, $\mathrm{Id} + \hat{\mathcal{K}}_\pm(s,z)$ is a holomorphic family of Fredholm operators acting on $\dot{\mathscr{E}}$. We can thus use the two-dimensional analytic Fredholm theory which implies that there is a meromorphic extension $D(0,\sigma) \times \big(]-R,R[\,+\mathrm{i}\,]-\varepsilon_0,\varepsilon_0[\big) \ni (s,z) \mapsto \big(\mathrm{Id} + \hat{\mathcal{K}}_\pm(s,z)\big)^{-1}$, and (22) is valid for $(s,z) \in D(0,\sigma) \times \big(]-R,R[\,+\mathrm{i}\,]-\varepsilon_0,\varepsilon_0[\big)$ with σ small. This shows that the poles of $w^\delta(\hat{H}(s) - z)^{-1}w^\delta$ are the poles of $(\mathrm{Id} + \hat{\mathcal{K}}_\pm(s,z))^{-1}$ and $w^\delta Q(s,z)w^\delta$, the last ones being the poles of $w^\delta(\hat{H}_\pm(s) - z)^{-1}w^\delta$.

The multidimensional analytic Fredholm theory also implies that there exists a (possibly empty) subvariety $S \subset D(0,\sigma) \times \big(]-R,R[\,+\mathrm{i}\,]-\varepsilon_0,\varepsilon_0[\big)$ such that $\mathrm{Id} + \hat{\mathcal{K}}(s,z)$ is invertible for $(s,z) \notin S$. We claim that we can take $\sigma > 0$ small enough so that

$$\Big(D(0,\sigma) \times \big(]-R,R[\,+\mathrm{i}\,]-\varepsilon_0,\varepsilon_0[\big)\Big) \cap S = \emptyset.$$

Otherwise, for every $n \in \mathbb{N} \setminus \{0\}$, there is a couple $(s_n,z_n) \in D(0,1/n) \times \big(]-R,R[\,+\mathrm{i}\,]-\varepsilon_0,\varepsilon_0[\big)$ such that $\mathrm{Id} + \hat{\mathcal{K}}(s_n,z_n)$ is not invertible. By compactness, we can assume that $(s_n,z_n) \to (0,z_0)$ as $n \to +\infty$ for some $z_0 \in [-R,R] + \mathrm{i}[-\varepsilon_0,\varepsilon_0]$. But $\mathrm{Id} + \hat{\mathcal{K}}(0,z_0) = \mathrm{Id}$ is invertible for all $z \in \mathbb{C}$, so $\mathrm{Id} + \hat{\mathcal{K}}_\pm(s,z)$ must be invertible too for all (s,z) in a small neighbourhood of $(0,z_0)$, a contradiction.

We now assume s small enough so that $\mathrm{Id} + \hat{\mathcal{K}}_\pm(s,z)$ is invertible on $\dot{\mathscr{E}}$ for $z \in \,]-R,R[\,+\mathrm{i}\,]-\varepsilon_0,\varepsilon_0[$. Using then the formula (22), we conclude that the poles of $w^\delta(\hat{H}(s) - z)^{-1}w^\delta$ are precisely the poles of $w^\delta Q(s,z)w^\delta$, which are the poles of $w^\delta(\hat{H}_\pm(s) - z)^{-1}w^\delta$. We then use Proposition 3.3 to conclude that there is no pole for $z \in \,]-R,R[\,+\mathrm{i}\,]-\varepsilon_0,\varepsilon_0[$. This completes the proof. □

As a first consequence, we deduce a holomorphy result for the resolvent.

Corollary 3.9. *Let $s \in \mathbb{R}$ such that $|s| < s_0$ with $Cs_0 < \varepsilon_0$. The resolvent $(\hat{H}(s)-z)^{-1}$ is holomorphic in $z \in \mathbb{C}^+$ for $s \in \mathbb{R}$ small enough.*

Proof. We know by Theorem 3.8 that the weighted resolvent $w^\delta(\hat{H}(s) - z)^{-1}w^\delta$ is holomorphic in $z \in (]-R,R[\,+\mathrm{i}\,]-\varepsilon_0,\varepsilon_0[) \cap \mathbb{C}^+ \subset D(0,R/2) \subset \mathbb{C}^+$ if we assume s sufficiently small. By Proposition3.4, $(\hat{H}(s) - z)^{-1}$ is holomorphic in $\mathbb{C}^+ \setminus Z$. Assume then that $z_0 \in \mathbb{C}^+ \cap Z$ is a pole of order $m_0 \in \mathbb{N}$: there exist some finite rank operators $A_1,\dots,A_{m_0} : \mathcal{H} \to \mathcal{H}$ such that

$$(\hat{H}(s) - z)^{-1} = \sum_{j=1}^{m_0} \frac{A_j}{(z-z_0)^j} + \text{holomorphic term} \qquad \forall z \in \mathbb{C}^+ \text{ near } z_0.$$

Since $R > R_0$, $Z \cap \mathbb{C}^+ \in D(0,R/2) \cap \mathbb{C}^+$ and then

$$w^\delta(\hat{H}(s) - z)^{-1}w^\delta = \sum_{j=1}^{m_0} \frac{w^\delta A_j w^\delta}{(z-z_0)^j} + \text{holomorphic term.}$$

is holomorphic in z near z_j, so that $A_1 = \ldots = A_{m_0} = 0$ and $(\hat{H}(s)-z)^{-1}$ is holomorphic in $z \in \mathbb{C}^+$. □

We finally deduce the existence of the cut-off inverse of the quadratic pencil and define resonances.

Corollary 3.10. *The operator $\chi p(z,s)^{-1}\chi : L^2(\mathbb{R}, \mathrm{d}x) \to H^2(\mathbb{R}, \mathrm{d}x)$ defines for any $\chi \in \mathcal{C}_c^\infty(\mathbb{R}, \mathbb{R})$ a meromorphic function of $z \in \{\omega \in \mathbb{C} \mid \Im\omega > -\varepsilon_{\kappa_-/2}\}$ and analytic if $\Im z > -\varepsilon_0$, with $\varepsilon_{\kappa_-/2}, \varepsilon_0 > 0$ given in Lemma 3.1 (for $\delta = \kappa_-$) and Proposition 3.3 respectively.*
The poles z of this extension are exactly the poles of the cut-off resolvent $\chi(\hat{H}(s) - z)^{-1}\chi$ and are independent of the choice of χ. We call them resonances of p and write $z \in \operatorname{Res}(p)$. Similarly, we define $\operatorname{Res}(p_\ell)$ as the poles of $\chi p_\ell(z,s)^{-1}\chi$ for all $\ell \in \mathbb{N}$.

Proof. Let $R > 0$ and let $z \in \mathbb{C}$ with $-R \leq \Re z \leq R$ and $\Im z > -\varepsilon_{\kappa_-/2}$. The meromorphic extension $w(\hat{H}(s) - z)^{-1}w : \dot{\mathscr{E}} \to \dot{\mathscr{E}}$ entails the meromorphic extension $w(\hat{K}(s) - z)^{-1}w : \dot{\mathcal{E}} \to \dot{\mathcal{E}}$ since $\hat{H}(s)$ and $\hat{K}(s)$ are equivalent on $\dot{\mathcal{E}}$ modulo the isomorphism $\Phi(sV)$ introduced in Subsection 3.1. Since $w(x)$ is exponentially decaying by (6), the operator $\chi(\hat{K}(s) - z)^{-1}\chi : \dot{\mathcal{E}} \to \dot{\mathcal{E}}$ has a meromorphic continuation in a strip in $\mathbb{C}^-$ for any cut-off $\chi \in \mathcal{C}_c^\infty(\mathbb{R}, \mathbb{R})$.

By formula (14) and the discussion below, we see that we can define the operator $\chi p(z,s)^{-1}\chi : L^2(\mathbb{R}, \mathrm{d}x) \to H^2(\mathbb{R}, \mathrm{d}x)$ for any $\chi \in \mathcal{C}_c^\infty(\mathbb{R}, \mathbb{R})$ as a meromorphic function of z, and its poles are precisely the poles of $\chi(\hat{K}(s) - z)^{-1}\chi$.

To conclude the proof, it remains to prove the analyticity in the whole strip $\{\Im z > -\varepsilon_0\}$: this follows from Theorem 4.1 below. □

4 Resonance expansion for the charged Klein-Gordon equation

We present in this section the main result of this paper which is an extension of Theorem 1.3 in [5]. By using the formula (13) and (14) as well as (3.21) in [10] and the local equivalence (12) of the norms $\|.\|_{\dot{\mathcal{E}}_\ell}$ and $\|.\|_{\mathcal{E}_\ell}$ if $z \in \mathbb{R}$, we can define for $\Im z > -\varepsilon_1$ (ε_1 as in Theorem 3.8) the meromorphic extension of the cut-off resolvent $\hat{R}_{\chi,\ell}(z) := \chi(\hat{K}_\ell - z)^{-1}\chi$. For all resonance $z_0 \in \operatorname{Res}(p_\ell)$, denote by $m(z_0) \in \mathbb{N}$ its multiplicity and set

$$\Pi_{j,k}^{\chi,\ell} := \frac{1}{2\pi \mathrm{i}} \oint_{\partial\gamma} \frac{(-\mathrm{i})^k}{k!} \hat{R}_{\chi,\ell}(z)(z - z_0)^k \mathrm{d}z$$

defined for all integer $k \geq -(m(z0) + 1)$ with γ a small positively oriented circle enclosing z_0 and no other resonance. We will denote by $\hat{R}_\chi(z)$ and $\Pi_{j,k}^\chi$ the cut-off resolvent of the full operator $\hat{K}$ and the corresponding generalized projector, respectively. Recall that $\operatorname{Res}(p)$ is introduced in Corollary 3.10.

We first introduce the set of pseudo-poles of P whose points approximate high frequency resonances. The proof is given in Appendix B.

Theorem 4.1. *There exist $K > 0$ and $\theta > 0$ such that, for any $C > 0$, there exists an injective map $\tilde{b} : \Gamma \to \operatorname{Res}(p)$ with*

$$\Gamma = \frac{\sqrt{F(\mathfrak{r})}}{\mathfrak{r}} \left(\pm\mathbb{N} \setminus \{0\} \pm \frac{1}{2} \pm \frac{qQ}{\sqrt{F(\mathfrak{r})}} - \frac{\mathrm{i}}{2} \sqrt{\left| 3 - \frac{12M}{\mathfrak{r}} + \frac{10Q^2}{\mathfrak{r}^2} \right|} \left(\mathbb{N} + \frac{1}{2} \right) \right).$$

the set of pseudo-poles, such that all the poles in

$$\Omega_C = \{\lambda \in \mathbb{C} \mid |\lambda| > K, \Im\lambda > -\max\{C, \theta|\Re\lambda|\}\}$$

are in the image of $\tilde{b}$. Furthermore, if $\mu \in \Gamma$ and $\tilde{b}(\mu) \in \Omega_C$, then

$$\lim_{|\mu| \to +\infty} (\tilde{b}(\mu) - \mu) = 0.$$

If $\Re\mu = \frac{\sqrt{F(\mathfrak{r})}}{\mathfrak{r}} \left(\pm\ell \setminus \{0\} \pm \frac{1}{2} \pm \frac{qQ}{\sqrt{F(\mathfrak{r})}} \right)$ for $\ell \in \mathbb{N} \setminus \{0\}$, then the corresponding pole $\tilde{b}(\mu)$ has multiplicity $2\ell + 1$.

We can now state our main result :

Theorem 4.2. *Let $\chi \in \mathcal{C}_c^\infty(\mathbb{R}, \mathbb{R})$.*

(i) Let $\nu > 0$ such that $\nu \notin \Gamma$ (Γ is the set of pseudo-poles as in Theorem 4.1), $\nu < \varepsilon_{\kappa_-/2}$ ($\varepsilon_{\kappa_-/2}$ as in Theorem 3.8) and $\mathrm{Res}(p) \cap \{\lambda \in \mathbb{C} \mid \Im\lambda = -\nu\} = \emptyset$. There exists $N > 0$ such that, for all $u \in \dot{\mathcal{E}}$ with $\langle -\Delta_{\mathbb{S}^2}\rangle^N u \in \dot{\mathcal{E}}$ and s small enough, we have

$$\chi \mathrm{e}^{-\mathrm{i}t\hat{K}} \chi u = \sum_{\substack{z_j \in \mathrm{Res}(p) \\ \Im z_j > -\nu}} \sum_{k=0}^{m(z_j)} \mathrm{e}^{-\mathrm{i}z_j t}\, t^k\, \Pi_{j,k}^\chi u + E(t)u \tag{23}$$

for $t > 0$ sufficiently large, with

$$\|E(t)u\|_{\dot{\mathcal{E}}} \lesssim \mathrm{e}^{-\nu t} \|\langle -\Delta_{\mathbb{S}^2}\rangle^N u\|_{\dot{\mathcal{E}}}$$

and the sum is absolutely convergent in the sense that

$$\sum_{\substack{z_j \in \mathrm{Res}(p) \\ \Im z_j > -\nu}} \sum_{k=0}^{m(z_j)} \|\Pi_{j,k}^\chi \langle -\Delta_{\mathbb{S}^2}\rangle^{-N}\|_{\dot{\mathcal{E}} \to \dot{\mathcal{E}}} < +\infty.$$

(ii) There exists $\varepsilon > 0$ such that, for any increasing positive function g with $\lim_{x\to+\infty} g(x) = +\infty$ and $g(x) \le x$ for $x \gg 0$, for all $u \in \dot{\mathcal{E}}$ with $g(-\Delta_{\mathbb{S}^2})u \in \dot{\mathcal{E}}$ and s small enough, we have

$$\|\chi \mathrm{e}^{-\mathrm{i}t\hat{K}} \chi u\|_{\dot{\mathcal{E}}} \lesssim (g(\mathrm{e}^{\varepsilon t}))^{-1} \|g(-\Delta_{\mathbb{S}^2})u\|_{\dot{\mathcal{E}}}$$

for $t > 0$ sufficiently large.

Remark 4.3. *1. Formula (23) provides a physical interpretation of resonances : they are the frequencies and dumping rates of charged Klein-Gordon field in presence of the charged black hole (see Chapter 4.35 in [7] for a discussion on the interpretation of resonances).*

2. Part (ii) of Theorem 4.2 shows that a logarithmic derivative loss in the angular direction $(\ln\langle -\Delta_{\mathbb{S}^2}\rangle)^\alpha u \in \dot{\mathcal{E}}$ with $\alpha > 1$ entails the integrability of the local energy :

$$\left\| \int_0^{+\infty} \chi \mathrm{e}^{-\mathrm{i}t(\hat{K}-z)} \chi\, u \,\mathrm{d}t \right\|_{\dot{\mathcal{E}}} \lesssim \|(\ln\langle -\Delta_{\mathbb{S}^2}\rangle)^\alpha\|_{\dot{\mathcal{E}}}.$$

3. When $Q = \Lambda = 0$, $\mathfrak{r} = 3M$ and $\kappa_- = 1/4M$. Denote by $\tilde{\Gamma}$ the set of pseudo-poles of Theorem 4.1 for $Q = \Lambda = 0$. For any pseudo-pole $\lambda \in \tilde{\Gamma}$ with negative imaginary part, $|\Im\lambda| \ge \frac{1}{12\sqrt{3}}$. By Theorem 3.8, we have a meromorphic extension of the weighted charge Klein-Gordon operator for $\Im\lambda > -\frac{\kappa_-}{2}$. Thus, if $M < \frac{3\sqrt{3}}{2}$, then $\tilde{\Gamma} \cap \{\omega \in \mathbb{C} \mid \Im\omega > -\kappa_-/2\} \neq \emptyset$. Since F is a smooth function, this holds true with Γ when Q and Λ are small and in this case the resonance expansion (23) is not only an exponential decay of solutions.

5 Estimates for the cut-off inverse of the quadratic pencil

In this section, we show some estimates on the cut-off inverse of the quadratic pencil. We can work with $\ell \in \mathbb{N}$ fixed but our estimates have to be uniform in ℓ. Since $\big(\chi p(-\bar{z} + 2sV, s)\chi\big)^* = \chi p(z, s)\chi$, we can restrict ourselves to consider $z \in \mathbb{C}$ with $\Re z > -2s_0\|V\|_{L^\infty}$ for some fixed $s_0 > 0$ such that $0 < |s| < s_0$. In the following, we are simply denoting by L^2 the space $L^2(\mathbb{R}, \mathrm{d}x)$. For some real numbers $R, C_0, C_1 > 0$ (determined by Theorem 5.1 below), we define the

- zone I as $[-R, R] + \mathrm{i}\,[-C_0, C_0]$,
- zone II as $[R, \ell/R] + \mathrm{i}\,[-C_0, C_0]$,

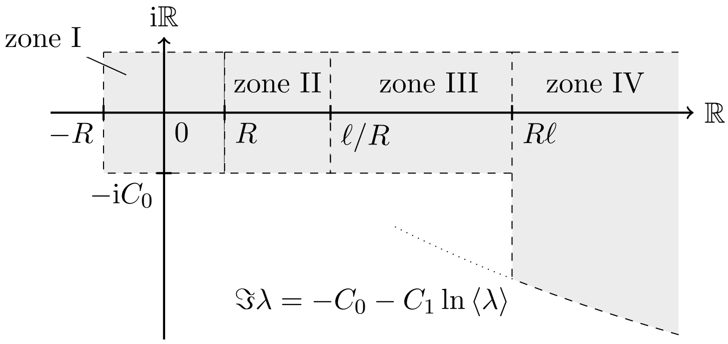

Figure 2: The four zones.

- zone III as $[\ell/R, R\ell] + \mathrm{i}\,[-C_0, C_0]$,
- zone IV as $([R\ell, +\infty[+ \mathrm{i}\,]-\infty, C_0]) \cap \{\lambda \in \mathbb{C} \mid \Im\lambda \geq -C_0 - C_1 \ln\langle\lambda\rangle\} \cap \Omega_\varepsilon$

with $\Omega_\varepsilon := \left\{\omega \in \mathbb{C} \mid \Im\omega > -\varepsilon\right\}$ and $\varepsilon = \varepsilon_{\kappa_-/2} > 0$ (see Part 1. of Theorem 3.8.
We quote here all the estimates that we are going to show in this chapter in the following theorem, which is similar to Theorem 2.1 in [5].

Theorem 5.1. *Let $\chi \in \mathcal{C}_c^\infty(\mathbb{R}, \mathbb{R})$, $s \in \mathbb{R}$ and ε, Ω_ε as above. If s is small enough, then the following estimates hold uniformly in $\ell \in \mathbb{N}$:*

1. *For all $R > 0$, $C > 0$ and $0 < C_0 < \varepsilon$, $\mathrm{Res}(p) \cap ([-R, R] + \mathrm{i}\,[-C_0, C]) = \emptyset$ and the operator*
$$\chi p\,(z, s)^{-1}\,\chi : L^2 \to L^2 \tag{24}$$
exists and is bounded uniformly in $z \in [-R, R] + \mathrm{i}\,[-C_0, C]$. Moreover, we have
$$\|\chi p_\ell\,(z, s)^{-1}\,\chi\|_{L^2 \to L^2} \leq \|\chi p\,(z, s)^{-1}\,\chi\|_{L^2 \to L^2} \lesssim \prod_{\substack{z_j \in \mathrm{Res}(p) \\ |z_j| < 2R}} \frac{1}{|z - z_j|}. \tag{25}$$

2. *There exist $R > 0$ and $0 < C_0 < \varepsilon$ such that there is no resonance in $[R, \ell/R] + \mathrm{i}\,[-C_0, C_0]$. Furthermore, for all $z \in [R, \ell/R] + \mathrm{i}\,[-C_0, C_0]$, we have*
$$\|\chi p_\ell(z, s)^{-1}\chi\|_{L^2 \to L^2} \lesssim \frac{1}{\langle z\rangle^2}. \tag{26}$$

3. *Let $R > 0$ and $0 < C_0 < \varepsilon$ be fixed and suppose that $\ell \gg 0$. The number of resonances of p_ℓ in $[\ell/R, R\ell] + \mathrm{i}[-C_0, C_0]$ is bounded uniformly in ℓ and there exists $C > 0$ such that, for all $z \in [\ell/R, R\ell] + \mathrm{i}[-C_0, C_0]$,*
$$\|\chi p_\ell(z, s)^{-1}\chi\|_{L^2 \to L^2} \lesssim \langle z\rangle^C \prod_{\substack{z_j \in \mathrm{Res}(p_\ell) \\ |z - z_j| < 1}} \frac{1}{|z - z_j|}. \tag{27}$$
Furthermore, there exists $\varepsilon > 0$ such that there is no resonance in $[\ell/R, R\ell] + \mathrm{i}[-\varepsilon, 0]$ and we have for all $z \in [\ell/R, R\ell] + \mathrm{i}[-\varepsilon, 0]$
$$\|\chi p_\ell(z, s)^{-1}\chi\|_{L^2 \to L^2} \lesssim \frac{\ln\langle z\rangle}{\langle z\rangle}\, \mathrm{e}^{|\Im z| \ln\langle z\rangle}. \tag{28}$$

4. *Let $R \gg 0$, $C_0 > 0$ and $C_1 > 0$. Set*
$$\tilde{\Omega}_\ell := \left([R\ell, +\infty[+ \mathrm{i}\,] - \infty, C_0]\right) \cap \left\{\lambda \in \mathbb{C} \mid \Im\lambda \geq -C_0 - C_1 \ln\langle\lambda\rangle\right\} \cap \Omega_\varepsilon.$$
There is no resonance in $\tilde{\Omega}_\ell$ and there exists $C > 0$ such that for all z in this set,
$$\|\chi p_\ell(z, s)^{-1}\chi\|_{L^2 \to L^2} \leq C\langle z\rangle^{-1} \mathrm{e}^{C|\Im z|}. \tag{29}$$

Remark 5.2. *High frequency resonances of the zone III (i.e. resonances whose real part are of order $\ell \gg 0$) are localized in Theorem 4.1.*

The announced estimate in the zone I is a direct application of results of Section 3 (see Theorem 3.8). We thus show the estimates for the zones II, III and IV.

5.1 Estimates in the zone II

We prove part 2. of Theorem 5.1 using the complex scaling introduced in Section 4 of [33]. Observe that the zone II does not exist if $\ell = 0$, so that we can assume that $\ell \geq 1$. Let $z \in [R, \ell/R] + \mathrm{i}\,[-C_0, C_0]$ and choose $N \in [R, \ell/R]$ such that $z \in [N, 2N] + \mathrm{i}\,[-C_0, C_0]$. We introduce the semiclassical parameter

$$h := N^{-1}$$

and the new spectral parameter

$$\lambda := h^2 z^2 \in [1/4, 4] + \mathrm{i}\,[-4C_0 h, 4C_0 h]\,.$$

In this setting, we define the operator

$$\tilde{p}_h(\sqrt{\lambda}, s) := h^2 p_\ell\,(z, s) = \underbrace{-h^2\partial_x^2 + \alpha^2 W_0\,(x)}_{=:Q_h} - \lambda + \underbrace{h^2 W_1\,(x) + 2h\sqrt{\lambda} s V\,(x) - h^2 s^2 V\,(x)^2}_{=:R_h(\lambda)}$$

where $\alpha := h(\ell(\ell+1))^{1/2} \gg 2\mathscr{A} > 0$, $\mathscr{A}$ as in Proposition 2.1.

We now use the (α-dependent) contour $\Gamma_\theta := \Gamma_\theta^- \cup \Gamma_\theta^+$ for $0 < \theta < \pi/2$, with[4]

$$\Gamma_\theta^\pm := \left\{x + \mathrm{i} f_\theta^\pm(x, \ln(g_\infty^\pm)/\kappa_\pm) \mid x \in \mathbb{R}_\pm\right\}$$

where (using estimate (6) for W_0)

$$g_\infty^\pm := \lim_{x \to \pm\infty} \mathrm{e}^{2\kappa_\pm x} W_0(x)$$

and

$$f_\theta^\pm(x, \beta) := \begin{cases} 0 & \text{if } |x| \leq \beta/2 - C_1 \\ \theta(x - \beta/2) & \text{if } |x| \geq \beta/2 + C_2 \end{cases}$$

with constants $C_1, C_2 > 0$ as in (4.4) in Section 4 of [33] (see Figure 3 for the behaviour of Γ_θ). Define next $L^2(\Gamma_\theta)$ and $H^2(\Gamma_\theta)$ as the associated Lebesgue and Sobolev spaces. Using the analytic extension of $x \mapsto r(x)$ on the set $\Sigma := \{\eta \in \mathbb{C} \mid |\Re \eta| > \mathscr{A}\}$, we extend V, W_0 and W_1 on Σ (and still denote them V, W_0 and W_1). We then define the distorted operators

$$\tilde{p}_{h,\theta}(\sqrt{\lambda}, s) = \tilde{p}_h(\sqrt{\lambda}, s) \restriction_{\Gamma_\theta}, \qquad Q_{h,\theta} := Q_h \restriction_{\Gamma_\theta}, \qquad R_{h,\theta}\,(\lambda) = R_h\,(\lambda) \restriction_{\Gamma_\theta}\,.$$

If $q_{h,\theta}$ denotes the symbol of $Q_{h,\theta}$, then Lemma 4.3 in [33] shows that there exists $0 < c < 1$ and $\theta_0 > 0$ such that

$$|q_{h,\theta}\,(x, \xi) - \lambda| \gtrsim \theta(\langle \xi \rangle^2 + \mathrm{e}^{\kappa_\pm x} \langle \alpha \rangle^2) \qquad \pm x \geq 0$$

provided that $\Re(\lambda) > 0$, $\Im(\lambda) < c$ and $0 < \theta < \theta_0$ (recall that $\pm\kappa_\pm < 0$). For h small enough (that is R large enough), we can apply Proposition 4.1 in [33] to get

$$\|(Q_{h,\theta} - \lambda)^{-1}\|_{L^2(\Gamma_\theta) \to H^2(\Gamma_\theta)} = \mathcal{O}(\theta^{-1}).$$

In order to invert the distorted quadratic pencil, we use a Neumann series argument by showing that $R_{h,\theta}(\lambda) = \mathcal{O}_{L^2(\Gamma_\theta)}(h)$ (as a multiplication operator). In view of the form of $R_{h,\theta}$ in (5.1) and

[4] The factor $1/\kappa_\pm$ in the second argument of $f_\theta^\pm$ comes from the fact that $\kappa_p m x$ for us corresponds to Zworski's variable r.

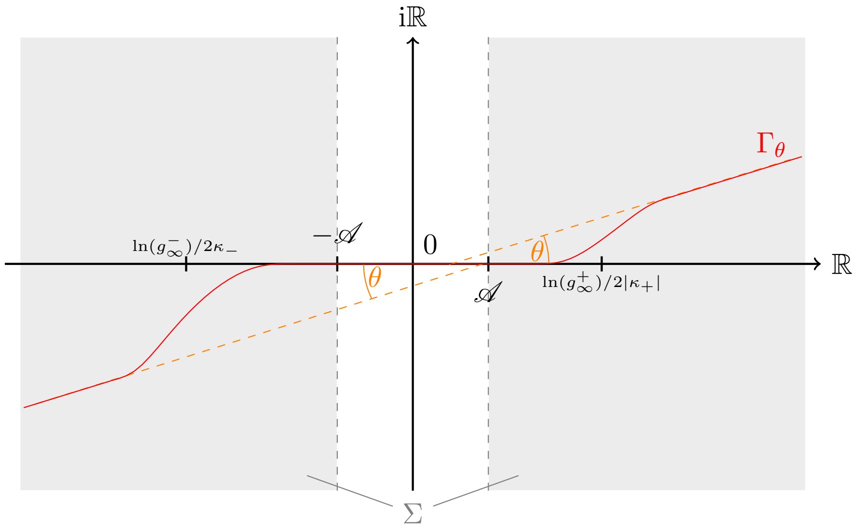

Figure 3: The contour Γ_θ.

because the extension of r is analytic, it is enough to bound $x \mapsto r(x + \mathrm{i} f_\theta^\pm(x))$ below and above for $|x| > 2\mathscr{A}$. By Lagrange inversion formula (41), we can write

$$|r(x + \mathrm{i} f_\theta^\pm(x)) - r_\pm| \leq \sum_{k=1}^{+\infty} c_k \frac{\mathrm{e}^{2k\kappa_\pm x}}{k!}$$

for some coefficients $c_k > 0$ (recall from Subsection 2.2 that $2\kappa_\pm = \frac{\Lambda}{3A_\pm r_\pm^2}$), and the series converges (because (41) converges uniformly when $|\Re x| > \mathscr{A}$). Since the sum is decreasing with respect to x, we deduce $r < +\infty$ on Γ_θ. On the other hand,

$$|r - r_\pm| = C(r - r_n)^{-\frac{A_n r_n^2}{A_\pm r_\pm^2}} (r - r_c)^{-\frac{A_c r_c^2}{A_\pm r_\pm^2}} |r - r_\mp|^{-\frac{A_\mp r_\mp^2}{A_\pm r_\pm^2}} \mathrm{e}^{2\kappa_\pm x}$$

with $C \in \mathbb{R}$. Since no terms on the right-hand side can blow up when restricted on Γ_θ and since the exponential goes to zero when $|x| \to +\infty$, it follows that $r \to r_\pm > 0$ as $x \to \pm\infty$. We therefore conclude that the restriction of r on $\Gamma_\theta \cap D(0, R_0)^\complement$ is bounded from below and above for $R_0 \gg 0$, giving $R_{h,\theta}(\lambda) = \mathcal{O}_{L^2(\Gamma_\theta)}(h)$. Thus,

$$\tilde{p}_{h,\theta}(\sqrt{\lambda}, s)^{-1} = \left(1 + (Q_{h,\theta} - \lambda)^{-1} R_{h,\theta}(\lambda)\right)^{-1} (Q_{h,\theta} - \lambda)^{-1}.$$

We finally choose $\chi \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R})$ and increase if necessary the value of the number $\mathscr{A}$ so that $\mathrm{Supp}\chi \subset [-\mathscr{A}, \mathscr{A}]$. From Lemma 3.5 in [28], we have in the L^2 sense

$$\chi \tilde{p}_{h,\theta}(\sqrt{\lambda}, s)^{-1} \chi = \chi \tilde{p}_h(\sqrt{\lambda}, s)^{-1} \chi$$

whence

$$\|\chi p_\ell(z, s)^{-1} \chi\|_{L^2 \to L^2} = h^2 \|\chi \tilde{p}_h(\sqrt{\lambda}, s)^{-1} \chi\|_{L^2 \to L^2} \lesssim \langle z \rangle^{-2}.$$

5.2 Estimates in the zone III

We turn to the proof of part 3. of Theorem 5.1. We define the semiclassical parameter

$$h := (\ell(\ell+1))^{-1/2}$$

with again $\ell > 0$ since the zone does not exist for $\ell = 0$. For $z \in [\ell/R, R\ell] + \mathrm{i}[-C_0, C_0]$, we define a new spectral parameter

$$\lambda := h^2 z^2 \in \left[\frac{1}{3R^2}, R^2\right] + \mathrm{i}[-\sqrt{2}\, C_0 R h, \sqrt{2}\, C_0 R h] \subset [a, b] + \mathrm{i}[-ch, ch]$$

for some $0 < a < b$ and $c > 0$. Finally, we set

$$\tilde{P}_h := h^2 P_\ell = -h^2\partial_x^2 + W_0 + h^2 W_1, \qquad \tilde{p}_h(\sqrt{\lambda}, s) := \tilde{P}_h - (\sqrt{\lambda} - hsV)^2$$

and write $\tilde{P}_\theta$ and $\tilde{p}_\theta$ for the corresponding distorted operators on the contour Γ_θ as we did in the paragraph 5.1. We are still using the subscript L^2 when we work with the distorted operators.

As W_0 admits a non-degenerate maximum at $x = 0$ (see Section 2, Figure 1), $(x, \xi) = (0, 0)$ is a trivial solution of the Hamilton equations associated to the principal symbol of $\tilde{P}_h$:

$$\begin{cases} \dot{x} &= 2\xi \\ \dot{\xi} &= -W_0'(x) \end{cases}.$$

Therefore the energy level $\mathfrak{E}_0 := W_0(0)$ is trapping. For this reason, the zone III is called the trapping zone.

We first show an adaptation of Lemma 6.5 of [6] to our setting.

Proposition 5.3. *For $\theta = Nh$ with $N > 0$ large enough and $s \in \mathbb{R}$ sufficiently small, there exist $C \equiv C(N) > 0$ and $\varepsilon > 0$ such that, for all $E \in [\mathfrak{E}_0 - \varepsilon, \mathfrak{E}_0 + \varepsilon]$ and $|\lambda - E| \leq \varepsilon\theta/2$, it holds*

$$\|(\tilde{P}_\theta - (\sqrt{\lambda} - hsV)^2)^{-1}\|_{L^2 \to L^2} = \mathcal{O}(h^{-C}) \prod_{\substack{\lambda_j \in \mathrm{Res}(\tilde{p}) \\ |\lambda - \lambda_j| < \varepsilon\theta}} \frac{h}{|\lambda - \lambda_j|}.$$

Proof. The announced estimate is known for the resolvent $(\tilde{P}_\theta - \lambda)^{-1} = \tilde{p}_\theta(\sqrt{\lambda}, 0)^{-1}$ corresponding to the case $s = 0$. The argument can be found in [29] which uses techniques developed in [27], and the authors of [6] adapted it for the one dimensional case of a non degenerate trapping energy level $\mathfrak{E}_0$. More precisely, for $\theta = Nh$ with $N \gg 0$ large enough, one can construct a bounded operator $\tilde{K} \in \mathcal{L}\left(L^2, L^2\right)$ (see (6.15) in [6]) satisfying the following properties :

(*i*) $\|\tilde{K}\|_{L^2 \to L^2} = \mathcal{O}(1)$,

(*ii*) $r := \operatorname{rank} \tilde{K} \leq \mathcal{O}(\theta h^{-1} \ln(1/\theta))$,

(*iii*) for h small enough, there exists $\varepsilon > 0$ such that, for all $E \in [\mathfrak{E}_0 - \varepsilon, \mathfrak{E}_0 + \varepsilon]$ and $\lambda \in [E - \varepsilon\theta, E + \varepsilon\theta]$,

$$\|(\tilde{P}_\theta - \mathrm{i}\theta\tilde{K} - \lambda)^{-1}\|_{L^2 \to \mathscr{D}} \leq \mathcal{O}(\theta^{-1}), \qquad \mathscr{D} := \mathscr{D}(\tilde{P}_\theta).$$

In [28] Lemma 3.2, it is shown that $\tilde{P}_\theta - \lambda$ is a Fredholm operator from its domain $\mathscr{D}$ to L^2, so we can construct a well-posed Grushin problem

$$\mathcal{P}(\lambda) := \begin{pmatrix} \tilde{P}_\theta - \lambda & R_- \\ R_+ & 0_{\mathbb{C}^r \to \mathbb{C}^r} \end{pmatrix} : \mathscr{D} \oplus \mathbb{C}^r \to L^2 \oplus \mathbb{C}^r \tag{30}$$

where R_- and R_+ are constructed with $\tilde{P}_\theta - \mathrm{i}\theta\tilde{K} - \lambda$ (see [27], page 401, below (6.12) for the construction).

Now consider $s \neq 0$. If s is small enough, $(\tilde{P}_\theta - \mathrm{i}\theta\tilde{K} - (\sqrt{\lambda} - hsV)^2)^{-1}$ is invertible by pseudodifferential calculus[5] as for the case $s = 0$. By the resolvent identity, one can show that

$$\|(\tilde{P}_\theta - \mathrm{i}\theta\tilde{K} - (\sqrt{\lambda} - hsV)^2)^{-1}\|_{L^2 \to \mathscr{D}} \leq \mathcal{O}(\theta^{-1}) + \mathcal{O}(h|s|)\|(\tilde{P}_\theta - \mathrm{i}\theta\tilde{K} - (\sqrt{\lambda} - hsV)^2)^{-1}\|_{L^2 \to \mathscr{D}}\mathcal{O}(\theta^{-1})$$

since $|\lambda| \leq h(|\Re z| + |\Im z|) \leq \mathcal{O}(1) + \mathcal{O}(h)$. Hence for s sufficiently small, we have

$$\|(\tilde{P}_\theta - \mathrm{i}\theta\tilde{K} - (\sqrt{\lambda} - hsV)^2)^{-1}\|_{L^2 \to \mathscr{D}} \leq \mathcal{O}(\theta^{-1}) \qquad \lambda \in [E - \varepsilon\theta, E + \varepsilon\theta].$$

[5] See the definition of the operator K at the beginning of the proof of Lemma 6.5 in [6].

Because the quadratic pencil remains a Fredholm operator provided that $\|hsV\|_{L^\infty}$ is sufficiently small[6], we can write a new well-posed Grushin problem

$$\mathcal{P}(\lambda) := \begin{pmatrix} \tilde{p}_\theta(\sqrt{\lambda}, s) & R_- \\ R_+ & 0_{\mathbb{C}^r \to \mathbb{C}^r} \end{pmatrix} : \mathscr{D} \oplus \mathbb{C}^r \to L^2 \oplus \mathbb{C}^r.$$

where this time R_+ and R_- are constructed with $\tilde{P}_\theta - \mathrm{i}\theta\tilde{K} - (\sqrt{\lambda} - hsV)^2$. If we note

$$\mathcal{E}(\lambda) := \mathcal{P}(\lambda)^{-1} = \begin{pmatrix} E(\lambda) & E_+(\lambda) \\ E_-(\lambda) & E_0(\lambda) \end{pmatrix} : L^2 \oplus \mathbb{C}^r \to \mathscr{D} \oplus \mathbb{C}^r,$$

then the relations $\mathcal{E}(\lambda)\mathcal{P}(\lambda) = \mathcal{P}(\lambda)\mathcal{E}(\lambda) = \mathrm{Id}$ as well as the following estimate (which results of properties (i) and (iii) above)

$$(\tilde{P}_\theta - \mathrm{i}\theta\tilde{K} - (\sqrt{\lambda} - hsV)^2)^{-1}(\tilde{P}_\theta - (\sqrt{\lambda} - hsV)^2) = \mathcal{O}(1)$$

imply as in [6] that $\|E(\lambda)\|_{L^2\to\mathscr{D}}, \|E_-(\lambda)\|_{L^2\to\mathbb{C}^r} = \mathcal{O}(\theta^{-1})$ and $\|E_+(\lambda)\|_{\mathbb{C}^r\to\mathscr{D}}, \|E_0(\lambda)\|_{\mathbb{C}^r\to\mathbb{C}^r} = \mathcal{O}(1)$. Applying formula (8.11) in [27], we obtain

$$(\tilde{P}_\theta - (\sqrt{\lambda} - hsV)^2)^{-1} = E(\lambda) - E_+(\lambda)E_0(\lambda)^{-1}E_-(\lambda)$$

which implies

$$\|(\tilde{P}_\theta - (\sqrt{\lambda} - hsV)^2)^{-1}\|_{L^2\to\mathscr{D}} = \mathcal{O}(\theta^{-1})(1 + \|E_0(\lambda)^{-1}\|_{\mathbb{C}^r\to\mathbb{C}^r}).$$

as in Lemma 6.5 of [6], and we then follow the end of its proof to conclude. □

We can now follow the arguments below Lemma 2.2 in [5]. The set of pseudo-poles (2.28) and the injective map (2.29) in this reference exist in our setting by Theorem 4.1 (but are quite different). This implies there is no resonance in $\Omega(h) := [a/2, 2b] + \mathrm{i}[-\varepsilon h, ch]$ provided that h and s are small enough. As a result, (27) holds true. As for the estimate (28), we use Burq's Lemma :

Lemma 5.4 (Lemma 2.3 in [5]). *Suppose that $f(\lambda, h)$ is a family of holomorphic functions defined for $0 < h < 1$ in a neighbourhood of $\Omega(h) := [a/2, 2b] + \mathrm{i}[-ch, ch]$ with $0 < a < b$ and $c > 0$, such that*

$$|f(\lambda, h)| \lesssim \begin{cases} h^{-C} & \text{in } \Omega(h) \\ \dfrac{1}{|\Im\lambda|} & \text{in } \Omega(h) \cap \mathbb{C}^+ \end{cases}$$

Then, there exists h_0 , $C > 0$ such that, for any $0 < h < h_0$ and any $\lambda \in [a, b] + \mathrm{i}[-ch, 0]$,

$$|f(\lambda, h)| \leq C\frac{|\ln h|}{h}\mathrm{e}^{C|\Im\lambda||\ln h|/h}.$$

We apply this result to the function $f(\lambda, h) := \|\chi(\tilde{P}_h - (\sqrt{\lambda} - hsV)^2)^{-1}\chi\|$, observing that for all $\lambda \in \Omega(h) \cap \mathbb{C}^+$ the resolvent identity gives

$$\|(\tilde{P}_h - (\sqrt{\lambda} - hsV)^2)^{-1}\| \leq \frac{1}{|\Im\lambda|} + \|(\tilde{P}_h - (\sqrt{\lambda} - hsV)^2)^{-1}\|\frac{\mathcal{O}(h|s|)}{|\Im\lambda|} \lesssim \frac{1}{|\Im\lambda|}$$

because $\|(\tilde{P}_h - (\sqrt{\lambda} - hsV)^2)^{-1}\|$ is uniformly bounded on this set for h and s small enough.

[6] Recall that the set of Fredholm operators in $\mathcal{L}(\mathscr{D}, L^2)$ is open for the norm topology.

5.3 Estimates in the zone IV

This last paragraph is devoted to the proof of part 4. of Theorem 5.1. For $z \in ([R\ell, +\infty[+ \mathrm{i}\,]-\infty, C_0]) \cap \{\lambda \in \mathbb{C} \mid \Im\lambda \geq -C_0 - C_1 \ln\langle\lambda\rangle\}$, there exists a number $N > R\ell > 0$ such that $z \in [N, 2N] + \mathrm{i}\,[-C \ln N, C_0]$. We introduce the semiclassical parameters

$$h := \frac{1}{N}, \qquad \mu := \ell\,(\ell+1)\,h^2, \qquad \nu := h^2.$$

Observe that these parameters are very small when $N \gg 0$. Moreover, we can consider that $h \leq 1$ even if $\ell = 0$, simply by taking $R \geq 1$ in the zone I if it was not the case (R as in Theorem 5.1). We then define a new spectral parameter

$$\lambda := z^2 h^2 \in [1,2] + \mathrm{i}\,[Ch \ln h, C_0 h] \subset [a,b] + \mathrm{i}\,[-ch\,|\ln h|\,, ch]$$

where $0 < a \leq 1 < 2 \leq b < +\infty$ and $\max\{C, C_0\} < c < +\infty$ (observe that a and b do not depend on h). Let $J := [a,b]$ and set

$$J^+ := \{\eta \in \mathbb{C}^+ \mid \Re(\eta) \in J\}.$$

Define then

$$\tilde{P}_h := h^2 P_\ell = -h^2\partial_x^2 + \mu W_0 + \nu W_1, \qquad \tilde{p}_h(\sqrt{\lambda}, s) := h^2 p_\ell\,(z,s) = \tilde{P}_h - (\sqrt{\lambda} - hsV)^2.$$

Semiclassical limiting absorption principle for the quadratic pencil. As in [5], we first get a control until the real line by using a semiclassical limiting absorption principle for the semiclassical quadratic pencil. The appendix C provides a proof, close to the idea developed by Gérard [11], of such a result for a class of perturbed resolvents, so we only have to check if the required abstract assumptions are satisfied.

Introduce the generator of dilations $A := -\mathrm{i}h\,(x\partial_x + \partial_x x)$ with domain $\mathscr{D}(A) := \{u \in L^2 \mid Au \in L^2\}$. We then pick $\rho \in \mathcal{C}_c^\infty(\mathbb{R}, [0,1])$ such that $\operatorname{Supp}\rho \subset [a/3, 3b]$ and $\rho \equiv 1$ on $I := [a/2, 2b]$, and we define $\mathcal{A}$ as the closure of the operator $\rho(P)A\rho(P)$. In this setting, $\rho(P)A\rho(P)$ is well-defined on $\mathscr{D}(A)$, $\mathcal{A}$ is self-adjoint and we have $P \in \mathcal{C}^2(\mathcal{A})$ (see [5], Section 2.4) so that (I) holds. A direct computation shows that

$$\mathrm{i}h^{-1}[P, A] = 4P - 4\mu W_0 - 4\nu W_1 - 2\mu x W_0' - 2\nu x W_1'$$

so that, for μ and ν sufficiently small, we get the Mourre estimate (M) (uniform in μ, ν)

$$\mathbb{1}_I(P)[P, \mathrm{i}\mathcal{A}]\mathbb{1}_I(P) \geq ah\mathbb{1}_I(P).$$

Since $V \in \mathcal{B}(\mathscr{D}(P), L^2)$ it is clear that $V \in L^\infty_{\ell oc}(\tilde{P}_h)$. Moreover, assumption (C) is fulfilled for $f(z,B) := (\sqrt{z} - sB)^2$.

It remains to show that assumption (A) is satisfied for $B = hV$. Observe that this abstract assumption is particularly well adapted to semiclassical pseudodifferential calculus framework, especially the commutator estimate which provides the supplementary term h. In [14], it is shown that $\mathcal{A} \in \Psi^{-\infty,1}$ ($\mathcal{A}$ is the operator $c_{\tilde{\chi}}$ in [14], see above Lemma 3.3). We are going to use it to show the following result :

Lemma 5.5. *For all $0 \leq \sigma \leq 1$, $V \in \mathcal{B}(\mathscr{D}(\langle\mathcal{A}\rangle^\sigma))$ and $[V, \chi(\tilde{P}_h)] \in h\mathcal{B}(\mathscr{D}(\langle\mathcal{A}\rangle^\sigma))$.*

Proof. Let $\Omega := [0,1] + \mathrm{i}\mathbb{R}$ and let $z \in \Omega$. On $\mathscr{D}(\langle\mathcal{A}\rangle^2) \times \mathscr{D}(\langle\mathcal{A}\rangle^2)$, we define the sesquilinear form

$$Q_z(\varphi, \psi) := \langle V\langle\mathcal{A}\rangle^{-2z}\varphi, \langle\mathcal{A}\rangle^{2z}\psi\rangle \qquad \forall\varphi, \psi \in \mathscr{D}(\langle\mathcal{A}\rangle^2).$$

By functional calculus, Q_z is well-defined and analytic in $z \in \Omega$. When $z \in \{0\} + \mathrm{i}\mathbb{R}$, $|(1+\lambda^2)^{z/2}| = 1$ for all $\lambda \in \mathbb{R}$ so that functional calculus first applied to $\langle\mathcal{A}\rangle^{2z}$ and then to $\langle\mathcal{A}\rangle^{-2z}$ gives

$$\begin{aligned}
|Q_z(\varphi, \psi)| &\leq |\langle V\langle\mathcal{A}\rangle^{-2z}\varphi, \psi\rangle| \\
&= |\langle\langle\mathcal{A}\rangle^{-2z}\varphi, V\psi\rangle| \\
&\leq |\langle\varphi, V\psi\rangle| \\
&\leq \|V\|_{L^\infty}\|\varphi\|_{L^2}\|\psi\|_{L^2}.
\end{aligned}$$

When $z = 1$, pseudodifferential calculus shows that $\langle \mathcal{A}\rangle^2 V \langle \mathcal{A}\rangle^{-2} \in \Psi^{0,0}$, so that for all $z \in \{1\} + \mathrm{i}\mathbb{R}$ (using again functional calculus for $\langle \mathcal{A}\rangle^{\pm 2\mathrm{i}\Im z}$),

$$
\begin{aligned}
|Q_z(\varphi,\psi)| &\le |\langle \langle \mathcal{A}\rangle^2 V \langle \mathcal{A}\rangle^{-2} \langle \mathcal{A}\rangle^{-2\mathrm{i}\Im z}\varphi, \psi\rangle| \\
&\le \|\langle \mathcal{A}\rangle^2 V \langle \mathcal{A}\rangle^{-2}\|_{L^2\to L^2} \|\langle \mathcal{A}\rangle^{-2\mathrm{i}\Im z}\varphi\|_{L^2} \|\psi\|_{L^2} \\
&\le \|\langle \mathcal{A}\rangle^2 V \langle \mathcal{A}\rangle^{-2}\|_{L^2\to L^2} \|\varphi\|_{L^2} \|\psi\|_{L^2}.
\end{aligned}
$$

By the maximum principle, there exists a constant $C > 0$ such that Q_z is bounded by C for all $0 \le \Re z \le 1$. In particular, we can extend $Q_{\sigma/2}$ on $L^2 \times \mathscr{D}(\langle \mathcal{A}\rangle^2)$ as a bounded sesquilinear form and for $\sigma \in [0,1]$ and $\varphi \in L^2$, we have

$$
|Q_{\sigma/2}(\varphi,\psi)| \le C\|\varphi\|_{L^2}\|\psi\|_{L^2}.
$$

This means that the map $\mathscr{D}(\langle \mathcal{A}\rangle^\sigma) \ni \psi \mapsto \langle V\langle \mathcal{A}\rangle^{-\sigma}\varphi, \langle \mathcal{A}\rangle^\sigma\psi\rangle$ is continuous. By definition of the adjoint operator and because $\langle \mathcal{A}\rangle^\sigma$ is self-adjoint, this implies that $V\langle \mathcal{A}\rangle^{-\sigma}\varphi \in \mathscr{D}(\langle \mathcal{A}\rangle^\sigma)$ for all $\varphi \in L^2$.

Consider now the sesquilinear form

$$
\tilde{Q}_z(\varphi,\psi) := \langle [V, \chi(\tilde{P}_h)]\langle \mathcal{A}\rangle^{-2z}\varphi, \langle \mathcal{A}\rangle^{2z}\psi\rangle \qquad \forall \varphi,\psi \in \mathscr{D}(\langle \mathcal{A}\rangle^2).
$$

By semiclassical pseudodifferential calculus, we have (see e.g. (4.4.19) in [34])

$$
\begin{aligned}
[V, \chi(\tilde{P}_h)] &= \frac{h}{\mathrm{i}}\{V(x), \chi(\xi^2 + \mu W_0(x) + \nu W_1(x))\}^{\mathrm{w}} + h^3\Psi^{-\infty,0} \\
&= h\Psi^{-\infty,-\infty} + h^3\Psi^{-\infty,0}
\end{aligned}
$$

because $V(x) \in \Psi^{0,0}$, $V'(x) \in \Psi^{0,-\infty}$ and $\chi(\tilde{P}_h) \in \Psi^{-\infty,0}$. Despite the fact that the error term above looks less regular than the main term, it is in fact more regular as it can be shown using expansion (4.4.15) in [34] (but we will not need such a regularity). Now we can proceed as above with Q_z and V to conclude. $\square$

Now that all assumptions in appendix C have been checked, we can use Theorem C.1 as well as the fact that $\|\langle x\rangle^{-\sigma}\langle \mathcal{A}\rangle^\sigma\| \lesssim 1$ for all $\sigma \le 1$[7] : for $\sigma \in]1/2,1]$ and h small enough, we have uniformly in μ,ν

$$
\sup_{\lambda\in J^+} \left\| \langle x\rangle^{-\sigma}\tilde{p}_h(\sqrt{\lambda},s)^{-1}\langle x\rangle^{-\sigma}\right\| \le \left\|\langle x\rangle^{-\sigma}\langle \mathcal{A}\rangle^\sigma\right\| \left(\sup_{\lambda\in J^+}\left\|\langle \mathcal{A}\rangle^{-\sigma}\tilde{p}_h(\sqrt{\lambda},s)^{-1}\langle \mathcal{A}\rangle^{-\sigma}\right\|\right)\left\|\langle \mathcal{A}\rangle^\sigma\langle x\rangle^{-\sigma}\right\| \lesssim h^{-1}.
$$

Estimates below the real axis. Next, we can use the work of Martinez [20] to get a bound under the real line. Indeed, Section 4 of the last reference applies in our setting because $\tilde{p}_h(\sqrt{\lambda},s)$ is a differential operator (so that Proposition 3.1 and Corollary 3.2 of [20] apply) and because $(\lambda - hsV(x))^2 \in [\lambda-\delta,\lambda+\delta] + \mathrm{i}[ch\ln h, 0]$ for all λ in the zone IV and all $x \in \mathbb{R}$ if s is small enough (so that the estimate (4.6) in [20] still holds). It follows that equation (4.13) holds with $\tilde{p}_h(\sqrt{\lambda},s)$ instead of $P_\theta - \rho$[8]. In our setting, this reads

$$
\|\chi\tilde{p}_h(\sqrt{\lambda},s)^{-1}\chi\| \le Ch^{-C} \tag{31}
$$

for some $C > 0$.

To get (29), we reproduce the argument at the end of the proof of Lemma 2.4 in [5]. Choose f holomorphic satisfying the following conditions :

$$
\begin{cases}
|f| < 1 & \text{for } \lambda \in [a/2, 2b] + \mathrm{i}[ch\ln h, 0], \\
|f| \ge 1 & \text{for } \lambda \in [a, b] + \mathrm{i}[ch\ln h, 0], \\
|f| \le h^C & \text{for } \lambda \in [a/2, 2b] \setminus [2a/3, 3b/2] + \mathrm{i}[ch\ln h, 0]
\end{cases}
$$

[7] We show it using the sesquilinear form $(\varphi,\psi) \mapsto \langle\langle x\rangle^{-\sigma}\langle \mathcal{A}\rangle^\sigma\varphi,\psi\rangle$ first well-defined on $\mathscr{D}(\mathcal{A}^2) \times \mathscr{D}(\mathcal{A}^2)$ because $\langle x\rangle^{-2} \in \Psi^{0,-2}$, and then extended to $L^2 \times L^2$ by maximum principle.

[8] We can in fact insert any pseudodifferential operator here provided that hypotheses of Section 2 in [20] are verified.

where $C > 0$ is the constant in (31). Since f is holomorphic, the function

$$g(\lambda) := \ln \|\chi\tilde{p}(\sqrt{\lambda}, s)^{-1}\chi\|_{L^2\to L^2} + \ln|f(\lambda)| + \frac{C}{ch}\Im\lambda$$

is subharmonic. We can check that $g(\lambda) \lesssim \ln(h^{-1})$ on the boundary of $[a/2, 2b] + \mathrm{i}[ch\ln h, 0]$. By the maximum principle, this estimate holds for all $\lambda \in [a/2, 2b] + \mathrm{i}[ch\ln h, 0]$, whence

$$\|\chi\tilde{p}_h(\sqrt{\lambda}, s)^{-1}\chi\|_{L^2\to L^2} \lesssim h^{-1}\mathrm{e}^{\frac{C}{ch}|\Im\lambda|}.$$

The desired estimate (29) then follows.

6 Proof of the main theorem

We prove in this section Theorem 4.2. The resonance expansion (23) follows from the theory of resonances as presented in Section 3 of [5], and we can follow the proof of this paper. We only have to adapt the estimate for the first order operator :

Proposition 6.1 (Proposition 3.1 in [5]). *Let $\ell \in \mathbb{N}$ and let $\chi \in \mathcal{C}^\infty_c(\mathbb{R}, \mathbb{R})$. There exists $\tilde{\chi} \in \mathcal{C}^\infty_c(\mathbb{R}, \mathbb{R})$ satisfying $\tilde{\chi}\chi = \chi$ such that for all $z \in \mathbb{C} \setminus \mathrm{Res}(p_\ell)$, the cut-off resolvent $\chi(\hat{K}_\ell - z)^{-1}\chi$ is a bounded operator on $\dot{\mathcal{E}}_\ell$ and satisfies uniformly in ℓ*

$$\|\hat{R}_{\chi,\ell}(z)\|_{\dot{\mathcal{E}}_\ell\to\dot{\mathcal{E}}_\ell} \lesssim \langle z\rangle \|\tilde{\chi}p_\ell(z, s)^{-1}\tilde{\chi}\|_{L^2\to L^2}.$$

Proof. Since the norms $\|.\|_{\mathcal{E}_\ell}$ and $\|.\|_{\dot{\mathcal{E}}_\ell}$ are locally equivalent thanks to the Hardy type estimate $\|\chi.\|_{L^2} \lesssim \|P_\ell^{1/2}.\|_{L^2}$ uniformly in ℓ (see Lemma 9.5 in [10]), we can work on $(\mathcal{E}_\ell, \|.\|_{\mathcal{E}_\ell})$. For $(u_0, u_1) \in \mathcal{E}_\ell$, we have

$$\hat{R}_{\chi,\ell}(z)\begin{pmatrix}u_0\\u_1\end{pmatrix} = \begin{pmatrix}\chi p_\ell(z,s)^{-1}\chi((z - sV)u_0 + u_1)\\ \chi(1 + (z - sV)p_\ell(z, s)^{-1}(z - sV))\chi u_0 + (z - sV)\chi p_\ell(z, s)^{-1}\chi u_1\end{pmatrix} \tag{32}$$

and since it holds

$$\|(z - sV)\chi p_\ell(z, s)^{-1}\chi u_1\|_{L^2} \le (1 + |s|\|V\|_{L^\infty})\langle z\rangle\|\chi p_\ell(z, s)^{-1}\chi\|_{L^2\to L^2}\|u_1\|_{L^2},$$

the $\mathcal{E}_\ell$-norm of (32) can be bounded if we show the following estimates :

$$\|P_\ell^{1/2}\chi p_\ell(z, s)^{-1}\chi(z - sV)u_0\|_{L^2} \le C_\mathrm{a}\langle z\rangle\|\tilde{\chi}p_\ell(z, s)^{-1}\tilde{\chi}\|_{L^2\to L^2}\|P_\ell^{1/2}u_0\|_{L^2}, \tag{33a}$$

$$\|P_\ell^{1/2}\chi p_\ell(z, s)^{-1}\chi u_1\|_{L^2} \le C_\mathrm{b}\langle z\rangle\|\tilde{\chi}p_\ell(z, s)^{-1}\tilde{\chi}\|_{L^2\to L^2}\|u_1\|_{L^2}, \tag{33b}$$

$$\|\chi(1 + (z - sV)p_\ell(z, s)^{-1}(z - sV))\chi u_0\|_{L^2} \le C_\mathrm{c}\langle z\rangle\|\tilde{\chi}p_\ell(z, s)^{-1}\tilde{\chi}\|_{L^2\to L^2}\|P_\ell^{1/2}u_0\|_{L^2}. \tag{33c}$$

We use complex interpolation.

Estimate (33a). Let us define

$$\Lambda_\mathrm{a}(\theta) := \langle z\rangle^{-2\theta}P_\ell^\theta\chi p_\ell(z, s)^{-1}\chi P_\ell^{-\theta}.$$

By functional calculus, Λ_a is analytic from $[0, 1] + \mathrm{i}\mathbb{R}$ to $\mathcal{L}(L^2, L^2)$ because $P_\ell > 0$ and $\langle z\rangle > 0$. We want to show that

$$\|\Lambda_\mathrm{a}(1/2)u\|_{L^2} \le C_\mathrm{a}\|\tilde{\chi}p_\ell(z, s)^{-1}\tilde{\chi}\|_{L^2\to L^2}\|u\|_{L^2} \qquad \forall u \in L^2$$

for some $C_{\mathrm{a}} > 0$. By the maximum principle, it is sufficient to bound $\Lambda_{\mathrm{a}}(\theta)$ for $\theta \in \{0,1\} + \mathrm{i}\mathbb{R}$, and since P_ℓ is self-adjoint, it is sufficient by functional calculus to restrict ourselves to $\Im\theta = 0$. If $\theta = 0$, there is nothing to do. Now for $\theta = 1$, we put $u = (z - sV)u_0$ and try to show that

$$\|P_\ell \chi p_\ell(z,s)^{-1}\chi u\|_{L^2\to L^2} \leq C_{\mathrm{a}}\langle z\rangle \|\tilde{\chi} p_\ell(z,s)^{-1}\tilde{\chi}\|_{L^2\to L^2}\|P_\ell u\|_{L^2}. \tag{34}$$

Write

$$P_\ell\chi p_\ell(z,s)^{-1}\chi = \underbrace{[P_\ell,\chi]p_\ell(z,s)^{-1}\chi}_{=:A} + \underbrace{\chi P_\ell p_\ell(z,s)^{-1}\chi}_{=:B}. \tag{35}$$

We first deal with A. Pick $z_0 \in \rho(\hat{K}_\ell) \cap \mathbb{C}^+$ so that $p_\ell(z_0,s)^{-1}$ exists (*cf.* (13)). Then

$$[P_\ell,\chi]p_\ell(z,s)^{-1} = p_\ell(z_0,s)^{-1}\big[p_\ell(z_0,s),[P_\ell,\chi]\big]p_\ell(z,s)^{-1} + p_\ell(z_0,s)^{-1}[P_\ell,\chi]p_\ell(z_0,s)p_\ell(z,s)^{-1}$$

with

$$\begin{aligned}[P_\ell,\chi] &= -\chi\partial_x - \chi',\\ \big[p_\ell(z_0,s),[P_\ell,\chi]\big] &= 2\chi'\partial_x^2 + (\chi'+\chi'')\partial_x + 2z_0 sV'\chi - 2s^2VV'\chi.\end{aligned}$$

By pseudodifferential calculus, we get :

$$p_\ell(z_0,s)^{-1} \in \Psi^{-2,0}, \qquad [P_\ell,\chi] \in \Psi^{1,-\infty}, \qquad \big[p_\ell(z_0,s),[P_\ell,\chi]\big] \in \Psi^{2,-\infty}.$$

On the other hand, we have

$$\begin{aligned}p_\ell(z_0,s)p_\ell(z,s)^{-1} &= \big(p_\ell(z,s) + (z^2 - z_0^2) - 2(z-z_0)sV)\big)p_\ell(z,s)^{-1}\\ &= 1 + p_\ell(z,s)^{-1}(z^2 - z_0^2) - 2(z-z_0)sVp_\ell(z,s)^{-1}\\ &= p_\ell(z,s)^{-1}(P_\ell - (z_0^2 - 2zsV + s^2V^2)) - 2(z-z_0)sVp_\ell(z,s)^{-1}.\end{aligned} \tag{36}$$

Using the identity

$$\chi p_\ell(z,s)^{-1}P_\ell\chi = \chi p_\ell(z,s)^{-1}[P_\ell,\chi] + \chi p_\ell(z,s)^{-1}\chi P_\ell$$

and the uniform bound in ℓ

$$\|\chi' u'\|_{L^2} \lesssim \|\chi_1 v\|_{L^2} + \|\chi_2 u''\|_{L^2} \qquad \chi_j \in \mathcal{C}_c^\infty(\mathbb{R},\mathbb{R}),\ \operatorname{Supp}\chi_j = \operatorname{Supp}\chi, \tag{37}$$

we obtain from (35)

$$\|Au\|_{L^2} \leq \tilde{C}_{\mathrm{a}}\langle z\rangle\|\tilde{\chi}p_\ell(z,s)^{-1}\tilde{\chi}\|_{L^2\to L^2}\|u\|_{L^2} \tag{38}$$

where the constant $\tilde{C}_{\mathrm{a}}$ only depends on $z_0, s, V, V', \chi, \chi', \chi'', \chi_1$ and χ_2.

We now turn to B. Using again (36), we see that

$$\begin{aligned}\|\chi P_\ell p_\ell(z,s)^{-1}\chi u\|_{L^2} &\leq \|\chi p_\ell(z_0,s)p_\ell(z,s)^{-1}\chi u\|_{L^2}\\ &\quad + \|\chi(z_0^2 - 2z_0sV + s^2V^2)p_\ell(z,s)^{-1}\chi u\|_{L^2}\\ &\leq \|\chi p_\ell(z,s)^{-1}(P_\ell - (z_0^2 - 2zsV + s^2V^2))\chi u\|_{L^2} + 2\big(|z|+|z_0|\big)|s|\|V\|_{L^\infty}\|\chi p_\ell(z,s)^{-1}\chi u\|_{L^2}\\ &\quad + \big(|z_0|^2 + 2|z_0||s|\|V\|_{L^\infty} + s^2\|V\|_{L^\infty}^2\big)\|\chi p_\ell(z,s)^{-1}\chi u\|_{L^2}\\ &\leq \|\chi p_\ell(z,s)^{-1}P_\ell\chi v\|_{L^2}\\ &\quad + \underbrace{2\big\langle |z_0| + 2|s|\|V\|_{L^\infty}\big\rangle^2}_{=:\tilde{\tilde{C}}_{\mathrm{a}}}\langle z\rangle\|\chi p_\ell(z,s)^{-1}\chi\|_{L^2\to L^2}\|u\|_{L^2}.\end{aligned}$$

Commuting P_ℓ with χ and using (37), we get (34) with $C_{\mathrm{a}} = \max\big\{\tilde{C}_{\mathrm{a}}, 1 + \tilde{\tilde{C}}_{\mathrm{a}}\big\}$.

Estimate (33b). Let us define

$$\Lambda_{\mathrm{b}}(\theta) = \langle z\rangle^{-2\theta} P_\ell^\theta \chi p_\ell(z,s)^{-1}\chi \qquad\qquad \theta \in [0,1] + \mathrm{i}\mathbb{R}.$$

Λ_{b} is analytic from $[0,1]+\mathrm{i}\mathbb{R}$ to $\mathcal{L}(L^2, L^2)$. As the above estimate, it is sufficient to show a bound on $\Lambda_{\mathrm{b}}(1)$, the imaginary part of θ playing no role and the case $\theta = 0$ being trivial. We get (33b) if we show that

$$\|P_\ell \chi p_\ell(z,s)^{-1}\chi u\|_{L^2} \le C_{\mathrm{b}} \langle z\rangle^2 \|\tilde{\chi} p_\ell(z,s)^{-1}\tilde{\chi}\|_{L^2} \|u\|_{L^2} \forall u \in L^2 \tag{39}$$

for some $C_{\mathrm{b}} > 0$. Using the identity (35) and the estimate (38), we obtain

$$\|P_\ell \chi p_\ell(z,s)^{-1}\chi u\|_{L^2\to L^2} \le \tilde{C}_{\mathrm{a}} \langle z\rangle \|\tilde{\chi} p_\ell(z,s)^{-1}\tilde{\chi}\|_{L^2\to L^2} \|u\|_{L^2} + \|\chi p_\ell(z,s)^{-1} P_\ell \chi u\|_{L^2}$$

but this time we ask for L^2 norm of u. Hence, we use that

$$p_\ell(z,s)^{-1} P_\ell = 1 + p_\ell(z,s)^{-1}(z - sV)^2$$

which yields (39) with $C_{\mathrm{b}} = \max\left\{\tilde{C}_{\mathrm{a}}, 2\langle s\|V\|_{L^\infty}\rangle^2\right\}$.

Estimate (33c). Let us define

$$\Lambda_{\mathrm{c}}(\theta) := \langle z\rangle^{2(\theta-1)}\chi(1 + (z-(sV)^{2(1-\theta)})p_\ell(z,s)^{-1}(z - 2^{2\theta-1}sV))\chi P_\ell^{-\theta}.$$

Once again, Λ_{c} is analytic from $[0,1]+\mathrm{i}\mathbb{R}$ to $\mathcal{L}(L^2, L^2)$ and (dropping the imaginary part)

$$\|\Lambda_{\mathrm{c}}(0)\|_{L^2\to L^2} \le (2 + |s|\|V\|_{L^\infty})^3 \|\chi p_\ell(z,s)^{-1}\chi\|_{L^2\to L^2}.$$

We then get a bound on $\Lambda_c(1)$: we prove

$$\|\chi(1 + z p_\ell(z,s)^{-1}(z - 2sV))\chi u\|_{L^2} \le C_{\mathrm{c}} \|\tilde{\chi} p_\ell(z,s)^{-1}\tilde{\chi}\|_{L^2\to L^2} \|P_\ell u\|_{L^2} \qquad \forall u \in L^2.$$

We have

$$\|\chi(1 + p_\ell(z,s)^{-1} z(z - 2sV))\chi u\|_{L^2} \le \|\chi(1 + p_\ell(z,s)^{-1}(z - sV)^2\chi)u\|_{L^2} + \|\chi p_\ell(z,s)^{-1} s^2 V^2 \chi u\|_{L^2}$$

and

$$\chi(1 + p_\ell(z,s)^{-1}(z - sV)^2)\chi = \chi p_\ell(z,s)^{-1} P_\ell \chi.$$

Commuting P_ℓ with χ and using (37) gives us

$$\|\chi p_\ell(z,s)^{-1} z(z - 2sV)\chi v\|_{L^2} \le C_{\mathrm{c}} \|\tilde{\chi} p_\ell(z,s)^{-1}\tilde{\chi}\|_{L^2\to L^2} \|\langle P_\ell\rangle v\|_{L^2}$$

with $C_{\mathrm{c}} = \max\left\{(1 + |s|\|V\|_{L^\infty})^3, \|\chi\|_{L^\infty}, \|\chi_1\|_{L^\infty} + \|\chi_2\|_{L^\infty}\right\}$. □

The proof is now straightforward. For $\nu > 0$ fixed and for $\ell \in \mathbb{N}$, we define $L^2_\nu(\mathbb{R}, \dot{\mathcal{E}}_\ell)$ as the class of functions $t \mapsto v(t)$ with values in $\dot{\mathcal{E}}_\ell$ such that $t \mapsto \mathrm{e}^{-\nu t} v(t) \in L^2(\mathbb{R}, \dot{\mathcal{E}}_\ell)$. For $u \in \dot{\mathcal{E}}_\ell$, the componentwise defined function

$$v(t) = \begin{cases} \mathrm{e}^{-\mathrm{i}t\hat{K}_\ell} u & \text{for } t \ge 0 \\ 0 & \text{for } t < 0 \end{cases}.$$

is in $L^2_\nu(\mathbb{R}, \dot{\mathcal{E}}_\ell)$ if ν is sufficiently large. For all $t \ge 0$, this means that

$$\mathrm{e}^{-\mathrm{i}t\hat{K}_\ell} u = \frac{1}{2\pi \mathrm{i}} \int_{-\infty+\mathrm{i}\nu}^{+\infty+\mathrm{i}\nu} \mathrm{e}^{-\mathrm{i}zt}(\hat{K}_\ell - z)^{-1} u \,\mathrm{d}z.$$

in the $L^2_\nu(\mathbb{R}, \dot{\mathcal{E}}_\ell)$ sense. We then use the following result :

Lemma 6.2 (Lemma 3.2 in [5]). *Let $N \in \mathbb{N}$, $\chi \in \mathcal{C}_c^\infty(\mathbb{R}, \mathbb{R})$ and define for all $j \in \mathbb{N}$ the spaces $\dot{\mathcal{E}}_\ell^{-j} := (\hat{K}_\ell - \mathrm{i})^j \dot{\mathcal{E}}_\ell$. Then for all $k \in \{0, \ldots, N\}$, there exist bounded operators $B_j \in \mathcal{L}(\dot{\mathcal{E}}_\ell^{-k}, \dot{\mathcal{E}}_\ell^{-k-j})$ and $B \in \mathcal{L}(\dot{\mathcal{E}}_\ell^{-k}, \dot{\mathcal{E}}_\ell^{-k-N-1})$ such that*

$$\hat{R}_{\chi,\ell}(z) = \sum_{j=0}^{N} \frac{B_j}{(z - \mathrm{i}(\nu+1))^{j+1}} + \frac{B \hat{R}_{\tilde{\chi},\ell}(z)\chi}{(z - \mathrm{i}(\nu+1))^{N+1}}$$

for some $\tilde{\chi} \in \mathcal{C}_c^\infty(\mathbb{R}, \mathbb{R})$ satisfying $\tilde{\chi}\chi = \chi$.

Now define

$$\tilde{R}_{\chi,\ell}(z) := \hat{R}_{\chi,\ell}(z) - \sum_{j=0}^{1} \frac{B_j}{(z - \mathrm{i}(\nu+1))^{j+1}}$$

with $B_j \in \mathcal{L}(\dot{\mathcal{E}}_\ell, \dot{\mathcal{E}}_\ell^{-j})$ as in Lemma 6.2. We can show that

$$\chi \mathrm{e}^{-\mathrm{i}t\hat{K}_\ell} \chi u = \frac{1}{2\pi \mathrm{i}} \int_{-\infty+\mathrm{i}\nu}^{+\infty+\mathrm{i}\nu} \mathrm{e}^{-\mathrm{i}zt} \tilde{R}_{\chi,\ell}\, u \, \mathrm{d}z \tag{40}$$

and the integral absolutely converges in $\mathcal{L}(\dot{\mathcal{E}}_\ell, \dot{\mathcal{E}}_\ell^{-2})$. On the other hand, if we integrate $\mathrm{e}^{-\mathrm{i}zt}\tilde{R}_{\chi,\ell}\, u$

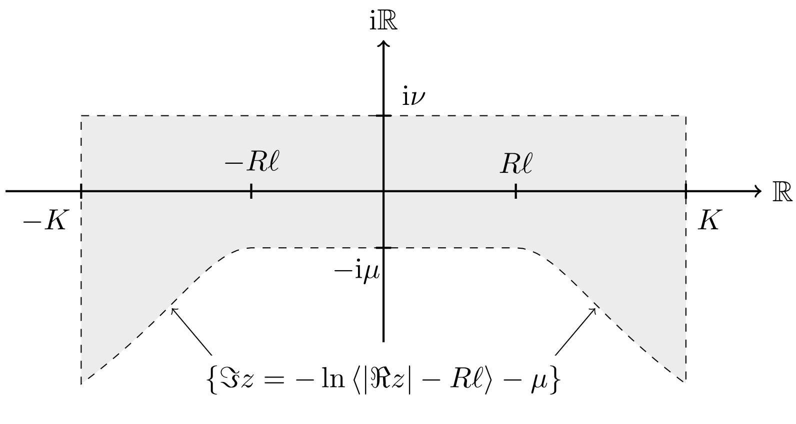

Figure 4: The contour used for the derivation of the resonance expansion.

other the contour described in Figure 4 (defined for $K, \mu > 0$), then letting $K \to +\infty$ and using residue theorem as well as estimates collected in Theorem 5.1 yield the equality between (40) and the resonance expansion of Theorem 4.2 in the space $\dot{\mathcal{E}}_\ell$. We refer to [5] Section 3.2 for more details.

A Analytic extension of the coordinate r

In this appendix, we prove Proposition 2.1 which is the equivalent to Proposition IV.2 in [4]. Let $r \in]r_-, r_+[$. By equation (5), we have

$$\exp\left(-\frac{\Lambda}{3A_\pm r_\pm^2} x\right) = \prod_{\alpha\in I} \left|\frac{r - r_\alpha}{\mathfrak{r} - r_\alpha}\right|^{\frac{A_\alpha r_\alpha^2}{A_\pm r_\pm^2}}.$$

Call the left-hand side z and the right-hand side $g_\pm(r)$. Observe that $g_\pm(r_\pm) = 0$. Since $r \mapsto x(r)$ is increasing and analytic, we can apply the Lagrange's inversion theorem (see for example [8], paragraph 2.2 and reference therein) to write

$$r = r_\pm + \sum_{\ell=1}^{+\infty} \frac{z^\ell}{\ell!} \left[\frac{\mathrm{d}^{\ell-1}}{\mathrm{d}r^{\ell-1}} \left(\frac{r - r_\pm}{g_\pm(r)}\right)^\ell\right]_{r=r_\pm}. \tag{41}$$

Let us introduce Kronecker's symbol

$$\delta_{\alpha,\pm} := \begin{cases} 1 & \text{if } \alpha = \pm \\ 0 & \text{otherwise} \end{cases}$$

and the notation

$$B_{\pm,\alpha} := \frac{A_\alpha r_\alpha^2}{A_\pm r_\pm^2} - \delta_{\alpha,\pm}.$$

Observe that $B_{-,-} = B_{+,+} = 0$. We then have

$$\frac{\mathrm{d}^{\ell-1}}{\mathrm{d}r^{\ell-1}} \left(\frac{r - r_\pm}{g_\pm(r)}\right)^\ell = \left(\prod_{\alpha\in I\setminus\{\pm\}} |\mathfrak{r} - r_\alpha|^{\ell B_{\pm,\alpha}}\right) \frac{\mathrm{d}^{\ell-1}}{\mathrm{d}r^{\ell-1}} \left(\prod_{\alpha\in I\setminus\{\pm\}} |r - r_\alpha|^{-\ell B_{\pm,\alpha}}\right).$$

We now fix $\pm = +$ (the conclusion will not be changed if we choose $\pm = -$). Then

$$\begin{aligned}\frac{\mathrm{d}^{\ell-1}}{\mathrm{d}r^{\ell-1}} \left(\prod_{\alpha\in I\setminus\{+\}} (r - r_\alpha)^{-\ell B_{+,\alpha}}\right) = &\sum_{0\le k_2\le k_1\le \ell} C_{\ell,k_1,k_2} \left(\frac{\mathrm{d}^{\ell-k_1}}{\mathrm{d}r^{\ell-k_1}} (r - r_n)^{-\ell B_{+,n}}\right) \times \\ &\times \left(\frac{\mathrm{d}^{k_1-k_2}}{\mathrm{d}r^{k_1-k_2}} (r - r_c)^{-\ell B_{+,c}}\right) \left(\frac{\mathrm{d}^{k_2}}{\mathrm{d}r^{k_2}} (r - r_-)^{-\ell B_{+,-}}\right)\end{aligned}$$

where

$$C_{\ell,k_1,k_2} = \binom{\ell}{k_1}\binom{k_1}{k_2}.$$

Direct computation shows that

$$\frac{\mathrm{d}^p}{\mathrm{d}r^p} (r - r_\alpha)^{-\ell B_{+,\alpha}} = (-1)^p (\ell B_{+,\alpha})(\ell B_{+,\alpha} + 1)\dots(\ell B_{+,\alpha} + p - 1)(r - r_\alpha)^{-\ell B_{+,\alpha} - p}.$$

If we let

$$K := \prod_{\alpha\in I\setminus\{+\}} (\mathfrak{r} - r_\alpha)^{B_{+,\alpha}}, \qquad B_+ := \max_{\alpha\in I\setminus\{+\}} \{|B_{+,\alpha}|\},$$

then it follows that

$$\begin{aligned}\frac{\mathrm{d}^{\ell-1}}{\mathrm{d}r^{\ell-1}} \left(\frac{r - r_+}{g_+(r)}\right)^\ell = K^\ell &\sum_{0\le k_2\le k_1\le \ell} C_{\ell,k_1,k_2} (-1)^\ell \times \\ &\times (\ell B_{+,n})(\ell B_{+,n} + 1)\dots(\ell B_{+,n} + (\ell - k_1) - 1)(r - r_n)^{-\ell B_{+,n} - (\ell - k_1)} \times \\ &\times (\ell B_{+,c})(\ell B_{+,c} + 1)\dots(\ell B_{+,c} + (k_1 - k_2) - 1)(r - r_c)^{-\ell B_{+,c} - (k_1 - k_2)} \times \\ &\times (\ell B_{+,-})(\ell B_{+,-} + 1)\dots(\ell B_{+,-} + k_2 - 1)(r - r_\alpha)^{-\ell B_{+,-} - k_2}\end{aligned}$$

and thus

$$\begin{aligned}\left|\frac{\mathrm{d}^{\ell-1}}{\mathrm{d}r^{\ell-1}}\left(\frac{r-r_+}{g_+(r)}\right)^{\ell}\right| &\leq K^{\ell}\ell^{\ell}(B_++1)^{\ell}\left(\prod_{\alpha\in I\setminus\{+\}}(r_+-r_\alpha)^{-B_{+,\alpha}}\right)^{\ell}\times\\ &\times\sum_{0\leq k_2\leq k_1\leq\ell}C_{\ell,k_1,k_2}(r_+-r_n)^{-(\ell-k_1)}(r_+-r_c)^{-(k_1-k_2)}(r_+-r_-)^{-k_2}\\ &=K^{\ell}\ell^{\ell}(B_++1)^{\ell}\left(\prod_{\alpha\in I\setminus\{+\}}(r_+-r_\alpha)^{-B_{+,\alpha}}\right)^{\ell}\left(\sum_{\alpha\in I\setminus\{+\}}(r_+-r_\alpha)^{-1}\right)^{\ell}\\ &=\left(K(B_++1)\prod_{\alpha\in I\setminus\{+\}}(r_+-r_\alpha)^{-B_{+,\alpha}}\sum_{\alpha\in I\setminus\{+\}}(r_+-r_\alpha)^{-1}\right)^{\ell}\ell^{\ell}\\ &=:\tilde{K}^{\ell}\ell^{\ell}.\end{aligned}$$

Therefore, the convergence of the original series is absolute for $z\in\mathbb{C}$ if

$$\frac{(|z|\ell\tilde{K})^{\ell}}{\ell!}<\ell^{-(1+\varepsilon)}$$

for any $\varepsilon>0$. Using Stirling approximation $\ell!\sim\sqrt{2\pi}\ell^{\ell+1/2}$ for large values of ℓ, we see that it is sufficient to have

$$\tilde{K}|z|<\frac{\mathrm{e}^{-(1/2+\varepsilon)\ln\ell/\ell}}{\sqrt{2\pi}^{\ell}}<1.$$

and this condition is fulfilled if

$$\Re x>\frac{3A_+r_+^2}{\Lambda}\ln\tilde{K}.$$

B Localization of high frequency resonances

We provide in this section an asymptotic approximation of resonances near the maximal energy $W_0(0)=\max_{x\in\mathbb{R}}\{W_0(x)\}$ as $h\to0$. This a generalization of the main Theorem in [24] to the case $Q\neq0$. More precisely, we show that the resonances associated to the meromorphic extension of $p(z,s)^{-1}$ are close to the ones associated with the extension of $(P-z^2)^{-1}$, provided that Q is sufficiently small. This is a direct consequence of the fact that the extra term hsV in the semiclassical quadratic pencil is $\mathcal{O}(hs)$.

As in the paragraph 5.2, we set $h:=(\ell(\ell+1))^{-1/2}$ with $\ell>0$ and consider $z\in[\ell/R,R\ell]+\mathrm{i}\,[-C_0,C_0]$. We then define the spectral parameter $\lambda:=h^2z^2$ and also $\tilde{P}_h$ the semicalssical operator associated to P_ℓ. Recall also that $\mathfrak{r}=\frac{3M}{2}\left(1+\sqrt{1-\frac{8Q^2}{9M^2}}\right)$ is the radius of the photon sphere and $W_0(0)=F(\mathfrak{r})/\mathfrak{r}^2$ with our definition of the Regge-Wheeler coordinate x (see (5)).

Theorem B.1. *Let*

$$\Gamma_0(h):=\left\{W_0(0)+h\left(2\sqrt{W_0(0)}sV(0)+\mathrm{i}^{-1}\sqrt{W_0''(0)/2}\left(k+\frac{1}{2}\right)\right)\mid k\in\mathbb{N}\right\}.$$

For all $C_0>0$ such that $\partial D(W_0(0),C_0h)\cap\Gamma_0(h)=\emptyset$, there is a bijection $b\equiv b(h)$ from $\Gamma_0(h)$ onto the set of resonances of $\tilde{P}_h$ in $D(W_0(0),C_0h)$ (counted with their natural multiplicity) such that

$$b(h)(\mu)-\mu=o_{h\to0}(h)\qquad\qquad\text{uniformly for } \mu\in\Gamma_0(h).$$

Proof. This is a direct application of the results of Sá Barreto-Zworski [24] which are based on the work of Sjöstrand [26] (see Theorem 0.1), the latter dealing with resonances generated by non-degenerate critical points when the trapping set is reduced to a single point (the difference for us is $W_0(0) \neq 0$).

We recall that in the zone III the symbol of the semiclassical quadratic pencil is the function $(x,\xi) \mapsto \xi^2 + W_0(x) + h^2 W_1(x) - (\sqrt{\lambda} - hsV(x))^2 =: p(x,\xi) - \lambda$. We also recall the hypothesis in [26] for the case of a Schrödinger operator of the form (0.1) in the reference:

- The trapping set is reduced to the point $\{(0,0)\}$ ((0.3) in [26]),
- 0 is a non-degenerate critical point ((0.4) in [26], which implies in the Schrödinger case the more general assumptions (0.7) and (0.9) in the reference).

Although the symbol p depends on λ, its principal part p_0 and subprincipal part p_{-1} do not : indeed, for $\lambda \in D\left(W_0(0), C_0 h\right)$ with $C_0 > 0$, we can write when $h \ll 1$

$$p(x,\xi) = \underbrace{\xi^2 + W_0(x)}_{p_0(x,\xi)} + h\underbrace{2\sqrt{W_0(0)}sV(x)}_{p_{-1}(x,\xi)} + \text{lower order terms in } h.$$

This is enough to apply [26], Theorem 0.1 : using formula (0.14) in the reference, we get the result for the set

$$\left\{p_0(0,0) + h\left(p_{-1}(0,0) + \mathrm{i}^{-1}\sqrt{W_0''(0)/2}\left(k + \frac{1}{2}\right)\right) \mid k \in \mathbb{N}\right\}$$

which is $\Gamma_0(h)$. $\square$

Approximation of high frequency resonances $\Gamma_0(h) \ni z^2 = \lambda/h^2$ is obtained as in [24], by taking the square root of any element of $\Gamma_0(h)$ and using Taylor expansion for $0 < h \ll 1$ (corresponding to $\ell \gg 0$) as well as symmetry with respect to the imaginary axis (for the choice of the sign of the square root). In our setting, we obtain the set Γ of Theorem 4.1.

Remark B.2. 1. *Let Γ_{DSS} be the set of pseudo-poles in the De Sitter-Schwarzschild case (see the Theorem at the end of [24]). Then Γ_{DSS} is the limit of Γ as $Q \to 0$ in the sense of the sets, i.e. for all $z \in \Gamma$, there exists $z_0 \in \Gamma_{\mathrm{DSS}}$ such that $z \to z_0$ as $Q \to 0$. In particular, the set of pseudo-poles does not depend on the only Klein-Gordon field's charge q.*

2. *The pseudo-poles in the charged case are shifted with respect to the uncharged case. If the Klein-Gordon field charge and the black hole one have the same sign (that is if $qQ > 0$), then all the pseudo-poles go to infinity with a real part which never vanishes. However, if the charges have opposite sign ($qQ < 0$), then all the pseudo-poles real part cancels precisely when $qQ = -(k+1/2)\sqrt{F(\mathfrak{r})}$, $k \in \mathbb{N}\setminus\{0\}$, before going to infinity. Notice that no pseudo-pole goes to $\mathbb{C}^+$ as $|s| \to +\infty$.*

3. *We can provide a physical interpretation of the set of pseudo-poles. First observe that $\sqrt{F(\mathfrak{r})}/\mathfrak{r}$ is nothing but the inverse of the impact parameter $b = |E/L|$ of trapped null geodesics (see the end of the paragraph 2.2). Theorem 4.1 shows that resonances near the real line in the zone III are qQ-dependent multiples of this quantity : they thus correspond to impact parameters of trapped photons with high energy and angular momentum. This is in adequation with the first remark above.*

4. *Observe that in Newtonian mechanics, the electric and gravitation effects exerted on chargeless and massless photons by the black hole's charge and mass are null. As a consequence, photons are not deviated and only ones with impact parameter $|b| \leq r_-$ can "fall" in the black hole. Hence, high frequency resonances in zone III are expected to be multiple of r_-^{-1}. As $r_- \to 0$, all resonances go to infinity : the trajectory are now classical straight lines as there is no obstacle anymore.*

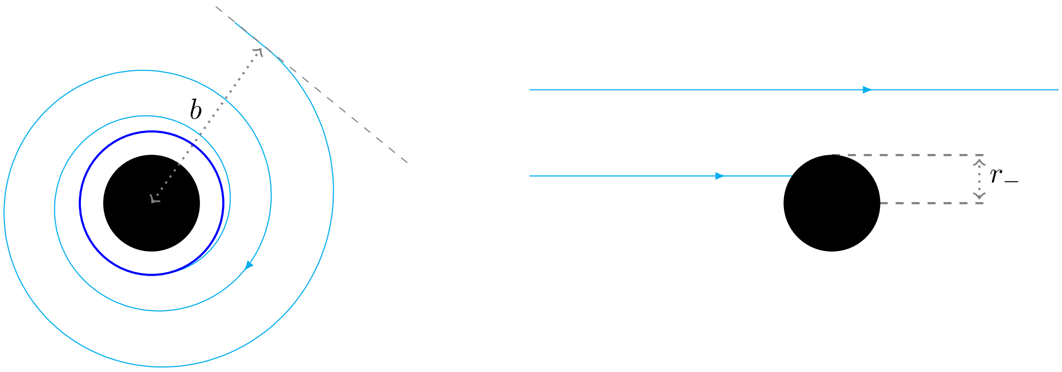

Figure 5: On the left : a relativistic trapped null geodesic. On the right : classical null geodesic trajectories.

C Abstract Semiclassical Limiting Absorption Principle for a class of Generalized Resolvents

We show in this section an abstract semiclassical limiting absorption principle for perturbed resolvents.

Abstract setting. Let $\big(\mathcal{H},\langle\cdot,\cdot\rangle\big)$ be a Hilbert space, $J:=[a,b]\subset\mathbb{R}$, $J_\mu^+:=\{\omega\in\mathbb{C}^+\mid\Re\omega\in J,\,\Im\omega<\mu\}$ for some $\mu>0$ fixed and $h_0>0$. The norm associated to $\langle\cdot,\cdot\rangle$ will be denoted by $\|\cdot\|$. We consider families of self-adjoint operators $P\equiv P(h)$ and $\mathcal{A}\equiv\mathcal{A}(h)$ acting on $\mathcal{H}$ for $0<h<h_0$. We set

$$L^\infty_{\ell\mathrm{oc}}(P):=\big\{A:\mathcal{H}\to\mathcal{H}\text{ linear}\mid\forall\chi\in\mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R},\mathbb{R}),\,\forall u\in\mathscr{D}(P),\,\|\chi(P)Au\|<+\infty\big\}$$

and $\|.\|_P$ will be the operator norm on $\mathcal{B}(\mathscr{D}(P),\mathcal{H})$. We also define the local version of the operator P :

$$P_\tau:=\tau(P)P\qquad\qquad\tau\in\mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R},\mathbb{R}).$$

Let then and $f:\mathbb{C}\times L^\infty_{\ell\mathrm{oc}}(P)\to L^\infty_{\ell\mathrm{oc}}(P)$ satisfying the following continuity type relation near $0_{L^\infty_{\ell\mathrm{oc}}(P)}$: there exist $\delta_{J,\mu}:\mathbb{R}_+\to\mathbb{R}$ satisfying $\delta_{J,\mu}(r)\to0$ as $r\to0$ and $\varepsilon_{J,\mu}:L^\infty_{\ell\mathrm{oc}}(P)\to L^\infty_{\ell\mathrm{oc}}(P)$ such that, for all $(z,A)\in J_\mu^+\times L^\infty_{\ell\mathrm{oc}}(P)$ with $\|A\|_P$ small,

$$f(z,A)=z+\delta_{J,\mu}(\|A\|_P)\,\varepsilon_{J,\mu}(A).\tag{C}$$

We make the following assumptions :

$$(P-f(z,hA))^{-1}\text{ exists for all }z\in J_\mu^+\text{ and }A\in L^\infty_{\ell\mathrm{oc}}(P)\text{ if }h\le h_0\tag{I}$$
$$P\in\mathcal{C}^2(\mathcal{A})\tag{P}$$
$$\mathbb{1}_I(P)[P,i\mathcal{A}]\mathbb{1}_I(P)\ge ch\mathbb{1}_I(P)\qquad\text{for some }c>0\text{ and }J\Subset I:=\,]\alpha,\beta[\,\subset\mathbb{R}\tag{M}$$
$$\mathrm{ad}^k_{\chi(P)}(\varepsilon_{J,\mu}(A))\in h^k\mathcal{B}(\mathscr{D}(\mathcal{A}))\qquad\text{for all }k\in\{0,1\},\,\chi\in\mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R},\mathbb{R}),\text{ uniformly in }A\text{ near }0_{L^\infty_{\ell\mathrm{oc}}(P)}.\tag{A}$$

Recall that $P\in\mathcal{C}^2(\mathcal{A})$ means for all $z\in\mathbb{C}\setminus\sigma(P)$ that the map

$$\mathbb{R}\ni t\mapsto\mathrm{e}^{\mathrm{i}t\mathcal{A}}(P-z)^{-1}\mathrm{e}^{-\mathrm{i}t\mathcal{A}}$$

is $\mathcal{C}^2$ for the strong topology of L^2. Recall also that for all linear operators L_1,L_2 acting on $\mathcal{H}$, $\mathrm{ad}^0_{L_1}(L_2):=L_2$ and $\mathrm{ad}^{k+1}_{L_1}(L_2):=[L_1,\mathrm{ad}^k_{L_1}(L_2)]$. Our goal is to show the following result :

Theorem C.1. *Assume hypotheses* (C), (I), (P), (M) *and* (A). *Then for all* $\sigma\in\,]1/2,1]$,

$$\sup_{z\in J_\mu^+}\|\langle\mathcal{A}\rangle^{-\sigma}(P-f(z,hB))^{-1}\langle\mathcal{A}\rangle^{-\sigma}\|\lesssim h^{-1}.\tag{42}$$

In the sequel, we will note $R(z,hB):=(P-f(z,hB))^{-1}$ and call it the generalized resolvent (of P). Also, since J and μ are now fixed, we will simply note J, δ and ε instead of J_μ, $\delta_{J,\mu}$ and $\varepsilon_{J,\mu}$.

Preliminary results. The purpose of this paragraph is to show preliminary results used to prove Theorem C.1. We first prove an adapted version of Lemma 2.1 in [11] to our situation.

Lemma C.2. *Let $0 \leq \sigma \leq 1$, $z \in J^+$ and let $\chi \in \mathcal{C}_c^\infty(\mathbb{R}, \mathbb{R})$. If h is small enough, then $R(z, hB)$ and $\chi(P)$ are bounded on $\mathscr{D}(\langle \mathcal{A} \rangle^\sigma)$.*

Proof. The result is true for $(P-z)^{-1}$ and $\chi(P)$ by Lemma 2.1 in [11]. Let us show that $R(z, hB)\mathscr{D}(\langle \mathcal{A} \rangle^\sigma) \subset \mathscr{D}(\langle \mathcal{A} \rangle^\sigma)$:

$$\begin{aligned}\|\langle \mathcal{A} \rangle^\sigma R(z, hB) \langle \mathcal{A} \rangle^{-\sigma}\| &\leq \|\langle \mathcal{A} \rangle^\sigma (P-z)^{-1} \langle \mathcal{A} \rangle^{-\sigma}\| + \|\langle \mathcal{A} \rangle^\sigma (R(z, hB) - (P-z)^{-1}) \langle \mathcal{A} \rangle^{-\sigma}\| \\ &\lesssim 1 + \|\langle \mathcal{A} \rangle^\sigma R(z, hB)(z - f(z, hB))(P-z)^{-1} \langle \mathcal{A} \rangle^{-\sigma}\|\end{aligned}$$

and (using that $\varepsilon(hB) \in \mathcal{B}(\mathscr{D}(\mathcal{A}))$ by Assumption (A) for $k = 0$)

$$\begin{aligned}&\|\langle \mathcal{A} \rangle^\sigma R(z, hB)(z - f(z, hB))(P-z)^{-1} \langle \mathcal{A} \rangle^{-\sigma}\| \\ &\leq \|\langle \mathcal{A} \rangle^\sigma R(z, hB) \langle \mathcal{A} \rangle^{-\sigma}\| \|\langle \mathcal{A} \rangle^\sigma (z - f(z, hB)) \langle \mathcal{A} \rangle^{-\sigma}\| \|\langle \mathcal{A} \rangle^\sigma (P-z)^{-1} \langle \mathcal{A} \rangle^{-\sigma}\| \\ &\lesssim \delta(h\|B\|_P) \|\langle \mathcal{A} \rangle^\sigma \varepsilon(hB) \langle \mathcal{A} \rangle^{-\sigma}\| \|\langle \mathcal{A} \rangle^\sigma R(z, hB) \langle \mathcal{A} \rangle^{-\sigma}\|.\end{aligned}$$

We then use the uniformity in assumption (A) for $k = 0$ to write for h very small

$$\delta(h\|B\|_P) \|\langle \mathcal{A} \rangle^\sigma \varepsilon(hB) \langle \mathcal{A} \rangle^{-\sigma}\| \|\langle \mathcal{A} \rangle^\sigma R(z, hB) \langle \mathcal{A} \rangle^{-\sigma}\| < \frac{1}{2} \|\langle \mathcal{A} \rangle^\sigma R(z, hB) \langle \mathcal{A} \rangle^{-\sigma}\|.$$

The proof is complete. □

Corollary C.3. *Let $0 \leq \sigma \leq 1$, $z \in J^+$ and $\tau, \chi \in \mathcal{C}_c^\infty(\mathbb{R}, [0,1])$ such that $\chi \equiv 1$ on I and $\tau\chi = \chi$. If h is small enough, then $(P_\tau - f(z, hB))\chi(P)$, $(P_\tau - f(z, hB))\chi(P)(P+\mathrm{i})^{-1}$ and $(P - f(z, hB))(P+\mathrm{i})^{-1}$ preserve $\mathscr{D}(\langle \mathcal{A} \rangle^\sigma)$.*

Proof. We have

$$\begin{aligned}\langle \mathcal{A} \rangle^\sigma (P_\tau - f(z, hB))\chi(P) \langle \mathcal{A} \rangle^{-\sigma} &= \langle \mathcal{A} \rangle^\sigma (P_\tau - z)\chi(P) \langle \mathcal{A} \rangle^{-\sigma} \\ &\quad + \langle \mathcal{A} \rangle^\sigma (z - f(z, hB)) \langle \mathcal{A} \rangle^{-\sigma} \langle \mathcal{A} \rangle^\sigma \chi(P) \langle \mathcal{A} \rangle^{-\sigma}\end{aligned}$$

which is bounded by assumption (A) for $k = 0$, Lemma C.2 and the fact that $P_\tau \chi(P) = \varphi(P)$ with $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}, \mathbb{R})$ by functional calculus. Next, Lemma 2.1 in [11] implies that $(P+\mathrm{i})^{-1}$ preserves $\mathscr{D}(\mathcal{A})$, so we can write

$$\langle \mathcal{A} \rangle^\sigma (P_\tau - f(z, hB))\chi(P)(P+\mathrm{i})^{-1} \langle \mathcal{A} \rangle^{-\sigma} = \langle \mathcal{A} \rangle^\sigma (P_\tau - f(z, hB))\chi(P) \langle \mathcal{A} \rangle^{-\sigma} \langle \mathcal{A} \rangle^\sigma (P+\mathrm{i})^{-1} \langle \mathcal{A} \rangle^{-\sigma}$$

which is clearly bounded thanks to the above computation. Finally,

$$\begin{aligned}\langle \mathcal{A} \rangle^\sigma (P - f(z, hB))(P+\mathrm{i})^{-1} \langle \mathcal{A} \rangle^{-\sigma} &= \langle \mathcal{A} \rangle^\sigma (P + \mathrm{i} - \mathrm{i} - z + z - f(z, hB))(P+\mathrm{i})^{-1} \langle \mathcal{A} \rangle^{-\sigma} \\ &= \mathrm{Id} - (\mathrm{i} + z) \langle \mathcal{A} \rangle^\sigma (P+\mathrm{i})^{-1} \langle \mathcal{A} \rangle^{-\sigma} \\ &\quad + \langle \mathcal{A} \rangle^\sigma (z - f(z, hB)) \langle \mathcal{A} \rangle^{-\sigma} \langle \mathcal{A} \rangle^\sigma (P+\mathrm{i})^{-1} \langle \mathcal{A} \rangle^{-\sigma}\end{aligned}$$

and we again use Lemma 2.1 in [11] and assumption (A) for $k = 0$. □

The next result is an adaptation of Lemma 3.1 in [11] to our setting.

Lemma C.4. *Let $0 < \sigma \leq 1$ and let $\tau, \chi \in \mathcal{C}_c^\infty(\mathbb{R}, [0,1])$ such that $\chi \equiv 1$ on I and $\tau\chi = \chi$. Consider the following three statements :*

(i)
$$\sup_{z \in J^+} \|\langle \mathcal{A} \rangle^{-\sigma} R(z, hB) \langle \mathcal{A} \rangle^{-\sigma}\| \lesssim h^{-1};$$

(ii) For all $z \in J^+$ and all $u \in (P+\mathrm{i})^{-1}\mathscr{D}(\langle \mathcal{A} \rangle^\sigma)$,
$$\|\langle \mathcal{A} \rangle^{-\sigma} u\| \lesssim h^{-1} \|(P - f(z, hB))u\| + h^{-1} \|\langle \mathcal{A} \rangle^\sigma (P - f(z, hB))\chi(P)u\|;$$

(iii) For all $z \in J^+$ and all $u \in \mathscr{D}(\langle \mathcal{A} \rangle^\sigma)$,
$$\|\langle \mathcal{A} \rangle^{-\sigma} \chi(P)u\| \lesssim h^{-1} \|\langle \mathcal{A} \rangle^\sigma (P_\tau - f(z, hB))\chi(P)u\|.$$

If h is sufficiently small, then (iii) implies (ii) and (ii) implies (i).

Proof. First of all, observe that (i) makes sense by Lemma (C.2), and (ii), (iii) make sense by Corollary C.3 and because $P\chi(P) = P_\tau\chi(P)$.

- We show that (ii) implies (i). Let $u \in \mathcal{H}$ and let $v := R(z, hB)\langle\mathcal{A}\rangle^{-\sigma}u$. Then

$$w := u - \langle\mathcal{A}\rangle^{\sigma}(f(z, hB) - \mathrm{i})R(z, hB)\langle\mathcal{A}\rangle^{-\sigma}u \in \mathcal{H}.$$

This makes sense if h is small enough because $R(z, hB)$ preserves $\mathscr{D}(\langle\mathcal{A}\rangle^{\sigma})$ by Lemma C.2 and because

$$\langle\mathcal{A}\rangle^{\sigma}(f(z, hB) - \mathrm{i})\langle\mathcal{A}\rangle^{-\sigma} = \langle\mathcal{A}\rangle^{\sigma}(f(z, hB) - z)\langle\mathcal{A}\rangle^{-\sigma} + (z - \mathrm{i})$$

is bounded by assumption (A) for $k = 0$. Next, using the resolvent identity $(P+\mathrm{i})^{-1} - R(z, hB) = (P + \mathrm{i})^{-1}(f(z, hB) - \mathrm{i})R(z, hB)$, we see that

$$\begin{aligned}(P + \mathrm{i})^{-1}\langle\mathcal{A}\rangle^{-\sigma}w &= \big((P + \mathrm{i})^{-1} - (P + \mathrm{i})^{-1}(f(z, hB) - \mathrm{i})\, R(z, hB)\big)\langle\mathcal{A}\rangle^{-\sigma}u\\ &= R(z, hB)\langle\mathcal{A}\rangle^{-\sigma}u\\ &= v\end{aligned}$$

so that $v \in (P + \mathrm{i})^{-1}\mathscr{D}(\langle\mathcal{A}\rangle^{\sigma})$. Hence, applying (ii) to v yields

$$\begin{aligned}\|\langle\mathcal{A}\rangle^{-\sigma}R(z, hB)\langle\mathcal{A}\rangle^{-\sigma}u\| &= \|\langle\mathcal{A}\rangle^{-\sigma}v\|\\ &\lesssim h^{-1}\|\langle\mathcal{A}\rangle^{-\sigma}u\| + h^{-1}\|\langle\mathcal{A}\rangle^{\sigma}(P - f(z, hB))\chi(P)R(z, hB)\langle\mathcal{A}\rangle^{-\sigma}u\|\\ &\lesssim h^{-1}\|\langle\mathcal{A}\rangle^{-\sigma}u\| + h^{-1}\|\langle\mathcal{A}\rangle^{\sigma}[P - f(z, hB), \chi(P)]R(z, hB)\langle\mathcal{A}\rangle^{-\sigma}u\|\\ &\quad + h^{-1}\|\langle\mathcal{A}\rangle^{\sigma}\chi(P)\langle\mathcal{A}\rangle^{-\sigma}u\|.\end{aligned}$$

By assumption (A) for $k = 1$ and Lemma C.2, we have

$$\begin{aligned}&\|\langle\mathcal{A}\rangle^{\sigma}[P - f(z, hB), \chi(P)]R(z, hB)\langle\mathcal{A}\rangle^{-\sigma}u\|\\ &= \|\langle\mathcal{A}\rangle^{\sigma}[z - f(z, hB), \chi(P)]R(z, hB)\langle\mathcal{A}\rangle^{-\sigma}u\|\\ &\le \delta(h\|B\|_{P})\|\langle\mathcal{A}\rangle^{\sigma}[\varepsilon(hB), \chi(P)]\langle\mathcal{A}\rangle^{-\sigma}\|\|\langle\mathcal{A}\rangle^{\sigma}R(z, hB)\langle\mathcal{A}\rangle^{-\sigma}u\|\\ &\lesssim h\delta(h\|B\|_{P}).\end{aligned}$$

Therefore, (i) follows from (ii) if h is small enough.

- We show that (iii) implies (ii). Let $\tilde{\chi} := 1 - \chi$ and let $u \in (P + \mathrm{i})^{-1}\mathscr{D}(\langle\mathcal{A}\rangle^{\sigma})$. We write

$$\|\langle\mathcal{A}\rangle^{-\sigma}u\| \le \|\langle\mathcal{A}\rangle^{-\sigma}\chi(P)u\| + \|\langle\mathcal{A}\rangle^{-\sigma}\tilde{\chi}(P)u\| \tag{43}$$

and (iii) implies that

$$\|\langle\mathcal{A}\rangle^{-\sigma}\chi(P)u\| \lesssim h^{-1}\|\langle\mathcal{A}\rangle^{\sigma}(P - f(z, hB))\chi(P)u\|$$

because $\tau \equiv 1$ on $\operatorname{Supp}\chi$. In order to control the term involving $\tilde{\chi}(P)$ in (43), we write $\tilde{\chi} = \psi_- + \psi_+$ with $\psi_\pm \in \mathcal{C}^\infty(\mathbb{R}, [0, 1])$ such that $\operatorname{Supp}\psi_- \subset]-\infty, \alpha]$ and $\operatorname{Supp}\psi_+ \subset [\beta, +\infty[$. We also pick $\rho \in \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}, \mathbb{R})$ such that $\rho\psi_- = \psi_-$. Since $B \in L^\infty_{\ell oc}(P)$, we have for any $v \in \mathscr{D}(P)$

$$\begin{aligned}&\Re\big\langle\psi_-(P)^2(f(z, hB) - P)v, v\big\rangle\\ &= \Re\big\langle\psi_-(P)^2 zv, v\big\rangle + \Re\big\langle\psi_-(P)^2\delta(h\|B\|_{P})\varepsilon(hB)v, v\big\rangle - \Re\big\langle\psi_-(P)^2 Pv, v\big\rangle\\ &\ge a\|\psi_-(P)v\|^2 - \delta(h\|B\|_{P})\|\rho(P)\varepsilon(hB)\|_{P}\|\psi_-(P)v\|^2 - \alpha\|\psi_-(P)^2 v\|^2\\ &\ge c_-\|\psi_-(P)v\|^2\end{aligned} \tag{44}$$

where $c_- > 0$ if h is sufficiently small. Using Cauchy-Schwarz inequality, we get $\|\psi_-(P)(P - f(z, hB))v\| \ge c_-\|\psi_-(P)v\|$ and thus $\|\psi_-(P)R(z, hB)v\| \lesssim \|\psi_-(P)v\|$. Similarly, one can show $\|\psi_+(P)R(z, hB)v\| \lesssim \|\psi_+(P)v\|$. These inequalities and $\tilde{\chi}^2 = (\psi_- + \psi_+)^2 = \psi_-^2 + \psi_+^2$ then imply

$$\|\tilde{\chi}(P)R(z, hB)v\| \lesssim \|\tilde{\chi}(P)v\|$$

which in turn implies for $u \in \mathscr{D}(P)$

$$\begin{aligned}\|\langle\mathcal{A}\rangle^{-\sigma}\tilde{\chi}(P)u\| &\lesssim \|\tilde{\chi}(P)u\| \\ &= \|\tilde{\chi}(P)R(z,hB)(P-f(z,hB))u\| \\ &\lesssim \|\tilde{\chi}(P)(P-f(z,hB))u\| \\ &\lesssim \|(P-f(z,hB))u\|.\end{aligned}$$

□

Proof of Theorem C.1. We show that the regularity (P) and the Mourre estimate (M) are enough to establish (42). As pointed out at the beginning of [11], the key point is the following energy estimate: for any self-adjoint operators H acting on $\mathcal{H}$, $u \in \mathscr{D}(H)$, $\tau \in \mathcal{C}_c^\infty(\mathbb{R},[0,1])$ and $P_\tau := \tau(P)P$, we have

$$2\Im\big\langle Hu,(P_\tau - f(z,hB))u\big\rangle = \big\langle u,[P_\tau,\mathrm{i}H]u\big\rangle - 2\Im\big\langle u, f(z,hB)Hu\big\rangle \tag{45}$$

where the commutator must be understood as a quadratic form on $\mathscr{D}(H)$.

We follow the proof of Theorem 1 in [11]. Let $\tau,\chi \in \mathcal{C}_c^\infty(\mathbb{R},[0,1])$ such that $\chi \equiv 1$ on I and $\tau\chi = \chi$ and let

$$F(\xi) := -\int_\xi^{+\infty} g(\zeta)^2 \mathrm{d}\zeta$$

with $g \in \mathcal{C}^\infty(\mathbb{R},[0,1])$ satisfying $g(\xi) = 0$ for $\xi \geq 2$ and $g(\xi) = 1$ for $\xi \leq 1$. By Lemma C.4, it is sufficient to prove the following estimate : for any $z \in J^+$ and $u \in \mathscr{D}(\langle\mathcal{A}\rangle^\sigma)$,

$$\|\langle\mathcal{A}\rangle^{-\sigma}\chi(P)u\| \lesssim h^{-1}\|\langle\mathcal{A}\rangle^{\sigma}(P_\tau - f(z,hB))\chi(P)u\|.$$

As $P \in \mathcal{C}^2(\mathcal{A})$, P and $\mathcal{A}$ are self-adjoint and satisfy the Mourre estimate (M) on I, we can apply the estimate (3.30) in the proof of Theorem 1 in [11] :

$$\chi(P)[P_\tau,\mathrm{i}F(\mathcal{A})]\chi(P) \gtrsim h\chi(P)\langle\mathcal{A}\rangle^{-2\sigma}\chi(P). \tag{46}$$

Now we apply the identity (45) with $H = F(\mathcal{A})$: for all $u \in \mathscr{D}(\mathcal{A})$,

$$2\Im\big\langle F(\mathcal{A})u,(P_\tau - f(z,hB))u\big\rangle = \big\langle u,[P_\tau,\mathrm{i}F(\mathcal{A})]u\big\rangle + 2\Im\big\langle f(z,hB)u, F(\mathcal{A})u\big\rangle.$$

Since $F < 0$ is bounded and $\Im z > 0$, we can write for all h sufficiently small

$$\begin{aligned}&2\Im\big\langle F(\mathcal{A})u,(P_\tau - f(z,hB))u\big\rangle \\ &= \big\langle u,[P_\tau,\mathrm{i}F(\mathcal{A})]u\big\rangle - 2(\Im z)\big\langle u,F(\mathcal{A})u\big\rangle - 2\delta(h\|B\|_P)\Im\big\langle u,\varepsilon(hB)F(\mathcal{A})u\big\rangle \\ &> \big\langle u,[P_\tau,\mathrm{i}F(\mathcal{A})]u\big\rangle - 2\delta(h\|B\|_P)\|\varepsilon(hB)u\|\|F(\mathcal{A})u\|\end{aligned}$$

where we used that that $\varepsilon(hB) \in \mathcal{B}(\mathscr{D}(\mathcal{A}))$ by Assumption (A). It thus follows

$$2\Im\big\langle F(\mathcal{A})u,(P_\tau - f(z,hB))u\big\rangle \geq \big\langle u,[P_\tau,\mathrm{i}F(\mathcal{A})]u\big\rangle. \tag{47}$$

Plugging the estimate (46) into inequality (47) and putting $\chi(P)u$ instead of u yield

$$\begin{aligned}\|\langle\mathcal{A}\rangle^{-\sigma}\chi(P)u\|^2 &= \langle u,\chi(P)\langle\mathcal{A}\rangle^{-2\sigma}\chi(P)u\rangle \\ &\lesssim h^{-1}\big\langle u,\chi(P)[P_\tau,\mathrm{i}F(\mathcal{A})]\chi(P)u\big\rangle \\ &\leq h^{-1}\big|\big\langle F(\mathcal{A})\chi(P)u,(P_\tau - f(z,hB))\chi(P)u\big\rangle\big|.\end{aligned}$$

Using again the boundedness of F, we get

$$\|\langle\mathcal{A}\rangle^{-\sigma}\chi(P)u\|^2 \lesssim h^{-1}\|\langle\mathcal{A}\rangle^{-\sigma}\chi(P)u\|\|\langle\mathcal{A}\rangle^{\sigma}(P_\tau - f(z,hB))\chi(P)u\|$$

which establishes the point (iii) and thus the point (i) in Lemma C.4.

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