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POLYGRAPHS AND DISCRETE CONDUCHÉ FUNCTORS

LÉONARD GUETTA

ABSTRACT. We define a class of ω -functors that generalize discrete Conduché functors between 1-categories and investigate their relation to polygraphs.

INTRODUCTION

In [Gir64, Theorem 4.4], Giraud introduced necessary and sufficient conditions for a functor $f : C \rightarrow D$ to be exponentiable in the category of (small) categories \mathbf{Cat} , i.e. such that the pullback functor

$$f^* : \mathbf{Cat}/D \rightarrow \mathbf{Cat}/C$$

induced by f admits a right adjoint. A functor satisfying these conditions is usually called a *Conduché functor* or *Conduché fibration* (named after Conduché who rediscovered Giraud's theorem in [Con72]). In the present article, we will focus on a variation of this notion.

Definition. A functor $f : C \rightarrow D$ is a *discrete Conduché functor* (or *discrete Conduché fibration*) if for every arrow $\gamma : x \rightarrow y$ in C and every factorization

$$f(\gamma) = f(x) \xrightarrow{\alpha} z \xrightarrow{\beta} f(y),$$

there exists a *unique* factorization

$$\gamma = x \xrightarrow{\bar{\alpha}} \bar{z} \xrightarrow{\bar{\beta}} y$$

such that $f(\bar{\alpha}) = \alpha$ and $f(\bar{\beta}) = \beta$.

Recall that a category C is free on a graph G if

$$C \simeq L(G)$$

where G is a graph and L is the left adjoint to the forgetful functor from \mathbf{Cat} to the category of graphs.

It was remarked in [Str96] that discrete Conduché functors, called *ulf functors* there, have some properties related to free categories on graphs. For example, the following theorem follows immediately from the first section of op. cit.

Theorem. *Let $f : C \rightarrow D$ be a discrete Conduché functor. If D is free on a graph then C is free on a graph.*

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In the setting of strict ω -categories (that we shall simply call ω -categories), the notion of free category on a graph can be generalized to the notion of *free ω -category on a polygraph* in the terminology of [Bur93] (or *free categories on a computad* in the terminology of [Str76] or [Mak05]).

In the present paper we shall introduce a notion of discrete Conduché functor between ω -categories and prove the following generalization of the previous theorem.

Theorem 1. *Let C and D be ω -categories and $f : C \rightarrow D$ be a discrete Conduché ω -functor. If D is free on a polygraph then C is free on a polygraph.*

We will even be more precise and explicitly construct the polygraph generating C from the one generating D . As a by-product we will also prove the following theorem.

Theorem 2. *Let C and D be ω -categories and $f : C \rightarrow D$ be a discrete Conduché ω -functor. If C is free on a polygraph and $f_n : C_n \rightarrow D_n$ is surjective for every $n \in \mathbb{N}$, then D is free on a polygraph.*

The original motivation for the present paper comes from a seemingly unrelated topic. Let $\mathcal{D}(\mathbb{Z})$ be the localization of the category of chain complexes with respect to quasi-isomorphisms and \mathbf{Cat}_ω the category of ω -categories and ω -functors. In [Mét03], the author defines a functor

$$H^{\text{pol}}(-) : \mathbf{Cat}_\omega \rightarrow \mathcal{D}(\mathbb{Z})$$

called the *polygraphic homology functor* by means of so-called *polygraphic resolution*. As it turns out, free ω -categories on polygraphs are the cofibrant objects of a "folk" model structure on \mathbf{Cat}_ω and the polygraphic homology functor can be understood as the left derived functor of a well-known abelianization functor (see [Mét03],[Mét08] and [LMW10]).

In [LM09], the authors prove that when we restrict this functor to the category of monoids (which can be considered as a subcategory of \mathbf{Cat} and hence of \mathbf{Cat}_ω) then it is isomorphic to the "classical" homology functor of monoids (which can be defined as the singular homology of the classifying space of the monoid).

While extending the previous result from monoids to 1-categories [Gue], I encountered the following question :

Let $f : P \rightarrow C$ be a ω -functor with P a polygraph, C a 1-category and let c be an object of C . Consider the ω -category P/c defined as the following fibred product in \mathbf{Cat}_ω :

$$\begin{array}{ccc} P/c & \longrightarrow & P \\ \downarrow & \lrcorner & \downarrow f \\ C/c & \longrightarrow & C \end{array}$$

where the anonymous arrow from the slice category C/c to C is the obvious forgetful functor.

Question: Is P/c a polygraph?

Now, it is straightforward to check that the arrow $C/c \rightarrow C$ is a discrete Conduché functor. Moreover, as we shall see, discrete Conduché functors

are stable by pullback. Hence, the arrow $P/c \rightarrow P$ is a discrete Conduché (ω -)functor. Then Theorem 1 provides a positive answer to the previous question.

The same strategy also yields an alternative proof of Proposition 6 of [LM09]. It suffices to notice that the so-called "unfolding" of a ω -functor

$$f : P \rightarrow M,$$

where M is a monoid (definition 13 of op. cit.), is just the category P/\star , with \star the only object of M when seen as a category.

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1. ω -CATEGORIES

This section is mainly devoted to fixing notations. Some facts are asserted and proofs are left to the reader.

1.1. A ω -graph consists of

- a sequence $(C_n)_{n \in \mathbb{N}}$ of sets,
- maps $s_n^k, t_n^k : C_n \rightarrow C_k$ for every $k < n \in \mathbb{N}$,

subject to the *globular identities*

$$\begin{aligned} s_n^l &= s_k^l \circ s_n^k = s_k^l \circ t_n^k, \\ t_n^l &= t_k^l \circ t_n^k = t_k^l \circ s_n^k, \end{aligned}$$

whenever $l < k < n \in \mathbb{N}$. When the context is clear, we often write s^k (resp. t^k) instead of s_n^k (resp. t_n^k).

Elements of C_n are called *n-cells* or *cells of dimension n*. For a *n-cell* x and $k < n$, $s^k(x)$ is its *k-source* and $t^k(x)$ its *k-target*.

Two cells x and y of dimension n are *parallel* if for every $k < n$,

$$s^k(x) = s^k(y) \text{ and } t^k(x) = t^k(y).$$

(Two 0-cells are always parallel.)

We define the set $C_n \times_{C_k} C_n$ as the following fibred product

$$\begin{array}{ccc} C_n \times_{C_k} C_n & \longrightarrow & C_n \\ \downarrow & \lrcorner & \downarrow t^k \\ C_n & \xrightarrow{s^k} & C_k. \end{array}$$

1.2. Given two ω -graphs C and D , a *morphism of ω -graphs*

$$f : C \rightarrow D$$

is a sequence

$$(f_n : C_n \rightarrow D_n)_{n \in \mathbb{N}}$$

such that, for all $k < n \in \mathbb{N}$, both diagrams

$$\begin{array}{ccc} C_n & \xrightarrow{f_n} & D_n \\ \downarrow s^k & & \downarrow s^k \\ C_k & \xrightarrow{f_k} & D_k \end{array} \quad \begin{array}{ccc} C_n & \xrightarrow{f_n} & D_n \\ \downarrow t^k & & \downarrow t^k \\ C_k & \xrightarrow{f_k} & D_k \end{array}$$

are commutative.

1.3. A ω -category consists of a ω -graph C together with maps

$$\begin{aligned} *_k^n : C_n \times_{C_k} C_n &\rightarrow C_n \\ 1_k^n : C_k &\rightarrow C_n \end{aligned}$$

for each pair $k < n$ subject to the following axioms:

- (1) For every $l \leq k < n \in \mathbb{N}$, for every $x, y \in C_n$ such that $s^k(x) = t^k(y)$,

$$s^l(x *_k^n y) = s^l(x),$$

$$t^l(x *_k^n y) = t^l(x).$$

- (2) For every $k < l < n \in \mathbb{N}$, for every $x, y \in C_n$ such that $s^k(x) = t^k(y)$,

$$s^l(x *_k^n y) = s^l(x) *_k^n s^l(y),$$

$$t^l(x *_k^n y) = t^l(x) *_k^n t^l(y).$$

- (3) For every $k < n \in \mathbb{N}$, for every $x \in C_k$,

$$s^k(1_k^n(x)) = x = t^k(1_k^n(x)).$$

- (4) For every $k < n$, for every $x, y, z \in C_n$ such that $s^k(x) = t^k(y)$ and $s^k(y) = t^k(z)$,

$$(x *_k^n y) *_k^n z = x *_k^n (y *_k^n z).$$

- (5) $\forall k < n \in \mathbb{N}, \forall x \in C_n$

$$x *_k^n 1_k^n(s^k(x)) = x = 1_k^n(t^k(x)) *_k^n x.$$

- (6) $\forall k < l < n \in \mathbb{N}, \forall x, y \in C_l$ such that $s^k(x) = t^k(y)$,

$$1_l^n(x *_k^l y) = 1_l^n(x) *_k^n 1_l^n(y).$$

- (7) For every $k < l < n \in \mathbb{N}$, for every $x, y, z, t \in C_n$ such that $s^k(x) = t^k(y)$, $s^k(z) = t^k(t)$, $s^l(x) = t^l(z)$, $s^l(y) = t^l(z)$,

$$((x *_k^n y) *_l^n (z *_k^n t)) = ((x *_l^n z) *_k^n (y *_l^n t)).$$

The same letter will refer to a ω -category and its underlying graph. We will almost always write $*_k$ instead of $*_k^n$, and, for a n -cell x , 1_x will sometimes be used as a synonym for $1_n^{n+1}(x)$. (Moreover, for consistency, we set $1_n^n(x) := x$ for any n -cell x .)

1.4. Given two ω -categories C and D , a ω -functor is a morphism of ω -graph $f : C \rightarrow D$ that satisfies the following axioms:

- (1) For every $k < n \in \mathbb{N}$, for every $x, y \in C_n$ such that $s^k(x) = t^k(y)$,

$$f_n(x *_k y) = f_n(x) *_k f_n(y).$$

- (2) For every $k < n \in \mathbb{N}$, for every $x \in C_k$,

$$f_n(1_k^n(x)) = 1_k^n(f(x)).$$

For a n -cell x , we will often write $f(x)$ instead of $f_n(x)$. \mathbf{Cat}_ω is the category of ω -categories and ω -functors.

1.5. Let x be a k -cell in an ω -category. We say that x is *degenerate* if there exists $x' \in C_{k'}$ with $k' < k$ such that

$$x = 1_{k'}^k(x').$$

(0-cells are never degenerate.)

For $n \in \mathbb{N}$, a n -category is a ω -category such that every k -cell with $k > n$ is degenerate. A n -functor is a ω -functor between two n -categories. \mathbf{Cat}_n is the category of n -categories and n -functors.

There is an obvious inclusion functor

$$\mathbf{Cat}_n \rightarrow \mathbf{Cat}_\omega.$$

This functor has a left and a right adjoint. In the sequel, we will only use the right adjoint that will be denoted by

$$\tau_{\leq n} : \mathbf{Cat}_\omega \rightarrow \mathbf{Cat}_n.$$

For a ω -category C , the ω -category $\tau_{\leq n}(C)$ is obtained by removing all non-identity cells of C of dimension strictly greater than n .

Remark 1.6. It follows from the axioms of ω -categories and ω -functors that, for n -categories and n -functors, everything involving cells of dimension strictly higher than n can be recovered from the rest. For example, we will often consider that the data of a n -category C only consists of

- $(C_k)_{0 \leq k \leq n}$
- $(s_l^k)_{0 \leq k < l \leq n}$
- $(t_l^k)_{0 \leq k < l \leq n}$
- $(*_k^l)_{0 \leq k < l \leq n}$
- $(1_k^l)_{0 \leq k < l \leq n}$.

1.7. Let $n \in \mathbb{N}$. It follows from the definition of ω -categories and ω -functors that we have a functor

$$\mathbf{Cell}_n : \mathbf{Cat}_\omega \rightarrow \mathbf{Set}$$

that associates to each ω -category C , the set C_n of its n -cells and to a ω -functor $f : C \rightarrow D$, the function $f_n : C_n \rightarrow D_n$.

This functor is representable and we define the n -globe \mathbb{D}_n to be the ω -category representing this functor. (\mathbb{D}_n is in fact a n -category.) We will make no distinction between a n -cell x and the ω -functor

$$x : \mathbb{D}_n \rightarrow C.$$

associated to it.

For $k < n \in \mathbb{N}$, the arrows s_n^k, t_n^k and 1_k^n induce natural transformations

$$\sigma_n^k, \tau_n^k : \mathbf{Cell}_n \Rightarrow \mathbf{Cell}_k$$

and

$$\kappa_k^n : \mathbf{Cell}_k \Rightarrow \mathbf{Cell}_n.$$

These natural transformations are in turn represented by ω -functors (denoted with the same letters):

$$\sigma_n^k, \tau_n^k : \mathbb{D}_k \rightarrow \mathbb{D}_n$$

and

$$\kappa_k^n : \mathbb{D}_n \rightarrow \mathbb{D}_k.$$

For example, having a commutative triangle

$$\begin{array}{ccc} \mathbb{D}_n & \xrightarrow{x} & C \\ \kappa_k^n \downarrow & \nearrow y & \\ \mathbb{D}_k & & \end{array}$$

means exactly that we have a n -cell x of C and a k -cell y of C such that

$$x = 1_k^n(y).$$

1.8. Similarly, for $k < n \in \mathbb{N}$, we have a functor

$$\text{Comp}_k^n : \mathbf{Cat}_\omega \rightarrow \mathbf{Set}$$

that associates to each ω -category C the set $C_n \times_{C_k} C_n$ and to a ω -functor $f : C \rightarrow D$ the canonically induced function

$$f_n \times_{f_k} f_n : C_n \times_{C_k} C_n \rightarrow D_n \times_{D_k} D_n.$$

This functor is represented by the category

$$\mathbb{D}_n \coprod_{\mathbb{D}_k} \mathbb{D}_n$$

(which is also a n -category) defined as the following amalgated sum

$$\begin{array}{ccc} \mathbb{D}_k & \xrightarrow{\sigma_n^k} & \mathbb{D}_n \\ \downarrow \tau_n^k & \lrcorner & \downarrow \\ \mathbb{D}_n & \longrightarrow & \mathbb{D}_n \coprod_{\mathbb{D}_k} \mathbb{D}_n. \end{array}$$

The arrow $*_k^n$ induces a natural transformation

$$\nabla_k^n : \text{Comp}_k^n \rightarrow \text{Cell}_n$$

which in turn is represented by a ω -functor

$$\nabla_k^n : \mathbb{D}_n \rightarrow \mathbb{D}_n \coprod_{\mathbb{D}_k} \mathbb{D}_n.$$

For example, having a commutative triangle

$$\begin{array}{ccc} \mathbb{D}_n & \xrightarrow{x} & C \\ \nabla_k^n \downarrow & \nearrow (y, y') & \\ \mathbb{D}_n \coprod_{\mathbb{D}_k} \mathbb{D}_n & & \end{array}$$

means exactly that we have n -cells x, y, z of C such that $s^k(y) = t^k(y')$ and $x = y *_k y'$.

2. CELLULAR EXTENSIONS

Definition 2.1. A *cellular extension* of a n -category C is a quadruplet $(C, \Sigma, \sigma, \tau)$ where:

- C is a n -category,
- Σ is a set,
- σ and τ are maps $\Sigma \rightarrow C_n$ such that if $n \geq 1$ the following equalities hold

$$\begin{aligned} s^{n-1} \circ \sigma &= s^{n-1} \circ \tau, \\ t^{n-1} \circ \sigma &= t^{n-1} \circ \tau. \end{aligned}$$

When the natural number n is understood, a *cellular extension* means a cellular extension of some n -category.

For an element $\alpha \in \Sigma$, we often write $\alpha : x \rightarrow y$ to say that $\sigma(\alpha) = x$ and $\tau(\alpha) = y$.

Definition 2.2. Let $E = (C, \Sigma, \sigma, \tau)$ and $E' = (C', \Sigma', \sigma', \tau')$ be two cellular extensions of n -categories. A *morphism of cellular extensions* from E to E' is a pair (f, φ) where

- f is n -functor from C to C' ,
- φ is a map $\Sigma \rightarrow \Sigma'$,
- the following squares are commutative

$$\begin{array}{ccc} \Sigma & \xrightarrow{\varphi} & \Sigma' \\ \downarrow \sigma & & \downarrow \sigma' \\ C_n & \xrightarrow{f_n} & C'_n \end{array} \quad \begin{array}{ccc} \Sigma & \xrightarrow{\varphi} & \Sigma' \\ \downarrow \tau & & \downarrow \tau' \\ C_n & \xrightarrow{f_n} & C'_n \end{array}$$

Cellular extensions of n -categories and morphisms between them form a category Cat_n^+ . There is an obvious functor $U_n : \text{Cat}_{n+1} \rightarrow \text{Cat}_n^+$ that sends a $(n+1)$ -category C to the cellular extension $(\tau_{\leq n}(C), C_{n+1}, s^n, t^n)$. We shall now explicitly construct a left adjoint to this functor.

2.3. Let $E = (C, \Sigma, \sigma, \tau)$ be a cellular extension of a n -category. We consider the alphabet that has:

- a symbol \mathbf{c}_α for each $\alpha \in \Sigma$,
- a symbol \mathbf{i}_x for each $x \in C_n$,
- a symbol $*_k$ for each $0 \leq k \leq n$,
- a symbol of opening parenthesis $($,
- a symbol of closing parenthesis $)$.

We write $\mathcal{W}[E]$ for the set of (finite) words on this alphabet. If w and w' are elements of $\mathcal{W}[E]$, we write ww' for their concatenation.

The *length* of a word w , denoted by $\mathcal{L}(w)$, is the number of symbols that appear in w .

2.4. We now define recursively the set $\mathcal{T}[E] \subseteq \mathcal{W}[E]$ of *well formed words* (or *terms*) on this alphabet together with maps $s^n, t^n : \mathcal{T}[E] \rightarrow C_n$ that satisfies the globular conditions.

- $(\mathbf{c}_\alpha) \in \mathcal{T}[E]$ with $s^n((\mathbf{c}_\alpha)) = \sigma(\alpha)$ and $t^n((\mathbf{c}_\alpha)) = \tau(\alpha)$ for each $\alpha \in \Sigma$,
- $(\mathbf{i}_x) \in \mathcal{T}[E]$ with $s^n((\mathbf{i}_x)) = t^n((\mathbf{i}_x)) = x$ for each $x \in C_n$,

- $(v *_n w) \in \mathcal{T}[E]$ with $s^n((v *_n w)) = s^n(w)$ and $t^n((v *_n w)) = t^n(v)$ for $v, w \in \mathcal{T}[E]$ such that $s^n(v) = t^n(w)$,
- $(v *_k w) \in \mathcal{T}[E]$ with

$$s^n((v *_k w)) = s^n(v) *_k s^n(w)$$

and

$$t^n((v *_k w)) = t^n(v) *_k t^n(w)$$

for $v, w \in \mathcal{T}[E]$ and $0 \leq k < n$, such that $s^k(s^n(v)) = t^k(t^n(w))$.

As usual, if $w \in \mathcal{T}[E]$ we often write $w : x \rightarrow y$ to say that $s^n(w) = x$ and $t^n(w) = y$. We also define $s^k, t^k : \mathcal{T}[E] \rightarrow C_k$ as iterated source and target.

Definition 2.5. Let $w \in \mathcal{T}[E]$. The *size* of w , denoted by $|w|$, is the number of symbols $*_k$ for any $0 \leq k \leq n$ that appear in the well formed word w .

Definition 2.6. Let $w, v \in \mathcal{W}[E]$. v is a *subword* of w if there exist words $a, b \in \mathcal{W}[E]$ such that $w = avb$.

Remark 2.7. Beware that in the previous definition, none of the words were supposed to be well formed. In particular, a subword of a well formed word is not necessarily well formed.

Lemma 2.8. Let $w \in \mathcal{T}[E]$ and suppose that it is of the form

$$w = (w_1 *_k w_2)$$

with $w_1, w_2 \in \mathcal{T}[E]$.

For any $v \in \mathcal{T}[E]$, if v is a subword of w , then we are in one of the following cases:

- (1) $v = w$,
- (2) v is a subword of w_1 ,
- (3) v is a subword of w_2 .

Proof. See appendix. □

Corollary 2.9. Let $u \in \mathcal{T}[E]$ be of the form

$$vew$$

with $v, w \in \mathcal{W}[E]$ and $e \in \mathcal{T}[E]$.

If e' is an element of $\mathcal{T}[E]$ with same n -source and n -target as e , then the word

$$ve'w$$

belongs to $\mathcal{T}[E]$ as well.

Proof. We proceed by induction on $|u|$.

Base case: If $|u| = 0$, then necessarily v and w are the empty words and the assertion is trivial.

Inductive step: If $|u| \geq 1$, then

$$u = (u_1 *_k u_2)$$

with $u_1, u_2 \in \mathcal{T}[E]$ such that $|u_1|, |u_2| < |u|$. By hypothesis, e is a subword of u and from Lemma 2.8, we are in one of the following cases.

- $u = e$ and in that case the assertion is trivial.

- e is a subword of u_1 , which means that there exists $\tilde{v}, \tilde{w} \in \mathcal{T}[E]$ such that

$$u_1 = \tilde{v}e\tilde{w}.$$

Moreover, we have

$$v = (\tilde{v}$$

and

$$w = *_k u_2).$$

By induction hypothesis, the word

$$\tilde{v}e'\tilde{w}$$

is well formed and thus

$$(\tilde{v}e'\tilde{w} *_k u_2) = vew$$

is well formed.

- e is a subword of u_2 , which is symmetric to the previous case. \square

Lemma 2.10. *Let E be a cellular extension of a n -category and $w_1, w'_1, w_2, w'_2 \in \mathcal{T}[E]$, $0 \leq k \leq n$ and $0 \leq k' \leq n$ such that*

$$s^k(w_1) = t^k(w_2) \text{ and } s^{k'}(w'_1) = t^{k'}(w'_2).$$

If

$$(w_1 *_k w_2) = (w'_1 *_k w'_2)$$

then

$$w_1 = w'_1, w_2 = w'_2 \text{ and } k = k'.$$

for $j \in \{1, 2\}$.

Proof. See appendix \square

Corollary 2.11. *Let $w \in \mathcal{T}[E]$ and suppose that it can be written as*

$$w = (w_1 *_k w_2)$$

with $0 \leq k \leq n$ and $w_1, w_2 \in \mathcal{T}[E]$. Then $s^k(w_1) = t^k(w_2)$.

Proof. By hypothesis, $|w| \geq 1$. From the definition of $\mathcal{T}[E]$, we know that w is of the form

$$(w'_1 *_k w'_2)$$

with $w'_1, w'_2 \in \mathcal{T}[E]$, $0 \leq k \leq n$ and

$$s^{k'}(w'_1) = t^{k'}(w'_2).$$

From Lemma 2.10, we have that $w'_1 = w_1$, $w'_2 = w_2$ and $k = k'$. \square

Definition 2.12. Let $E = (C, \Sigma, \sigma, \tau)$ be a cellular extension of a n -category and let $u, u' \in \mathcal{T}[E]$. An *elementary movement* from u to u' is a quadruplet $\mu = (v, w, e, e')$ with $v, w \in \mathcal{W}[E]$ and $e, e' \in \mathcal{T}[E]$ such that

$$u = vew,$$

$$u' = ve'w,$$

and one of the following holds

(1) e is of the form

$$((x *_k y) *_k z)$$

e' is of the form

$$(x *_k (y *_k z))$$

and with $x, y, z \in \mathcal{T}[E]$ and $0 \leq k \leq n$,

(2) e is of the form

$$((\mathbf{i}_c) *_k x)$$

and e' is of the form

$$x$$

with $x \in \mathcal{T}[E]$, $0 \leq k \leq n$ and $c = 1_k^n(t^k(x))$

(3) e is of the form

$$(x *_k (\mathbf{i}_c))$$

and e' is of the form

$$x$$

with $x \in \mathcal{T}[E]$, $0 \leq k \leq n$ and $c = 1_k^n(s^k(x))$

(4) e is of the form

$$((\mathbf{i}_c) *_k (\mathbf{i}_d))$$

and e' is of the form

$$(\mathbf{i}_{c*_k d})$$

with $c, d \in C_n$ and $0 \leq k < n$,

(5) e is of the form

$$((x *_k y) *_l (z *_k t))$$

and e' is of the form

$$((x *_l z) *_k (y *_l t))$$

with $x, y, z, t \in \mathcal{T}[E]$, $0 \leq l < k \leq n$.

2.13. We will use the notation

$$\mu : u \rightarrow u'$$

to say that μ is an elementary movement from u to u' .

We now define an oriented graph¹ $\mathcal{G}[E]$ with

- $\mathcal{T}[E]$ as its set of objects,
- For all u, u' in $\mathcal{T}[E]$, the set of elementary movements from u to u' as its set of arrows from u to u' .

We will use the categorical notation

$$\mathcal{G}[E](u, u')$$

to denote the set of arrows from u to u' .

We will also sometimes write

$$u \leftrightarrow u'$$

to say that there exists an elementary movement from u to u' or from u' to u .

¹Here *oriented graph* is to be understood in the same way as the underlying (oriented) graph of a category.

Definition 2.14. Let $E = (C, \Sigma, \sigma, \tau)$ be a cellular extension of a n -category and $u, u' \in \mathcal{T}[E]$. We say that the well formed words u and u' are *equivalent* and write

$$u \sim u'$$

if they are in the same connected component of $\mathcal{G}[E]$. More precisely, it means that there exists a finite sequence $(u_j)_{0 \leq j \leq N}$ of well formed words with $u_0 = u$, $u_N = u'$ and $u_j \leftrightarrow u_{j+1}$ for $0 \leq j < N$.

The equivalence class of u will be denoted by $[u]$.

Lemma 2.15. *Let $w, w' \in T[E]$.*

If $u \sim u'$ then $s^n(u) = s^n(u')$ and $t^n(u) = t^n(u')$.

Proof. Let

$$\mu = (v, w, e, e') : u \rightarrow u'$$

be an elementary movement from w to w' . We are going to prove that $s^n(u) = s^n(u')$ and $t^n(u) = t^n(u')$ with an induction on $\mathcal{L}(v) + \mathcal{L}(w)$ (cf. 2.3). Notice first that, by definition of elementary movements, $|u| \geq 1$ and thus

$$u = (u_1 *_k u_2)$$

with $u_1, u_2 \in \mathcal{T}[E]$.

Base case: If $\mathcal{L}(v) + \mathcal{L}(w) = 0$, it means that both v and w are the empty words. It is then straightforward to check the desired property using Definition 2.12.

Inductive step: Suppose now that $\mathcal{L}(v) + \mathcal{L}(w) \geq 0$. Since e is a subword of u that is well formed, from Lemma 2.8 we are in one the following cases.

- $e = u$, which is exactly the base case.
- e is a subword of u_1 , which means that there exists $\tilde{v}, \tilde{w} \in \mathcal{T}[E]$ such that

$$u_1 = \tilde{v}e\tilde{w}.$$

Moreover, we have

$$v = (\tilde{v}$$

and

$$w = \tilde{w} *_k u_2).$$

From corollary 2.9, the word

$$u'_1 := \tilde{v}e'\tilde{w}$$

is well formed. Therefore we can use the induction hypothesis on

$$\tilde{\mu} := (\tilde{v}, \tilde{w}, e, e') : u_1 \rightarrow u'_1.$$

That shows that $s^n(u_1) = s^n(u'_1)$ and $t^n(u_1) = t^n(u'_1)$ and since

$$u = (u_1 *_k u_2) \text{ and } u' = (u'_1 *_k u_2)$$

it follows easily that $s^n(u) = s^n(u')$ and $t^n(u) = t^n(u')$.

- e is a subword of u_2 , which is symmetric to the previous case.

By definition of \sim , this suffices to show the desired properties. \square

Lemma 2.16. *Let $u, u' \in \mathcal{T}[E]$ be respectively of the form*

$$(v_1 *_k v_2)$$

and

$$(v'_1 *_k v'_2)$$

with $v_1, v_2, v'_1, v'_2 \in \mathcal{T}[E]$ and $0 \leq k \leq n$.

If $v_1 \sim v_2$ and $v'_1 \sim v'_2$ then $u \sim u'$.

Proof. Let

$$\mu = (v, w, e, e') : v_1 \rightarrow v'_1$$

be an elementary movement. Set

$$\tilde{v} := (v$$

and

$$\tilde{w} := w *_k v_2).$$

Then, by definition, $(\tilde{v}, \tilde{w}, e, e')$ is an elementary movement from $(v_1 *_k v_2)$ to $(v'_1 *_k v_2)$. Similarly, if we have an elementary movement from v_2 to v'_2 , we obtain an elementary movement from $(v_1 *_k v_2)$ to $(v_1 *_k v'_2)$.

By definition of \sim , this suffices to show the desired property. \square

2.17. Let $E = (C, \Sigma, \sigma, \tau)$ be a cellular extension of a n -category.

From Lemma 2.15, we deduce that $s^n, t^n : \mathcal{T}[E] \rightarrow C_n$ induce maps

$$s^n, t^n : \mathcal{T}[E]/\sim \rightarrow C_n.$$

Let $[v]$ and $[w]$ be two elements of $\mathcal{T}[E]/\sim$ such that $s^k([v]) = t^k([w])$ for $0 \leq k \leq n$. From Lemma 2.16, we can define without ambiguity:

$$[v] *_k [w] := [v *_k w].$$

The reader is left to show that these data add up to a $(n+1)$ -category E^* with $\tau_{\leq n}(E^*) = C$ and $E_{n+1}^* = \mathcal{T}[E]/\sim$.

Note that we have a canonical map

$$\begin{aligned} j_E : \Sigma &\rightarrow E_{n+1}^* \\ \alpha &\mapsto [(\mathbf{c}_\alpha)] \end{aligned}$$

and the following two triangles are commutative

$$\begin{array}{ccc} \Sigma & \xrightarrow{j_E} & E_{n+1}^* \\ & \searrow \sigma & \downarrow s^n \\ & & E_n \end{array} \quad \begin{array}{ccc} \Sigma & \xrightarrow{j_E} & E_{n+1}^* \\ & \searrow \tau & \downarrow t^n \\ & & E_n. \end{array}$$

Remark 2.18. For any $\alpha \in \Sigma$, it is straightforward to check that the number of occurrences of \mathbf{c}_α in a well formed word w only depends on its equivalence class $[w]$. In particular, for $\alpha \neq \beta$ in Σ , $[(\mathbf{c}_\alpha)] \neq [(\mathbf{c}_\beta)]$. In other words, the map j_E defined above is injective.

Proposition 2.19. *Let $E = (C, \Sigma, \sigma, \tau)$ be a cellular extension of a n -category. The $(n+1)$ -category E^* satisfies the following universal property:*

For every $(n + 1)$ -category D , n -functor $f : C \rightarrow \tau_{\leq n}(D)$ and map $\varphi : \Sigma \rightarrow D_{n+1}$ such that the two squares

$$\begin{array}{ccc} \Sigma & \xrightarrow{\varphi} & D_{n+1} \\ \downarrow \sigma & & \downarrow s^n \\ E_n & \xrightarrow{f_n} & D_n \end{array} \quad \begin{array}{ccc} \Sigma & \xrightarrow{\varphi} & D_{n+1} \\ \downarrow \tau & & \downarrow t^n \\ E_n & \xrightarrow{f_n} & D_n \end{array}$$

are commutative, there exists a unique $(n + 1)$ -functor $g : E^* \rightarrow D$ such that $\tau_{\leq n}(g) = f$ and $g_{n+1} \circ j_E = \varphi$.

Proof. The only thing we need to care about is the existence and unicity of $g_{n+1} : E_{n+1}^* \rightarrow D_{n+1}$.

Existence: First we extend φ to a map

$$\bar{\varphi} : \mathcal{T}[E] \rightarrow D_{n+1}$$

by saying that

- If $|w| = 1$, i.e. $w \in \Sigma$, then

$$\bar{\varphi}(w) := \varphi(w).$$

- If $w = (w_1 *_k w_2)$ with $w_1, w_2 \in \mathcal{T}[E]$, then

$$\bar{\varphi}(w) := \bar{\varphi}(w_1) *_k \bar{\varphi}(w_2).$$

Since D is a $(n + 1)$ -category, it is straightforward to see that if $w \sim w'$, then $\bar{\varphi}(w) = \bar{\varphi}(w')$. Therefore, we can define g_{n+1} as the only map that makes the following triangle commutative:

$$\begin{array}{ccc} \mathcal{T}[E] & \xrightarrow{\bar{\varphi}} & D_{n+1} \\ \downarrow & \nearrow g_{n+1} & \\ E_{n+1}^* & & \end{array}$$

where the anonymous arrow is the quotient map $\mathcal{T}[E] \rightarrow \mathcal{T}[E]/\sim$.

The fact that g_{n+1} respects sources and targets and the operations $*_k$ comes from the fact that $\bar{\varphi}$ does (immediate induction left to the reader) and what we have said in paragraph 2.17.

By definition of φ , we have $g_{n+1} \circ j_E = \varphi$. Since for any $x \in C_n$, $1_x = [(i_x)]$, it follows that

$$g_{n+1}(1_x) = \varphi((i_x)) = 1_{f(x)}.$$

(The last equality comes from the fact that φ respects sources and targets by hypothesis.)

This shows everything we needed to define $g : E^* \rightarrow D$ with the desired properties.

Unicity: Let $g_{n+1}, g'_{n+1} : E_{n+1}^* \rightarrow D_{n+1}$ be two solutions of the problem.

Let $x \in E_{n+1}^*$. By definition, there exists $w \in \mathcal{T}[E]$ such that

$$[w] = x.$$

If $|w| = 0$, it means that $w \in \Sigma$ and by hypothesis

$$g_{n+1}([w]) = g_{n+1}(j_E(w)) = g'_{n+1}(j_E(w)) = g'_{n+1}([w]).$$

If $|w| \geq 1$, then $w = (w_1 *_k w_2)$ with $w_1, w_2 \in \mathcal{T}[E]$ and $|w_1|, |w_2| < |w|$. By hypothesis

$$g_{n+1}([w]) = g_{n+1}([w_1] *_k [w_2]) = g_{n+1}([w_1]) *_k g_{n+1}([w_2])$$

and the same equalities hold with g'_{n+1} instead of g_{n+1} . By immediate induction, this shows that

$$g_{n+1} = g'_{n+1}.$$

□

In other words, the correspondence $E \mapsto E^*$ can be extended to a functor $\text{Cat}_n^+ \rightarrow \text{Cat}_{n+1}$ that is left adjoint to $U_n : \text{Cat}_{n+1} \rightarrow \text{Cat}_n^+$.

Remark 2.20. Note that this left adjoint was already explicitly constructed in [Mét08] and our construction is greatly inspired from it. However, it differs in one subtle point. In loc. cit., the author define an elementary relation on parallel well formed words and then takes the congruence generated by it, whereas we directly defined an explicit equivalence relation (definition 2.14) and then show that that two equivalent well formed words necessarily are parallels (Lemma 2.15) and that it is in fact a congruence (Lemma 2.16).

3. POLYGRAPHS

3.1. Let C be a ω -category and Σ a subset of C_{n+1} . We define the cellular extension

$$E_\Sigma = (\tau_{\leq n}(C), \Sigma, s^n, t^n)$$

where s^n and t^n in reality means the restrictions of $s^n, t^n : C_{n+1} \rightarrow C_n$ to $\Sigma \subseteq C_{n+1}$.

In order to simplify the notations, we will allow ourselves to write Σ instead of E_Σ when there is no ambiguity on the rest of the data.

From Proposition 2.19 we deduce that there is a canonical functor

$$g : \Sigma^* \rightarrow \tau_{\leq n+1}(C)$$

with $\tau_{\leq n}(g) = \text{id}_{\tau_{\leq n}(C)}$ and such that the triangle

$$(1) \quad \begin{array}{ccc} \Sigma & \xrightarrow{j_\Sigma} & \Sigma_{n+1}^* \\ & \searrow & \downarrow g_{n+1} \\ & & C_{n+1} \end{array}$$

(where the anonymous arrow is the canonical inclusion) is commutative.

Definition 3.2. Let C be a ω -category and $\Sigma \subseteq C_k$ with $k \in \mathbb{N}$. We say that Σ is a k -basis if either $k = 0$ and

$$\Sigma = C_0$$

or $k = n + 1$ with $n \in \mathbb{N}$ and the canonical map

$$\Sigma_{n+1}^* \rightarrow C_{n+1}$$

described above is a bijection.

Note that a n -category has a k -basis for any $k > n$. Namely, the empty set.

Example 3.3. For any cellular extension $E = (C, \Sigma, \sigma, \tau)$ of a n -category, we have already seen that the canonical map $j_E : \Sigma \rightarrow E_{n+1}^*$ is injective (remark 2.18) and therefore Σ can be identified with a subset of E_{n+1}^* . With this identification, Σ is a $(n+1)$ -basis of E^* .

3.4. Let C be a ω -category and $\Sigma \subseteq C_{n+1}$. We define recursively a map

$$\rho_\Sigma : \mathcal{T}[\Sigma] \rightarrow C_{n+1}$$

(recall that we write Σ instead of E_Σ) such that

- $\rho_\Sigma((\mathbf{c}_\alpha)) = \alpha$ for $\alpha \in \Sigma$,
- $\rho_\Sigma((\mathbf{i}_x)) = 1_x$ for $x \in C_n$,
- $\rho_\Sigma((v *_k w)) = \rho_\Sigma(v) *_k \rho_\Sigma(w)$ for $0 \leq k \leq n$, v and w two well formed words such that $s^k(v) = t^k(w)$.

An immediate induction shows that ρ_Σ commutes with sources and targets, i.e. $s^k \circ \rho_\Sigma = s_n^k$ and $t_n^k \circ \rho_\Sigma = t^k$ for any $0 \leq k < n$.

Lemma 3.5. *The triangle*

$$\begin{array}{ccc} \mathcal{T}[\Sigma] & \longrightarrow & \mathcal{T}[\Sigma]/\sim \\ & \searrow \rho_\Sigma & \downarrow \\ & & C_{n+1} \end{array}$$

(where the horizontal arrow is $w \mapsto [w]$ and the vertical arrow is the $(n+1)$ -level of the canonical functor $\Sigma^* \rightarrow C$) is commutative.

Proof. It is a formal consequence of the recursive definition of ρ_Σ and the commutativity of (1). Details are left to the reader. \square

3.6. Let a be an element of C_{n+1} , we define $\mathcal{T}[\Sigma]_a$ to be

$$\mathcal{T}[\Sigma]_a = \{w \in \mathcal{T}[\Sigma] \mid \rho_\Sigma(w) = a\}.$$

The previous lemma implies that if $v \in \mathcal{T}[\Sigma]_a$ and $v \sim w$, then $w \in \mathcal{T}[\Sigma]_a$.

We define $\mathcal{G}[\Sigma]_a$ to be the (full) subgraph of $\mathcal{G}[\Sigma]$ whose set of objects is $\mathcal{T}[\Sigma]_a$.

Proposition 3.7. *Let C be a ω -category and $\Sigma \subseteq C_{n+1}$. Then Σ is a $(n+1)$ -basis of C if and only if for every $a \in C_{n+1}$, the graph $\mathcal{G}[\Sigma]_a$ is 0-connected (i.e. non-empty and connected).*

More precisely this means that for every $a \in C_{n+1}$:

- there exists $w \in \mathcal{T}[\Sigma]$ such that $\rho_\Sigma(w) = a$,
- for every $v, w \in \mathcal{T}[\Sigma]$, if $\rho_\Sigma(v) = a = \rho_\Sigma(w)$ then $v \sim w$.

Proof. By definition Σ is $(n+1)$ -basis if and only if the canonical map

$$g_{n+1} : \mathcal{T}[\Sigma]/\sim \rightarrow C_{n+1}$$

is a bijection. The surjectivity means exactly that for every $a \in C_{n+1}$ there exists a $w \in \mathcal{T}[\Sigma]$ such that $g_{n+1}(w) = a$. The injectivity means exactly that if $g_{n+1}([v]) = g_{n+1}([w])$ then $v \sim w$. By Lemma 3.5 this means exactly that the graph $\mathcal{T}[\Sigma]_a$ is 0-connected for every $a \in C_{n+1}$. \square

Definition 3.8. A ω -category C is a *free ω -category* if it has a k -basis for every $k \in \mathbb{N}$.

Definition 3.9. Let C be a ω -category and $n \in \mathbb{N}$. A n -cell x of C is *indecomposable* if

- (1) x is not degenerate,
- (2) for any $0 \leq k < n$, if $x = x_1 *_k x_2$ with $x_1, x_2 \in C_n$ then

$$x_1 = 1_k^n(t^k(x))$$

or

$$x_2 = 1_k^n(s^k(x)).$$

In particular, any 0-cell is indecomposable.

Proposition 3.10. Let C be a free ω -category. For any $k \in \mathbb{N}$, a subset $\Sigma_k \subseteq C_k$ is a k -basis if and only if it is the set of indecomposable k -cells.

Proof. See [Mak05, section 4, proposition 8.3]. \square

3.11. In particular, Proposition 3.10 says that for a free category C , there is a unique k -basis for any $k \in \mathbb{N}$. This allows us to talk about *the* k -basis a free category C . The sequence

$$(\Sigma_k \subseteq C_k)_{k \in \mathbb{N}}$$

of the k -basis of a free category C is simply called the *basis* of C .

Definition 3.12. A ω -functor $f : C \rightarrow D$ between two free ω -categories is *rigid* if for every $k \in \mathbb{N}$, we have

$$f_k(\Sigma_k^C) \subseteq \Sigma_k^D$$

where Σ_k^C (resp. Σ_k^D) is the k -basis of C (resp. D).

Free ω -categories and rigid ω -functors form a category denoted by **Pol**.

Remark 3.13. Objects of **Pol** are commonly called *polygraphs* and morphisms of **Pol** are commonly called *morphisms of polygraphs* and we will sometimes do so too. Although the terms “polygraph” and “free ω -category” are synonyms, we prefer to use the former one when we think of them as objects of the category **Pol** and the latter one when we think of them as objects of the category **Cat** $_\omega$.

4. DISCRETE CONDUCHÉ FUNCTORS

4.1. Recall that given a category \mathcal{C} and M a class of arrows of \mathcal{C} , an arrow $f : X \rightarrow Y$ of \mathcal{C} is said to be *right orthogonal to* M if for every $m : A \rightarrow B \in M$ and every commutative square

$$\begin{array}{ccc} A & \longrightarrow & X \\ m \downarrow & & \downarrow f \\ B & \longrightarrow & Y. \end{array}$$

there exists a *unique* $l : B \rightarrow X$ (referred to as a *lifting*) such that the diagram

$$\begin{array}{ccc} A & \longrightarrow & X \\ m \downarrow & \nearrow l & \downarrow f \\ B & \longrightarrow & Y. \end{array}$$

is commutative.

Definition 4.2. Let $f : C \rightarrow D$ be a ω -functor and $n \in \mathbb{N}$. We say that f is a *n-discrete Conduché functor* if it is right orthogonal to the arrows

$$\begin{array}{c} \mathbb{D}_n \\ \downarrow \kappa_k^n \\ \mathbb{D}_k \end{array}$$

and

$$\begin{array}{c} \mathbb{D}_n \\ \downarrow \nabla_k^n \\ \mathbb{D}_n \amalg_{\mathbb{D}_k} \mathbb{D}_n \end{array}$$

for any k such that $0 \leq k < n$.

Unfolding the previous definition, the right orthogonality to κ_k^n means that for any $x \in C_n$, for any $y \in D_k$ such that

$$f(x) = 1_k^n(y)$$

there exists a unique² $x' \in C_k$ such that

$$x = 1_k^n(x')$$

and

$$f(x') = y.$$

Similarly, the right orthogonality to ∇_k^n means that for any $x \in C_n$, if

$$f(x) = y_1 *_k y_2$$

with $0 \leq k < n$ and $y_1, y_2 \in C_n$ such that $s^k(y_1) = t^k(y_2)$, then there exists a *unique* pair (x_1, x_2) of elements of C_n such that

- (1) $s_n^k(x_1) = t_n^k(x_2)$ and $x = x_1 *_k x_2$,
- (2) $f(x_1) = y_1$ and $f(x_2) = y_2$.

Note that any ω -functor is a 0-Conduché discrete functor.

As it turns out, the right orthogonality to κ_k^n comes for free with the right orthogonality to ∇_k^n .

Lemma 4.3. *Let $f : C \rightarrow D$ be a ω -functor and $n \in \mathbb{N}$. f is a n-Conduché discrete functor if and only if it is right orthogonal to the arrows*

$$\begin{array}{c} \mathbb{D}_n \\ \downarrow \nabla_k^n \\ \mathbb{D}_n \amalg_{\mathbb{D}_k} \mathbb{D}_n \end{array}$$

for any k such that $0 \leq k < n$.

Proof. If $n = 0$ there is nothing to show. Suppose now that $n \geq 1$. If $f(x) = 1_k^n(y')$ with $y' \in D_k$ then $f(x) = 1_k(y')^n *_k 1_k(y')^n$. Notice that

$$x = x *_k 1_k^n(s^k(x)) = 1_k^n(t^k(x)) *_k x$$

and

$$f(1_k^n(s^k(x))) = 1_k^n(s^k(f(x))) = 1_k^n(y') = 1_k^n(t^k(f(x))) = f(1_k^n(t^k(x))).$$

²Note that since the map $z \mapsto 1_k^n(z)$ is injective, the uniqueness comes for free.

Using the fact that f is right orthogonal to ∇_k^n , we deduce that $x = 1_k^n(s^k(x)) = 1_k^n(t^k(x))$ and $f(s^k(x)) = f(t^k(x)) = y'$. Thus we can choose $x' = 1_k^n(s^k(x)) = 1_k^n(t^k(x))$. This shows the right orthogonality to

$$\begin{array}{c} \mathbb{D}_n \\ \downarrow \kappa_k^n \\ \mathbb{D}_k. \end{array}$$

□

Definition 4.4. A ω -functor $f : C \rightarrow D$ is a *discrete Conduché functor* if it is a n -discrete Conduché functor for every $n \in \mathbb{N}$.

Remark 4.5. Since the class of discrete Conduché functors is a right orthogonal class, it has many good properties. One of them is that discrete Conduché functors are stable by pullback.

Lemma 4.6. *Let $f : C \rightarrow D$ be a ω -functor and $n \in \mathbb{N}$. If f is a n -discrete Conduché functor, then it is a k -discrete Conduché functor for every $k \leq n$.*

Proof. If $n = 0$ the assertion is trivial. We suppose now that $n > 0$.

Let $x \in C_k$ and suppose that $f(x) = y_1 *_l y_2$ with $l < k$. Then $1_x^n \in C_n$ and $f(1_x^n) = 1_{y_1}^n *_l 1_{y_2}^n$.

By hypothesis, there exists $z_1, z_2 \in C_n$ such that $s^l(z_1) = t^l(z_2)$, $f(z_1) = 1_{y_1}^n$, $f(z_2) = 1_{y_2}^n$ and $1_x^n = z_1 *_l z_2$. From Lemma 4.3 we deduce that there exists $x_1, x_2 \in C_k$ such that $f(x_1) = y_1$, $f(x_2) = y_2$, $z_1 = 1_{x_1}^n$ and $z_2 = 1_{x_2}^n$. It follows that $s^l(x_1) = t^l(x_2)$ and $1_x^n = 1_{x_1}^n *_l 1_{x_2}^n = 1_{x_1 *_l x_2}^n$, hence $x = x_1 *_l x_2$. Now suppose that there are two pairs (x_1, x_2) and (x'_1, x'_2) that lift the pair (y_1, y_2) in the usual way. It follows that $(1_{x_1}^n, 1_{x_2}^n)$ and $(1_{x'_1}^n, 1_{x'_2}^n)$ lift the pair $(1_{y_1}^n, 1_{y_2}^n)$ in the usual way. Using the fact that f is a n -discrete Conduché functor, we deduce that $1_{x_1}^n = 1_{x'_1}^n$ and $1_{x_2}^n = 1_{x'_2}^n$, hence $x_1 = x'_1$ and $x_2 = x'_2$. With Lemma 4.3, that shows that f is a k -discrete Conduché functor. □

From the previous lemma we deduce the following corollary that we shall implicitly use in the sequel.

Corollary 4.7. *Let $f : C \rightarrow D$ be a n -functor. It is a discrete Conduché functor if and only if it is a n -discrete Conduché functor.*

Lemma 4.8. *Let $f : C \rightarrow D$ be a discrete Conduché functor and x a cell of C . The following holds :*

x is an indecomposable cell if and only if $f(x)$ is an indecomposable cell.

Proof. If x is a 0-cell, there is nothing to show since every 0-cell is indecomposable. We suppose now that x is a n -cell with $n > 0$.

Suppose that x is indecomposable. The right orthogonality to κ_k^n for any $0 \leq k < n$ implies that $f(x)$ is non-degenerate as if it were x would be too. Suppose that

$$f(x) = y_1 *_k y_2$$

with $y_1, y_2 \in D_n$ such that $s_k^n(y_1) = t_k^n(y_2)$. The right orthogonality to ∇_k^n implies that

$$x = x_1 *_k x_2$$

with (amongst other things) $f(x_1) = y_1$ and $f(x_2) = y_2$. Since x is indecomposable, x_1 or x_2 has to be of the form $1_k^n(z)$ with $z \in C_k$. Thus, y_1 or y_2 has to be of the form $1_k^n(z')$ with $z' \in D_k$. This proves that $f(x)$ is indecomposable.

Suppose that $f(x)$ is indecomposable. Then x is non-degenerate because otherwise $f(x)$ would be degenerate. Suppose that

$$x = x_1 *_k x_2$$

with $x_1, x_2 \in C_n$ such that $s_k^n(x_1) = s_k^n(x_2)$. Thus

$$f(x) = f(x_1) *_k f(x_2).$$

Since $f(x)$ is indecomposable, either $f(x_1)$ or $f(x_2)$ has to be of the form $1_k^n(z)$ with $z \in D_k$. From the right orthogonality to κ_k^n , it follows that either x_1 or x_2 has to be of the form $1_k^n(z')$ with $z' \in C_k$. This proves that x is indecomposable. \square

From the previous lemma and Proposition 3.10, we deduce the following proposition.

Proposition 4.9. *Let $f : C \rightarrow D$ be a ω -functor with C and D free ω -categories. If f is a discrete Conduché functor then f is rigid.*

5. DISCRETE CONDUCHÉ FUNCTORS AND POLYGRAPHS

5.1. Let $f : C \rightarrow D$ be a $(n+1)$ -functor, $\Sigma^C \subseteq C_{n+1}$ and $\Sigma^D \subseteq D_{n+1}$ such that $f_{n+1}(\Sigma^C) \subseteq \Sigma^D$. We define recursively a map

$$\tilde{f} : \mathcal{W}[\Sigma^C] \rightarrow \mathcal{W}[\Sigma^D]$$

with

- $\tilde{f}(\mathbf{c}_\alpha) = \mathbf{c}_{f(\alpha)}$ for $\alpha \in \Sigma^C$,
- $\tilde{f}(\mathbf{i}_x) = \mathbf{i}_{f(x)}$ for $x \in C_n$,
- $\tilde{f}(*_k) = *_k$ for $0 \leq k \leq n$,
- $\tilde{f}(\text{()}) = (\text{,})$,
- $\tilde{f}(\text{()}) = \text{()})$.

Notice that for any word $w \in \mathcal{W}[\Sigma^C]$, $|\tilde{f}(w)| = |w|$ and $\mathcal{L}(\tilde{f}(w)) = \mathcal{L}(w)$.

Lemma 5.2. *Let $f : C \rightarrow D$ be an ω -functor, $\Sigma^C \subseteq C_{n+1}$ and $\Sigma^D \subseteq D_{n+1}$ such that $f_{n+1}(\Sigma^C) \subseteq \Sigma^D$. For every $u \in \mathcal{W}[\Sigma^C]$,*

- (1) *if $u \in \mathcal{T}[\Sigma^C]$ then $\tilde{f}(u) \in \mathcal{T}[\Sigma^D]$,*
- (2) *if $\tilde{f}(u) \in \mathcal{T}[\Sigma^D]$ and if u is a subword (2.6) of a well formed word then $u \in \mathcal{T}[\Sigma^C]$.*

Proof. The first part of the previous lemma is proved with a short induction left to the reader. For the second part, see appendix. \square

5.3. The first part of Lemma 5.2 show that \tilde{f} induces a map

$$\tilde{f} : \mathcal{T}[\Sigma^C] \rightarrow \mathcal{T}[\Sigma^D].$$

Moreover, we have a commutative square

$$\begin{array}{ccc} \mathcal{T}[\Sigma^C] & \xrightarrow{\rho_C} & C_{n+1} \\ \downarrow \tilde{f} & & \downarrow f_{n+1} \\ \mathcal{T}[\Sigma^D] & \xrightarrow{\rho_D} & D_{n+1} \end{array}$$

where ρ_C and ρ_D respectively stand for ρ_{Σ^C} and ρ_{Σ^D} .

Thus for every $a \in C_{n+1}$ we can define a map:

$$\begin{aligned} \tilde{f}_a : \mathcal{T}[\Sigma^C]_a &\rightarrow \mathcal{T}[\Sigma^D]_{f(a)} \\ w &\mapsto \tilde{f}(w). \end{aligned}$$

Proposition 5.4. *Let $f : C \rightarrow D$ be a ω -functor and $n \in \mathbb{N}$, the following conditions are equivalent:*

- (1) f is a $(n+1)$ -discrete Conduché functor,
- (2) for every $\Sigma^D \subseteq D_{n+1}$ and $\Sigma^C := f^{-1}(\Sigma^D)$ and for every $a \in C_{n+1}$ the map

$$\tilde{f}_a : \mathcal{T}[\Sigma^C]_a \rightarrow \mathcal{T}[\Sigma^D]_{f(a)}$$

defined above is bijective,

Proof. We begin with $1 \Rightarrow 2$.

Surjectivity: We are going to prove the following assertion:

$$\begin{aligned} \forall l \in \mathbb{N}, \forall a \in C_{n+1}, \forall w \in \mathcal{T}[\Sigma^D]_{f(a)} \text{ such that } |w| \leq l \\ \exists v \in \mathcal{T}[\Sigma^C]_a \text{ such that } \tilde{f}_a(v) = w. \end{aligned}$$

We proceed by induction on l .

Suppose first that $l = 0$, we are necessarily in one of the two cases:

- (1) $w = (\mathbf{c}_\beta)$ with $\beta \in \Sigma^D$. By hypothesis, $\rho_D(w) = f(a)$ and by definition of ρ_D , $\rho_D(w) = \beta$ thus $f(a) = \beta$. By definition of Σ^C , $a \in \Sigma^C$ and we can choose $v = (\mathbf{c}_a)$.
- (2) $w = (\mathbf{i}_y)$ with $y \in \Sigma_D$. By hypothesis, $\rho_D(w) = f(a)$ and by definition of ρ_D , $\rho_D(w) = 1_y$ thus $f(a) = 1_y$. Using Lemma 4.3, we know that there exists $x \in C_n$ such that $a = 1_x$ and $f(x) = y$. We can then choose $v = (\mathbf{i}_x) \in \mathcal{T}[\Sigma^C]_a$.

Now suppose that the assertion is true for a fixed $l \in \mathbb{N}$ and let $w \in \mathcal{T}[\Sigma^D]_{f(a)}$ be such that $|w| = l + 1$.

By definition of well formed words, we have

$$w = (w_1 *_k w_2)$$

with $0 \leq k \leq n$ and $w_1, w_2 \in \mathcal{T}[\Sigma^D]$ such that $|w_1| \leq l$ and $|w_2| \leq l$.

By hypothesis, $\rho_D(w) = f(a)$ and by definition of ρ_D , $\rho_D(w) = \rho_D(w_1) *_k \rho_D(w_2)$, thus $\rho_D(w_1) *_k \rho_D(w_2) = f(a)$. Using that f is a $(n+1)$ -discrete Conduché functor, we know that there exists $a_1 \in C_{n+1}$ and $a_2 \in C_{n+1}$ such that $s_n^k(a_1) = t_n^k(a_2)$, $a = a_1 *_k a_2$, $f(a_1) = \rho_D(w_1)$ and $f(a_2) = \rho_D(w_2)$.

Since $|w_1| \leq l$ and $|w_2| \leq l$, we can apply the induction hypothesis. Hence there exist $v_1 \in \mathcal{T}[\Sigma^C]_{a_1}$ and $v_2 \in \mathcal{T}[\Sigma^C]_{a_2}$ such that $\tilde{f}_{a_1}(v_1) = \tilde{f}(v_1) = w_1$ and $\tilde{f}_{a_2}(v_2) = \tilde{f}(v_2) = w_2$. Since ρ_C commutes with

sources and targets as we've already noticed (remark 3.4), the word $(v_1 *_k v_2)$ is a well formed word. By definition of ρ_C ,

$$\rho_C((v_1 *_k v_2)) = \rho_C(v_1) *_k \rho_C(v_2) = a_1 *_k a_2 = a.$$

Thus $(v_1 *_k v_2) \in \mathcal{T}[\Sigma^C]_a$ and

$$\tilde{f}_a((v_1 *_k v_2)) = \tilde{f}((v_1 *_k v_2)) = (\tilde{f}(v_1) *_k \tilde{f}(v_2)) = (w_1 *_k w_2) = w.$$

Injectivity: We are going to prove the following assertion:

$$\forall l \in \mathbb{N}, \forall v \in \mathcal{T}[\Sigma^C]_a, w \in \mathcal{T}[\Sigma^C]_a \text{ such that } |v| = |w| \leq l$$

$$\tilde{f}_a(v) = \tilde{f}_a(w) \Rightarrow v = w$$

We proceed by induction on l .

Suppose first that $l = 0$, we are necessarily in one of the three cases:

- (1) $v = (\mathbf{c}_\alpha)$ and $w = (\mathbf{c}_\beta)$ with α and β in Σ^C . By definition of ρ_C , $\alpha = \rho_C(v) = a = \rho_C(w) = \beta$. Hence $v = w$.
- (2) $v = (\mathbf{i}_x)$ and $w = (\mathbf{i}_y)$ with x and y in C_n . By hypothesis $\rho_C(v) = a = \rho_C(w)$ and by definition of ρ_C , $1_x = \rho_C(v) = a = \rho_C(w) = 1_y$, thus $x = y$ and $v = w$.
- (3) $v = (\mathbf{c}_\alpha)$ and $w = (\mathbf{i}_x)$ with $\alpha \in \Sigma^C$ and $x \in C_n$. By hypothesis, $(\mathbf{c}_{f(\alpha)}) = \tilde{f}(v) = \tilde{f}(w) = (\mathbf{i}_{f(x)})$ which is impossible.

Now suppose that the assertion is true for a fixed $l \in \mathbb{N}$ and let $v, w \in \mathcal{T}[\Sigma^C]$ such that $|v| = |w| = l + 1$ and $\tilde{f}(v) = \tilde{f}(w)$. By definition of well formed words, we have

$$v = (v_1 *_k v_2)$$

and

$$w = (w_1 *_k w_2)$$

with $|v_1|, |v_2|, |w_1|, |w_2| \leq l$.

By hypothesis

$$(\tilde{f}(v_1) *_k \tilde{f}(v_2)) = \tilde{f}(v) = \tilde{f}(w) = (\tilde{f}(w_1) *_k \tilde{f}(w_2)).$$

From the unicity part of Corollary 2.11, we deduce that $*_k = *_k'$ and $\tilde{f}(v_j) = \tilde{f}(w_j)$ for $j \in \{1, 2\}$.

In order to apply the induction hypothesis, we need to show that $\rho_C(v_j) = \rho_C(w_j)$ for $j \in \{1, 2\}$.

By hypothesis,

$$\rho_C(v_1) *_k \rho_C(v_2) = \rho_C(v) = a = \rho_C(w) = \rho_C(w_1) *_k \rho_C(w_2).$$

Hence,

$$f(\rho_C(v_1)) *_k f(\rho_C(v_2)) = f(a) = f(\rho_C(w_1)) *_k f(\rho_C(w_2)).$$

Besides, $f(\rho_C(v_j)) = \rho_D(\tilde{f}(v_j)) = \rho_D(\tilde{f}(w_j)) = f(\rho_C(w_j))$. We deduce from that fact that f is a $(n + 1)$ -Conduché discrete functor that $\rho_C(v_j) = \rho_C(w_j)$ for $j \in \{1, 2\}$.

From the induction hypothesis we have $v_j = w_j$ for $j \in \{1, 2\}$, hence $v = w$.

Now we prove $2 \Rightarrow 1$.

Let $a \in C_{n+1}$ and suppose that $f(a) = b_1 *_{k'} b_2$. We set $\Sigma^D = \{b_1, b_2\}$. By definition, $((\mathbf{c}_{b_1}) *_{k'} (\mathbf{c}_{b_2})) \in \mathcal{T}[\Sigma^D]_{f(a)}$ and by hypothesis there exists a unique $v \in \mathcal{T}[\Sigma^C]_a$ such that $\tilde{f}_a(v) = ((\mathbf{c}_{b_1}) *_{k'} (\mathbf{c}_{b_2}))$. Since $|\tilde{f}_a(v)| = |v| = 1$, we have

$$v = (v_1 *_{k'} v_2)$$

with $|v_1| = |v_2| = 0$, $s^{k'}(v_1) = t^{k'}(v_2)$ and $0 \leq k' \leq n$. Thus

$$(\tilde{f}(v_1) *_{k'} \tilde{f}(v_2)) = \tilde{f}(v) = ((\mathbf{c}_{b_1}) *_{k'} (\mathbf{c}_{b_2})).$$

Using corollary 2.11, we deduce that $k = k'$ and $\tilde{f}(v_j) = (\mathbf{c}_{b_j})$ for $j \in \{1, 2\}$.

We set $a_1 = \rho_C(v_1)$, $a_2 = \rho_C(v_2)$ and we have $s^k(a_1) = t^k(a_2)$, $a = \rho_C(v) = \rho_C(v_1) *_{k'} \rho_C(v_2) = a_1 *_{k'} a_2$ and $f(a_j) = f(\rho_C(v_j)) = \rho_D(\tilde{f}(v_j)) = \rho_D(\mathbf{c}_{b_j}) = b_j$ for $j \in \{1, 2\}$ which proves the existence part of the right orthogonality to ∇_k^n .

Now suppose that we have $a_1, a'_1, a_2, a'_2 \in C_{n+1}$ with $s^k(a_1) = t^k(a_2)$, $s^k(a'_1) = t^k(a'_2)$, $a_1 *_{k'} a_2 = a'_1 *_{k'} a'_2 = a$, $f(a_1) = f(a'_1) = b_1$ and $f(a_2) = f(a'_2) = b_2$.

By definition of $\Sigma^C = f^{-1}(\Sigma^D)$, we have $a_1, a'_1, a_2, a'_2 \in \Sigma^C$. We set $w = ((\mathbf{c}_{a_1}) *_{k'} (\mathbf{c}_{a_2}))$ and $w' = ((\mathbf{c}_{a'_1}) *_{k'} (\mathbf{c}_{a'_2}))$. We have $\rho_C(w) = \rho_C(w') = a$ and $\tilde{f}(w) = ((\mathbf{c}_{b_1}) *_{k'} (\mathbf{c}_{b_2})) = \tilde{f}(w')$. The injectivity of \tilde{f}_a implies that $w = w'$, hence $a_1 = a'_1$ and $a_2 = a'_2$ which proves the uniqueness part of the right orthogonality to ∇_k^n . From Lemma 4.3, this shows that f is a $(n+1)$ -discrete Conduché functor. \square

5.5. Let $f : C \rightarrow D$ be a ω -functor, $n \in \mathbb{N}$, $\Sigma^C \subseteq C_{n+1}$ and $\Sigma^D \subseteq D_{n+1}$ such that $f(\Sigma^C) \subseteq \Sigma^D$. It follows from the definition of $\tilde{f} : \mathcal{T}[\Sigma^C] \rightarrow \mathcal{T}[\Sigma^D]$ and the definition of elementary movement (2.12) that for an elementary movement

$$\mu = (v, w, e, e') : u \rightarrow u'$$

with $u, u' \in \mathcal{T}[\Sigma^C]$, the quadruplet

$$(\tilde{f}(v), \tilde{f}(w), \tilde{f}(e), \tilde{f}(e'))$$

is an elementary movement from $\tilde{f}(u)$ to $\tilde{f}(u')$. Thus, we have defined a map

$$\mathcal{G}[\Sigma^C](u, u') \rightarrow \mathcal{G}[\Sigma^D](\tilde{f}(u), \tilde{f}(u')).$$

Together with the map $\tilde{f} : \mathcal{T}[\Sigma^C] \rightarrow \mathcal{T}[\Sigma^D]$, this defines a morphism of graphs

$$\tilde{f} : \mathcal{G}[\Sigma^C] \rightarrow \mathcal{G}[\Sigma^D]$$

and, by restriction, a morphism of graphs

$$\tilde{f}_a : \mathcal{G}[\Sigma^C]_a \rightarrow \mathcal{G}[\Sigma^D]_{f(a)}$$

for any $a \in C_{n+1}$.

Lemma 5.6. *With the notations of the above paragraph, the map*

$$\mathcal{G}[\Sigma^C](u, u') \rightarrow \mathcal{G}[\Sigma^D](\tilde{f}(u), \tilde{f}(u'))$$

is injective.

Proof. Let (v_1, w_1, e_1, e'_1) and (v_2, w_2, e_2, e'_2) be two elementary movements from u to u' such that

$$(\tilde{f}(v_1), \tilde{f}(w_1), \tilde{f}(e_1), \tilde{f}(e'_1)) = (\tilde{f}(v_2), \tilde{f}(w_2), \tilde{f}(e_2), \tilde{f}(e'_2)))$$

then we have in particular

$$\mathcal{L}(v_1) = \mathcal{L}(v_2) , \mathcal{L}(w_1) = \mathcal{L}(w_2) , \mathcal{L}(e_1) = \mathcal{L}(e_2) , \mathcal{L}(e'_1) = \mathcal{L}(e'_2).$$

Since

$$v_1 e_1 w_1 = u = v_2 e_2 w_2 \text{ and } v_1 e'_1 w_1 = u' = v_2 e'_2 w_2,$$

we have

$$v_1 = v_2 , w_1 = w_2 , e_1 = e_2 , e'_1 = e'_2.$$

□

Lemma 5.7. *Let $f : C \rightarrow D$ be a ω -functor, $n \in \mathbb{N}$, $\Sigma^C \subseteq C_{n+1}$ and $\Sigma^D \subseteq D_{n+1}$ such that $f(\Sigma^C) \subseteq \Sigma^D$ and f is a n -discrete Conduché functor.*

Let

$$\mu : v \rightarrow v'$$

be an elementary movement in $\mathcal{T}[\Sigma^D]$.

If there exists $\bar{u} \in \mathcal{T}[\Sigma^C]$ such that

$$\tilde{f}(\bar{u}) = v$$

then there exist $\bar{u}' \in \mathcal{T}[\Sigma^C]$ and an elementary movement

$$\bar{\mu} : \bar{u} \rightarrow \bar{u}'$$

such that

$$\tilde{f}(\bar{u}') = v' \text{ and } \tilde{f}(\bar{\mu}) = \mu.$$

Remark 5.8. Notice that we only supposed that f was a n -discrete Conduché functor and not a $(n + 1)$ -discrete Conduché functor.

Proof. The proof is long and tedious as we have to check all the different cases of elementary movements (2.12). For the sake of clarity, we first outline a sketch of the proof that is common to all the cases of elementary movements and then we proceed to fill in the blanks successively for every case.

Let

$$\mu = (v_1, v_2, e, e') : v \rightarrow v'$$

be an elementary movement. Since, by definition,

$$\tilde{f}(u) = v_1 e v_2$$

u is necessarily of the form

$$u = u_1 \bar{e} u_2$$

with $\bar{e}, u_1, u_2 \in \mathcal{W}[E]$ such that

$$\tilde{f}(\bar{e}) = e$$

and

$$\tilde{f}(u_j) = v_j$$

for $j \in \{1, 2\}$. From Lemma 5.2, we deduce that \bar{e} is well formed. In each different case, we will prove the existence of a well formed word \bar{e}' with same sources and targets as \bar{e} and such that

$$\tilde{f}(\bar{e}') = e'.$$

From corollary 2.9, we deduce that the word

$$\bar{u}' := u_1 e' u_2$$

is well formed. By definition, we have

$$\tilde{f}(\bar{u}') = v'.$$

Moreover, in each case, it will be immediate that the pair (\bar{e}, \bar{e}') is such that the quadruplet

$$\nu := (w_1, w_2, \bar{e}, \bar{e}')$$

and that

$$\tilde{f}(\bar{\mu}) = \mu.$$

All that is left now is to prove the existence of \bar{e}' with the desired properties.

First case: e is of the form

$$((x *_k y) *_k z)$$

and e' is of the form

$$(x *_k (y *_k z))$$

with $x, y, z \in \mathcal{T}[\Sigma^D]$. The word \bar{e} is then necessarily of the form

$$((\bar{x} *_k \bar{y}) *_k \bar{z}).$$

Since $\tilde{f}(\bar{e}) = e$, we deduce from Lemma 5.2 that $\bar{x}, \bar{y}, \bar{z}$ and $(\bar{x} *_k \bar{y})$ are well formed. From corollary 2.11, we deduce that

$$s_k(\bar{x}) = t_k(\bar{y})$$

and

$$s_k(\bar{y}) = t_k(\bar{z}).$$

Thus the word

$$\bar{e}' := (\bar{x} *_k (\bar{y} *_k \bar{z}))$$

is well formed and it satisfies the desired properties.

Second case: e is of the form

$$(x *_k (\mathbf{i}_{1_k^n(z)}))$$

and e' is of the form

$$x$$

with $x \in \mathcal{T}[\Sigma^C]$, $0 \leq k \leq n$ and $z \in D_k$.³

Necessarily \bar{e} is of the form

$$(\bar{x} *_k (\mathbf{i}_y))$$

with $\bar{x} \in \mathcal{T}[E]$ and $y \in C_n$ such that

$$\tilde{f}(\bar{x}) = x$$

³Notice that since e is well formed, we deduce from corollary 2.11 that $z = s^k(x)$.

and

$$\tilde{f}(y) = 1_k^n(z).$$

Then we set

$$\bar{e}' := x.$$

The only thing left to show is that $y = 1_k^n(s^k(\bar{x}))$. If $k = n$, this follows from corollary 2.11 and the fact that \bar{e} is well formed. If $k < n$, we need first to use the fact that f is a n -discrete Conduché functor to deduce that

$$y = 1_k^n(\bar{z})$$

for some $\bar{z} \in C_k$ such that $f(\bar{z}) = z$ and then use corollary 2.11 and the fact that \bar{e} is well formed.

Third case: Similar to the second one with unit on the left.

Fourth case: e is of the form

$$((\mathbf{i}_x) *_k (\mathbf{i}_y))$$

and e' is of the form

$$(\mathbf{i}_{x *_k y})$$

with $x, y \in D_n$ such that $s^k(x) = t^k(y)$. Necessarily, \bar{e} is of the form

$$((\bar{\mathbf{i}}_{\bar{x}}) *_k (\bar{\mathbf{i}}_{\bar{y}}))$$

with $\bar{x}, \bar{y} \in C_n$ such that

$$f(\bar{x}) = x \text{ and } f(\bar{y}) = y.$$

Using corollary 2.11 and the fact that e is well formed, we deduce that $s^k(\bar{x}) = t^k(\bar{y})$. Thus the word

$$\bar{e}' := (\bar{\mathbf{i}}_{\bar{x} *_k \bar{y}})$$

is well formed. It satisfies all the desired properties.

Fifth case: e is of the form

$$((x *_k y) *_l (z *_k t))$$

and e' is of the form

$$((x *_l z) *_k (y *_l t))$$

with $x, y, z, t \in \mathcal{T}[\Sigma^D]$ and $0 \leq l < k \leq n$ such that all the compatibilities of sources and targets needed are satisfied.

Necessarily, \bar{e} is of the form

$$((\bar{x} *_k \bar{y}) *_l (\bar{z} *_k \bar{t}))$$

with $\bar{x}, \bar{y}, \bar{z}, \bar{t} \in \mathcal{W}[\Sigma^C]$ such that

$$\tilde{f}(\bar{x}) = x,$$

$$\tilde{f}(\bar{y}) = y,$$

$$\tilde{f}(\bar{z}) = z,$$

$$\tilde{f}(\bar{t}) = t.$$

From Lemma 5.2 and the fact that \bar{e} is well formed, we deduce that $\bar{x}, \bar{y}, \bar{z}, \bar{t}, (\bar{x} *_k \bar{y}), (\bar{z} *_k \bar{t})$ are well formed and from corollary 2.11, we deduce that

$$s^k(\bar{x}) = t^k(\bar{y}),$$

$$s^k(\bar{z}) = t^k(\bar{t})$$

and

$$s^l((\bar{x} *_k \bar{y})) = t^l((\bar{z} *_k \bar{t})).$$

Since $l < k$, we deduce from this last equality that

$$s^l(\bar{x}) = s^l(\bar{y}) = t^l(\bar{z}) = t^l(\bar{t}).$$

Thus the word

$$\bar{e}' := ((\bar{x} *_l \bar{z}) *_k (\bar{y} *_l \bar{t}))$$

is well formed. It satisfies all the desired properties. \square

Remark 5.9. In the proof of the previous theorem, we only used the hypothesis that f is right orthogonal to the κ_k^n for any k such that $0 \leq k < n$.

Corollary 5.10. *Let $f : C \rightarrow D$ be a ω -functor, $n \in \mathbb{N}$, $\Sigma^D \subseteq D_{n+1}$ and $\Sigma^C = f^{-1}(\Sigma^D)$. If f is a n -discrete Conduché functor, then for every $a \in C_{n+1}$*

$$\tilde{f}_a : \mathcal{G}[\Sigma^C]_a \rightarrow \mathcal{G}[\Sigma^D]_{f(a)}$$

is an isomorphism of graphs.

Proof. Proposition 5.4 exactly says that the map

$$\tilde{f}_a : \mathcal{G}[\Sigma^C]_a \rightarrow \mathcal{G}[\Sigma^D]_{f(a)}$$

is an isomorphism on objects and we know from Lemma 5.6 that it is a faithful morphism of graphs (same definition as for functors). All that is left to show is that it is also full.

In other words, we have to show that for any $u, u' \in \mathcal{T}[\Sigma^C]$ the map

$$\mathcal{G}[\Sigma^C](u, u') \rightarrow \mathcal{G}[\Sigma^D](\tilde{f}(u), \tilde{f}(u'))$$

is surjective.

Let $\mu : \tilde{f}(u) \rightarrow \tilde{f}(u')$ be an element of the codomain. From Lemma 5.7 we know that there exists

$$\bar{\mu} : u \rightarrow v$$

in $\mathcal{G}[\Sigma^C]$ such that

$$\tilde{f}(\bar{\mu}) = \mu.$$

In particular, we have

$$\tilde{f}(v) = \tilde{f}(u').$$

Since we have an elementary movement from u to v and by hypothesis $u \in \mathcal{T}[\Sigma^C]_a$, we also have $v \in \mathcal{T}[\Sigma^C]_a$ (see Lemma 3.5). Using the injectivity of the map

$$\tilde{f}_a : \mathcal{T}[\Sigma^C]_a \rightarrow \mathcal{T}[\Sigma^D]_{f(a)}$$

we conclude that $v = u'$. \square

Proposition 5.11. *Let $f : C \rightarrow D$ be a ω -functor, $n \in \mathbb{N}$, $\Sigma^D \subseteq D_n$ and $\Sigma^C = f^{-1}(\Sigma^D)$. If f is a n -discrete Conduché functor, then:*

- (1) if Σ^D is a n -basis then so is Σ^C ,
- (2) if $f_n : C_n \rightarrow D_n$ is surjective and Σ^C is a n -basis then so is Σ^D .

Proof. The case $n = 0$ is trivial. We know suppose that $n = k + 1$ with $k \in \mathbb{N}$. From corollary 5.10 we have that for every $a \in C_{n+1}$, the map

$$\tilde{f}_a : \mathcal{G}[\Sigma^C]_a \rightarrow \mathcal{G}[\Sigma^D]_{f(a)}$$

is an isomorphism of graphs. In particular, $\mathcal{G}[\Sigma^C]_a$ is 0-connected if and only if $\mathcal{G}[\Sigma^D]_{f(a)}$ is 0-connected. We conclude with Proposition 3.7. \square

Theorem 5.12. *Let $f : C \rightarrow D$ be a discrete Conduché functor.*

- (1) *If D is a free ω -category with basis $(\Sigma_n^D)_{n \in \mathbb{N}}$, then C is a free ω -category with basis $(f^{-1}(\Sigma_n^D))_{n \in \mathbb{N}}$.*
- (2) *If for every $n \in \mathbb{N}$, $f_n : C_n \rightarrow D_n$ is surjective and if C is a free ω -category with basis $(\Sigma_n^C)_{n \in \mathbb{N}}$, then D is a free ω -category with basis $(f(\Sigma_n^C))_{n \in \mathbb{N}}$.*

Proof. The first property follow directly from the previous proposition. For the second property, it follows from Lemma 4.8 and Proposition 3.10 that

$$f(f^{-1}(\Sigma_n^D)) = \Sigma_n^D$$

and we can use the the previous proposition. \square

APPENDIX A. PROOF OF LEMMAS 2.10, 2.8 AND 5.2

A.1. Let $E = (C, \Sigma, \sigma, \tau)$ be a cellular extension of a n -categories. Recall that we define the length of a word $w \in \mathcal{W}[E]$ as the number of symbols that appears in w . It is denoted by $\mathcal{L}(w)$. Since a word is just a finite sequence of symbols, it makes sense to write $w(i)$ for the symbol at position i (with $0 \leq i \leq \mathcal{L}(w) - 1$).

For any $0 \leq i \leq \mathcal{L}(w) - 1$, define $P_w(i)$ to be the number of opening parenthesis in w with position $\leq i$ minus the number of closing parenthesis in w with position $\leq i$. This defines a function

$$P_w : \{0, \dots, \mathcal{L}(w) - 1\} \rightarrow \mathbb{Z}.$$

Remark A.2. Such a counting function is standard in the literature about formal languages. For example see [HU79, chapter 1, exercice 1.4].

Definition A.3. A *well parenthesized word* is a word $w \in \mathcal{W}[E]$ such that

- (1) It is not empty,
- (2) $P_w(i) \geq 0$ for any $0 \leq i \leq \mathcal{L}(w) - 1$,
- (3) $P_w(i) = 0$ if and only if $i = \mathcal{L}(w) - 1$.

A.4. It follows from this definition the first letter of a well parenthesized word is necessarily an opening parenthesis and that last letter is necessarily a closing parenthesis. Thus, the length of a well parenthesized word is not less than 2.

Moreover, it is immediate that if w_1 and w_2 are well parenthesized words then

$$(w_1 *_k w_2)$$

is well parenthesized (for any $0 \leq k \leq n$).

Lemma A.5. *Let w be well formed word, then it is a well parenthesized word.*

Proof. We proceed by induction on $|w|$. If $|w| = 0$, then w is either of the form

$$(\mathbf{c}_\alpha)$$

or of the form

$$(\mathbf{i}_x).$$

In either case, the assertion is trivial. Now suppose that $|w| > 0$, we know by definition that

$$w = (w_1 *_k w_2)$$

with w_1, w_2 well formed words such that $|w_1|, |w_2| < |w|$. The desired properties follow easily from the induction hypothesis. Details are left to the reader. \square

The converse of the previous lemma is obviously not true. However, corollary A.7 below is a partial converse. Recall that word x is a subword of a word y if there exist words a and b such that $x = ayb$.

Lemma A.6. *Let w be a well parenthesized word of the form*

$$w = (w_1 *_k w_2)$$

with w_1 and w_2 well parenthesized words and $0 \leq k \leq n$ and let v be a subword of w . If v is well parenthesized then one the following holds.

- (1) $v = w$,
- (2) v is a subword of w_1 ,
- (3) v is a subword of w_2 .

Proof. Let a and b be words such that

$$avb = w = (w_1 *_k w_2).$$

Let $l_1, l_2, l, l_a, l_b, l_v$ respectively be the length of w_1, w_2, w, a, b, v . Notice that

$$l_a + l_v + l_b = l = l_1 + l_2 + 3.$$

Notice that since v is well parenthesized, the following cases are forbidden.

- (1) $l_1 \leq l_a \leq l_1 + 1$,
- (2) $l_2 \leq l_b \leq l_2 + 1$,
- (3) $l_a \geq l - 1$,
- (4) $l_b \geq l - 1$.

Indeed, the first case would imply that the first letter of v is a closing parenthesis or the symbol $*_k$. Similarly, the second case would imply that the last letter of v is an opening parenthesis or the symbol $*_k$. The third and fourth case would imply that $l_v < 2$ is empty which is also impossible.

That leaves us with the following cases:

- (1) $l_a = 0$
- (2) $l_b = 0$
- (3) $0 < l_a < l_1$ and $0 < l_b < l_2$
- (4) $0 < l_a < l_1$ and $l_b > l_2 + 1$
- (5) $l_1 + 1 < l_a$ and $0 < l_b < l_2$

If we are in the first case, then

$$P_w(j) = P_v(j)$$

for $0 \leq j \leq l_v - 1$. That implies that $P_w(l_v - 1) = 0$ which means that $l = l_v$, hence $w = v$.

By a similar argument that we leave to the reader, we can show that the second case implies that $w = v$.

If we are in the fourth (resp. fifth) case, then it is clear that v is a subword of w_1 (resp. w_2).

Suppose now that we are in the third case. (Intuitively, it means that the first letter of v is inside w_1 and the last letter of v is inside w_2 .) Notice first that

$$(2) \quad l_a < l_1 < l_a + l_v - 3.$$

(This last equality comes from the fact that since v is well formed, $l_v \geq 2$.)

Besides, by definition of P_w ,

$$P_w(j) = P_v(j - l_a) + P_w(l_a)$$

for $l_a \leq j < l_v + l_a$. In particular, we have

$$1 = P_{w_1}(l_1 - 1) + 1 = P_w(l_1 - 1) = P_v(l_1 - 1) + P_w(l_a).$$

From (2) and since v is well parenthesized, deduce that

$$P_v(l_1 - 1) > 0.$$

Hence $P_w(l_a) \leq 0$ which is impossible because w is well formed and $l_a < l - 1$. \square

Corollary A.7. *Let w be a well parenthesized word. If w is a subword of a well formed word, then it is also well formed.*

Proof. Let u be a well formed word such that w is a subword of u . We proceed by induction on $|u|$. If $|u| = 0$, then either u is of the form

$$(\mathbf{c}_\alpha)$$

or of the form

$$(\mathbf{i}_x).$$

In both case, $w = u$ since the only well parenthesized subword of u is u itself.

Suppose now that $|u| > 0$. By definition,

$$u = (u_1 *_{k'} u_2)$$

with $|u_1|, |u_2| < |u|$. By lemmas A.5 and A.6, we have that either:

- $w = u$ and in that case w is well formed by hypothesis,
- w is a subword of u_1 and from the induction hypothesis we deduce that w is well formed,
- w is a subword of u_2 which is similar to previous case.

\square

Lemma A.8. *Let w_1, w_2, w'_1, w'_2 be well parenthesized words, $0 \leq k \leq n$ and $0 \leq k' \leq n$ such that*

$$(w_1 *_{k'} w_2) = (w'_1 *_{k'} w'_2).$$

Then $w_1 = w'_1$, $w_2 = w'_2$ and $k = k'$.

Proof. Let us define $l := \min(\mathcal{L}(w_1), \mathcal{L}(w'_1))$. Notice that

$$P_w(j) = P_{w_1}(j-1) + 1 = P_{w'_1}(j-1) + 1$$

for $0 < j \leq l$ hence

$$P_{w_1}(l-1) = P_{w'_1}(l-1).$$

Since w_1 and w'_1 are well parenthesized, one of the member of the last equality (and thus both) is equal to 0. That implies that $\mathcal{L}(w_1) = \mathcal{L}(w'_1)$ and the desired properties follow immediately from that. \square

A.9 (Proof of Lemma 2.8 and Lemma 2.10). To prove Lemma 2.8, it suffices to apply Lemma A.5, then Lemma A.6 and then corollary A.7.

To prove Lemma 2.10, it suffices to apply Lemma A.5, then Lemma A.8 and then corollary A.7.

A.10 (Proof of Lemma 5.2). First notice that the map

$$\tilde{f} : \mathcal{W}[\Sigma^C] \rightarrow \mathcal{W}[\Sigma^D]$$

satisfies the following property:

For any $w \in \mathcal{W}[\Sigma^C]$, w is well parenthesized if and only if $\tilde{f}(w)$ is well parenthesized.

It suffices then to apply Lemma A.5 and then corollary A.7.

APPENDIX B. COMPLEMENTS: RIGID FUNCTORS AND DISCRETE CONDUCHÉ FUNCTORS

B.1. We know from Proposition 4.9 that if

$$f : C \rightarrow D$$

is a discrete Conduché fibration and if C and D are free ω -categories then f is rigid. However, the converse does not hold. This phenomenon was already noticed for 2-categories between the lines of [Str96, section 5]. We shall now give a simple counter-example.

B.2 (Counter-example). Let e be terminal 1-category and let \star be its unique object. Let $E = (e, \Sigma, \sigma, \tau)$ be the cellular extension of e such that Σ has two elements $a, b : \star \rightarrow \star$ and let $C := E^*$. By the Eckmann-Hilton argument, we have that

$$a *_0 b = a *_1 b = b *_0 a.$$

Let $E' = (e, \Sigma', \sigma, \tau)$ be the cellular extension of e such that Σ' has one element $c : \star \rightarrow \star$ and let $C' := E'^*$. Let $f : C \rightarrow C'$ be the unique rigid functor such that $f(a) = f(b) = c$. Now set $x := a *_0 b$ and consider the decomposition

$$f(x) = c *_0 c.$$

The fact that $a *_0 b = b *_0 b$ but that $a \neq b$ shows that the unicity of the lifting of the previous decomposition of $f(x)$ fails.

Remark B.3. While in the previous counter-example, the unicity part of the definition of Conduché functor fails, the existence still holds. The author of these notes believes that there should be examples where the existence part fails as well.

B.4. Since the category **Pol** is cocomplete (see [Mak05, section 5]) it admits a terminal object \top . Hence, a rigid ω -functor $f : C \rightarrow D$ between two free ω -category always fits in a commutative triangle

$$\begin{array}{ccc} C & \xrightarrow{f} & D \\ & \searrow & \swarrow \\ & \top & \end{array}$$

where the anonymous arrows are the canonical rigid functors to the terminal polygraph. Since the class of Conduché functors is a right orthogonal class, it has the following cancellation property: for $f : C \rightarrow D$ and $g : D \rightarrow E$ two ω -functors, if g and $g \circ f$ are Conduché functor then so is f .

Following the terminology of [Str96, section 5], we say that a free ω -category C is *tight* if the canonical rigid functor $C \rightarrow \top$ is a Conduché functor. Putting all the pieces together, we obtain the following partial converse of Proposition 4.9.

Proposition B.5. *Let C and D be two free ω -categories and $f : C \rightarrow D$ a rigid ω -functor. If C and D are tight then f is a Conduché functor.*

B.6. The terminal object of **Pol** is a rather complicated object (see [Str96, section 4] for an explicit description of the 2-cells of that polygraph) and the previous criterion seems hard to use in practise.

However, it can be checked that every free 1-category is tight and the previous proposition implies that a 1-functor $f : C \rightarrow D$ between free 1-categories is rigid if and only if it is a Conduché functor. This fact can also be directly proved "by hand". We leave the details to the reader.

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