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New refinements of two well-known inequalities

Christophe Chesneau¹, Yogesh J. Bagul²,

¹LMNO, University of Caen Normandie, France

Email : christophe.chesneau@unicaen.fr

²Department of Mathematics, K. K. M. College Manwath,

Dist: Parbhani(M.S.) - 431505, India.

Email : yjbagul@gmail.com

Abstract. In this note, we improve the useful inequalities $\sin(x)\sinh(x) \leq x^2$ and $\cos(x)\cosh(x) \leq 1$ by the use of infinite products.

Keywords: Trigonometric function; infinite product; exponential bounds.

Mathematics Subject Classification(2010): 26D07, 33B10, 33B20.

1 Introduction

During the past several years, sharp inequalities involving trigonometric and hyperbolic functions have received a lot of attention. Thanks to their usefulness in all areas of mathematics. Old and new such inequalities, as well as refinements of the so-called Jordan's, Cusa-Huygens and Wilker inequalities, can be found in [2–11], and the references therein.

In this note, we focus our attention on the following famous results proved by [9, Lemma 1 page 67] and [11, Lemma 1 page 148].

Lemma 1. [9, Lemma 1], [11, Lemma 1] For $x \in (0, \pi/2)$, we have

$$\sin(x)\sinh(x) \leq x^2, \quad \cos(x)\cosh(x) \leq 1.$$

The proof of Lemma 1 is based on the studies of appropriate functions. The first inequality is also known in the form $\sin(x)/x \leq x/\sinh(x)$ or $\sin(x) \leq x^2/\sinh(x)$. In this note, we develop refinements of the two inequalities in Lemma 1 by using infinite products and applying the methodology developed in [1]. Sharp lower bounds of various nature are also established, with graphical supports.

2 Main results

In the following result, we propose a double inequality for $\sin(x)\sinh(x)$, implying $\sin(x)\sinh(x) \leq x^2$ for the upper bound.

Proposition 1. *For $x \in (0, \alpha)$ and $\alpha \in (0, \pi)$, we have*

$$x^2 \exp(-\beta x^4) \leq \sin(x)\sinh(x) \leq x^2 \exp\left(-\frac{x^4}{90}\right),$$

with $\beta = -\ln(\sin(\alpha)\sinh(\alpha)/\alpha^2)/\alpha^4$.

Proof of Proposition 1. We have the following infinite products:

$$\frac{\sin(x)}{x} = \prod_{k=1}^{+\infty} \left(1 - \frac{x^2}{\pi^2 k^2}\right), \quad \frac{\sinh(x)}{x} = \prod_{k=1}^{+\infty} \left(1 + \frac{x^2}{\pi^2 k^2}\right).$$

Therefore, using the inequality $1 + y \leq e^y$ for $y \in \mathbb{R}$ and $\zeta(4) = \sum_{k=1}^{+\infty} (1/k^4) = \pi^4/90$, we have

$$\begin{aligned} \sin(x)\sinh(x) &= x^2 \prod_{k=1}^{+\infty} \left(1 - \frac{x^2}{\pi^2 k^2}\right) \prod_{k=1}^{+\infty} \left(1 + \frac{x^2}{\pi^2 k^2}\right) \\ &= x^2 \prod_{k=1}^{+\infty} \left(1 - \frac{x^4}{\pi^4 k^4}\right) \leq x^2 \prod_{k=1}^{+\infty} \exp\left(-\frac{x^4}{\pi^4 k^4}\right) \\ &= x^2 \exp\left(-\frac{x^4}{\pi^4} \zeta(4)\right) = x^2 \exp\left(-\frac{x^4}{90}\right). \end{aligned}$$

For the lower bound, using the Bernoulli inequality, i.e. for $u, v \in (0, 1)$, we have $1 - uv \geq (1 - v)^u$, we obtain, for $x \in (0, \alpha)$,

$$\begin{aligned} \sin(x)\sinh(x) &= x^2 \prod_{k=1}^{+\infty} \left(1 - \frac{x^4}{\alpha^4} \frac{\alpha^4}{\pi^4 k^4}\right) \geq x^2 \prod_{k=1}^{+\infty} \left(1 - \frac{\alpha^4}{\pi^4 k^4}\right)^{x^4/\alpha^4} \\ &= x^2 \left[\prod_{k=1}^{+\infty} \left(1 - \frac{\alpha^4}{\pi^4 k^4}\right) \right]^{x^4/\alpha^4} = x^2 \left[\frac{\sin(\alpha)\sinh(\alpha)}{\alpha^2} \right]^{x^4/\alpha^4} \\ &= x^2 \exp(-\beta x^4). \end{aligned}$$

This ends the proof of Proposition 1. □

The proposition below proposes a tight polynomial lower bound for $\sin(x)\sinh(x)$.

Proposition 2. For $x \in (0, \pi)$, we have

$$x^2 \left(1 - \frac{x^4}{\pi^4}\right)^{\pi^4/90} \leq \sin(x) \sinh(x).$$

Proof of Proposition 2. Using the Bernoulli inequality, i.e. for $u, v \in (0, 1)$, we have $1 - uv \geq (1 - v)^u$, and $\zeta(4) = \pi^4/90$, we have

$$\begin{aligned} \sin(x) \sinh(x) &= x^2 \prod_{k=1}^{+\infty} \left(1 - \frac{x^4}{\pi^4 k^4}\right) \geq x^2 \prod_{k=1}^{+\infty} \left(1 - \frac{x^4}{\pi^4}\right)^{1/k^4} \\ &= x^2 \left(1 - \frac{x^4}{\pi^4}\right)^{\zeta(4)} = x^2 \left(1 - \frac{x^4}{\pi^4}\right)^{\pi^4/90}. \end{aligned}$$

This ends the proof of Proposition 2. □

Figure 1 illustrates the sharpness of the bounds in Propositions 1 and 2 for $x \in (0, 3)$. We see that these inequalities are sharp, particularly for $x \in (0, 1.5)$. At least for $x \in (1.5, 1)$, the lower bound in Proposition 2 is clearly more sharp to the one in Proposition 1.

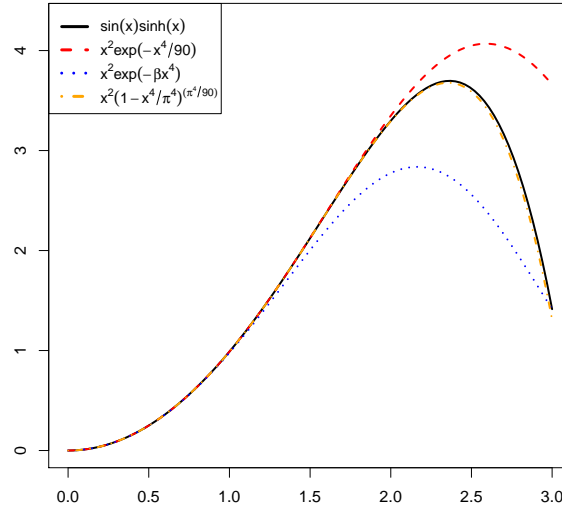


Figure 1: Graphs of the functions in Propositions 1 and 2 for $x \in (0, 3)$.

Similarly, the following result determines a double inequality for $\cos(x) \cosh(x)$, implying $\cos(x) \cosh(x) \leq 1$ for the upper bound.

Proposition 3. For $x \in (0, \alpha)$ and $\alpha \in (0, \pi/2)$, we have

$$\exp(-\gamma x^4) \leq \cos(x)\cosh(x) \leq \exp\left(-\frac{x^4}{6}\right),$$

with $\gamma = -\ln(\cos(\alpha)\cosh(\alpha))/\alpha^4$.

Proof of Proposition 3. We have the following infinite products:

$$\cos(x) = \prod_{k=1}^{+\infty} \left(1 - \frac{4x^2}{\pi^2(2k-1)^2}\right), \quad \cosh(x) = \prod_{k=1}^{+\infty} \left(1 + \frac{4x^2}{\pi^2(2k-1)^2}\right).$$

Therefore, using the inequality $1 + y \leq e^y$ for $y \in \mathbb{R}$ and $\sum_{k=1}^{+\infty} (1/(2k-1)^4) = (15/16)\zeta(4) = (15/16)(\pi^4/90) = (1/16)(\pi^4/6)$, we have

$$\begin{aligned} \cos(x)\cosh(x) &= \prod_{k=1}^{+\infty} \left(1 - \frac{4x^2}{\pi^2(2k-1)^2}\right) \prod_{k=1}^{+\infty} \left(1 + \frac{4x^2}{\pi^2(2k-1)^2}\right) \\ &= \prod_{k=1}^{+\infty} \left(1 - \frac{16x^4}{\pi^4(2k-1)^4}\right) \leq \prod_{k=1}^{+\infty} \exp\left(-\frac{16x^4}{\pi^4(2k-1)^4}\right) \\ &= \exp\left(-x^4 \frac{16}{\pi^4} \sum_{k=1}^{+\infty} \frac{1}{(2k-1)^4}\right) = \exp\left(-\frac{x^4}{6}\right). \end{aligned}$$

Using the Bernoulli inequality, i.e. for $u, v \in (0, 1)$, we have $1 - uv \geq (1-v)^u$, we obtain, for $x \in (0, \alpha)$,

$$\begin{aligned} \cos(x)\cosh(x) &= \prod_{k=1}^{+\infty} \left(1 - \frac{x^4}{\alpha^4} \frac{16\alpha^4}{\pi^4(2k-1)^4}\right) \geq \prod_{k=1}^{+\infty} \left(1 - \frac{16\alpha^4}{\pi^4(2k-1)^4}\right)^{x^4/\alpha^4} \\ &= \left[\prod_{k=1}^{+\infty} \left(1 - \frac{16\alpha^4}{\pi^4(2k-1)^4}\right) \right]^{x^4/\alpha^4} = [\cos(\alpha)\cosh(\alpha)]^{x^4/\alpha^4} \\ &= \exp(-\gamma x^4). \end{aligned}$$

The proof of Proposition 3 is completed. \square

The proposition below presents a sharp polynomial lower bound for $\cos(x)\cosh(x)$.

Proposition 4. For $x \in (0, \pi/2)$, we have

$$\left(1 - \frac{16x^4}{\pi^4}\right)^{\pi^4/96} \leq \cos(x)\cosh(x).$$

Proof of Proposition 4. Using the Bernoulli inequality, i.e. for $u, v \in (0, 1)$, we have $1 - uv \geq (1 - v)^u$, and $\sum_{k=1}^{+\infty} (1/(2k-1)^4) = \pi^4/96$, we have

$$\begin{aligned} \cos(x)\cosh(x) &= \prod_{k=1}^{+\infty} \left(1 - \frac{16x^4}{\pi^4(2k-1)^4}\right) \geq \prod_{k=1}^{+\infty} \left(1 - \frac{16x^4}{\pi^4}\right)^{1/(2k-1)^4} \\ &= \left(1 - \frac{16x^4}{\pi^4}\right)^{\sum_{k=1}^{+\infty} (1/(2k-1)^4)} = \left(1 - \frac{16x^4}{\pi^4}\right)^{\pi^4/96}. \end{aligned}$$

This ends the proof of Proposition 4. □

Figure 2 illustrates the sharpness of the bounds in Propositions 3 and 4 for $x \in (0, 1.5)$. At least for $x \in (0.5, 1.5)$, we see that the lower bound in Proposition 4 is clearly more sharp to the one in Proposition 3.

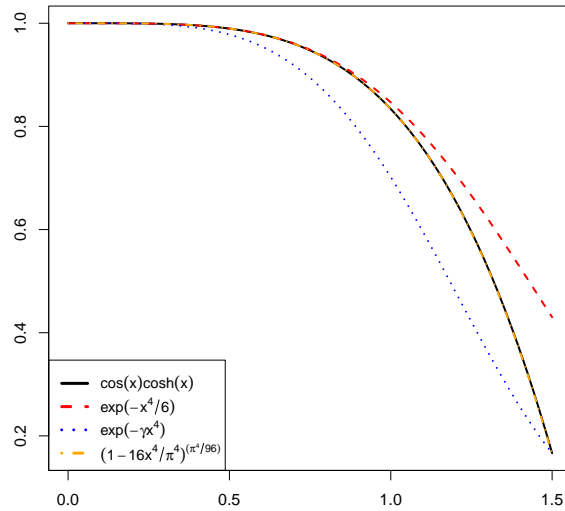


Figure 2: Graphs of the functions in Propositions 3 and 4 for $x \in (0, 1.5)$.

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