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# APPROXIMATION OF SEMI-GROUPS IN THE SENSE OF TROTTER AND ASYMPTOTIC MATHEMATICAL MODELING IN PHYSICS OF CONTINUOUS MEDIA

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ABSTRACT. We derive several models in Physics of continuous media using Trotter theory of convergence of semi-groups of operators acting on variable spaces.

Alain Léger and Christian Licht met for the first time on August 1993 in Beijing during the second International Conference on Nonlinear Mechanics. Ironically, each of their communications was scheduled on the same time. Thus Alain Léger was not directly aware of the use of (a nonlinear extension of) Trotter theory applied on this occasion to the asymptotic behavior of a thin dissipative layer [21]. On October 2007, Alain Léger, one of the leaders of the English/French GDR 2501 “Research on Ultrasound Propagation for Non Destructive Tests” invited Christian Licht to join it and was made aware of Trotter theory - a tool that Christian Licht and Thibaut Weller were using for several years - in supplying a joint communication [23] on the next meeting of the GDR on June 2008 devoted to the Dynamics of elastic bodies connected by a thin adhesive layer. Afterwards a close collaboration using Trotter theory remained by studying the case of a thin viscoelastic adhesive layer [24]...

## 1. Asymptotic mathematical modeling in physics of continuous media.

As steady state cases should be considered as particular cases of transient phenomena, we will only focus on transient situations which can be mathematically formulated in terms of evolution equations in Hilbert spaces. First, such Hilbert spaces, each element of which corresponds to a possible state of the system having finite energy, in short being spaces of possible states with finite energy, this means that energy is (or is close to) a positive definite quadratic form of the state variables. So, as far as Solid Mechanics is involved, we are confined to small strains assumption, but there are still interesting things to deal with in this framework... The term “possible” is the generalization to (Multi-)Physics of the classical kinematically admissible fields. Note that there exist rather technical extensions to

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Banach spaces of the theory of semi-groups of operators and consequently of their approximations. This could imply to consider convex energies defined on Orlicz-Sobolev spaces, but they are mere formal extensions and not fundamental ones as realistic strain energy densities may not be convex. Second, and this is capital in our purpose, the considered evolution has to be governed by a time-independent  $m$ -dissipative operator, hence generating a strongly continuous semi-group of operators<sup>1</sup>. Independency with respect to the time strongly simplifies the analysis but if some particular cases are appropriate, it could be relaxed (we will see that it is not the semi-group property which is essential but the contraction and convergence properties of the resolvent of the evolution operator). But  $m$ -dissipative property is capital: it corresponds, for example, to the formalism of Generalized Standard Materials [13, 12] which is adequately representative of reality.

Mathematical modeling is the creation and/or the use of *ad-hoc* tools that make it possible to lead a rigorous analysis of a phenomenon through a thorough, detailed and generally short way. Then asymptotic mathematical modeling obtains models by a rigorous mathematical analysis of approximation or convergence processes, these last two notions being intimately related. Given a physical problem ( $\mathcal{P}$ ), the approximation process consists in considering and setting up a sequence of problems ( $\mathcal{P}_s$ ) on which experiments or numerical studies are made easier and whose solutions are as close as wanted to the one to ( $\mathcal{P}$ ). An example treated further (see section 3.2) will be the small vibrations of a structure immersed in an open sea, where the open sea is approximated by a sequence of bounded fluid domains where experiments or numerical computations can be done. The convergence process is rooted in the conversion of a set of data into parameters so that the genuine physical problem associated with a set of data becomes a family of problems ( $\mathcal{P}_s$ ) associated with a set  $s$  of parameters. It is then possible to perform an asymptotic analysis of the solution to ( $\mathcal{P}_s$ ) when  $s$  goes to its natural limit (*i.e.* a small parameter may tend to zero while another large one may tend to  $+\infty$ ). This approach is based on the hope that from this asymptotic study will arise a limit problem whose mainly quantitative study is easier or more tractable. An example is bonding (see section 3.5) where a thin adhesive layer is replaced by either a mechanical constraint or a material surface between the sole adherents. In both cases however, we therefore propose a simplified but accurate model. It is worthwhile to observe that these two points of view may operate simultaneously. A purely phenomenological modeling likes to show its *bien fondé* as a limit of a complex model at a different scale of description. We will see (see section 3.3) that a limit model may stem from neglecting a physical phenomenon, while the same problem can be approximated by a sequence of more tractable problems from the mathematical and numerical point of view!

**2. Approximation of semi-groups of operators acting on variable Hilbert spaces.** The mathematical tool we will use to realize these modelings will be a theory of approximation/convergence of semi-groups. Historical irony, the first study devoted to this topic, due to H. F. Trotter [29], considers sequences of variable Hilbert spaces  $\mathcal{H}_n$ . Actually in most of the lectures concerned with this subject (actually dying out), the variable character of the Hilbert spaces is set aside: the sequence of evolution equations is posed in a sole Hilbert space. This is a point

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<sup>1</sup>As Trotter theory was first set in the linear case, we use the vocabulary of linear operators theory, we will see that most of the considerations may be extended to nonlinear multivoque maximal monotone operators: if  $A$  is said  $m$ -dissipative, then  $-A$  is said maximal monotone!...

of view of Mathematicians, it does not seem to us suitable to the modeling of the real world. Quite all boundary value problems stemming from Physics are parameterized, for instance by:

- i) the domain where the problem is posed, whose size or shape may vary or is not exactly known, which implies to deal with functional spaces  $V(\Omega_n)$  defined on sequences of domains  $\Omega_n$ ,
- ii) the physical coefficients which may be very high or low or strongly oscillating, which leads, even if the domain is fixed, to sequences of functional spaces  $V_n(\Omega)$ , because their natural energy-norms vary,
- iii) both previous items acting simultaneously:  $V_n(\Omega_n)$ ,
- iv) some physical effects we want to test the importance, here the number of significant state variables may increase or decrease at the limit!...

Thus, this is the most radical approach, the one of H. F. Trotter, that we will consider.

First the framework of this theory is as follows. On one hand is a sequence of evolution equations

$$(\mathcal{P}_n) \quad \begin{cases} \frac{du_n}{dt} - A_n u_n = f_n \text{ in } \mathcal{H}_n \\ u_n(0) = u_n^0 \end{cases}$$

set in a sequence of Hilbert spaces  $\mathcal{H}_n$  with norm  $|\cdot|_n$ , governed by a sequence of  $m$ -dissipative operators  $A_n$  with domains  $D(A_n)$  and with data  $(u_n^0, f_n)$  in  $D(A_n) \times C^{0,1}([0, T]; \mathcal{H}_n)$  in order that  $(\mathcal{P}_n)$  has a unique solution in  $C^1([0, T]; \mathcal{H}_n) \cap C^0([0, T]; D(A_n))$ . On the other hand is an evolution equation

$$(\mathcal{P}) \quad \begin{cases} \frac{du}{dt} - A u = f \text{ in } \mathcal{H} \\ u(0) = u^0 \end{cases}$$

set in an Hilbert space  $\mathcal{H}$  with norm  $|\cdot|$ , governed by a  $m$ -dissipative operator  $A$  with domain  $D(A)$  and with data  $(u_0, f)$  in  $D(A) \times C^{0,1}([0, T]; \mathcal{H})$  in order that  $(\mathcal{P})$  has a unique solution in  $C^1([0, T]; \mathcal{H}) \cap C^0([0, T]; D(A))$ . As already suggested, and it is essential in the applications to Physics of continuous media, the nature of the spaces  $\mathcal{H}_n$  and  $\mathcal{H}$  may be very different, but the sequence of spaces  $\mathcal{H}_n$  has to approach/converge to  $\mathcal{H}$ ! It will be in - what we allow ourselves to call - the sense of Trotter through a linear continuous operator from  $\mathcal{H}$  to  $\mathcal{H}_n$  which to a certain extent makes it possible to compare an element of  $\mathcal{H}$  with an element of  $\mathcal{H}_n$ , indeed, more precisely, allows to associate a suitable representative  $P_n u$  in  $\mathcal{H}_n$  with any  $u$  in  $\mathcal{H}$ . Operator  $P_n$  has to satisfy two conditions of uniform continuity  $(T_1)$  and of good energetic representation  $(T_2)$ :

$$\begin{aligned} (T_1) \quad & \text{there exists } C > 0 \text{ such that } |P_n u|_n \leq C|u|, \forall u \in \mathcal{H} \\ (T_2) \quad & \lim_{n \rightarrow \infty} |P_n u|_n = |u|, \forall u \in \mathcal{H} \end{aligned}$$

Note that  $P_n$  is not necessarily onto or one-to-one. Then the notion of comparison, defining the Trotter convergence of a sequence, may be introduced:

**Definition 2.1.** A sequence  $(u_n)_{n \in \mathbb{N}}$  in  $\mathcal{H}_n$  converges in the sense of Trotter toward an element  $u$  of  $\mathcal{H}$  if and only if:

$$\lim_{n \rightarrow \infty} |P_n u - u_n|_n = 0$$

Eventually, it is in these terms that the result of convergence of the solution to  $(\mathcal{P}_n)$  toward the one to  $(\mathcal{P})$  may be conveyed:

**Theorem 2.2.** *Let  $u, u_n$  the solutions to  $(\mathcal{P})$  and  $(\mathcal{P}_n)$ , if*

$$i) |P_n u^0 - u_n^0|_n \rightarrow 0$$

$$ii) \int_0^T |P_n f(t) - f_n(t)|_n dt \rightarrow 0$$

$$iii) \forall y \in X, \text{ dense in } \mathcal{H}, |P_n(I - A)^{-1}y - (I - A_n)^{-1}P_n y|_n \rightarrow 0$$

*then, uniformly on  $[0, T]$ ,  $|P_n u(t) - u_n(t)|_n \rightarrow 0$  and  $|u_n(t)|_n \rightarrow |u(t)|$ .*

The first two conditions concern the data (initial states and second members) of the problems, the last one<sup>2</sup> deals with the resolvents of operators  $A_n$  and  $A$ . We will see in the examples, and it is a general fact, that the resolvents correspond to the solution to a steady state problem satisfied by the Laplace transform of  $u$  or  $u_n$ . So that, roughly speaking, when one knows how to solve convergence in the steady state version (actually a slight kinetic perturbation) of the transient problem, one knows how to solve convergence in the genuine transient problem. This implies very short proofs confining to the implementation of the Trotter theory: guessing  $\mathcal{H}$  or  $\mathcal{H}_n$ , building  $P_n$  satisfying  $(T_1)$  and  $(T_2)$  because the Trotter convergence of the resolvents easily ensues, no additional efforts concerning the transient problem has to be supplied! In comparison to the variational evolution equations techniques, we do not have to study the static problem first and next to deal with pages full of estimations and limits of integrals from 0 to  $T$ ! For us, this implementation of the right framework reminds the one of the right definition of the right domain of the operator to formulate a transient problem in terms of evolution equations such as  $(P_n)$  or  $(P)$ . It seems to us more classy than the steamroller of variational formulations.

## 2.1. Some remarks.

**2.1.1. Nonlinear extension.** As already said, linearity is not essential in these considerations and semi-groups theory has been extended to nonlinear evolution equations in Hilbert spaces governed by maximal (mutivoque) operators [8]. To study the limit quasi-static behavior of a thin dissipative layer, an extension to this nonlinear case of Trotter theory has been done in [20] and published in [14]. It suffices to judiciously insert  $P_n$  in the proof given in [8] dealing with the case of fixed Hilbert spaces.

**2.1.2. Convergence in the sense of Trotter.** Previous assumptions and conclusions are formulated in terms of Trotter convergence, which is natural as  $\mathcal{H}$  and  $\mathcal{H}_n$  are different. Even if this is the right notion from the physical point of view, it could be of interest to consider this convergence with respect to some classical notions to avoid the wrath of applied mathematicians who ignore physical problems that they pretend to solve, or of physicists for whom reality confines to what they know: strong, even weak, convergences in Sobolev spaces. Generally, Trotter convergence

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<sup>2</sup>Recall that  $I$  stands for the identity operator.

means strong convergence in an appropriate and identified functional space augmented by a convergence of the energies!

### 3. Some examples.

**3.1. Presentation.** “Tant il est vrai que sur le marché des idées molles, la moindre raideur formelle produit de la plus-value”, so we prefer to illustrate our arguments by pertinent examples rather than speculate on an abstract point of view telling roughly that each time one has variational convergence (say à la Mosco) for the steady state problems, Trotter framework of sequences of Hilbert spaces  $\mathcal{H}_n$  approaching “energetically”  $\mathcal{H}$  may be settled. We apologize for only presenting examples where we have participated in. Indeed, except two examples in Biology (especially Genetics) proposed in [4, 5, 33], we did not find in the literature any analysis of convergence or approximation using Trotter theory, even after formulating the genuine problem in terms of evolution equation, which, however - we hope the following examples will convince the reader - is particularly efficient and so flexible and simple to use. To such a point that we asked ourself if Trotter theorem of convergence was true! Actually, T. Kato showed in [15] that the proof given by H. F. Trotter in [29] was not complete and he completed it, so that one speaks of Trotter-Kato theorem! Indeed, the genuine proof by H. F. Trotter uses another of his results (this one perfectly correct!) about the exponential approximation formula. But there exists a simpler proof using these two obvious lemmas:

**Lemma 3.1.** *If  $B$  is a  $m$ -dissipative operator in a Hilbert space  $\mathcal{H}$ , and  $S_B$  the semi-group generated by  $B$ , we have*

$$\begin{aligned} B S_B(t)x &= S_B(t) Bx, \quad \forall x \in D(B), \quad \forall t \geq 0 \\ (I - \lambda B)^{-1} S_B(t)x &= S_B(t)(I - \lambda B)^{-1}x, \quad \forall x \in \mathcal{H}, \quad \forall t \geq 0, \quad \forall \lambda > 0 \end{aligned}$$

**Lemma 3.2.** *Let  $A_n$ ,  $A$  be  $m$ -dissipative operators in  $\mathcal{H}_n$  and  $\mathcal{H}$ , and  $S_n$ ,  $S$  the semi-groups associated with, then*

$$\begin{aligned} (I - \lambda A_n)^{-1} \left( P_n S(t) - S_n(t) P_n \right) (I - \lambda A)^{-1} &= \\ &= \frac{1}{\lambda} \int_0^t S_n(t-s) \left( P_n (I - \lambda A)^{-1} - (I - \lambda A_n)^{-1} P_n \right) S(s) ds \end{aligned}$$

with  $P_n$  satysfying  $(T_1)$  and  $(T_2)$ .

So that we prefer to refer to Trotter theory!

We will consider the following examples which illustrate the necessity of considering variable spaces:

- i) gravity surface water waves,
- ii) more general water waves,
- iii) dynamics of linearly piezoelectric plates,
- iv) dynamics of two linearly elastic bodies connected by a thin anelastic layer.

First example is related to modeling by approximation, the others concern convergence. In the first example, an unbounded domain is approached through truncations by a sequence of bounded subdomains. Neglecting a physical effect is the subject of the second (which can also be considered from the point of view of approximation...), hence a state variable disappear at the limit. The third example deals with a coupled multi-physics problem set in variable domains that can be transformed by a change of coordinates into a fixed domain but, consequently, with

variable energy-norms. In the last example, both variable domains and variable mechanical coefficients are present.

Let  $\{e_1, e_2, e_3\}$  be an orthonormal basis of  $\mathbb{R}^3$  assimilated to the euclidean physical space. For all  $\xi = (\xi_1, \xi_2, \xi_3)$  in  $\mathbb{R}^3$ ,  $\widehat{\xi}$  stands for  $(\xi_1, \xi_2)$ . The space of all  $(n \times n)$  symmetric matrices is denoted by  $\mathbb{S}^n$  and equipped with the usual inner product and norm denoted by  $\cdot$  and  $|\cdot|$ , as in  $\mathbb{R}^3$ . For all  $e$  in  $\mathbb{S}^3$ , we set

$$e = \widehat{e} + e^\perp \quad (1)$$

where  $(\widehat{e})_{\alpha\beta} = e_{\alpha\beta}$  and  $(e^\perp)_{\alpha\beta} = 0$ ,  $1 \leq \alpha, \beta \leq 2$ , and  $(\widehat{e})_{i3} = 0$ ,  $(e^\perp)_{i3} = e_{i3}$ ,  $1 \leq i \leq 3$ . For all  $a, b$  in  $\mathbb{R}^3$ ,  $a \otimes_s b$  stands for the symmetrized tensor product of  $a$  by  $b$ . Given a vector space  $E$ , we will denote by  $\text{Lin}(E)$  the space of linear mappings from  $E$  to  $E$ . Moreover, for all subset  $\mathcal{O}$  of  $\mathbb{R}^N$ ,  $\chi_{\mathcal{O}}$  is the characteristic function of  $\mathcal{O}$ . Finally we will use the symbol  $\mathfrak{h}_n$  to denote  $n$ -dimensional Hausdorff measure and the letter  $C$  to introduce various constants which may vary from line to line.

**3.2. Gravity surface water waves.** On the end of the seventies, due to the importance of transient phenomena in Offshore Engineering, the Institut Français du Pétrole sponsored the PhD thesis of Christian Licht devoted to the theoretical and numerical study of the evolution of a system fluid-floating body ([16, 18]). In order to assess physical experiments done in towing tanks and numerical experiments through a finite element method computer code, the essential idea was to approximate the unbounded open real ocean by a sequence of bounded lakes. Here to simplify the presentation we confine to a simplified problem where the body is assumed to be rigid and at rest.

**3.2.1. Setting the problem.** Let  $G \subset \{x \in \mathbb{R}^3; x_3 < 0\}$  be the unbounded domain occupied by the fluid whose boundary is made of three parts:  $S_B$  an unbounded locally Lipschitz-continuous connected surface lying at a positive and bounded distance from  $\{x_3 = 0\}$  represents the bottom of the ocean,  $S_F$  the complementary in  $\{x_3 = 0\}$  of a compact subset of  $\{x_3 = 0\}$  (possibly empty if the body is fully immersed) represents the free surface of the ocean at rest, while  $S_I$  a Lipschitz compact surface included in  $x_3 < 0$  represents the immersed surface of the body at rest. Let  $S_W = S_B \cup S_I$  be the wet surface. If  $\phi$  denotes the potential of velocity in the assumed incompressible fluid and  $\eta$  the free surface elevation, the equations describing the linearized evolution of the fluid read as:

$$(\mathcal{P}) \quad \begin{cases} \Delta \phi = 0 \text{ in } G; \dot{\phi} = -g\eta, \dot{\eta} = \partial_n \phi \text{ on } S_F; \partial_n \phi = 0 \text{ on } S_W \\ \phi|_{t=0} = \phi^0 \text{ in } G, \eta|_{t=0} = \eta^0 \text{ in } S_F \end{cases}$$

where  $\Delta$ ,  $\partial_n$  respectively stand for the Laplacian operator and the normal derivative, while upper dot denotes the derivative with respect to the time  $t$  and  $g$  a positive constant representing the gravity acceleration.

In a beautiful paper [2], J.Thomas Beale showed that the problem may be formulated in the form of an evolution equation in a Hilbert space  $\mathcal{H}$  of possible states with finite mechanical energy governed by a skew-adjoint operator according to:

$$(\mathcal{P}) \quad \begin{cases} \frac{du}{dt} - A u = 0 \text{ in } \mathcal{H} \\ u(0) = u^0 := (\phi^0, \eta^0) \end{cases}$$

$$\mathcal{H} := \left\{ u = (\phi, \eta) \in H_D(G) \times L^2(S_F); \int_G \nabla \phi \cdot \nabla \varphi \, dx = 0, \forall \varphi \in H_{S_F}^1(\Omega) \right\}$$

the Dirichlet space  $H_D(G)$  being the closure of the space of infinitely differentiable functions with compact support in  $\bar{G}$  with respect to the norm  $\left(\int_G |\nabla \phi|^2 dx\right)^{\frac{1}{2}}$  and throughout the paper, for all domains  $\mathcal{O}$  we denote the set of elements of  $H^1(\mathcal{O})$  with vanishing trace on  $\Gamma \subset \partial\mathcal{O}$  by  $H_\Gamma^1(\mathcal{O})$ ,

$$|u|_{\mathcal{H}}^2 = \langle u, u \rangle_{\mathcal{H}} := \int_{\Omega} |\nabla \phi|^2 dx + g \int_{S_F} \eta^2 d\hat{x}$$

$$D(A) = \left\{ u = (\phi, \eta) \in \mathcal{H} ; \partial_n \phi \in L^2(S_F), \exists! B\eta \in H^1(G) \text{ such that} \right.$$

$$\Delta B\eta = 0 \text{ in } G, B\eta|_{S_F} = \eta, \partial_n B\eta = 0 \text{ on } S_W \left. \right\}$$

$$Au = (-gB\eta, \partial_n \phi)$$

Operator  $A$  being obviously closed and conservative:

$$\langle Au, u \rangle_{\mathcal{H}} = \int_{\Omega} -g \nabla B\eta \cdot \nabla \phi dx + g \int_{S_F} \partial_n \phi \cdot \eta d\hat{x} = 0, \forall u \in D(A)$$

and, as straightforwardly

$$\begin{aligned} \bar{u} - A\bar{u} = f, \forall f \in X = \{(f^1, f^2) \in \mathcal{H} ; f^1 \in H^1(G)\} \\ \Leftrightarrow \begin{cases} \phi \in H^1(G) ; \int_G \nabla \phi \cdot \nabla \varphi dx + g \int_{S_F} \phi \cdot \varphi ds \\ \quad = \int_{S_F} (f^1 - g f^2) \cdot \varphi d\hat{x}, \forall \varphi \in H^1(G) \\ \eta = \frac{f^1 - \phi}{g} \end{cases} \end{aligned}$$

$X$  being dense in  $\mathcal{H}$ ,  $A$  is skew-adjoint. Hence  $(\mathcal{P})$  has a unique solution in  $C^1([0, \infty); \mathcal{H}) \cap C^0([0, \infty); D(A))$  if  $u^0$  belongs to  $D(A)$ .

As previously said, we intend to approximate the solution  $u$  to  $(\mathcal{P})$  by truncation of the fluid domain (and of the free surface!).

### 3.2.2. Approximation by truncation.

3.2.2.1. *Problems  $(\mathcal{P}_n)$ .* For  $n$  large enough let be

$$G_n := \{x \in G ; |\hat{x}| < n\}$$

$$S_{F_n} := \{x \in S_F ; |\hat{x}| < n\}$$

$$S_{W_n} := \{x \in S_W ; |\hat{x}| < n\}$$

$$S_n := \{x \in G ; |\hat{x}| = n\}$$

and  $(\mathcal{P}_n)$  be defined with the same equations as  $(\mathcal{P})$  but set in  $G_n$ ,  $S_{F_n}$  and  $S_{W_n}$  with the additional condition on  $S_n$ :

$$\dot{\phi} + \nu \partial_n \phi = 0, \quad \nu \in [0, +\infty]$$

$\nu = 0$  corresponds to the Dirichlet condition  $\phi = 0$  and the fact that the dynamic pressure on  $S_n$  should be hydrostatic,

$0 < \nu < +\infty$  corresponds to a condition of absorption of energy which is more or less done in towing tanks to avoid reflection of waves,



$\nu = +\infty$ , which reads  $\partial_n \phi = 0$ , corresponds to rigid wall for the tank and will generates unwanted reflections.

3.2.2.2. *Entering the framework of Trotter theory*  $(\mathcal{H}_n, P_n, \mathcal{H})$ . First, due to the various conditions on  $S_n$  we introduce two kinds of sequences of Hilbert spaces  $\mathcal{H}_n$  approximating  $\mathcal{H}$  (in the sense of Trotter!). For  $\nu = 0$  we set:

$$\mathcal{H}_n := \left\{ (\phi, \eta) \in H_{S_n}^1(G_n) \times L^2(S_{F_n}) ; \int_{G_n} \nabla \phi \cdot \nabla \varphi \, dx = 0, \forall \varphi \in H_{S_n \cup S_{F_n}}^1(G_n) \right\}$$

For  $0 < \nu \leq +\infty$  we set:

$$\mathcal{H}_n := \left\{ (\check{\phi}, \eta) \in H^1(G_n)/\mathbb{R} \times L^2(S_{F_n}) ; \int_{G_n} \nabla \phi \cdot \nabla \varphi \, dx = 0, \forall \varphi \in H_{S_n \cup S_{F_n}}^1(G_n) \right\}$$

where  $\check{\phi}$  denotes an equivalent class in  $H^1(G_n)/\mathbb{R}$  whose one representative is  $\phi$ . These Hilbert spaces are equipped with the norm

$$|u_n|_n^2 := \int_{G_n} |\nabla \phi_n|^2 \, dx + g \int_{S_{F_n}} |\eta_n|^2 \, d\hat{x}.$$

As previously said the natures of  $\mathcal{H}$  and  $\mathcal{H}_n$  may be very different, which is not exactly the case when  $\nu > 0$  because it is well known that  $H_D(G)$  can be identified as the quotient by  $\mathbb{R}$  of a weighted Sobolev space in  $G$ . Nevertheless we neither have  $\mathcal{H}_n \subset \mathcal{H}$  nor  $\mathcal{H}_n \supset \mathcal{H}$  due to the side condition defining  $\mathcal{H}$  and  $\mathcal{H}_n$ .

Next, for all  $u = (\phi, \eta)$  in  $\mathcal{H}$ ,  $P_n u = (\phi'_n, \eta'_n)$  (or  $(\check{\phi}'_n, \eta'_n)$ ) is naturally defined as follows:

i)  $\phi'_n, \check{\phi}'_n :$

- If  $\nu = 0$ ,  $\int_{G_n} \nabla \phi'_n \cdot \nabla v \, dx = \int_G \nabla \phi \cdot \nabla \tilde{v} \, dx, \forall v \in H_{S_n}^1(G_n)$ ,  $\tilde{v}$  the extension by 0 of  $v$  which belongs to  $H_D(G)$ !
- If  $0 < \nu \leq \infty$ ,  $\phi \mapsto \check{\phi}'_n$  is the extension by continuity of the  $H_D(G)$ -continuous mapping:

$$\phi \in H^1(G) \mapsto \check{\phi}'_n := \widetilde{\phi|_{G_n}} \in H^1(G_n)/\mathbb{R}$$

ii)  $\eta'_n = \eta|_{S_{F_n}}$

Therefore one immediately deduces that  $(\mathcal{H}_n, P_n, \mathcal{H})$  satisfies the conditions  $(T_1)$  and  $(T_2)$  for sequences  $\mathcal{H}_n$  approaching  $\mathcal{H}$  (in the sense of Trotter!) and that

$$|P_n u - u_n|_n \rightarrow 0 \Leftrightarrow \begin{cases} |\nabla \phi - \nabla \phi_n|_{L^2(G_n)} \rightarrow 0, |\eta - \eta_n|_{L^2(S_{F_n})} \rightarrow 0 \\ \text{and} \\ \int_{G_n} |\nabla \phi_n|^2 \, dx + g \int_{S_{F_n}} \eta_n^2 \, d\hat{x} \rightarrow \int_G |\nabla \phi|^2 \, dx + g \int_{S_F} \eta^2 \, d\hat{x} \end{cases}$$

(Trotter convergence)

Note that  $P_n$  is onto but not one-to-one.

Eventually, in the Dirichlet or Neumann cases,  $A_n$  is defined by similar formula as  $A$  but with index  $n$  and conditions  $\phi_n = 0$  or  $\partial_n \phi_n = 0$  on  $S_n$ . In the absorption case, one has  $A_n u_n = (\hat{\psi}_n, \partial_n \phi_n)$  with  $\psi_n \in H^1(G_n)$  such that  $\Delta \psi_n = 0$  in  $G_n$ ,  $\partial_n \psi_n = 0$  on  $S_{W_n}$ ,  $\psi_n = -g\eta_n$  on  $S_{F_n}$  and  $\psi_n = -\nu \partial_n \psi_n$  on  $S_n$ .

Proceeding as for  $A$ , it is straightforward to verify that  $A_n$  is skew-adjoint in the Dirichlet or Neumann cases and  $m$ -dissipative in the absorption case, so that  $(P_n)$

is formally equivalent to

$$(\mathcal{P}_n) \begin{cases} \frac{du_n}{dt} - A_n u_n = 0 \text{ in } \mathcal{H}_n \\ u_n(0) = u_n^0 \end{cases}$$

and has a unique solution in  $C^1([0, +\infty); \mathcal{H}_n) \cap C^0([0, +\infty); D(A_n))$  if  $u_n^0 \in D(A_n)$ .

**3.2.2.3. Is approximation by truncation a good approximation?** A positive answer stems from the claim that uniformly on bounded time intervals  $u_n$  does Trotter converge (i.e. according to definition 2.1) toward  $u$ ! This is implied by

$$\begin{aligned} i) \quad & \forall u^0 \in D(A), \exists u_n^0 \in D(A_n) \text{ such that } |P_n u^0 - u_n^0|_n \rightarrow 0 \\ ii) \quad & \forall f \in X, |P_n(I - A)^{-1}f - (I - A_n)^{-1}P_n f|_n \rightarrow 0 \end{aligned}$$

Point *ii*) yields that point *i*) is satisfied by, for instance,

$$u_n^0 := (I - A_n)^{-1} P_n (I - A)u^0$$

As, for all  $f = (f^1, f^2)$  in  $X$ ,

$$\begin{aligned} \bar{u} = (\bar{\phi}, \bar{\eta}) := (I - A)^{-1}f &\Leftrightarrow \begin{cases} \bar{\phi} \in H^1(G) ; \Delta \bar{\phi} = 0 \text{ in } G \\ \partial_n \bar{\phi} + \frac{1}{g} \bar{\phi} = \frac{f^1}{g} - f^2 \text{ on } S_F \\ \partial_n \bar{\phi} = 0 \text{ on } S_W \\ \bar{\eta} = \frac{f^1 - \bar{\phi}}{g} \end{cases} \\ \bar{u}_n = (\bar{\phi}_n, \bar{\eta}_n) := (I - A_n)^{-1}f_n &\Leftrightarrow \begin{cases} \bar{\phi}_n \in H^1(G) ; \Delta \bar{\phi}_n = 0 \text{ in } G_n \\ \partial_n \bar{\phi}_n + \frac{1}{g} \bar{\phi}_n = \frac{f_n^1}{g} - f_n^2 \text{ on } S_{F_n} \\ \partial_n \bar{\phi}_n = 0 \text{ on } S_{W_n} \\ \nu \partial_n \bar{\phi}_n + \bar{\phi}_n = f_n^1 \text{ on } S_n \\ \bar{\eta}_n = \frac{f_n^1 - \bar{\phi}_n}{g} \end{cases} \end{aligned}$$

point *ii*) comes from standard convergence results on approximation by truncation of elliptic boundary value problems set in unbounded open sets.

Hence are justified transient experiments in towing tanks or numerical experiment through a code using finite element approximations in  $G_n$  and  $S_{F_n}$  (of course fully validated by using again Trotter theory, see [16, 18]). In [16, 18] is quantitatively illustrated the better efficiency of  $0 < \nu < +\infty$  especially with  $\nu = (gd)^{\frac{1}{2}}$  if  $d$  is the assumed constant depth of the ocean outside  $G_{n_0}$  for a fixed  $n_0$ .

**3.3. More general water waves.** In previous example, we did not take the low compressibility of the liquid (salty water) of the ocean into account. It is not fully neglectable and for instance is used by naval officers who listen to the motions of nuclear submarines. Hence we are led to (see [17, 19]):

$$(\mathcal{P}_{gc}) \begin{cases} \dot{\phi} = -\frac{p}{\rho}, \dot{p} = -\rho c^2 \Delta \phi \text{ in } G \\ \dot{\eta} = \partial_n \phi, p = \rho g \eta \text{ on } S_F \\ \partial_n \phi = 0 \text{ on } S_W \\ u^0 = (\phi^0, \eta^0, p^0) \end{cases} \Leftrightarrow \begin{cases} \frac{du_{gc}}{dt} - A_{gc} u_{gc} = 0 \text{ in } \mathcal{H}_{gc} \\ u_{gc}(0) = u_{gc}^0 \in D(A_{gc}) \end{cases}$$

where  $p$ ,  $\rho$  and  $c$  denote the pressure, density and celerity of the sound in the fluid, respectively, and set

$$\begin{aligned}\mathcal{H}_{gc} &:= H_D(G) \times L^2(S_F) \times L^2(G) \\ |u|_{gc}^2 &:= \langle u, u \rangle_{gc} := \rho \int_G |\nabla \phi|^2 dx + \rho g \int_{S_F} \eta^2 d\hat{x} + \frac{1}{\rho c^2} \int_G p^2 dx, \\ &\quad \forall u = (\phi, \eta, p) \in \mathcal{H}_{gc} \\ D(A_{gc}) &= \left\{ u = (\phi, \eta, p) \in \mathcal{H}_{gc} ; p \in H^1(G), \boxed{p = \rho g \eta \text{ on } S_F}, \right. \\ &\quad \left. \Delta \phi \in L^2(G), \partial_n \phi \in L^2(S_F), \partial_n \phi = 0 \text{ on } S_W \right\} \\ A_{gc} u &= \left( -\frac{p}{\rho}, \partial_n \phi, -\rho c^2 \Delta \phi \right)\end{aligned}$$

As it is obvious that  $A_{gc}$  is conservative

$$\begin{aligned}\langle A_{gc} u, u \rangle_{gc} &= \rho \int_\Omega -\frac{\nabla p}{\rho} \cdot \nabla \phi dx + \rho g \int_{S_F} \partial_n \phi \eta d\hat{x} + \frac{1}{\rho c^2} \int_G \Delta \phi p dx \\ &= - \int_{S_F} \rho g \eta \partial_n \phi d\hat{x} + \rho g \int_{S_F} \partial_n \phi \eta d\hat{x} = 0, \quad \forall u \in D(A_{gc})\end{aligned}$$

but also straightforwardly closed and does satisfy

$$\begin{aligned}\bar{u}_{gc} - A_{gc} \bar{u}_{gc} &= f, \quad \forall f \in X_{gc} = \{f = (f^1, f^2, f^3) \in \mathcal{H}_{gc}; f^1 \in H^1(G)\} \\ \Leftrightarrow &\begin{cases} \bar{\phi}_{gc} \in H^1(G) ; -\Delta \bar{\phi}_{gc} + \frac{1}{c^2} \bar{\phi}_{gc} = \frac{1}{c^2} (f^1 - \frac{1}{\rho} f^3) \text{ in } G \\ \partial_n \bar{\phi}_{gc} + \frac{1}{g} \bar{\phi}_{gc} = \frac{f^1}{g} - f^2 \text{ on } S_F \\ \partial_n \bar{\phi}_{gc} = 0 \text{ on } S_W \\ \bar{\eta}_{gc} = \frac{f^1 - \bar{\phi}}{g}, \quad \bar{p}_{gc} = \rho (f^1 - \bar{\phi}_{gc}) \end{cases}\end{aligned}$$

$X_{gc}$  being dense in  $\mathcal{H}_{gc}$ ,  $A_{gc}$  is skew-adjoint. Hence  $(\mathcal{P}_{gc})$  has a unique solution in  $C^1([0, +\infty); \mathcal{H}_{gc}) \cap C^0([0, +\infty); D(A_{gc}))$  if  $u_{gc}^0$  belongs to  $D(A_{gc})$ .

Here we intend to derive a simpler model by neglecting the compressibility effects, that is to say by studying the asymptotic behavior when  $c$  goes to infinity.

**3.3.1. Asymptotic behavior when  $c$  goes to infinity.** If we want to use Trotter theory, here we have to exhibit what could be the limit space  $\mathcal{H}_g$  such that  $\mathcal{H}_{gc}$  be a sequence approximating  $\mathcal{H}_g$  in the sense of Trotter.

**3.3.1.1. Entering the framework of Trotter theory.** From physical reasons it is intuitive to guess what could be  $\mathcal{H}_g$ . However, if it is not, it suffices to consider the behavior of sequences with bounded energy and satisfying  $(\mathcal{P}_{gc})$ : for example, the boundedness of  $|u_{gc}|_{gc}^2$  lets us guess that the limit state  $u_g$  may be described only by  $(\phi_g, \eta_g)$  and that  $|u_g|_g^2 = \rho \int_G |\nabla \phi_g|^2 dx + \rho g \int_{S_F} |\eta_g|^2 d\hat{x}$ . Moreover,  $p_{gc} = -\rho c^2 \Delta \phi_{gc}$  let us guess that  $\Delta \phi_g = 0$ . Hence, the following proposal is in order:

$$\begin{aligned}\mathcal{H}_g &:= \left\{ u = (\phi_g, \eta_g) \in H_D(G) \times L^2(S_F) ; \Delta \phi = 0 \text{ in } G, \partial_n \phi = 0 \text{ on } S_W \right\} \\ |u|_g^2 &:= \rho \int_G |\nabla \phi|^2 dx + \rho g \int_{S_F} \eta^2 d\hat{x}\end{aligned}$$

which is exactly the previous (pure) gravity surface water waves situation!

$P_{gc}$  defined by

$$u_g = (\phi_g, \eta_g) \in \mathcal{H}_g \mapsto P_{gc} u_g := (\phi_g, \eta_g, 0) \in \mathcal{H}_{gc}$$

being isometric, it satisfies the Trotter conditions  $(T_1)$  and  $(T_2)$ .

Note that  $\mathcal{H}_{gc}$  strictly contains  $\mathcal{H}_g \times \{0\}$  and that  $P_{gc}$  is not onto! Moreover one has:

$$|P_{gc} u_g - u_{gc}|_{gc} \rightarrow 0 \Leftrightarrow |\nabla \phi_{gc} - \nabla \phi_g|_{L^2(G; \mathbb{R}^3)}^2 + |\eta_{gc} - \eta_g|_{L^2(S_F)}^2 + \frac{1}{c^2} |p_{gc}|_{L^2(G)}^2 \rightarrow 0$$

(Trotter convergence)

Lastly the expected limit model will involve operator  $A_g$  defined exactly as in the pure gravity water waves case.

**3.3.1.2. Convergence result.** The uniform convergence in the sense of Trotter on bounded time intervals of  $u_{gc}$  toward  $u_g$  will be the consequence of

i) a good choice of  $u_{gc}^0$  (for instance  $u_{gc}^0 = (I - A_{gc})^{-1} P_{gc} (I - A_g) u_g^0$ ) satisfying

$$\lim_{c \rightarrow \infty} |u_{gc}^0 - P_{gc} u_g^0|_{gc} = 0,$$

ii)  $\forall f \in X_g = \left\{ u \in \mathcal{H}_g \text{ such that } \phi \in H^1(G) \right\},$

$$\lim_{c \rightarrow \infty} |P_{gc} (I - A_g)^{-1} f - (I - A_{gc})^{-1} P_{gc} f|_{gc} = 0,$$

which, as

$$\bar{u}_{gc} = (\bar{\phi}_{gc}, \bar{\eta}_{gc}, \bar{p}_{gc}) := (I - A_{gc})^{-1} (f, 0) \Leftrightarrow \begin{cases} -\Delta \bar{\phi}_{gc} + \frac{1}{c^2} \bar{\phi}_{gc} = \frac{1}{c^2} f^1 \text{ in } G; \\ \partial_n \bar{\phi}_{gc} + \frac{1}{g} \bar{\phi}_{gc} = \frac{f^1}{g} - f^2 \text{ on } S_F \\ \partial_n \bar{\phi}_{gc} = 0 \text{ on } S_W \\ \bar{\eta}_{gc} = \frac{f^1 - \bar{\phi}_{gc}}{g}; \bar{p}_{gc} = \rho(f^1 - \bar{\phi}_{gc}) \end{cases}$$

$$\bar{u}_g = (\bar{\phi}_g, \bar{\eta}_g, \bar{p}_g) := (I - A_g)^{-1} f \Leftrightarrow \begin{cases} -\Delta \bar{\phi}_g = 0 \text{ in } G; \\ \partial_n \bar{\phi}_g + \frac{1}{g} \bar{\phi}_g = \frac{f^1}{g} - f^2 \text{ on } S_F \\ \partial_n \bar{\phi}_{gc} = 0 \text{ on } S_W \\ \bar{\eta}_g = \frac{f^1 - \bar{\phi}_g}{g} \end{cases}$$

stems from a standard result on (not even singular) perturbation of elliptic boundary value problems. Moreover [17, 19] the following estimates stand:

i)  $|P_{gc} u_g - u_{gc}|_{gc} \rightarrow 0$  uniformly on bounded time intervals

ii)  $c |\Delta \phi_{gc}|_{L^2(G)} \rightarrow 0$

iii)  $|\phi_g(t) - \phi_{gc}(t)|_{H^1(G)} \leq \frac{C}{c^2} (1 + |t|^3)$

iv)  $|P_{gc} u_g(t) - u_{gc}(t)|_{gc} \leq \frac{C}{c^2} (1 + |t|^2)$

Therefore, Trotter theory permits to qualitatively and quantitatively study the influence of compressibility of water on transient water waves in ocean.

Of course a similar result occurs [17, 19] when “ $g \rightarrow 0$ ” i.e when neglecting the gravity effects with respect to the compressibility ones.

We presented this example through the point of view of “convergence in asymptotic modeling” but it can be also considered from the point of view of approximation. It is the view of mathematicians such as J.-L. Lions in [26, 27]. Noticing that  $(\mathcal{P}_g)$ , already solved by Garipov in [11], was not of Cauchy-Kowaleska type, he introduced an artificial time derivative  $\ddot{\phi}_{gc} = c^2 \Delta \phi_{gc}$  in  $G$  and showed that the well posed problem  $(\mathcal{P}_{gc})$  has a limit when  $c$  goes to infinity which solves  $(\mathcal{P}_g)$ . Note that semi-group theory directly supplies existence as well as uniqueness [2] but also numerical approximation for  $(\mathcal{P}_g)$  [16].

**3.4. Dynamics of linearly piezoelectric plates in the “quasi-electrostatic approximation”.** Thin plates are extensively used in technological applications and a large amount of studies have been devoted to their transient responses to given loadings. However the main difficulty lies in quantitative analysis through numerical computations because of the rather low thickness of these structures. So asymptotic mathematical modeling may be useful in supplying a simpler but accurate enough model by studying the asymptotic behavior when the thickness, considered as a parameter, tends to zero. In [28] it has been shown that the problem for purely elastic plates was well posed through a quite long variational evolution equations technique. Even if the same point was proved in [3] for linearly thermoelastic plates by a semi-groups technique, the method used in this paper to get strong convergence results follows that of [28]. It is based on the convergence of the norms in the Hilbert space  $L^2(0, T; L^2(\Omega))$  and involves several pages of estimations and convergence of integrals. We show in this section that Trotter theory provides the same kind of result without much efforts comparing to the static case.

**3.4.1. Setting the problem.** Here we consider linearly piezoelectric thin plates on which lives a physical state  $(u, \varphi)$  which is a couple (displacement field, electrical potential). As usual, we proceed to a change of coordinates denoted by  $S'_\varepsilon$  (see section 3.5.3) in order to transform transient problem posed on the variable domain  $\Omega^\varepsilon = \omega \times (-\varepsilon, \varepsilon)$  occupied by the true physical plate into a “scaled problem” posed on a fixed domain  $\Omega = \omega \times (-1, 1)$  where  $\omega$  denotes a domain of  $\mathbb{R}^2$  with a Lipschitz-continuous boundary. The thickness and the density of the genuine plate are the two main parameters of this scaled problem and are denoted by a couple  $\eta = (\varepsilon, \rho)$ . We therefore have to study

$$(\mathcal{P}_\eta) \left\{ \begin{array}{l} \text{Find a smooth enough physical state } (u_\eta, \varphi_\eta) \text{ such that:} \\ \int_{\Omega} \left( \rho(\ddot{u}_\eta \cdot \widehat{v'} + \frac{1}{\varepsilon^2} \ddot{u}_{\eta_3} v'_3) + M(x) k_p(\varepsilon, u_\eta, \varphi_\eta) \cdot k_p(\varepsilon, v', \varphi') \right) dx \\ \quad = L((v', \varphi')), \quad \forall (v', \varphi') \in H_{\Gamma_{mD}}^1(\Omega; \mathbb{R}^3) \times H_{\Gamma_{eD}}^1(\Omega; \mathbb{R}) \\ (u_\eta, \dot{u})|_{t=0} = (u_\eta^0, v_\eta^0) \\ M(x) = \begin{bmatrix} a & -b \\ b^T & c \end{bmatrix} \in L^\infty(\Omega; \text{Lin}(\mathbb{K})), \quad M k \cdot k \geq c |k|_{\mathbb{K}}^2, \quad \forall k \in \mathbb{K} := \mathbb{S}^3 \times \mathbb{R}^3 \end{array} \right.$$

with

$$k_p(\varepsilon, v', \varphi') = \left( e'(\varepsilon, v'), \nabla_p(\varepsilon, \varphi') \right) \\ \widehat{e'(\varepsilon, v')} = \widehat{e(v')}$$

$$\begin{aligned}
\left(e'(\varepsilon, v')^\perp\right)_{\alpha 3} &= \frac{1}{\varepsilon} \left(e(v')\right)_{\alpha 3}, \quad 1 \leq \alpha \leq 2, \quad \left(e'(\varepsilon, v')^\perp\right)_{33} = \frac{1}{\varepsilon^2} e(v')_{33}, \\
e(v') &= (\nabla v')_{\text{sym}} \text{ the strain associated with the displacement } v' \\
\widehat{\nabla_p(\varepsilon, \varphi')} &= \varepsilon^{p-1} \widehat{\nabla \varphi'}, \quad \left(\nabla_p(\varepsilon, \varphi')\right)_3 = \varepsilon^{p-2} \partial_3 \varphi', \quad p = 1, 2
\end{aligned}$$

For the clarity of the exposure, we confine to a vanishing electromechanical loading; the non-vanishing case can be easily handled by a suitable decomposition involving the steady state response (see [30, 32]).

As explained in [30, 31], the “quasi-static approximation” means that the electrical potential may be neglected through the introduction of the operator  $\mathcal{S}_\varepsilon$ :

$$\begin{aligned}
\varphi_\eta &= \mathcal{S}_\varepsilon u_\eta \\
\begin{cases} \mathcal{S}_\varepsilon u_\eta \in H_{\Gamma_{eD}}^1(\Omega) \text{ such that} \\ m_p(\varepsilon)((u_\eta, \mathcal{S}_\varepsilon u_\eta), (0, \varphi')) = 0, \quad \forall \varphi' \in H_{\Gamma_{eD}}^1(\Omega), \quad u_\eta \in H_{\Gamma_{mD}}^1(\Omega; \mathbb{R}^3), \end{cases} \\
m_p(\varepsilon)((u, \varphi), (u', \varphi')) &:= \int_\Omega M(x) k_p(\varepsilon, u, \varphi) \cdot k_p(\varepsilon, u', \varphi') dx, \\
\forall (u, \varphi), (u', \varphi') &\in H_{\Gamma_{mD}}^1(\Omega; \mathbb{R}^3) \times H_{\Gamma_{eD}}^1(\Omega)
\end{aligned}$$

so that  $(\mathcal{P}_\eta)$  reads as an abstract (more precisely non-local) problem in elastodynamics equivalent to the usual one:

$$(\mathcal{P}_\eta) \quad \begin{cases} \mathcal{E}_\varepsilon(u_\eta, v') + \mathcal{K}_\eta(\ddot{u}_\eta, v') = 0, \quad \forall v' \in H_{\Gamma_{mD}}^1(\Omega; \mathbb{R}^3) \\ u_\eta(0) = u^0 \end{cases}$$

with

$$\mathcal{E}_\varepsilon(u, v') := m_p(\varepsilon)((u, \mathcal{S}_\varepsilon u), (v', 0))$$

an inner product on  $H_{\Gamma_{mD}}^1(\Omega; \mathbb{R}^3)$  (due to the structure of  $M$ !) and

$$\mathcal{K}_\eta(v, v') := \int_\Omega \rho(\widehat{v} \cdot \widehat{v'} + \frac{1}{\varepsilon^2} v_3 v'_3) dx$$

In a standard way,  $(\mathcal{P}_\eta)$  can be formulated in an evolution equation in a Hilbert space  $\mathcal{H}_\eta$  of possible states with finite energy governed by a skew-adjoint operator  $A_\eta$ :

$$\begin{aligned}
\mathcal{H}_\eta &:= H_{\Gamma_{mD}}^1(\Omega; \mathbb{R}^3) \times L^2(\Omega; \mathbb{R}^3) \\
|U|_\eta^2 &= \langle U, U \rangle_\eta \quad \text{with} \\
\langle U, U' \rangle_\eta &:= \mathcal{E}_\varepsilon(u, u') + \mathcal{K}_\eta(v, v'), \quad \forall U = (u, v), U' = (u', v') \in \mathcal{H}_\eta \\
D(A_\eta) &= \left\{ U_\eta = (u_\eta, v_\eta) \in \mathcal{H}_\eta \text{ such that} \right. \\
&\quad i) \quad v_\eta \in H_{\Gamma_{mD}}^1(\Omega; \mathbb{R}^3), \\
&\quad ii) \quad \exists! z_\eta \in L^2(\Omega; \mathbb{R}^3); \quad \langle (u_\eta, z_\eta), (v', v') \rangle_\eta = 0, \quad \forall v' \in H_{\Gamma_{mD}}^1(\Omega; \mathbb{R}^3) \left. \right\} \\
A_\eta U_\eta &= (v_\eta, z_\eta)
\end{aligned}$$

because  $(\mathcal{P}_\eta)$  is equivalent to

$$\begin{cases} \frac{du_\eta}{dt} - v_\eta = 0 \\ \langle (u_\eta, \frac{dv_\eta}{dt}), (v', v') \rangle_\eta = 0, \quad \forall v' \in H_{\Gamma_{mD}}^1(\Omega; \mathbb{R}^3) \end{cases}$$

Note that  $A_\eta$  is  $m$ -conservative as

$$\forall U_\eta \in D(A_\eta) \quad \langle A_\eta U_\eta, U_\eta \rangle_\eta = \langle (v_\eta, z_\eta), (u_\eta, v_\eta) \rangle_\eta = \langle (u_\eta, z_\eta), (v_\eta, v_\eta) \rangle_\eta = 0!$$

and

$$\begin{aligned} \bar{U}_\eta - A_\eta \bar{U}_\eta = F_\eta &\Leftrightarrow \begin{cases} \bar{u}_\eta = \bar{v}_\eta + F_\eta^1 \\ \langle (\bar{v}_\eta + F_\eta^1, \bar{v}_\eta - F_\eta^2), (v, v') \rangle_\eta = 0 \end{cases} \\ &\Leftrightarrow \begin{cases} \bar{u}_\eta = \bar{v}_\eta + F_\eta^1 \\ \bar{v}_\eta \text{ minimizes } \frac{1}{2} |(v, v)|_\eta^2 + \langle (F_\eta^1, -F_\eta^2), (v, v) \rangle_\eta \end{cases} \\ &\hspace{15em} \text{on } H_{\Gamma_{mD}}^1(\Omega; \mathbb{R}^3) \end{aligned}$$

**3.4.2. Convergence when  $\eta \rightarrow 0$ .** For brevity, we confine to the case  $\rho \sim \varepsilon^2$ , the most treated case in the litterature and refer to [30, 32] for the other cases (interesting in fact from the physical point of view). Hence one has

$$|(u, v)|_\eta^2 \sim \int_\Omega m_p(\varepsilon) ((u, \mathcal{S}_\varepsilon u), (u, 0)) dx + \int_\Omega (\varepsilon^2 |\widehat{v}|^2 + |v_3|^2) dx \quad (2)$$

**3.4.2.1. Entering the framework of Trotter theory**

**3.4.2.1.1.  $(\mathcal{H}_n, P_\eta, \mathcal{H})$ .** Due to 2 and results in the static cases, it is natural to propose:

$$\begin{aligned} \mathcal{H} &:= V_{KL}^F(\Omega) \times L_3^2(\Omega) \\ V_{KL}(\Omega) &:= \{u \in H_{\Gamma_{mD}}^1(\Omega; \mathbb{R}^3) ; e(u)^\perp = 0\} \\ V_{KL}^F(\Omega) &:= \left\{ u \in H_{\Gamma_{mD}}^1(\Omega; \mathbb{R}^3) ; u_3 \in H^2(\omega), \widehat{u}(x) = -x_3 \widehat{\nabla} u_3 \right\} \quad \text{the subspace of} \\ &\quad \text{Kirchhoff-Love displacements with vanishing membrane component} \\ &\quad \text{displacements} \\ L_3^2(\Omega) &:= \left\{ v \in H^{-1}(\Omega; \mathbb{R}^3) ; v_3 \in L^2(\omega), \widehat{v}(x) = -x_3 \widehat{\nabla} v_3 \right\} \end{aligned}$$

$$|(u, v)|^2 = \langle (u, v), (u, v) \rangle := \tilde{m}_p((u, \mathcal{S} u), (u, 0)) + \int_\Omega |v_3|^2 dx$$

where  $\tilde{m}_p((u, \mathcal{S} u), (u, 0))$  is the limit elastic energy involved in the limit static case (see [30, 32])<sup>3</sup>.

Note that  $\mathcal{H} \not\subset \mathcal{H}_\eta$  and  $\mathcal{H}_\eta \not\subset \mathcal{H}$  while  $P_\eta$  is defined in the following variational way:

$$P_\eta U \in \mathcal{H}_\eta ; \langle P_\eta U, U' \rangle_\eta = \langle U, U' \rangle, \quad \forall U' \in \mathcal{H}_\eta.$$

---

<sup>3</sup>If  $u_\eta$  in  $H_{\Gamma_{mD}}^1(\Omega; \mathbb{R}^3)$  satisfies  $m_p(\varepsilon)((u_\eta, \mathcal{S}_\varepsilon u_\eta), (v', 0)) = L(v')$ , for all  $v'$  in  $H_{\Gamma_{mD}}^1(\Omega; \mathbb{R}^3)$  then  $u_\eta$  converges in  $H^1(\Omega; \mathbb{R}^3)$  toward  $u$  in  $V_{KL}(\Omega)$  such that

$$\tilde{m}_p((u, \mathcal{S} u), (u, 0)) = L(v'), \quad \forall v' \in V_{KL}(\Omega)$$

and

$$\tilde{m}_p((u, \mathcal{S} u), (0, \varphi')) = 0, \quad \forall \varphi' \text{ admissible}$$

An immediate consequence of the study of the static case is that  $(P_\eta, \mathcal{H}_\eta, \mathcal{H})$  satisfies  $(T_1)$  and  $(T_2)$  and that

$$\text{Trotter convergence} \Leftrightarrow \begin{cases} (u_\eta, v_{\eta_3}) \rightarrow (u, v) \text{ in } H_{\Gamma_{mD}}^1(\Omega; \mathbb{R}^3) \times L^2(\Omega; \mathbb{R}^3) \\ |U_\eta|_\eta^2 \rightarrow |U|^2 \end{cases}$$

Lastly, operator  $A$  is defined by formulae similar to those of  $A_\eta$  because

$$\langle (u, \ddot{u}), (v', v') \rangle = 0, \forall v' \in V_{KL}^F(\Omega)$$

3.4.2.1.2. *Convergence of  $U_\eta(t)$  toward  $U(t)$ .* If we assume that

$$\exists U^0 \in D(A) \text{ such that } |P_\eta U^0 - U_\eta^0|_\eta \rightarrow 0,$$

to prove that  $U_\eta$  converges uniformly on bounded time intervals toward the solution to

$$(\mathcal{P}) \begin{cases} \frac{dU}{dt} - AU = 0 \text{ in } \mathcal{H} \\ U(0) = U^0 \end{cases}$$

it remains to establish

$$\lim_{\eta \rightarrow 0} |P_\eta \bar{U} - \bar{U}_\eta|_\eta = 0 \quad (3)$$

where

$$\begin{cases} \bar{U}_\eta = (\bar{u}_\eta, \bar{v}_\eta) := (I - A_\eta)^{-1} P_\eta F \\ \bar{U} = (\bar{u}, \bar{v}) := (I - A)^{-1} F \end{cases} \Leftrightarrow \begin{cases} \bar{u}_\eta = \bar{v}_\eta + F_\eta^1, \bar{v}_\eta \text{ minimizes} \\ \frac{1}{2} |(v, v)|_\eta^2 + \langle F^1 - F^2, (v, v) \rangle_\eta \\ \bar{u} = \bar{v} + F^1, \bar{v} \text{ minimizes} \\ \frac{1}{2} |(v, v)|^2 + \langle F^1 - F^2, (v, v) \rangle \end{cases}$$

But our good choices of  $|\cdot|$  and of  $P_\eta$  obviously implies:

$$\begin{aligned} |\bar{U}_\eta - P_\eta U|_\eta^2 &= |\bar{U}_\eta|^2 - 2 \langle P_\eta \bar{U}, \bar{U}_\eta \rangle_\eta + |P_\eta \bar{U}|_\eta^2 \\ &= |\bar{U}_\eta|^2 - 2 \langle \bar{U}, \bar{U}_\eta \rangle + |P_\eta \bar{U}|_\eta^2 \end{aligned}$$

First, condition  $(T_2)$  implies that  $|P_\eta \bar{U}|_\eta^2$  converges toward  $|\bar{U}|^2$ , while the study of the stationary problem yields  $\lim_{\eta \rightarrow 0} |\bar{U}_\eta|^2_\eta - 2 \langle \bar{U}, \bar{U}_\eta \rangle_\eta = |\bar{U}|^2 - 2 \langle \bar{U}, \bar{U} \rangle$  which establishes 3. When  $\eta$  goes to zero, we then have that  $(u_\eta(t), v_{\eta_3}(t)) \rightarrow (u(t), v_3(t))$  in  $H^1(\Omega; \mathbb{R}^3) \times L^2(\Omega; \mathbb{R}^3)$  and  $|U_\eta(t)|_\eta \rightarrow |U(t)|$  uniformly on bounded time intervals, so that  $(\mathcal{P})$  supplies a simplified and accurate model involving essentially bidimensional mid surface  $\omega$  and only flexural motions.

**3.5. Dynamics of two linearly elastic bodies connected by a thin anelastic layer.** The proper choice of an adequate bonding is essential for a productive engineering process. It is therefore important to convey a good mathematical modeling to ensure a high quality of the resulting structure. As in the previous example, the main difficulty stems from numerical aspects due to the meshing of the very thin layer occupied by the adhesive and its very different mechanical behavior from the ones of the adherents which yields very ill-conditioned systems. Our asymptotic modeling will replace the very thin adhesive by either a material surface or a mechanical constraint along the surface the adhesive layer shrinks to.



If  $\Omega$  denotes the reference configuration of the structure made of two adherents and the adhesive, let  $\Omega_\varepsilon^\pm = \{x \in \Omega; \pm x_3 > \varepsilon\}$ ,  $B_\varepsilon = \{x \in \Omega; |x_3| < \varepsilon\}$  be the domains occupied by each adherent and the adhesive, respectively ; we set  $S = \{x \in \Omega; x_3 = 0\}$ . Four data are essential:

- i)  $\varepsilon$ , the low thickness of the adhesive layer,
- ii)  $\mu$ , the stiffness of the adhesive in the sense that its strain energy density is  $\mu W_I(e(u))$ , with  $C_m|e(u)|^2 \leq W_I(e(u)) \leq C_M|e(u)|^2$  and  $C_m, C_M$  fixed positive numbers,
- iii)  $b$ , the dissipation coefficient of the adhesive in the sense that the density of the dissipation potential in the adhesive is  $b \mathcal{D}(e(\dot{u}))$  with  $C_m|e(\dot{u})|^p \leq \mathcal{D}(e(\dot{u})) \leq C_M|e(\dot{u})|^p$ ,  $1 \leq p \leq 2$ ,
- iv)  $\rho$ , the density of the adhesive.

As usual our asymptotic modeling will be obtained by considering these data as parameters and studying the asymptotic behavior when  $s = (\varepsilon, \mu, b, \rho)$  goes to its natural limit  $\bar{s}$  will be the occurrence to use our nonlinear extension of Trotter theory (cf. [14]).

We assume that the structure is clamped on  $\Gamma_0$ ,  $\mathfrak{h}_2(\Gamma_0) > 0$  and  $\text{dist}(\Gamma_0, S) > \varepsilon_0$  fixed. One more time, to go to the heart of the matter, we confine to the autonomous problem, *i.e.* without loading, which reads as:

$$(\mathcal{P}_s) \begin{cases} \int_{\Omega_\varepsilon} \gamma \ddot{u}_s \cdot v' dx + \rho \int_{B_\varepsilon} \ddot{u}_s \cdot v' dx + \int_{\Omega_\varepsilon} a e(u_s) \cdot e(v') dx \\ \quad + \int_{B_\varepsilon} (\mu a_I(e(u_s)) + b \zeta_s) \cdot e(v') dx = 0, \forall v' \in H_{\Gamma_0}^1(\Omega; \mathbb{R}^3) \\ (u_s, v_s)(0) = U_s^0 := (u_s^0, v_s^0) \end{cases}$$

where  $\gamma \in L^\infty(\Omega)$  with  $\gamma > C_m > 0$ ,  $a \in L^\infty(\Omega; \text{Lin}(\mathbb{S}^3))$ ,  $a_I \in L^\infty(S; \text{Lin}(\mathbb{S}^3))$ ,  $a \xi \cdot \xi \geq C_m |\xi|^2$ ,  $a_I \xi \cdot \xi \geq 2W_I(\xi)$ , for all  $\xi$  in  $\mathbb{S}^3$ ,  $\zeta_s \in \partial \mathcal{D}(e(\dot{u}_s))$ . The problem  $(\mathcal{P}_s)$  is formally equivalent to the nonlinear evolution equation

$$\begin{cases} \frac{dU_s}{dt} + A_s U_s \ni 0 \text{ in } \mathcal{H}_s \\ U_s(0) = U_s^0 \end{cases}$$

with

$$\begin{aligned} \mathcal{H}_s &:= H_{\Gamma_0}^1(\Omega; \mathbb{R}^3) \times L^2(\Omega; \mathbb{R}^3) \\ |U_s|_s^2 &= \langle (u_s, v_s), (u_s, v_s) \rangle_s := \mathcal{E}_{\varepsilon\mu}(u_s, u_s) + \mathcal{K}_{\varepsilon\rho}(v_s, v_s) \\ \mathcal{E}_{\varepsilon\mu}(u, u') &:= \int_{\Omega_\varepsilon} a e(u) \cdot e(u') dx + \mu \int_{B_\varepsilon} a_I e(u) \cdot e(u') dx \\ \mathcal{K}_{\varepsilon\rho}(v, v') &:= \int_{\Omega_\varepsilon} \gamma v \cdot v' dx + \rho \int_{B_\varepsilon} v \cdot v' dx \\ D(A_s) &= \left\{ U_s \in \mathcal{H}_s ; i) v_s \in H_{\Gamma_0}^1(\Omega; \mathbb{R}^3), \right. \\ &\quad ii) \exists z_s \in L^2(\Omega; \mathbb{R}^3), \exists \xi_s \in \partial \mathcal{D}(e(v_s)) \text{ such that} \\ &\quad \left. \langle (u_s, z_s), (v', v') \rangle_s + b \int_{B_\varepsilon} \xi_s \cdot e(v') dx = 0 \right\} \\ A_s U_s &= (-v_s, 0) + \{(0, -z_s) \text{ satisfying } ii)\} \end{aligned}$$

It is straightforward to verify that  $A_s$  is maximal monotone and

$$\bar{U}_s + A_s \bar{U}_s \ni F_s \Leftrightarrow \begin{cases} \bar{u}_s = \bar{v}_s + F_s^1 \\ \bar{v}_s \text{ minimizes } \frac{1}{2}|(v, v)|_s^2 + b \int_{B_\varepsilon} \mathcal{D}(e(v)) dx \\ \quad + \langle F_s^1 - F_s^2, (v, v) \rangle_s \text{ on } H_{\Gamma_0}^1(\Omega; \mathbb{R}^3) \end{cases}$$

so that  $(\mathcal{P}_s)$  has a unique solution in  $W^{1,\infty}([0, +\infty); \mathcal{H}_s)$  whose limit behavior will be studied in the following three cases:

- i) soft (actually not too hard) and light adhesive :  $\mu\varepsilon \rightarrow 0$ ,  $\rho\varepsilon \rightarrow 0$ , cf. [24],
- ii) soft (actually not too hard) and heavy adhesive :  $\mu\varepsilon \rightarrow 0$ ,  $\rho\varepsilon \rightarrow 1$ , cf. [6],
- iii) hard and light adhesive :  $\lim \mu\varepsilon \in (0, +\infty]$ ,  $\rho\varepsilon \rightarrow 0$ , cf. [25].

3.5.1. *Light and not too hard adhesive.* For the sake of brevity we confine to the subcase  $\mu \sim 2\bar{\mu}\varepsilon$ ,  $\bar{\mu} \in (0, +\infty)$  with  $\rho \leq C$  so that

$$\begin{aligned} \mathcal{E}_{\varepsilon\mu}(w_s, w_s) &\sim \int_{\Omega_\varepsilon} a e(w_s) \cdot e(w_s) dx + 2\bar{\mu}\varepsilon \int_{B_\varepsilon} a_I e(w_s) \cdot e(w_s) dx \\ \mathcal{K}_{\varepsilon\rho}(w_s, w_s) &= \int_{\Omega_\varepsilon} \gamma |w_s|^2 dx + \rho \int_{B_\varepsilon} |w_s|^2 dx \end{aligned}$$

3.5.1.1. *Finding  $\mathcal{H}$  and  $P_s$ .* To enter the framework of Trotter theory, that is to say finding  $\mathcal{H}$  and  $P_s$  satisfying  $(T_1)$  and  $(T_2)$ , it suffices to systematically examine the behavior of sequences with uniformly bounded energies:

a)  $\mathcal{E}_{\varepsilon\mu}(w_s, w_s) \leq C$  implies

- i) there exists  $w$  in  $H_{\Gamma_0}^1(\Omega \setminus S; \mathbb{R}^3)$  such that up to a not relabeled sequence  $T_\varepsilon w_s$  weakly converges toward  $w$  in  $H_{\Gamma_0}^1(\Omega \setminus S; \mathbb{R}^3)$ , where  $(T_\varepsilon w_s)(x) = \zeta_0(x_3)w_s(x) + (1 - \zeta_0)(x_3)w_s(x, x_3 \pm \varepsilon)$  (as soon as  $\pm x_3 > 0$ ) with  $\zeta_0$  an element of  $\mathcal{D}(\mathbb{R})$  such that  $\zeta_0(x_3) = 1$  if  $|x_3| \geq 2\varepsilon_0$ ,  $\zeta_0(x_3) = 0$  if  $|x_3| \leq \varepsilon_0$ , and  $\int_{\Omega} a e(w) \cdot e(w) dx \leq \lim_{s \rightarrow \bar{s}} \int_{\Omega_\varepsilon} a e(w_s) \cdot e(w_s) dx$ ,

- ii) there exists  $j \in L^2(S; \mathbb{S}^3)$  such that up to a not relabeled sequence  $\int_{-\varepsilon}^\varepsilon e(w_s) dx_3$  weakly converges in  $L^2(S; \mathbb{S}^3)$  toward  $j$ , classically identified as  $[w] \otimes_s e_3$ ,  $[w] := \gamma_S(w^+) - \gamma_S(w^-)$ ,  $\gamma_S(w^\pm)$  the trace on  $S$  of the restriction  $w^\pm$  of  $w$  to  $\{\pm x_3 > 0\}$ , by going to the limit in  $\int_{B_\varepsilon} \tau(\hat{x}) \cdot e(w_s) dx$ ,  $\tau$  arbitrary in  $\mathcal{D}(S)$ . Hence  $\bar{\mu} \int_S W_I([w] \otimes_s e_3) dx \leq \lim_{s \rightarrow \bar{s}} \bar{\mu} \int_{B_\varepsilon} W_I(e(w_s)) dx$ .

- b)  $\mathcal{K}_{\varepsilon\rho}(w_s, w_s) \leq C$  implies that there exists  $w$  in  $L^2(\Omega; \mathbb{R}^3)$  such that up to a not relabeled sequence  $\chi_{\Omega_\varepsilon} w$  weakly converges in  $L^2(\Omega; \mathbb{R}^3)$  toward  $w$  with  $\int_{\Omega} \gamma |w|^2 dx \leq \lim_{s \rightarrow \bar{s}} \mathcal{K}_{\varepsilon\rho}(w_s)$ .

We therefore propose

$$\begin{aligned} \mathcal{H} &:= H_{\Gamma_0}^1(\Omega \setminus S; \mathbb{R}^3) \times L^2(\Omega; \mathbb{R}^3) \\ |(u, v)|^2 &= \langle (u, v), (u, v) \rangle := \mathcal{E}(u, u) + \mathcal{K}(v, v) \end{aligned}$$

$$\mathcal{E}(u, u') := \int_{\Omega \setminus S} ae(u) \cdot e(u') dx + \bar{\mu} \int_S a_I([u] \otimes_s e_3) \cdot ([u'] \otimes_s e_3) d\hat{x}$$

$$\mathcal{K}(v, v') := \int_{\Omega} \gamma v \cdot v' dx$$

$$P_s U = (u'_s, v'_s) \text{ such that}$$

a)  $v \in L^2(\Omega; \mathbb{R}^3) \mapsto v'_s = v$  (or any other obvious variant!)

b)  $u'_s$  in  $H_{\Gamma_0}^1(\Omega; \mathbb{R}^3)$  may be defined in two ways:

i) smoothing:

$$u \in H_{\Gamma_0}^1(\Omega \setminus S; \mathbb{R}^3) \mapsto u'_s := \chi_{B^\varepsilon} (u^{\text{sym}} + \text{Min}(1, \frac{|x_3|}{\varepsilon}) u^{\text{skew}}) + \chi_{\Omega^\varepsilon} u,$$

with  $2u^{\text{sym}} = u(\hat{x}, x_3) + u(\hat{x}, -x_3)$ ,  $2u^{\text{skew}} = u(\hat{x}, x_3) - u(\hat{x}, -x_3)$

ii) variational:

$$u \in H_{\Gamma_0}^1(\Omega \setminus S; \mathbb{R}^3) \mapsto u'_s \text{ s.t. } \mathcal{E}_{\varepsilon\mu}(u'_s, u') = \mathcal{E}(u, u'), \forall u' \in H_{\Gamma_0}^1(\Omega; \mathbb{R}^3)$$

Clearly such  $(\mathcal{H}_s, P_s, \mathcal{H})$  satisfies  $(T_1)$  and  $(T_2)$  and Trotter convergence is equivalent to

$$|e(u_s - u)|_{L^2(\Omega_\varepsilon; \mathbb{S}^3)}^2 + \left| \int_{-\varepsilon}^{\varepsilon} e(u_s) dx_3 - [u] \otimes_s e_3 \right|_{L^2(S)}^2 + |v_s - v|_{L^2(\Omega_\varepsilon)}^2 \rightarrow 0$$

3.5.1.2. *Operator A, the limit problem.* This operator will have the same structure as  $A_s$  but with  $\langle \cdot, \cdot \rangle_s$  replaced by  $\langle \cdot, \cdot \rangle$  and  $b\mathcal{D}$  by  $\bar{b}\bar{\mathcal{D}}$  where  $\bar{b} = \lim_{\varepsilon \rightarrow 0} \frac{b}{2\varepsilon^{p-1}}$ ,

$\bar{\mathcal{D}}(\dot{e}) = \overline{\lim}_{\tau \rightarrow \infty} \frac{\mathcal{D}(\tau \dot{e})}{\tau^p}$ . Hence the guessed problem

$$(\mathcal{P}) \begin{cases} \frac{dU}{dt} - AU = 0 \text{ in } \mathcal{H} \\ U(0) = U^0 \end{cases}$$

describes the dynamics of two sole adherents linked along  $S$  according to

$$-\sigma e_3 \in \bar{\mu} a_I([u] \otimes_s e_3) + \bar{b} \partial \bar{\mathcal{D}}([\dot{u}] \otimes_s e_3)$$

where  $\sigma$  denotes the stress tensor in the adherents.

The convergence in the sense of Trotter uniformly on bounded time intervals stems from the fact that the resolvents involves minimizations of linear perturbations of the total energy functionals  $\mathcal{E}_{\varepsilon\mu} + \mathcal{K}_{\varepsilon\rho}$  which variationally (more precisely Mosco-) converges toward  $\mathcal{E} + \mathcal{K}$  and the assumption that there exists  $U^0 \in \mathcal{H}$  such that  $\lim_{s \rightarrow \bar{s}} |P_s U^0 - U_s^0|_s = 0$ . This is again an accurate and simplified model since it avoids the meshing of the adhesive layer.

3.5.2. *Heavy and not too hard adhesive.* Again, for the sake of brevity, we confine to the more interesting case when  $\rho \sim \frac{\bar{\rho}}{\varepsilon}$ ,  $\mu \sim 2\bar{\mu}\varepsilon$ ;  $\bar{\rho}, \bar{\mu} \in (0, +\infty)$ , so that  $\mathcal{E}_{\varepsilon\mu}$  and  $\mathcal{K}_{\varepsilon\rho}$  read as:

$$\mathcal{E}_{\varepsilon\mu}(u, u) \sim \int_{\Omega_\varepsilon} ae(u) \cdot e(u) dx + 2\bar{\mu}\varepsilon \int_{B_\varepsilon} a_I e(u) \cdot e(u) dx$$

$$\mathcal{K}_{\varepsilon\rho}(v, v) \sim \int_{\Omega_\varepsilon} \gamma |v|^2 dx + \frac{\bar{\rho}}{\varepsilon} \int_{B_\varepsilon} |v|^2 dx$$

$\mathcal{K}_{\varepsilon\rho}(v_s, v_s) \leq C$  implies that

i) there exists  $v^\Omega$  in  $L^2(\Omega; \mathbb{R}^3)$  such that  $\chi_{\Omega_\varepsilon} v_s$  weakly converges in  $L^2(\Omega; \mathbb{R}^3)$  to  $v$ ,

ii) if  $\mathcal{S}_\varepsilon$  is the mapping defined by

$$w \in L^2(B_\varepsilon; \mathbb{R}^3) \mapsto \mathcal{S}_\varepsilon w \in L^2(B; \mathbb{R}^3), \quad B := S \times (-1, 1)$$

$$(\mathcal{S}_\varepsilon w)(\hat{x}, x_3) = w(\hat{x}, \frac{x_3}{\varepsilon})$$

then  $\mathcal{K}_{\varepsilon\rho}(v_s, v_s) \sim \int_{\Omega_\varepsilon} \gamma |v_s|^2 dx + \bar{\rho} \int_B |\mathcal{S}_\varepsilon v_s|^2 dx$  so that there exists  $v^B$  in  $L^2(B; \mathbb{R}^3)$  such that  $\mathcal{S}_\varepsilon v_s$  weakly converges in  $L^2(B; \mathbb{R}^3)$  toward  $v^B$ .

The limit kinetic state has therefore to be described by *two fields*  $(v^\Omega, v^B)$  in  $L^2(\Omega; \mathbb{R}^3) \times L^2(B; \mathbb{R}^3)$ . But as  $v_s = \frac{du_s}{dt}$ , one must introduce  $\mathcal{S}_\varepsilon u_s$  also. Moreover we have

$$2\bar{\mu}\varepsilon \int_{B_\varepsilon} |e(u_s)|^2 dx = \bar{\mu} \int_B |e(\varepsilon, \mathcal{S}_\varepsilon u_s)|^2 dx$$

where

$$e_{\alpha\beta}(\varepsilon, w) = \varepsilon e_{\alpha\beta}(w), \quad e_{\alpha 3}(\varepsilon, w) = \frac{1}{2}(\varepsilon \partial_\alpha w_3 + \partial_3 w_\alpha), \quad e_{33}(\varepsilon, w) = \partial_3 w_3$$

so that the boundedness of  $\mathcal{E}_{\varepsilon\mu}(u_s, u_s)$  implies:

i)  $\mathcal{S}_\varepsilon u_s$  converges weakly to some  $u^B$  in  $H_{\partial_3}(B; \mathbb{R}^N) = \left\{ u \in L^2(B; \mathbb{R}^N); \partial_3 u \in L^2(B; \mathbb{R}^N) \right\}$ , for  $N = 3$ ,

ii)  $e(\varepsilon, \mathcal{S}_\varepsilon u_s)$  weakly converges in  $L^2(B; \mathbb{S}^3)$  toward  $\partial_3 u^B \otimes_s e_3$ ,

iii) the traces  $\gamma_{S^\pm}(u^B)$  on  $S^\pm := S \times (-1, 1)$  of  $u^B$  are equal to the traces  $\gamma_S((u^\Omega)^\pm)$  on  $S$  of  $(u^\Omega)^\pm$ , the restrictions on  $\Omega^\pm := \Omega \cap \{ \pm x_3 > 0 \}$  of the weak limit  $u^\Omega$  in  $H_{\Gamma_0}^1(\Omega \setminus S; \mathbb{R}^3)$  of  $T^\varepsilon u^s$  as in previous section [3.5.1](#).

The staple limit space of possible states with finite energy then arises as:

$$\mathcal{H} := \mathcal{U} \times L^2(\Omega; \mathbb{R}^3) \times L^2(B; \mathbb{R}^3)$$

$$\mathcal{U} = \left\{ u = (u^\Omega, u^B) \in H_{\Gamma_0}^1(\Omega \setminus S; \mathbb{R}^3) \times H_{\partial_3}(B; \mathbb{R}^3); \gamma_S((u^\Omega)^\pm) = \gamma_{S^\pm}(u^B) \right\}$$

$$\langle U, U' \rangle := \int_{\Omega \setminus S} a e(u^\Omega) \cdot e(u'^\Omega) dx + \bar{\mu} \int_B a_I (\partial_3 u^B \otimes_s e_3) \cdot (\partial_3 u'^B \otimes_s e_3) d\hat{x}$$

$$+ \int_\Omega \gamma v^\Omega \cdot v'^\Omega dx + \bar{\rho} \int_B v^B \cdot v'^B dx$$

The choice of  $P_s$  will reflect how a field like  $u^B$  or  $v^B$  does appear in the asymptotic behavior of  $u^s$  or  $v^s$ :

$$P_s U = (P_s^\mathcal{U}(u^\Omega, u^B), P_s^\mathcal{V}(v^\Omega, v^B))$$

$$P_s^\mathcal{V}(v^\Omega, v^B) := \chi_{\Omega^\varepsilon} v^\Omega + (1 - \chi_{\Omega^\varepsilon})(\mathcal{S}_\varepsilon)^{-1}(v^B)$$

$P_s^{\mathcal{U}}(u^\Omega, u^B)$  is defined through:

$$\begin{aligned} i) (u^\Omega, u^B) &\mapsto \phi_s \in H^1(B; \mathbb{R}^3) \text{ such that } \gamma_{S^\pm}(\phi_s) = \gamma_S((u^\Omega)^\pm); \\ \int_B a_I e(\varepsilon, \phi_s) \cdot e(\varepsilon, \varphi) dx &= \int_B a_I (\partial_3 u_B \otimes_s e_3) \cdot e(\varepsilon, \varphi) dx, \forall \varphi \in H_{S^+ \cup S^-}^1(B; \mathbb{R}^3) \\ ii) P_s^{\mathcal{U}}(u^\Omega, u^B) &= \begin{cases} (\mathcal{S}_\varepsilon)^{-1} \phi_s & \text{in } B_\varepsilon \\ (1 - \zeta_0) u^\Omega(\cdot \pm \varepsilon e_3) + \zeta_0 u^\Omega & \text{in } \Omega_\varepsilon^\pm \end{cases} \end{aligned}$$

$\zeta_0$  being already defined in section 3.5.1

Note that Trotter convergence implies:

$$\begin{aligned} \chi_{\Omega^\varepsilon} e(u_s) &\rightarrow e(u^\Omega) \text{ in } L^2(\Omega \setminus S; \mathbb{S}^3) \\ \widehat{\mathcal{S}_\varepsilon u_s} &\rightharpoonup \widehat{u^B} \text{ in } L^2(B; \mathbb{R}^2), \\ ((\mathcal{S}_\varepsilon u_s)_3, e(\varepsilon, \mathcal{S}_\varepsilon u_s)) &\rightarrow (u_3^B, \partial_3 u^B \otimes_s e_3) \text{ in } H_{\partial_3}(B) \times L^2(B; \mathbb{S}^3) \\ \chi_{\Omega^\varepsilon} v_s &\rightarrow v^\Omega \text{ in } L^2(\Omega; \mathbb{R}^3), \mathcal{S}_\varepsilon v_s \rightarrow v^B \text{ in } L^2(B; \mathbb{R}^3) \end{aligned}$$

Obviously (*sic!*)  $(\mathcal{H}_s, P_s, \mathcal{H})$  satisfies  $(T_1)$  and  $(T_2)$  and everything is in place to get (without additional effort) the limit behavior which reads as:

$$\begin{aligned} \exists \zeta \in \partial \bar{\mathcal{D}}(\partial_3 \frac{dv^B}{dt} \otimes_s e_3); \\ \int_{\Omega \setminus S} \gamma \frac{d^2 u^\Omega}{dt^2} \cdot \psi^\Omega dx + \int_{\Omega \setminus S} a e(u^\Omega) \cdot e(\psi) dx + \bar{\rho} \int_B \frac{d^2 u^B}{dt^2} \cdot \psi^B dx \\ + \int_B (\bar{\mu} a_I (\partial_3 u^B \otimes_s e_3) + \bar{b} \zeta) \cdot (\partial_3 \psi^B \otimes_s e_3) dx = 0 \\ \forall \psi = (\psi^\Omega, \psi^B) \in H_{\Gamma_0}^1(\Omega; \mathbb{R}^3) \times H_{\partial_3}(B; \mathbb{R}^3); \gamma_{S^\pm}(\psi^B) = \gamma_S((\psi^\Omega)^\pm) \\ \sigma^\Omega = a e(u^\Omega) \text{ in } \Omega \setminus S \\ \text{div } \sigma^\Omega = \gamma \ddot{u} \text{ in } \Omega \setminus S \\ \pm \sigma^\Omega e_3 = \frac{1}{2} \int_{-1}^1 (1 \pm x_3) \frac{d^2 u^B}{dt^2} \pm (\bar{\mu} a_I (\partial_3 u^B \otimes_s e_3) + \bar{b} \zeta) dx_3 \text{ on } S \end{aligned}$$

This corresponds to the evolution of each adherent clamped on  $\Gamma_0^\pm = \Gamma_0 \cap \{\pm x_3 > 0\}$  and linked by a mechanical constraint along  $S$ . On the contrary to the previous case of an adhesive layer with an evanescent total mass, the contact may not only be described by the traces  $\gamma_S((u^\Omega)^\pm)$ ,  $\gamma_S((v^\Omega)^\pm)$  of the displacement and velocity of the sole adherents, but also by the additional variables  $(u^B, v^B = \frac{du^B}{dt})$  which keep the memory of the dynamics of the adhesive layer. These variables satisfy the following equations

$$\begin{aligned} \sigma^B &\in \bar{\mu} a_I (\partial_3 u^B \otimes_s e_3) + \bar{b} \partial \bar{\mathcal{D}}(\partial_3 \dot{u}^B \otimes_s e_3) \text{ in } B \\ \bar{\rho} \ddot{u}^B + \partial_3(\sigma^B e_3) &= 0 \\ \gamma_{S^\pm}(u^B) &= \gamma_S((u^\Omega)^\pm) \end{aligned}$$

they are of the same type as the ones in the genuine layer. Of course  $(u^B, v^B)$  may be eliminated and, consequently, the contact condition along  $S$  between the two adherents is a non-local (in time only!) relationship between the stress vector  $(\sigma^\Omega)^\pm(\hat{x}, t) e_3$  and the history of  $\gamma_S((u^\Omega)^\pm)(\hat{x}, \tau)$ ,  $0 \leq \tau \leq t$ .

3.5.3. *Hard and light adhesive.* We assume that  $\rho\varepsilon \rightarrow 0$  and confine to the two essential cases:

$$\begin{aligned}\mu &\sim \bar{\mu}_q / \left( 2\varepsilon \left( \frac{\varepsilon^{2(q-1)}}{2q-1} \right) \right) \\ b &\sim \bar{b}_q / \left( 2\varepsilon \left( \frac{\varepsilon^{p(q-1)}}{1+(q-1)p} \right) \right) \\ \bar{\mu}_q, \bar{b}_q &\in (0, +\infty), \quad q = 1, 2\end{aligned}$$

Moreover  $W_I$  is an even function of  $x_3$ . A comprehensive study is done in [25].

As previously, it is first obvious that  $\mathcal{K}_{\varepsilon\rho}(w_s, w_s) \leq C$  implies the existence of  $w$  in  $L^2(\Omega; \mathbb{R}^3)$ , the weak limit in  $L^2(\Omega; \mathbb{R}^3)$  of  $\chi_{\Omega_\varepsilon} w_s$ , satisfying

$$\int_{\Omega} \gamma |w|^2 dx \leq \mathcal{K}_{\varepsilon\rho}(w_s, w_s)$$

Next,  $\mathcal{E}_{\varepsilon\mu}(w_s, w_s) \leq C$  implies that  $w_s$  weakly converges in  $H^1(\Omega \setminus S; \mathbb{R}^3)$  toward some  $w$  which belongs to  ${}^q\mathcal{H}_d$  with

$$\begin{aligned}{}^1\mathcal{H}_d &:= \left\{ w \in H^1(\Omega; \mathbb{R}^3) ; \widehat{w} \in H^1(S; \mathbb{R}^2) \right\} \\ {}^2\mathcal{H}_d &:= \left\{ w \in H^1(\Omega; \mathbb{R}^3) ; \widehat{e(w)} = 0 \text{ in } S \text{ and } w_3 \in H^2(S) \right\}\end{aligned}$$

This last point stems from

$$\frac{1}{2\varepsilon} \int_{-\varepsilon}^{\varepsilon} \widehat{w_s} dx_3 = \gamma_s(\widehat{(T^\varepsilon w_s)^+}) - \frac{1}{2\varepsilon} \int_{-\varepsilon}^{\varepsilon} \int_{x_3}^{\varepsilon} \partial_3 \widehat{u_s} dt dx_3, \text{ when } q = 1$$

When  $q = 2$ , the boundedness of  $\mathcal{K}_{\varepsilon\rho}(w_s, w_s)$  implies the one of

$$\int_B |e(\varepsilon, \mathcal{S}'_\varepsilon w_s)|^2 dx$$

where, for all  $x = (\hat{x}, x_3)$  in  $B$

$$\widehat{\mathcal{S}'_\varepsilon w}(\hat{x}, x_3) = \varepsilon \widehat{w}(\hat{x}, \varepsilon x_3), \quad (\mathcal{S}'_\varepsilon w)_3(\hat{x}, x_3) = w_3(\hat{x}, \varepsilon x_3)$$

so that there exists an element  $\mathbf{w}$  in the space

$$\begin{aligned}V_{KL}(B) &= \left\{ w \in H^1(B) ; \exists (w^M, w^F) \in H^1(S; \mathbb{R}^2) \times H^2(S) ; \right. \\ &\quad \widehat{w}(y) = w^M(\hat{y}) - y_3 \widehat{\nabla w^F}(y), \quad w_3(y) = w^F(\hat{y}), \\ &\quad \left. \forall y = (\hat{y}, y_3) \in B \right\}\end{aligned}$$

and a rigid displacement  $\rho_s$  such that, up to a subsequence,  $\mathcal{S}'_\varepsilon w_s + \rho_s$  weakly converges toward  $\mathbf{w}$  in  $H^1(B; \mathbb{R}^3)$ . Moreover, as

$$\int_S \tau(\hat{x}) \cdot \frac{1}{\varepsilon^3} \int_{-\varepsilon}^{\varepsilon} \widehat{e(w_s)} dx_3 d\hat{x} = \int_B \tau(\hat{x}) \cdot x_3 \widehat{e(\mathcal{S}'_\varepsilon w_s)} dx, \quad \forall \tau \in L^2(S; \mathbb{S}^2)$$

one deduces that  $\frac{1}{\varepsilon^3} \int_{-\varepsilon}^{\varepsilon} x_3 \widehat{e(w_s)} dx_3$  converges weakly in  $L^2(S; \mathbb{S}^2)$  toward

$$\int_{-1}^1 \widehat{e(\mathbf{w})} dx = \int_{-1}^1 x_3 (\widehat{e(\mathbf{w}^M)} - x_3 D^2 \mathbf{w}^F) dx_3 = -\frac{2}{3} D^2(\mathbf{w}^F)$$

But the trace on  $S + e_3$  of  $(\mathcal{S}'_\varepsilon)_3$  being equal to the one on  $S$  of  $(T^\varepsilon w_s)^+$  one deduces that  $\mathbf{w}^F = \gamma_s(w)_3$ .

Lastly we get

$$\int_{\Omega} ae(w) \cdot e(w) dx + \bar{\mu}_q \int_S W_I^{KL}(\widehat{e_q(w)}) d\hat{x} \leq \lim_{s \rightarrow \bar{s}} \mathcal{E}_{\varepsilon\mu}(w_s)$$

where

$$W_I^{KL}(\xi) := \inf \left\{ W_I(q) ; \widehat{q} = \xi \right\},$$

$$\forall \xi \in \mathbb{S}^2 \text{ such that } \widehat{e_1(w)} = \widehat{e(w)}, \widehat{e_2(w)} = \widehat{D^2(w_3)}, \forall w \in {}^q\mathcal{H}_d$$

We therefore propose

$${}^q\mathcal{H} := {}^q\mathcal{H}_d \times L^2(\Omega; \mathbb{R}^3)$$

$${}^q|U|^2 = \langle (u, v), (u, v) \rangle_q:$$

$$= \int_{\Omega} ae(u) \cdot e(u) dx + 2\bar{\mu}_q \int_S W_I^{KL}(\widehat{e_q(u)}) d\hat{x} + \int_{\Omega} \gamma|v|^2 dx$$

and the previous derivation of  ${}^q\mathcal{H}$  justifies the following “variational” definition of  ${}^qP_s$  by:

$$(u, v) \in {}^q\mathcal{H} \mapsto {}^qP_s(u, v) = (u'_s, v'_s) \in \mathcal{H}_s$$

with

- $u'_s \in H_{\Gamma_0}^1(\Omega; \mathbb{R}^3)$  such that  $\forall w \in {}^q\mathcal{H}_d$   
 $\mathcal{E}_{\varepsilon\mu}(u'_s, w) = \int_{\Omega} ae(u) \cdot e(w) dx + \bar{\mu}_q \int_S DW_I^{KL}(\widehat{e({}^q\nu)}) \cdot \widehat{e(w)} dx$   
 where  ${}^1\nu, {}^2\nu \in V_{KL}(B^\varepsilon)$  with  $({}^1\nu^M, {}^1\nu^F) = (\gamma_S(\widehat{u}), 0)$  and  $({}^2\nu^M, {}^2\nu^F) = (0, \gamma_S(u_3))$
- $v'_s = \chi_{\Omega^\varepsilon} v$

and consequently  $(T_1)$  and  $(T_2)$  conditions are satisfied while

$$\text{Trotter convergence of } (w_s^1, w_s^2) \text{ toward } (w^1, w^2) \iff \begin{cases} T^\varepsilon w_s^1 \text{ converges strongly in } H^1(\Omega \setminus S; \mathbb{R}^3) \text{ to } w^1 \\ \frac{1}{2\varepsilon} \int_\varepsilon^\varepsilon \widehat{w_s^1} dx_3 \text{ converges strongly in } H^1(S; \mathbb{R}^2) \text{ to } \widehat{w^1} \\ \int_{\Omega} ae(w^1) \cdot e(w^1) dx + \bar{\mu}_q \int_S W_I^{KL}(\widehat{e_q(w^1)}) d\hat{x} \\ \hspace{15em} = \lim_{s \rightarrow \bar{s}} \mathcal{E}_{\varepsilon\mu}(w_s^1) \\ T^\varepsilon w_s^2 \text{ converges strongly in } L^2(\Omega; \mathbb{R}^2) \text{ to } w^2 \\ \int_{\Omega} \gamma|w^2|^2 dx = \lim_{s \rightarrow \bar{s}} \mathcal{K}_{\varepsilon\rho}(w_s^2) \end{cases}$$

and the obtention of the limit behavior which reads as<sup>4</sup>

$$\begin{aligned} \exists \zeta \in \partial \mathcal{D}^{KL}(\widehat{e_q(\frac{d^q u}{dt})}) \text{ such that} \\ \int_{\Omega} \gamma \frac{d^2 {}^q u}{dt^2} \cdot \varphi \, dx + \int_{\Omega} a e({}^q u) \cdot e(\varphi) \, dx + \bar{\mu}_q \int_S DW_I^{KL}(\widehat{e_q({}^q u)}) \cdot (\widehat{e_q(\varphi)}) \, d\hat{x} \\ + \bar{b}_q \int_S \zeta \cdot (\widehat{e_q(\varphi)}) \, d\hat{x} = 0, \quad \forall \varphi \in {}^q \mathcal{H}. \end{aligned}$$

does not need more efforts. Hence, the limit behavior describes the evolution of a structure consisting of two linearly elastic adherents occupying  $\Omega_{\pm}$ , which are perfectly bonded to a material deformable flat surface whose behavior is of the same kind as the genuine adhesive (*i.e.* nonlinear viscoelasticity of Kelvin-Voigt generalized type). Moreover, the mass of the adhesive being evanescent, there is no inertial term in the interface condition. Case  $q = 1$  corresponds to membrane deformations whereas case  $q = 2$  corresponds to flexural deformations.

### 3.6. Important remarks.

**3.6.1. Second members.** To simplify the presentation, we choose to consider autonomous problems. To take into account non vanishing external loadings, it suffices to split (see the details in [30, 32]) the state  $u_n$  into

$$u_n = u_n^e + u_n^r$$

where  $u_n^e$  solves the steady state problem associated with the transient problem under consideration and involving the complete external loading. The study of the convergence of  $u_n^e$  stems directly from the settlement  $(\mathcal{H}_n, P_n, \mathcal{H})$  of Trotter framework. Then  $u_n^r$  does solve an evolution equation with a second member  $f_n$  continuous function of  $u_n^e$  hence continuous function of the loading. The same is done for  $u$  solution to the limit problem. So the condition

$$\lim_{n \rightarrow \infty} \int_0^T |P_n f(t) - f_n(t)|_n \, dt = 0$$

reduces to a condition on the loadings easy to formulate.

**3.6.2. Initial conditions.** In the case of modelings by approximation, the data is  $u^0$  (in  $D(A)$ !) and if the Trotter convergence of the resolvents is established then

$$u_n^0 := (I - A_n)^{-1} P_n (I - A) u^0$$

satisfies both  $u_n^0$  in  $D(A_n)$  and

$$\lim_{n \rightarrow \infty} |P_n u^0 - u_n^0|_n = 0$$

On the contrary, in modelings by convergence, the data is the sequence  $u_n^0$  and we have to state an additional assumption

$$\exists u^0 \in D(A) \text{ such that } \lim_{n \rightarrow \infty} |P_n u^0 - u_n^0|_n = 0$$

on the data  $u_n^0$  in order to have the Trotter convergence of  $u_n(t)$  toward  $u(t)$ !

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<sup>4</sup>We define  $\mathcal{D}^{KL}$  similarly as  $W_I^{KL}$ .



**4. An interesting insight through Trotter theory.** We consider three other examples concerning linearized (visco)elasticity already treated by variational evolution equations or Laplace transforms. This section starts with a genuine study through Trotter theory, its purpose is to introduce the common mechanical framework, to recall the fact that a state variable (here the velocity) may disappear at the limit, and suggests the noteworthiness of initial conditions which will be emphasized in the second example (see section 4.2).

**4.1. Quasi-static evolution of a linearly viscoelastic body of Kelvin-Voigt type.** The problem of the autonomous evolution of such a body involves the density  $\rho$  as main data and reads as:

$$(\mathcal{P}_\rho) \begin{cases} u_\rho \in H_{\Gamma_0}^1(\Omega; \mathbb{R}^3); \int_{\Omega} \rho \ddot{u}_\rho \cdot v' dx + \int_{\Omega} be(\dot{u}_\rho) \cdot e(v') dx \\ \quad + \int_{\Omega} ae(u_\rho) \cdot e(v') dx, \forall v' \in H_{\Gamma_0}^1(\Omega; \mathbb{R}^3) \\ (u_\rho, \dot{u}_\rho)(0) = (u_\rho^0, v_\rho^0) =: U_\rho^0 \end{cases}$$

$a, b$ , being the coefficients of elasticity and viscosity which are assumed to satisfy the classical conditions of symmetry, boundedness and ellipticity and may be termed as:

$$(\mathcal{P}_\rho) \begin{cases} \frac{dU_\rho}{dt} - A_\rho U_\rho = 0 \text{ in } \mathcal{H}_\rho \\ U_\rho(0) = U_\rho^0 \end{cases}$$

where

$$\begin{aligned} \mathcal{H}_\rho &:= H_{\Gamma_0}^1(\Omega; \mathbb{R}^3) \times L^2(\Omega; \mathbb{R}^3) \\ |(u_\rho, v_\rho)|_\rho^2 &= (u_\rho, v_\rho), (u_\rho, v_\rho) >_\rho := \int_{\Omega} ae(u_\rho) \cdot e(v_\rho) dx + \int_{\Omega} \rho |v_\rho|^2 dx \\ D(A_\rho) &= \left\{ U_\rho = (u_\rho, v_\rho) \in \mathcal{H}_\rho \text{ s.t. } v_\rho \in H_{\Gamma_0}^1(\Omega), \exists! w_\rho \in L^2(\Omega; \mathbb{R}^3) \text{ s.t.} \right. \\ &\quad \left. (u_\rho, z_\rho), (v', v') >_\rho + \int_{\Omega} be(v_\rho) \cdot e(v') dx = 0, \right. \\ &\quad \left. \forall v' \in H_{\Gamma_0}^1(\Omega; \mathbb{R}^3) \right\} \\ A_\rho U_\rho &= (v_\rho, z_\rho) \end{aligned}$$

$A_\rho$  is obviously  $m$ -dissipative and

$$\begin{aligned} \bar{U}_\rho - A_\rho \bar{U}_\rho &= F_\rho \\ \Leftrightarrow \begin{cases} \bar{v}_\rho = \bar{u}_\rho - F_\rho^1 \\ \bar{u}_\rho \text{ minimizes } \frac{1}{2} |(u', u')|_\rho^2 + \frac{1}{2} \int_{\Omega} be(u') \cdot e(u') dx - \int_{\Omega} be(F_\rho^1) \cdot e(v') dx \\ \quad - (0, F_\rho^1 + F_\rho^2), (u', u') >_\rho \text{ on } H_{\Gamma_0}^1(\Omega; \mathbb{R}^3) \end{cases} \end{aligned}$$

so that if  $U_\rho^0$  belongs to  $D(A_\rho)$ ,  $(\mathcal{P}_\rho)$  has a unique solution in

$$C^1([0, \infty); \mathcal{H}_\rho) \cap C^0([0, +\infty); D(A_\rho))$$

The very questions are what does happen when  $\rho$  goes to zero? In what extent can we neglect “inertial terms”? Trotter theory may supply rigorous answers. We set

$$\begin{aligned}
\mathcal{H} &:= H_{\Gamma_0}^1(\Omega; \mathbb{R})^3 \\
|U|^2 &= \langle U, U \rangle := \int_{\Omega} ae(U) \cdot e(U) \, dx \\
P_{\rho} U &= (U, 0) \in \mathcal{H}_{\rho} \text{ which clearly satisfies } (T_1) \text{ and } (T_2) \\
D(A) &= \left\{ U \in H_{\Gamma_0}^1(\Omega; \mathbb{R}^3) ; \exists ! z \in H_{\Gamma_0}^1(\Omega; \mathbb{R}^3) \text{ such that} \right. \\
&\quad \left. \langle U, v' \rangle + \int_{\Omega} be(z) \cdot e(v') \, dx = 0, \forall v' \in H_{\Gamma_0}^1(\Omega; \mathbb{R}^3) \right\} \\
&= \mathcal{H}! \\
AU &= z
\end{aligned}$$

$A$  is bounded, self adjoint and

$$\begin{aligned}
\bar{U} - A\bar{U} &= F \\
\Leftrightarrow \bar{U} \text{ minimizes } &\frac{1}{2}|u'|^2 + \frac{1}{2} \int_{\Omega} be(u') \cdot e(u') \, dx \\
&\quad - \int_{\Omega} be(F) \cdot e(v') \, dx \text{ on } H_{\Gamma_0}^1(\Omega; \mathbb{R}^3)
\end{aligned}$$

Let  $U$  be the solution to

$$(\mathcal{P}) \quad \begin{cases} \frac{dU}{dt} - AU = 0 \text{ in } \mathcal{H} \\ U(0) = U^0 \end{cases}$$

Hence, obviously,

$$\begin{cases} u_{\rho} \rightarrow U \text{ in } H_{\Gamma_0}^1(\Omega; \mathbb{R}^3) \\ \rho \int_{\Omega} |v_{\rho}|^2 \, dx \rightarrow 0 \end{cases} \quad \text{uniformly on bounded time intervals}$$

is equivalent to

$$\lim_{\rho \rightarrow 0} \int_{\Omega} |e(u_{\rho}^0 - U^0)|^2 \, dx + \rho \int_{\Omega} |v_{\rho}^0|^2 \, dx = 0$$

That is to say quasi-static evolution

$$\int_{\Omega} ae(U) \cdot e(v') \, dx + \int_{\Omega} be(\dot{U}) \cdot e(v') \, dx = 0, \forall v' \in H_{\Gamma_0}^1(\Omega; \mathbb{R}^3)$$

is a good approximation if the sequence of kinetic energies of the initial data goes to zero with, of course, a good choice of  $u_{\rho}^0$ ; convergence toward zero of the density is not enough!

**4.2. Homogenization in elastodynamics.** S. Brahim-Otsmane, G. Francfort and F. Murat in a famous paper [7] considered homogenization of waves equation. In order to maintain a mechanical unity in this section, we will rephrase it in terms of elastodynamics with density  $\rho_{\varepsilon}$  uniformly bounded from above and below with respect to  $\varepsilon$ , and elasticity coefficients  $a_{\varepsilon}$  satisfying usual properties of symmetry and of uniform boundedness and ellipticity with respect to  $\varepsilon$ . By using variational

evolution equation theory it may be shown that the following problem:

$$(\mathcal{P}_\varepsilon) \begin{cases} u_\varepsilon \in H_{\Gamma_0}^1(\Omega; \mathbb{R}^3) ; \int_{\Omega} \rho_\varepsilon \ddot{u}_\varepsilon \cdot v' dx + \int_{\Omega} a_\varepsilon e(u_\varepsilon) \cdot e(v') dx = 0, \\ (u_\varepsilon, \dot{u}_\varepsilon)(0) = (u_\varepsilon^0, v_\varepsilon^0) \text{ given in } H_{\Gamma_0}^1(\Omega; \mathbb{R}^3) \times L^2(\Omega; \mathbb{R}^3) \end{cases} \quad \forall v' \in H_{\Gamma_0}^1(\Omega; \mathbb{R}^3)$$

under assumption<sup>5</sup>:

$$(H_1) \begin{cases} a_\varepsilon \text{ } H\text{-converges toward } a^{\text{hom}} \\ \rho_\varepsilon \text{ weak star converges in } L^\infty(\Omega) \text{ toward } \bar{\rho} \\ (u_\varepsilon^0, v_\varepsilon^0) \text{ weakly converges in } H_{\Gamma_0}^1(\Omega; \mathbb{R}^3) \times L^2(\Omega; \mathbb{R}^3) \text{ toward } (u^0, \bar{v}) \\ \rho_\varepsilon v_\varepsilon \text{ weakly converges in } L^2(\Omega; \mathbb{R}^3) \text{ toward } \bar{\rho} \bar{v} \\ \text{let } v^0 := \frac{\bar{\rho} \bar{v}}{\bar{\rho}} \end{cases}$$

has a unique solution which converges in the following sense:

$$\begin{aligned} & (u_\varepsilon, \dot{u}_\varepsilon) \text{ weak star converges in } L^\infty(0, T; H_{\Gamma_0}^1(\Omega; \mathbb{R}^3) \times L^2(\Omega; \mathbb{R}^3)) \\ & \text{toward } (u, \dot{u}) \text{ such that} \\ (\mathcal{P}) \quad & \begin{cases} \int_{\Omega} \bar{\rho} \ddot{u} \cdot v' dx + \int_{\Omega} a^{\text{hom}} e(u) \cdot e(v') dx = 0, \quad \forall v' \in H_{\Gamma_0}^1(\Omega; \mathbb{R}^3) \\ (u, \dot{u})(0) = (u^0, v^0) \end{cases} \end{aligned}$$

But in general we do not have

$$|(u_\varepsilon, \dot{u}_\varepsilon)|_\varepsilon^2(t) \rightarrow |(u, \dot{u})|^2(t) \quad (4)$$

where

$$\begin{aligned} |(u_\varepsilon, \dot{u}_\varepsilon)|_\varepsilon^2 &= \int_{\Omega} a_\varepsilon e(u_\varepsilon) \cdot e(u_\varepsilon) dx + \int_{\Omega} \rho_\varepsilon |\dot{u}_\varepsilon|^2 dx \\ |(u, \dot{u})|^2 &= \int_{\Omega} a^{\text{hom}} e(u) \cdot e(u) dx + \int_{\Omega} \bar{\rho} |\dot{u}|^2 dx \end{aligned}$$

Nevertheless, it can be shown as in [7] that there exist “well prepared” initial data such that 4 occurs!

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<sup>5</sup>Recall that  $a_\varepsilon$  is said to  $H$ -converge to  $a^{\text{hom}}$  if and only if for any  $f$  in  $H^{-1}(\Omega; \mathbb{R}^3)$  the sequence  $v^\varepsilon$  of solutions to

$$\begin{aligned} -\operatorname{div} (a^\varepsilon e(v^\varepsilon)) &= f, \text{ in } \Omega \\ v^\varepsilon &= 0 \text{ on } \partial\Omega \end{aligned}$$

satisfies

$$\begin{aligned} v^\varepsilon &\rightharpoonup v^{\text{hom}} \text{ weakly in } H_0^1(\Omega; \mathbb{R}^3) \\ a^\varepsilon e(v^\varepsilon) &\rightharpoonup a^{\text{hom}} e(v^{\text{hom}}) \text{ weakly in } L^2(\Omega; \mathbb{S}^3) \end{aligned}$$

where  $v^{\text{hom}}$  is the solution to

$$\begin{aligned} -\operatorname{div} (a^{\text{hom}} e(v^{\text{hom}})) &= f, \text{ in } \Omega \\ v^{\text{hom}} &= 0 \text{ on } \partial\Omega \end{aligned}$$

Let  $\tilde{u}_\varepsilon$  and  $\check{u}_\varepsilon$  solution to  $(\mathcal{P}_\varepsilon)$  with

$$\begin{aligned}(\tilde{u}_\varepsilon, \dot{\tilde{u}}_\varepsilon)(0) &= (\tilde{u}_\varepsilon^0, v^0) \\(\check{u}_\varepsilon, \dot{\check{u}}_\varepsilon)(0) &= (u_\varepsilon^0 - \tilde{u}_\varepsilon^0, v_\varepsilon^0 - v^0) \\ \tilde{u}_\varepsilon^0 &\in H_{\Gamma_0}^1(\Omega; \mathbb{R}^3) \text{ such that} \\ \int_{\Omega} a_\varepsilon e(\tilde{u}_\varepsilon^0) \cdot e(v') dx &= \int_{\Omega} a^{\text{hom}} e(u^0) \cdot e(v') dx, \forall v' \in H_{\Gamma_0}^1(\Omega; \mathbb{R}^3)\end{aligned}$$

then

i)  $(\tilde{u}_\varepsilon, \dot{\tilde{u}}_\varepsilon)$  weak star converges in  $L^\infty(0, T; H_{\Gamma_0}^1(\Omega; \mathbb{R}^3) \times L^2(\Omega; \mathbb{R}^3))$  and  $\dot{\tilde{u}}_\varepsilon$  converges strongly in  $C^0([0, T]; L^2(\Omega; \mathbb{R}^3))$  toward  $\dot{u}$  with  $|(\tilde{u}_\varepsilon, \dot{\tilde{u}}_\varepsilon)|_\varepsilon \rightarrow |(u, \dot{u})|$  in  $C^0([0, T])$ ,

ii)  $(\check{u}_\varepsilon, \dot{\check{u}}_\varepsilon)$  weak star converges in  $L^\infty(0, T; H_{\Gamma_0}^1(\Omega; \mathbb{R}^3) \times L^2(\Omega; \mathbb{R}^3))$  toward 0.

This nice result can be obtained very easily by using Trotter theory, illustrating the noteworthiness of the assumption on the data in the Trotter convergence theorem.

Obviously,  $(\mathcal{P}_\varepsilon)$  and  $(\mathcal{P})$  may be formulated as:

$$(\mathcal{P}_\varepsilon) \quad \begin{cases} \frac{dU^\varepsilon}{dt} - A_\varepsilon U_\varepsilon = 0 \text{ in } \mathcal{H}_\varepsilon \\ U_\varepsilon(0) = U_\varepsilon^0 \end{cases}, \quad (\mathcal{P}) \quad \begin{cases} \frac{dU}{dt} - A U = 0 \text{ in } \mathcal{H} \\ U(0) = U^0 \end{cases}$$

with

$$\begin{aligned}\mathcal{H}_\varepsilon &:= H_{\Gamma_0}^1(\Omega; \mathbb{R}^3) \times L^2(\Omega; \mathbb{R}^3), \\ \mathcal{H} &:= H_{\Gamma_0}^1(\Omega; \mathbb{R}^3) \times L^2(\Omega; \mathbb{R}^3) \\ |(u_\varepsilon, v_\varepsilon)|_\varepsilon^2 &= \langle (u_\varepsilon, v_\varepsilon), (u_\varepsilon, v_\varepsilon) \rangle_\varepsilon := \int_{\Omega} a_\varepsilon e(u_\varepsilon) \cdot e(u_\varepsilon) dx + \int_{\Omega} \rho_\varepsilon |v_\varepsilon|^2 dx \\ |(u, v)|^2 &= \langle (u, v), (u, v) \rangle := \int_{\Omega} a^{\text{hom}} e(u) \cdot e(u) dx + \int_{\Omega} \bar{\rho} |v|^2 dx \\ D(A_\varepsilon) &= \left\{ U_\varepsilon = (u_\varepsilon, v_\varepsilon) \in \mathcal{H}_\varepsilon ; v_\varepsilon \in H_{\Gamma_0}^1(\Omega; \mathbb{R}^3) ; \exists ! z_\varepsilon \in L^2(\Omega; \mathbb{R}^3) \text{ such that} \right. \\ &\quad \left. \langle (u_\varepsilon, z_\varepsilon), (v', v') \rangle_\varepsilon = 0, \forall v' \in H_{\Gamma_0}^1(\Omega; \mathbb{R}^3) \right\} \\ D(A) &= \left\{ U = (u, v) \in \mathcal{H} ; v \in H_{\Gamma_0}^1(\Omega; \mathbb{R}^3) ; \exists ! z \in L^2(\Omega; \mathbb{R}^3) \text{ such that} \right. \\ &\quad \left. \langle (u, z), (v', v') \rangle = 0, \forall v' \in H_{\Gamma_0}^1(\Omega; \mathbb{R}^3) \right\}\end{aligned}$$

and let  $P_\varepsilon$  be defined by

$P_\varepsilon U = (u'_\varepsilon, v'_\varepsilon)$  such that

$$\begin{aligned}u'_\varepsilon &\in H_{\Gamma_0}^1(\Omega; \mathbb{R}^3) ; \int_{\Omega} a_\varepsilon e(u'_\varepsilon) \cdot e(u') dx = \int_{\Omega} a^{\text{hom}} e(u) \cdot e(u') dx, \forall u' \in H_{\Gamma_0}^1(\Omega; \mathbb{R}^3) \\ v'_\varepsilon &= v\end{aligned}$$

Here the concept that  $P_\varepsilon U$  represents  $U$  in  $\mathcal{H}_\varepsilon$  takes its full meaning:  $u'_\varepsilon$  is a microscopic state corresponding to the macroscopic state  $u$ !

Theory of homogenization immediately yields that  $(\mathcal{H}_\varepsilon, P_\varepsilon, \mathcal{H})$  satisfies  $(T_1)$  and  $(T_2)$  and

$$\text{Trotter convergence} \Leftrightarrow \begin{cases} u_\varepsilon \text{ weakly converges in } H_{\Gamma_0}^1(\Omega; \mathbb{R}^3) \text{ toward } u \\ \int_{\Omega} a_\varepsilon e(u_\varepsilon) \cdot e(u_\varepsilon) dx \rightarrow \int_{\Omega} a^{\text{hom}} e(u) \cdot e(u) dx \\ v_\varepsilon \text{ strongly converges in } L^2(\Omega; \mathbb{R}^3) \text{ toward } v \end{cases}$$

As Trotter convergence of the resolvents obviously stems from results of homogenization in elasticity, to get the Trotter convergence uniformly on  $[0, T]$

$$\lim_{\varepsilon \rightarrow 0} |P_\varepsilon U(t) - U_\varepsilon(t)|_\varepsilon = 0$$

(and consequently  $|U_\varepsilon(t)|_\varepsilon \rightarrow |U(t)|$ ), it suffices that

$$\lim_{\varepsilon \rightarrow 0} |P_\varepsilon U^0 - U_\varepsilon^0|_\varepsilon = 0$$

that is to say  $((u^0)'_\varepsilon - u_\varepsilon^0, v^0 - v_\varepsilon^0)$  converges strongly to  $(0, 0)$  in  $H^1(\Omega; \mathbb{R}^3) \times L^2(\Omega; \mathbb{R}^3)$  which is satisfied by the initial data  $\tilde{u}_\varepsilon^0 = (u^0)'_\varepsilon$  and  $\tilde{v}_\varepsilon^0 = v_0$  of [7]!

**4.3. Periodic homogenization in visco-elasticity of Kelvin-Voigt type.** In another famous paper [10] is, *inter alia*, studied the following problem of homogenization:

$$(\mathcal{P}_\varepsilon) \begin{cases} u_\varepsilon \in H_0^1(\Omega; \mathbb{R}^3) ; \int_{\Omega} b(x/\varepsilon) e(\dot{u}_\varepsilon) \cdot e(v') dx \\ \quad + \int_{\Omega} a(x/\varepsilon) e(\dot{u}_\varepsilon) \cdot e(v') dx = 0, \forall v' \in H_0^1(\Omega; \mathbb{R}^3) \\ u_\varepsilon(0) = u_\varepsilon^0 \end{cases}$$

where the viscoelasticity coefficients are deduced from  $Y$ -periodic functions  $a$  and  $b$ ,  $Y := (0, 1)^3$ , satisfying the usual properties of symmetry, boundedness and ellipticity. Problem  $(\mathcal{P}_\varepsilon)$  can be formulated as

$$(\mathcal{P}_\varepsilon) \begin{cases} \frac{du_\varepsilon}{dt} - A_\varepsilon u_\varepsilon = 0 \text{ in } \mathcal{H}_\varepsilon \\ u_\varepsilon(0) = u_\varepsilon^0 \end{cases}$$

where

$$\mathcal{H}_\varepsilon := H_0^1(\Omega; \mathbb{R}^3)$$

$$|u_\varepsilon|_\varepsilon^2 = \langle u_\varepsilon, u_\varepsilon \rangle_\varepsilon := \int_{\Omega} a(x/\varepsilon) e(u_\varepsilon) \cdot e(u_\varepsilon) dx$$

$$D(A_\varepsilon) = \mathcal{H}_\varepsilon$$

$$A_\varepsilon u_\varepsilon = z_\varepsilon \text{ such that } \int_{\Omega} b(x/\varepsilon) e(z_\varepsilon) \cdot e(v') dx + \int_{\Omega} a(x/\varepsilon) e(u_\varepsilon) \cdot e(v') dx = 0$$

$A_\varepsilon$  is bounded self adjoint on  $\mathcal{H}_\varepsilon$  and for all positive  $\lambda$  one has:

$$\begin{aligned} \bar{u}_\varepsilon - \lambda A_\varepsilon \bar{u}_\varepsilon &= f_\varepsilon \\ \Leftrightarrow \int_{\Omega} (a(x/\varepsilon) + \frac{1}{\lambda} b(x/\varepsilon)) e(\bar{u}_\varepsilon) \cdot e(v') dx \\ &= \frac{1}{\lambda} \int_{\Omega} b(x/\varepsilon) e(f_\varepsilon) \cdot e(v') dx, \forall v' \in H_0^1(\Omega; \mathbb{R}^3) \end{aligned}$$

which is the same equation as the one satisfied by the Laplace transform

$$\mathcal{L} \bar{u}_\varepsilon(\lambda) := \int_0^{+\infty} e^{-\frac{t}{\lambda}} \bar{u}_\varepsilon(t) dt$$

with  $u_\varepsilon^0 = f_\varepsilon$ !

The macroscopic (or homogenized) behavior is obtained by letting  $\varepsilon$  tend to zero. The resolvent of  $A_\varepsilon$  needs the  $H$ -limit of  $a_\varepsilon + \frac{1}{\lambda} b_\varepsilon$  since  $\bar{u}_\varepsilon$  involves a problem of periodic homogenization in elasticity. There exist  $a^{\text{hom}}$  and  $b^{\text{hom}}$ ,  $H$ -limits of  $a_\varepsilon = a(\cdot/\varepsilon)$  and  $b_\varepsilon = b(\cdot/\varepsilon)$ , but the  $H$ -limit is not an additive process as  $(a + \frac{1}{\lambda} b)^{\text{hom}}$  is different from  $a^{\text{hom}} + \frac{1}{\lambda} b^{\text{hom}}$ ! Hence the limit of  $\bar{u}_\varepsilon$  seems not involving a resolvent! G. Francfort and P. Suquet showed that

$$(a + \frac{1}{\lambda} b)^{\text{hom}} = a^{\text{hom}} + \frac{1}{\lambda} b^{\text{hom}} + \mathcal{L} K(\lambda)$$

so that the limit behavior as  $\varepsilon$  goes to zero of the solution to  $(\mathcal{P}_\varepsilon)$  is  $u$  solution to:

$$(\mathcal{P}) \begin{cases} \int_{\Omega} \left( a^{\text{hom}} e(u) + b^{\text{hom}} e(\dot{u}) + \int_0^t K(t-\tau) e(\dot{u}(\tau)) d\tau \right) \cdot e(v') dx = 0, \\ u(0) = u^0 \end{cases} \quad \forall v' \in H_0^1(\Omega; \mathbb{R}^3)$$

The limit behavior is no longer of Kelvin-Voigt type but involves viscoelasticity with memory! And it seems that here is an example “of a sequence of semi-groups whose limit is not a semi-group”.

The key point in trying to operate Trotter theory is the tool of two-scale convergence [1], the most suitable and efficient tool for mathematical analysis in periodic homogenization: if  $d$  is any  $Y$ -periodic function satisfying the usual properties of boundedness and ellipticity and if  $f \in L^2(\Omega; \mathbb{R}^3)$  then

$$\begin{aligned} w_\varepsilon \in H_0^1(\Omega; \mathbb{R}^3) ; \quad \int_{\Omega} d(x/\varepsilon) e(w_\varepsilon) \cdot e(w') dx &= \int_{\Omega} f \cdot w' dx, \quad \forall w' \in H_0^1(\Omega; \mathbb{R}^3) \\ \Leftrightarrow \begin{cases} \exists! (w, w^1) \in H_0^1(\Omega; \mathbb{R}^3) \times L^2(\Omega; H_{\text{per}}^1(Y; \mathbb{R}^3)/\mathbb{R}) =: \mathcal{H} \text{ such that} \\ \int_{\Omega \times Y} d(y) (e_x w + e_y w^1) \cdot (e_x w' + e_y w^{1'}) dx dy = \int_{\Omega} f w' dx, \\ \lim_{\varepsilon \rightarrow 0} \int_{\Omega} d(x/\varepsilon) (e(w_\varepsilon)(x) - e_x(w)(x) - e_y(w^1)(x, x/\varepsilon)) \\ \cdot (e(w_\varepsilon)(x) - e_x(w)(x) - e_y(w^1)(x, x/\varepsilon)) dx = 0 \end{cases} \quad \forall (w', w^{1'}) \in \mathcal{H} ; \end{aligned}$$

At that point, we simply cannot resist the impulse to let:

$$\begin{aligned} \mathcal{H} &:= \left\{ (u, u^1) \in H_0^1(\Omega; \mathbb{R}^3) \times L^2(\Omega; H_{\text{per}}^1(Y; \mathbb{R}^3)/\mathbb{R}) ; \right. \\ &\quad \left. < (u, u^1), (0, u^{1'}) > = 0, \quad \forall u^{1'} \in L^2(\Omega; H_{\text{per}}^1(Y; \mathbb{R}^3)/\mathbb{R}) \right\} \text{ with} \\ |U|^2 &= < (u, u^1), (u, u^1) > = \int_{\Omega \times Y} a(y) (e_x(u) + e_y(u^1)) \cdot \\ &\quad \cdot (e_x(u) + e_y(u^1)) dx dy \end{aligned}$$

$$\begin{aligned}
P_\varepsilon(u, u^1) &= u'_\varepsilon \in \mathcal{H}_\varepsilon \text{ s.t. } \langle u'_\varepsilon, w' \rangle_\varepsilon = \langle (u, u^1), (w', w^{1'}) \rangle, \forall (w', w^{1'}) \in \mathcal{H} \\
D(A) &= \mathcal{H} \\
AU &= (z, z^1) \in \mathcal{H} \text{ s.t. } \langle (u, u^1), (w', w^{1'}) \rangle + \\
&\quad + \int_{\Omega \times Y} b(y)(e_x(z) + e_y(z^1)) \cdot (e_x(w') + e_y(w^{1'})) dx dy = 0, \\
&\quad \forall (w', w^{1'}) \in \mathcal{H}
\end{aligned}$$

and

$$(\mathcal{P}) \quad \begin{cases} \frac{dU}{dt} - AU = 0 \text{ in } \mathcal{H} \\ U(0) = U^0 \end{cases},$$

as candidate to describe the homogeneous macroscopic behavior!

First note that Trotter convergence of  $u_\varepsilon$  toward  $U$  is equivalent to:

$$\begin{aligned}
\lim_{\varepsilon \rightarrow 0} \int_{\Omega} d(x/\varepsilon) (e_x(u^\varepsilon)(x) - e_x(u)(x) - e_y(u^1)(x, x/\varepsilon)) \cdot \\
\cdot (e_x(u^\varepsilon)(x) - e_x(u)(x) - e_y(u^1)(x, x/\varepsilon)) dx = 0
\end{aligned}$$

and, second, that Trotter convergence of the resolvent of  $A_\varepsilon$  toward the one of  $A$  stems obviously from two-scale convergence theory applied to  $c(\lambda) = a(y) + \frac{1}{\lambda}b(y)$ . Hence, if we assume

$$\lim_{\varepsilon \rightarrow 0} \|P_\varepsilon U^0 - u_\varepsilon^0\|_\varepsilon = 0 \quad (5)$$

the limit behavior is effectively described by  $(\mathcal{P})$ . It has the same structure as the genuine one *but with  $(u, u^1)$  as unknowns*. The sequence of semi-groups generated by  $A_\varepsilon$  converges in the sense of Trotter to the semi-group generated by  $A$ . Certainly, one can eliminate  $u^1$  (the additional microscopic state variable) to get an integro-differential equation involving the sole macroscopic state variable  $u$ . A possibility for 5 is of course  $u_\varepsilon^0 = P_\varepsilon U^0$  that is to say  $u_\varepsilon^0$  has to be a faithful microscopic representative of  $U^0 = (u^0, u^{01})$  and consequently does satisfy  $\text{div } a(x/\varepsilon) e(u_\varepsilon^0)$  independent of  $\varepsilon$ , a condition already stated in [10].

The idea of using two-scale convergence tool to treat the problem through Trotter theory was inspired to us by the deep insight of [33].

**4.4. Thin linear viscoelastic Kelvin-Voigt type plates.** After the classical change of coordinates and unknowns  $\mathcal{S}'_\varepsilon$  (see subsection 3.5.3), the autonomous quasi-static evolution of a thin linearly viscoelastic plate of Kelvin-Voigt type occupying  $\Omega_\varepsilon = \omega \times (-\varepsilon, \varepsilon)$ , where  $\omega$  is a domain of  $\mathbb{R}^2$  with a Lipschitz boundary, reads as:

$$(\mathcal{P}_\varepsilon) \quad \int_{\Omega} \left( a e'(\varepsilon, u_\varepsilon) + b e'(\varepsilon, \dot{u}_\varepsilon) \right) \cdot e'(\varepsilon, v') dx = 0, \forall v' \in H_{\Gamma_0}^1(\Omega; \mathbb{R}^3)$$

where  $\Omega = \omega \times (-1, 1)$  and  $\Gamma_0 = \gamma_0 \times (-\varepsilon, \varepsilon)$ ,  $\gamma_0 \subset \partial\omega$ ,  $\mathfrak{h}_1(\gamma_0) > 0$ , and may be formulated as:

$$(\mathcal{P}_\varepsilon) \quad \begin{cases} \frac{du_\varepsilon}{dt} - A_\varepsilon u_\varepsilon = 0 \text{ in } \mathcal{H}_\varepsilon := H_{\Gamma_0}^1(\Omega; \mathbb{R}^3) \\ u_\varepsilon(0) = u_\varepsilon^0 \end{cases}$$

with

$$D(A_\varepsilon) = \mathcal{H}_\varepsilon$$

$$A_\varepsilon u_\varepsilon = z_\varepsilon ; z_\varepsilon \in H_{\Gamma_0}^1(\Omega; \mathbb{R}^3) \text{ such that}$$

$$\int_{\Omega} \left( a e'(\varepsilon, u_\varepsilon) + b e'(\varepsilon, z_\varepsilon) \right) \cdot e'(\varepsilon, v') dx = 0, \forall v' \in H_{\Gamma_0}^1(\Omega; \mathbb{R}^3)$$

$A_\varepsilon$  is bounded self-adjoint and one has:

$$\begin{aligned} \bar{u}_\varepsilon - \lambda A_\varepsilon \bar{u}_\varepsilon = f_\varepsilon &\Leftrightarrow \int_{\Omega} \left( a + \frac{1}{\lambda} b \right) e'(\varepsilon, \bar{u}_\varepsilon) \cdot e'(\varepsilon, v') dx \\ &= \int_{\Omega} \frac{1}{\lambda} b e'(\varepsilon, f_\varepsilon) \cdot e'(\varepsilon, v') dx, \forall v' \in H_{\Gamma_0}^1(\Omega; \mathbb{R}^3) \end{aligned} \quad (6)$$

The limit behavior of  $\bar{u}_\varepsilon$  when  $\varepsilon$  goes to zero can be found in the mathematical justification of Kirchhoff-Love theory of plates [9, 14].

Recall that according to 1, for all  $e$  in  $\mathbb{S}^3$  one writes

$$e = \widehat{e} + e^\perp$$

so that we introduce  $c_{\wedge\wedge}, c_{\wedge\perp}, c_{\perp\wedge}, c_{\perp\perp}$  the canonical decomposition of  $c \in \text{Lin}(\mathbb{S}^3)$ . We set

$$c^{KL} := c_{\wedge\wedge} - c_{\wedge\perp} (c_{\perp\perp})^{-1} c_{\perp\wedge} \quad (7)$$

Then, as  $\varepsilon$  goes to zero,  $\bar{u}_\varepsilon$  converges strongly in  $H_{\Gamma_0}^1(\Omega; \mathbb{R}^3)$  toward  $\bar{u}$  such that:

$$\begin{aligned} \bar{u} \in V_{KL}(\Omega) ; \int_{\Omega} \left( a + \frac{b}{\lambda} \right)^{KL} \widehat{e(\bar{u})} \cdot \widehat{e(v')} dx &= \int_{\Omega} \frac{b}{\lambda} \widehat{e(f)} \cdot \widehat{e(v')} dx, \\ \forall v' \in V_{KL}(\Omega) &= \left\{ v \in H_{\Gamma_0}^1(\Omega; \mathbb{R}^3) ; e(v)^\perp = 0 \right\} \end{aligned}$$

if one assumes that  $f_\varepsilon$  weakly converges in  $H_{\Gamma_0}^1(\Omega; \mathbb{R}^3)$  toward  $f$  in  $V_{KL}(\Omega)$ . But due to 7,  $(a + \frac{b}{\lambda})^{KL}$  is different from  $a^{KL} + \frac{b^{KL}}{\lambda}$  so that the equation supplying  $\bar{u}$  does not look as the one stemming from a resolvent like 6. So, proceeding as [10] it is easy to show [22] the existence of  $K$  such that

$$\left( a + \frac{1}{\lambda} b \right)^{KL} = a^{KL} + \frac{1}{\lambda} b^{KL} + \mathcal{L} K(\lambda)$$

so that  $u_\varepsilon(t)$  converges uniformly on bounded time intervals strongly in  $H_{\Gamma_0}^1(\Omega; \mathbb{R}^3)$  toward some  $u(t)$  with a Kirchhoff-Love kinematics but with a mechanical behavior which is no longer of Kelvin-Voigt type but viscoelastic with memory!

This frustrating result is due to an incomplete vision of the reduction dimension process, like in subsection 4.3. Actually the convergence result of  $\bar{u}_\varepsilon$  has to be completed as follows:

$$\begin{aligned} w_\varepsilon \in H_{\Gamma_0}^1(\Omega; \mathbb{R}^3) ; \int_{\Omega} d(x) e(\varepsilon, w_\varepsilon) \cdot e(\varepsilon, w') dx \\ = \int_{\Omega} f \cdot w' dx, \forall w' \in H_{\Gamma_0}^1(\Omega; \mathbb{R}^3), f \in L^2(\Omega; \mathbb{R}^3) \\ \Leftrightarrow \exists! (w, w^1) \in V_{KL}(\Omega) \times H_m^1(-1, 1; L^2(\omega; \mathbb{R}^3)) =: \mathcal{H} \text{ such that} \end{aligned}$$



$$\begin{aligned}
& \int_{\Omega \times Y} d\left(e(w) + \partial_3 w^1 \otimes_s e_3\right) \cdot \left(e(w') + \partial_3 w'^1 \otimes_s e_3\right) dx dy \\
& = \int_{\Omega} f w' dx, \forall (w', w'^1) \in \mathcal{H} \\
& \lim_{\varepsilon \rightarrow 0} \int_{\Omega} d\left(e(\varepsilon, w_\varepsilon) - ew - \partial_3 w^1 \otimes_s e_3\right) \cdot \\
& \quad \cdot \left(e(\varepsilon, w_\varepsilon) - e(w) - \partial_3 w^1 \otimes_s e_3\right) dx = 0
\end{aligned}$$

where

$$H_m^1(-1, 1; L^2(\omega; \mathbb{R}^3)) = \left\{ w \in H^1(-1, 1; L^2(\omega; \mathbb{R}^3)) ; \int_{-1}^1 w(\cdot, x_3) dx_3 = 0 \right\}$$

Therefore we can run the Trotter apparatus by choosing a limit space  $\mathcal{H}$  constituted by *couples of displacement fields* so that the limit behavior described by  $u$  and  $u^1$  is still viscoelastic of Kelvin-Voigt type!

**4.5. Final remark.** In the context of the mathematical modeling in Physics which most of the time involves parameterized problems, it appears clearly that a theory of convergence of semi-groups has to be formulated in a variable spaces framework. Situations considered in sections 4.3 and 4.4 let us guess that it is appropriate to operate in such a framework so that a sequence of semi-groups converges to a semi-group.

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