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► To cite this version:

Thierry Cazenave, Yvan Martel, Lifeng Zhao. Solutions blowing up on any given compact set for the energy subcritical nonlinear wave equation. *Journal of Differential Equations*, 2020, <10.1016/j.jde.2019.08.030>. <hal-01950804>

HAL Id: hal-01950804

<https://hal.science/hal-01950804v1>

Submitted on 21 Dec 2021

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SOLUTIONS BLOWING UP ON ANY GIVEN COMPACT SET FOR THE ENERGY SUBCRITICAL WAVE EQUATION

THIERRY CAZENAVE¹, YVAN MARTEL², AND LIFENG ZHAO³

ABSTRACT. We consider the focusing energy subcritical nonlinear wave equation $\partial_{tt}u - \Delta u = |u|^{p-1}u$ in \mathbb{R}^N , $N \geq 1$. Given any compact set $E \subset \mathbb{R}^N$, we construct finite energy solutions which blow up at $t = 0$ exactly on E .

The construction is based on an appropriate ansatz. The initial ansatz is simply $U_0(t, x) = \kappa(t + A(x))^{-\frac{2}{p-1}}$, where $A \geq 0$ vanishes exactly on E , which is a solution of the ODE $h'' = h^p$. We refine this first ansatz inductively using only ODE techniques and taking advantage of the fact that (for suitably chosen A), space derivatives are negligible with respect to time derivatives. We complete the proof by an energy argument and a compactness method.

1. INTRODUCTION

We consider the focusing nonlinear wave equation on \mathbb{R}^N

$$\partial_{tt}u - \Delta u = |u|^{p-1}u, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^N, \quad (1.1)$$

for any space dimension $N \geq 1$, and energy subcritical nonlinearities, *i.e.*

$$1 < p < \infty \text{ if } N = 1, 2 \text{ and } 1 < p < \frac{N+2}{N-2} \text{ if } N \geq 3. \quad (1.2)$$

It is well-known that under such condition on p the Cauchy problem for (1.1) is locally well-posed in the energy space $H^1(\mathbb{R}^N) \times L^2(\mathbb{R}^N)$ (see [8, 9, 25]). For $H^1 \times L^2$ solutions, the energy

$$E(u(t), \partial_t u(t)) = \int \left\{ \frac{1}{2} |\partial_t u(t, x)|^2 + \frac{1}{2} |\nabla u(t, x)|^2 - \frac{1}{p+1} |u(t, x)|^{p+1} \right\} dx$$

is conserved through time. Moreover, it is known how to produce solutions blowing up in finite time (see *e.g.* [10, 18]).

Our main result states that for any given compact set E of \mathbb{R}^N , there exists a finite-energy solution of (1.1) which blows up in finite time exactly on E .

Theorem 1.1. *Let p satisfy (1.2) and let E be any compact set of \mathbb{R}^N . There exists $\delta_0 > 0$ and a solution $(u, \partial_t u) \in \mathcal{C}((0, \delta_0]; H^1(\mathbb{R}^N) \times L^2(\mathbb{R}^N))$ of (1.1) which blows up at time 0 exactly on E in the following sense.*

- If $x_0 \in E$ then for any $r > 0$,

$$\lim_{t \downarrow 0} \|u(t)\|_{L^2(|x-x_0|<r)} = \infty \quad \text{and} \quad \lim_{t \downarrow 0} \|\partial_t u(t)\|_{L^2(|x-x_0|<r)} = \infty. \quad (1.3)$$

- If $x_0 \notin E$ then there exists $r > 0$ such that

$$\sup_{t \in (0, \delta_0]} \left\{ \|u(t)\|_{L^2(|x-x_0|<r)} + \|\nabla u(t)\|_{L^2(|x-x_0|<r)} + \|\partial_t u(t)\|_{L^2(|x-x_0|<r)} \right\} < \infty. \quad (1.4)$$

2010 *Mathematics Subject Classification.* Primary 35L05; secondary 35B44, 35B40.
Key words and phrases. nonlinear wave equation, finite-time blowup, blow-up set.

Remark 1.2. For $t > 0$, the function

$$h(t) = \kappa t^{-\frac{2}{p-1}} \quad \text{where} \quad \kappa = \left[\frac{2(p+1)}{(p-1)^2} \right]^{\frac{1}{p-1}} \quad (1.5)$$

is a solution of the ordinary differential equation $h'' = h^p$ which blows up at time 0. It is also a solution of (1.1), but of course it fails to be in the energy space. The function h is the building block for our construction, it is thus relevant to compare it with the blow-up rate of the solutions constructed in Theorem 1.1. It follows from the proof that for any $0 < \mu < \frac{2}{p-1}$ there exist solutions u as in the statement of Theorem 1.1 satisfying in addition the following estimates: for any $x_0 \in E$, $r > 0$, and all $t \in (0, \delta_0]$,

$$t^{-\mu} \lesssim \|u(t)\|_{L^2(|x-x_0|<r)} \lesssim t^{-\frac{2}{p-1}}, \quad (1.6)$$

$$t^{-\mu-1} \lesssim \|\partial_t u(t)\|_{L^2(|x-x_0|<r)} \lesssim t^{-\frac{2}{p-1}-1}. \quad (1.7)$$

Moreover, if $x_0 \in E$ and E contains a neighborhood of x_0 then it also holds, for any $r > 0$, and all $t \in (0, \delta_0]$,

$$\|u(t)\|_{L^2(|x-x_0|<r)} \gtrsim t^{-\frac{2}{p-1}} \quad \text{and} \quad \|\partial_t u(t)\|_{L^2(|x-x_0|<r)} \gtrsim t^{-\frac{2}{p-1}-1}. \quad (1.8)$$

In contrast, if x_0 is an isolated point of the compact set E , solutions u as in Theorem 1.1 can be chosen so that, for a small $r > 0$,

$$\lim_{t \downarrow 0} \left\{ t^{\frac{2}{p-1}} \|u(t)\|_{L^2(|x-x_0|<r)} + t^{\frac{2}{p-1}+1} \|\partial_t u(t)\|_{L^2(|x-x_0|<r)} \right\} = 0.$$

To prove Theorem 1.1, we follow the strategy developed in [4] to construct blow-up solutions of ODE type for a class of semilinear Schrödinger equations. First, we construct an approximate solution to the blow-up problem based on the explicit blow-up solution h defined by (1.5). The main order term of the approximate solution is $U_0(t, x) = \kappa(t + A(x))^{-\frac{2}{p-1}}$, where A is a suitable nonnegative function which vanishes exactly on E and whose behavior at ∞ ensures that U_0 belongs to the energy space. Typically, to obtain blowup at only one point x_0 , it suffices to consider $A(x) = |x - x_0|^k$ for k large enough. Compared to [4] where a simple ansatz such as U_0 is sufficient, at least for strong enough nonlinearities, the wave equation requires to introduce iterated refinements U_J of this ansatz (the number of iterations $J \geq 1$ depends on p , see Remark 2.4). The basic idea is that for such blow-up profiles, the space derivatives are of lower order compared to time derivatives and to nonlinear terms. This allows to use only elementary arguments of ordinary differential equations for the construction of the refined ansatz $U_J(t, x)$, at fixed x . See Section 2.

Second, we consider the sequence (u_n) of solutions of the wave equation (1.1) with initial data $u_n(\frac{1}{n}) = U_J(\frac{1}{n})$. Using energy method in $H^1 \times L^2$, we prove uniform estimates on this sequence on intervals $[\frac{1}{n}, \delta_0]$, where $\delta_0 > 0$ is uniform in n (see Section 3). Passing to the limit $n \rightarrow \infty$ yields the solution u of Theorem 1.1.

We point out that this strategy by approximate solution and compactness is also reminiscent to [19, 20, 24] where global or blow-up solutions with special asymptotic behavior are constructed using the reversibility of the equation and suitable uniform estimates on backwards solutions.

For stability results concerning the solution h (1.2), we refer to [7]. For ODE-type blowup for quasilinear wave equations, see [26] and the references therein. We also refer to [5] where an ODE blow-up profile similar to U_0 is used to construct blow-up solutions of the nonlinear heat equation with applications to the Burgers equation.

In this article, we restrict ourselves to energy subcritical power nonlinearities for simplicity, since this framework allows us to use the energy method at the level of regularity $H^1 \times L^2$ only. However, the approximate solutions constructed in Section 2 are relevant for any power nonlinearity, and we expect that a higher order energy method (to estimate higher order Sobolev norms) should be sufficient to extend the construction to energy critical or supercritical nonlinearities (at least for integer powers to avoid regularity issues).

Remark 1.3. A more general question for nonlinear wave equations concerns the blow-up surface. For a solution of (1.1) with initial data at $t = 0$, which is assumed to blow up in finite time, there exists a 1-Lipschitz function $x \mapsto \phi(x) > 0$ such that the solution is well-defined in a suitable sense in the maximal domain of influence $D = \{(t, x) : 0 \leq t < \phi(x)\}$, see *e.g.* [1], Sections III.2 and III.3. The surface $\{(\phi(x), x) : x \in \mathbb{R}^N\}$ is called the blow-up surface. The question of the regularity of blow-up surface is addressed in [1, 2, 3, 21, 22]. The question of constructing solutions of the nonlinear wave equation with prescribed blow-up surface (with sufficient regularity and satisfying the space-like condition $\|\nabla\phi\|_{L^\infty} < 1$) is also a classical question, addressed in several articles and books, notably [15, 16], [11, 12, 13], [17] and [1]. The approach by Fuschian reduction is especially well-described in the book [13]. First developed for analytic surfaces and exponential nonlinearity, this method was later extended to surfaces with Sobolev regularity and to some power nonlinearities. However, it is not clear to us whether the strategy described in [13] for constructing solutions with given blow-up surface can be extended to power nonlinearities $|u|^{p-1}u$ for any $p > 1$, or to more general nonlinearities.

Prescribing the blow-up set of a blow-up solution can be seen as a sub-product of prescribing its blow-up surface. This issue is discussed in [13, 14, 17]. However, the solutions constructed in [13, 14, 17] may only exist in a space-time region around the blow-up surface, which does not guarantee that the solution is globally defined in space at any one specific time.

We also would like to point out a difference between the above mentioned articles and our approach. Here, we resolutely work with finite energy solutions and the initial value problem for (1.1). It is often argued that finite speed of propagation and cut-off arguments allow to reduce to finite energy solutions. For example, the function (1.5) is used to claim that ODE-type blowup is easy to reach for finite energy solutions. However, the cut-off necessary to localize the initial data could lead to blowup in an earlier time. Our method deals with these issues by constructing directly a finite energy solution with initial data from a finite energy ansatz. Moreover, we hope that our somehow elementary approach can be of interest for its simplicity and its large range of applicability to other more complicated problems where ODE blowup is relevant.

Notation. We fix a smooth, even function $\chi : \mathbb{R} \rightarrow \mathbb{R}$ satisfying:

$$\chi \equiv 1 \text{ on } [0, 1], \chi \equiv 0 \text{ on } [2, \infty) \text{ and } \chi' \leq 0 \leq \chi \leq 1 \text{ on } [0, \infty). \quad (1.9)$$

For $p > 1$ satisfying (1.2), recall the well-known inequality, for any $u \in H^1$,

$$\|u\|_{L^{p+1}}^{p+1} \lesssim \|u\|_{L^2}^{p+1 - \frac{N}{2}(p-1)} \|\nabla u\|_{L^2}^{\frac{N}{2}(p-1)}. \quad (1.10)$$

Let $f(u) = |u|^{p-1}u$ and $F(u) = \int_0^u f(v)dv$. For future reference, we recall Taylor's formulas involving the functions F and f . Let $\bar{p} = \min(2, p)$. First, we claim that for any $u > 0$ and $v \in \mathbb{R}$,

$$\left| F(u+v) - F(v) - F'(u)v - \frac{1}{2}F''(u)v^2 \right| \lesssim |v|^{p+1} + u^{p-\bar{p}}|v|^{\bar{p}+1}. \quad (1.11)$$

Indeed, in the region $|v| \geq \frac{1}{2}u$, each term on the left-hand side is bounded by $|v|^{p+1}$. In the region $|v| \leq \frac{1}{2}u$, we use Taylor's expansion to write

$$\left| F(u+v) - F(u) - F'(u)v - \frac{1}{2}F''(u)v^2 \right| \lesssim u^{p-2}|v|^3.$$

If $p \geq 2$, then $\bar{p} = 2$ and (1.11) is proved. If $1 < p < 2$, we finish by saying that in this case $u^{p-2}|v|^3 \lesssim |v|^{p+1}$. The same argument shows that

$$|(f(u+v) - f(u) - f'(u)v)v| \lesssim |v|^{p+1} + u^{p-\bar{p}}|v|^{\bar{p}+1}. \quad (1.12)$$

Next, we claim that for any $u > 0$ and $v \in \mathbb{R}$,

$$\left| f(u+v) - f(u) - f'(u)v - \frac{1}{2}f''(u)v^2 \right| \lesssim u^{-1}|v|^{p+1} + u^{p-\bar{p}-1}|v|^{\bar{p}+1}. \quad (1.13)$$

Indeed, in the region $|v| \geq \frac{1}{2}u$, each term on the left-hand side is bounded by $u^{-1}|v|^{p+1}$, and (1.13) follows. In the region $|v| \leq \frac{1}{2}u$, we use Taylor's expansion to write

$$\left| f(u+v) - f(u) - f'(u)v - \frac{1}{2}f''(u)v^2 \right| \lesssim u^{p-3}|v|^3.$$

If $p \geq 2$, then $\bar{p} = 2$ and (1.13) is proved. If $1 < p < 2$, we finish by saying that in this case $u^{p-3}|v|^3 \lesssim u^{-1}|v|^{p+1}$.

In this article, we will use multi-variate notation and results from [6]. For any $\beta = (\beta_1, \dots, \beta_N) \in \mathbb{N}^N$, $x = (x_1, \dots, x_N) \in \mathbb{R}^N$, we set

$$|\beta| = \sum_{j=1}^N \beta_j, \quad \beta! = \prod_{j=1}^N (\beta_j!), \quad x^\beta = \prod_{j=1}^N x_j^{\beta_j},$$

$$\partial_x^0 = \text{Id} \quad \text{and} \quad \partial_x^\beta = \frac{\partial^{|\beta|}}{\partial_{x_1}^{\beta_1} \dots \partial_{x_N}^{\beta_N}} \quad \text{for } |\beta| > 0.$$

For $\beta, \beta' \in \mathbb{N}^N$, we write $\beta' \leq \beta$ if $\beta'_j \leq \beta_j$ for all $j = 1, \dots, N$. When $\beta' \leq \beta$, we set

$$\binom{\beta}{\beta'} = \prod_{j=1}^N \binom{\beta_j}{\beta'_j} = \frac{\beta!}{\beta'!(\beta - \beta')!}.$$

With this notation, given two functions $a, b : \mathbb{R}^N \rightarrow \mathbb{R}$, Leibniz's formula writes:

$$\partial_x^\beta (ab) = \sum_{\beta' \leq \beta} \binom{\beta}{\beta'} (\partial_x^{\beta'} a) (\partial_x^{\beta - \beta'} b). \quad (1.14)$$

We write $\beta' \prec \beta$ if one of the following holds

- $|\beta'| < |\beta|$;
- $|\beta'| = |\beta|$ and $\beta'_1 < \beta_1$;
- $|\beta'| = |\beta|$, $\beta'_1 = \beta_1, \dots, \beta'_\ell = \beta_\ell$ and $\beta'_{\ell+1} < \beta_{\ell+1}$ for some $1 \leq \ell < N$.

Finally, we recall the Faà di Bruno formula (see Corollary 2.10 in [6]). Let $n = |\beta| \geq 1$. Then, for functions $q : \mathbb{R} \rightarrow \mathbb{R}$, $a : \mathbb{R}^N \rightarrow \mathbb{R}$,

$$\partial_x^\beta (q \circ a) = \sum_{\nu=1}^n (q^{(\nu)} \circ a) \sum_{P(\beta, \nu)} (\beta!) \prod_{\ell=1}^n \frac{(\partial_x^{\beta_\ell} a)^{\nu_\ell}}{(\nu_\ell!)(\beta_\ell!)^{\nu_\ell}} \quad (1.15)$$

where

$$P(\beta, \nu) = \left\{ (\nu_1, \dots, \nu_n; \beta_1, \dots, \beta_n) : \text{there exists } 1 \leq m \leq n \text{ such that} \right.$$

$$\left. \begin{aligned} &\nu_\ell = 0 \text{ and } \beta_\ell = 0 \text{ for } 1 \leq \ell \leq n - m; \nu_\ell > 0 \text{ for } n - m + 1 \leq \ell \leq n; \\ &\text{and } 0 \prec \beta_{n-m+1} \prec \dots \prec \beta_n \text{ are such that } \sum_{\ell=1}^n \nu_\ell = \nu, \sum_{\ell=1}^n \nu_\ell \beta_\ell = \beta \right\}.$$

2. THE BLOW-UP ANSATZ

2.1. Preliminary. Recall that h is the explicit solution (1.5) of the equation $h'' = h^p$ which blows up at 0. The linearization of this equation around the solution h yields the linear equation

$$g'' = ph^{p-1}g = \frac{2p(p+1)}{(p-1)^2}t^{-2}g$$

which admits the following two independent solutions

$$g_1(t) = t^{-\frac{p+1}{p-1}}, \quad g_2(t) = t^{\frac{2p}{p-1}}, \quad \text{for } t > 0.$$

Since $\frac{p+1}{p-1} > \frac{2}{p-1}$, the function g_1 , related to time invariance, is more singular at 0 than the function h . Note also that for a function G satisfying $\int_0^1 s^{\frac{2p}{p-1}}|G(s)|ds < \infty$, a solution of the following linearized equation with source G

$$g'' = \frac{2p(p+1)}{(p-1)^2}t^{-2}g + G,$$

is given by

$$g(t) = -\frac{p-1}{3p+1} \left(t^{-\frac{p+1}{p-1}} \int_0^t s^{\frac{2p}{p-1}} G(s) ds + t^{\frac{2p}{p-1}} \int_t^1 s^{-\frac{p+1}{p-1}} G(s) ds \right).$$

2.2. First blow-up ansatz. Set

$$J = \left\lfloor \frac{p+1}{p-1} \right\rfloor \quad \text{and} \quad k \geq 2J + 2 \quad (2.1)$$

where $x \mapsto \lfloor x \rfloor$ is the floor function which maps x to the greatest integer less than or equal to x . (See Remark 2.4 below for the explanation of the numbers J and k .) We consider a function $A : \mathbb{R}^N \rightarrow \mathbb{R}$ of class C^{k-1} on \mathbb{R}^N and of class C^k piecewise on \mathbb{R}^N such that, for any $\beta \in \mathbb{N}^N$, with $|\beta| \leq k-1$, the following hold

$$\begin{cases} A \geq 0 \text{ and } |\partial_x^\beta A| \lesssim A^{1-\frac{|\beta|}{k}} & \text{on } \mathbb{R}^N, \\ A(x) = |x|^k & \text{for } x \in \mathbb{R}^N, |x| \geq 2. \end{cases} \quad (2.2)$$

Remark 2.1. Typical examples of such functions are $A(x) := |x|^k$, which vanishes at 0 and

$$A(x) := \begin{cases} 0 & \text{if } |x| \leq 1 \\ (|x| - \chi(x))^k & \text{if } 1 < |x| \leq 2 \\ |x|^k & \text{if } |x| > 2 \end{cases}$$

(where χ is given by (1.9)) which vanishes on the closed ball of center 0 and radius 1. Another example, important for the proof of Theorem 1.1 is given in Section 4: for any compact set E of \mathbb{R}^N included in the open ball of center 0 and radius 1, there exists a function A satisfying (2.2) which vanishes exactly on E .

For $t > 0$ and $x \in \mathbb{R}^N$, set

$$U_0(t, x) = \kappa(t + A(x))^{-\frac{2}{p-1}} = h(W(t, x)) \quad \text{where} \quad W(t, x) = t + A(x),$$

so that U_0 satisfies $\partial_{tt}U_0 = f(U_0)$ on $(0, \infty) \times \mathbb{R}^N$. Let

$$\mathcal{E}_0 = -\partial_{tt}U_0 + \Delta U_0 - f(U_0) = \Delta U_0.$$

We gather in the next lemma some estimates for U_0 and \mathcal{E}_0 .

Lemma 2.2. *The function U_0 satisfies*

$$\partial_t U_0 = -\left(\frac{2}{p+1}U_0^{p+1}\right)^{\frac{1}{2}}, \quad (\partial_t U_0)^2 = \frac{2}{p+1}U_0^{p+1}, \quad \partial_{tt}U_0 = U_0^p. \quad (2.3)$$

Moreover, for any $\beta \in \mathbb{N}^N$, $\rho \in \mathbb{R}$, $0 < t \leq 1$, $x \in \mathbb{R}^N$, the following hold.

(i) If $0 \leq |\beta| \leq k-1$ and $|x| \leq 2$,

$$|\partial_x^\beta(U_0^\rho)| \lesssim U_0^{\rho + \frac{|\beta|}{k} \frac{p-1}{2}}, \quad |\partial_t \partial_x^\beta(U_0^\rho)| \lesssim U_0^{\rho + (1 + \frac{|\beta|}{k}) \frac{p-1}{2}}. \quad (2.4)$$

(ii) If $0 \leq |\beta| \leq k-3$ and $|x| \leq 2$,

$$|\partial_x^\beta \mathcal{E}_0| \lesssim U_0^{1 + \frac{2+|\beta|}{k} \frac{p-1}{2}}. \quad (2.5)$$

(iii) If $|x| > 2$,

$$|\partial_x^\beta U_0| \lesssim |x|^{-\frac{2k}{p-1} - |\beta|}, \quad |\partial_x^\beta \mathcal{E}_0| \lesssim |x|^{-\frac{2k}{p-1} - |\beta| - 2}. \quad (2.6)$$

Furthermore, for any $x_0 \in \mathbb{R}^N$ such that $A(x_0) = 0$, for any $r > 0$, $0 < t \leq 1$,

$$t^{-\frac{2}{p-1} + \frac{N}{2k}} \lesssim \|U_0(t)\|_{L^2(|x-x_0| < r)} \lesssim t^{-\frac{2}{p-1}}, \quad (2.7)$$

$$t^{-\frac{2}{p-1} - 1 + \frac{N}{2k}} \lesssim \|\partial_t U_0(t)\|_{L^2(|x-x_0| < r)} \lesssim t^{-\frac{2}{p-1} - 1}, \quad (2.8)$$

where the implicit constants in (2.7) and (2.8) depend on r .

Proof. The identities in (2.3) follow from the definition of U_0 and direct calculations.

Proof of (2.4)-(2.5). For $0 < t \leq 1$ and $|x| \leq 2$, one has $0 < t + A(x) \lesssim 1$ and thus $U_0 \gtrsim 1$. From $U_0 = h \circ W$, setting $n = |\beta|$ and using (1.15), one has

$$\partial_x^\beta U_0 = \sum_{\nu=1}^n \left(h^{(\nu)} \circ W \right) \sum_{P(\beta, \nu)} (\beta!) \prod_{\ell=1}^n \frac{(\partial_x^{\beta_\ell} W)^{\nu_\ell}}{(\nu_\ell!)(\beta_\ell!)^{\nu_\ell}}.$$

For $\nu \geq 1$, we have $|h^{(\nu)} \circ W| \lesssim W^{-\frac{2}{p-1} - \nu}$. Moreover, using the assumption (2.2), we have, for $1 \leq |\beta_\ell| \leq k-1$,

$$|\partial_x^{\beta_\ell} W| \lesssim |\partial_x^{\beta_\ell} A| \lesssim A^{1 - \frac{|\beta_\ell|}{k}}.$$

Since $\sum_{\ell=1}^n \nu_\ell = \nu$, $\sum_{\ell=1}^n \nu_\ell |\beta_\ell| = |\beta|$ and $|\beta| \leq k-1$, we obtain

$$\begin{aligned} |\partial_x^\beta U_0| &\lesssim \sum_{\nu=1}^n W^{-\frac{2}{p-1} - \nu} \sum_{P(\beta, \nu)} \prod_{\ell=1}^n \left(A^{1 - \frac{|\beta_\ell|}{k}} \right)^{\nu_\ell} \\ &\lesssim \sum_{\nu=1}^n W^{-\frac{2}{p-1} - \nu} A^{\nu - \frac{|\beta|}{k}} \lesssim W^{-\frac{2}{p-1} - \frac{|\beta|}{k}} \lesssim U_0^{1 + \frac{|\beta|}{k} \frac{p-1}{2}}, \end{aligned}$$

which proves the first estimate of (2.4) for $\rho = 1$. For $\rho \in \mathbb{R}$, using (1.15), we also have, for $1 \leq n = |\beta| \leq k-1$,

$$\partial_x^\beta (U_0^\rho) = \sum_{\nu=1}^n [\rho \cdots (\rho - \nu + 1)] U_0^{\rho - \nu} \sum_{P(\beta, \nu)} (\beta!) \prod_{\ell=1}^n \frac{(\partial_x^{\beta_\ell} U_0)^{\nu_\ell}}{(\nu_\ell!)(\beta_\ell!)^{\nu_\ell}}.$$

Using the above estimate on $|\partial_x^{\beta_\ell} U_0|$ and $\sum_{\ell=1}^n \nu_\ell = \nu$, $\sum_{\ell=1}^n \nu_\ell |\beta_\ell| = |\beta|$, we obtain

$$|\partial_x^\beta (U_0^\rho)| \lesssim \sum_{\nu=1}^n U_0^{\rho - \nu} \sum_{P(\beta, \nu)} \prod_{\ell=1}^n U_0^{\nu_\ell \left[1 + \frac{|\beta_\ell|}{k} \frac{p-1}{2} \right]} \lesssim \sum_{\nu=1}^n U_0^{\rho - \nu} U_0^{\nu + \frac{|\beta|}{k} \frac{p-1}{2}} \lesssim U_0^{\rho + \frac{|\beta|}{k} \frac{p-1}{2}}.$$

Next, using the first identity in (2.3), we see that $\partial_t U_0^\rho = -\rho \left(\frac{2}{p+1} \right)^{\frac{1}{2}} U_0^{\rho + \frac{p-1}{2}}$; and so the second estimate in (2.4) follows from the first. Since $\mathcal{E}_0 = \Delta U_0$, (2.5) is an immediate consequence of the first estimate in (2.4).

Estimate (2.6) is a direct consequence of the definitions of U_0 and \mathcal{E}_0 and of the fact that $A(x) = |x|^k$ for $|x| > 2$.

Proof of (2.7)-(2.8). For any $x_0 \in \mathbb{R}^N$ and $r > 0$, the upper bounds in (2.7) and (2.8) are direct consequences of the estimates $0 \leq U_0 \lesssim t^{-\frac{2}{p-1}}$ and $|\partial_t U_0| \lesssim t^{-\frac{2}{p-1} - 1}$. Let $x_0 \in \mathbb{R}^N$ be such that $A(x_0) = 0$ and $r > 0$. By (2.2) and the fact that the function A is of class \mathcal{C}^k piecewise, the Taylor formula implies that for any

x such that $|x - x_0| < r$, $|A(x)| \leq C(r)|x - x_0|^k$. It follows that for such x , and for any $t \in (0, 1]$, $U_0^2(t, x) = \kappa^2(t + A(x))^{-\frac{4}{p-1}} \gtrsim (t + |x - x_0|^k)^{-\frac{4}{p-1}}$. The lower estimate in (2.7) then follows from

$$\begin{aligned} \int_{|y| < r} (t + |y|^k)^{-\frac{4}{p-1}} dy &= t^{-\frac{4}{p-1} + \frac{N}{k}} \int_{|z| < t^{-\frac{1}{k}} r} (1 + |z|^k)^{-\frac{4}{p-1}} dz \\ &\gtrsim t^{-\frac{4}{p-1} + \frac{N}{k}} \int_{|z| < r} (1 + |z|^k)^{-\frac{4}{p-1}} dz \gtrsim t^{-\frac{4}{p-1} + \frac{N}{k}}. \end{aligned} \quad (2.9)$$

Estimate (2.8) is proved similarly. \square

2.3. Refined blow-up ansatz. Starting from U_0 , we define by induction a refined ansatz to the nonlinear wave equation. Let $t_0 = 1$ and for any $j \in \{1, \dots, J\}$, let $0 < a_j \leq 1$ and $0 < t_j \leq 1$ to be chosen later. Let

$$\begin{aligned} w_j &= -\kappa^{\frac{p-1}{2}} \frac{p-1}{3p+1} \left(U_0^{\frac{p+1}{2}} \int_0^t U_0^{-p} \mathcal{E}_{j-1} ds + U_0^{-p} \int_t^{t_j-1} U_0^{\frac{p+1}{2}} \mathcal{E}_{j-1} ds \right), \\ U_j &= U_0 + \sum_{\ell=1}^j \chi_\ell w_\ell, \quad \mathcal{E}_j = -\partial_{tt} U_j + \Delta U_j + f(U_j), \end{aligned}$$

where $\chi_j(x) = \chi(A(x)/a_j)$ and χ satisfies (1.9).

Lemma 2.3. *There exist $0 < a_j \leq \dots \leq a_1 \leq 1$ and $0 < t_j \leq \dots \leq t_1 \leq 1$ such that for any $0 \leq j \leq J$, for any $\beta \in \mathbb{N}^N$, $0 < t \leq t_j$ and $x \in \mathbb{R}^N$, the following hold.*

(i) *If $1 \leq j \leq J$, $0 \leq |\beta| \leq k - 1 - 2j$, $|x| \leq 2$, then*

$$|\partial_x^\beta w_j| \lesssim U_0^{1-j(p-1) + \frac{2j+|\beta|}{k} \frac{p-1}{2}}, \quad (2.10)$$

$$|\partial_t \partial_x^\beta w_j| \lesssim U_0^{\frac{p+1}{2} - j(p-1) + \frac{2j+|\beta|}{k} \frac{p-1}{2}}. \quad (2.11)$$

(ii) *If $1 \leq j \leq J$, then*

$$|U_j - U_0| \leq \frac{1}{4} (1 - 2^{-j}) U_0, \quad |U_j - U_0| \leq (1 - 2^{-j}) (1 + U_0)^{-\frac{p-1}{2}} U_0, \quad (2.12)$$

$$|\partial_t U_j - \partial_t U_0| \lesssim U_0. \quad (2.13)$$

(iii) *If $0 \leq |\beta| \leq k - 3 - 2j$, $|x| \leq 2$, then*

$$|\partial_x^\beta \mathcal{E}_j| \lesssim U_0^{1-j(p-1) + \frac{2j+2+|\beta|}{k} \frac{p-1}{2}}. \quad (2.14)$$

(iv) *If $|x| \geq 2$, then*

$$|\partial_x^\beta U_j| \lesssim |x|^{-\frac{2k}{p-1} - |\beta|}, \quad |\partial_x^\beta \mathcal{E}_j| \lesssim |x|^{-\frac{2k}{p-1} - 2 - |\beta|}. \quad (2.15)$$

Remark 2.4. We comment on the mechanism of the refined ansatz. For the energy control which we establish in the next section, we need an estimate on the error term $\|\mathcal{E}_j\|_{L^2} \lesssim t^{(\frac{2}{p-1})^+}$. (See formulas (3.20) and (3.21).) By formula (2.14), this is achieved if $J > \frac{2}{p-1}$, which is the first condition in (2.1), and then k sufficiently large (once J is chosen), which is the second condition in (2.1). Note that for $p > 3$, $J = 1$ is enough, but one can never choose $J = 0$, so a refined ansatz is always needed. We see on formula (2.14) that at each step, the error estimate improves by a factor $U_0^{-(p-1)(1-\frac{1}{k})} \sim t^{2(1-\frac{1}{k})}$. It is clear then that the number of steps goes to ∞ as $p \rightarrow 1$.

Proof of Lemma 2.3. Observe that (2.14) for $j = 0$ is exactly (2.5) in Lemma 2.2. Now, we proceed by induction on j : for any $1 \leq j \leq J$, we prove that estimate (2.14) for \mathcal{E}_{j-1} implies estimates (2.10)–(2.14) for w_j , U_j and \mathcal{E}_j , for an appropriate choice of a_j and t_j .

Proof of (2.10)–(2.11). First, assuming (2.14) for \mathcal{E}_{j-1} , we show the following estimates related to the two components of w_j : for $|\beta| \leq k - 1 - 2j$, $0 < t \leq t_{j-1}$ and $|x| \leq 2$,

$$\left| \partial_x^\beta \left(\int_0^t U_0^{-p} \mathcal{E}_{j-1} ds \right) \right| \lesssim U_0^{-\frac{p-1}{2} - j(p-1) + \frac{2j+|\beta|}{k} \frac{p-1}{2}}, \quad (2.16)$$

$$\left| \partial_x^\beta \left(\int_t^{t_{j-1}} U_0^{\frac{p+1}{2}} \mathcal{E}_{j-1} ds \right) \right| \lesssim U_0^{p+1-j(p-1) + \frac{2j+|\beta|}{k} \frac{p-1}{2}}. \quad (2.17)$$

Indeed, we have by the Leibniz's formula (1.14)

$$\partial_x^\beta (U_0^{-p} \mathcal{E}_{j-1}) = \sum_{\beta' \leq \beta} \binom{\beta}{\beta'} \left(\partial_x^{\beta'} (U_0^{-p}) \right) \left(\partial_x^{\beta-\beta'} \mathcal{E}_{j-1} \right),$$

and thus, using (2.4) and (2.14),

$$\begin{aligned} |\partial_x^\beta (U_0^{-p} \mathcal{E}_{j-1})| &\lesssim \sum_{\beta' \leq \beta} \binom{\beta}{\beta'} U_0^{-p + \frac{|\beta'|}{k} \frac{p-1}{2}} U_0^{1-(j-1)(p-1) + \frac{2j+|\beta|-|\beta'|}{k} \frac{p-1}{2}} \\ &\lesssim U_0^{-j(p-1) + \frac{2j+|\beta|}{k} \frac{p-1}{2}} \lesssim (t+A)^\gamma, \end{aligned}$$

where for $j \geq 1$, $|\beta| \leq k$,

$$\gamma := 2j - \frac{2j+|\beta|}{k} = 2j \left(1 - \frac{1}{k} \right) - \frac{|\beta|}{k} \geq 0.$$

Integrating on $(0, t)$ for $t \in (0, t_{j-1}]$, we obtain

$$\left| \partial_x^\beta \left(\int_0^t U_0^{-p} \mathcal{E}_{j-1} ds \right) \right| \lesssim (t+A)^{\gamma+1} \leq U_0^{-\frac{p-1}{2} - j(1-\frac{1}{k})(p-1) + \frac{|\beta|}{k} \frac{p-1}{2}},$$

which is (2.16). Similarly, using Leibniz's formula, we check the following estimate

$$\left| \partial_x^\beta (U_0^{\frac{p+1}{2}} \mathcal{E}_{j-1}) \right| \lesssim U_0^{\frac{3p+1}{2} - j(p-1) + \frac{2j+|\beta|}{k} \frac{p-1}{2}} \lesssim (t+A)^{-\gamma'},$$

where, using $0 < j \leq J \leq \frac{p+1}{p-1}$,

$$\gamma' := \frac{3p+1}{p-1} - 2j + \frac{2j+|\beta|}{k} > 1.$$

Thus, by time integration, for $t \in (0, t_{j-1}]$,

$$\left| \int_t^{t_{j-1}} \partial_x^\beta (U_0^{\frac{p+1}{2}} \mathcal{E}_{j-1}) ds \right| \lesssim (t+A)^{-\gamma'+1} \lesssim U_0^{p+1-j(p-1) + \frac{2j+|\beta|}{k} \frac{p-1}{2}},$$

which is (2.17).

Using Leibniz's formula, (2.4), and (2.16)–(2.17), we deduce easily that, for any $\beta \in \mathbb{N}^N$, $|\beta| \leq k - 1 - 2j$,

$$\begin{aligned} \left| \partial_x^\beta \left(U_0^{\frac{p+1}{2}} \int_0^t U_0^{-p} \mathcal{E}_{j-1} ds \right) \right| &\lesssim U_0^{1-j(p-1) + \frac{2j+|\beta|}{k} \frac{p-1}{2}}, \\ \left| \partial_x^\beta \left(U_0^{-p} \int_t^{t_{j-1}} U_0^{\frac{p+1}{2}} \mathcal{E}_{j-1} ds \right) \right| &\lesssim U_0^{1-j(p-1) + \frac{2j+|\beta|}{k} \frac{p-1}{2}}. \end{aligned}$$

Estimate (2.10) follows. Moreover, by the definition of w_j and setting $b = \kappa \frac{p-1}{2} \frac{p-1}{3p+1}$,

$$\partial_t w_j = -b \left(\partial_t (U_0^{\frac{p+1}{2}}) \int_0^t U_0^{-p} \mathcal{E}_{j-1} ds + \partial_t (U_0^{-p}) \int_t^{t_{j-1}} U_0^{\frac{p+1}{2}} \mathcal{E}_{j-1} ds \right). \quad (2.18)$$

Similarly as above, Leibniz's formula, (2.4), and (2.16)-(2.17) yield (2.11). Note that we have proved estimates (2.10) and (2.11) for all $0 < t \leq t_{j-1}$.

Proof of (2.12)-(2.13). For $0 < t \leq t_{j-1}$ and $|x| \leq 2$, by the estimate (2.4) on w_j for $\beta = 0$, the property $U_0 \gtrsim 1$ for $|x| \leq 2$, and the definition of χ_j , we have

$$\chi_j |w_j| \lesssim \chi_j U_0^{1-j(1-\frac{1}{k})(p-1)} \lesssim \chi_j U_0^{1-(1-\frac{1}{k})(p-1)} \lesssim \chi_j (t+A)^{2-\frac{2}{k}} U_0 \lesssim (t+a_j) U_0.$$

Choosing $0 < a_j \leq 1$ and $0 < t_j \leq t_{j-1}$ sufficiently small, for all $t \in (0, t_j]$,

$$\chi_j |w_j| \leq 2^{-j-2} U_0 \quad \text{and} \quad \chi_j |w_j| \leq 2^{-j} (1+U_0)^{-\frac{p-1}{2}} U_0.$$

From now on, a_j and t_j are fixed to such values. In the case $j = 1$, this proves (2.12) for $|x| \leq 2$. For $2 \leq j \leq J$, combining this estimate with (2.12) for $j-1$, we find, for all $t \in (0, t_j]$ and $|x| \leq 2$,

$$\sum_{\ell=1}^j \chi_\ell |w_\ell| \leq \frac{1}{4} (1-2^{-j}) U_0 \quad \text{and} \quad \sum_{\ell=1}^j \chi_\ell |w_\ell| \leq (1-2^{-j}) (1+U_0)^{-\frac{p-1}{2}} U_0, \quad (2.19)$$

which implies (2.12) for U_j and for $|x| \leq 2$.

To prove (2.13) for $|x| \leq 2$, we note that by (2.11) with $\beta = 0$ and $U_0 \gtrsim 1$,

$$\sum_{\ell=1}^j \chi_\ell |\partial_t w_\ell| \lesssim \sum_{\ell=1}^j \chi_\ell U_0^{\frac{p+1}{2}-\ell(1-\frac{1}{k})(p-1)} = U_0 \sum_{\ell=1}^j \chi_\ell U_0^{\frac{p-1}{2}(1-2\ell(1-\frac{1}{k}))} \lesssim U_0.$$

For $|x| \geq 2$, (2.2) implies that $A(x) \geq 2^k \geq 2a_1 \geq \dots \geq 2a_j$ and thus $\chi_j(x) = 0$ and $U_j(t, x) = U_0(t, x)$. The same applies to $\partial_t U_j$.

Proof of (2.14). Differentiating (2.18) with respect to t , using the relations (2.3), $\partial_{tt}(U_0^{\frac{p+1}{2}}) = f'(U_0)U_0^{\frac{p+1}{2}}$ and $\partial_{tt}(U_0^{-p}) = f'(U_0)U_0^{-p}$ (these calculations are related to observations made in Section 2.1), we check that w_j satisfies

$$\partial_{tt} w_j = f'(U_0)w_j + \mathcal{E}_{j-1}.$$

Using also $U_j = U_{j-1} + \chi_j w_j$ and the definition of \mathcal{E}_{j-1} , we obtain

$$\begin{aligned} \mathcal{E}_j &= \mathcal{E}_{j-1} - \chi_j \partial_{tt} w_j + \Delta(\chi_j w_j) + f(U_j) - f(U_{j-1}) \\ &= (1 - \chi_j) \mathcal{E}_{j-1} + \Delta(\chi_j w_j) + f(U_j) - f(U_{j-1}) - f'(U_0) \chi_j w_j. \end{aligned}$$

We estimate ∂_x^β of each term on the right-hand side above for $|\beta| \leq k-3-2j$ and $|x| \leq 2$. For the first term, recall that for x such that $A(x) \leq a_j$, it holds $1 - \chi_j(x) = 0$ and for any β , $\partial_x^\beta \chi_j(x) = 0$. Moreover, for $0 < t \leq 1$, for $0 \leq x \leq 2$ such that $A(x) \geq a_j$, it holds $A(x) \approx 1$ and so $U_0(t, x) \approx 1$. Thus, using the Leibniz formula and (2.14) for \mathcal{E}_{j-1} , we find

$$|\partial_x^\beta [(1 - \chi_j) \mathcal{E}_{j-1}]| \lesssim U_0^{1-(j-1)(p-1) + \frac{2j+|\beta|}{k} \frac{p-1}{2}} \lesssim U_0^{1-j(p-1) + \frac{2j+2+|\beta|}{k} \frac{p-1}{2}}.$$

Next, by the Leibniz's formula, the properties of χ and χ_j , the estimate (2.10) on w_j and then $U_0 \gtrsim 1$, we have, for $0 < t < t_j$ and $|x| \leq 2$,

$$|\partial_x^\beta \Delta(\chi_j w_j)| \lesssim \sum_{|\beta'| \leq |\beta|+2} |\partial_x^{\beta'} w_j| \lesssim U_0^{1-j(p-1) + \frac{2j+2+|\beta|}{k} \frac{p-1}{2}}.$$

Last, we estimate $\partial_x^\beta [f(U_j) - f(U_{j-1}) - f'(U_0) \chi_j w_j]$. We begin with the case $\beta = 0$. Recall that by (2.12), we have $0 < \frac{3}{4} U_0 \leq U_j \leq \frac{5}{4} U_0$, so that by elementary calculations

$$|f(U_j) - f(U_{j-1}) - f'(U_{j-1}) \chi_j w_j| \lesssim \chi_j U_0^{p-2} w_j^2$$

and

$$|f'(U_{j-1}) - f'(U_0)| \lesssim U_0^{p-2} \sum_{\ell=1}^{j-1} \chi_\ell |w_\ell|.$$

These estimates imply

$$|f(U_j) - f(U_{j-1}) - f'(U_0)\chi_j w_j| \lesssim \chi_j U_0^{p-2} |w_j| \sum_{\ell=1}^j \chi_\ell |w_\ell|.$$

For $1 \leq \ell \leq j$, using (2.10) and $U_0 \gtrsim 1$, we have

$$\begin{aligned} U_0^{p-2} |w_j| |w_\ell| &\lesssim U_0^{p-2} U_0^{1-j(1-\frac{1}{k})(p-1)} U_0^{1-\ell(1-\frac{1}{k})(p-1)} \lesssim U_0^{p-(j+\ell)(1-\frac{1}{k})(p-1)} \\ &\lesssim U_0^{p-(j+1)(1-\frac{1}{k})(p-1)} \lesssim U_0^{1-j(1-\frac{1}{k})(p-1) + \frac{1}{k}(p-1)}. \end{aligned}$$

Thus, $|f(U_j) - f(U_{j-1}) - f'(U_0)\chi_j w_j| \lesssim U_0^{1-j(1-\frac{1}{k})(p-1) + \frac{1}{k}(p-1)}$ is proved.

Now, we deal with the case $1 \leq |\beta| \leq k-3-2j$. By the Taylor formula with integral remainder, we have, for any U and w ,

$$f(U+w) - f(U) - f'(U)w = w^2 \int_0^1 (1-\theta) f''(U+\theta w) d\theta.$$

Thus, by the Leibniz's formula (1.14)

$$\begin{aligned} \partial_x^\beta [f(U+w) - f(U) - f'(U)w] &= \sum_{\beta' \leq \beta} \binom{\beta}{\beta'} \left(\partial_x^{\beta-\beta'} (w^2) \right) \int_0^1 (1-\theta) \partial_x^{\beta'} [f''(U+\theta w)] d\theta. \end{aligned}$$

Moreover, by the Faa di Bruno's formula (1.15), for $\beta' \neq 0$, denoting $n' = |\beta'|$,

$$\partial_x^{\beta'} [f''(U+\theta w)] = \sum_{\nu=1}^{n'} f^{(\nu+2)}(U+\theta w) \sum_{P(\beta', \nu)} (\beta'!) \prod_{\ell=1}^{\nu} \frac{(\partial_x^{\beta_\ell}(U+\theta w))^{\nu_\ell}}{(\nu_\ell!)(\beta_\ell!)^{\nu_\ell}}.$$

To estimate the term $\partial_x^\beta [f(U_j) - f(U_{j-1}) - f'(U_0)\chi_j w_j]$, we apply these formulas to $U = U_j$ and $w = \chi_j w_j$. For $\beta' \leq \beta$, using (2.10) and the properties of χ , we have

$$\begin{aligned} \left| \partial_x^{\beta-\beta'} [(\chi_j w_j)^2] \right| &\lesssim \sum_{\beta'' \leq \beta-\beta'} \left| \partial_x^{\beta''} (\chi_j w_j) \right| \left| \partial_x^{\beta-\beta'-\beta''} (\chi_j w_j) \right| \\ &\lesssim U_0^{2-2j(p-1) + \frac{4j+|\beta|-|\beta'|}{k} \frac{p-1}{2}}. \end{aligned}$$

For $\beta' = 0$ and $\theta \in [0, 1]$, using also (2.19), we obtain

$$\begin{aligned} \left| (\partial_x^\beta [(\chi_j w_j)^2]) f''(U_0 + \theta \chi_j w_j) \right| &\lesssim U_0^{2-2j(p-1) + \frac{4j+|\beta|}{k} \frac{p-1}{2}} U_0^{p-2} \\ &\lesssim U_0^{1-j(p-1) + \frac{2j+2+|\beta|}{k} \frac{p-1}{2}}. \end{aligned}$$

For $\beta' \neq 0$, $\beta' \leq \beta$ and $\theta \in [0, 1]$, using (2.4), (2.10) and (2.19), we have (recall that the definition of $P(\beta', \nu)$ implies that $\sum_{\ell=1}^{\nu} \nu_\ell = \nu$ and $\sum_{\ell=1}^{\nu} \nu_\ell |\beta_\ell| = |\beta'|$)

$$\begin{aligned} \left| \partial_x^{\beta'} [f''(U_{j-1} + \theta \chi_j w_j)] \right| &\lesssim \sum_{\nu=1}^{n'} U_0^{p-\nu-2} \sum_{P(\beta', \nu)} \prod_{\ell=1}^{\nu} \left(U_0^{1 + \frac{|\beta_\ell|}{k} \frac{p-1}{2}} \right)^{\nu_\ell} \\ &\lesssim \sum_{\nu=1}^{n'} U_0^{p-\nu-2} U_0^{\nu + \frac{|\beta'|}{k} \frac{p-1}{2}} \lesssim U_0^{p-2 + \frac{|\beta'|}{k} \frac{p-1}{2}}. \end{aligned}$$

Thus, similarly as before, it holds

$$\left| \partial_x^{\beta-\beta'} [(\chi_j w_j)^2] \partial_x^{\beta'} [f''(U_{j-1} + \theta \chi_j w_j)] \right| \lesssim U_0^{1-j(p-1) + \frac{2j+2+|\beta|}{k} \frac{p-1}{2}}.$$

Integrating these estimates in $\theta \in [0, 1]$, we obtain

$$\left| \partial_x^\beta [f(U_j) - f(U_{j-1}) - f'(U_{j-1})\chi_j w_j] \right| \lesssim U_0^{1-j(p-1) + \frac{2j+2+|\beta|}{k} \frac{p-1}{2}}. \quad (2.20)$$

By similar arguments, for any U, W, w , we have

$$f'(U) - f'(W) = (U - W) \int_0^1 f''(W + \theta(U - W)) d\theta,$$

and thus

$$\begin{aligned} & \partial_x^\beta [w(f'(U) - f'(W))] \\ &= \sum_{\beta' \leq \beta} \binom{\beta}{\beta'} \left(\partial_x^{\beta - \beta'} [w(U - W)] \right) \int_0^1 \partial_x^{\beta'} [f''(W + \theta(U - W))] d\theta. \end{aligned}$$

Moreover, for $\beta' \neq 0$,

$$\begin{aligned} & \partial_x^{\beta'} [f''(W + \theta(U - W))] \\ &= \sum_{\nu=1}^{n'} f^{(\nu+2)}(W + \theta(U - W)) \sum_{P(\beta', \nu)} (\beta'!) \prod_{\ell=1}^{n'} \frac{(\partial_x^{\beta_\ell}(W + \theta(U - W)))^{\nu_\ell}}{(\nu_\ell!)(\beta_\ell!)^{\nu_\ell}}. \end{aligned}$$

To estimate the term $\partial_x^\beta [\chi_j w_j (f'(U_{j-1}) - f'(U_0))]$, we apply these formulas to $U = U_{j-1}$, $W = U_0$ and $w = \chi_j w_j$.

For $\beta' \leq \beta$, using (2.10) and the properties of χ , we have, for $1 \leq \ell \leq j-1$,

$$\left| \partial_x^{\beta - \beta'} [\chi_j w_j \chi_\ell w_\ell] \right| \lesssim U_0^{2 - (j+\ell)(p-1) + \frac{2j+2\ell+|\beta|-|\beta'|}{k} \frac{p-1}{2}}.$$

For $\beta' = 0$ and $\theta \in [0, 1]$, from (2.19), we obtain

$$\left| \partial_x^\beta [\chi_j w_j (U_{j-1} - U_0)] f''(U_0 + \theta(U_{j-1} - U_0)) \right| \lesssim U_0^{1-j(p-1) + \frac{2j+2+|\beta|}{k} \frac{p-1}{2}}.$$

For $\beta' \neq 0$, $\beta' \leq \beta$ and $\theta \in [0, 1]$, by the formula above, using (2.4), (2.10) and (2.19), we have as before

$$\begin{aligned} \left| \partial_x^{\beta'} [f''(U_0 + \theta(U_{j-1} - U_0))] \right| &\lesssim \sum_{\nu=1}^{n'} U_0^{p-\nu-2} \sum_{P(\beta', \nu)} \prod_{\ell=1}^{n'} \left(U_0^{1 + \frac{|\beta_\ell|}{k} \frac{p-1}{2}} \right)^{\nu_\ell} \\ &\lesssim \sum_{\nu=1}^{n'} U_0^{p-\nu-2} U_0^{\nu + \frac{|\beta'|}{k} \frac{p-1}{2}} \lesssim U_0^{p-2 + \frac{|\beta'|}{k} \frac{p-1}{2}}. \end{aligned}$$

Thus, we obtain

$$\left| \partial_x^{\beta - \beta'} [\chi_j w_j (U_{j-1} - U_0)] \partial_x^{\beta'} [f''(U_0 + \theta(U_{j-1} - U_0))] \right| \lesssim U_0^{1-j(p-1) + \frac{2j+2+|\beta|}{k} \frac{p-1}{2}}.$$

Integrating in $\theta \in [0, 1]$ and summing in $\beta' \leq \beta$, we obtain

$$\left| \partial^\beta [\chi_j w_j (f'(U_{j-1}) - f'(U_0))] \right| \lesssim U_0^{1-j(p-1) + \frac{2j+2+|\beta|}{k} \frac{p-1}{2}}. \quad (2.21)$$

Combining (2.20) and (2.21), we have proved for $t \in (0, t_j]$, $|x| \leq 2$,

$$\left| \partial^\beta [f(U_j) - f(U_{j-1}) - f'(U_0) \chi_j w_j] \right| \lesssim U_0^{1-j(p-1) + \frac{2j+2+|\beta|}{k} \frac{p-1}{2}}.$$

In conclusion, we have estimated all terms in the expression of $\partial_x^\beta \mathcal{E}_j$ and (2.14) is now proved.

Finally, for $|x| \geq 2$, (2.2) implies that $A(x) \geq 2^k \geq 2a_1 \geq \dots \geq 2a_j$ and thus $\chi_j(x) = 0$, $U_j(t, x) = U_0(t, x)$ and $\mathcal{E}_j(t, x) = \mathcal{E}_0(t, x)$, so that (2.15) follows from (2.6). \square

3. UNIFORM BOUNDS ON APPROXIMATE SOLUTIONS

Let the function χ be given by (1.9) and U_J be defined as in §2.3 with J and k as in (2.1). Set

$$\lambda = \min \left(J - \frac{2}{p-1}, \frac{1}{2} \right) \in \left(0, \frac{1}{2} \right], \quad (3.1)$$

and impose the following additional condition on k

$$k \geq \frac{2(p+1)}{\lambda(p-1)} + 2. \quad (3.2)$$

For any n large, let $T_n = \frac{1}{n} < t_J$ and

$$B_n = \sup_{t \in [T_n, t_J]} \|U_J(t)\|_{L^\infty} \quad \text{so that} \quad \lim_{n \rightarrow \infty} B_n = \infty. \quad (3.3)$$

We let n be sufficiently large so that $B_n > 1$, and we define the function $f_n : \mathbb{R} \rightarrow [0, \infty)$ by

$$f_n(u) = f(u)\chi\left(\frac{u}{B_n}\right) \quad \text{so that} \quad f_n(u) = \begin{cases} f(u) & \text{for } |u| < B_n \\ 0 & \text{for } u > 2B_n \end{cases}. \quad (3.4)$$

Let $F_n(v) = \int_0^v f_n(w)dw$. It follows from elementary calculations that for every $\alpha \in \mathbb{N}$, there exists a constant $C_\alpha > 0$ independent of n , such that for all $u > 0$,

$$|f_n^{(\alpha)}(u)| \leq C_\alpha u^{p-\alpha}. \quad (3.5)$$

In particular, we observe that Taylor's estimates such as (1.11)–(1.13) still hold for F_n and f_n with constants independent of n . We will refer to these inequalities for F_n and f_n with the same numbers (1.11), (1.12) and (1.13). In this proof, any implicit constant related the symbol \lesssim is independent of n .

We define the sequence of solutions u_n of

$$\begin{cases} \partial_{tt}u_n - \Delta u_n = f_n(u_n) \\ u_n(T_n) = U_J(T_n), \quad \partial_t u_n(T_n) = \partial_t U_J(T_n). \end{cases} \quad (3.6)$$

The nonlinearity f_n being globally Lipschitz, the existence of a global solution $(u_n, \partial_t u_n)$ in the energy space is a consequence of standard arguments from semi-group theory. Using energy estimates, we prove uniform bounds on u_n in the energy space. For this we set, for all $t \in [T_n, t_J]$,

$$u_n(t) = U_J(t) + \varepsilon_n(t), \quad (3.7)$$

so that $(\varepsilon_n, \partial_t \varepsilon_n) \in \mathcal{C}([T_n, t_J], H^1(\mathbb{R}^N) \times L^2(\mathbb{R}^N))$.

Proposition 3.1. *There exist $C > 0$, $n_0 > 0$ and $0 < \delta_0 < 1$ such that*

$$\|(\varepsilon_n(t), \partial_t \varepsilon_n(t))\|_{H^1 \times L^2} \leq C(t - T_n)^{\frac{\lambda}{2}}. \quad (3.8)$$

for all $n \geq n_0$ and $t \in [T_n, T_n + \delta_0]$, where λ is given by (3.1).

Proof. The equation of ε_n on $[T_n, t_J] \times \mathbb{R}^N$ is

$$\begin{cases} \partial_{tt}\varepsilon_n - \Delta \varepsilon_n = f_n(U_J + \varepsilon_n) - f_n(U_J) + \mathcal{E}_J \\ \varepsilon_n(T_n) = 0, \quad \partial_t \varepsilon_n(T_n) = 0 \end{cases} \quad (3.9)$$

where we have used from (3.3) and (3.4) that $f(U_J) = f_n(U_J)$ on $[T_n, t_J] \times \mathbb{R}^N$.

Define the auxiliary function z as follows

$$\varepsilon_n = Q^{\frac{1}{2}}z \quad \text{where} \quad Q = (1 - \chi + U_0)^{p+1}.$$

We note that $Q \gtrsim 1$, $Q \lesssim t^{-\frac{2(p+1)}{p-1}}$. Moreover, it follows from (2.4) that $|\nabla U_0| \lesssim Q^{\frac{1}{p+1} + \frac{p-1}{2k(p+1)}}$, from which we deduce easily that $|\nabla Q| \lesssim t^{-\frac{1}{k}}Q$. One proves similarly that $|\Delta Q| \lesssim t^{-\frac{2}{k}}Q$. To write the equation of z , we compute

$$\begin{aligned} \partial_{tt}\varepsilon_n &= \partial_{tt}(Q^{\frac{1}{2}}z) = (\partial_{tt}Q^{\frac{1}{2}})z + (p+1)(1-\chi+U_0)^{\frac{p-1}{2}}\partial_t U_0 \partial_t z + Q^{\frac{1}{2}}\partial_{tt}z \\ &= (\partial_{tt}Q^{\frac{1}{2}})z + Q^{-\frac{1}{2}}\partial_t(Q\partial_t z). \end{aligned}$$

Thus, setting $G = Q^{\frac{1}{2}}(f'(U_0)Q^{\frac{1}{2}} - \partial_{tt}Q^{\frac{1}{2}})$, we obtain

$$\begin{aligned} \partial_t(Q\partial_t z) &= Q^{\frac{1}{2}}\Delta(Q^{\frac{1}{2}}z) \\ &\quad + Q^{\frac{1}{2}}\left(f_n(U_J + Q^{\frac{1}{2}}z) - f_n(U_J) - f'_n(U_0)Q^{\frac{1}{2}}z\right) + Gz + Q^{\frac{1}{2}}\mathcal{E}_J. \end{aligned} \quad (3.10)$$

Let $\sigma = \frac{3}{4}$. We define the following weighted norm and energy functional for z ,

$$\begin{aligned} \mathcal{N} &= \left(\int (Q\partial_t z)^2 + Q^2|\nabla z|^2 + t^{-2\sigma}Q^2z^2 \right)^{\frac{1}{2}}, \\ \mathcal{H} &= \int \left[(Q\partial_t z)^2 + Q^2|\nabla z|^2 + t^{-2\sigma}Q^2z^2 \right. \\ &\quad \left. - Q\left(2F_n(U_J + Q^{\frac{1}{2}}z) - 2F_n(U_J) - 2F'_n(U_J)Q^{\frac{1}{2}}z - F''_n(U_0)Qz^2\right) \right]. \end{aligned}$$

We remark that the first two terms in \mathcal{H} are the energy for the linear part of equation (3.10). The third term yields the control of a weighted L^2 norm, and the last term is associated with the nonlinear terms in the equation.

Step 1. Coercivity of the energy. We claim that, for $0 < \delta \leq t_J$ and $0 < \omega \leq 1$ sufficiently small, for n large, if $\mathcal{N} \leq \omega$ and $T_n \leq t \leq \delta$ then

$$\mathcal{N}^2 \leq 2\mathcal{H}, \quad (3.11)$$

and

$$\|(\varepsilon_n(t), \partial_t \varepsilon_n(t))\|_{\dot{H}^1 \times L^2} \lesssim \mathcal{N}, \quad \|\varepsilon_n(t)\|_{L^2} \lesssim t^\sigma \mathcal{N}. \quad (3.12)$$

Proof of (3.11). Let

$$A_1 = |2F_n(U_J + Q^{\frac{1}{2}}z) - 2F_n(U_J) - 2F'_n(U_J)Q^{\frac{1}{2}}z - F''_n(U_0)Qz^2|. \quad (3.13)$$

The triangle inequality and the Taylor inequality (1.11) yield

$$\begin{aligned} A_1 &\lesssim |2F_n(U_J + Q^{\frac{1}{2}}z) - 2F_n(U_J) - 2F'_n(U_J)Q^{\frac{1}{2}}z - F''_n(U_J)Qz^2| \\ &\quad + |F''_n(U_0) - F''_n(U_J)|Qz^2 \\ &\lesssim \Lambda_1 \end{aligned} \quad (3.14)$$

where

$$\Lambda_1 = Q^{\frac{p+1}{2}}|z|^{p+1} + U_J^{p-\bar{p}}Q^{\frac{\bar{p}+1}{2}}|z|^{\bar{p}+1} + U_0^{p-2}|U_0 - U_J|Qz^2. \quad (3.15)$$

Using $U_J \lesssim U_0$ and $U_0 \lesssim Q^{\frac{1}{p+1}}$, we see that $U_J^{p-\bar{p}} \lesssim Q^{\frac{p-\bar{p}}{p+1}}$. Moreover, since $|U_0 - U_J| \lesssim (1+U_0)^{-\frac{p-1}{2}}U_0 \lesssim Q^{-\frac{p-1}{2(p+1)}}U_0$ (see (2.12)), we obtain

$$U_0^{p-2}|U_0 - U_J| \lesssim U_0^{p-1}Q^{-\frac{p-1}{2(p+1)}} \lesssim Q^{\frac{p-1}{2(p+1)}},$$

and so

$$\Lambda_1 \lesssim Q^{\frac{p+1}{2}}|z|^{p+1} + Q^{\frac{\bar{p}+1}{2} + \frac{p-\bar{p}}{p+1}}|z|^{\bar{p}+1} + Q^{\frac{p-1}{2(p+1)}+1}z^2. \quad (3.16)$$

It follows that

$$\int Q\Lambda_1 \lesssim \int Q^{\frac{p+3}{2}}|z|^{p+1} + \int Q^{\frac{\bar{p}+3}{2} + \frac{p-\bar{p}}{p+1}}|z|^{\bar{p}+1} + \int Q^{\frac{p-1}{2(p+1)}+2}z^2.$$

For the first term on the right-hand side above, we use $Q \gtrsim 1$, thus

$$\int Q^{\frac{p+3}{2}}|z|^{p+1} = \int Q^{-\frac{p-1}{2}}|Qz|^{p+1} \lesssim \int |Qz|^{p+1}. \quad (3.17)$$

Applying now (1.10), $|\nabla Q| \lesssim t^{-\frac{1}{k}}Q$, and the definition of \mathcal{N} ,

$$\begin{aligned} \int |Qz|^{p+1} &\lesssim \left(\int |\nabla(Qz)|^2 \right)^{\frac{N}{4}(p-1)} \left(\int Q^2 z^2 \right)^{\frac{p+1}{2} - \frac{N}{4}(p-1)} \\ &\lesssim \left(\int Q^2 |\nabla z|^2 + t^{-\frac{2}{k}} \int Q^2 z^2 \right)^{\frac{N}{4}(p-1)} \left(\int Q^2 z^2 \right)^{\frac{p+1}{2} - \frac{N}{4}(p-1)} \quad (3.18) \\ &\lesssim \mathcal{N}^{p+1}. \end{aligned}$$

In the case $1 < p \leq 2$, one has $\bar{p} = p$ and the second term is identical to the first one. In the case $p > \bar{p} = 2$, the second term $Q^{\frac{5}{2} + \frac{p-2}{p+1}} |z|^3 = Q^{\frac{7p+1}{2(p+1)}} |z|^3$ is estimated as follows (using $|z|^3 \lesssim a^{p-2} |z|^{p+1} + \frac{1}{a} z^2$ with $a = Q^{\frac{p-1}{p+1}}$, $Q^{-1} \lesssim 1$ and $Q^{\frac{p-1}{2(p+1)}} \lesssim t^{-1}$)

$$\begin{aligned} Q^{\frac{7p+1}{2(p+1)}} |z|^3 &\lesssim Q^{-\frac{3}{2} \frac{p-1}{p+1}} Q^{p+1} |z|^{p+1} + Q^{\frac{5p+3}{2(p+1)}} z^2 \\ &\lesssim Q^{p+1} |z|^{p+1} + Q^{\frac{p-1}{2(p+1)}} Q^2 z^2 \lesssim Q^{p+1} |z|^{p+1} + t^{-1} Q^2 z^2, \end{aligned}$$

and so

$$\int Q^{\frac{\bar{p}+3}{2} + \frac{p-\bar{p}}{p+1}} |z|^{\bar{p}+1} \lesssim \mathcal{N}^{p+1} + t^{2\sigma-1} \mathcal{N}^2. \quad (3.19)$$

Last, since $Q^{\frac{p-1}{2(p+1)}} \lesssim t^{-1}$, we observe that

$$\int Q^{\frac{p-1}{2(p+1)}} Q^2 z^2 \lesssim t^{-1} \int Q^2 z^2 \lesssim t^{2\sigma-1} \mathcal{N}^2.$$

In conclusion, we have obtained $\int QA_1 \lesssim \int Q\Lambda_1 \lesssim t^{2\sigma-1} \mathcal{N}^2 + \mathcal{N}^{p+1}$, which implies that for t and \mathcal{N} small enough, $\mathcal{H} \geq \frac{1}{2} \mathcal{N}^2$.

Proof of (3.12). Since $\varepsilon_n = Q^{\frac{1}{2}}z$, the inequality $\|\varepsilon_n\|_{L^2} \lesssim t^{-1} \mathcal{N}$ follows readily from the definition of \mathcal{N} and $Q \gtrsim 1$. Next, using $|\nabla Q| \lesssim t^{-\frac{1}{k}}Q$, we see that

$$\int |\nabla \varepsilon_n|^2 = \int \left| Q^{\frac{1}{2}} \nabla z + \frac{1}{2} Q^{-\frac{1}{2}} z \nabla Q \right|^2 \lesssim \int Q |\nabla z|^2 + t^{-\frac{2}{k}} \int Q z^2 \lesssim \mathcal{N}^2.$$

Last, using $|\partial_t Q| \lesssim Q^{\frac{p-1}{2(p+1)}} Q \lesssim t^{-\frac{1}{2}} Q^{\frac{5p+3}{4(p+1)}}$, we have

$$\begin{aligned} \int |\partial_t \varepsilon_n|^2 &= \int \left| Q^{\frac{1}{2}} \partial_t z + \frac{1}{2} Q^{-\frac{1}{2}} z \partial_t Q \right|^2 \lesssim \int Q |\partial_t z|^2 + \int |\partial_t Q|^2 Q^{-1} z^2 \\ &\lesssim \int Q^2 |\partial_t z|^2 + t^{-1} \int Q^{\frac{3p+1}{2(p+1)}} z^2 \lesssim \mathcal{N}^2. \end{aligned}$$

This completes the proof of (3.12).

Step 2. Energy control. We claim that for $0 < \delta \leq t_J$ small enough and $C > 0$ large enough, for any n large and for all $t \in [T_n, T_n + \delta]$

$$\frac{d}{dt} \mathcal{H} \leq C \left[t^{-1+\lambda} \mathcal{N} + t^{-\frac{1}{2}} \mathcal{N}^2 + \mathcal{N}^{p+1} \right]. \quad (3.20)$$

Proof of (3.20). Taking the time-derivative of all the terms in \mathcal{H} , we obtain

$$\begin{aligned}
\frac{1}{2} \frac{d}{dt} \mathcal{H} &= \int (Q \partial_t z \partial_t (Q \partial_t z) + Q^2 \nabla z \cdot \nabla \partial_t z + t^{-2\sigma} Q^2 z \partial_t z) \\
&\quad - \int Q^{\frac{3}{2}} \left(f_n(U_J + Q^{\frac{1}{2}} z) - f_n(U_J) - f'_n(U_0) Q^{\frac{1}{2}} z \right) \partial_t z \\
&\quad + \int Q \partial_t Q |\nabla z|^2 + t^{-2\sigma} \int Q \partial_t Q z^2 - \sigma t^{-2\sigma-1} \int Q^2 z^2 \\
&\quad - \frac{1}{2} \int \partial_t Q \left(2F_n(U_J + Q^{\frac{1}{2}} z) - 2F_n(U_J) - 2F'_n(U_J) Q^{\frac{1}{2}} z - F''_n(U_0) Q z^2 \right) \\
&\quad - \frac{1}{2} \int \partial_t Q \left(f_n(U_J + Q^{\frac{1}{2}} z) - f_n(U_J) - f'_n(U_0) Q^{\frac{1}{2}} z \right) Q^{\frac{1}{2}} z \\
&\quad - \frac{1}{2} \int Q \partial_t U_0 \left(2f_n(U_J + Q^{\frac{1}{2}} z) - 2f_n(U_J) - 2f'_n(U_J) Q^{\frac{1}{2}} z - f''_n(U_0) Q z^2 \right) \\
&\quad - \frac{1}{2} \int Q \partial_t (U_J - U_0) \left(2f_n(U_J + Q^{\frac{1}{2}} z) - 2f_n(U_J) - 2f'_n(U_J) Q^{\frac{1}{2}} z \right) \\
&= I_1 + I_2 + I_3 + I_4 + I_5 + I_6 + I_7.
\end{aligned}$$

First, we note that $\partial_t Q = (p+1)Q^{\frac{p}{p+1}} \partial_t U_0 \leq 0$, so that

$$I_3 \leq -\sqrt{2(p+1)} \int U_0^{\frac{p+1}{2}} Q^{\frac{2p+1}{p+1}} |\nabla z|^2 - \sigma t^{-2\sigma-1} \int Q^2 z^2.$$

We now use equation (3.10) to replace the term $\partial_t(Q \partial_t z)$ in I_1 , and we obtain

$$\begin{aligned}
I_1 + I_2 &= \int \left(Q^{\frac{3}{2}} \partial_t z \Delta(Q^{\frac{1}{2}} z) + Q^2 \nabla z \cdot \nabla \partial_t z \right) \\
&\quad + \int \left(Gz + Q^{\frac{1}{2}} \mathcal{E}_J \right) Q \partial_t z + t^{-2\sigma} \int Q^2 z \partial_t z = I_8 + I_9 + I_{10}.
\end{aligned}$$

The term I_{10} is controlled using the Cauchy-Schwarz inequality,

$$|I_{10}| \leq \frac{1}{10} |I_3| + C t^{-2\sigma+1} \int (Q \partial_t z)^2 \leq \frac{1}{10} |I_3| + C t^{-2\sigma+1} \mathcal{N}^2 \leq \frac{1}{10} |I_3| + C t^{-\frac{1}{2}} \mathcal{N}^2.$$

Next, integrating by parts,

$$\begin{aligned}
I_8 &= - \int \nabla(Q^{\frac{1}{2}} z) \cdot \nabla(Q^{\frac{3}{2}} \partial_t z) + Q^2 \nabla z \cdot \nabla(\partial_t z) \\
&= - \int z \nabla(Q^{\frac{1}{2}}) \cdot \nabla(Q^{\frac{3}{2}} \partial_t z) - \int Q^{\frac{1}{2}} (\partial_t z) \nabla z \cdot \nabla(Q^{\frac{3}{2}}) \\
&= - \int Q \partial_t z \nabla z \cdot \nabla Q + \int \Delta(Q^{\frac{1}{2}}) Q^{\frac{3}{2}} z \partial_t z.
\end{aligned}$$

By $|\nabla Q| \lesssim t^{-\frac{1}{k}} Q$ and the Cauchy-Schwarz inequality,

$$\left| \int Q \partial_t z \nabla z \cdot \nabla Q \right| \lesssim t^{-\frac{1}{k}} \mathcal{N}^2.$$

Similarly, $|\Delta(Q^{\frac{1}{2}}) Q^{\frac{3}{2}}| \lesssim |\nabla Q|^2 + |\Delta Q| Q \lesssim t^{-\frac{2}{k}} Q^2$, and so

$$\left| \int \Delta(Q^{\frac{1}{2}}) Q^{\frac{3}{2}} z \partial_t z \right| \leq \int Q^2 |\partial_t z|^2 + t^{-\frac{4}{k}} \int Q^2 z^2 \lesssim \mathcal{N}^2.$$

We note that by Cauchy-Schwarz,

$$|I_9| \lesssim \|G\|_{L^\infty} \mathcal{N}^2 + \|Q^{\frac{1}{2}} \mathcal{E}_J\|_{L^2} \mathcal{N},$$

and so, we only have to bound the L^∞ norm of G and the L^2 norm of $Q^{\frac{1}{2}}\mathcal{E}_J$. We begin with $G = Q^{\frac{1}{2}}\left(pU_0^{p-1}Q^{\frac{1}{2}} - \partial_{tt}Q^{\frac{1}{2}}\right)$. Using $Q = (1 - \chi + U_0)^{p+1}$ and the expressions of $\partial_{tt}U_0$ and $(\partial_t U_0)^2$, we observe that

$$\partial_{tt}Q^{\frac{1}{2}} = \frac{p+1}{2}U_0^p Q^{\frac{p-1}{2(p+1)}} + \frac{p-1}{2}U_0^{p+1}Q^{\frac{p-3}{2(p+1)}}.$$

Thus,

$$\begin{aligned} pU_0^{p-1}Q^{\frac{1}{2}} - \partial_{tt}Q^{\frac{1}{2}} &= \frac{p+1}{2}U_0^{p-1}Q^{\frac{p-1}{2(p+1)}}\left(Q^{\frac{1}{p+1}} - U_0\right) + \frac{p-1}{2}U_0^{p-1}Q^{\frac{p-3}{2(p+1)}}\left(Q^{\frac{2}{p+1}} - U_0^2\right) \\ &= \frac{p+1}{2}U_0^{p-1}Q^{\frac{p-1}{2(p+1)}}(1-\chi) + \frac{p-1}{2}U_0^{p-1}Q^{\frac{p-3}{2(p+1)}}(1-\chi)(1-\chi+2U_0). \end{aligned}$$

Since for $|x| > 1$, we have $U_0 \lesssim 1$ and $Q \lesssim 1$, we obtain $\|G\|_{L^\infty} \lesssim 1$.

Now, we estimate $\|Q^{\frac{1}{2}}\mathcal{E}_J\|_{L^2}$ from Lemma 2.3. For $|x| \geq 2$, it follows from (2.15) that

$$Q^{\frac{1}{2}}|\mathcal{E}_J| \lesssim |\mathcal{E}_J| \lesssim |x|^{-\frac{2k}{p-1}-2}.$$

Note that for $N \geq 3$, $p-1 \leq \frac{4}{N-2}$ and so $\frac{2k}{p-1} + 2 \geq \frac{N-2}{2}k + 2 \geq N$. Thus, the following bound holds $\|Q^{\frac{1}{2}}\mathcal{E}_J\|_{L^2(|x|>2)} \lesssim 1$.

Next, using (2.14), we have for $|x| \lesssim 2$

$$\begin{aligned} Q^{\frac{1}{2}}|\mathcal{E}_J| &\lesssim Q^{\frac{1}{2}}U_0^{1-J(p-1)+\frac{(J+1)(p-1)}{k}} \lesssim U_0^{\frac{p+3}{2}-J(p-1)+\frac{(J+1)(p-1)}{k}} \\ &\lesssim U_0^{\frac{p+3}{2}-J(1-\frac{1}{k})(p-1)+\frac{p-1}{k}}. \end{aligned}$$

Note that by (3.1), $J \geq \lambda + \frac{2}{p-1}$, so that $-J(1 - \frac{1}{k}) + \frac{p-1}{k} \leq -2(1 - \frac{1}{k}) + \frac{p-1}{k} - \lambda(p-1)(1 - \frac{1}{k})$; and so

$$Q^{\frac{1}{2}}|\mathcal{E}_J| \lesssim U_0^{\frac{p+3}{2}-2(1-\frac{1}{k})+\frac{p-1}{k}-\lambda(p-1)(1-\frac{1}{k})} \lesssim U_0^{\frac{p-1}{2}+\frac{p+1}{k}-\lambda(p-1)(1-\frac{1}{k})}.$$

Moreover, the additional condition (3.2) is equivalent to $\frac{p+1}{k} - \lambda(p-1)(1 - \frac{1}{k}) \leq -\frac{\lambda(p-1)}{2}$. Thus, for $|x| \lesssim 2$,

$$Q^{\frac{1}{2}}|\mathcal{E}_J| \lesssim U_0^{(1-\lambda)\frac{p-1}{2}} \lesssim (t + A(x))^{-1+\lambda} \lesssim t^{-1+\lambda}. \quad (3.21)$$

Therefore, one obtains $\|Q^{\frac{1}{2}}\mathcal{E}_J\|_{L^2} \lesssim t^{-1+\lambda}$.

To complete the proof of (3.20), we estimate I_4, I_5, I_6 and I_7 . First, using (3.13)–(3.16), and $|\partial_t Q| \lesssim |\partial_t U_0|Q^{\frac{p}{p+1}} \lesssim U_0^{\frac{p+1}{2}}Q^{\frac{p}{p+1}} \lesssim Q^{\frac{3p+1}{2(p+1)}}$, we obtain

$$\begin{aligned} |\partial_t Q|A_1 &\lesssim Q^{\frac{3p+1}{2(p+1)}}\Lambda_1 \\ &\lesssim Q^{p+1-\frac{p(p-1)}{2(p+1)}}|z|^{p+1} + Q^{\frac{3p+1}{2(p+1)}+\frac{\bar{p}+1}{2}+\frac{p-\bar{p}}{p+1}}|z|^{\bar{p}+1} + Q^{2+\frac{p-1}{p+1}}z^2. \end{aligned}$$

Using $U_0 \gtrsim 1$ and the estimate (3.18), we treat the first term above as follows

$$\int Q^{p+1-\frac{p(p-1)}{2(p+1)}}|z|^{p+1} \lesssim \int |Qz|^{p+1} \lesssim \mathcal{N}^{p+1}.$$

In the case $1 < p \leq 2$, one has $\bar{p} = p$ and the second term is identical to the first one. In the case $p > \bar{p} = 2$, the second term $Q^{\frac{4p}{p+1}}|z|^3$ is estimated as follows (using $|z|^3 \lesssim a^{p-2}|z|^{p+1} + \frac{1}{a}z^2$ with $a = Q^{\frac{p-1}{p+1}}$, $Q^{-1} \lesssim 1$ and $Q^{\frac{p-1}{p+1}} \lesssim t^{-2}$)

$$Q^{\frac{4p}{p+1}}|z|^3 \lesssim Q^{-n\frac{p-1}{p+1}}Q^{p+1}|z|^{p+1} + Q^{\frac{p-1}{p+1}}Q^2z^2 \lesssim Q^{p+1}|z|^{p+1} + t^{-2}Q^2z^2.$$

Therefore

$$\int Q^{\frac{4p}{p+1}}|z|^3 \lesssim \mathcal{N}^{p+1} + t^{-2(1-\sigma)}\mathcal{N}^2 \lesssim \mathcal{N}^{p+1} + t^{-\frac{1}{2}}\mathcal{N}^2.$$

Since $Q^{2+\frac{p-1}{p+1}}z^2 \lesssim t^{-2}Q^2z^2$, we have proved

$$|I_4| \lesssim \int |\partial_t Q| \Lambda_1 \lesssim \int Q^{\frac{3p+1}{2(p+1)}} \Lambda_1 \lesssim \mathcal{N}^{p+1} + t^{-\frac{1}{2}} \mathcal{N}^2. \quad (3.22)$$

We proceed similarly for I_5 . Indeed, setting

$$\begin{aligned} A_2 &= \left| f_n(U_J + Q^{\frac{1}{2}}z) - f_n(U_J) - f'_n(U_0)Q^{\frac{1}{2}}z \right| Q^{\frac{1}{2}}|z| \\ &\leq \left| f_n(U_J + Q^{\frac{1}{2}}z) - f_n(U_J) - f'_n(U_J)Q^{\frac{1}{2}}z \right| Q^{\frac{1}{2}}|z| + |f'_n(U_0) - f'_n(U_J)|Qz^2 \end{aligned}$$

we deduce from (1.12) and Taylor's inequality that, with the notation (3.15),

$$A_2 \lesssim Q^{\frac{p+1}{2}}|z|^{p+1} + U_J^{p-\bar{p}}Q^{\frac{\bar{p}+1}{2}}|z|^{\bar{p}+1} + U_0^{p-2}|U_0 - U_J|Q|z|^2 \lesssim \Lambda_1.$$

Using the last two inequalities in (3.22), we conclude that $|I_5| \lesssim \mathcal{N}^{p+1} + t^{-\frac{1}{2}}\mathcal{N}^2$.

Now, we estimate I_6 , and we set

$$A_3 = |2f_n(U_J + Q^{\frac{1}{2}}z) - 2f_n(U_J) - 2f'_n(U_J)Q^{\frac{1}{2}}z - f''_n(U_J)Qz^2|.$$

By the triangle inequality, Taylor's inequality (1.13), and $U_J^{-1} \lesssim U_0^{-1}$ (see (2.12)),

$$\begin{aligned} A_3 &\lesssim |2f_n(U_J + Q^{\frac{1}{2}}z) - 2f_n(U_J) - 2f'_n(U_J)Q^{\frac{1}{2}}z - f''_n(U_0)Qz^2| \\ &\quad + |f''_n(U_0) - f''_n(U_J)|Qz^2 \\ &\lesssim U_J^{-1}Q^{\frac{p+1}{2}}|z|^{p+1} + U_J^{p-\bar{p}-1}Q^{\frac{\bar{p}+1}{2}}|z|^{\bar{p}+1} + U_0^{p-3}|U_0 - U_J|Qz^2 \\ &\lesssim U_0^{-1}[Q^{\frac{p+1}{2}}|z|^{p+1} + U_J^{p-\bar{p}}Q^{\frac{\bar{p}+1}{2}}|z|^{\bar{p}+1} + U_0^{p-2}|U_0 - U_J|Qz^2] \\ &\lesssim U_0^{-1}\Lambda_1 \end{aligned}$$

with the notation (3.15). Using $|\partial_t U_0| \lesssim U_0^{\frac{p+1}{2}}$ and $U_0 \lesssim Q^{\frac{1}{p+1}}$, we see that $Q|\partial_t U_0| \lesssim QU_0^{\frac{p+1}{2}} \lesssim QU_0^{\frac{p-1}{2}}U_0 \lesssim Q^{\frac{3p+1}{2(p+1)}}U_0$, hence $Q|\partial_t U_0|A_3 \lesssim Q^{\frac{3p+1}{2(p+1)}}\Lambda_1$. The last inequality in (3.22) yields $|I_6| \lesssim \mathcal{N}^{p+1} + t^{-\frac{1}{2}}\mathcal{N}^2$.

Finally, we estimate I_7 and we set

$$A_4 = |f_n(U_J + Q^{\frac{1}{2}}z) - f_n(U_J) - f'_n(U_J)Q^{\frac{1}{2}}z|.$$

By the triangle inequality Taylor's expansion (1.13),

$$\begin{aligned} A_4 &\lesssim \left| f_n(U_J + Q^{\frac{1}{2}}z) - f_n(U_J) - f'_n(U_J)Q^{\frac{1}{2}}z - \frac{1}{2}f''_n(U_J)Qz^2 \right| \\ &\quad + \frac{1}{2}|f''_n(U_J)|Qz^2 \\ &\lesssim U_J^{-1}Q^{\frac{p+1}{2}}|z|^{p+1} + U_J^{p-\bar{p}-1}Q^{\frac{\bar{p}+1}{2}}|z|^{\bar{p}+1} + U_J^{p-2}Qz^2 \end{aligned}$$

Using $Q|\partial_t(U_J - U_0)| \lesssim QU_0$ (see (2.13)), $U_J^{-1} \lesssim U_0^{-1}$, and $U_J \lesssim U_0$, we obtain

$$Q|\partial_t(U_J - U_0)|A_4 \lesssim Q^{\frac{p+3}{2}}|z|^{p+1} + U_0^{p-\bar{p}}Q^{\frac{\bar{p}+3}{2}}|z|^{\bar{p}+1} + U_0^{p-1}Q^2z^2.$$

Since $U_0 \lesssim Q^{\frac{1}{p+1}}$ and $U_0^{p-1} \lesssim t^{-2}$, we deduce that

$$Q|\partial_t(U_J - U_0)|A_4 \lesssim Q^{\frac{p+3}{2}}|z|^{p+1} + Q^{\frac{p-\bar{p}}{p+1} + \frac{\bar{p}+3}{2}}|z|^{\bar{p}+1} + t^{-2}Q^2z^2.$$

Applying (3.17)-(3.18) for the first term and (3.19) for the second term, we see that $|I_7| \lesssim \mathcal{N}^{p+1} + t^{-\frac{1}{2}}\mathcal{N}^2$. Collecting the above estimates, we have proved (3.20).

Step 3. Conclusion. The values of $\delta \in (0, t_J]$ and $0 < \omega \leq 1$ are now fixed so that (3.11), (3.12) and (3.20) hold. Since $\mathcal{N}(T_n) = 0$, the following is well-defined

$$T_n^* = \sup\{t \in [T_n, \delta] : \text{for all } s \in [T_n, t], \mathcal{N}(s) \leq \omega\}$$

and by continuity, $T_n^* \in (T_n, \delta]$. For all $t \in [T_n, T_n^*]$, using (3.20), we find (recall that $\lambda \in (0, \frac{1}{2}]$)

$$\frac{d}{dt} \mathcal{H} \leq C \left[t^{-1+\lambda} + t^{-\frac{1}{2}} + 1 \right] \leq C t^{-1+\lambda}.$$

Let $t \in [T_n, T_n^*]$. Since $\mathcal{H}(T_n) = 0$, we obtain by integration on $[T_n, t]$

$$\mathcal{H}(t) \leq C(t^\lambda - T_n^\lambda) \leq C(t - T_n)^\lambda.$$

Therefore, using the definition of T_n^* and (3.11), for all $t \in [T_n, T_n^*]$,

$$\mathcal{N}(t) \leq C(t - T_n)^{\frac{\lambda}{2}}.$$

In particular, there exists $\delta_0 > 0$ independent of n such that, for n large, it holds $T_n^* \geq T_n + \delta_0$. Moreover, using (3.12), for all $t \in [T_n, T_n + \delta_0]$,

$$\|(\varepsilon_n(t), \partial_t \varepsilon_n(t))\|_{H^1 \times L^2} \lesssim \mathcal{N}(t) \lesssim (t - T_n)^{\frac{\lambda}{2}},$$

which completes the proof of Proposition 3.1. \square

4. END OF THE PROOF OF THEOREM 1.1

Let E be any compact set of \mathbb{R}^N included in the ball of center 0 and radius 1 (by the scaling invariance of equation (1.1), this assumption does not restrict the generality). It is well-known that there exists a smooth function $Z : \mathbb{R}^N \rightarrow [0, \infty)$ which vanishes exactly on E (see *e.g.* Lemma 1.4, page 20 of [23]). For p as in (1.2), choose J and k satisfying (2.1) and (3.2). Define the function $A : \mathbb{R}^N \rightarrow [0, \infty)$ by

$$A(x) = (Z(x)\chi(x) + (1 - \chi(x))|x|)^k,$$

where χ is given by (1.9). It follows that the function A satisfies (2.2) and vanishes exactly on E .

We consider the global solutions u_n of equation (3.6), ε_n defined by (3.7) and we set for $0 \leq t \leq t_J - T_n$,

$$V_n(t) = U_J(T_n + t), \quad \eta_n(t) = \varepsilon_n(T_n + t), \quad \mathcal{F}_n(t) = \mathcal{E}_J(T_n + t).$$

It follows from Proposition 3.1 that there exist $0 < \delta_0 < t_J$, $0 < \lambda \leq \frac{1}{2}$, and $C > 0$ such that, for n large and for all $t \in [0, \delta_0]$,

$$\|(\eta_n(t), \partial_t \eta_n(t))\|_{H^1 \times L^2} \leq C t^{\frac{\lambda}{2}}. \quad (4.1)$$

Moreover, it follows from (3.9) that

$$\partial_{tt} \eta_n - \Delta \eta_n = f_n(V_n + \eta_n) - f_n(V_n) + \mathcal{F}_n. \quad (4.2)$$

Using the estimate $|f_n(u + v) - f_n(u)| \lesssim (|u|^{p-1} + |v|^{p-1})|v|$ and the embeddings $H^1(\mathbb{R}^N) \hookrightarrow L^{p+1}(\mathbb{R}^N)$, $L^{\frac{p+1}{p}}(\mathbb{R}^N) \hookrightarrow H^{-1}(\mathbb{R}^N)$, we deduce that

$$\|\partial_{tt} \eta_n\|_{H^{-1}} \lesssim \|\eta_n\|_{H^1} + \|V_n\|_{H^1}^{p-1} \|\eta_n\|_{H^1} + \|\eta_n\|_{H^1}^p + \|\mathcal{F}_n\|_{L^2}$$

so that by the estimates of Lemmas 2.2 and 2.3, there exist $C, c > 0$ such that, for all $t \in (0, \delta_0]$,

$$\|\partial_{tt} \eta_n\|_{H^{-1}} \leq C t^{-c}. \quad (4.3)$$

Given $\tau \in (0, \delta_0)$, it follows from (4.1) and (4.3) that the sequence $(\eta_n)_{n \geq 1}$ is bounded in $L^\infty((\tau, \delta_0), H^1(\mathbb{R}^N)) \cap W^{1,\infty}((\tau, \delta_0), L^2(\mathbb{R}^N)) \cap W^{2,\infty}((\tau, \delta_0), H^{-1}(\mathbb{R}^N))$. Therefore, after possibly extracting a subsequence (still denoted by η_n), there exists

$\eta \in L^\infty((\tau, \delta_0), H^1(\mathbb{R}^N)) \cap W^{1,\infty}((\tau, \delta_0), L^2(\mathbb{R}^N)) \cap W^{2,\infty}((\tau, \delta_0), H^{-1}(\mathbb{R}^N))$ such that

$$\eta_n \xrightarrow{n \rightarrow \infty} \eta \quad \text{in } L^\infty((\tau, \delta_0), H^1(\mathbb{R}^N)) \text{ weak}^* \quad (4.4)$$

$$\partial_t \eta_n \xrightarrow{n \rightarrow \infty} \partial_t \eta \quad \text{in } L^\infty((\tau, \delta_0), L^2(\mathbb{R}^N)) \text{ weak}^* \quad (4.5)$$

$$\partial_{tt} \eta_n \xrightarrow{n \rightarrow \infty} \partial_{tt} \eta \quad \text{in } L^\infty((\tau, \delta_0), H^{-1}(\mathbb{R}^N)) \text{ weak}^* \quad (4.6)$$

$$\eta_n \xrightarrow{n \rightarrow \infty} \eta \quad \text{weakly in } H^1(\mathbb{R}^N), \text{ for all } t \in [\tau, \delta_0] \quad (4.7)$$

$$\partial_t \eta_n \xrightarrow{n \rightarrow \infty} \partial_t \eta \quad \text{weakly in } L^2(\mathbb{R}^N), \text{ for all } t \in [\tau, \delta_0]. \quad (4.8)$$

Since $\tau \in (0, \delta_0)$ is arbitrary, a standard argument of diagonal extraction shows that there exists a function $\eta \in L_{\text{loc}}^\infty((0, \delta_0), H^1(\mathbb{R}^N)) \cap W_{\text{loc}}^{1,\infty}((0, \delta_0), L^2(\mathbb{R}^N)) \cap W_{\text{loc}}^{2,\infty}((0, \delta_0), H^{-1}(\mathbb{R}^N))$ such that (after extraction of a subsequence) (4.4)–(4.8) hold for all $0 < \tau < \delta_0$. Moreover, (4.1) and (4.7)–(4.8) imply that

$$\|(\eta(t), \partial_t \eta(t))\|_{H^1 \times L^2} \leq Ct^{\frac{\lambda}{2}}, \quad t \in (0, \delta_0), \quad (4.9)$$

and (4.3) and (4.6) imply that

$$\|\partial_{tt} \eta\|_{L^\infty((\tau, \delta_0), H^{-1})} \leq C\tau^{-c}, \quad \tau \in (0, \delta_0). \quad (4.10)$$

In addition, it follows easily from (4.2), (3.3), (3.4) and the convergence properties (4.4)–(4.8) that

$$\partial_{tt} \eta - \Delta \eta = f(U_J + \eta) - f(U_J) + \mathcal{E}_J \quad (4.11)$$

in $L_{\text{loc}}^\infty((0, \delta_0), H^{-1}(\mathbb{R}^N))$. Therefore, setting

$$u(t) = U_J(t) + \eta(t), \quad t \in (0, \delta_0),$$

we observe that the function $u \in L_{\text{loc}}^\infty((0, \delta_0), H^1(\mathbb{R}^N)) \cap W_{\text{loc}}^{1,\infty}((0, \delta_0), L^2(\mathbb{R}^N)) \cap W_{\text{loc}}^{2,\infty}((0, \delta_0), H^{-1}(\mathbb{R}^N))$ and satisfies $\partial_{tt} u - \Delta u = f(u)$ in $L_{\text{loc}}^\infty((0, \delta_0), H^{-1}(\mathbb{R}^N))$. It is a well-known property of the energy subcritical wave equation (corresponding to assumption (1.2)) that then it holds the stronger property

$$u \in \mathcal{C}((0, \delta_0), H^1(\mathbb{R}^N)) \cap \mathcal{C}^1((0, \delta_0), L^2(\mathbb{R}^N)) \cap \mathcal{C}^2((0, \delta_0), H^{-1}(\mathbb{R}^N)). \quad (4.12)$$

We refer for example to Proposition 3.1 and Lemma 2.1 in [8].

Finally, we prove estimates (1.3) and (1.4). For $x_0 \notin E$, there exist $r > 0$ and $C > 0$ such that $A(x) \geq C$ for all $x \in \mathbb{R}^N$ such that $|x - x_0| < r$. In particular, for such x , by (2.12) and (2.13), $|U_J(x)| + |\partial_t U_J(x)| \leq C'$ for some constant $C' > 0$. Estimate (1.4) then follows from (4.9). For $x_0 \in E$, (2.7), (2.8), (2.12) and (2.13) imply, for $t \in (0, \delta_0)$,

$$\begin{aligned} t^{-\mu} &\lesssim \|U_J(t)\|_{L^2(|x-x_0|<r)} \lesssim t^{-\frac{2}{p-1}}, \\ t^{-\mu-1} &\lesssim \|\partial_t U_J(t)\|_{L^2(|x-x_0|<r)} \lesssim t^{-\frac{2}{p-1}-1}, \end{aligned}$$

where $\mu = \frac{2}{p-1} - \frac{N}{2k}$. Estimate (1.3), and more precisely estimates (1.6) and (1.7) then follow from (4.9).

Now, we justify the last part of Remark 1.2. If $x_0 \in E$ and E contains a neighborhood of x_0 then $A(x) = 0$ on this neighborhood and the lower estimate easily follows. In the case where $x_0 \in E$ is isolated, the function A can be chosen so that $A(x) = |x|^k$ in a neighbourhood of x_0 (see Remark 2.1). In particular, by (2.9) and a similar estimate for $\partial_t U_0$, we obtain for small $r > 0$, $\|u(t)\|_{L^2(|x-x_0|<r)} \lesssim t^{-\frac{2}{p-1} + \frac{N}{2k}}$ and $\|\partial_t u(t)\|_{L^2(|x-x_0|<r)} \lesssim t^{-1 - \frac{2}{p-1} + \frac{N}{2k}}$.

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