

Enactive metaphors in mathematical problem solving

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We are interested in exploring the role of enactive metaphorizing in mathematical thinking, especially in the context of problem posing and solving, not only as a means to foster and enhance the learner's ability to think mathematically but also as a mean to alleviate the cognitive abuse that the teaching of mathematics has turned out to be for most children and adolescents in the world. We present some illustrative examples to this end besides describing our theoretical framework.

Keywords: Metaphors, enaction, representation, visualisation, cognitive bullying.

Introduction

Our concern in this paper is the role of metaphor, more precisely enactive metaphor, in the teaching and learning of mathematics, particularly in mathematical thinking arising in, or triggered by, problem posing and solving. Fostering mathematical thinking in the classroom is a widespread aim in mathematics education indeed (OECD, 2014), but in our viewpoint we have a much more severe and dramatic issue to address: not only mathematical thinking is not fostered in our classrooms (Chilean and worldwide), but mathematics has turned out to be a tool of torture for millions of children, who cannot escape from it. This has been recently acknowledged as “cognitive bullying” or “cognitive abuse” in the English literature (Watson, 2008; Johnston-Wilder & Lee, 2010). a practice that is “at best marginally productive and at worst emotionally damaging” (Watson, 2008: p. 165). We thus echo Tillich’s famous statement: “the fatal pedagogical error is to throw answers like stones at the heads of those who have not yet asked the questions”. To tackle this complex and systemic problem, a multidisciplinary approach is most wanted, where a first diagnosis emerges: traditional (and abusive) teaching of mathematics tends to thwart cognitive brain mechanisms installed during millions of years of evolution, that we would need on the contrary to recognise, appreciate and tap into in the context of learning, to wit: metaphorising, enacting, collaborating...

Our main hypothesis is that practice of metaphorising, especially enactive metaphorising, in the classroom, might be a fundamental means to contribute to alleviating this situation of cognitive abuse towards students without forsaking their access to mathematical thinking, but on the contrary fostering it. It is our hypothesis that the way a mathematical situation is metaphorised and enacted by the learners strongly determines the ideas and insights that may emerge in them and may help to bridge the gap between the “mathematically gifted” and those apparently not so gifted or mathematically oriented. A big challenge is then trying to figure out under which conditions enaction and metaphorising, more precisely enactive metaphorising, impact on mathematical thinking processes as hypothesised above.

We intend here to pursue our recent research on metaphorising and enacting (Diaz-Rojas & Soto-Andrade, 2015, 2016; Soto-Andrade, 2015). Our earlier work on metaphor in the learning of mathematics was presented already in CERME5 (Soto-Andrade, 2007).

In this paper we focus on some examples and case studies that illustrate the role that metaphorising and enacting may play in the spectrum of mathematical thinking elicited by problem posing and solving. The contextual background of our case studies involves a variety of learners in Chile: prospective secondary math teachers, in service primary and secondary math teachers, first year university students majoring in social sciences and humanities, undergraduate and graduate students in mathematics, primary and secondary students.

Research questions

Which is the role of metaphorising and enacting in mathematical problem posing and solving at various levels? To which extent do they influence mathematical thinking elicited by the problematic situation, in particular moving amongst various mathematical registers of representation to change the problem? Do they shape our understanding of the processes involved, notably the relation between problem and learner, emotional overtones included?

How does the interplay between affect and metaphoring helps in alleviating cognitive bullying in the teaching of mathematics, and even lead to enjoyment of learning and doing mathematics?

Theoretical framework and state of the art

Metaphorising in cognitive science and mathematics education.

Widespread agreement has been reached in cognitive science that our ordinary conceptual system, in terms of which we both think and act, is fundamentally metaphorical in nature (Gibbs, 2008; Johnson & Lakoff, 2003). In mathematics education proper it has been progressively recognized during the last decades (English, 1997; Lakoff & Núñez, 2000; Sfard, 2009; Soto-Andrade, 2007, 2014, and many others) that metaphors are not just rhetorical devices, but powerful cognitive tools, that help us in grasping or even building new concepts, as well as in solving problems in an efficient and friendly way. See Soto-Andrade (2014) for a recent survey. We may visualize (conceptual) metaphors (Lakoff & Núñez, 2000) as mappings from a more down-to-earth “source domain” into a more abstract “target domain”, carrying the inferential structure of the former into that of the latter. For the learning of mathematics we emphasize the “poietic” role of metaphor that brings concepts into existence, through “reification” (Sfard, 2009). In view, here lies the main difference between representation and metaphor: we *re-present* something given beforehand but we *metaphorise* to try to fathom something unknown or a concept not yet constructed. For instance, we construct the concept of probability when, studying a symmetric random walk on the integers (a frog jumping on a row of stones in a pond), we see the walker *splitting* into 2 equal halves instead of going equally likely right or left (Soto-Andrade, 2007, 2015). In what follows we will use metaphorical language as a meta-language to describe cognitive or didactic theories of interest to us, since – we claim – a theory is essentially the unfolding of a metaphor (Soto-Andrade, 2014).

Enaction in cognitive science and mathematics education

Varela metaphorized enaction as the laying down of a path in walking (Varela, 1987, p. 63), as in Machado’s famous poem, when he introduced the enactive approach in cognitive science (Varela,

Thompson, & Rosch, 1991). In his own words: “The world is not something that is given to us but something we engage in by moving, touching, breathing, and eating. This is what I call cognition as enaction since enaction connotes this bringing forth by concrete handling” (loc. cit). Less radical enaction in mathematics education may be traced back to Bruner (1966), inspired by Dewey’s “learning by doing” (Dewey, 1997), who characterised enactive, iconic and symbolic modes of representation. For recent significant theoretical and practical developments see Proulx (2013). In what follows we are especially interested in enactive metaphors, where the learner is whole bodily engaged, as opposed to “sitting metaphors” in the sense of Gallagher and Lindgren (2015).

(A) didactic situations and didactic contract

The theory of didactical situations (Brousseau & Warfield, 2014) might be described as the unfolding of the *emergence metaphor* in math education: mathematical concepts we intend to teach should emerge in a suitable challenging situation the learner is enmeshed in, as the only means to “save his life”. No real learning is possible if mathematical concepts are airborne from Olympus. Such a situation is called a *didactic* situation, because of the didactical intent of the teacher who set it up. It becomes an *adidactic* situation when the teacher definitely steps back to let the learners interact on their own with the setting, with no hope of fathoming beforehand her didactical design or the mathematical content she is aiming at. Key metaphors are likely to emerge, as sparking voltaic arcs, in and among the learners, when enough “didactical tension” is built up in an adidactic situation for them. The notion of *didactical contract* (Brousseau, Sarrazy, & Novotna, 2014) is also of interest to us, in the context of the teacher-student relation. It is in fact a keen metaphoric description of the mainly implicit and unspoken mutual expectations, beliefs and commitments regarding the actions and reactions of the partners involved in a didactic or adidactic situation.

Affect in mathematical problem solving

The role of affect in mathematical problem solving is often neglected in spite of its significant incidence in learner’s performance (Mason, Burton, & Stacey, 2003; Hannula, 2014). Here we are specially concerned by the role of negative emotions that trigger a learner’s emergent metaphorising that can transform a problem that is a tool of cognitive bullying into a friendlier one. The outcome of this may be, for most learners, a positive feeling of liberation from the Procrustean bed of arithmetic and algebra, for instance (see example 2 below).

Methodology and experimental background

Our research includes an experimental facet, where our methodology mainly relies on qualitative approaches, to wit: Case Studies (Stake, 1995), Participant Observation techniques and Ethnographic methods (Spradley, 1980).

In all, 4 cohorts of learners have been involved in our teaching and learning according to our metaphoric and enactivist approach from 2014 to 2016. They include prospective secondary school physics and mathematics teachers in a one-semester course in number theory at the University of Chile; students in a one semester first year mathematics course in the social sciences and humanities option of the Baccalaureate Programme of the same University; in service primary and secondary school teachers engaged in one week professional development workshop in the South of Chile, in service primary school teachers engaged in a 15 month professional development programme (mathematics option) at the University of Chile at Santiago; graduate students working towards a

Ph. D. in Didactics of Mathematics, at the Catholic University of Valparaiso (UCV), most of them secondary school math teachers holding a Master in Didactics of Mathematics. They were chosen because they constituted a broad spectrum of learners we had access to while performing our usual teaching duties at the University of Chile, besides some invited workshops elsewhere, with whom our overarching approach could be tested. Learners, working most of the time in (random or spontaneous) groups of 2 to 4, were observed by the teacher or facilitator and an assistant, the latter assuming the role of participant observer or ethnographer (Spradley, 1980; Brewer & Firmin, 2006). Among aspects observed were: level of participation, questions and answers, horizontal (peer) interaction, emergence of metaphors, especially idiosyncratic ones, spontaneously or under prompting, gestural language of learners and teacher, expression and explicit acknowledgement of affective reactions from the learners. Snapshots of their written output in problem solving activities were taken and videos of their enacting moments were recorded.

Illustrative examples and case studies

We present and discuss here two paradigmatic examples, in geometry and arithmetics, that we have come across during our teaching at the University of Chile, to illustrating important aspects of our theoretical perspective, often neglected in usual approaches. The case of randomness has been dealt elsewhere (Diaz-Rojas & Soto-Andrade, 2015). Our geometrical example deals with the exterior angles of a polygon and their sum: a typical geometrical notion often abusively and gratuitously introduced, with no context or motivation. In arithmetic, we recall the consecutive sums of positive integers problem, thoroughly discussed in the literature (e.g. Mason et al., 2003)

Example 1. The sum of the exterior angles of a polygon

We have observed that almost every in service and prospective secondary mathematics school teacher in our country, after introducing exterior angles coming out of the blue *after* inner angles and explaining them in terms of the latter, calculates dutifully their sum from the sum of the inner angles, that depends on the number of sides of the polygon. Doing a bit of algebra they finally wind up discovering that the sum of all exterior angles is 360° , independently of the number of sides! Surprising! This traditional way to “get into” the task (Proulx, 2013), is not very appealing for most students, that experience it as “blind calculation” (a case for cognitive bullying). When trying to fathom out exterior angles of a polygon however a first thing to do – from our perspective – would be to metaphorise it, to get into the task in a more transparent way. Not just reciting its formal definition, of course. Among the metaphors emerging amongst the learners we work with, the most frequent are “a polygon is an enclosure between crossing sticks” (most popular among primary school teachers) and “a polygon is a closed path, made out of straight segments”. Enacting the first metaphor triggers the idea of manipulating the sticks, as to make clearly visible the exterior angles first and then shifting them parallel to themselves to get smaller and smaller homothetic polygons. In this way teachers *see* that the sum of all exterior angles is 360° , instead of blindly calculating. Enaction of the metaphor “polygons are closed paths” by the learners themselves, literally lying down a polygonal path in walking, enables them to immediately “see” that the sum of the polygon’s exterior angles corresponds to a complete revolution (Diaz-Rojas & Soto-Andrade, 2015). In this way they realise that exterior angles, not inner angles, are the convenient data for the walker to inflect or bend his path as wanted. Analogously for the sum of all acute angles of a pointed star... We noticed that metaphorising a polygon is an unusual challenge, almost a violation of the

didactical contract, for both students and teachers. But once they feel they are allowed to, even prompted to, metaphors begin to arise, shyly at first. The enactive metaphoric approach conveys here a completely different experience of mathematics than the traditional one, including the role of gestures, movements and, more broadly embodiment, in the learning of mathematics, particularly in problem solving (Libedinsky & Soto-Andrade, 2015).

Example 2. Which numbers are consecutive sums? Just an arithmetic problem?

The question is: Which numbers are sums of a string of consecutive (positive) integers. An unexpected question to many learners, however familiar with Gauss well known trick to sum $1 + 2 + \dots + 100$ in a wink. From our perspective it is interesting to observe how easily this question (or any question) emerges in the learners, once their attention is drawn to this sort of sums. Our hypothesis is that learners' reaction here is heavily dependent on their previous schooling and the amount of cognitive abuse they have endured. As a typical example we recall an informal short meeting to chat about "the mathematical experience" with a class of 12th graders from a Waldorf school, to whom we told about consecutive sums (just what they are), period. After a few seconds, a girl said: Which series of numbers do you obtain that way? We claim than in usual problem solving this is an often neglected aspect: enactively letting questions emerge instead of asking them... Another often neglected aspect in problem solving is the affective reaction a problem elicits in the learner. This sort of arithmetic problem quite often triggers a feeling of distaste, especially in adolescents. This negative emotion may have the immediate positive effect of stimulating the learner to metaphorise, to transform the problem into a more attractive or friendly one, i. e. a creative reaction! This is very rarely observed in our prospective teachers and Ph. D. students. Apparently didactical contract weights heavily here: learners are not supposed to transform or metaphorise the problems they receive, nor are they supposed to *like* or *dislike* mathematical objects or procedures, just to understand them or not.

We observed that every learner tackled the problem arithmetically first, doing some experimenting (some calculating small consecutive sums, others following the opposite path: checking whether 2, 3, 4, 5, etc. might be consecutive sums). Some got a closed formula for a consecutive sum but did not see what numbers are so obtained. Those who checked numbers one by one arrived quickly to the (surprising) conjecture that powers of 2 cannot be reached. In fact they re-traced Mason et al. (2003). The proof of the conjecture remained elusive until some noticed that powers of 2 do not have odd divisors and so devised an algebraic proof of the conjecture. The fact that conversely a number which is not a power of two must be a consecutive sum remained in the shadow for 45 minutes or so. At his point, we asked prospective teachers whether they liked the way we were tackling and solving the problem. Two of them said that the conjecture was interesting and that they understood the algebraic procedure but that they were not very happy about it. Fernanda said that she was not fond of this algebraic yoga, although she was able to carry it out. For Enrique this algebraic approach was easy but he was unhappy because he had noticed (giving private lessons to secondary school students) that for most students algebraic calculations are not appealing at all. So both were motivated to look for different, may be geometric, approaches.

For Ph. D. students, didactical contract played in the opposite direction: after working on metaphorising some months before, when asked now whether they were satisfied with their discovery regarding powers of 2, several students thought about metaphorising the problem, looking

at the numbers as quantities of dots arranged in clever ways. Andrea, an insightful female student, drew a trapezoidal house with a slanted roof (of slope 1) and so transformed the problem to a question about the area of this trapezoid. Some tried to remember the area formula, but others, like three clever prospective teachers, saw by rearrangement or compensation that their trapezoid could be turned into a rectangle with the same base. But then they realised that this worked only “half of the time”: for an odd basis! In that case the area has an obvious odd divisor. For the even case, some conjectured that they could get a two-step horizontal roof, each step of equal length and so the idea emerged of slicing vertically the trapezoid into two “halves” of equal base and putting one on top of the other. Some went into distinguishing the cases: half base odd or even. But others had the idea of putting one “half trapezoid” topsy-turvy on top of the other, getting in this way a rectangle of odd height and half base. Then, a prospective teacher had the idea of borrowing from scratch a copy of the original trapezium and coupling both to obtain a rectangle with either odd base or odd height and whose area is twice the original one! The proof of the converse conjecture was left open. Regarding liking or disliking, graduate students at UCV were more enthusiastic about the geometric approach than prospective secondary teachers. Roughly two thirds of the latter said that they did not feel comfortable with geometry so that they preferred calculating algebraically! In fact even when trying to think in geometric terms, they quickly reverted to algebraic calculation. On the other hand happy visualizers realised that the trapezium area may have any value in the continuous case but not in the discrete one, because something pops up that has no continuous analogue: Parity! A surprising fact for them, who knew, from their laptop screens, that the discrete models the continuum well. The question remained open as to whether we can see geometrically that the area of a rectangular trapezium cannot be the volume of a hypercube.

From our perspective this is an emblematic example of the possible “unfolding” of an “arithmetic” problem that that can be solved by some algebraic yoga (that many students do not appreciate at all) but can also be metaphorised as a geometric problem, more appealing to others. This metaphorising “prompts” us to jump naturally into the continuous world and get some inspiration there. We realise also the hard way that there is a tricky property of discrete shapes with no analogue in the continuous world: parity! Discrete lengths, areas or volumes may be odd or even, although asymptotically however parity vanishes... Remarkably, even insightful discussions of this problem found in the literature (e. g. Mason et al., 2003) remain confined in the arithmetic-algebraic realm, not taking advantage of the avenues and possible generalisations that metaphorisation may open up.

Discussion

We have shown several important aspects of the role of metaphorising and enacting in mathematical thinking elicited by problem posing and solving. First, we have seen that the way we metaphorise and enact determines the ideas and insights we may have when tackling a problem. Then, how metaphorisation triggered by distaste of the problem may allow the learners to move from one realm to another, instead of remaining confined in just one. By so doing, they may take advantage of different intuitions and handlings, eventually much friendlier to them, that enhance their mathematical thinking and also alleviate the cognitive abuse they have been exposed to. Indeed, an acknowledged negative affective reaction to a proposed problem may trigger creative metaphorisation to change it. In this way metaphorisation appears as a means to empower students to transform an unappealing problem given to them, something especially relevant for adolescents

who otherwise have the feeling of being abused by being forced to follow prescribed rules to solve nonsensical tasks (Watson, 2008). Also visualization appears as concatenation of metaphors: In the case of consecutive sums: “numbers are quantities”, “summing is putting together”, “factorizing is rearranging to form a rectangle” etc. Furthermore, it appears that usual problem solving, as found in the literature, tends to neglect, the important role of metaphorisation and enaction, as a learner’s first reaction when tackling a problem that looks opaque to him or her. Not only because this may allow the learner to solve an otherwise unyielding problem but also because it may allow him or her not just to solve the problem but to “see” a solution, turning a hitherto blind calculation into pellucid insight. Finally the enactive and metaphoric perspective reshapes our understanding of the relation between problem and learner in problem posing and solving, that appears as a quite more circular and entangled process than usually acknowledged, where each one codetermines the other.

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